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The Zero-Temperature Limit of the Free Energy Density in Many-Electron Systems at Half-Filling

By Yohei Kashima

Abstract. We prove by means of a renormalization group method that in weakly interacting many-electron systems at half-filling on a periodic hyper-cubic lattice, the free energy density uniformly converges to an analytic function of the coupling constants in the infinite-volume, zero-temperature limit if the external magnetic field has a chessboard-like flux configuration. The spatial dimension is allowed to be any number larger than 1. The system covers the Hubbard model with a nearest-neighbor hopping term, on-site interactions, exponentially decaying density-density interactions and exponentially decaying spin-spin interactions. The magnetic field must be included in the kinetic term by the Peierls substitution. The flux configuration and the sign of the nearest-neighbor density-density/spin-spin interactions can be adjusted so that the free energy density is minimum among all the flux configurations. Consequently, the minimum free energy density is proved to converge to an analytic function of the coupling constants in the infinite-volume, zero-temperature limit. These are extension of the results on a square lattice in the preceding work ([Kashima, Y., "The special issue for the 20th anniversary", J. Math. Sci. Univ. Tokyo. **23** (2016), 1–288]). We refer to lemmas proved in the reference in order to complete the proof of the main results of this paper. So this work is a continuation of the preceding work.

Contents

1. Many-Electron Systems and the Main Results 2
1.1. Introduction 2
1.2. Hamiltonians 6
1.3. Examples 14
1.4. The main results 20
2. Multi-Band Formulation 24

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2.1. Multi-band Hamiltonian 24
2.2. Grassmann integral formulation 36
3. The Matsubara Ultra-Violet Integration 48
3.1. Covariances with the Matsubara ultra-violet cut-off 49
3.2. Isothermal bounds 56
3.3. Anisothermal bounds 68
4. The Infrared Integration 81
4.1. General lemmas 82
4.2. Completion of the infrared integration 114
Appendix A. Reordering in a Non-Commutative $\mathbb{C}$-Algebra 132
Appendix B. The Flux Phase Problem on a Periodic Hyper-Cubic Lattice 135
Appendix C. Lemmas for the Time-Continuum, Infinite-Volume Limit 141
Acknowledgments 154
Supplementary List of Notations 154
Parameters and constants 154
Sets and spaces 155
Functions and maps 156
Other notations 156
References 156

1. Many-Electron Systems and the Main Results

1.1. Introduction

Rigorous construction of many-electron systems in low temperature is a frontier of mathematical physics. Especially reaching the infinite-volume, zero-temperature limit from a formulation in finite volume and positive temperature appears to be a mathematical challenge. As considered as the simplest possible model of interacting electrons, the Hubbard models have been the central objectives in the constructive theories based on multi-scale Grassmann integration. Among them, substantial progress has been made in the zero-temperature construction of the 1-dimensional models. See [5], [6] for the latest results. As for the 2-dimensional Hubbard models, there have been attempts to develop low-temperature theories since the 2000s (see [26], [2], [3], [4], [25]). There was also a thorough construction of 2-dimensional Fermion systems in spatial continuum at zero temperature by Feldman, Knörrer and Trubowitz [10], [11]. As yet we have seen few examples of reaching the zero-temperature limit in the concrete lattice models in spatial dimension ≥ 2. One pioneering example of taking the zero-temperature limit in 2 dimension was reported by Giuliani and Mastropietro in [13] where the half-filled Hubbard model on the honeycomb lattice was specifically considered. Beneath the model-dependent details, the work of Giuliani and Mastropietro seems to suggest an effective remedy for the temperature-dependency of the constructive theories. The hint from [13] was explored and another example of the 2-dimensional Hubbard model which admits the infinite-volume, zero-temperature limit was given in our previous work [19]. In more detail the model studied in [19] was the half-filled Hubbard model on a square lattice, containing an external magnetic field whose flux is π (mod 2π) per plaquette and 0 (mod 2π) through the large circles around the periodic lattice. Recently, Giuliani and Jauslin reported a zero-temperature construction of the free energy density and the two-point Schwinger function of an interacting Fermion model on a bilayer honeycomb lattice in [12].

Since the focus of [19] was on presenting a pile of lemmas leading to the zero-temperature limit in a self-contained manner, possibility of applying its framework to other models was not fully investigated there. As a continuation of [19], here we focus on providing other examples of many-electron systems where the analyticity at zero-temperature can be proven essentially within the same framework. The main results of this paper can be seen as a generalization of the results of [19]. We will establish a theorem stating that the free energy density of a weakly interacting many-electron system at half-filling uniformly converges with respect to the amplitude of interaction in the infinite-volume, zero-temperature limit. Here we allow the spatial dimension to be any number larger than 1. The system is defined on a periodic hyper-cubic lattice. The kinetic term of the Hamiltonian is determined by the nearest-neighbor hopping of electrons and contains an external magnetic field by means of the Peierls substitution. The magnetic flux is assumed to change its sign at plaquette alternately like a chessboard. The flux π (mod 2π) per plaquette is a special case of such configurations. The magnetic flux through the large circles winding around the periodic lattice is assumed to be either uniformly 0 (mod 2π) or uniformly π (mod

2π). The interacting part of the Hamiltonian has a general form satisfying a number of invariant properties and a decay property which is faster than any polynomial order and slower than an exponential order. The interaction covers on-site interactions, exponentially decaying density-density interactions and exponentially decaying spin-spin interactions as special cases. The whole Hamiltonian has a symmetry which ensures that the system is at half-filling. The magnetic flux and the interacting term can be chosen so that the free energy density of the system is minimum among all flux configurations, according to Lieb's result on the flux phase problem ([20]). Thus, it follows that the minimum free energy density in the flux phase problem on a hyper-cubic lattice uniformly converges in the infinite-volume, zero-temperature limit. We will explain how these results generalize the main results of [19] in Remark 1.8 after officially stating the main theorem and its corollary in Subsection 1.4.

The key strategy of our construction is to view the hyper-cubic lattice as a composition of some sparser hyper-cubic lattices. The original one-band Hamiltonian is accordingly formulated into a multi-band Hamiltonian. More precisely, we transform the one-band Hamiltonian on a d-dimensional hyper-cubic lattice into a 2^d-band Hamiltonian. This procedure is a generalization of the formulation in [19] where the one-band Hamiltonian on a square lattice was formulated into a 4-band Hamiltonian. The multi-band formulation makes it feasible to study symmetric properties and spectral properties of the hopping matrix. We prove that the modulus of the band spectrum of the hopping matrix is bounded from below by a non-negative function of momentum variable vanishing at a single point. In fact the hopping matrix in momentum space fails to be invertible only at the point. Therefore, this point times zero time-momentum is the only singular point of the free covariance in the zero-temperature limit. The Hamiltonian has sufficient symmetries to guarantee that the singular point of the free covariance remains to be the singular point of the effective covariance during infrared (IR) integration. Therefore, the same renormalization technique as in [19], which was motivated by [13], applies to this model as well. The power-counting in the IR integration depends on the spatial dimension quantitatively. The power in the norm estimation of Grassmann polynomials contains the spatial dimension d as a parameter. By substituting $d = 2$ we can recover the same power-counting as in the IR integration process [19, Section 7].

However, our multi-scale integration is qualitatively unaffected by the generalization of the spatial dimension in the sense that Grassmann monomials of degree ≥ 4 are irrelevant at every iteration of the IR integration if the spatial dimension is larger than 1. We follow steps, which are seen essentially parallel to the stories of [19] in the eyes of abstraction, to complete the proof of the main theorem. We will refer to the relevant parts of [19] from time to time to fill the proofs of necessary lemmas. For this reason this work should be strictly considered as a continuation of [19].

Nonetheless the generalization of the spatial dimension and the generalization of the interaction cause some technical details to be different from the previous construction in [19]. The generalization in terms of the spatial dimension requires the multi-band formulation to be constructed inductively. This part is explained in Subsection 2.1. In addition to the new 2^d-band formulation procedure in Subsection 2.1, we will present other sections which are largely affected by the generalization of the interaction without significant omission. These are the symmetric Grassmann integral formulation in Subsection 2.2, the Matsubara ultra-violet (UV) integration in Section 3 and the time-continuum, infinite-volume limit of the truncated Grassmann integral formulation in Appendix C. Moreover, in the belief that the inductive arguments in [19, Section 7] are not seen trivial at present, we make this occasion to present a more organized version of the IR integration process than [19, Section 7] in order to convince the readers of the true validity of the mathematical renormalization group method.

As for a relevance to the contemporary physical research, one can find the Fermionic Hamiltonian with magnetic flux in a mean-field theory of the Heisenberg-Hubbard model simulating the high-Tc superconducting materials ([1]). More recently, the half-filled Hubbard model with flux π per plaquette together with the half-filled Hubbard model on the honeycomb lattice tends to be studied by means of numerical computation in order to describe the metal-insulator transition driven by the electron-electron interaction ([23], [8], [15], [28], [9], [24] and so on). These numerical studies commonly start with a speculation that in the π-flux Hubbard model at half-filling, unlike in the 0-flux Hubbard model at half-filling, the semi-metal phase remains in a weak-coupling region so that the metal-insulator transition is detectable in a middle (not the edge) of the phase diagram with the horizontal axis of the coupling strength. The main result of this paper suggests that there is no phase transition caused by the weak electron interaction not only in the π-flux Hubbard model but also in a class of electron models with staggered flux. This should provide a theoretical support for numerical studies into the metal-insulator transition away from the edge of the phase diagram in these models yet to appear in physical literature.

The contents of this paper are outlined as follows. In the rest of this section we define the Hamiltonian operators, see what kind of interaction is actually covered by our general definition and state the main results of this paper. In Section 2 we transform the one-band Hamiltonian into a multi-band Hamiltonian and formulate the multi-band Hamiltonian by means of finite-dimensional Grassmann integration. In Section 3 we construct the Matsubara UV integration both at a fixed temperature and at 2 different temperatures. In Section 4 we carry out the IR integration and complete the proof of the main theorem. In Appendix A we provide a lemma concerning reordering in a non-commutative $\mathbb{C}$-algebra, which is conveniently used in the proof that our many-electron system is at half-filling in Subsection 1.2. In Appendix B we restate Lieb's result on a d-dimensional flux phase problem in order to facilitate the derivation of the corollary about the minimum free energy density from the main theorem. Finally in Appendix C we prove that each truncation of the Taylor series of the Grassmann integral formulation of the free energy density converges in the time-continuum, infinite-volume limit. A flow chart of our construction showing the dependency between the sections of this paper and the lemmas of the previous work [19] is given in Figure 1. We also attach a list of notations for sake of the readers in the end. However, this list only contains notations which were not used in [19] or were used in [19] with different meanings and thus need additional remarks. The readers should refer to the more comprehensive list in [19] for notations which are not contained in the supplementary list of this paper.

1.2. Hamiltonians

We let the number d (≥ 2) denote the spatial dimension throughout this paper. For $L \in \mathbb{N}$ we define the d-dimensional spatial lattice $\Gamma(L)$ by $\Gamma(L) := \{0, 1, \cdots, L-1\}^d$. In this subsection we introduce a class of Hamiltonians on the Fermionic Fock space $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$. For a technical reason we define the Hamiltonians in the spatial lattice of even length $2L$. Our Hamiltonians contain an external magnetic field by means

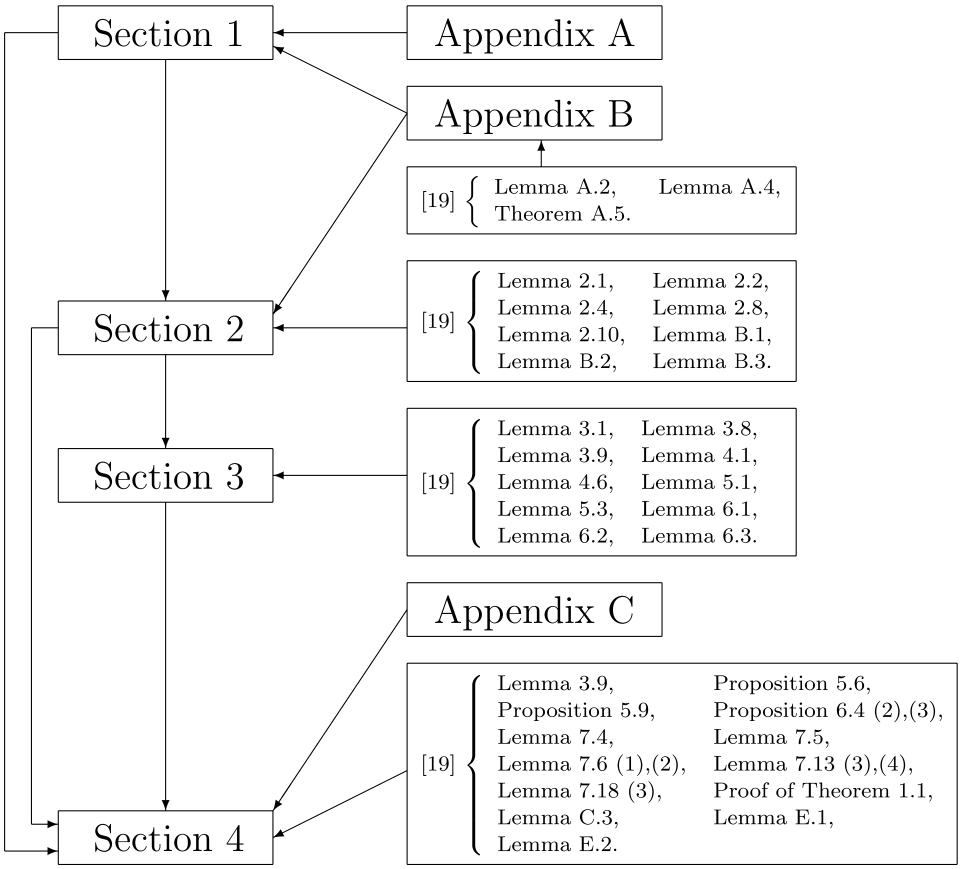

Fig. 1. Flow chart of our construction, where the arrows mean major dependency.

of the Peierls substitution. The phase $\theta_L : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is assumed to satisfy that

$$
\begin{aligned}
(1.1)\qquad & \theta_L(\mathbf{x},\mathbf{y}) = -\theta_L(\mathbf{y},\mathbf{x}) \pmod{2\pi},\\
& \theta_L\left(\mathbf{x} + 2L\sum_{j=1}^{d} m_j\mathbf{e}_j, \mathbf{y}\right) = \theta_L(\mathbf{x},\mathbf{y}) \pmod{2\pi},\\
& (\forall \mathbf{x},\mathbf{y} \in \mathbb{Z}^d, m_j \in \mathbb{Z}\ (j = 1, 2, \cdots, d)).
\end{aligned}
$$

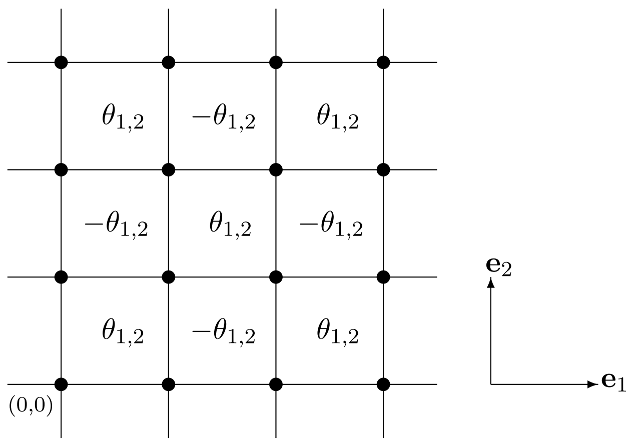

Fig. 2. The chessboard-like flux configuration.

Here $\mathbf{e}_j$ is the vector of $\mathbb{Z}^d$ whose j-th entry is 1 and the other entries are 0. The free energy of the system with the periodic boundary condition is known to be dependent on the magnetic field only by the flux per plaquette and the flux through large circles winding around the periodic lattice. Thus, it is important to specify these fluxes in advance. Let $\theta_{j,k} \in \mathbb{R}$, $\varepsilon_l^L \in \{0,1\}$ for $j,k,l \in \{1,2,\cdots,d\}$ with $j<k$. We allow ε_l^L to change its value depending on L and assume that $\varepsilon_l^1 = 0$ ($\forall l \in \{1,2,\cdots,d\}$). We assume that

$$\begin{aligned}(1.2)\qquad &\theta_L(\mathbf{x}+\mathbf{e}_j,\mathbf{x}) + \theta_L(\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j)\\ &+\theta_L(\mathbf{x}+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k)+\theta_L(\mathbf{x},\mathbf{x}+\mathbf{e}_k)\\ &=(-1)^{x_j+x_k}\theta_{j,k} \pmod{2\pi},\\ (1.3)\qquad &\sum_{m=0}^{2L-1}\theta_L(\mathbf{x}+(m+1)\mathbf{e}_l,\mathbf{x}+m\mathbf{e}_l) = \varepsilon_l^L\pi \pmod{2\pi},\\ &(\forall \mathbf{x}=(x_1,x_2,\cdots,x_d)\in\mathbb{Z}^d, j,k,l\in\{1,2,\cdots,d\} \text{ with } j<k).\end{aligned}$$

The condition (1.2) determines the flux per plaquette. When $d=2$, the condition (1.2) requires the flux per plaquette to be arranged like a chessboard as pictured in Figure 2. The condition (1.3) states that the flux through the closed contour parallel to $\mathbf{e}_l$ is $\varepsilon_l^L\pi$ (mod 2π) for any $l \in \{1,2,\cdots,d\}$.

Our analysis will be made on the quantitative assumption that

$$\frac{1}{2}\max_{m\in\{1,2,\cdots,d\}}\left(\sum_{j=1}^{m-1}|1+e^{i\theta_{j,m}}|+\sum_{j=m+1}^{d}|1+e^{i\theta_{m,j}}|\right)<1. \tag{1.4}$$

Let $t_j \in \mathbb{R}_{>0}$ $(j = 1, 2, \cdots, d)$ be the hopping amplitudes. The free Hamiltonian H_0 is defined by

$$\begin{aligned}\mathsf{H}_0 := \sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)}\sum_{\sigma\in\{\uparrow,\downarrow\}} & 1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ or }-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}\\ & \cdot t_j e^{i\theta_L(\mathbf{x},\mathbf{y})}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{y}\sigma},\end{aligned}$$

where $\psi_{\mathbf{x}\sigma}$ is the annihilation operator and $\psi^*_{\mathbf{x}\sigma}$ is its adjoint operator called creation operator. The function 1_P returns 1 if a proposition P is true, 0 otherwise. For any $\mathbf{x} \in \mathbb{Z}^d$ we define $\psi_{\mathbf{x}\sigma}$, $\psi^*_{\mathbf{x}\sigma}$ by identifying $\mathbf{x}$ with the site $\mathbf{x}'$ of $\Gamma(2L)$ satisfying that $\mathbf{x}' = \mathbf{x}$ in $(\mathbb{Z}/2L\mathbb{Z})^d$. The condition (1.1) ensures that H_0 is self-adjoint.

To define the interacting part, we introduce the kernel functions. For any set A, B let $\mathrm{Map}(A, B)$ denote the set of maps from A to B. Take $n_v \in \mathbb{N}$, $N_v \in \mathbb{N}_{\geq 2}$. We assume that $V^L_0 \in \mathrm{Map}(\mathbb{C}^{n_v}, \mathbb{C})$, $V^L_m \in \mathrm{Map}(\mathbb{C}^{n_v}, \mathrm{Map}((\mathbb{Z}^d \times \{\uparrow,\downarrow\})^m \times (\mathbb{Z}^d \times \{\uparrow,\downarrow\})^m, \mathbb{C}))$ $(m = 1, 2, \cdots, N_v)$ satisfy the following conditions.

(i)

$$\begin{aligned}&\mathbf{U} \mapsto V^L_0(\mathbf{U}) : \mathbb{C}^{n_v} \to \mathbb{C},\\ &\mathbf{U} \mapsto V^L_m(\mathbf{U}) : \mathbb{C}^{n_v} \to \mathrm{Map}((\mathbb{Z}^d \times \{\uparrow,\downarrow\})^m \times (\mathbb{Z}^d \times \{\uparrow,\downarrow\})^m, \mathbb{C})\end{aligned}$$

are linear.

(ii)

$$\begin{aligned}&V^L_m(\mathbf{U})((X_1, X_2, \cdots, X_m), (Y_1, Y_2, \cdots, Y_m))\\ &= \mathrm{sgn}(\eta)\,\mathrm{sgn}(\xi)\\ &\quad\cdot V^L_m(\mathbf{U})((X_{\eta(1)}, X_{\eta(2)}, \cdots, X_{\eta(m)}), (Y_{\xi(1)}, Y_{\xi(2)}, \cdots, Y_{\xi(m)})),\\ &(\forall X_j, Y_j \in \mathbb{Z}^d \times \{\uparrow,\downarrow\}\ (j = 1, 2, \cdots, m), \mathbf{U} \in \mathbb{C}^{n_v}, \eta, \xi \in \mathbb{S}_m),\end{aligned} \tag{1.5}$$

where $\mathbb{S}_m$ is the set of all permutations over $\{1,2,\cdots,m\}$.

(iii)

$$V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m))) \\ = (-1)^{\sum_{j=1}^m(1_{\sigma_j=\uparrow}+1_{\tau_j=\uparrow})} \\ \cdot V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m))), \\ (\forall(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\mathbb{Z}^d\times\{\uparrow,\downarrow\}\ (j=1,2,\cdots,m),\mathbf{U}\in\mathbb{C}^{n_v}). \tag{1.6}$$

(iv)

$$V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m))) \\ = V_m^L(\mathbf{U})(((\mathbf{x}_1,-\sigma_1),\cdots,(\mathbf{x}_m,-\sigma_m)),((\mathbf{y}_1,-\tau_1),\cdots,(\mathbf{y}_m,-\tau_m))), \\ (\forall(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\mathbb{Z}^d\times\{\uparrow,\downarrow\}\ (j=1,2,\cdots,m),\mathbf{U}\in\mathbb{C}^{n_v}). \tag{1.7}$$

(v)

$$V_m^L(\mathbf{U})(((\mathbf{x}_1^1,\sigma_1),\cdots,(\mathbf{x}_m^1,\sigma_m)),((\mathbf{y}_1^1,\tau_1),\cdots,(\mathbf{y}_m^1,\tau_m))) \\ = V_m^L(\mathbf{U})(((\mathbf{x}_1^2,\sigma_1),\cdots,(\mathbf{x}_m^2,\sigma_m)),((\mathbf{y}_1^2,\tau_1),\cdots,(\mathbf{y}_m^2,\tau_m))), \\ (\forall\mathbf{x}_j^1,\mathbf{x}_j^2,\mathbf{y}_j^1,\mathbf{y}_j^2\in\mathbb{Z}^d \text{ satisfying } \mathbf{x}_j^1=\mathbf{x}_j^2,\mathbf{y}_j^1=\mathbf{y}_j^2 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d, \\ \sigma_j,\tau_j\in\{\uparrow,\downarrow\}\ (j=1,2,\cdots,m),\mathbf{U}\in\mathbb{C}^{n_v}). \tag{1.8}$$

(vi)

$$V_m^L(\mathbf{U})(((\mathbf{x}_1+2\mathbf{z},\sigma_1),\cdots,(\mathbf{x}_m+2\mathbf{z},\sigma_m)), \\ ((\mathbf{y}_1+2\mathbf{z},\tau_1),\cdots,(\mathbf{y}_m+2\mathbf{z},\tau_m))) \\ = V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m))), \\ (\forall(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\mathbb{Z}^d\times\{\uparrow,\downarrow\}\ (j=1,2,\cdots,m),\mathbf{z}\in\mathbb{Z}^d,\mathbf{U}\in\mathbb{C}^{n_v}). \tag{1.9}$$

(vii)

$$V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m))) \\ = V_m^L(\mathbf{U})(((-\mathbf{x}_1,\sigma_1),\cdots,(-\mathbf{x}_m,\sigma_m)), \\ ((-\mathbf{y}_1,\tau_1),\cdots,(-\mathbf{y}_m,\tau_m))), \\ (\forall(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\mathbb{Z}^d\times\{\uparrow,\downarrow\}\ (j=1,2,\cdots,m),\mathbf{U}\in\mathbb{C}^{n_v}). \tag{1.10}$$

(viii) For any $\theta\in \mathrm{Map}(\mathbb{Z}^d,\mathbb{R})$ satisfying that $\theta(\mathbf{x})=\theta(\mathbf{y})$ ($\forall \mathbf{x},\mathbf{y}\in\mathbb{Z}^d$ with $\mathbf{x}=\mathbf{y}$ in $(\mathbb{Z}/2L\mathbb{Z})^d$),

$$\begin{aligned}(1.11)\qquad & V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m)))\\ &= e^{i(\sum_{j=1}^m\theta(\mathbf{x}_j)-\sum_{j=1}^m\theta(\mathbf{y}_j))}\\ &\quad\cdot V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m))),\\ &(\forall(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\mathbb{Z}^d\times\{\uparrow,\downarrow\}\ (j=1,2,\cdots,m),\mathbf{U}\in\mathbb{C}^{n_v}).\end{aligned}$$

(ix)

$$\begin{aligned}(1.12)\qquad & V_0^L(\mathbf{U})=\overline{V_0^L(\overline{\mathbf{U}})},\quad V_m^L(\mathbf{U})(\mathbf{X},\mathbf{Y})=\overline{V_m^L(\overline{\mathbf{U}})(\mathbf{Y},\mathbf{X})},\\ &(\forall \mathbf{X},\mathbf{Y}\in(\mathbb{Z}^d\times\{\uparrow,\downarrow\})^m,\mathbf{U}\in\mathbb{C}^{n_v}).\end{aligned}$$

(x)

$$\begin{aligned}(1.13)\qquad & V_m^L(\mathbf{U})(\mathbf{X},\mathbf{Y})\\ &=(-1)^m\sum_{l=0}^{N_v-m}\binom{m+l}{l}^2 l!\sum_{\substack{(\mathbf{z}_j,\eta_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\ (j=1,2,\cdots,l)}}\\ &\quad\cdot V_{m+l}^L(\mathbf{U})((\mathbf{X},((\mathbf{z}_1,\eta_1),(\mathbf{z}_2,\eta_2),\cdots,(\mathbf{z}_l,\eta_l))),\\ &\qquad\qquad(((\mathbf{z}_l,\eta_l),(\mathbf{z}_{l-1},\eta_{l-1}),\cdots,(\mathbf{z}_1,\eta_1)),\mathbf{Y})),\\ &(\forall \mathbf{X},\mathbf{Y}\in(\mathbb{Z}^d\times\{\uparrow,\downarrow\})^m,\mathbf{U}\in\mathbb{C}^{n_v}).\end{aligned}$$

(xi) For any $j\in\{1,2,\cdots,n_v\}$, $\mathbf{X},\mathbf{Y}\in(\mathbb{Z}^d\times\{\uparrow,\downarrow\})^m$,

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{1}{L^d}\frac{\partial}{\partial U_j}V_0^L(\mathbf{U}),\quad \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{\partial}{\partial U_j}V_m^L(\mathbf{U})(\mathbf{X},\mathbf{Y})$$

converge.

(xii) For any $c\in\mathbb{R}_{\ge 0}$,

$$(1.14)\quad \sup_{L\in\mathbb{N}}\ \sup_{\substack{\mathbf{U}\in\mathbb{C}^{n_v}\text{ with }|U_j|\le 1\\ (j=1,2,\cdots,n_v)}}\ \sup_{\substack{p,q\in\{1,2,\cdots,2m-1\}\\ k\in\{1,2,\cdots,d\}}}\ \sup_{(\mathbf{x},\sigma)\in\Gamma(2L)\times\{\uparrow,\downarrow\}}$$

$$\cdot \sum_{\substack{(\mathbf{x}_j,\sigma_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,2m-1)}}$$
$$\cdot \left(\frac{L}{\pi}|e^{i\frac{\pi}{L}\langle\mathbf{x}-\mathbf{x}_q,\mathbf{e}_k\rangle}-1|+1\right)e^{\sum_{j=1}^{d}(c\frac{L}{\pi}|e^{i\frac{\pi}{L}\langle\mathbf{x}-\mathbf{x}_p,\mathbf{e}_j\rangle}-1|)^{1/2}}$$
$$\cdot |V_m^L(\mathbf{U})(((\mathbf{x},\sigma),(\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_{m-1},\sigma_{m-1})),$$
$$((\mathbf{x}_m,\sigma_m),(\mathbf{x}_{m+1},\sigma_{m+1}),\cdots,(\mathbf{x}_{2m-1},\sigma_{2m-1})))|<\infty,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product.

For $\mathbf{U}\in\mathbb{R}^{n_v}$ we define the interacting part of the Hamiltonian by

$$\mathsf{V}:=\sum_{m=0}^{N_v}\sum_{\substack{(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,m)}} \cdot V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m))) \cdot \psi^*_{\mathbf{x}_1\sigma_1}\cdots\psi^*_{\mathbf{x}_m\sigma_m}\psi_{\mathbf{y}_1\tau_1}\cdots\psi_{\mathbf{y}_m\tau_m}. \tag{1.15}$$

By the property (1.12) the operator V is self-adjoint. The Hamiltonian H is defined by $\mathsf{H}:=\mathsf{H}_0+\mathsf{V}$. Note that H is a self-adjoint operator in the Fermionic Fock space $F_f(L^2(\Gamma(2L)\times\{\uparrow,\downarrow\}))$.

The main results of this paper concern analyticity and convergent properties of the free energy density

$$-\frac{1}{\beta(2L)^d}\log(\mathrm{Tr}\,e^{-\beta\mathsf{H}}),$$

where $\beta\in\mathbb{R}_{>0}$ is the inverse temperature. Since the phase is an important parameter, we sometimes write $\mathsf{H}_0(\theta_L)$, $\mathsf{H}(\theta_L)$ in place of H_0, H respectively. The many-electron system is half-filled in the following sense.

LEMMA 1.1. *For any* $(\mathbf{x},\sigma)\in\Gamma(2L)\times\{\uparrow,\downarrow\}$,

$$\frac{\mathrm{Tr}(e^{-\beta\mathsf{H}}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma})}{\mathrm{Tr}\,e^{-\beta\mathsf{H}}}=\frac{1}{2}.$$

PROOF. Let Ω_{2L} denote the vacuum of the Fock space $F_f(L^2(\Gamma(2L)\times\{\uparrow,\downarrow\}))$. Define the transform A on $F_f(L^2(\Gamma(2L)\times\{\uparrow,\downarrow\}))$ by

$$\begin{aligned}&A\Omega_{2L} := \prod_{\mathbf{x}\in\Gamma(2L)}(\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow})\Omega_{2L},\\ &A(\psi^*_{\mathbf{x}_1\sigma_1}\cdots\psi^*_{\mathbf{x}_n\sigma_n}\Omega_{2L})\\ &:= (-1)^{\sum_{j=1}^n\sum_{k=1}^d\langle\mathbf{x}_j,\mathbf{e}_k\rangle}\psi_{\mathbf{x}_1\sigma_1}\cdots\psi_{\mathbf{x}_n\sigma_n}\prod_{\mathbf{x}\in\Gamma(2L)}(\psi^*_{\mathbf{x}\uparrow}\psi^*_{\mathbf{x}\downarrow})\Omega_{2L}\end{aligned}$$

for any $(\mathbf{x}_j,\sigma_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}$ $(j=1,2,\cdots,n)$, and by linearity. We can see that A is a unitary transform and $A\mathsf{H}_0(\theta_L)A^*=\mathsf{H}_0(-\theta_L)$. Moreover, by using the properties (1.11), (1.13), (1.12), (1.5) and Lemma A.1 proved in Appendix A in this order,

$$\begin{aligned}&A\mathsf{V}A^*\\ &=\sum_{m=0}^{N_v}\sum_{\substack{(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,m)}}(-1)^{\sum_{j=1}^m\sum_{k=1}^d\langle\mathbf{x}_j+\mathbf{y}_j,\mathbf{e}_k\rangle}\\ &\quad\cdot V_m^L(\mathbf{U})(((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m)))\\ &\quad\cdot\psi_{\mathbf{x}_1\sigma_1}\cdots\psi_{\mathbf{x}_m\sigma_m}\psi^*_{\mathbf{y}_1\tau_1}\cdots\psi^*_{\mathbf{y}_m\tau_m}\\ &=\sum_{m=0}^{N_v}\sum_{l=0}^{N_v-m}\sum_{\substack{(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,m)}}\sum_{\substack{(\mathbf{z}_j,\eta_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,l)}}(-1)^m\begin{pmatrix}m+l\\l\end{pmatrix}^2 l!\\ &\quad\cdot V_{m+l}^L(\mathbf{U})((((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{z}_1,\eta_1),\cdots,(\mathbf{z}_l,\eta_l))),\\ &\qquad\qquad(((\mathbf{z}_l,\eta_l),\cdots,(\mathbf{z}_1,\eta_1)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m))))\\ &\quad\cdot\psi_{\mathbf{x}_1\sigma_1}\cdots\psi_{\mathbf{x}_m\sigma_m}\psi^*_{\mathbf{y}_1\tau_1}\cdots\psi^*_{\mathbf{y}_m\tau_m}\\ &=\sum_{m=0}^{N_v}\sum_{l=0}^{m}\sum_{\substack{(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,m-l)}}\sum_{\substack{(\mathbf{z}_j,\eta_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,l)}}(-1)^{m-l}\begin{pmatrix}m\\l\end{pmatrix}^2 l!\\ &\quad\cdot\overline{V_m^L(\mathbf{U})((((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_{m-l},\tau_{m-l})),((\mathbf{z}_1,\eta_1),\cdots,(\mathbf{z}_l,\eta_l))),}\\ &\qquad\qquad\overline{(((\mathbf{z}_l,\eta_l),\cdots,(\mathbf{z}_1,\eta_1)),((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_{m-l},\sigma_{m-l}))))}\\ &\quad\cdot\psi_{\mathbf{x}_1\sigma_1}\cdots\psi_{\mathbf{x}_{m-l}\sigma_{m-l}}\psi^*_{\mathbf{y}_1\tau_1}\cdots\psi^*_{\mathbf{y}_{m-l}\tau_{m-l}}\\ &=\overline{\mathsf{V}},\end{aligned}$$

where we set

$$\overline{\mathsf{V}} := \sum_{m=0}^{N_v} \sum_{\substack{(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\ (j=1,2,\cdots,m)}} \cdot \overline{V_m^L(\mathbf{U})((((\mathbf{x}_1,\sigma_1),\cdots,(\mathbf{x}_m,\sigma_m)),((\mathbf{y}_1,\tau_1),\cdots,(\mathbf{y}_m,\tau_m)))} \cdot \psi^*_{\mathbf{x}_1\sigma_1}\cdots\psi^*_{\mathbf{x}_m\sigma_m}\psi_{\mathbf{y}_1\tau_1}\cdots\psi_{\mathbf{y}_m\tau_m}.$$

Thus, we have for any $(\mathbf{x},\sigma)\in\Gamma(2L)\times\{\uparrow,\downarrow\}$ that

$$\begin{aligned}\frac{\mathrm{Tr}(e^{-\beta\mathsf{H}}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma})}{\mathrm{Tr}\,e^{-\beta\mathsf{H}}} &= \frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}_0(-\theta_L)+\overline{\mathsf{V}})}A\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma}A^*)}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}_0(-\theta_L)+\overline{\mathsf{V}})}}\\ &= 1-\frac{\mathrm{Tr}(e^{-\beta(\mathsf{H}_0(-\theta_L)+\overline{\mathsf{V}})}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma})}{\mathrm{Tr}\,e^{-\beta(\mathsf{H}_0(-\theta_L)+\overline{\mathsf{V}})}}.\end{aligned}$$

Then, by considering that

$$\begin{aligned}&\mathrm{Tr}\,e^{-\beta(\mathsf{H}_0(-\theta_L)+\overline{\mathsf{V}})} = \overline{\mathrm{Tr}\,e^{-\beta\mathsf{H}}} = \mathrm{Tr}\,e^{-\beta\mathsf{H}},\\ &\mathrm{Tr}(e^{-\beta(\mathsf{H}_0(-\theta_L)+\overline{\mathsf{V}})}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma}) = \overline{\mathrm{Tr}(e^{-\beta\mathsf{H}}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma})} = \mathrm{Tr}(e^{-\beta\mathsf{H}}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{x}\sigma}),\end{aligned}$$

we obtain the claimed equality. □

REMARK 1.2. There was unfortunately a flaw in the definition of the unitary transform in [19, Remark 1.4] which was intended to demonstrate a proof of the same claim as the above lemma. By using the unitary transform A we can correct [19, Remark 1.4]. It is simpler to confirm the equalities $A\mathsf{H}_0(\theta_L)A^* = \mathsf{H}_0(-\theta_L)$, $A\mathsf{V}A^* = \mathsf{V}$ for the free Hamiltonian $\mathsf{H}_0(\theta_L)$ and the on-site interaction V of [19]. Then, the conclusion of [19, Remark 1.4] follows from the same argument as the last part of the above proof.

1.3. Examples

Let us see that the interaction V covers some relevant models of interacting electrons. To shorten formulas, let $v_m(c)$ denote the left-hand side of the inequality (1.14) for $m\in\{1,2,\cdots,N_v\}$ and $c\in\mathbb{R}_{\ge 0}$. Moreover, set

$$v_0 := \sup_{L\in\mathbb{N}} \sup_{\substack{\mathbf{U}\in\mathbb{C}^{n_v}\text{ with}\\ |U_j|\le 1(j=1,2,\cdots,n_v)}} \frac{1}{L^d}|V_0^L(\mathbf{U})|.$$

Example 1.3 (The on-site interaction). Let $g \in \mathrm{Map}(\{1,-1\}^d, \{1,2,3,\cdots,2^d\})$. With the coupling constants $\mathbf{U}_o = (U_o(1), U_o(2), U_o(3), \cdots, U_o(2^d))$ the on-site interaction V_o is defined by

$$\mathsf{V}_o := \sum_{\mathbf{x}\in\Gamma(2L)} U_o(g((-1)^{x_1}, (-1)^{x_2}, \cdots, (-1)^{x_d})) \left(\psi^*_{\mathbf{x}\uparrow}\psi_{\mathbf{x}\uparrow} - \frac{1}{2}\right)\left(\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\downarrow} - \frac{1}{2}\right).$$

The operator V_o is equivalently written as follows.

$$\begin{aligned}\mathsf{V}_o = &\sum_{\substack{X_j,Y_j\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2)}} V^L_{o,2}(\mathbf{U}_o)((X_1,X_2),(Y_1,Y_2))\psi^*_{X_1}\psi^*_{X_2}\psi_{Y_1}\psi_{Y_2}\\ &+ \sum_{X,Y\in\Gamma(2L)\times\{\uparrow,\downarrow\}} V^L_{o,1}(\mathbf{U}_o)(X,Y)\psi^*_X\psi_Y + V^L_{o,0}(\mathbf{U}_o)\end{aligned}$$

with

$$\begin{aligned}&V^L_{o,2}(\mathbf{U}_o)(((\mathbf{x}_1,\sigma_1),(\mathbf{x}_2,\sigma_2)),((\mathbf{y}_1,\tau_1),(\mathbf{y}_2,\tau_2)))\\ &:= \frac{1}{4}U_o(g((-1)^{x_{1,1}},(-1)^{x_{1,2}},\cdots,(-1)^{x_{1,d}}))1_{\mathbf{x}_1=\mathbf{x}_2=\mathbf{y}_1=\mathbf{y}_2 \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d}\\ &\quad\cdot (1_{(\sigma_1,\sigma_2)=(\uparrow,\downarrow)} - 1_{(\sigma_1,\sigma_2)=(\downarrow,\uparrow)})(1_{(\tau_1,\tau_2)=(\downarrow,\uparrow)} - 1_{(\tau_1,\tau_2)=(\uparrow,\downarrow)}),\\ &V^L_{o,1}(\mathbf{U}_o)((\mathbf{x},\sigma),(\mathbf{y},\tau))\\ &:= -\frac{1}{2}U_o(g((-1)^{x_1},(-1)^{x_2},\cdots,(-1)^{x_d}))1_{(\mathbf{x},\sigma)=(\mathbf{y},\tau) \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d\times\{\uparrow,\downarrow\}},\\ &V^L_{o,0}(\mathbf{U}_o) := \frac{L^d}{4}\sum_{\mathbf{x}\in\{1,-1\}^d} U_o(g(\mathbf{x})).\end{aligned}$$

We can check that the kernels $V^L_{o,j}$ ($j = 0,1,2$) satisfy the conditions (i), (ii), $\cdots$, (xi) with $N_v = 2$, $n_v = 2^d$. We can estimate the factors v_0, $v_m(c)$ ($m = 1,2$) for this interaction as follows.

$$v_2(c) \le \frac{1}{2}, \quad v_1(c) \le \frac{1}{2}, \quad v_0 \le 2^{d-2}.$$

The operator $\mathsf{V}_o - V^L_{o,0}(\mathbf{U}_o)$ is also one example of the interaction V and it is equal to the interaction treated in [19] when $d = 2$ and g is bijective.

Example 1.4 (The density-density interaction). Let f_d be a real-valued continuous function on $\mathbb{R}^d$ satisfying that

$$f_d(\mathbf{0})=0,\quad |f_d(\mathbf{x})|\le c_1e^{-c_2\sum_{j=1}^d|x_j|},\ (\forall\mathbf{x}\in\mathbb{R}^d),$$

where $c_1,c_2\in\mathbb{R}_{>0}$ are fixed constants. We define the periodic function f_d^L on $\mathbb{R}^d$ by

$$f_d^L(\mathbf{x}):=f_d\left(\frac{L}{\pi}|e^{i\frac{\pi}{L}x_1}-1|,\frac{L}{\pi}|e^{i\frac{\pi}{L}x_2}-1|,\cdots,\frac{L}{\pi}|e^{i\frac{\pi}{L}x_d}-1|\right)$$

and the density-density interaction V_d by

$$\mathsf{V}_d:=U_d\sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)}f_d^L(\mathbf{x}-\mathbf{y})(\psi^*_{\mathbf{x}\uparrow}\psi_{\mathbf{x}\uparrow}+\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\downarrow}-1)(\psi^*_{\mathbf{y}\uparrow}\psi_{\mathbf{y}\uparrow}+\psi^*_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\downarrow}-1),$$

where U_d is the coupling constant. We can write as follow.

$$\begin{aligned}\mathsf{V}_d=&\sum_{\substack{X_j,Y_j\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\(j=1,2)}}V^L_{d,2}(U_d)((X_1,X_2),(Y_1,Y_2))\psi^*_{X_1}\psi^*_{X_2}\psi_{Y_1}\psi_{Y_2}\\&+\sum_{X,Y\in\Gamma(2L)\times\{\uparrow,\downarrow\}}V^L_{d,1}(U_d)(X,Y)\psi^*_X\psi_Y+V^L_{d,0}(U_d)\end{aligned}$$

with the bi-anti-symmetric kernels $V^L_{d,j}$ $(j=0,1,2)$ defined by

$$\begin{aligned}&V^L_{d,2}(U_d)((((\mathbf{x}_1,\sigma_1),(\mathbf{x}_2,\sigma_2)),((\mathbf{y}_1,\tau_1),(\mathbf{y}_2,\tau_2)))\\&:=\frac{1}{4}U_df_d^L(\mathbf{x}_1-\mathbf{x}_2)\sum_{\eta,\xi\in\mathbb{S}_2}\mathrm{sgn}(\eta)\,\mathrm{sgn}(\xi)\\&\quad\cdot 1_{(\mathbf{x}_{\eta(1)},\sigma_{\eta(1)})=(\mathbf{y}_{\xi(2)},\tau_{\xi(2)})\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d\times\{\uparrow,\downarrow\}}\\&\quad\cdot 1_{(\mathbf{x}_{\eta(2)},\sigma_{\eta(2)})=(\mathbf{y}_{\xi(1)},\tau_{\xi(1)})\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d\times\{\uparrow,\downarrow\}},\\&V^L_{d,1}(U_d)((\mathbf{x},\sigma),(\mathbf{y},\tau)):=-2U_d1_{(\mathbf{x},\sigma)=(\mathbf{y},\tau)\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d\times\{\uparrow,\downarrow\}}\sum_{\mathbf{z}\in\Gamma(2L)}f_d^L(\mathbf{z}),\\&V^L_{d,0}(U_d):=(2L)^dU_d\sum_{\mathbf{z}\in\Gamma(2L)}f_d^L(\mathbf{z}).\end{aligned}$$

The kernels $V^L_{d,j}$ $(j=0,1,2)$ satisfy the conditions (i), (ii), $\cdots$, (xi) with $N_v=2$, $n_v=1$. The factors v_0, $v_m(c)$ $(m=1,2)$ can be estimated as follows.

$$\begin{aligned}
v_2(c) &\le 2\sup_{L\in\mathbb{N}}\sup_{j'\in\{1,2,\cdots,d\}}\sum_{\mathbf{x}\in\Gamma(2L)}\\
&\quad\cdot\left(\frac{L}{\pi}|e^{i\frac{\pi}{L}x_{j'}}-1|+1\right)e^{\sum_{j=1}^{d}(c\frac{L}{\pi}|e^{i\frac{\pi}{L}x_j}-1|)^{1/2}}|f_d^L(\mathbf{x})|\\
&\le 2c_1\left(\sum_{x\in\mathbb{Z}}(|x|+1)e^{(c|x|)^{1/2}-c_2\frac{2}{\pi}|x|}\right)^d,\\
v_1(c) &\le 2\sum_{\mathbf{z}\in\Gamma(2L)}|f_d^L(\mathbf{z})|\le 2c_1\left(\sum_{x\in\mathbb{Z}}e^{-c_2\frac{2}{\pi}|x|}\right)^d,\\
v_0 &\le 2^dc_1\left(\sum_{x\in\mathbb{Z}}e^{-c_2\frac{2}{\pi}|x|}\right)^d.
\end{aligned}$$

The density-density interaction only between nearest-neighbor sites has particular importance for the flux phase problem, since it can be dealt within the framework of repeated reflection. Such a model is one special case of the interactions introduced above. To see this, let us choose a continuous function f on $[0,\infty)$ satisfying that

$$f(x)\in[0,1]\ (\forall x\in[0,\infty)),\qquad f(x)=\begin{cases}1 & \text{if } x\in[\frac{2}{\pi},1],\\ 0 & \text{if } x\in\{0\}\cup[\frac{4}{\pi},\infty)\end{cases}\tag{1.16}$$

and set

$$f_d(\mathbf{x}):=f\left(\sum_{j=1}^{d}|x_j|\right),\ (\mathbf{x}\in\mathbb{Z}^d).$$

It follows that

$$f_d(\mathbf{0})=0,\quad |f_d(\mathbf{x})|\le e^{\frac{4}{\pi}}e^{-\sum_{j=1}^{d}|x_j|},\ (\forall\mathbf{x}\in\mathbb{Z}^d).\tag{1.17}$$

Moreover, for any $\mathbf{x}\in\mathbb{Z}^d$,

$$f_d^L(\mathbf{x})=\begin{cases}1 & \text{if } \exists j\in\{1,2,\cdots,d\} \text{ s.t. } \mathbf{x}=\mathbf{e}_j \text{ or } -\mathbf{e}_j \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d,\\ 0 & \text{otherwise}\end{cases}$$

and thus,

$$\mathsf{V}_d = U_d \sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)} 1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ or }-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d} \cdot (\psi^*_{\mathbf{x}\uparrow}\psi_{\mathbf{x}\uparrow} + \psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\downarrow} - 1)(\psi^*_{\mathbf{y}\uparrow}\psi_{\mathbf{y}\uparrow} + \psi^*_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\downarrow} - 1).$$

In this case the above estimation of $v_2(c)$, $v_1(c)$, v_0 holds with $c_1 = e^{\frac{4}{\pi}}$, $c_2 = 1$.

Example 1.5 (The spin-spin interaction). Let us choose real-valued continuous functions $f_{s,j}$ $(j = 1, 2, 3)$ on $\mathbb{R}^d$ satisfying that

$$f_{s,j}(\mathbf{0}) = 0, \quad |f_{s,j}(\mathbf{x})| \le c_1 e^{-c_2\sum_{k=1}^d |x_k|},\ (\forall \mathbf{x}\in\mathbb{R}^d),$$

with constants $c_1, c_2 \in \mathbb{R}_{>0}$. Then, set

$$f^L_{s,j}(\mathbf{x}) := f_{s,j}\left(\frac{L}{\pi}|e^{i\frac{\pi}{L}x_1} - 1|, \frac{L}{\pi}|e^{i\frac{\pi}{L}x_2} - 1|, \cdots, \frac{L}{\pi}|e^{i\frac{\pi}{L}x_d} - 1|\right)$$

for $\mathbf{x}\in\mathbb{R}^d$. With the Pauli matrices

$$P^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad P^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the coupling constants $U_{s,j}$ $(j = 1, 2, 3)$, the spin-spin interaction V_s is defined as follows.

$$\begin{aligned} &\mathsf{V}_s := \sum_{j=1}^3 \mathsf{V}_{s,j}, \\ &\mathsf{V}_{s,j} := U_{s,j} \sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)} \sum_{\sigma,\tau,\mu,\lambda\in\{\uparrow,\downarrow\}} f^L_{s,j}(\mathbf{x}-\mathbf{y}) \\ &\qquad\qquad \cdot (\psi^*_{\mathbf{x}\sigma} P^{(j)}_{\sigma,\tau}\psi_{\mathbf{x}\tau})(\psi^*_{\mathbf{y}\mu} P^{(j)}_{\mu,\lambda}\psi_{\mathbf{y}\lambda}), \\ &(j = 1, 2, 3). \end{aligned} \tag{1.18}$$

The operators $\mathsf{V}_{s,j}$ $(j = 1, 2, 3)$ can be rewritten as

$$\mathsf{V}_{s,j} = \sum_{\substack{X_k, Y_k\in\Gamma(2L)\times\{\uparrow,\downarrow\} \\ (k=1,2)}} V^L_{s,j,2}(U_{s,j})((X_1, X_2), (Y_1, Y_2))\psi^*_{X_1}\psi^*_{X_2}\psi_{Y_1}\psi_{Y_2}$$

with the kernels $V^L_{s,j,2}$ $(j = 1, 2, 3)$ defined by

$$\begin{aligned}
&V^L_{s,j,2}(U_{s,j})(((\mathbf{x}_1, \sigma_1), (\mathbf{x}_2, \sigma_2)), ((\mathbf{y}_1, \tau_1), (\mathbf{y}_2, \tau_2)))\\
&:= \frac{1}{4} U_{s,j} f^L_{s,j}(\mathbf{x}_1 - \mathbf{x}_2)\\
&\quad\cdot \sum_{\eta,\xi \in \mathbb{S}_2} \mathrm{sgn}(\eta)\,\mathrm{sgn}(\xi) 1_{\mathbf{x}_{\eta(1)}=\mathbf{y}_{\xi(2)} \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d} 1_{\mathbf{x}_{\eta(2)}=\mathbf{y}_{\xi(1)} \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d}\\
&\quad\cdot P^{(j)}_{\sigma_{\eta(1)},\tau_{\xi(2)}} P^{(j)}_{\sigma_{\eta(2)},\tau_{\xi(1)}}.
\end{aligned}$$

The kernel $V^L_{s,j,2}$ satisfies (i), (ii), $\cdots$, (xi) with $N_v = 2$, $n_v = 1$ and so does the whole kernel $V^L_{s,2} = \sum_{j=1}^3 V^L_{s,j,2}$ with $N_v = 2$, $n_v = 3$. For $V^L_{s,2}$ an upper bound on $v_2(c)$ is obtained as follows.

$$\begin{aligned}
&v_2(c)\\
&\le 6 \sup_{L\in\mathbb{N}} \sup_{\substack{j'\in\{1,\cdots,d\}\\ k\in\{1,2,3\}}} \sum_{\mathbf{x}\in\Gamma(2L)} \left(\frac{L}{\pi} |e^{i\frac{\pi}{L}x_{j'}} - 1| + 1 \right) e^{\sum_{j=1}^d (c\frac{L}{\pi}|e^{i\frac{\pi}{L}x_j}-1|)^{1/2}} |f^L_{s,k}(\mathbf{x})|\\
&\le 6c_1 \left(\sum_{x\in\mathbb{Z}} (|x| + 1) e^{(c|x|)^{\frac{1}{2}} - c_2\frac{2}{\pi}|x|} \right)^d.
\end{aligned}$$

Again by using a continuous non-negative function f on $[0, \infty)$ satisfying (1.16) we can formulate the spin-spin interaction between nearest-neighbor sites. By setting

$$\begin{aligned}
&f_{s,j}(\mathbf{x}) = f\left(\sum_{k=1}^d |x_k| \right),\\
&f^L_{s,j}(\mathbf{x}) = f_{s,j}\left(\frac{L}{\pi}|e^{i\frac{\pi}{L}x_1} - 1|, \frac{L}{\pi}|e^{i\frac{\pi}{L}x_2} - 1|, \cdots, \frac{L}{\pi}|e^{i\frac{\pi}{L}x_d} - 1| \right)\\
&(j = 1, 2, 3),
\end{aligned}$$

the operator $\mathsf{V}_{s,j}$ defined in (1.18) reads

$$\begin{aligned}
\mathsf{V}_{s,j} =& U_{s,j} \sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)} \sum_{\sigma,\tau,\mu,\lambda\in\{\uparrow,\downarrow\}} 1_{\exists k\in\{1,2,\cdots,d\} \text{ s.t. } \mathbf{x}-\mathbf{y}=\mathbf{e}_k \text{ or } -\mathbf{e}_k \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d}\\
&\cdot (\psi^*_{\mathbf{x}\sigma} P^{(j)}_{\sigma,\tau} \psi_{\mathbf{x}\tau})(\psi^*_{\mathbf{y}\mu} P^{(j)}_{\mu,\lambda} \psi_{\mathbf{y}\lambda}).
\end{aligned}$$

Moreover, the upper bound on $v_2(c)$ derived above holds with $c_1 = e^{\frac{4}{\pi}}$, $c_2 = 1$ since $f_{s,j}$ satisfies (1.17) in this case.

In summary, the operator $\mathsf{V}_o + \mathsf{V}_d + \mathsf{V}_s$ is one example of the interactions treated in this paper.

1.4. The main results

For $c \in \mathbb{R}_{>0}$ let $D(c)$ denote the disk $\{z \in \mathbb{C} \mid |z| < c\}$. Recall that for $m \in \{1, 2, \cdots, N_v\}$, $v_m(c)$ denotes the left-hand side of the inequality (1.14). For any non-empty compact set K of $\mathbb{C}^{n_v}$, $C(K; \mathbb{C})$ denotes the Banach space of all complex-valued continuous functions on K, equipped with the uniform norm. Remind us that the norm of $f \in C(K; \mathbb{C})$ is equal to $\sup_{\mathbf{z} \in K} |f(\mathbf{z})|$. The following theorem is the main result of this paper.

THEOREM 1.6. *There exists a constant $c(d, N_v) \in \mathbb{R}_{>0}$ depending only on d and N_v such that the following statements hold true with the quantity R defined by*

$$
\begin{aligned}
R := & \left(\sum_{l=1}^{N_v} c(d, N_v)^l v_l(c(d, N_v)) \right)^{-1} \\
& \cdot \left(1 - \frac{1}{2} \max_{m \in \{1,2,\cdots,d\}} \left(\sum_{j=1}^{m-1} |1 + e^{i\theta_{j,m}}| + \sum_{j=m+1}^{d} |1 + e^{i\theta_{m,j}}| \right) \right)^{\frac{N_v d}{2}} \\
& \cdot \left(\min_{j \in \{1,2,\cdots,d\}} t_j \right)^{N_v d} \left(\max_{j \in \{1,2,\cdots,d\}} t_j \right)^{1 - N_v d}.
\end{aligned}
$$

(1) *There exists $F(\beta, L) \in C(\overline{D(R)}^{n_v}; \mathbb{C})$ parameterized by $\beta \in \mathbb{R}_{>0}$ and $L \in \mathbb{N}$ satisfying $L \geq \max\{t_1, t_2, \cdots, t_d\}\beta$ such that $F(\beta, L)$ is analytic in $D(R)^{n_v}$ and*

$$
\begin{aligned}
& F(\beta, L)(\mathbf{U}) = -\frac{1}{\beta(2L)^d} \log(\mathrm{Tr}\, e^{-\beta \mathsf{H}}), \\
& (\forall \mathbf{U} \in \overline{D(R)}^{n_v} \cap \mathbb{R}^{n_v}, \beta \in \mathbb{R}_{>0}, \\
& \quad L \in \mathbb{N} \text{ satisfying } L \geq \max\{t_1, t_2, \cdots, t_d\}\beta).
\end{aligned}
$$

(2) *There exists* $F(\beta)\in C(\overline{D(R)}^{n_v};\mathbb{C})$ *parameterized by* $\beta\in\mathbb{R}_{>0}$ *such that*

$$\lim_{\substack{L\to\infty,L\in\mathbb{N}\\ \text{with } L\ge\max\{t_1,t_2,\cdots,t_d\}\beta}}F(\beta,L)=F(\beta)\ \text{in}\ C(\overline{D(R)}^{n_v};\mathbb{C}).$$

(3) *There exists* $F\in C(\overline{D(R)}^{n_v};\mathbb{C})$ *such that*

$$\lim_{\beta\to\infty,\beta\in\mathbb{R}_{>0}}F(\beta)=F\ \text{in}\ C(\overline{D(R)}^{n_v};\mathbb{C}).$$

If we restrict the interaction V to have a special form and choose the phase θ_L to satisfy a certain condition, the free energy density considered in Theorem 1.6 becomes the minimum free energy in the flux phase problem. More precisely, we assume that

$$\begin{aligned}(1.19)\qquad \mathsf{V}=&U_o\sum_{\mathbf{x}\in\Gamma(2L)}\left(\psi^*_{\mathbf{x}\uparrow}\psi_{\mathbf{x}\uparrow}-\frac{1}{2}\right)\left(\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\downarrow}-\frac{1}{2}\right)\\ &+U_d\sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)}1_{\exists k\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_k\text{ or }-\mathbf{e}_k\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}\\ &\qquad\cdot(\psi^*_{\mathbf{x}\uparrow}\psi_{\mathbf{x}\uparrow}+\psi^*_{\mathbf{x}\downarrow}\psi_{\mathbf{x}\downarrow}-1)(\psi^*_{\mathbf{y}\uparrow}\psi_{\mathbf{y}\uparrow}+\psi^*_{\mathbf{y}\downarrow}\psi_{\mathbf{y}\downarrow}-1)\\ &+\sum_{j=1}^{3}U_{s,j}\sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)}\sum_{\sigma,\tau,\mu,\lambda\in\{\uparrow,\downarrow\}}\\ &\qquad\cdot 1_{\exists k\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_k\text{ or }-\mathbf{e}_k\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}\\ &\qquad\cdot(\psi^*_{\mathbf{x}\sigma}P^{(j)}_{\sigma,\tau}\psi_{\mathbf{x}\tau})(\psi^*_{\mathbf{y}\mu}P^{(j)}_{\mu,\lambda}\psi_{\mathbf{y}\lambda})\end{aligned}$$

with the Pauli matrices $P^{(j)}$ ($j=1,2,3$) and $U_o\in\mathbb{R}$, $U_d,U_{s,j}\in\mathbb{R}_{\ge 0}$ ($j=1,2,3$). The interaction V has a form to which the reflection positivity lemma [20, Lemma] is applicable. As studied in the previous subsection, the factors v_0, $v_1(c)$, $v_2(c)$ for this interaction are bounded from above by a constant depending only on c and d.

Recall that for a phase $\varphi:\mathbb{Z}^d\times\mathbb{Z}^d\to\mathbb{R}$ satisfying (1.1) we set

$$\begin{aligned}(1.20)\qquad \mathsf{H}_0(\varphi)=&\sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)}\sum_{\sigma\in\{\uparrow,\downarrow\}}1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ or }-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}\\ &\cdot t_je^{i\varphi(\mathbf{x},\mathbf{y})}\psi^*_{\mathbf{x}\sigma}\psi_{\mathbf{y}\sigma},\end{aligned}$$

and $\mathsf{H}(\varphi) = \mathsf{H}_0(\varphi) + \mathsf{V}$. The flux phase problem is to find a phase φ which minimizes the free energy $-(1/\beta)\log(\mathrm{Tr}\, e^{-\beta \mathsf{H}(\varphi)})$. Theorem B.4, which is a simple extension of Lieb's theorem [20], stated in Appendix B implies that if the phase θ_L satisfies (1.1), (1.2) with $\theta_{j,k} = \pi$ for all $j, k \in \{1, 2, \cdots, d\}$ with $j < k$ and (1.3) with $\varepsilon_l^L = 1_{L\in 2\mathbb{Z}}$ for all $l \in \{1, 2, \cdots, d\}$, then

$$\begin{aligned}&-\frac{1}{\beta}\log(\mathrm{Tr}\, e^{-\beta \mathsf{H}(\theta_L)})\\ &= \min\left\{-\frac{1}{\beta}\log(\mathrm{Tr}\, e^{-\beta \mathsf{H}(\varphi)}) \;\Big|\; \varphi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R} \text{ satisfying } (1.1)\right\}.\end{aligned}$$

Combined with Theorem 1.6, we obtain the following corollary.

COROLLARY 1.7. *There exists a constant $c(d) \in \mathbb{R}_{>0}$ depending only on d such that the following statements hold with the quantity R defined by*

$$R := c(d)\left(\min_{j\in\{1,2,\cdots,d\}} t_j\right)^{2d}\left(\max_{j\in\{1,2,\cdots,d\}} t_j\right)^{1-2d}.$$

(1) *There exists $F(\beta, L) \in C(\overline{D(R)}^5; \mathbb{C})$ parameterized by $\beta \in \mathbb{R}_{>0}$ and $L \in \mathbb{N}$ satisfying $L \ge \max\{t_1, t_2, \cdots, t_d\}\beta$ such that $F(\beta, L)$ is analytic in $D(R)^5$ and*

$$\begin{aligned}&F(\beta, L)(U_o, U_d, U_{s,1}, U_{s,2}, U_{s,3})\\ &= \min\left\{-\frac{1}{\beta(2L)^d}\log(\mathrm{Tr}\, e^{-\beta \mathsf{H}(\varphi)}) \;\Big|\; \varphi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R} \textit{ satisfying } (1.1)\right\},\\ &(\forall U_o \in \overline{D(R)} \cap \mathbb{R}, (U_d, U_{s,1}, U_{s,2}, U_{s,3}) \in \overline{D(R)}^4 \cap \mathbb{R}^4_{\ge 0},\\ &\quad \beta \in \mathbb{R}_{>0}, L \in \mathbb{N} \textit{ satisfying } L \ge \max\{t_1, t_2, \cdots, t_d\}\beta).\end{aligned}$$

(2) *There exists $F(\beta) \in C(\overline{D(R)}^5; \mathbb{C})$ parameterized by $\beta \in \mathbb{R}_{>0}$ such that*

$$\lim_{\substack{L\to\infty, L\in\mathbb{N}\\ \textit{with } L \ge \max\{t_1,t_2,\cdots,t_d\}\beta}} F(\beta, L) = F(\beta) \textit{ in } C(\overline{D(R)}^5; \mathbb{C}).$$

(3) *There exists $F \in C(\overline{D(R)}^5; \mathbb{C})$ such that*

$$\lim_{\beta\to\infty, \beta\in\mathbb{R}_{>0}} F(\beta) = F \textit{ in } C(\overline{D(R)}^5; \mathbb{C}).$$

REMARK 1.8. Let us explain how Theorem 1.6 and Corollary 1.7 generalize [19, Theorem 1.1, Corollary 1.2]. Both in Theorem 1.6 and Corollary 1.7 the spatial dimension d is any number larger than 1, while it was fixed to be 2 in [19, Theorem 1.1, Corollary 1.2]. In Theorem 1.6 we assume the flux conditions (1.2), (1.3), which are more general than the conditions [19, (1,2)] requiring that the flux per plaquette is π (mod 2π) and the flux through the large circles around the periodic square lattice is 0 (mod 2π). As we saw in Example 1.3, the interaction V covers the on-site interaction considered in [19, Theorem 1.1] as a special case. Concerning the spatial dimension and the flux configuration, therefore, Theorem 1.6 is more general than [19, Theorem 1.1]. However, here the hopping amplitude depends only on the direction and thus the whole hopping amplitudes are described by the d parameters $t_1, t_2, \cdots, t_d$, while in [19, Theorem 1.1] the hopping amplitude is constant in each direction and is allowed to vary alternately and thus the whole hopping amplitudes are described by the 4 parameters "$t_{h,e}, t_{h,o}, t_{v,e}, t_{v,o}$" as it was 2-dimensional. See [19, Figure 2] for the configuration of the hopping amplitudes. Theorem 1.6 is less general than [19, Theorem 1.1] only in this sense. In this paper we do not stick to the generalization of the hopping amplitudes in the interest of simplicity. If we assume that the hopping amplitude depends only on the direction in [19, Theorem 1.1], then the factor "$f_{\mathbf{t}}^2$" determining the possible magnitude of the coupling in [19, Theorem 1.1] becomes the factor $(\min\{t_1, t_2\})^4(\max\{t_1, t_2\})^{-3}$ included in R in Theorem 1.6. In this setting, therefore, Theorem 1.6 naturally extends [19, Theorem 1.1]. As for Corollary 1.7, the apparent generality is that the interaction includes not only the on-site interaction but also the density-density interaction and the spin-spin interaction as defined in (1.19). Moreover, the number L can be both odd and even, while it was restricted to be odd in [19, Corollary 1.2]. This generalization is due to the fact that here the magnetic flux through the large circles around the lattice can be uniformly 0 (mod 2π) or uniformly π (mod 2π) depending on the parity of L and thus the free energy density in Theorem 1.6 can be the minimum in the flux phase problem in both cases, according to the known sufficient condition restated in Theorem B.4.

REMARK 1.9. It is not trivial to make explicit the dependency of the constants $c(d, N_v)$, $c(d)$ on d, N_v. We can see from our construction that it would require a wide range of additional calculations to do so. Not to lengthen the paper further, we decide not to tackle this clarification.

Remark 1.10. The condition (1.4) requires the flux per plaquette $\theta_{j,k}$ not to vanish for any $j, k \in \{1, 2, \cdots, d\}$ with $j < k$. In 2-dimensional case the constraint (1.4) is fulfilled if $\theta_{1,2} \neq 0$ (mod 2π). This means that the infinite-volume, zero-temperature limit of the free energy density can be taken if the system contains an arbitrarily thin magnetic field having a chessboard-like flux pattern over the square lattice and the interaction is accordingly weak.

Remark 1.11. The exponent $1/2$ in (1.14) stems from the fact that we use a Gevrey-class cut-off function ϕ satisfying that

$$\sup_{x\in\mathbb{R}} |\phi^{(n)}(x)| \le 2^n (n!)^2, \ (\forall n \in \mathbb{N} \cup \{0\})$$

(see the beginning of Subsection 3.1). We can prove the similar results for the interactions satisfying (1.14) with the exponent $r \in (0, 1)$ in place of $1/2$ by using a cut-off function ϕ satisfying that

$$\sup_{x\in\mathbb{R}} |\phi^{(n)}(x)| \le const^n (n!)^{\frac{1}{r}}, \ (\forall n \in \mathbb{N} \cup \{0\}).$$

However, this generalization will bring the extra parameter r into the major part of the construction since other parameters need to be tuned depending on r. In this paper we choose not to pursue this generalization for simplicity.

2. Multi-Band Formulation

In this section we introduce a 2^d-band Hamiltonian operator whose free energy density is equal to that governed by the 1-band Hamiltonian H. Then, we will focus on the 2^d-band model and derive the finite-dimensional Grassmann integral formulation of the partition function. The Grassmann integral formulation of the 2^d-band model will be the major objective of our multi-scale analysis in the following sections.

2.1. Multi-band Hamiltonian

We will define the hopping matrix of the multi-band Hamiltonian by induction with respect to the spatial dimension. To this end, we need some notations. For $n \in \mathbb{N}$ let $\mathrm{Mat}(n, \mathbb{C})$ denote the set of all $n \times n$ complex matrices and let I_n denote the $n \times n$ unit matrix. Set

$$\Gamma_n(L) := \{0, 1, \cdots, L-1\}^n, \quad \mathcal{B}_n := \{1, 2, 3, \cdots, 2^n\}.$$

Note that for any $\rho \in \mathcal{B}_n$ there uniquely exists $(\rho_1, \rho_2, \cdots, \rho_n) \in \{0,1\}^n$ such that $\rho = \sum_{j=0}^{n-1} \rho_{j+1} 2^j + 1$. Thus, we can define $b_n \in \mathrm{Map}(\mathcal{B}_n, \{0,1\}^n)$ by $b_n(\rho) := (\rho_1, \rho_2, \cdots, \rho_n)$. The map b_n is bijective. We will suppress the index n of $\Gamma_n(L)$, $\mathcal{B}_n$, b_n after fixing n to be the spatial dimension d. We keep showing the dependency on n while we argue inductively with respect to n. For $n \in \mathbb{N}$ and $(\xi_j)_{1 \le j \le n} \in \mathbb{R}^n$ we define the matrix $U_n((\xi_j)_{1 \le j \le n}) \in \mathrm{Mat}(2^n, \mathbb{C})$ parameterized by $(\xi_j)_{1 \le j \le n}$ as follows. Set

$$U_1(\xi_1) := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\xi_1} \end{pmatrix}.$$

Assume that we have defined $U_m((\xi_j)_{1 \le j \le m}) \in \mathrm{Mat}(2^m, \mathbb{C})$. Then, define $U_{m+1}((\xi_j)_{1 \le j \le m+1}) \in \mathrm{Mat}(2^{m+1}, \mathbb{C})$ by

$$U_{m+1}((\xi_j)_{1 \le j \le m+1}) := \begin{pmatrix} U_m((\xi_j)_{1 \le j \le m}) & O \\ O & e^{i\xi_{m+1}} U_m((\xi_j)_{1 \le j \le m}) \end{pmatrix}.$$

LEMMA 2.1. *For any* $n \in \mathbb{N}$,

$$U_n((\xi_j)_{1 \le j \le n})(\rho, \eta) = e^{i \sum_{j=1}^n b_n(\rho)(j)\xi_j} \delta_{\rho,\eta}, \ (\forall \rho, \eta \in \mathcal{B}_n).$$

PROOF. The claim holds for $n = 1$ by definition. Assume that it holds for some $n \in \mathbb{N}$. Let $\rho, \eta \in \mathcal{B}_{n+1}$. If $b_{n+1}(\rho)(n+1) \neq b_{n+1}(\eta)(n+1)$, $U_{n+1}((\xi_j)_{1 \le j \le n+1})(\rho, \eta) = 0$ by definition. If $b_{n+1}(\rho)(n+1) = b_{n+1}(\eta)(n+1) = 0$, by the hypothesis of induction,

$$\begin{aligned} U_{n+1}((\xi_j)_{1 \le j \le n+1})(\rho, \eta) &= U_n((\xi_j)_{1 \le j \le n})(\rho, \eta) = e^{i \sum_{j=1}^n b_n(\rho)(j)\xi_j} \delta_{\rho,\eta} \\ &= e^{i \sum_{j=1}^{n+1} b_{n+1}(\rho)(j)\xi_j} \delta_{\rho,\eta}. \end{aligned}$$

If $b_{n+1}(\rho)(n+1) = b_{n+1}(\eta)(n+1) = 1$, by the hypothesis of induction,

$$U_{n+1}((\xi_j)_{1 \le j \le n+1})(\rho, \eta) = e^{i\xi_{n+1}} U_n((\xi_j)_{1 \le j \le n})(\rho - 2^n, \eta - 2^n)$$

$$= e^{i\sum_{j=1}^{n} b_n(\rho-2^n)(j)\xi_j + i\xi_{n+1}}\delta_{\rho-2^n,\eta-2^n}$$
$$= e^{i\sum_{j=1}^{n+1} b_{n+1}(\rho)(j)\xi_j}\delta_{\rho,\eta}.$$

Thus, the result holds for $n+1$. By induction, the claim holds for any $n \in \mathbb{N}$. □

Let $\gamma_{j,k} \in \mathbb{R}$ for $j,k \in \{1,2,\cdots,n\}$ with $j<k$. Then, let $(\gamma_{j,k})_{1\le j<k\le n}$ denote the vector

$$(\gamma_{1,2},\gamma_{1,3},\gamma_{2,3},\gamma_{1,4},\gamma_{2,4},\gamma_{3,4},\cdots,\gamma_{n-1,n}) \in \mathbb{R}^{\frac{n(n-1)}{2}}.$$

For $n \in \mathbb{N}$ we define $M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n}) \in \mathrm{Mat}(2^n,\mathbb{C})$ parameterized by $(a_j)_{1\le j\le n} \in \mathbb{C}^n$, $(\gamma_{j,k})_{1\le j<k\le n} \in \mathbb{R}^{\frac{n(n-1)}{2}}$ as follows.

$$M_1(a_1) := \begin{pmatrix} 0 & a_1 \\ \overline{a_1} & 0 \end{pmatrix}.$$

Assume that we have defined $M_m((a_j)_{1\le j\le m},(\gamma_{j,k})_{1\le j<k\le m}) \in \mathrm{Mat}(2^m,\mathbb{C})$. Then, define $M_{m+1}((a_j)_{1\le j\le m+1},(\gamma_{j,k})_{1\le j<k\le m+1}) \in \mathrm{Mat}(2^{m+1},\mathbb{C})$ by

$$M_{m+1}((a_j)_{1\le j\le m+1},(\gamma_{j,k})_{1\le j<k\le m+1})$$
$$:= \begin{pmatrix} M_m((a_j)_{1\le j\le m},(\gamma_{j,k})_{1\le j<k\le m}) & a_{m+1}U_m((\gamma_{j,m+1})_{1\le j\le m}) \\ \overline{a_{m+1}}U_m((\gamma_{j,m+1})_{1\le j\le m})^* & M_m((a_j)_{1\le j\le m},(\gamma_{j,k})_{1\le j<k\le m}) \end{pmatrix}.$$

We can see from the definition that $M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})$ is hermitian. The matrix $M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})$ is meant to be a generalization of the hopping matrix in momentum space. Before substituting the physical parameters, let us summarize its general properties. For any $M \in \mathrm{Mat}(n,\mathbb{C})$ let $\|M\|_{n\times n}$ denote its operator norm $\sup_{\mathbf{v}\in\mathbb{C}^n \text{ with } \|\mathbf{v}\|_{\mathbb{C}^n}=1}\|M\mathbf{v}\|_{\mathbb{C}^n}$.

LEMMA 2.2.

(1) *For any* $\rho,\eta \in \mathcal{B}_n$,

$$(2.1)\quad M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})(\rho,\eta)$$
$$= 1_{\exists j\in\{1,2,\cdots,n\} \text{ s.t. } b_n(\rho)(j)<b_n(\eta)(j)\wedge\ b_n(\rho)(k)=b_n(\eta)(k)\ (\forall k\in\{1,2,\cdots,n\}\setminus\{j\})}$$
$$\cdot e^{i1_{j\ge 2}\sum_{l=1}^{j-1}b_n(\rho)(l)\gamma_{l,j}}a_j$$
$$+ 1_{\exists j\in\{1,2,\cdots,n\} \text{ s.t. } b_n(\rho)(j)>b_n(\eta)(j)\wedge\ b_n(\rho)(k)=b_n(\eta)(k)\ (\forall k\in\{1,2,\cdots,n\}\setminus\{j\})}$$
$$\cdot e^{-i1_{j\ge 2}\sum_{l=1}^{j-1}b_n(\eta)(l)\gamma_{l,j}}\overline{a_j}.$$

(2) *For any* $(\xi_j)_{1\le j\le n}\in\mathbb{R}^n$,

$$U_n((\xi_j)_{1\le j\le n})M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})U_n((\xi_j)_{1\le j\le n})^*\\ = M_n((e^{-i\xi_j}a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n}).$$

(3)

$$\|M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})\|_{2^n\times 2^n}\le\sum_{j=1}^n|a_j|.$$

(4)

$$\inf_{\substack{\mathbf{v}\in\mathbb{C}^{2^n}\ with\\ \|\mathbf{v}\|_{\mathbb{C}^{2^n}}=1}}\|M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})\mathbf{v}\|^2_{\mathbb{C}^{2^n}}\\ \ge\left(1-1_{n\ge 2}\frac{1}{2}\max_{m\in\{1,2,\cdots,n\}}\left(\sum_{j=1}^{m-1}|1+e^{i\gamma_{j,m}}|+\sum_{j=m+1}^{n}|1+e^{i\gamma_{m,j}}|\right)\right)\\ \cdot\sum_{j=1}^n|a_j|^2.$$

Proof. (1): Assume that the result is true for $\rho,\eta\in\mathcal{B}_n$ with $\rho\le\eta$. Then, the result for $\rho,\eta\in\mathcal{B}_n$ with $\rho>\eta$ follows from the hermiticity of M_n. Thus, it suffices to prove the equality for $\rho,\eta\in\mathcal{B}_n$ with $\rho\le\eta$. It holds for $n=1$ by definition. Assume that it is true for some $n\in\mathbb{N}$. Take $\rho,\eta\in\mathcal{B}_{n+1}$ satisfying $\rho\le\eta$. It follows that $b_{n+1}(\rho)(n+1)\le b_{n+1}(\eta)(n+1)$. If $b_{n+1}(\rho)(n+1)=b_{n+1}(\eta)(n+1)$, by setting $m:=b_{n+1}(\rho)(n+1)2^n$ we see that

$$M_{n+1}((a_j)_{1\le j\le n+1},(\gamma_{j,k})_{1\le j<k\le n+1})(\rho,\eta)\\ =M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})(\rho-m,\eta-m)\\ =1_{\exists j\in\{1,2,\cdots,n\}\text{ s.t. }b_n(\rho-m)(j)<b_n(\eta-m)(j)\wedge\ b_n(\rho-m)(k)=b_n(\eta-m)(k)\ (\forall k\in\{1,2,\cdots,n\}\setminus\{j\})}\\ \cdot e^{i1_{j\ge 2}\sum_{l=1}^{j-1}b_n(\rho-m)(l)\gamma_{l,j}}a_j\\ =1_{\exists j\in\{1,2,\cdots,n+1\}\text{ s.t. }b_{n+1}(\rho)(j)<b_{n+1}(\eta)(j)\wedge\ b_{n+1}(\rho)(k)=b_{n+1}(\eta)(k)\ (\forall k\in\{1,2,\cdots,n+1\}\setminus\{j\})}\\ \cdot e^{i1_{j\ge 2}\sum_{l=1}^{j-1}b_{n+1}(\rho)(l)\gamma_{l,j}}a_j\\ =(\text{the right-hand side of }(2.1)).$$

If $b_{n+1}(\rho)(n+1) < b_{n+1}(\eta)(n+1)$, by Lemma 2.1,

$$\begin{aligned}
&M_{n+1}((a_j)_{1\le j\le n+1},(\gamma_{j,k})_{1\le j<k\le n+1})(\rho,\eta)\\
&= a_{n+1}U_n((\gamma_{j,n+1})_{1\le j\le n})(\rho,\eta-b_{n+1}(\eta)(n+1)2^n)\\
&= e^{i\sum_{l=1}^n b_n(\rho)(l)\gamma_{l,n+1}}\delta_{\rho,\eta-b_{n+1}(\eta)(n+1)2^n}a_{n+1}\\
&= 1_{b_{n+1}(\rho)(k)=b_{n+1}(\eta)(k)\ (\forall k\in\{1,2,\cdots,n\})}e^{i\sum_{l=1}^n b_n(\rho)(l)\gamma_{l,n+1}}a_{n+1}\\
&= (\text{the right-hand side of (2.1)}).
\end{aligned}$$

Thus, the results hold for $n+1$. The induction with n proves the claim for any $n\in\mathbb{N}$.

(2): The equality for $n=1$ can be confirmed by a direct calculation. Assume that it is true for some $n\in\mathbb{N}$. By the definition and the hypothesis of induction,

$$\begin{aligned}
&U_{n+1}((\xi_j)_{1\le j\le n+1})M_{n+1}((a_j)_{1\le j\le n+1},(\gamma_{j,k})_{1\le j<k\le n+1})U_{n+1}((\xi_j)_{1\le j\le n+1})^*\\
&=\begin{pmatrix} U_n((\xi_j)_{1\le j\le n})M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})U_n((\xi_j)_{1\le j\le n})^* & e^{-i\xi_{n+1}}a_{n+1}U_n((\gamma_{j,n+1})_{1\le j\le n})\\ e^{i\xi_{n+1}}\overline{a_{n+1}}U_n((\gamma_{j,n+1})_{1\le j\le n})^* & U_n((\xi_j)_{1\le j\le n})M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})U_n((\xi_j)_{1\le j\le n})^*\end{pmatrix}\\
&= M_{n+1}((e^{-i\xi_j}a_j)_{1\le j\le n+1},(\gamma_{j,k})_{1\le j<k\le n+1}).
\end{aligned}$$

Thus, by induction the equality holds for any $n\in\mathbb{N}$.

(3): We can see from the definition that the inequality holds for $n=1$. Assume that it holds for some $n\in\mathbb{N}$. By the unitary property of $U_n((\gamma_{j,n+1})_{1\le j\le n})$ and the claim (2) we have that

$$\begin{aligned}
&M_{n+1}((a_j)_{1\le j\le n+1},(\gamma_{j,k})_{1\le j<k\le n+1})^2 =\\
&\begin{pmatrix} M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})^2+|a_{n+1}|^2I_{2^n} & a_{n+1}M_n(((1+e^{-i\gamma_{j,n+1}})a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})U_n((\gamma_{j,n+1})_{1\le j\le n})\\ \overline{a_{n+1}}U_n((\gamma_{j,n+1})_{1\le j\le n})^*M_n(((1+e^{-i\gamma_{j,n+1}})a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n}) & M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})^2+|a_{n+1}|^2I_{2^n}\end{pmatrix}.
\end{aligned}$$

Thus, for any $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{C}^{2^n}$,

$$\left\|M_{n+1}((a_j)_{1\le j\le n+1},(\gamma_{j,k})_{1\le j<k\le n+1})\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}\right\|_{\mathbb{C}^{2^{n+1}}}^2 = \tag{2.2}$$

$$\sum_{l=1}^{2}\langle \mathbf{v}_l, M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})^2\mathbf{v}_l + |a_{n+1}|^2\mathbf{v}_l\rangle_{\mathbb{C}^{2^n}}$$
$$+ 2\,\mathrm{Re}\,\langle \mathbf{v}_1,$$
$$a_{n+1}M_n(((1+e^{-i\gamma_{j,n+1}})a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})$$
$$\cdot U_n((\gamma_{j,n+1})_{1\le j\le n})\mathbf{v}_2\rangle_{\mathbb{C}^{2^n}}.$$

It follows from (2.2) and the hypothesis of induction that

$$\begin{aligned}
&\|M_{n+1}((a_j)_{1\le j\le n+1},(\gamma_{j,k})_{1\le j<k\le n+1})\|_{2^{n+1}\times 2^{n+1}}^2\\
&\le \|M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})\|_{2^n\times 2^n}^2 + |a_{n+1}|^2\\
&\quad + 2|a_{n+1}|\|M_n(((1+e^{-i\gamma_{j,n+1}})a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})\|_{2^n\times 2^n}\\
&\qquad \cdot \sup_{\substack{\mathbf{v}_1,\mathbf{v}_2\in\mathbb{C}^{2^n}\text{ with}\\ \|\mathbf{v}_1\|_{\mathbb{C}^{2^n}}^2+\|\mathbf{v}_2\|_{\mathbb{C}^{2^n}}^2=1}}\|\mathbf{v}_1\|_{\mathbb{C}^{2^n}}\|\mathbf{v}_2\|_{\mathbb{C}^{2^n}}\\
&\le \|M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})\|_{2^n\times 2^n}^2 + |a_{n+1}|^2\\
&\quad + |a_{n+1}|\|M_n(((1+e^{-i\gamma_{j,n+1}})a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})\|_{2^n\times 2^n}\\
&\le \left(\sum_{j=1}^{n}|a_j|\right)^2 + |a_{n+1}|^2 + |a_{n+1}|\sum_{j=1}^{n}|1+e^{i\gamma_{j,n+1}}||a_j|\\
&\le \left(\sum_{j=1}^{n+1}|a_j|\right)^2.
\end{aligned}$$

Thus, the inequality holds for $n+1$. The induction with n ensures the result.

(4): First let us prove that

$$\inf_{\substack{\mathbf{v}\in\mathbb{C}^{2^n}\text{ with}\\ \|\mathbf{v}\|_{\mathbb{C}^{2^n}}=1}}\|M_n((a_j)_{1\le j\le n},(\gamma_{j,k})_{1\le j<k\le n})\mathbf{v}\|_{\mathbb{C}^{2^n}}^2 \ge \sum_{j=1}^{n}|a_j|^2 - 1_{n\ge 2}\sum_{m=2}^{n}|a_m|\sum_{j=1}^{m-1}|1+e^{i\gamma_{j,m}}||a_j|. \tag{2.3}$$

We can check that the inequality (2.3) holds for $n=1$. Assume that it holds for some $n\in\mathbb{N}$. By (2.2), the induction hypothesis and the claim (3),

$$\inf_{\mathbf{v}\in\mathbb{C}^{2^{n+1}}\text{ with }\|\mathbf{v}\|_{\mathbb{C}^{2^{n+1}}}=1}\|M_{n+1}((a_j)_{1\le j\le n+1},(\gamma_{j,k})_{1\le j<k\le n+1})\mathbf{v}\|_{\mathbb{C}^{2^{n+1}}}^2$$

$$\geq \inf_{\substack{\mathbf{v}\in\mathbb{C}^{2^n} \text{ with} \\ \|\mathbf{v}\|_{\mathbb{C}^{2^n}}=1}} \|M_n((a_j)_{1\leq j\leq n}, (\gamma_{j,k})_{1\leq j<k\leq n})\mathbf{v}\|^2_{\mathbb{C}^{2^n}}$$
$$+ |a_{n+1}|^2 - |a_{n+1}|\|M_n(((1+e^{-i\gamma_{j,n+1}})a_j)_{1\leq j\leq n}, (\gamma_{j,k})_{1\leq j<k\leq n})\|_{2^n\times 2^n}$$
$$\geq \sum_{j=1}^n |a_j|^2 - 1_{n\geq 2}\sum_{m=2}^n |a_m|\sum_{j=1}^{m-1}|1+e^{i\gamma_{j,m}}||a_j|$$
$$+ |a_{n+1}|^2 - |a_{n+1}|\sum_{j=1}^n |1+e^{i\gamma_{j,n+1}}||a_j|$$
$$= \sum_{j=1}^{n+1} |a_j|^2 - \sum_{m=2}^{n+1} |a_m|\sum_{j=1}^{m-1}|1+e^{i\gamma_{j,m}}||a_j|.$$

Thus, the inequality (2.3) holds for $n+1$. By induction it holds true for any $n\in\mathbb{N}$.

Define $S\in \mathrm{Mat}(n,\mathbb{C})$ by

$$S(j,k) := \begin{cases} \frac{1}{2}|1+e^{i\gamma_{j,k}}| & \text{if } j<k, \\ \frac{1}{2}|1+e^{i\gamma_{k,j}}| & \text{if } j>k, \\ 0 & \text{if } j=k. \end{cases}$$

It follows from the inequality (2.3) that

$$\inf_{\substack{\mathbf{v}\in\mathbb{C}^{2^n} \text{ with} \\ \|\mathbf{v}\|_{\mathbb{C}^{2^n}}=1}} \|M_n((a_j)_{1\leq j\leq n}, (\gamma_{j,k})_{1\leq j<k\leq n})\mathbf{v}\|^2_{\mathbb{C}^{2^n}}$$
$$\geq \sum_{j=1}^n |a_j|^2 - 1_{n\geq 2}\sum_{j=1}^n\sum_{k=1}^n S(j,k)|a_j||a_k| \geq (1-1_{n\geq 2}\|S\|_{n\times n})\sum_{j=1}^n |a_j|^2.$$

It remains to prove that

$$\|S\|_{n\times n} \leq \frac{1}{2}\max_{m\in\{1,2,\cdots,n\}}\left(\sum_{j=1}^{m-1}|1+e^{i\gamma_{j,m}}| + \sum_{j=m+1}^{n}|1+e^{i\gamma_{m,j}}|\right). \tag{2.4}$$

Though the inequality of this form is well-known (see e.g. [7, Lemma 3.1.1]), we give the proof for completeness. Let $\alpha\in\mathbb{R}$ be an eigen value of S such that $|\alpha| = \|S\|_{n\times n}$. Let $\mathbf{v} = (v_1,\cdots,v_n)^t \in \mathbb{C}^n$ be its eigen vector. We can choose $l \in \{1,2,\cdots,n\}$ so that $|v_l| = \max_{j\in\{1,2,\cdots,n\}}|v_j|$. Then,

$$\|S\|_{n\times n} = \left|\frac{1}{v_l}\sum_{j=1}^{n} S(l,j)v_j\right| \le \sum_{j=1}^{n}|S(l,j)| \le \max_{m\in\{1,2,\cdots,n\}}\sum_{j=1}^{n}|S(m,j)|,$$

which is (2.4). □

Now we fix $d \in \mathbb{N}_{\ge 2}$ and use the notations $\Gamma(L)$, $\mathcal{B}$, b instead of $\Gamma_d(L)$, $\mathcal{B}_d$, b_d respectively. Here we formulate the hopping matrix of our multi-band model. Set $\boldsymbol{\pi} := (\pi,\pi,\cdots,\pi) \in \mathbb{R}^d$. For parameters $\boldsymbol{\varepsilon} = (\varepsilon_j)_{1\le j\le d} \in \mathbb{R}^d$, $\boldsymbol{\gamma} = (\gamma_{j,k})_{1\le j<k\le d} \in \mathbb{R}^{d(d-1)/2}$ we define $E(\boldsymbol{\varepsilon},\boldsymbol{\gamma}) \in \mathrm{Map}(\mathbb{R}^d, \mathrm{Mat}(2^d,\mathbb{C}))$ by

$$E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})(\mathbf{k}) := M_d\left(\left(t_j(1+e^{i\frac{\pi}{L}\varepsilon_j - ik_j})\left(\frac{1}{2}\right)^{1_{L=1}}\right)_{1\le j\le d}, -\boldsymbol{\gamma}\right), \quad (\mathbf{k}\in\mathbb{R}^d).$$

We will see that $E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})$ is equal to the hopping matrix of our multi-band Hamiltonian in momentum space if we replace $\boldsymbol{\varepsilon},\boldsymbol{\gamma}$ by the actual parameters. The next lemma follows from Lemma 2.2 and the definition of $E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})$.

LEMMA 2.3. *The following statements hold for any* $\mathbf{k}\in\mathbb{R}^d$, $\boldsymbol{\varepsilon}\in\mathbb{R}^d$, $\boldsymbol{\gamma}\in\mathbb{R}^{d(d-1)/2}$.

(1)

$$U_d(\boldsymbol{\pi})E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})(\mathbf{k})U_d(\boldsymbol{\pi})^* = -E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})(\mathbf{k}).$$

(2)

$$U_d\left(-\frac{\pi}{L}\boldsymbol{\varepsilon}\right)U_d(\mathbf{k})E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})\left(-\mathbf{k}+\frac{2\pi}{L}\boldsymbol{\varepsilon}\right)U_d(\mathbf{k})^*U_d\left(-\frac{\pi}{L}\boldsymbol{\varepsilon}\right)^*$$
$$= E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})(\mathbf{k}).$$

(3)

$$\|E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})(\mathbf{k})\|_{2^d\times 2^d} \le 2\sum_{j=1}^{d} t_j.$$

(4)

$$\left\|\left(\frac{\partial}{\partial k_j}\right)^m E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})(\mathbf{k})\right\|_{2^d\times 2^d} \le t_j,\ \ (\forall j\in\{1,2,\cdots,d\}, m\in\mathbb{N}_{\ge 1}).$$

(5)

$$\begin{aligned}&\inf_{\substack{\mathbf{v}\in\mathbb{C}^{2^d}\ with\\ \|\mathbf{v}\|_{\mathbb{C}^{2^d}}=1}}\|E(\boldsymbol{\varepsilon},\boldsymbol{\gamma})(\mathbf{k})\mathbf{v}\|_{\mathbb{C}^{2^d}}\\
&\ge\left(1-\frac{1}{2}\max_{m\in\{1,2,\cdots,d\}}\left(\sum_{j=1}^{m-1}|1+e^{i\gamma_{j,m}}|+\sum_{j=m+1}^{d}|1+e^{i\gamma_{m,j}}|\right)\right)^{\frac{1}{2}}\\
&\quad\cdot\frac{1}{2}\min_{j\in\{1,2,\cdots,d\}}t_j\cdot\left(\sum_{l=1}^{d}|1+e^{i\frac{\pi}{L}\varepsilon_l-ik_l}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

We define $\nu\in\mathrm{Map}(\mathcal{B}\times\Gamma(L),\Gamma(2L))$ by $\nu((\rho,\mathbf{x})):=2\mathbf{x}+b(\rho)$. Note that ν is bijective. The momentum lattice $\Gamma(L)^*$, dual to $\Gamma(L)$, is defined by

$$\Gamma(L)^*:=\left\{0,\frac{2\pi}{L},\cdots,\frac{2\pi}{L}(L-1)\right\}^d.$$

With the physical parameters $\varepsilon_l^L\in\{0,1\}$, $\theta_{j,k}\in\mathbb{R}$ ($l,j,k\in\{1,2,\cdots,d\}$ with $j<k$) introduced in Subsection 1.2, we set $\boldsymbol{\varepsilon}^L:=(\varepsilon_j^L)_{1\le j\le d}$, $\boldsymbol{\theta}:=(\theta_{j,k})_{1\le j<k\le d}$. Then, we define $F(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})\in\mathrm{Map}((\mathcal{B}\times\Gamma(L))^2,\mathbb{C})$, $G(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})\in\mathrm{Map}(\Gamma(2L)^2,\mathbb{C})$, which formulate the hopping matrices, as follows.

$$\begin{aligned}&F(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})((\rho,\mathbf{x}),(\eta,\mathbf{y})):=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}E(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{k})(\rho,\eta),\\
&(\forall(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma(L)),\\
&G(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{x},\mathbf{y}):=F(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\nu^{-1}(\mathbf{x}),\nu^{-1}(\mathbf{y})),\quad(\forall\mathbf{x},\mathbf{y}\in\Gamma(2L)).\end{aligned}$$

Moreover, we define $\varphi\in\mathrm{Map}(\mathbb{Z}^d\times\mathbb{Z}^d,\mathbb{R})$ by

$$\varphi(\mathbf{x},\mathbf{y}):=\begin{cases}(-1)^{x_j+1}1_{j\ge2}\sum_{l=1}^{j-1}1_{x_l\in2\mathbb{Z}+1}\theta_{l,j}+1_{x_j\in2\mathbb{Z}}\frac{\pi}{L}\varepsilon_j^L\\ \quad\text{if }\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d,\\ (-1)^{x_j+1}1_{j\ge2}\sum_{l=1}^{j-1}1_{x_l\in2\mathbb{Z}+1}\theta_{l,j}-1_{x_j\in2\mathbb{Z}+1}\frac{\pi}{L}\varepsilon_j^L\\ \quad\text{if }\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d,\\ 0\text{ otherwise.}\end{cases}$$

Note that φ satisfies (1.1).

LEMMA 2.4.

(1)
$$G(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{x},\mathbf{y}) = |G(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{x},\mathbf{y})|e^{i\varphi(\mathbf{x},\mathbf{y})},\ (\forall \mathbf{x},\mathbf{y}\in\Gamma(2L)).$$

(2)
$$\begin{aligned}&|G(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{x},\mathbf{y})|\\ &=\begin{cases} t_j & \text{if } \exists j\in\{1,2,\cdots,d\} \text{ s.t. } \mathbf{x}-\mathbf{y}=\mathbf{e}_j \text{ or } -\mathbf{e}_j \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d,\\ 0 & \text{otherwise,}\end{cases}\\ &(\forall \mathbf{x},\mathbf{y}\in\Gamma(2L)).\end{aligned}$$

(3)
$$\begin{aligned}&\varphi(\mathbf{x}+\mathbf{e}_j,\mathbf{x})+\varphi(\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j)+\varphi(\mathbf{x}+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k)\\ &\quad+\varphi(\mathbf{x},\mathbf{x}+\mathbf{e}_k)\\ &=(-1)^{x_j+x_k}\theta_{j,k},\quad (\forall \mathbf{x}\in\mathbb{Z}^d, j,k\in\{1,2,\cdots,d\} \text{ with } j<k).\end{aligned}$$

(4)
$$\sum_{m=0}^{2L-1}\varphi(\mathbf{x}+(m+1)\mathbf{e}_j,\mathbf{x}+m\mathbf{e}_j)=\varepsilon_j^L\pi,\quad (\forall \mathbf{x}\in\mathbb{Z}^d, j\in\{1,2,\cdots,d\}).$$

PROOF. (1), (2): Take $\mathbf{x},\mathbf{y}\in\Gamma(2L)$. Let $(\rho,\hat{\mathbf{x}}),(\eta,\hat{\mathbf{y}})$ $(\in\mathcal{B}\times\Gamma(L))$ be such that $(\rho,\hat{\mathbf{x}})=\nu^{-1}(\mathbf{x})$, $(\eta,\hat{\mathbf{y}})=\nu^{-1}(\mathbf{y})$. Moreover, let $b(\rho)=(\rho_1,\rho_2,\cdots,\rho_d)$, $b(\eta)=(\eta_1,\eta_2,\cdots,\eta_d)$. By Lemma 2.2 (1) and the assumption that $\varepsilon_l^1=0$ $(\forall l\in\{1,2,\cdots,d\})$,

$$\begin{aligned}&G(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{x},\mathbf{y})\\ &=\frac{1}{L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}e^{i\langle\hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{k}\rangle}\Bigg(1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\rho_j<\eta_j\wedge\rho_m=\eta_m\ (\forall m\in\{1,2,\cdots,d\}\setminus\{j\})}\\ &\quad\cdot e^{-i1_{j\ge2}\sum_{l=1}^{j-1}\rho_l\theta_{l,j}}t_j(1+e^{i\frac{\pi}{L}\varepsilon_j^L-ik_j})\left(\frac{1}{2}\right)^{1_{L=1}}\\ &\quad+1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\rho_j>\eta_j\wedge\rho_m=\eta_m\ (\forall m\in\{1,2,\cdots,d\}\setminus\{j\})}\end{aligned}$$

$$\cdot e^{i1_{j\ge 2}\sum_{l=1}^{j-1}\eta_l\theta_{l,j}}t_j(1+e^{-i\frac{\pi}{L}\varepsilon_j^L+ik_j})\left(\frac{1}{2}\right)^{1_{L=1}}\Bigg)$$

$$\begin{aligned}
&= 1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\rho_j<\eta_j\wedge\rho_m=\eta_m\ (\forall m\in\{1,2,\cdots,d\}\backslash\{j\})}\\
&\quad\cdot e^{-i1_{j\ge 2}\sum_{l=1}^{j-1}\rho_l\theta_{l,j}}t_j\left(1_{\hat{x}_j=\hat{y}_j}+e^{i\frac{\pi}{L}\varepsilon_j^L}1_{\hat{x}_j=\hat{y}_j+1\ (\mathrm{mod}L)}\right)\left(\frac{1}{2}\right)^{1_{L=1}}\\
&\quad\cdot\prod_{\substack{m=1\\m\neq j}}^{d}1_{\hat{x}_m=\hat{y}_m}\\
&\quad+1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\rho_j>\eta_j\wedge\rho_m=\eta_m\ (\forall m\in\{1,2,\cdots,d\}\backslash\{j\})}\\
&\quad\cdot e^{i1_{j\ge 2}\sum_{l=1}^{j-1}\eta_l\theta_{l,j}}t_j\left(1_{\hat{x}_j=\hat{y}_j}+e^{-i\frac{\pi}{L}\varepsilon_j^L}1_{\hat{x}_j=\hat{y}_j-1\ (\mathrm{mod}L)}\right)\left(\frac{1}{2}\right)^{1_{L=1}}\\
&\quad\cdot\prod_{\substack{m=1\\m\neq j}}^{d}1_{\hat{x}_m=\hat{y}_m}\\
&= 1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ or }-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d\wedge x_j\in 2\mathbb{Z}}e^{-i1_{j\ge 2}\sum_{l=1}^{j-1}1_{x_l\in 2\mathbb{Z}+1}\theta_{l,j}}t_j\\
&\quad\cdot\left(1_{\mathbf{x}-\mathbf{y}=-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}+1_{\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}e^{i\frac{\pi}{L}\varepsilon_j^L}\right)\left(\frac{1}{2}\right)^{1_{L=1}}\\
&\quad+1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ or }-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d\wedge x_j\in 2\mathbb{Z}+1}e^{i1_{j\ge 2}\sum_{l=1}^{j-1}1_{x_l\in 2\mathbb{Z}+1}\theta_{l,j}}t_j\\
&\quad\cdot\left(1_{\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}+1_{\mathbf{x}-\mathbf{y}=-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}e^{-i\frac{\pi}{L}\varepsilon_j^L}\right)\left(\frac{1}{2}\right)^{1_{L=1}}\\
&= 1_{L=1}1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}e^{i(-1)^{x_j+1}1_{j\ge 2}\sum_{l=1}^{j-1}1_{x_l\in 2\mathbb{Z}+1}\theta_{l,j}}t_j\\
&\quad+1_{L\ge 2}\Big(1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}\\
&\qquad\qquad\cdot e^{i(-1)^{x_j+1}1_{j\ge 2}\sum_{l=1}^{j-1}1_{x_l\in 2\mathbb{Z}+1}\theta_{l,j}+i1_{x_j\in 2\mathbb{Z}}\frac{\pi}{L}\varepsilon_j^L}t_j\\
&\qquad\quad+1_{\exists j\in\{1,2,\cdots,d\}\text{ s.t. }\mathbf{x}-\mathbf{y}=-\mathbf{e}_j\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d}\\
&\qquad\qquad\cdot e^{i(-1)^{x_j+1}1_{j\ge 2}\sum_{l=1}^{j-1}1_{x_l\in 2\mathbb{Z}+1}\theta_{l,j}-i1_{x_j\in 2\mathbb{Z}+1}\frac{\pi}{L}\varepsilon_j^L}t_j\Big).
\end{aligned}$$

This implies the claims (1), (2).

(3): Take $j,k\in\{1,2,\cdots,d\}$ with $j<k$ and $\mathbf{x}\in\mathbb{Z}^d$. By definition,

$$\begin{aligned}
&\varphi(\mathbf{x}+\mathbf{e}_j,\mathbf{x})+\varphi(\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j)+\varphi(\mathbf{x}+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k)\\
&\quad+\varphi(\mathbf{x},\mathbf{x}+\mathbf{e}_k)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{x_j+2} 1_{j\ge 2} \sum_{l=1}^{j-1} 1_{x_l \in 2\mathbb{Z}+1} \theta_{l,j} + 1_{x_j+1\in 2\mathbb{Z}} \frac{\pi}{L} \varepsilon_j^L \\
&\quad + (-1)^{x_k+2} \left(\sum_{l=1}^{j-1} 1_{x_l\in 2\mathbb{Z}+1} \theta_{l,k} + 1_{x_j+1\in 2\mathbb{Z}+1} \theta_{j,k} + \sum_{l=j+1}^{k-1} 1_{x_l \in 2\mathbb{Z}+1} \theta_{l,k} \right) \\
&\quad + 1_{x_k+1\in 2\mathbb{Z}} \frac{\pi}{L} \varepsilon_k^L + (-1)^{x_j+1} 1_{j\ge 2} \sum_{l=1}^{j-1} 1_{x_l\in 2\mathbb{Z}+1} \theta_{l,j} - 1_{x_j\in 2\mathbb{Z}+1} \frac{\pi}{L} \varepsilon_j^L \\
&\quad + (-1)^{x_k+1} \sum_{l=1}^{k-1} 1_{x_l\in 2\mathbb{Z}+1} \theta_{l,k} - 1_{x_k\in 2\mathbb{Z}+1} \frac{\pi}{L} \varepsilon_k^L \\
&= (-1)^{x_k+2} 1_{x_j+1\in 2\mathbb{Z}+1} \theta_{j,k} + (-1)^{x_k+1} 1_{x_j\in 2\mathbb{Z}+1} \theta_{j,k} \\
&= (-1)^{x_j+x_k} \theta_{j,k}.
\end{aligned}
$$

Thus, the claim (3) holds.

(4): The equality follows from the definition of φ. □

Since we have constructed the hopping matrix, we can readily define the 2^d-band Hamiltonian. Using the creation, annihilation operators on the Fermionic Fock space $F_f(L^2(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}))$, we set

$$
\begin{aligned}
&H_0 := \sum_{(\rho,\mathbf{x}),(\eta,\mathbf{y})\in\mathcal{B}\times\Gamma(L)} \sum_{\sigma\in\{\uparrow,\downarrow\}} F(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})((\rho,\mathbf{x}),(\eta,\mathbf{y}))\psi^*_{\rho\mathbf{x}\sigma}\psi_{\eta\mathbf{y}\sigma}, \\
&V := \sum_{m=0}^{N_v} \sum_{\substack{(\rho_j,\mathbf{x}_j,\sigma_j),(\eta_j,\mathbf{y}_j,\tau_j)\in\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\} \\ (j=1,2,\cdots,m)}} \\
&\cdot V_m^L(\mathbf{U})((\nu(\rho_1,\mathbf{x}_1)\sigma_1,\cdots,\nu(\rho_m,\mathbf{x}_m)\sigma_m),(\nu(\eta_1,\mathbf{y}_1)\tau_1,\cdots,\nu(\eta_m,\mathbf{y}_m)\tau_m)) \\
&\qquad \cdot \psi^*_{\rho_1\mathbf{x}_1\sigma_1}\cdots\psi^*_{\rho_m\mathbf{x}_m\sigma_m}\psi_{\eta_1\mathbf{y}_1\tau_1}\cdots\psi_{\eta_m\mathbf{y}_m\tau_m}, \\
&H := H_0 + V
\end{aligned}
$$

for $\mathbf{U}\in\mathbb{R}^{n_v}$. The operator H is defined in $F_f(L^2(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}))$ and self-adjoint. The following lemma suggests that we can focus on the free energy density governed by the Hamiltonian H in order to prove Theorem 1.6.

Lemma 2.5.

$$\operatorname{Tr} e^{-\beta \mathsf{H}} = \operatorname{Tr} e^{-\beta H}.$$

Proof. Let us define the operators H_0', H' on $F_f(L^2(\Gamma(2L)\times\{\uparrow,\downarrow\}))$ by

$$\mathsf{H}_0' := \sum_{\mathbf{x},\mathbf{y}\in\Gamma(2L)}\sum_{\sigma\in\{\uparrow,\downarrow\}} G(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{x},\mathbf{y})\psi_{\mathbf{x}\sigma}^*\psi_{\mathbf{y}\sigma},$$
$$\mathsf{H}' := \mathsf{H}_0' + \mathsf{V}.$$

Moreover, define the map W from $F_f(L^2(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}))$ to $F_f(L^2(\Gamma(2L)\times\{\uparrow,\downarrow\}))$ by

$$W(\Omega_L) := \Omega_{2L},$$
$$W(\psi_{\rho_1\mathbf{x}_1\sigma_1}^*\cdots\psi_{\rho_n\mathbf{x}_n\sigma_n}^*\Omega_L) := \psi_{\nu(\rho_1\mathbf{x}_1)\sigma_1}^*\cdots\psi_{\nu(\rho_n\mathbf{x}_n)\sigma_n}^*\Omega_{2L}$$

and by linearity. Here Ω_L denotes the vacuum of $F_f(L^2(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}))$. We can see that W is unitary, $WHW^* = \mathsf{H}'$ and thus $\mathrm{Tr}\, e^{-\beta\mathsf{H}'} = \mathrm{Tr}\, e^{-\beta H}$. Since the phases θ_L, φ satisfy (1.1), (1.2) and (1.3), Lemma B.3 in Appendix B ensures that $\mathrm{Tr}\, e^{-\beta\mathsf{H}} = \mathrm{Tr}\, e^{-\beta\mathsf{H}'}$. Thus, we obtain the claimed equality. □

From here until the proof of Theorem 1.6 in Subsection 4.2 we mainly study $\mathrm{Tr}\, e^{-\beta H}$ instead of $\mathrm{Tr}\, e^{-\beta\mathsf{H}}$.

2.2. Grassmann integral formulation

In this subsection we derive finite-dimensional Grassmann integral formulations of the quantity $\log(\mathrm{Tr}\, e^{-\beta H}/\mathrm{Tr}\, e^{-\beta H_0})$. Most of the lemmas in this subsection are based on the same ideas as in [19, Subsection 2.2, 2.3, 2.4, 2.5]. To avoid unnecessary repetition, we only provide parts of the proofs which need to be clarified.

With the parameter $h\in(2/\beta)\mathbb{N}$ the index set I of the basis of Grassmann algebra is defined by

$$I := \mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\times[0,\beta)_h\times\{1,-1\},$$

where $[0,\beta)_h := \{0, 1/h, 2/h,\cdots,\beta-1/h\}$, a discrete version of the interval $[0,\beta)$. Let N denote $2^{d+2}L^d\beta h$, the cardinality of I. Let $\mathcal{V}$ be the complex vector space spanned by the abstract basis $\{\psi_X\}_{X\in I}$. Then, let $\bigwedge\mathcal{V}$ denote the direct sum of anti-symmetric tensor products of $\mathcal{V}$. We call $\bigwedge\mathcal{V}$ Grassmann algebra generated by $\{\psi_X\}_{X\in I}$. Apart from minor differences between the index sets, the basic description of finite-dimensional

Grassmann integral in [19, Subsection 2.2] applies in this paper as well. We follow the same notational rules concerning Grassmann polynomials set in [19, Subsection 2.2]. The Grassmann polynomial $V(\psi)$, the analogue of the interaction V in $\bigwedge \mathcal{V}$ is defined by

$$
\begin{aligned}
V(\psi) := \sum_{m=0}^{N_v} & \sum_{\substack{(\rho_j,\mathbf{x}_j,\sigma_j),(\eta_j,\mathbf{y}_j,\tau_j)\in\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\\ (j=1,2,\cdots,m)}} \frac{1}{h}\sum_{s\in[0,\beta)_h} \\
& \cdot V_m^L(\mathbf{U})((\nu(\rho_1,\mathbf{x}_1)\sigma_1,\cdots,\nu(\rho_m,\mathbf{x}_m)\sigma_m), \\
& \qquad (\nu(\eta_1,\mathbf{y}_1)\tau_1,\cdots,\nu(\eta_m,\mathbf{y}_m)\tau_m)) \\
& \cdot \overline{\psi}_{\rho_1\mathbf{x}_1\sigma_1 s}\cdots\overline{\psi}_{\rho_m\mathbf{x}_m\sigma_m s}\psi_{\eta_1\mathbf{y}_1\tau_1 s}\cdots\psi_{\eta_m\mathbf{y}_m\tau_m s}
\end{aligned}
\tag{2.5}
$$

with $\mathbf{U} \in \mathbb{C}^{n_v}$. We can expand a Grassmann polynomial $f(\psi) \in \bigwedge \mathcal{V}$ by using the anti-symmetric kernels $f_m : I^m \to \mathbb{C}$ $(m = 1, 2, \cdots, N)$ as follows.

$$f(\psi) = f_0 + \sum_{m=1}^{N}\left(\frac{1}{h}\right)^m \sum_{\mathbf{X}\in I^m} f_m(\mathbf{X})\psi_{\mathbf{X}},$$

where $f_0 \in \mathbb{C}$, $\psi_{\mathbf{X}} := \psi_{X_1}\psi_{X_2}\cdots\psi_{X_m}$ for $\mathbf{X} = (X_1, X_2, \cdots, X_m) \in I^m$. For any function g on I^m its L^1-norm $\|g\|_{L^1}$ is defined by

$$\|g\|_{L^1} := \left(\frac{1}{h}\right)^m \sum_{\mathbf{X}\in I^m} |g(\mathbf{X})|.$$

It will be convenient to let $\|g_0\|_{L^1}$ denote $|g_0|$ for $g_0 \in \mathbb{C}$ as well. Set $U_{max} := \max_{j\in\{1,2,\cdots,n_v\}} |U_j|$. The anti-symmetric kernels of $V(\psi)$ can be estimated as follows.

Lemma 2.6.

$$
\begin{aligned}
&|V_0| \le \beta L^d U_{max} v_0, \\
&\|V_{2m}\|_{L^1} \le 2^{d+1}\beta L^d U_{max} v_m(0), \quad (\forall m \in \{1, 2, \cdots, N_v\}), \\
&\|V_m\|_{L^1} = 0, \quad (\forall m \in \{1, 2, \cdots, N\}\backslash\{2, 4, \cdots, 2N_v\}).
\end{aligned}
$$

PROOF. The bounds on $|V_0|$, $\|V_m\|_{L^1}$ $(m \in \{1,2,\cdots,N\}\backslash\{2,4,\cdots,2N_v\})$ follow from definition. Let $m \in \{1,2,\cdots,N_v\}$. By [19, Lemma B.1], the bijectivity of ν and the definition of $v_m(0)$ we have that

$$\begin{aligned}
&\|V_{2m}\|_{L^1}\\
&\le \beta \sum_{\substack{(\mathbf{x}_j,\sigma_j),(\mathbf{y}_j,\tau_j)\in\Gamma(2L)\times\{\uparrow,\downarrow\}\\ (j=1,2,\cdots,m)}} |V_m^L(\mathbf{U})((\mathbf{x}_1\sigma_1,\cdots,\mathbf{x}_m\sigma_m),(\mathbf{y}_1\tau_1,\cdots,\mathbf{y}_m\tau_m))|\\
&\le 2^{d+1}\beta L^d U_{max} v_m(0). \ \square
\end{aligned}$$

The free covariance C is defined as follows. For $(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y) \in \mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\times[0,\beta)$,

$$\begin{aligned}
(2.6)\qquad &C(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)\\
&:= \frac{\mathrm{Tr}(e^{-\beta H_0}(1_{x\ge y}\psi^*_{\rho\mathbf{x}\sigma}(x)\psi_{\eta\mathbf{y}\tau}(y) - 1_{x<y}\psi_{\eta\mathbf{y}\tau}(y)\psi^*_{\rho\mathbf{x}\sigma}(x)))}{\mathrm{Tr}\, e^{-\beta H_0}},
\end{aligned}$$

where $\psi^{(*)}_{\rho\mathbf{x}\sigma}(x) := e^{xH_0}\psi^{(*)}_{\rho\mathbf{x}\sigma}e^{-xH_0}$. Let $\mathcal{M}$ denote the set of the Matsubara frequency $(\pi/\beta)(2\mathbb{Z}+1)$. We introduce the finite subset $\mathcal{M}_h$ of $\mathcal{M}$ by

$$\mathcal{M}_h := \{\omega \in \mathcal{M} \mid |\omega| < \pi h\}.$$

If we restrict the time variables to the discrete set $[0,\beta)_h$, the covariance can be written as a sum over $\mathcal{M}_h\times\Gamma(L)^*$. Set

$$I_0 := \mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\times[0,\beta)_h.$$

For $(\rho,\mathbf{x},\sigma,x)$, $(\eta,\mathbf{y},\tau,y) \in I_0$,

$$\begin{aligned}
(2.7)\qquad C(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y) = \frac{\delta_{\sigma,\tau}}{\beta L^d} &\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*} e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega}\\
&\cdot h^{-1}(I_{2^d} - e^{-i\frac{\omega}{h}I_{2^d} + \frac{1}{h}\mathcal{E}(\mathbf{k})})^{-1}(\rho,\eta),
\end{aligned}$$

where $\mathcal{E} \in \mathrm{Map}(\mathbb{R}^d, \mathrm{Mat}(2^d,\mathbb{C}))$ is defined by

$$\mathcal{E}(\mathbf{k}) := E(-\boldsymbol{\varepsilon}^L, -\boldsymbol{\theta})(\mathbf{k}).$$

Let us briefly explain how to derive (2.7). It is implied by [19, Lemma 2.1] that

$$C(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)=\frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*}e^{-i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle+i(x-y)\omega}$$
$$\cdot h^{-1}\big(I_{2^d}-e^{-i\frac{\omega}{h}I_{2^d}+\frac{1}{h}\overline{E(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{k})}}\big)^{-1}(\rho,\eta).$$

Then, by using that $\overline{E(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{k})}=E(-\boldsymbol{\varepsilon}^L,-\boldsymbol{\theta})(-\mathbf{k})$ we obtain (2.7).

The next lemma states that the quantity $\log(\mathrm{Tr}\,e^{-\beta H}/\,\mathrm{Tr}\,e^{-\beta H_0})$ is equal to the time-continuum limit of the Grassmann Gaussian integral with the covariance C. Despite the generalization of the interaction, its proof is parallel to [19, Lemma 2.2], which was built upon the idea that the discretization of the integrals over $[0,\beta)$ inside the perturbative expansion of $\mathrm{Tr}\,e^{-\beta H}/\,\mathrm{Tr}\,e^{-\beta H_0}$ converges well as the step size is sent to zero. For any $z\in\mathbb{C}\backslash\mathbb{R}_{\le 0}$ we define $\log z\in\mathbb{C}$ by the principal value $\log|z|+i\theta$ with $\theta\in(-\pi,\pi)$ satisfying $z=|z|e^{i\theta}$. See [19, Subsection 2.2] for the definition of the Grassmann Gaussian integral $\int\cdot d\mu_C(\psi)$.

Lemma 2.7.

(1) *For any* $r\in\mathbb{R}_{>0}$ *there exists* $h_0\in\mathbb{R}_{>0}$ *such that*

$$\mathrm{Re}\int e^{-V(\psi)}d\mu_C(\psi)>0,$$
$$(\forall\mathbf{U}\in\overline{D(r)}^{n_v}\cap\mathbb{R}^{n_v},h\in(2/\beta)\mathbb{N}\ \textit{with}\ h\ge h_0).$$

(2) *For any* $r\in\mathbb{R}_{>0}$,

$$\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}}\sup_{\mathbf{U}\in\overline{D(r)}^{n_v}\cap\mathbb{R}^{n_v}}\Bigg|\log\left(\frac{\mathrm{Tr}\,e^{-\beta H}}{\mathrm{Tr}\,e^{-\beta H_0}}\right)$$
$$-\log\left(\int e^{-V(\psi)}d\mu_C(\psi)\right)\Bigg|=0.$$

Next we connect the above formulation to another Grassmann integral formulation which has better symmetric properties from a technical view point of infrared integration process. The general estimation in [19,

Appendix B] underlies the analysis in the rest of this section. Let χ be a compactly supported smooth function on $\mathbb{R}$ satisfying that $\chi(x) \in [0,1]$ $(\forall x \in \mathbb{R})$. This section proceeds without imposing more conditions on χ. The function χ will be specified after this section. Using χ as a cut-off function, we introduce the covariances $C_{\le 0}^{+}$, $C_{>0}^{+}$, $C_{\le 0}^{\infty}$, $C_{>0}^{-}$, $C_{>0}^{+(h)}$, $\mathcal{I} \in \mathrm{Map}(I_0^2, \mathbb{C})$ as follows. For $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in I_0$,

$$\begin{aligned}
&C_{\le 0}^{+}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)\\
&:= \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*} e^{i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega}\\
&\quad \cdot \chi(h|1-e^{i\frac{\omega}{h}}|)h^{-1}(I_{2^d} - e^{-i\frac{\omega}{h}I_{2^d} + \frac{1}{h}\mathcal{E}(\mathbf{k})})^{-1}(\rho,\eta),
\end{aligned}$$

$$\begin{aligned}
&C_{>0}^{+}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)\\
&:= \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*} e^{i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega}\\
&\quad \cdot (1-\chi(h|1-e^{i\frac{\omega}{h}}|))h^{-1}(I_{2^d} - e^{-i\frac{\omega}{h}I_{2^d} + \frac{1}{h}\mathcal{E}(\mathbf{k})})^{-1}(\rho,\eta),
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&C_{\le 0}^{\infty}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)\\
&:= \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*} e^{i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega}\\
&\quad \cdot \chi(|\omega|)(i\omega I_{2^d} - \mathcal{E}(\mathbf{k}))^{-1}(\rho,\eta),
\end{aligned}$$

$$\begin{aligned}
&C_{>0}^{-}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)\\
&:= \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*} e^{i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega}\\
&\quad \cdot (1-\chi(h|1-e^{i\frac{\omega}{h}}|))h^{-1}(e^{i\frac{\omega}{h}I_{2^d} - \frac{1}{h}\mathcal{E}(\mathbf{k})} - I_{2^d})^{-1}(\rho,\eta),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&C_{>0}^{+(h)}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)\\
&:= C_{>0}^{+}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)\\
&\quad + \frac{1_{(\rho,\mathbf{x},\sigma)=(\eta,\mathbf{y},\tau)}}{\beta h} \sum_{\omega\in\mathcal{M}_h} e^{i(x-y)\omega}\chi(h|1-e^{i\frac{\omega}{h}}|),\\
&\mathcal{I}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y) := 1_{(\rho,\mathbf{x},\sigma,x)=(\eta,\mathbf{y},\tau,y)}.
\end{aligned}$$

One can derive from the definitions that

$$C_{>0}^{+(h)}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y) = C_{>0}^{-}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y) + \mathcal{I}(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y), \tag{2.10}$$

$$(\forall(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in I_0).$$

The next lemma can be proved by applying Gram's inequality and the Cauchy-Binet formula in the same way as in the proof of [19, Lemma 2.4].

LEMMA 2.8. *There exist $(\beta,L,d,\chi,\mathcal{E})$-dependent, h-independent constants h_0, $c_1\in\mathbb{R}_{>0}$ such that the following inequalities hold for any $h\in(2/\beta)\mathbb{N}$ with $h\ge h_0$.*

$$\begin{aligned}
&|\det(C_o(X_i,Y_j))_{1\le i,j\le n}|\le c_1^n,\\
&|\det(C_{>0}^{+(h)}(X_i,Y_j)-C_{>0}^{+}(X_i,Y_j))_{1\le i,j\le n}|\le\frac{1}{h}c_1^n,\\
&|\det(C_{\le 0}^{+}(X_i,Y_j)-C_{\le 0}^{\infty}(X_i,Y_j))_{1\le i,j\le n}|\le\frac{1}{h}c_1^n,\\
&(\forall n\in\mathbb{N},X_j,Y_j\in I_0\ (j=1,2,\cdots,n))
\end{aligned}$$

for $C_o=C$, $C_{\le 0}^{+}$, $C_{>0}^{+}$, $C_{\le 0}^{\infty}$, $C_{>0}^{-}$, $C_{>0}^{+(h)}$.

In the following we assume that $h\ge h_0$ so that the results of Lemma 2.8 are available. Define the Grassmann polynomials $V^+(\psi)$, $V^-(\psi)$, $S^+(\psi)$, $S^-(\psi)$, $S^0(\psi)\in\bigwedge\mathcal{V}$ by

$$\begin{aligned}
(2.11)\qquad &V^+(\psi):=V(\psi),\\
&V^-(\psi):=\sum_{m=0}^{N_v}\sum_{\substack{(\rho_j,\mathbf{x}_j,\sigma_j),(\eta_j,\mathbf{y}_j,\tau_j)\in\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,m)}}\frac{1}{h}\sum_{s\in[0,\beta)_h}(-1)^m\\
&\qquad\cdot V_m^L((\nu(\rho_1,\mathbf{x}_1)\sigma_1,\cdots,\nu(\rho_m,\mathbf{x}_m)\sigma_m),\\
&\qquad\qquad(\nu(\eta_1,\mathbf{y}_1)\tau_1,\cdots,\nu(\eta_m,\mathbf{y}_m)\tau_m))\\
&\qquad\cdot\overline{\psi}_{\rho_1\mathbf{x}_1\sigma_1 s}\cdots\overline{\psi}_{\rho_m\mathbf{x}_m\sigma_m s}\psi_{\eta_1\mathbf{y}_1\tau_1 s}\cdots\psi_{\eta_m\mathbf{y}_m\tau_m s},\\
&S^\delta(\psi):=\int e^{-V^\delta(\psi+\psi^1)}d\mu_{C_{>0}^{\delta}}(\psi^1),\ (\delta\in\{+,-\}),\\
&S^0(\psi):=\int e^{-V^+(\psi+\psi^1)}d\mu_{C_{>0}^{+(h)}}(\psi^1).
\end{aligned}$$

For conciseness let $g(\alpha)$ denote

$$\beta L^d v_0+2^{d+1}\beta L^d\sum_{m=1}^{N_v}(\alpha+1)^{2m}c_1^m v_m(0)$$

for $\alpha \in \mathbb{R}_{\ge 0}$, where c_1 is the constant appearing in Lemma 2.8.

LEMMA 2.9.

(1)

$$|S_0^\delta - e^{-V_0}| \le e^{U_{max} g(0)} - e^{U_{max}\beta L^d v_0}, \quad (\forall \delta \in \{+,-,0\}).$$

(2)

$$\sum_{m=0}^{N} \alpha^m c_1^{\frac{m}{2}} \|S_m^\delta\|_{L^1} \le e^{U_{max} g(\alpha)}, \quad (\forall \alpha \in \mathbb{R}_{\ge 0}, \delta \in \{+,-,0\}).$$

(3)

$$\sum_{m=0}^{N} \alpha^m c_1^{\frac{m}{2}} \|S_m^+ - S_m^0\|_{L^1} \le \frac{1}{h}(e^{U_{max} g(\alpha+1)} - e^{U_{max}\beta L^d v_0}), \quad (\forall \alpha \in \mathbb{R}_{\ge 0}).$$

(4)

$$\sum_{m=0}^{N} \alpha^m c_1^{\frac{m}{2}} \|S_m^- - S_m^0\|_{L^1} \le \frac{1}{\beta h}(U_{max} g(\alpha) - U_{max}\beta L^d v_0)^2 e^{U_{max} g(\alpha)},$$
$(\forall \alpha \in \mathbb{R}_{\ge 0})$.

PROOF. Combination of Lemma 2.6, Lemma 2.8 and [19, Lemma B.2 (1),(2),(4)] yields the inequalities in (1), (2), (3).

Let us prove the inequality in (4), which is a generalization of [19, Lemma 2.6]. Define the functions W_m^δ ($\delta = +,-$, $m = 1,2,\cdots,N_v$) on $(\mathcal{B} \times \Gamma(L) \times \{\uparrow,\downarrow\})^m \times (\mathcal{B} \times \Gamma(L) \times \{\uparrow,\downarrow\})^m$ by

$$\begin{aligned} &W_m^\delta(((\rho_1,\mathbf{x}_1,\sigma_1),\cdots,(\rho_m,\mathbf{x}_m,\sigma_m)),((\eta_1,\mathbf{y}_1,\tau_1),\cdots,(\eta_m,\mathbf{y}_m,\tau_m))) \\ &:= (1_{\delta=+} + 1_{\delta=-}(-1)^m) \\ &\quad \cdot V_m^L((\nu(\rho_1,\mathbf{x}_1)\sigma_1,\cdots,\nu(\rho_m,\mathbf{x}_m)\sigma_m),(\nu(\eta_1,\mathbf{y}_1)\tau_1,\cdots,\nu(\eta_m,\mathbf{y}_m)\tau_m)), \\ &(\forall (\rho_j,\mathbf{x}_j,\sigma_j),(\eta_j,\mathbf{y}_j,\tau_j) \in \mathcal{B} \times \Gamma(L) \times \{\uparrow,\downarrow\} \ (j=1,2,\cdots,m)). \end{aligned}$$

For any $s \in [0,\beta)_h$, $\mathbf{X} = (X_1,X_2,\cdots,X_m) \in (\mathcal{B} \times \Gamma(L) \times \{\uparrow,\downarrow\})^m$ we abbreviate $(X_m,X_{m-1},\cdots,X_1)$, $((X_1,s),(X_2,s),\cdots,(X_m,s))$, $\overline{\psi}_{X_1 s}\overline{\psi}_{X_2 s}$

$\cdots\overline{\psi}_{X_m s}$, $\psi_{X_1 s}\psi_{X_2 s}\cdots\psi_{X_m s}$ to $\widetilde{\mathbf{X}}$, $\mathbf{X}s$, $\overline{\psi}_{\mathbf{X}s}$, $\psi_{\mathbf{X}s}$ respectively. For $s \in [0,\beta)_h$ we define $W_s^{\delta}(\psi) \in \bigwedge \mathcal{V}$ $(\delta = +,-)$ by

$$W_s^{\delta}(\psi) := \sum_{m=1}^{N_v} \sum_{\mathbf{X},\mathbf{Y}\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^m} W_m^{\delta}(\mathbf{X},\mathbf{Y})\overline{\psi}_{\mathbf{X}s}\psi_{\mathbf{Y}s}.$$

Take $s_1, s_2, \cdots, s_n \in [0,\beta)_h$ satisfying $s_j \neq s_k$ ($\forall j,k \in \{1,2,\cdots,n\}$ with $j \neq k$). By the invariance (1.13), the equality (2.10) and anti-symmetry,

$$\begin{aligned}
&\int \prod_{j=1}^{n} W_{s_j}^{+}(\psi+\psi^1)d\mu_{C_{>0}^{+(h)}}(\psi^1) \\
&= \int \prod_{j=1}^{n} W_{s_j}^{+}(\psi+\psi^1+\psi^2)d\mu_{\mathcal{I}}(\psi^2)d\mu_{C_{>0}^{-}}(\psi^1) \\
&= \int \prod_{j=1}^{n} \Bigg(\sum_{m_j=1}^{N_v}\sum_{l_j=0}^{m_j} \sum_{\substack{\mathbf{X}_j,\mathbf{Y}_j\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^{m_j-l_j}\\ \mathbf{W}_j\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^{l_j}}} \begin{pmatrix} m_j \\ l_j \end{pmatrix}^2 l_j! \\
&\qquad \cdot W_{m_j}^{+}((\mathbf{X}_j,\mathbf{W}_j),(\widetilde{\mathbf{W}_j},\mathbf{Y}_j)) \\
&\qquad \cdot (\overline{\psi}+\overline{\psi}^1)_{\mathbf{X}_j s_j}(\psi+\psi^1)_{\mathbf{Y}_j s_j}\overline{\psi}^2_{\mathbf{W}_j s_j}\psi^2_{\widetilde{\mathbf{W}}_j s_j}\Bigg) \\
&\quad \cdot d\mu_{\mathcal{I}}(\psi^2)d\mu_{C_{>0}^{-}}(\psi^1) \\
&= \int \prod_{j=1}^{n} \Bigg(\sum_{m_j=1}^{N_v}\sum_{l_j=0}^{m_j} \sum_{\substack{\mathbf{X}_j,\mathbf{Y}_j\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^{m_j-l_j}\\ \mathbf{W}_j\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^{l_j}}} \begin{pmatrix} m_j \\ l_j \end{pmatrix}^2 l_j! \\
&\qquad \cdot W_{m_j}^{+}((\mathbf{X}_j,\mathbf{W}_j),(\widetilde{\mathbf{W}_j},\mathbf{Y}_j))(\overline{\psi}+\overline{\psi}^1)_{\mathbf{X}_j s_j}(\psi+\psi^1)_{\mathbf{Y}_j s_j}\Bigg) \\
&\quad \cdot d\mu_{C_{>0}^{-}}(\psi^1) \\
&= \int \prod_{j=1}^{n} W_{s_j}^{-}(\psi+\psi^1)d\mu_{C_{>0}^{-}}(\psi^1).
\end{aligned} \tag{2.12}$$

Define $Q^{\delta}(\psi)\in\bigwedge\mathcal{V}$ ($\delta=+,-$) by

$$Q^{\delta}(\psi):=e^{-V_0}+e^{-V_0}\sum_{n=1}^{N}\frac{(-1)^n}{n!}\prod_{j=1}^{n}\left(\frac{1}{h}\sum_{s_j\in[0,\beta)_h}\right)\cdot 1_{\forall j\forall k\in\{1,2,\cdots,n\}(j\neq k\to s_j\neq s_k)}\prod_{j=1}^{n}W_{s_j}^{\delta}(\psi).$$

Then, the equality (2.12) implies that

$$\int Q^{+}(\psi+\psi^1)d\mu_{C_{>0}^{+(h)}}(\psi^1)=\int Q^{-}(\psi+\psi^1)d\mu_{C_{>0}^{-}}(\psi^1).$$

Therefore,

$$\begin{aligned}S^0(\psi)-S^-(\psi)=&\int(e^{-V^+(\psi+\psi^1)}-Q^+(\psi+\psi^1))d\mu_{C_{>0}^{+(h)}}(\psi^1)\\&-\int(e^{-V^-(\psi+\psi^1)}-Q^-(\psi+\psi^1))d\mu_{C_{>0}^{-}}(\psi^1).\end{aligned}\tag{2.13}$$

Let us set

$$\tilde{S}^0(\psi):=\int(e^{-V^+(\psi+\psi^1)}-Q^+(\psi+\psi^1))d\mu_{C_{>0}^{+(h)}}(\psi^1).$$

For any $m\in\{0,1,\cdots,N\}$ we can characterize $\tilde{S}^0_m(\psi)$, the m-th order part of $\tilde{S}^0(\psi)$ as follows.

$$\begin{aligned}&\tilde{S}^0_m(\psi)\\&=e^{-V_0}\sum_{n=2}^{N}\frac{(-1)^n}{n!}\prod_{j=1}^{n}\Bigg(\frac{1}{h}\sum_{s_j\in[0,\beta)_h}\sum_{m_j=1}^{N_v}\sum_{k_j=0}^{m_j}\sum_{l_j=0}^{m_j}\begin{pmatrix}m_j\\k_j\end{pmatrix}\begin{pmatrix}m_j\\l_j\end{pmatrix}\\&\quad\cdot\sum_{\substack{\mathbf{X}_j\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^{m_j-k_j},\mathbf{X}'_j\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^{k_j}\\\mathbf{Y}_j\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^{m_j-l_j},\mathbf{Y}'_j\in(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})^{l_j}}}W_{m_j}^{+}((\mathbf{X}_j,\mathbf{X}'_j),(\mathbf{Y}_j,\mathbf{Y}'_j))\Bigg)\\&\quad\cdot 1_{\exists j\exists k\in\{1,2,\cdots,n\}(j\neq k\wedge s_j=s_k)}1_{\sum_{j=1}^{n}k_j=\sum_{j=1}^{n}l_j=\frac{m}{2}}\varepsilon_{\pm}\\&\quad\cdot\int\overline{\psi}^1_{\mathbf{X}_1s_1}\psi^1_{\mathbf{Y}_1s_1}\overline{\psi}^1_{\mathbf{X}_2s_2}\psi^1_{\mathbf{Y}_2s_2}\cdots\overline{\psi}^1_{\mathbf{X}_ns_n}\psi^1_{\mathbf{Y}_ns_n}d\mu_{C_{>0}^{+(h)}}(\psi^1)\end{aligned}$$

$$\cdot \overline{\psi}_{\mathbf{X}'_1 s_1} \psi_{\mathbf{Y}'_1 s_1} \overline{\psi}_{\mathbf{X}'_2 s_2} \psi_{\mathbf{Y}'_2 s_2} \cdots \overline{\psi}_{\mathbf{X}'_n s_n} \psi_{\mathbf{Y}'_n s_n},$$

where the factor $\varepsilon_{\pm} \in \{1, -1\}$ depends only on m_j, k_j, l_j $(j = 1, 2, \cdots, n)$. From this equality, Lemma 2.8, [19, Lemma B.1] and the inequality that

$$\prod_{j=1}^{n} \left(\frac{1}{h} \sum_{s_j \in [0,\beta)_h} \right) 1_{\exists j \exists k \in \{1,2,\cdots,n\}(j \neq k \wedge s_j = s_k)} \le \left(\begin{array}{c} n \\ 2 \end{array} \right) \frac{\beta^{n-1}}{h}, \quad (\forall n \in \mathbb{N}_{\ge 2}),$$

we can deduce that

$$\begin{aligned} \|\tilde{S}^0_m\|_{L^1} \le e^{|V_0|} \sum_{n=2}^{N} \frac{1}{n!} \left(\begin{array}{c} n \\ 2 \end{array} \right) \frac{\beta^{n-1}}{h} \prod_{j=1}^{n} \Bigg(\sum_{m_j=1}^{N_v} \sum_{k_j=0}^{m_j} \sum_{l_j=0}^{m_j} \left(\begin{array}{c} m_j \\ k_j \end{array} \right) \left(\begin{array}{c} m_j \\ l_j \end{array} \right) \\ \cdot \sum_{\mathbf{X},\mathbf{Y} \in (\Gamma(2L) \times \{\uparrow,\downarrow\})^{m_j}} |V^L_{m_j}(\mathbf{X}, \mathbf{Y})| c_1^{m_j} \Bigg) c_1^{-\frac{m}{2}} 1_{\sum_{j=1}^{n} k_j = \sum_{j=1}^{n} l_j = \frac{m}{2}}. \end{aligned}$$

Thus, for any $\alpha \in \mathbb{R}_{\ge 0}$,

$$\begin{aligned} \sum_{m=0}^{N} \alpha^m c_1^{\frac{m}{2}} \|\tilde{S}^0_m\|_{L^1} &= \sum_{m=0}^{N/2} \alpha^{2m} c_1^m \|\tilde{S}^0_{2m}\|_{L^1} \\ &\le \frac{1}{2\beta h} e^{U_{max} \beta L^d v_0} \sum_{n=2}^{\infty} \frac{1}{(n-2)!} (U_{max} g(\alpha) - U_{max} \beta L^d v_0)^n \\ &= \frac{1}{2\beta h} (U_{max} g(\alpha) - U_{max} \beta L^d v_0)^2 e^{U_{max} g(\alpha)}. \end{aligned}$$

The Grassmann polynomial

$$\int (e^{-V^-(\psi + \psi^1)} - Q^-(\psi + \psi^1)) d\mu_{C^-_{>0}}(\psi^1)$$

can be estimated in the same way as above. By combining these bounds with the equality (2.13) we can derive the claimed inequality. □

LEMMA 2.10. *Let $\alpha \in \mathbb{R}_{\ge 0}$ and $\varepsilon \in (0, 1)$. Assume that*

$$U_{max} g(\alpha) \le \log \left(\frac{2(\varepsilon + 1)}{\varepsilon + 2} \right).$$

Then, the following inequalities hold.

(1)

$$|S_0^\delta - e^{-V_0}| \le \frac{\varepsilon}{\varepsilon+2}, \quad (\forall \delta \in \{+,-,0\}).$$

(2)

$$\sup_{\delta\in\{+,-,0\}} \sum_{m=1}^{N} \alpha^m c_1^{\frac{m}{2}} \|S_m^\delta\|_{L^1} \le \varepsilon \inf_{\delta\in\{+,-,0\}} |S_0^\delta|.$$

PROOF. (1): It follows from Lemma 2.9 (1) and the assumption that for $\delta \in \{+,-,0\}$,

$$|S_0^\delta - e^{-V_0}| \le e^{U_{max} g(0)} - 1 \le \frac{\varepsilon}{\varepsilon+2}.$$

(2): The assumption implies that

$$e^{U_{max} g(\alpha)} \le (\varepsilon+1)(2 - e^{U_{max} g(\alpha)}). \tag{2.14}$$

Moreover, by Lemma 2.9 (1) and the inequality that

$$|e^{-V_0} - 1| \le e^{U_{max}\beta L^d v_0} - 1$$

we see that

$$|S_0^\delta - 1| \le |S_0^\delta - e^{-V_0}| + |e^{-V_0} - 1| \le e^{U_{max} g(0)} - 1 \tag{2.15}$$

for any $\delta \in \{+,-,0\}$. Thus,

$$2 - e^{U_{max} g(\alpha)} \le \inf_{\delta\in\{+,-,0\}} |S_0^\delta|. \tag{2.16}$$

Using Lemma 2.9 (2), (2.14) and (2.16), we have that

$$\sup_{\delta\in\{+,-,0\}} \sum_{m=1}^{N} \alpha^m c_1^{\frac{m}{2}} \|S_m^\delta\|_{L^1} \le e^{U_{max} g(\alpha)} - \inf_{\delta\in\{+,-,0\}} |S_0^\delta| \le \varepsilon \inf_{\delta\in\{+,-,0\}} |S_0^\delta|. \ \square$$

LEMMA 2.11. *Let $\alpha \in \mathbb{R}_{\ge 0}$, $\varepsilon \in (0,1)$. Assume that*

$$U_{max} g(\alpha+1) \le \log\left(\frac{2(\varepsilon+1)}{\varepsilon+2}\right).$$

Set $R^\delta(\psi) := \log S^\delta(\psi)$, $(\delta \in \{+,-,0\})$. Then, the following inequalities hold for any $h \in (2/\beta)\mathbb{N}$ satisfying $h > \max\{1/2, 2/\beta, h_0\}$.

(1)
$$|R_0^{\delta}| \le \log\left(\frac{\varepsilon+2}{2}\right), \quad (\forall \delta \in \{+,-,0\}).$$

(2)
$$\sum_{m=1}^{N} \alpha^m c_1^{\frac{m}{2}} \|R_m^{\delta}\|_{L^1} \le -\log(1-\varepsilon), \quad (\forall \delta \in \{+,-,0\}).$$

(3)
$$|R_0^{\delta} - R_0^0| \le -\log\left(1 - \max\left\{1, \frac{4}{\beta}\right\}\frac{1}{2h}\right), \quad (\forall \delta \in \{+,-\}).$$

(4)
$$\sum_{m=1}^{N} \alpha^m c_1^{\frac{m}{2}} \|R_m^{\delta} - R_m^0\|_{L^1} \le \max\left\{1, \frac{4}{\beta}\right\}\frac{1}{2(1-\varepsilon)h}, \quad (\forall \delta \in \{+,-\}).$$

PROOF. It follows from (2.15) and the assumption that

$$|S_0^{\delta} - 1| \le \frac{\varepsilon}{\varepsilon+2} < 1, \quad (\forall \delta \in \{+,-,0\}).$$

This means that the assumption of [19, Lemma B.3] is satisfied and thus we can apply it. The claims can be proved in a way close to the proof of [19, Lemma 2.8]. We only explain which lemmas are necessary to prove each claim. We use the assumption, (2.15) and [19, Lemma B.3 (1)] to prove the claim (1). The assumption and Lemma 2.10 (2) enable us to apply [19, Lemma B.3 (2)] to prove the claim (2). We use the assumption, Lemma 2.9 (3),(4), (2.16) and [19, Lemma B.3 (3)] to prove the claim (3). By combining the assumption, Lemma 2.9 (3),(4), Lemma 2.10 (2) and (2.16) with [19, Lemma B.3 (4)] we can deduce the claim (4). □

Here we reach the lemma stating that the Grassmann integral formulation in Lemma 2.7 can be approximated by another formulation which will turn out to have a desirable symmetry later in Section 4. We will mainly deal with this formulation in the infrared multi-scale analysis in Section 4.

LEMMA 2.12. *There exist* $(\beta, L, d, g(2), \chi, \mathcal{E})$*-dependent, h-independent constants* $h_0, c_2, c_3 \in \mathbb{R}_{>0}$ *such that the following statements hold for any*

$h \in (2/\beta)\mathbb{N}$ satisfying $h \ge h_0$ and $\mathbf{U} \in \mathbb{C}^{n_v}$ satisfying $|U_j| \le c_2$ ($\forall j \in \{1, 2, \cdots, n_v\}$).

(1)

$$\mathrm{Re} \int e^{-V(\psi)} d\mu_C(\psi) > 0,$$
$$\mathrm{Re} \int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C^\infty_{\le 0}}(\psi) > 0.$$

(2)

$$\left| \log \left(\int e^{-V(\psi)} d\mu_C(\psi) \right) - \log \left(\int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))} d\mu_{C^\infty_{\le 0}}(\psi) \right) \right| \le \frac{1}{h} c_3.$$

PROOF. Take $\varepsilon \in (0, 2/5)$. Assume that

$$U_{max} \le g(3)^{-1} \log \left(\frac{2(\varepsilon + 1)}{\varepsilon + 2} \right).$$

Then, all the inequalities claimed in Lemma 2.11 hold with $\alpha = 2$ and $h \in (2/\beta)\mathbb{N}$ satisfying $h > \max\{1/2, 2/\beta, h_0\}$. Note that the inequalities proved in Lemma 2.11 have exactly the same form as those proved in [19, Lemma 2.8]. Based on these inequalities and [19, Lemma B.2], we only need to follow the same argument as in the proof of [19, Lemma 2.10] to obtain the results. □

3. The Matsubara Ultra-Violet Integration

In this section we carry out a multi-scale integration over the large Matsubara frequency. In the first subsection we summarize properties of the covariances with the Matsubara UV cut-off. Most of these properties have already been proved in [19, Lemma 6.2, Lemma 6.3]. We only provide proofs for claims which are not directly implied by [19, Lemma 6.2, Lemma 6.3]. Using these results, we will establish upper bounds on Grassmann polynomials produced by the Matsubara UV integration in Subsection 3.2 and Subsection 3.3. Though these subsections are aimed at achieving the same goal as in [19, Subsection 5.1, Subsection 5.2, Section 6], the generalization of the interaction creates different aspects which cannot be skipped without proof. We will provide the full construction of the Matsubara UV integration.

3.1. Covariances with the Matsubara ultra-violet cut-off

From now till the proof of Theorem 1.6 in Subsection 4.2 we assume that

$$\max_{j\in\{1,2,\cdots,d\}} t_j = 1. \tag{3.1}$$

Theorem 1.6, the main theorem of this paper, can be deduced from that proved under this condition. It follows from Lemma 2.3 (3),(4) and (3.1) that

$$\sup_{j\in\{1,2,\cdots,d\}} \sup_{\mathbf{k}\in\mathbb{R}^d} \left\| \left(\frac{\partial}{\partial k_j}\right)^n \mathcal{E}(\mathbf{k}) \right\|_{2^d\times 2^d} \le 2d, \quad (\forall n \in \mathbb{N}\cup\{0\}). \tag{3.2}$$

In [19, Lemma 6.1], which was based on [14, Theorem 1.3.5], we introduced a function $\phi \in C^\infty(\mathbb{R})$ satisfying that

$$\begin{aligned}
&\phi(x) = 1,\ (\forall x \in (-\infty, \pi^2/6]),\\
&\phi(x) = 0,\ (\forall x \in [\pi^2/3, \infty)),\\
&\frac{d}{dx}\phi(x) \le 0,\ (\forall x \in \mathbb{R}),\\
&\left|\left(\frac{d}{dx}\right)^k \phi(x)\right| \le 2^k (k!)^2,\ (\forall x \in \mathbb{R}, k \in \mathbb{N}\cup\{0\}).
\end{aligned}$$

We keep using this function to construct cut-off functions in this paper as well.

The inequality (3.2) suggests that the general results in [19, Subsection 6.1] hold with "$E_1 = 2d$", "$E_2 = 1$" for our covariances if we define the cut-off functions in the same manner as in [19, Subsection 6.1]. Let us do so for simplicity. With $M \in \mathbb{R}_{>1}$, set

$$\begin{aligned}
&M_{UV} := \frac{2\sqrt{6}}{\pi}(2d+1),\\
&N_h := \max\left\{ \left\lfloor \frac{\log\left(2h\left(\frac{\pi^2}{6}\right)^{-1/2} M_{UV}^{-1}\right)}{\log M} \right\rfloor + 1, 1 \right\}.
\end{aligned}$$

Here $\lfloor x \rfloor$ denotes the largest integer not exceeding x for $x \in \mathbb{R}$. It follows that

$$\phi(M_{UV}^{-2} M^{-2N_h} h^2 |1 - e^{i\frac{\omega}{h}}|^2) = 1, \quad (\forall \omega \in \mathbb{R}).$$

We define the cut-off function $\chi_{h,l}:\mathbb{R}\to\mathbb{R}_{\ge 0}$ ($l=0,1,\cdots,N_h$) by

$$\begin{aligned}&\chi_{h,0}(\omega):=\phi(M_{UV}^{-2}h^2|1-e^{i\frac{\omega}{h}}|^2),\\ &\chi_{h,l}(\omega):=\phi(M_{UV}^{-2}M^{-2l}h^2|1-e^{i\frac{\omega}{h}}|^2)-\phi(M_{UV}^{-2}M^{-2(l-1)}h^2|1-e^{i\frac{\omega}{h}}|^2),\\ &(\omega\in\mathbb{R},l\in\{1,2,\cdots,N_h\}).\end{aligned}$$

These functions have the properties described in [19, (6.3), (6.4)]. Using these functions, we define the covariances with the Matsubara UV cut-off C_l^+, $C_l^-:I_0^2\to\mathbb{C}$ ($l=0,1,\cdots,N_h$) as follows.

$$\begin{aligned}C_l^+(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)&:=\frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle+i(x-y)\omega}\chi_{h,l}(\omega)\\ &\quad\cdot h^{-1}(I_{2^d}-e^{-i\frac{\omega}{h}I_{2^d}+\frac{1}{h}\mathcal{E}(\mathbf{k})})^{-1}(\rho,\eta),\\ C_l^-(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)&:=\frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle+i(x-y)\omega}\chi_{h,l}(\omega)\\ &\quad\cdot h^{-1}(e^{i\frac{\omega}{h}I_{2^d}-\frac{1}{h}\mathcal{E}(\mathbf{k})}-I_{2^d})^{-1}(\rho,\eta),\\ &((\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in I_0).\end{aligned}$$

Here let us introduce some notations which will be used to study the decay properties of the covariances in this section and for many other purposes in the rest of this paper. For any $(\rho,\mathbf{x},\sigma,x,\theta),(\eta,\mathbf{y},\tau,y,\xi)\in I$, $j\in\{0,1,\cdots,d\}$, set

$$\begin{aligned}&d_j((\rho,\mathbf{x},\sigma,x,\theta),(\eta,\mathbf{y},\tau,y,\xi))\\ &:=\begin{cases}\frac{\beta}{2\pi}|e^{i\frac{2\pi}{\beta}x}-e^{i\frac{2\pi}{\beta}y}| & \text{if } j=0,\\ \frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle\mathbf{x},\mathbf{e}_j\rangle}-e^{i\frac{2\pi}{L}\langle\mathbf{y},\mathbf{e}_j\rangle}| & \text{if } j\in\{1,2,\cdots,d\}.\end{cases}\end{aligned}$$

For any $x\in(1/h)\mathbb{Z}$ let $r_\beta(x)\in[0,\beta)_h$, $n_\beta(x)\in\mathbb{Z}$ be such that $x=n_\beta(x)\beta+r_\beta(x)$. This defines the maps $r_\beta:(1/h)\mathbb{Z}\to[0,\beta)_h$, $n_\beta:(1/h)\mathbb{Z}\to\mathbb{Z}$. We will assume that

$$\beta_1,\beta_2\in\mathbb{N},\quad \beta_1\le\beta_2,\quad h\in 4\mathbb{N}, \tag{3.3}$$

when we need to estimate differences between anti-symmetric functions defined at 2 different temperatures. Here β_1, β_2 are meant to be the 2 different inverse temperatures. Though the inverse temperature originally belongs to

$\mathbb{R}_{>0}$, we will later see that the convergence property of the free energy density as $\beta \to \infty$ ($\beta \in \mathbb{R}_{>0}$) can be deduced from the convergent property as $\beta \to \infty$ ($\beta \in \mathbb{N}$). On the assumption (3.3), set

$$\left[-\frac{\beta_1}{4}, \frac{\beta_1}{4}\right)_h := \left\{-\frac{\beta_1}{4}, -\frac{\beta_1}{4} + \frac{1}{h}, -\frac{\beta_1}{4} + \frac{2}{h}, \cdots, \frac{\beta_1}{4} - \frac{1}{h}\right\}.$$

Note that $0 \in [-\beta_1/4, \beta_1/4)_h$. We define the index sets $\hat{I}_0$, $\hat{I}$, I_0^0, I^0 by

$$\hat{I}_0 := \mathcal{B} \times \Gamma(L) \times \{\uparrow, \downarrow\} \times \left[-\frac{\beta_1}{4}, \frac{\beta_1}{4}\right)_h, \quad \hat{I} := \hat{I}_0 \times \{1, -1\},$$
$$I_0^0 := \mathcal{B} \times \Gamma(L) \times \{\uparrow, \downarrow\} \times \{0\}, \quad I^0 := I_0^0 \times \{1, -1\}.$$

For any $(\rho, \mathbf{x}, \sigma, x, \theta)$, $(\eta, \mathbf{y}, \tau, y, \xi) \in \hat{I}$, $j \in \{0, 1, \cdots, d\}$, set

$$\hat{d}_j((\rho, \mathbf{x}, \sigma, x, \theta), (\eta, \mathbf{y}, \tau, y, \xi))$$
$$:= \begin{cases} |x - y| & \text{if } j = 0, \\ \frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle \mathbf{x}, \mathbf{e}_j\rangle} - e^{i\frac{2\pi}{L}\langle \mathbf{y}, \mathbf{e}_j\rangle}| & \text{if } j \in \{1, 2, \cdots, d\}. \end{cases}$$

In fact these notations were used in [19]. We add the notation (β) to the right side of a temperature-dependent object when we want to show its temperature dependency explicitly. For example we sometimes write $I_0(\beta)$ instead of I_0 and $C_l^+(\beta) : I_0(\beta)^2 \to \mathbb{C}$ instead of $C_l^+ : I_0^2 \to \mathbb{C}$.

LEMMA 3.1. *Assume that* $h \geq e^{4d}$. *There exists a constant* $c_0 \in \mathbb{R}_{\geq 1}$, *which depends only on* d, M, *and a constant* $c_w \in (0, 1]$ *independent of any parameter such that the following statements hold for any* $\delta \in \{+, -\}$, $l \in \{1, 2, \cdots, N_h\}$.

(1)

$$(3.4) \qquad |\det(\langle \mathbf{p}_i, \mathbf{q}_j\rangle_{\mathbb{C}^m} C_l^\delta(X_i, Y_j))_{1 \leq i,j \leq n}| \leq c_0^n,$$
$$(\forall m, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^m \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^m}, \|\mathbf{q}_i\|_{\mathbb{C}^m} \leq 1,$$
$$X_i, Y_i \in I_0 \ (i = 1, 2, \cdots, n)).$$

(2)

$$|\det(\langle \mathbf{p}_i,\mathbf{q}_j\rangle_{\mathbb{C}^m} C_l^{\delta}((X_i,s),(Y_j,s)))_{1\le i,j\le n}| \le (M^{-l}+M^{l-N_h})c_0^n, \quad (3.5)$$
$$(\forall m,n\in\mathbb{N},\mathbf{p}_i,\mathbf{q}_i\in\mathbb{C}^m \textit{ with } \|\mathbf{p}_i\|_{\mathbb{C}^m},\|\mathbf{q}_i\|_{\mathbb{C}^m}\le 1,$$
$$X_i,Y_i\in\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\ (i=1,2,\cdots,n), s\in[0,\beta)_h).$$

(3)

$$\sup_{j'\in\{0,1,\cdots,d\}}\sup_{X\in I}\frac{1}{h}\sum_{Y\in I}(d_{j'}(X,Y)+1)e^{\sum_{j=0}^d(c_w(d+1)^{-2}M^{-2}d_j(X,Y))^{1/2}} \quad (3.6)$$
$$\cdot|\widetilde{C_l^{\delta}}(X,Y)|$$
$$\le M^{-l}c_0.$$

(4) *On the assumption* (3.3),

$$|\det(\langle \mathbf{p}_i,\mathbf{q}_j\rangle_{\mathbb{C}^m} C_l^{\delta}(\beta_1)(\rho_i\mathbf{x}_i\sigma_i r_{\beta_1}(x_i),\eta_j\mathbf{y}_j\tau_j r_{\beta_1}(y_j)))_{1\le i,j\le n} \quad (3.7)$$
$$-\det(\langle \mathbf{p}_i,\mathbf{q}_j\rangle_{\mathbb{C}^m} C_l^{\delta}(\beta_2)(\rho_i\mathbf{x}_i\sigma_i r_{\beta_2}(x_i),\eta_j\mathbf{y}_j\tau_j r_{\beta_2}(y_j)))_{1\le i,j\le n}|$$
$$\le \beta_1^{-\frac{1}{2}}M^{-\frac{l}{2}}c_0^n,$$
$$(\forall m,n\in\mathbb{N},\mathbf{p}_i,\mathbf{q}_i\in\mathbb{C}^m \textit{ with } \|\mathbf{p}_i\|_{\mathbb{C}^m},\|\mathbf{q}_i\|_{\mathbb{C}^m}\le 1,$$
$$(\rho_i,\mathbf{x}_i,\sigma_i,x_i),(\eta_i,\mathbf{y}_i,\tau_i,y_i)\in\hat{I}_0\ (i=1,2,\cdots,n)).$$

(5) *On the assumption* (3.3),

$$\sup_{X\in I^0}\frac{1}{h}\sum_{(\eta,\mathbf{y},\tau,y,\xi)\in\hat{I}} e^{\sum_{j=0}^d(\frac{1}{\pi}c_w(d+1)^{-2}M^{-2}\hat{d}_j(X,(\eta,\mathbf{y},\tau,y,\xi)))^{1/2}} \quad (3.8)$$
$$\cdot|\widetilde{C_l^{\delta}}(\beta_1)(X,\eta\mathbf{y}\tau r_{\beta_1}(y)\xi)-\widetilde{C_l^{\delta}}(\beta_2)(X,\eta\mathbf{y}\tau r_{\beta_2}(y)\xi)|$$
$$\le \beta_1^{-\frac{1}{2}}M^{-l}c_0.$$

In (3), (5), $\widetilde{C_l^{\delta}}: I^2\to\mathbb{C}$ *denotes the anti-symmetric extension of* C_l^{δ} *defined by*

$$\widetilde{C_l^{\delta}}((X,\theta),(Y,\xi)) \quad (3.9)$$

$$:= \frac{1}{2}(1_{(\theta,\xi)=(1,-1)}C_l^{\delta}(X,Y) - 1_{(\theta,\xi)=(-1,1)}C_l^{\delta}(Y,X)),$$
$$(\forall X,Y \in I_0, \theta,\xi \in \{1,-1\}).$$

REMARK 3.2. There are unfortunately insufficiencies in the estimation of the difference between the determinants defined at β_1, β_2 in the proofs of [19, Lemma 6.3, Lemma 7.14], though the results themselves hold true. Here we prove (3.7) in a way that it recovers the insufficient parts of the proofs of the related inequalities in [19, Lemma 6.3, Lemma 7.14].

PROOF OF LEMMA 3.1. First of all, let us note that the condition "$h \ge e^{2E_1}$" required in [19, Lemma 6.2, Lemma 6.3] is equal to $h \ge e^{4d}$ in this case because of (3.2). Thus, we can refer to these lemmas in the following.

(1): This was proved in [19, Lemma 6.2].

(2): Let us confirm that there exists a constant $c(d,M) \in \mathbb{R}_{>0}$ depending only on d and M such that

$$|C_l^{\delta}(\rho\mathbf{x}\sigma s, \eta\mathbf{y}\tau s)| \le c(d,M)(M^{l-N_h} + M^{-l}), \tag{3.10}$$
$$(\forall(\rho,\mathbf{x},\sigma),(\eta,\mathbf{y},\tau) \in \mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}, s \in [0,\beta)_h).$$

By periodicity, for any $j \in \{1,2,\cdots,d\}$,

$$\begin{aligned}
&\frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle} - 1)C_l^{+}(\cdot\mathbf{x}\sigma x, \cdot\mathbf{y}\tau y)\\
&= \frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*} e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega}\chi_{h,l}(\omega)\frac{L}{2\pi}\int_0^{2\pi/L} dp\\
&\quad\cdot\frac{\partial}{\partial p}h^{-1}(I_{2^d} - e^{-i\frac{\omega}{h}I_{2^d} + \frac{1}{h}\mathcal{E}(\mathbf{k}+p\mathbf{e}_j)})^{-1}\\
&= \frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*} e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega}\chi_{h,l}(\omega)\frac{L}{2\pi}\int_0^{2\pi/L} dp\\
&\quad\cdot h^{-1}(I_{2^d} - e^{-i\frac{\omega}{h}I_{2^d} + \frac{1}{h}\mathcal{E}(\mathbf{k}+p\mathbf{e}_j)})^{-1}\left(\frac{\partial}{\partial p}h(e^{-i\frac{\omega}{h}I_{2^d} + \frac{1}{h}\mathcal{E}(\mathbf{k}+p\mathbf{e}_j)} - I_{2^d})\right)\\
&\quad\cdot h^{-1}(I_{2^d} - e^{-i\frac{\omega}{h}I_{2^d} + \frac{1}{h}\mathcal{E}(\mathbf{k}+p\mathbf{e}_j)})^{-1}.
\end{aligned}$$

Using the inequalities [19, (6.7), (6.10), (6.14)], we can derive from the above equality that

$$\left\| \frac{L}{2\pi}(e^{-i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1)C_l^{+}(\cdot\mathbf{x}\sigma x,\cdot\mathbf{y}\tau y)\right\|_{2^d\times 2^d}\le c(d,M)M^{-l}.$$

This inequality implies that

$$\begin{aligned}&|C_l^{+}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)|\le c(d,M)M^{-l},\\ &(\forall(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in I_0 \text{ with } \mathbf{x}\neq\mathbf{y}).\end{aligned}\tag{3.11}$$

In the final part of the proof of [19, Lemma 6.2] we proved that

$$\|C_l^{+}(\cdot\mathbf{0}\sigma 0,\cdot\mathbf{0}\sigma 0)\|_{2^d\times 2^d}\le c(M)(M^{l-N_h}+M^{-l})\tag{3.12}$$

with a constant $c(M)\in\mathbb{R}_{>0}$ depending only on M. The inequalities (3.11), (3.12) imply (3.10) for $\delta=+$. The proof for $\delta=-$ is parallel. The determinant bound (3.5) can be obtained by combining the determinant bound (3.4) with (3.10).

(3),(5): These were essentially proved in [19, Lemma 6.2, Lemma 6.3]. Recall that the weight "w(0)" was given by

$$c_w(d+1)^{-2}\min\{M_{UV},(E_2+1)^{-1}\}M^{-2}$$

with a constant $c_w\in(0,1]$ independent of any parameter in [19, Lemma 6.2]. Since $E_2=1$ in the present case, $\min\{M_{UV},(E_2+1)^{-1}\}=1/2$. We can replace $(1/2)c_w$ in the weight "w(0)" in [19, Lemma 6.2, Lemma 6.3] by c_w to obtain the weight $c_w(d+1)^{-2}M^{-2}$ with some $c_w\in(0,1]$ and thus (3.6) and (3.8) follow.

(4): The inequality [19, (6.27)] implies that

$$\begin{aligned}&|C_l^{+}(\beta_1)(\rho\mathbf{x}\sigma r_{\beta_1}(x),\eta\mathbf{y}\tau r_{\beta_1}(y))-C_l^{+}(\beta_2)(\rho\mathbf{x}\sigma r_{\beta_2}(x),\eta\mathbf{y}\tau r_{\beta_2}(y))|\\ &\le c(d,M)\beta_1^{-\frac{1}{2}}M^{-\frac{l}{2}},\quad(\forall(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in\hat{I}_0).\end{aligned}\tag{3.13}$$

Take any $(\rho_i,\mathbf{x}_i,\sigma_i,x_i),(\eta_i,\mathbf{y}_i,\tau_i,y_i)\in\hat{I}_0$ and $\mathbf{p}_i,\mathbf{q}_i\in\mathbb{C}^m$ satisfying $\|\mathbf{p}_i\|_{\mathbb{C}^m}$, $\|\mathbf{q}_i\|_{\mathbb{C}^m}\le 1$ $(i=1,2,\cdots,n)$. Define $C_1,C_2\in\mathrm{Mat}(n,\mathbb{C})$ by

$$C_a:=(\langle\mathbf{p}_i,\mathbf{q}_j\rangle_{\mathbb{C}^m}C_l^{+}(\beta_a)(\rho_i\mathbf{x}_i\sigma_i r_{\beta_a}(x_i),\eta_j\mathbf{y}_j\tau_j r_{\beta_a}(y_j)))_{1\le i,j\le n},\quad(a=1,2).$$

Since

$$C_1 - C_2 = \begin{pmatrix} C_1 & I_n \end{pmatrix} \begin{pmatrix} I_n \\ -C_2 \end{pmatrix},$$

the Cauchy-Binet formula yields that

$$\det(C_1 - C_2) = \sum_{\substack{\gamma:\{1,2,\cdots,n\}\to\{1,2,\cdots,2n\} \\ \text{with } \gamma(1)<\gamma(2)<\cdots<\gamma(n)}} \det\left(\begin{pmatrix} C_1 & I_n \end{pmatrix}(i,\gamma(j))\right)_{1\le i,j\le n} \cdot \det\left(\begin{pmatrix} I_n \\ -C_2 \end{pmatrix}(\gamma(i),j)\right)_{1\le i,j\le n}.$$

By using (3.4) and assuming that $\gamma(0) = n$, $\gamma(n+1) = n+1$ we see that

$$|\det(C_1 - C_2)| \le \sum_{m=0}^{n} \sum_{\substack{\gamma:\{1,2,\cdots,n\}\to\{1,2,\cdots,2n\} \\ \text{with } \gamma(1)<\gamma(2)<\cdots<\gamma(n)}} 1_{\gamma(m)\le n<\gamma(m+1)} c_0^m c_0^{n-m} = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} n \\ n-m \end{pmatrix} c_0^n \le 2^{2n} c_0^n. \tag{3.14}$$

By expanding along the 1st column and using (3.13), (3.14) we have

$$|\det(C_1 - C_2)| \le c(d,M)\beta_1^{-\frac{1}{2}} M^{-\frac{l}{2}} \sum_{s=1}^{n} \left| \det((C_1 - C_2)(i,j))_{\substack{1\le i,j\le n \\ i\neq s, j\neq 1}} \right| \le c(d,M)\beta_1^{-\frac{1}{2}} M^{-\frac{l}{2}} n 2^{2(n-1)} c_0^{n-1}. \tag{3.15}$$

By applying the Cauchy-Binet formula once more and substituting (3.4), (3.15),

$$\begin{aligned} &|\det C_1 - \det C_2| \\ &= \Bigg| \sum_{\substack{\gamma:\{1,2,\cdots,n\}\to\{1,2,\cdots,2n\} \\ \text{with } \gamma(1)<\gamma(2)<\cdots<\gamma(n),\ r(1)\le n}} \cdot \det\left(\begin{pmatrix} C_1 - C_2 & I_n \end{pmatrix}(i,\gamma(j))\right)_{1\le i,j\le n} \det\left(\begin{pmatrix} I_n \\ C_2 \end{pmatrix}(\gamma(i),j)\right)_{1\le i,j\le n} \Bigg| \\ &\le \sum_{m=1}^{n} \sum_{\substack{\gamma:\{1,2,\cdots,n\}\to\{1,2,\cdots,2n\} \\ \text{with } \gamma(1)<\gamma(2)<\cdots<\gamma(n)}} 1_{\gamma(m)\le n<\gamma(m+1)} c(d,M) \end{aligned}$$

$$\cdot \beta^{-\frac{1}{2}} M^{-\frac{l}{2}} m 2^{2(m-1)} c_0^{m-1} c_0^{n-m}$$
$$\le \beta^{-\frac{1}{2}} M^{-\frac{l}{2}} (c(d,M)c_0)^n.$$

Thus, we obtained the determinant bound of the form (3.7) for $\delta = +$. The bound for $\delta = -$ can be proved in the same way. □

3.2. Isothermal bounds

Our multi-scale analysis at fixed temperature is built on estimation of kernels of Grassmann polynomials with respect to scale-dependent (semi-)norms. Let us define the (semi-)norms at this point. Set

$$\mathrm{w}(0) := c_w (d+1)^{-2} M^{-2}$$

with the constant $c_w \in (0,1]$ appearing in Lemma 3.1. For $l \in \mathbb{Z}_{\le 0}$, set $\mathrm{w}(l) := \mathrm{w}(0) M^l$. For an anti-symmetric function f on I^m $(m \ge 2)$ we define $\|f\|_{l,0}$, $\|f\|_{l,1}$ by

$$(3.16)\quad \|f\|_{l,0} := \sup_{X \in I} \left(\frac{1}{h}\right)^{m-1} \sum_{\mathbf{Y}=(Y_1,Y_2,\cdots,Y_{m-1}) \in I^{m-1}} e^{\sum_{j=0}^{d} (\mathrm{w}(l) d_j(X,Y_1))^{1/2}} \cdot |f(X,\mathbf{Y})|,$$

$$\|f\|_{l,1} := \sup_{j' \in \{0,1,\cdots,d\}} \sup_{q \in \{1,2,\cdots,m-1\}} \sup_{X \in I} \cdot \left(\frac{1}{h}\right)^{m-1} \sum_{\mathbf{Y}=(Y_1,Y_2,\cdots,Y_{m-1}) \in I^{m-1}} d_{j'}(X,Y_q) e^{\sum_{j=0}^{d} (\mathrm{w}(l) d_j(X,Y_1))^{1/2}} \cdot |f(X,\mathbf{Y})|.$$

In our Matsubara UV integration, anti-symmetric kernels are measured by $\|\cdot\|_{0,t}$ $(t = 0,1)$. The measurement with $\|\cdot\|_{l,t}$ $(l < 0,\ t = 0,1)$ will be necessary in the infrared integration in Section 4. From now we assume that

$$h \ge e^{4d}$$

so that the results of Lemma 3.1 are available. The inequality (3.6) implies that

$$(3.17)\quad \|\widetilde{C_l^{\delta}}\|_{0,t} \le c_0 M^{-l}, \quad (\forall l \in \{1,2,\cdots,N_h\}, \delta \in \{+,-\}, t \in \{0,1\}).$$

Fix $\delta \in \{+,-\}$ and set

$$\begin{aligned}F^{N_h}(\psi) &:= -\sum_{m=1}^{N_v}\frac{1}{h}\sum_{s\in[0,\beta)_h}\sum_{\substack{(\rho_j,\mathbf{x}_j,\sigma_j),(\eta_j,\mathbf{y}_j,\tau_j)\in\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\\(j=1,2,\cdots,m)}}\\&\qquad\cdot(1_{\delta=+}+1_{\delta=-}(-1)^m)\\&\qquad\cdot V_m^L(\mathbf{U})((\nu(\rho_1,\mathbf{x}_1)\sigma_1,\cdots,\nu(\rho_m,\mathbf{x}_m)\sigma_m),\\&\qquad\qquad(\nu(\eta_1,\mathbf{y}_1)\tau_1,\cdots,\nu(\eta_m,\mathbf{y}_m)\tau_m))\\&\qquad\cdot\overline{\psi}_{\rho_1\mathbf{x}_1\sigma_1 s}\cdots\overline{\psi}_{\rho_m\mathbf{x}_m\sigma_m s}\psi_{\eta_1\mathbf{y}_1\tau_1 s}\cdots\psi_{\eta_m\mathbf{y}_m\tau_m s},\\T^{N_h}(\psi) &:= 0,\\J^{N_h}(\psi) &:= F^{N_h}(\psi)\end{aligned}$$

with $\mathbf{U}\in\mathbb{C}^{n_v}$. We input $J^{N_v}(\psi)$ into the Matsubara UV integration process as the initial data. We define $F^l(\psi)$, $T^l(\psi)$, $J^l(\psi)\in\bigwedge\mathcal{V}$ $(l=0,1,\cdots,N_h-1)$ inductively as follows. Assume that we have $J^{l+1}(\psi)\in\bigwedge\mathcal{V}$ for some $l\in\{0,1,\cdots,N_h-1\}$. Set

$$\begin{aligned}F^l(\psi) &:= \int J^{l+1}(\psi+\psi^1)d\mu_{C_{l+1}^\delta}(\psi^1),\\T^{l,(n)}(\psi) &:= \frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n\Big|_{z=0}\log\left(\int e^{zJ^{l+1}(\psi+\psi^1)}d\mu_{C_{l+1}^\delta}(\psi^1)\right)\end{aligned}$$

for $n\in\mathbb{N}_{\ge 2}$. Then, set

$$T^l(\psi) := \sum_{n=2}^{\infty}T^{l,(n)}(\psi),\quad J^l(\psi) := F^l(\psi)+T^l(\psi)$$

on the assumption that $\sum_{n=2}^\infty T^{l,(n)}(\psi)$ converges. See [19, Subsection 2.2] for the notion of convergence and differentiation of Grassmann polynomials. Note that $F^{N_h}(\psi)=-V^+(\psi)+\beta V_0^L$ if $\delta=+$, $F^{N_h}(\psi)=-V^-(\psi)+\beta V_0^L$ if $\delta=-$. Also, an inductive argument based on [19, Lemma 3.9 (1)], parallel to the proof of [19, Lemma 5.1] ensures that if $\sum_{n=2}^\infty T^{l,(n)}(\psi)$ converges for any $l\in\{0,1,\cdots,N_h-1\}$,

$$\begin{aligned}&T_m^{l,(n)}(\psi)=F_m^l(\psi)=0,\\&(\forall l\in\{0,1,\cdots,N_h\}, m\in\{0,1,\cdots,N\}\cap(2\mathbb{N}+1), n\in\mathbb{N}_{\ge 2}).\end{aligned}$$

Lemma 3.3.

$$\|F_{2m}^{N_h}\|_{0,t} \le \max_{k\in\{1,2,\cdots,n_v\}} |U_k| e^{d\mathrm{w}(0)^{1/2}} v_m(\mathrm{w}(0)),$$
$$(\forall m \in \{1,2,\cdots,N_v\}, t\in\{0,1\}).$$

Proof. By the uniqueness of the anti-symmetric kernel we have that for any $(\rho_j, \mathbf{x}_j, \sigma_j, s_j, \theta_j) \in I$ $(j = 1,2,\cdots,2m)$,

$$\begin{aligned}
&F_{2m}^{N_h}(\rho_1\mathbf{x}_1\sigma_1 s_1\theta_1,\cdots,\rho_{2m}\mathbf{x}_{2m}\sigma_{2m}s_{2m}\theta_{2m})\\
&= \frac{-1}{(2m)!}(1_{\delta=+} + 1_{\delta=-}(-1)^m)\\
&\quad \cdot \sum_{\xi\in\mathbb{S}_{2m}} \mathrm{sgn}(\xi) V_m^L(\nu(\rho_{\xi(1)},\mathbf{x}_{\xi(1)})\sigma_{\xi(1)},\cdots,\nu(\rho_{\xi(2m)},\mathbf{x}_{\xi(2m)})\sigma_{\xi(2m)})\\
&\quad \cdot h^{2m-1} 1_{s_1=\cdots=s_{2m}} 1_{(\theta_{\xi(1)},\cdots,\theta_{\xi(2m)})=(1,\cdots,1,-1,\cdots,-1)}.
\end{aligned}$$

If $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \Gamma(2L)$, $(\rho,\mathbf{x}),(\eta,\mathbf{y}) \in \mathcal{B}\times\Gamma(L)$ satisfy $\hat{\mathbf{x}} = \nu(\rho,\mathbf{x})$, $\hat{\mathbf{y}} = \nu(\eta,\mathbf{y})$, then

$$\frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle} - 1| \le \frac{L}{\pi}|e^{i\frac{\pi}{L}\langle \hat{\mathbf{x}}-\hat{\mathbf{y}},\mathbf{e}_j\rangle} - 1| + 1, \quad (\forall j \in \{1,2,\cdots,d\}).$$

Using this inequality and the invariances (1.5), (1.12), we observe that for $t \in \{0,1\}$,

$$\begin{aligned}
&\|F_{2m}^{N_h}\|_{0,t}\\
&\le \max_{k\in\{1,2,\cdots,n_v\}} |U_k| \sup_{\mathbf{U}\in\overline{D(1)}^{n_v}} \sup_{\substack{p,q\in\{1,2,\cdots,2m-1\}\\ j'\in\{1,2,\cdots,d\}}} \sup_{(\rho,\mathbf{x},\sigma)\in\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}}\\
&\quad \cdot \sum_{\substack{(\rho_j,\mathbf{x}_j,\sigma_j)\in\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\\ (j=1,2,\cdots,2m-1)}} \left(\frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{x}_q,\mathbf{e}_{j'}\rangle} - 1|\right)^t\\
&\quad \cdot e^{\sum_{j=1}^d (\mathrm{w}(0)\frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle \mathbf{x}-\mathbf{x}_p,\mathbf{e}_j\rangle}-1|)^{1/2}}\\
&\quad \cdot |V_m^L(\mathbf{U})((\nu(\rho,\mathbf{x})\sigma, \nu(\rho_1,\mathbf{x}_1)\sigma_1,\cdots,\nu(\rho_{m-1},\mathbf{x}_{m-1})\sigma_{m-1}),\\
&\qquad (\nu(\rho_m,\mathbf{x}_m)\sigma_m, \nu(\rho_{m+1},\mathbf{x}_{m+1})\sigma_{m+1},\cdots,\nu(\rho_{2m-1},\mathbf{x}_{2m-1})\sigma_{2m-1}))|
\end{aligned}$$

$$\leq \max_{k\in\{1,2,\cdots,n_v\}} |U_k| e^{d\mathrm{w}(0)^{1/2}} v_m(\mathrm{w}(0)). \square$$

The main purpose of this subsection is to prove the following lemma. We will refer to [19, Lemma 3.8] as the main tool in the proof.

LEMMA 3.4. *Let $\alpha \in \mathbb{R}_{\geq 1}$ and let c_0 be the constant appearing in Lemma 3.1. There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if*

$$M \geq c^{N_v^2}, \quad \alpha \geq c M^{\frac{1}{2}} \tag{3.18}$$

and

$$\max_{j\in\{1,2,\cdots,n_v\}} |U_j| e^{d\mathrm{w}(0)^{1/2}} \sum_{m=1}^{N_v} c_0^m \alpha^{2m} v_m(\mathrm{w}(0)) \leq \frac{1}{2}, \tag{3.19}$$

the following inequalities hold for any $l \in \{0,1,\cdots,N_h\}$, $t \in \{0,1\}$.

$$\frac{h}{N}(|F_0^l| + |T_0^l|) \leq \alpha^{-1}, \tag{3.20}$$

$$\sum_{m=1}^{2N_v} c_0^{\frac{m}{2}} \alpha^m (\|F_m^l\|_{0,t} + \|T_m^l\|_{0,t}) \leq 1, \tag{3.21}$$

$$M^{-\frac{N_v}{N_v-1}l} \sum_{m=1}^{N} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l}{2N_v-2}m} (\|F_m^l\|_{0,t} + \|T_m^l\|_{0,t}) \leq 1. \tag{3.22}$$

Moreover, for any $l \in \{0,1,\cdots,N_h-1\}$, $m \in \{1,2,\cdots,N\}$,

$$\sum_{n=2}^{\infty} \sup_{\substack{\mathbf{U}\in\mathbb{C}^{n_v} \text{ with} \\ |U_j|\leq U_{max}(\alpha,M)\ (j=1,\cdots,n_v)}} |T_0^{l,(n)}(\mathbf{U})| < \infty, \tag{3.23}$$

$$\sum_{n=2}^{\infty} \sup_{\substack{\mathbf{U}\in\mathbb{C}^{n_v} \text{ with} \\ |U_j|\leq U_{max}(\alpha,M)\ (j=1,\cdots,n_v)}} \|T_m^{l,(n)}(\mathbf{U})\|_{0,0} < \infty, \tag{3.24}$$

where

$$U_{max}(\alpha, M) := \left(2e^{d\mathrm{w}(0)^{1/2}} \sum_{m=1}^{N_v} c_0^m \alpha^{2m} v_m(\mathrm{w}(0)) \right)^{-1}.$$

Remark 3.5. We claim (3.23), (3.24) in order to emphasize the uniform convergent property of $\sum_{n=2}^{\infty} T^{l,(n)}(\psi)$ with respect to the coupling constants. We should have explicitly claimed the uniform convergent properties of the infinite series of the Grassmann polynomials produced by the tree expansions in [19, Proposition 5.2, Proposition 5.6, Proposition 6.4], though these properties are obvious from the proofs. Strictly speaking, the previous deduction of the regularity with the coupling constants [19, Proposition 6.4 (2)] from the point-wise convergent properties [19, Proposition 6.4 (1)] is incomplete. The claim [19, Proposition 6.4 (2)] is rigorously proved by additionally remarking the uniform convergent properties such as (3.23), (3.24) in [19, Proposition 6.4 (1)]. With the aim of convincing the readers of the validity of the construction, in this paper we intend to make clear the deduction of the regularity with the coupling constants from the uniform convergent properties. The clarification will be specifically made in the proof of Lemma 4.9 (1) and Lemma 4.10.

Proof of Lemma 3.4. During the proof the symbol c denotes a generic constant independent of any parameter. We replace c by a larger generic constant denoted by the same symbol from time to time without any comment. However, such replacements do not affect the conclusions of the proof. We prove the claimed inequalities by induction with $l \in \{0, 1, \cdots, N_h\}$. By assumption and Lemma 3.3,

$$\begin{aligned}
&\frac{h}{N}(|F_0^{N_h}| + |T_0^{N_h}|) = 0,\\
&\sum_{m=1}^{2N_v} c_0^{\frac{m}{2}} \alpha^m (\|F_m^{N_h}\|_{0,t} + \|T_m^{N_h}\|_{0,t}) \le \frac{1}{2},\\
&M^{-\frac{N_v}{N_v-1}N_h} \sum_{m=1}^{N} c_0^{\frac{m}{2}} \alpha^m M^{\frac{N_h}{2N_v-2}m} (\|F_m^{N_h}\|_{0,t} + \|T_m^{N_h}\|_{0,t})\\
&\le \sum_{m=1}^{2N_v} c_0^{\frac{m}{2}} \alpha^m \|F_m^{N_h}\|_{0,t} \le \frac{1}{2}, \quad (\forall t \in \{0,1\}).
\end{aligned} \tag{3.25}$$

Thus, the inequalities (3.20), (3.21), (3.22) hold for $l = N_h$.

Assume that $l \in \{0, 1, \cdots, N_h - 1\}$ and for any $j \in \{l+1, l+2, \cdots, N_h\}$, $t \in \{0, 1\}$ the inequalities (3.21), (3.22) hold.

Let us prepare a couple of inequalities. By the hypothesis of induction, for any $t \in \{0,1\}$,

$$\begin{aligned}(3.26)\quad &\sum_{m=2}^{N} 2^{3m} c_0^{\frac{m}{2}} \alpha^m \|J_m^{l+1}\|_{0,t}\\ &\le c^{N_v} \sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m \|J_m^{l+1}\|_{0,t}\\ &\quad + c^{N_v} M^{-\frac{N_v+1}{N_v-1}(l+1)} \sum_{m=2N_v+2}^{N} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l+1}{2N_v-2}m} \|J_m^{l+1}\|_{0,t}\\ &\le c^{N_v},\end{aligned}$$

$$\begin{aligned}(3.27)\quad &\sum_{m=2}^{N} 2^{2m} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l}{2N_v-2}m} \|J_m^{l+1}\|_{0,t}\\ &\le cM^{-\frac{1}{N_v-1}} \sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l+1}{2N_v-2}m} \|J_m^{l+1}\|_{0,t} \le cM^{-\frac{1}{N_v-1}+\frac{N_v}{N_v-1}(l+1)},\end{aligned}$$

where we especially used the condition that $M \ge c^{N_v}$.

By combining (3.4), (3.17), (3.26) with [19, Lemma 3.8 (1)] we obtain that for any $n \in \mathbb{N}_{\ge 2}$,

$$\begin{aligned}|T_0^{l,(n)}| &\le \frac{N}{h} c_0^{-n+1} (c_0 M^{-l-1})^{n-1} \left(\sum_{m=2}^{N} 2^{2m} c_0^{\frac{m}{2}} \|J_m^{l+1}\|_{0,0} \right)^n\\ &\le \frac{N}{h} M^{l+1} (c^{N_v} M^{-l-1} \alpha^{-2})^n.\end{aligned}$$

Thus, on the assumption $M \ge c^{N_v}$, (3.23) holds and

$$(3.28)\qquad \frac{h}{N} |T_0^l| \le c^{N_v} M^{-l-1} \alpha^{-4}.$$

By (3.4), (3.17) and [19, Lemma 3.8 (2)], for any $m \in \{2,3,\cdots,N\}$, $t \in \{0,1\}$, $n \in \mathbb{N}_{\ge 2}$,

$$\begin{aligned}(3.29)\quad &\|T_m^{l,(n)}\|_{0,t} \le 2^{-2m} c_0^{-\frac{m}{2}-n+1}\\ &\quad \cdot \prod_{i=1}^{n} \left(\sum_{q_i=0}^{1} \right) \prod_{j=2}^{n} \left(\sum_{r_j=0}^{1} \right) 1_{\sum_{i=1}^n q_i + \sum_{j=2}^n r_j = t}\end{aligned}$$

$$\cdot (c_0 M^{-l-1})^{n-1}$$
$$\cdot \prod_{k=1}^{n}\left(\sum_{m_k=2}^{N} 2^{3m_k} c_0^{\frac{m_k}{2}} \|J_{m_k}^{l+1}\|_{0,q_k}\right) 1_{\sum_{j=1}^{n} m_j-2n+2\ge m}.$$

Moreover, by substituting (3.26),

$$\begin{aligned}
&\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|T_m^{l,(n)}\|_{0,t}\\
&\le 2^{4n-4} M^{-(l+1)(n-1)} \alpha^{-2n+2} \prod_{i=1}^{n}\left(\sum_{q_i=0}^{1}\right) \prod_{j=2}^{n}\left(\sum_{r_j=0}^{1}\right) 1_{\sum_{i=1}^{n} q_i+\sum_{j=2}^{n} r_j=t}\\
&\quad\cdot \prod_{k=1}^{n}\left(\sum_{m_k=2}^{N} 2^{m_k} c_0^{\frac{m_k}{2}} \alpha^{m_k} \|J_{m_k}^{l+1}\|_{0,q_k}\right)\\
&\le M^{l+1}\alpha^2 (c^{N_v} M^{-l-1}\alpha^{-2})^n,
\end{aligned}$$

which implies on the assumption $M \ge c^{N_v}$ that (3.24) holds and

$$\sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m \|T_m^l\|_{0,t} \le c^{N_v} M^{-l-1}\alpha^{-2}, \quad (\forall t \in \{0,1\}). \tag{3.30}$$

Also, by (3.29) and (3.27),

$$\begin{aligned}
&M^{-\frac{N_v}{N_v-1}l} \sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l}{2N_v-2}m} \|T_m^{l,(n)}\|_{0,t}\\
&\le c^n M^{-\frac{N_v}{N_v-1}l-(l+1+\frac{l}{N_v-1})(n-1)} \alpha^{-2n+2}\\
&\quad\cdot \prod_{i=1}^{n}\left(\sum_{q_i=0}^{1}\right) \prod_{j=2}^{n}\left(\sum_{r_j=0}^{1}\right) 1_{\sum_{i=1}^{n} q_i+\sum_{j=2}^{n} r_j=t}\\
&\quad\cdot \prod_{k=1}^{n}\left(\sum_{m_k=2}^{N} 2^{m_k} c_0^{\frac{m_k}{2}} \alpha^{m_k} M^{\frac{l}{2N_v-2}m_k} \|J_{m_k}^{l+1}\|_{0,q_k}\right)\\
&\le M^{1-(l+1+\frac{l}{N_v-1})n} \alpha^{-2n+2} (c M^{-\frac{1}{N_v-1}+\frac{N_v}{N_v-1}(l+1)})^n\\
&= M\alpha^2 (c\alpha^{-2})^n.
\end{aligned}$$

Thus, on the assumption $\alpha \ge c$,

$$M^{-\frac{N_v}{N_v-1}l}\sum_{m=2}^{N} c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m}\|T_m^l\|_{0,t} \le cM\alpha^{-2}, \quad (\forall t\in\{0,1\}). \tag{3.31}$$

To establish upper bounds on the free part $F^l(\psi)$, we introduce the Grassmann polynomials $\hat{F}^j(\psi)$ $(j=l,l+1,\cdots,N_h)$ inductively as follows. Set $\hat{F}^{N_h}(\psi):=0$. Assume that $l'\in\{l,l+1,\cdots,N_h-1\}$ and we have $\hat{F}^j(\psi)$ $(j=l'+1,l'+2,\cdots,N_h)$. For any $m\in\{0,2N_v+1,2N_v+2,\cdots,N\}$, $\hat{F}_m^{l'}(\psi):=0$. For any $m\in\{1,2,\cdots,2N_v\}$,

$$\begin{aligned}\hat{F}_m^{l'}(\psi) :=& F_m^{l'}(\psi)-F_m^{N_h}(\psi)\\ &-\sum_{j=l'+1}^{N_h}\mathcal{P}_m\sum_{n=m+2}^{2N_v}\\ &\cdot\int (F_n^j(\psi+\psi^1)-\hat{F}_n^j(\psi+\psi^1))d\mu_{C_j^\delta}(\psi^1),\end{aligned}\tag{3.32}$$

where $\mathcal{P}_m:\bigwedge\mathcal{V}\to\bigwedge^m\mathcal{V}$ is the standard projection. It follows that for any $l'\in\{l,l+1,\cdots,N_h\}$, $m\in\{1,2,\cdots,2N_v\}$, $(\rho_j,\mathbf{x}_j,\sigma_j,s_j,\theta_j)\in I$ $(j=1,2,\cdots,m)$,

$$\begin{aligned}&1_{\exists j\exists k\in\{1,2,\cdots,m\}(j\neq k\wedge s_j\neq s_k)}\\ &\cdot(F_m^{l'}(\rho_1\mathbf{x}_1\sigma_1 s_1\theta_1,\cdots,\rho_m\mathbf{x}_m\sigma_m s_m\theta_m)\\ &\quad-\hat{F}_m^{l'}(\rho_1\mathbf{x}_1\sigma_1 s_1\theta_1,\cdots,\rho_m\mathbf{x}_m\sigma_m s_m\theta_m))=0.\end{aligned}\tag{3.33}$$

In fact, the equality (3.33) is true for $l'=N_h$ by definition. Assume that it holds true for any $j\in\{l'+1,l'+2,\cdots,N_h\}$. Since

$$\begin{aligned}F_m^{l'}(\mathbf{X})-\hat{F}_m^{l'}(\mathbf{X}) =& F_m^{N_h}(\mathbf{X})\\ &+\sum_{j=l'+1}^{N_h}\sum_{n=m+2}^{2N_v}\begin{pmatrix} n\\ m\end{pmatrix}\left(\frac{1}{h}\right)^{n-m}\sum_{\mathbf{Y}\in I^{n-m}}\\ &\cdot(F_n^j(\mathbf{X},\mathbf{Y})-\hat{F}_n^j(\mathbf{X},\mathbf{Y}))\int\psi_{\mathbf{Y}}^1 d\mu_{C_j^\delta}(\psi^1),\\ &(\forall m\in\{1,2,\cdots,2N_v\},\mathbf{X}\in I^m),\end{aligned}\tag{3.34}$$

the equality (3.33) holds for l' as well. Thus, by induction the equality (3.33) is true for any $l'\in\{l,l+1,\cdots,N_h\}$.

Let us prove that for any $l' \in \{l, l+1, \cdots, N_h\}$, $t \in \{0,1\}$,

$$\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m \|\hat{F}_m^{l'}\|_{0,t} \le \alpha^{-2} M^{-\frac{l'}{N_v-1}}. \tag{3.35}$$

This inequality is true for $l' = N_h$ by definition. Assume that $l' \in \{l, l+1, \cdots, N_h - 1\}$ and (3.35) holds for any $j \in \{l'+1, l'+2, \cdots, N_h\}$. Note that for any $m \in \{2, 3, \cdots, 2N_v\}$,

$$\begin{aligned} F_m^{l'}(\psi) = F_m^{l'+1}(\psi) + T_m^{l'+1}(\psi) &+ \mathcal{P}_m \sum_{n=m+2}^{2N_v} \int F_n^{l'+1}(\psi + \psi^1) d\mu_{C_{l'+1}^{\delta}}(\psi^1) \\ &+ \mathcal{P}_m \sum_{n=m+2}^{2N_v} \int T_n^{l'+1}(\psi + \psi^1) d\mu_{C_{l'+1}^{\delta}}(\psi^1) \\ &+ \mathcal{P}_m \sum_{n=2N_v+2}^{N} \int J_n^{l'+1}(\psi + \psi^1) d\mu_{C_{l'+1}^{\delta}}(\psi^1), \end{aligned}$$

and thus,

$$\begin{aligned} \hat{F}_m^{l'}(\psi) = \hat{F}_m^{l'+1}(\psi) &+ T_m^{l'+1}(\psi) \\ &+ \mathcal{P}_m \sum_{n=m+2}^{2N_v} \int \hat{F}_n^{l'+1}(\psi + \psi^1) d\mu_{C_{l'+1}^{\delta}}(\psi^1) \\ &+ \mathcal{P}_m \sum_{n=m+2}^{2N_v} \int T_n^{l'+1}(\psi + \psi^1) d\mu_{C_{l'+1}^{\delta}}(\psi^1) \\ &+ \mathcal{P}_m \sum_{n=2N_v+2}^{N} \int J_n^{l'+1}(\psi + \psi^1) d\mu_{C_{l'+1}^{\delta}}(\psi^1). \end{aligned} \tag{3.36}$$

It follows from this equality and an estimation similar to [19, Lemma 3.1] that

$$\begin{aligned} \|\hat{F}_m^{l'}\|_{0,t} \le& \sum_{n=m}^{2N_v} 2^n c_0^{\frac{n-m}{2}} \|\hat{F}_n^{l'+1}\|_{0,t} \\ &+ \sum_{n=m}^{2N_v} 2^n c_0^{\frac{n-m}{2}} \|T_n^{l'+1}\|_{0,t} + \sum_{n=2N_v+2}^{N} 2^n c_0^{\frac{n-m}{2}} \|J_n^{l'+1}\|_{0,t}. \end{aligned}$$

Using (3.22), (3.30), (3.35) for $l'+1$ and the conditions $\alpha \geq c$, $M \geq c^{N_v^2}$, we have that

$$\begin{aligned}
&\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m \|\hat{F}_m^{l'}\|_{0,t} \\
&\leq \sum_{m=2}^{2N_v} 2^m \sum_{n=m}^{2N_v} c_0^{\frac{n}{2}} \alpha^n \|\hat{F}_n^{l'+1}\|_{0,t} + \sum_{m=2}^{2N_v} 2^m \sum_{n=m}^{2N_v} c_0^{\frac{n}{2}} \alpha^n \|T_n^{l'+1}\|_{0,t} \\
&\quad + \sum_{m=2}^{2N_v} 2^{2N_v+2} \alpha^{m-2N_v-2} M^{-\frac{l'+1}{N_v-1}(N_v+1)} \sum_{n=2N_v+2}^{N} c_0^{\frac{n}{2}} \alpha^n M^{\frac{l'+1}{2N_v-2}n} \|J_n^{l'+1}\|_{0,t} \\
&\leq c^{N_v} \alpha^{-2} M^{-\frac{l'+1}{N_v-1}} \leq \alpha^{-2} M^{-\frac{l'}{N_v-1}}.
\end{aligned}$$

Thus, the inequality (3.35) for l' holds. By induction, (3.35) holds for all $l' \in \{l, l+1, \cdots, N_h\}$.

By (3.5), (3.33) and (3.34), for any $m \in \{2, 3, \cdots, 2N_v\}$, $t \in \{0, 1\}$,

$$\begin{aligned}
\|F_m^l\|_{0,t} \leq& \|\hat{F}_m^l\|_{0,t} + \|F_m^{N_h}\|_{0,t} \\
&+ \sum_{j=l+1}^{N_h} \sum_{n=m+2}^{2N_v} 2^n (M^{-j} + M^{j-N_h}) c_0^{\frac{n-m}{2}} (\|F_n^j\|_{0,t} + \|\hat{F}_n^j\|_{0,t}).
\end{aligned}$$

Moreover, by (3.21) for $l' \in \{l+1, l+2, \cdots, N_h\}$, (3.25), (3.35) for $l' \in \{l, l+1, \cdots, N_h\}$ and the assumption that $\alpha \geq 2$, $M \geq 2$,

$$\begin{aligned}
\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m \|F_m^l\|_{0,t} &\leq \sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m \|\hat{F}_m^l\|_{0,t} + \sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m \|F_m^{N_h}\|_{0,t} \\
&\quad + \sum_{m=2}^{2N_v} \sum_{j=l+1}^{N_h} 2^{m+2} \alpha^{-2} (M^{-j} + M^{j-N_h}) \\
&\quad \cdot \sum_{n=m+2}^{2N_v} c_0^{\frac{n}{2}} \alpha^n (\|F_n^j\|_{0,t} + \|\hat{F}_n^j\|_{0,t}) \\
&\leq \frac{1}{2} + c^{N_v} \alpha^{-2}.
\end{aligned} \tag{3.37}$$

This also yields that

$$M^{-\frac{N_v}{N_v-1}l}\sum_{m=2}^{2N_v}c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m}\|F_m^l\|_{0,t}\le\frac{1}{2}+c^{N_v}\alpha^{-2}. \tag{3.38}$$

On the other hand, for $m\in\{2N_v+2,2N_v+3,\cdots,N\}$,

$$F_m^l(\psi)=J_m^{l+1}(\psi)+\mathcal{P}_m\sum_{n=m+2}^{N}\int J_n^{l+1}(\psi+\psi^1)d\mu_{C_{l+1}^\delta}(\psi^1)$$

and thus by (3.4),

$$\|F_m^l\|_{0,t}\le\sum_{n=m}^{N}2^n c_0^{\frac{n-m}{2}}\|J_n^{l+1}\|_{0,t}.$$

Moreover, by (3.22) for $l+1$ and the condition $M\ge c^{N_v}$,

$$\begin{aligned}
&\sum_{m=2N_v+2}^{N}c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m}\|F_m^l\|_{0,t}\\
&\le\sum_{n=2N_v+2}^{N}\sum_{m=2N_v+2}^{n}\alpha^m M^{\frac{l}{2N_v-2}m}2^n c_0^{\frac{n}{2}}\|J_n^{l+1}\|_{0,t}\\
&\le c\sum_{n=2N_v+2}^{N}2^n c_0^{\frac{n}{2}}\alpha^n M^{\frac{l}{2N_v-2}n}\|J_n^{l+1}\|_{0,t}\\
&\le c^{N_v}M^{-\frac{N_v+1}{N_v-1}}\sum_{n=2N_v+2}^{N}c_0^{\frac{n}{2}}\alpha^n M^{\frac{l+1}{2N_v-2}n}\|J_n^{l+1}\|_{0,t}\\
&\le c^{N_v}M^{-\frac{N_v+1}{N_v-1}+\frac{N_v}{N_v-1}(l+1)},
\end{aligned}$$

or

$$M^{-\frac{N_v}{N_v-1}l}\sum_{m=2N_v+2}^{N}c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m}\|F_m^l\|_{0,t}\le c^{N_v}M^{-\frac{1}{N_v-1}}. \tag{3.39}$$

It remains to deal with F_0^l. Note that

$$F_0^l=F_0^{l+1}+T_0^{l+1}+\sum_{m=2}^{2N_v}\int\hat{F}_m^{l+1}(\psi)d\mu_{C_{l+1}^\delta}(\psi)$$

$$+\sum_{m=2}^{2N_v}\int(F_m^{l+1}(\psi)-\hat{F}_m^{l+1}(\psi))d\mu_{C_{l+1}^{\delta}}(\psi)+\sum_{m=2}^{2N_v}\int T_m^{l+1}(\psi)d\mu_{C_{l+1}^{\delta}}(\psi)$$
$$+\sum_{m=2N_v+2}^{N}\int J_m^{l+1}(\psi)d\mu_{C_{l+1}^{\delta}}(\psi).$$

Then, by (3.4), (3.5), (3.21), (3.22), (3.28), (3.30), (3.35) for $l'\in\{l+1,l+2,\cdots,N_h\}$ and (3.33),

$$\begin{aligned}(3.40)\qquad |F_0^l|&\le|F_0^{l+1}|+|T_0^{l+1}|+\frac{N}{h}\sum_{m=2}^{2N_v}c_0^{\frac{m}{2}}\|\hat{F}_m^{l+1}\|_{0,0}\\&\quad+\frac{N}{h}\sum_{m=2}^{2N_v}(M^{-l-1}+M^{l+1-N_h})c_0^{\frac{m}{2}}(\|\hat{F}_m^{l+1}\|_{0,0}+\|F_m^{l+1}\|_{0,0})\\&\quad+\frac{N}{h}\sum_{m=2}^{2N_v}c_0^{\frac{m}{2}}\|T_m^{l+1}\|_{0,0}+\frac{N}{h}\sum_{m=2N_v+2}^{N}c_0^{\frac{m}{2}}\|J_m^{l+1}\|_{0,0}\\&\le|F_0^{l+1}|+\frac{N}{h}c^{N_v}\alpha^{-2}(M^{-l-1}+M^{l+1-N_h}+M^{-\frac{l+1}{N_v-1}})\\&\le\frac{N}{h}c^{N_v}\alpha^{-2}\sum_{j=l}^{N_h-1}(M^{-j-1}+M^{j+1-N_h}+M^{-\frac{j+1}{N_v-1}})\\&\le\frac{N}{h}c^{N_v}\alpha^{-2}.\end{aligned}$$

Finally we sum up (3.28), (3.30), (3.31), (3.37), (3.38), (3.39) and (3.40) to deduce that for any $t\in\{0,1\}$,

$$\begin{aligned}&\frac{h}{N}(|F_0^l|+|T_0^l|)\le c^{N_v}\alpha^{-2},\\&\sum_{m=1}^{2N_v}c_0^{\frac{m}{2}}\alpha^m(\|F_m^l\|_{0,t}+\|T_m^l\|_{0,t})\le\frac{1}{2}+c^{N_v}\alpha^{-2},\\&M^{-\frac{N_v}{N_v-1}l}\sum_{m=1}^{N}c_0^{\frac{m}{2}}\alpha^mM^{\frac{l}{2N_v-2}m}(\|F_m^l\|_{0,t}+\|T_m^l\|_{0,t})\\&\le\frac{1}{2}+c^{N_v}\alpha^{-2}+cM\alpha^{-2}+c^{N_v}M^{-\frac{1}{N_v-1}}.\end{aligned}$$

Recall that so far we have used the conditions $\alpha \ge c$, $M \ge c^{N_v^2}$ and (3.19). Now we can see that under the conditions (3.18) with a sufficiently large generic constant c the inequalities above imply (3.20), (3.21), (3.22) for l. Therefore, by induction these inequalities hold true for all $l \in \{0, 1, \cdots, N_h\}$, $t \in \{0, 1\}$ on the conditions (3.18), (3.19). □

3.3. Anisothermal bounds

Our result concerning the existence of the zero-temperature limit of the free energy density is made out of a series of estimates on the differences between Grassmann polynomials defined at 2 different temperatures. As one part of these analysis, here we focus on establishing temperature-dependent upper bounds on Grassmann polynomials produced by the Matsubara UV integration. In addition to the notations already introduced in Subsection 3.1 and Subsection 3.2, let us define some notations necessary for our anisothermal measurements. These notations are essentially same as those introduced in the beginning of [19, Section 4].

For any $\mathbf{X} = ((\rho_1, \mathbf{x}_1, \sigma_1, s_1), (\rho_2, \mathbf{x}_2, \sigma_2, s_2), \cdots, (\rho_m, \mathbf{x}_m, \sigma_m, s_m)) \in (\mathcal{B} \times \Gamma(L) \times \{\uparrow, \downarrow\} \times (1/h)\mathbb{Z})^m$, we define $R_\beta(\mathbf{X}) \in I_0^m$, $N_\beta(\mathbf{X}) \in \mathbb{Z}$ by

$$
\begin{aligned}
&R_\beta(\mathbf{X}) := ((\rho_1, \mathbf{x}_1, \sigma_1, r_\beta(s_1)), \cdots, (\rho_m, \mathbf{x}_m, \sigma_m, r_\beta(s_m))), \\
&N_\beta(\mathbf{X}) := \sum_{j=1}^m n_\beta(s_j).
\end{aligned}
$$

Though this is admittedly abuse of notation, we let $R_\beta(\mathbf{X})$, $N_\beta(\mathbf{X})$ denote

$$
\begin{aligned}
&((\rho_1, \mathbf{x}_1, \sigma_1, r_\beta(s_1), \theta_1), \cdots, (\rho_m, \mathbf{x}_m, \sigma_m, r_\beta(s_m), \theta_m)) \ (\in I^m), \\
&\sum_{j=1}^m n_\beta(s_j) \ (\in \mathbb{Z})
\end{aligned}
$$

respectively, for $\mathbf{X} = ((\rho_1, \mathbf{x}_1, \sigma_1, s_1, \theta_1), \cdots, (\rho_m, \mathbf{x}_m, \sigma_m, s_m, \theta_m)) \in (\mathcal{B} \times \Gamma(L) \times \{\uparrow, \downarrow\} \times (1/h)\mathbb{Z} \times \{1, -1\})^m$ as well.

For $\mathbf{X} = ((\rho_1, \mathbf{x}_1, \sigma_1, s_1, \theta_1), \cdots, (\rho_m, \mathbf{x}_m, \sigma_m, s_m, \theta_m)) \in (\mathcal{B} \times \Gamma(L) \times \{\uparrow, \downarrow\} \times (1/h)\mathbb{Z} \times \{1, -1\})^m$, $s \in (1/h)\mathbb{Z}$, we let $\mathbf{X} + s$ denote

$$
((\rho_1, \mathbf{x}_1, \sigma_1, s_1 + s, \theta_1), \cdots, (\rho_m, \mathbf{x}_m, \sigma_m, s_m + s, \theta_m))
$$

in order to shorten formulas.

In the rest of this subsection we always assume (3.3). Set

$$\left[\frac{\beta_1}{4}, \beta_a - \frac{\beta_1}{4}\right)_h := \left\{\frac{\beta_1}{4}, \frac{\beta_1}{4} + \frac{1}{h}, \cdots, \beta_a - \frac{\beta_1}{4} - \frac{1}{h}\right\}, \quad (a = 1, 2).$$

We measure the difference between anti-symmetric functions $f(\beta_a) : I(\beta_a)^m \to \mathbb{C}$ $(a = 1, 2, m \in \mathbb{N}_{\ge 2})$ by the quantify $|f(\beta_1) - f(\beta_2)|_l$, which is defined by

$$\begin{aligned}
&|f(\beta_1) - f(\beta_2)|_l \\
&:= \sup_{X \in I^0} \left(\frac{1}{h}\right)^{m-1} \sum_{(Y_1, Y_2, \cdots, Y_{m-1}) \in \hat{I}^{m-1}} e^{\sum_{j=0}^{d} (\frac{1}{\pi} \mathrm{w}(l) \hat{d}_j(X, Y_1))^{1/2}} \\
&\quad \cdot |f(\beta_1)(R_{\beta_1}(X, Y_1, \cdots, Y_{m-1})) - f(\beta_2)(R_{\beta_2}(X, Y_1, \cdots, Y_{m-1}))|.
\end{aligned}$$

In this subsection we estimate Grassmann polynomials by using $|\cdot - \cdot|_0$. The infrared analysis in Section 4 will largely use $|\cdot - \cdot|_l$ with $l \in \mathbb{Z}_{<0}$.

With these notations we have for any $l \in \{1, 2, \cdots, N_h\}$, $\delta \in \{+, -\}$ that

$$\begin{aligned}
&\widetilde{C_l^\delta}(\beta_a)(\mathbf{X}) = (-1)^{N_{\beta_a}(\mathbf{X}+s)} \widetilde{C_l^\delta}(\beta_a)(R_{\beta_a}(\mathbf{X} + s)), \\
&(\forall \mathbf{X} \in I(\beta_a)^2, s \in (1/h)\mathbb{Z}, a \in \{1, 2\}),
\end{aligned} \tag{3.41}$$

$$|\widetilde{C_l^\delta}(\beta_1) - \widetilde{C_l^\delta}(\beta_2)|_0 \le \beta_1^{-\frac{1}{2}} M^{-l} c_0. \tag{3.42}$$

The inequality (3.42) is due to (3.8).

Fix $\delta \in \{+, -\}$ and for $a = 1, 2$ let $F^l(\beta_a)(\psi)$, $T^l(\beta_a)(\psi)$, $J^l(\beta_a)(\psi) \in \bigwedge \mathcal{V}(\beta_a)$ $(l = 0, 1, \cdots, N_h)$ be the Grassmann polynomials defined in the beginning of the previous subsection at the inverse temperature β_a.

By anti-symmetry, for any $f(\beta_1)(\psi) \in \bigwedge \mathcal{V}(\beta_1)$ and $m \in \{N(\beta_1) + 1, N(\beta_1) + 2, \cdots, N(\beta_2)\}$, $f_m(\beta_1)(\psi) = 0$. Keeping this fact in mind, we can write that $f(\beta_1)(\psi) = \sum_{m=0}^{N(\beta_2)} f_m(\beta_1)(\psi)$.

Lemma 3.6.

$$\begin{aligned}
&F_m^l(\beta_a)(\mathbf{X}) = (-1)^{N_{\beta_a}(\mathbf{X}+s)} F_m^l(\beta_a)(R_{\beta_a}(\mathbf{X} + s)), \\
&T_m^{l,(n)}(\beta_a)(\mathbf{X}) = (-1)^{N_{\beta_a}(\mathbf{X}+s)} T_m^{l,(n)}(\beta_a)(R_{\beta_a}(\mathbf{X} + s)), \\
&(\forall l \in \{0, 1, \cdots, N_h\}, n \in \mathbb{N}_{\ge 2}, a \in \{1, 2\}, m \in \{1, 2, \cdots, N(\beta_2)\},
\end{aligned}$$

$\mathbf{X} \in I(\beta_a)^m, s \in (1/h)\mathbb{Z})$.

Proof. We can see from the definition that the claimed equalities hold for $l = N_h$. Then, by (3.41) the same inductive argument based on [19, Lemma 3.9 (1)] as in the proof of [19, Lemma 5.3] ensures the results. □

The invariant property summarized in Lemma 3.6 is one of the basic assumptions in the general theory [19, Section 4]. The rest of the assumptions in [19, Section 4] are the bound properties of the covariances which we prepared in Lemma 3.1. Thus, we can apply [19, Lemma 4.1, Lemma 4.6] in the proof of the following lemma.

Lemma 3.7. *Let $\alpha \in \mathbb{R}_{\geq 1}$ and let c_0 be the constant appearing in Lemma 3.1. There exists a constant $c \in \mathbb{R}_{>0}$ independent of any parameter such that if* (3.18) *holds with c and* (3.19) *holds, the following inequalities hold for any $l \in \{0, 1, \cdots, N_h\}$.*

$$\left|\frac{h}{N(\beta_1)}F_0^l(\beta_1) - \frac{h}{N(\beta_2)}F_0^l(\beta_2)\right| + \left|\frac{h}{N(\beta_1)}T_0^l(\beta_1) - \frac{h}{N(\beta_2)}T_0^l(\beta_2)\right| \leq \beta_1^{-\frac{1}{2}}\alpha^{-1}, \tag{3.43}$$

$$\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}}\alpha^m(|F_m^l(\beta_1) - F_m^l(\beta_2)|_0 + |T_m^l(\beta_1) - T_m^l(\beta_2)|_0) \leq \beta_1^{-\frac{1}{2}}, \tag{3.44}$$

$$M^{-\frac{N_v}{N_v-1}l}\sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m} \cdot (|F_m^l(\beta_1) - F_m^l(\beta_2)|_0 + |T_m^l(\beta_1) - T_m^l(\beta_2)|_0) \leq \beta_1^{-\frac{1}{2}}. \tag{3.45}$$

Proof. We assume the conditions (3.18), (3.19) with a constant $c' \in \mathbb{R}_{>0}$ so that the results of Lemma 3.4 hold for β_1 and β_2. Let us make clear the logic. During the proof we do not touch the initial constant c'. In the end of the proof we will see that all the estimations are justified if the initial constant c' is sufficiently large. In the following we use the symbol

c to express a generic positive constant independent of any parameter and will replace it by a larger constant denoted by the same symbol from time to time. This notational convention helps to simplify the arguments. Not to confuse, we should stress that a constant denoted by c does not depend on c', either.

We prove the claims by induction with $l \in \{0, 1, \cdots, N_h\}$. By definition the left-hand sides of the claimed inequalities for $l = N_h$ vanish. Thus, the results hold for $l = N_h$.

Assume that $l \in \{0, 1, \cdots, N_h - 1\}$ and for any $j \in \{l+1, l+2, \cdots, N_h\}$ the inequalities (3.44), (3.45) hold. In the same way as in the derivation of (3.26), (3.27) we can derive from the hypothesis of induction that

$$\sum_{m=2}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} \alpha^m |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_0 \le c^{N_v} \beta_1^{-\frac{1}{2}}, \tag{3.46}$$

$$\begin{aligned}&\sum_{m=2}^{N(\beta_2)} 2^{2m} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l}{2N_v-2}m} |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_0 \\ &\le c M^{-\frac{1}{N_v-1} + \frac{N_v}{N_v-1}(l+1)} \beta_1^{-\frac{1}{2}},\end{aligned} \tag{3.47}$$

on the assumption that $M \ge c^{N_v}$.

Substitution of (3.4), (3.7), (3.17), (3.26), (3.42), (3.46) into the inequality in [19, Lemma 4.6 (1)] yields that for any $n \in \mathbb{N}_{\ge 2}$,

$$\begin{aligned}&\left| \frac{h}{N(\beta_1)} T_0^{l,(n)}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{l,(n)}(\beta_2) \right| \\ &\le c^n M^{-(l+1)(n-1)} \left(\sum_{m=2}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} \sum_{a=1}^{2} \|J_m^{l+1}(\beta_a)\|_{0,0} \right)^{n-1} \\ &\quad \cdot \sum_{m=2}^{N(\beta_2)} 2^{3m} c_0^{\frac{m}{2}} \left(\beta_1^{-\frac{1}{2}} \sum_{b=1}^{2} \sum_{t=0}^{1} \|J_m^{l+1}(\beta_b)\|_{0,t} + |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_0 \right) \\ &\le \beta_1^{-\frac{1}{2}} M^{l+1} (c^{N_v} M^{-l-1} \alpha^{-2})^n.\end{aligned}$$

Moreover, on the assumption that $M \ge c^{N_v}$,

$$\left| \frac{h}{N(\beta_1)} T_0^{l}(\beta_1) - \frac{h}{N(\beta_2)} T_0^{l}(\beta_2) \right| \le c^{N_v} \beta_1^{-\frac{1}{2}} M^{-l-1} \alpha^{-4}. \tag{3.48}$$

By (3.4), (3.7), (3.17), (3.42) and [19, Lemma 4.6 (2)] we have for any $m \in \{2, 3, \cdots, N(\beta_2)\}$, $n \in \mathbb{N}_{\geq 2}$ that

$$\begin{aligned}(3.49)\qquad &|T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \\ &\leq c^n \cdot 2^{-2m} c_0^{-\frac{m}{2}} M^{-(l+1)(n-1)} \\ &\quad \cdot \prod_{j=2}^{n} \left(\sum_{m_j=2}^{N(\beta_2)} 2^{4m_j} c_0^{\frac{m_j}{2}} \sum_{a=1}^{2} \|J_{m_j}^{l+1}(\beta_a)\|_{0,0} \right) \\ &\quad \cdot \sum_{m_1=2}^{N(\beta_2)} 2^{4m_1} c_0^{\frac{m_1}{2}} \Bigg(\beta_1^{-\frac{1}{2}} \sum_{b=1}^{2} \sum_{t=0}^{1} \|J_{m_1}^{l+1}(\beta_b)\|_{0,t} \\ &\qquad\qquad + |J_{m_1}^{l+1}(\beta_1) - J_{m_1}^{l+1}(\beta_2)|_0 \Bigg) \\ &\quad \cdot 1_{\sum_{j=1}^{n} m_j - 2n + 2 \geq m}.\end{aligned}$$

Then, by (3.26) and (3.46),

$$\begin{aligned}&\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0 \\ &\leq c^n M^{-(l+1)(n-1)} \alpha^{-2n+2} \left(\sum_{m=2}^{N(\beta_2)} 2^{2m} c_0^{\frac{m}{2}} \alpha^m \sum_{a=1}^{2} \|J_m^{l+1}(\beta_a)\|_{0,0} \right)^{n-1} \\ &\quad \cdot \sum_{m=2}^{N(\beta_2)} 2^{2m} c_0^{\frac{m}{2}} \alpha^m \left(\beta_1^{-\frac{1}{2}} \sum_{b=1}^{2} \sum_{t=0}^{1} \|J_m^{l+1}(\beta_b)\|_{0,t} + |J_m^{l+1}(\beta_1) - J_m^{l+1}(\beta_2)|_0 \right) \\ &\leq \beta_1^{-\frac{1}{2}} M^{l+1} \alpha^2 (c^{N_v} M^{-l-1} \alpha^{-2})^n.\end{aligned}$$

Thus, on the assumption $M \geq c^{N_v}$ we have that

$$(3.50)\qquad \sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m |T_m^l(\beta_1) - T_m^l(\beta_2)|_0 \leq \beta_1^{-\frac{1}{2}} c^{N_v} M^{-l-1} \alpha^{-2}.$$

Also, it follows from (3.49) and (3.27), (3.47) that

$$M^{-\frac{N_v}{N_v-1} l} \sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l}{2N_v-2} m} |T_m^{l,(n)}(\beta_1) - T_m^{l,(n)}(\beta_2)|_0$$

$$
\begin{aligned}
&\le c^n M^{-\frac{N_v}{N_v-1}l-(l+1+\frac{l}{N_v-1})(n-1)}\alpha^{-2n+2}\\
&\quad\cdot\left(\sum_{m=2}^{N(\beta_2)}2^{2m}\alpha^m c_0^{\frac{m}{2}}M^{\frac{l}{2N_v-2}m}\sum_{a=1}^{2}\|J_m^{l+1}(\beta_a)\|_{0,0}\right)^{n-1}\\
&\quad\cdot\sum_{m=2}^{N(\beta_2)}2^{2m}c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m}\\
&\quad\cdot\left(\beta_1^{-\frac{1}{2}}\sum_{b=1}^{2}\sum_{t=0}^{1}\|J_m^{l+1}(\beta_b)\|_{0,t}+|J_m^{l+1}(\beta_1)-J_m^{l+1}(\beta_2)|_0\right)\\
&\le \beta_1^{-\frac{1}{2}}M^{1-(l+1+\frac{l}{N_v-1})n}\alpha^{-2n+2}(cM^{-\frac{1}{N_v-1}+\frac{N_v}{N_v-1}(l+1)})^n\\
&=\beta_1^{-\frac{1}{2}}M\alpha^2(c\alpha^{-2})^n.
\end{aligned}
$$

Thus, by the assumption $\alpha \ge c$,

$$
M^{-\frac{N_v}{N_v-1}l}\sum_{m=2}^{N(\beta_2)}c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m}|T_m^l(\beta_1)-T_m^l(\beta_2)|_0\le c\beta_1^{-\frac{1}{2}}M\alpha^{-2}. \tag{3.51}
$$

In order to find upper bounds on the difference between $F^l(\beta_1)(\psi)$ and $F^l(\beta_2)(\psi)$, we need to establish upper bounds on the difference between $\hat{F}^l(\beta_1)(\psi)$ and $\hat{F}^l(\beta_2)(\psi)$. To this end, first we need to confirm that

$$
\begin{aligned}
&\hat{F}_m^{l'}(\beta_a)(\mathbf{X})=(-1)^{N_{\beta_a}(\mathbf{X}+s)}\hat{F}_m^{l'}(\beta_a)(R_{\beta_a}(\mathbf{X}+s)),\\
&(\forall l'\in\{l,l+1,\cdots,N_h\},a\in\{1,2\},m\in\{1,2,\cdots,2N_v\},\\
&\quad\mathbf{X}\in I(\beta_a)^m,s\in(1/h)\mathbb{Z}),
\end{aligned}
\tag{3.52}
$$

where $\hat{F}_m^{l'}(\beta_a)(\mathbf{X})$ $(m=1,2,\cdots,2N_v)$ are the kernels of $\hat{F}^{l'}(\beta_a)(\psi)\in\bigwedge\mathcal{V}(\beta_a)$ defined in (3.32). By definition, (3.52) holds for $l'=N_h$. Assume that $l'\in\{l,l+1,\cdots,N_h-1\}$ and (3.52) is true for $j\in\{l'+1,l'+2,\cdots,N_h\}$. Take any $s\in(1/h)\mathbb{Z}$. It follows from (3.41) that for any $a\in\{1,2\}$, $n\in\mathbb{N}$, $\mathbf{Y}\in I(\beta_a)^n$, $j\in\{l'+1,l'+2,\cdots,N_h\}$,

$$
\int\psi_{R_{\beta_a}(\mathbf{Y}+s)}d\mu_{C_j^\delta(\beta_a)}(\psi)=(-1)^{N_{\beta_a}(\mathbf{Y}+s)}\int\psi_{\mathbf{Y}}d\mu_{C_j^\delta(\beta_a)}(\psi). \tag{3.53}
$$

By using this equality, Lemma 3.6, (3.34) and the induction hypothesis we have that for any $\mathbf{X}\in I(\beta_a)^m$,

$$
(-1)^{N_{\beta_a}(\mathbf{X}+s)}\hat{F}_m^{l'}(\beta_a)(R_{\beta_a}(\mathbf{X}+s))
$$

$$
\begin{aligned}
&= F_m^{l'}(\beta_a)(\mathbf{X}) - F_m^{N_h}(\beta_a)(\mathbf{X})\\
&\quad - \sum_{j=l'+1}^{N_h}\sum_{n=m+2}^{2N_v}\left(\begin{array}{c} n\\ m\end{array}\right)\left(\frac{1}{h}\right)^{n-m}\sum_{\mathbf{Y}\in I(\beta_a)^{n-m}}(-1)^{N_{\beta_a}(\mathbf{X}+s)}\\
&\quad\quad \cdot (F_n^j(\beta_a)(R_{\beta_a}(\mathbf{X}+s),\mathbf{Y}) - \hat{F}_n^j(\beta_a)(R_{\beta_a}(\mathbf{X}+s),\mathbf{Y}))\\
&\quad\quad \cdot \int \psi_{\mathbf{Y}}^1 d\mu_{C_j^\delta(\beta_a)}(\psi^1)\\
&= F_m^{l'}(\beta_a)(\mathbf{X}) - F_m^{N_h}(\beta_a)(\mathbf{X})\\
&\quad - \sum_{j=l'+1}^{N_h}\sum_{n=m+2}^{2N_v}\left(\begin{array}{c} n\\ m\end{array}\right)\left(\frac{1}{h}\right)^{n-m}\sum_{\mathbf{Y}\in I(\beta_a)^{n-m}}(-1)^{N_{\beta_a}(\mathbf{X}+s)+N_{\beta_a}(\mathbf{Y}+s)}\\
&\quad\quad \cdot (F_n^j(\beta_a)(R_{\beta_a}(\mathbf{X}+s), R_{\beta_a}(\mathbf{Y}+s))\\
&\quad\quad\quad - \hat{F}_n^j(\beta_a)(R_{\beta_a}(\mathbf{X}+s), R_{\beta_a}(\mathbf{Y}+s)))\\
&\quad\quad \cdot \int \psi_{\mathbf{Y}}^1 d\mu_{C_j^\delta(\beta_a)}(\psi^1)\\
&= \hat{F}_m^{l'}(\beta_a)(\mathbf{X}).
\end{aligned}
$$

Thus, by induction the equality (3.52) holds for all $l' \in \{l, l+1, \cdots, N_h\}$.

Let us prove that for any $l' \in \{l, l+1, \cdots, N_h\}$,

$$
\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}}\alpha^m |\hat{F}_m^{l'}(\beta_1) - \hat{F}_m^{l'}(\beta_2)|_0 \le \beta_1^{-\frac{1}{2}}\alpha^{-2}M^{-\frac{l'}{N_v-1}}. \tag{3.54}
$$

For $l' = N_h$ the inequality (3.54) holds since its left-hand side is zero. Assume that $l' \in \{l, l+1, \cdots, N_h - 1\}$ and (3.54) holds for all $j \in \{l'+1, l'+2, \cdots, N_h\}$. By (3.4), (3.7), (3.36) and the estimation parallel to [19, Lemma 4.1 (2)],

$$
\begin{aligned}
&|\hat{F}_m^{l'}(\beta_1) - \hat{F}_m^{l'}(\beta_2)|_0\\
&\le |\hat{F}_m^{l'+1}(\beta_1) - \hat{F}_m^{l'+1}(\beta_2)|_0 + |T_m^{l'+1}(\beta_1) - T_m^{l'+1}(\beta_2)|_0\\
&\quad + c\sum_{n=m+2}^{2N_v} 2^{2n} c_0^{\frac{n-m}{2}}\Bigg(|\hat{F}_n^{l'+1}(\beta_1) - \hat{F}_n^{l'+1}(\beta_2)|_0\\
&\quad\quad\quad + \beta_1^{-\frac{1}{2}}\sum_{t=0}^{1}\sum_{a=1}^{2}\|\hat{F}_n^{l'+1}(\beta_a)\|_{0,t} + |T_n^{l'+1}(\beta_1) - T_n^{l'+1}(\beta_2)|_0
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \beta_1^{-\frac{1}{2}} \sum_{t=0}^{1}\sum_{a=1}^{2} \|T_n^{l'+1}(\beta_a)\|_{0,t}\Bigg)\\
&+ c \sum_{n=2N_v+2}^{N(\beta_2)} 2^{2n} c_0^{\frac{n-m}{2}} \Bigg(|J_n^{l'+1}(\beta_1) - J_n^{l'+1}(\beta_2)|_0\\
&\quad + \beta_1^{-\frac{1}{2}} \sum_{t=0}^{1}\sum_{a=1}^{2} \|J_n^{l'+1}(\beta_a)\|_{0,t}\Bigg).
\end{aligned}
$$

On the assumption $\alpha \ge c$, $M \ge c^{N_v^2}$, insertion of (3.22), (3.30), (3.35), (3.45), (3.50), (3.54) for $l'+1$ yields that

$$
\begin{aligned}
&\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}} \alpha^m |\hat{F}_m^{l'}(\beta_1) - \hat{F}_m^{l'}(\beta_2)|_0\\
&\le c \sum_{m=2}^{2N_v} 2^{2m} \sum_{n=m}^{2N_v} c_0^{\frac{n}{2}} \alpha^n \Bigg(|\hat{F}_n^{l'+1}(\beta_1) - \hat{F}_n^{l'+1}(\beta_2)|_0 + |T_n^{l'+1}(\beta_1) - T_n^{l'+1}(\beta_2)|_0\\
&\quad + \beta_1^{-\frac{1}{2}} \sum_{t=0}^{1}\sum_{a=1}^{2} \|\hat{F}_n^{l'+1}(\beta_a)\|_{0,t} + \beta_1^{-\frac{1}{2}} \sum_{t=0}^{1}\sum_{a=1}^{2} \|T_n^{l'+1}(\beta_a)\|_{0,t}\Bigg)\\
&\quad + c \sum_{m=2}^{2N_v} 2^{4N_v+4} \alpha^{m-2N_v-2} M^{-\frac{l'+1}{N_v-1}(N_v+1)} \sum_{n=2N_v+2}^{N(\beta_2)} c_0^{\frac{n}{2}} \alpha^n M^{\frac{l'+1}{2N_v-2}n}\\
&\quad\quad \cdot \Bigg(|J_n^{l'+1}(\beta_1) - J_n^{l'+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{t=0}^{1}\sum_{a=1}^{2} \|J_n^{l'+1}(\beta_a)\|_{0,t}\Bigg)\\
&\le c^{N_v} \beta_1^{-\frac{1}{2}} \alpha^{-2} M^{-\frac{l'+1}{N_v-1}} \le \beta_1^{-\frac{1}{2}} \alpha^{-2} M^{-\frac{l'}{N_v-1}}.
\end{aligned}
$$

Therefore, the induction concludes that (3.54) holds for all $l' \in \{l, l+1, \cdots, N_h\}$.

We can see from (3.33) and (3.34) that for any $X_0 \in I^0$, $\mathbf{X} \in \hat{I}^{m-1}$, $a \in \{1,2\}$, $m \in \{2,3,\cdots,2N_v\}$,

$$
\begin{aligned}
&F_m^l(\beta_a)(X_0, R_{\beta_a}(\mathbf{X}))\\
&= \hat{F}_m^l(\beta_a)(X_0, R_{\beta_a}(\mathbf{X})) + F_m^{N_h}(\beta_a)(X_0, R_{\beta_a}(\mathbf{X}))\\
&\quad + \sum_{j=l+1}^{N_h} \sum_{n=m+2}^{2N_v} \begin{pmatrix} n \\ m \end{pmatrix} \left(\frac{1}{h}\right)^{n-m} \sum_{\mathbf{Y}\in(I^0)^{n-m}}
\end{aligned}
$$

$$\cdot (F_n^j(\beta_a)(X_0, R_{\beta_a}(\mathbf{X},\mathbf{Y})) - \hat{F}_n^j(\beta_a)(X_0, R_{\beta_a}(\mathbf{X},\mathbf{Y})))$$
$$\cdot \int \psi_{\mathbf{Y}}^1 d\mu_{C_j^\delta(\beta_a)}(\psi^1).$$

Then, by (3.5), (3.7),

$$\begin{aligned}
&|F_m^l(\beta_1)(X_0, R_{\beta_1}(\mathbf{X})) - F_m^l(\beta_2)(X_0, R_{\beta_2}(\mathbf{X}))| \\
&\le |\hat{F}_m^l(\beta_1)(X_0, R_{\beta_1}(\mathbf{X})) - \hat{F}_m^l(\beta_2)(X_0, R_{\beta_2}(\mathbf{X}))| \\
&\quad + \sum_{j=l+1}^{N_h} \sum_{n=m+2}^{2N_v} \begin{pmatrix} n \\ m \end{pmatrix} \left(\frac{1}{h}\right)^{n-m} \sum_{\mathbf{Y}\in(I^0)^{n-m}} \\
&\qquad \cdot (|F_n^j(\beta_1)(X_0, R_{\beta_1}(\mathbf{X},\mathbf{Y})) - F_n^j(\beta_2)(X_0, R_{\beta_2}(\mathbf{X},\mathbf{Y}))| \\
&\qquad\quad + |\hat{F}_n^j(\beta_1)(X_0, R_{\beta_1}(\mathbf{X},\mathbf{Y})) - \hat{F}_n^j(\beta_2)(X_0, R_{\beta_2}(\mathbf{X},\mathbf{Y}))|) \\
&\qquad \cdot (M^{-j} + M^{j-N_h}) c_0^{\frac{n-m}{2}} \\
&\quad + \sum_{j=l+1}^{N_h} \sum_{n=m+2}^{2N_v} \begin{pmatrix} n \\ m \end{pmatrix} \left(\frac{1}{h}\right)^{n-m} \sum_{\mathbf{Y}\in(I^0)^{n-m}} \\
&\qquad \cdot (|F_n^j(\beta_2)(X_0, R_{\beta_2}(\mathbf{X},\mathbf{Y}))| + |\hat{F}_n^j(\beta_2)(X_0, R_{\beta_2}(\mathbf{X},\mathbf{Y}))|) \\
&\qquad \cdot \beta_1^{-\frac{1}{2}} M^{-\frac{j}{2}} c_0^{\frac{n-m}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|F_m^l(\beta_1) - F_m^l(\beta_2)|_0 \\
&\le |\hat{F}_m^l(\beta_1) - \hat{F}_m^l(\beta_2)|_0 \\
&\quad + \sum_{j=l+1}^{N_h} (M^{-j} + M^{j-N_h}) \\
&\qquad \cdot \sum_{n=m+2}^{2N_v} 2^n c_0^{\frac{n-m}{2}} (|F_n^j(\beta_1) - F_n^j(\beta_2)|_0 + |\hat{F}_n^j(\beta_1) - \hat{F}_n^j(\beta_2)|_0) \\
&\quad + \beta_1^{-\frac{1}{2}} \sum_{j=l+1}^{N_h} M^{-\frac{j}{2}} \sum_{n=m+2}^{2N_v} 2^n c_0^{\frac{n-m}{2}} (\|F_n^j(\beta_2)\|_{0,0} + \|\hat{F}_n^j(\beta_2)\|_{0,0}).
\end{aligned}$$

Moreover, by substituting (3.21), (3.35), (3.44) for $j \in \{l+1, l+2, \cdots, N_h\}$, (3.54) for $j \in \{l, l+1, \cdots, N_h\}$ and using the condition $\alpha \ge c$ we deduce that

$$\begin{aligned}(3.55)\qquad &\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}}\alpha^m |F_m^l(\beta_1)-F_m^l(\beta_2)|_0\\ &\le \sum_{m=2}^{2N_v} c_0^{\frac{m}{2}}\alpha^m |\hat{F}_m^l(\beta_1)-\hat{F}_m^l(\beta_2)|_0\\ &\quad+\sum_{m=2}^{2N_v}\sum_{j=l+1}^{N_h} 2^{m+2}\alpha^{-2}(M^{-j}+M^{j-N_h})\\ &\quad\cdot\sum_{n=m+2}^{2N_v} c_0^{\frac{n}{2}}\alpha^n(|F_n^j(\beta_1)-F_n^j(\beta_2)|_0+|\hat{F}_n^j(\beta_1)-\hat{F}_n^j(\beta_2)|_0)\\ &\quad+\beta_1^{-\frac{1}{2}}\sum_{m=2}^{2N_v}\sum_{j=l+1}^{N_h} 2^{m+2}\alpha^{-2}M^{-\frac{j}{2}}\\ &\quad\cdot\sum_{n=m+2}^{2N_v} c_0^{\frac{n}{2}}\alpha^n(\|F_n^j(\beta_2)\|_{0,0}+\|\hat{F}_n^j(\beta_2)\|_{0,0})\\ &\le \beta_1^{-\frac{1}{2}}c^{N_v}\alpha^{-2},\end{aligned}$$

which also implies that

$$(3.56)\qquad M^{-\frac{N_v}{N_v-1}l}\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m}|F_m^l(\beta_1)-F_m^l(\beta_2)|_0 \le \beta_1^{-\frac{1}{2}}c^{N_v}\alpha^{-2}.$$

It follows from (3.4), (3.7) and [19, Lemma 4.1 (2)] that for $m\in\{2N_v+2, 2N_v+3,\cdots,N(\beta_2)\}$,

$$\begin{aligned}&|F_m^l(\beta_1)-F_m^l(\beta_2)|_0\\ &\le |J_m^{l+1}(\beta_1)-J_m^{l+1}(\beta_2)|_0\\ &\quad+c\sum_{n=m+2}^{N(\beta_2)} 2^{2n}c_0^{\frac{n-m}{2}}\Bigg(|J_n^{l+1}(\beta_1)-J_n^{l+1}(\beta_2)|_0\\ &\qquad\qquad+\beta_1^{-\frac{1}{2}}\sum_{a=1}^{2}\sum_{t=0}^{1}\|J_n^{l+1}(\beta_a)\|_{0,t}\Bigg).\end{aligned}$$

Then, by (3.22), (3.45) for $l+1$ and the conditions $M \geq c^{N_v}$, $\alpha \geq c$,

$$
\begin{aligned}
&\sum_{m=2N_v+2}^{N(\beta_2)} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l}{2N_v-2}m} |F_m^l(\beta_1) - F_m^l(\beta_2)|_0 \\
&\leq c \sum_{n=2N_v+2}^{N(\beta_2)} \sum_{m=2N_v+2}^{n} 2^{2n} c_0^{\frac{n}{2}} \alpha^m M^{\frac{l}{2N_v-2}m} \\
&\quad \cdot \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{a=1}^{2} \sum_{t=0}^{1} \|J_n^{l+1}(\beta_a)\|_{0,t} \right) \\
&\leq c \sum_{n=2N_v+2}^{N(\beta_2)} 2^{2n} c_0^{\frac{n}{2}} \alpha^n M^{\frac{l}{2N_v-2}n} \\
&\quad \cdot \left(|J_n^{l+1}(\beta_1) - J_n^{l+1}(\beta_2)|_0 + \beta_1^{-\frac{1}{2}} \sum_{a=1}^{2} \sum_{t=0}^{1} \|J_n^{l+1}(\beta_a)\|_{0,t} \right) \\
&\leq c^{N_v} M^{-\frac{N_v+1}{N_v-1} + \frac{N_v}{N_v-1}(l+1)} \beta_1^{-\frac{1}{2}},
\end{aligned}
$$

and thus

$$
\begin{aligned}
(3.57) \qquad & M^{-\frac{N_v}{N_v-1}l} \sum_{m=2N_v+2}^{N(\beta_2)} c_0^{\frac{m}{2}} \alpha^m M^{\frac{l}{2N_v-2}m} |F_m^l(\beta_1) - F_m^l(\beta_2)|_0 \\
& \leq c^{N_v} M^{-\frac{1}{N_v-1}} \beta_1^{-\frac{1}{2}}.
\end{aligned}
$$

Finally, let us estimate the difference between $F_0^l(\beta_1)$ and $F_0^l(\beta_2)$. Note that

$$
\begin{aligned}
(3.58) \qquad F_0^l(\beta_a) &= F_0^{l+1}(\beta_a) + T_0^{l+1}(\beta_a) \\
&\quad + \sum_{m=2}^{2N_v} \int (F_m^{l+1}(\beta_a)(\psi) - \hat{F}_m^{l+1}(\beta_a)(\psi)) d\mu_{C_{l+1}^{\delta}(\beta_a)}(\psi) \\
&\quad + \sum_{m=2}^{2N_v} \int \hat{F}_m^{l+1}(\beta_a)(\psi) d\mu_{C_{l+1}^{\delta}(\beta_a)}(\psi) \\
&\quad + \sum_{m=2}^{2N_v} \int T_m^{l+1}(\beta_a)(\psi) d\mu_{C_{l+1}^{\delta}(\beta_a)}(\psi)
\end{aligned}
$$

$$+\sum_{m=2N_v+2}^{N(\beta_2)}\int J_m^{l+1}(\beta_a)(\psi)d\mu_{C_{l+1}^{\delta}(\beta_a)}(\psi).$$

By (3.33), Lemma 3.6, (3.52), (3.53),

$$\begin{aligned}
&\int(F_m^{l+1}(\beta_a)(\psi)-\hat{F}_m^{l+1}(\beta_a)(\psi))d\mu_{C_{l+1}^{\delta}(\beta_a)}(\psi)\\
&=\left(\frac{1}{h}\right)^m\sum_{s\in[0,\beta_a)_h}\sum_{X\in I^0}\sum_{\mathbf{Y}\in I(\beta_a)^{m-1}}(F_m^{l+1}(\beta_a)(X,R_{\beta_a}(\mathbf{Y}-s))\\
&\qquad-\hat{F}_m^{l+1}(\beta_a)(X,R_{\beta_a}(\mathbf{Y}-s)))\int\psi_X\psi_{R_{\beta_a}(\mathbf{Y}-s)}d\mu_{C_{l+1}^{\delta}(\beta_a)}(\psi)\\
&=\beta_a\left(\frac{1}{h}\right)^{m-1}\sum_{X\in I^0}\sum_{\mathbf{Y}\in(I^0)^{m-1}}(F_m^{l+1}(\beta_a)(X,\mathbf{Y})-\hat{F}_m^{l+1}(\beta_a)(X,\mathbf{Y}))\\
&\quad\cdot\int\psi_X\psi_{\mathbf{Y}}d\mu_{C_{l+1}^{\delta}(\beta_a)}(\psi).
\end{aligned}$$

Combined with (3.5) and (3.7), this equality implies that

$$\begin{aligned}
&\left|\frac{h}{N(\beta_1)}\int(F_m^{l+1}(\beta_1)(\psi)-\hat{F}_m^{l+1}(\beta_1)(\psi))d\mu_{C_{l+1}^{\delta}(\beta_1)}(\psi)\right.\\
&\left.-\frac{h}{N(\beta_2)}\int(F_m^{l+1}(\beta_2)(\psi)-\hat{F}_m^{l+1}(\beta_2)(\psi))d\mu_{C_{l+1}^{\delta}(\beta_2)}(\psi)\right|\\
&\le(|F_m^{l+1}(\beta_1)-F_m^{l+1}(\beta_2)|_0+|\hat{F}_m^{l+1}(\beta_1)-\hat{F}_m^{l+1}(\beta_2)|_0)\\
&\quad\cdot(M^{-l-1}+M^{l+1-N_h})c_0^{\frac{m}{2}}\\
&\quad+(\|F_m^{l+1}(\beta_2)\|_{0,0}+\|\hat{F}_m^{l+1}(\beta_2)\|_{0,0})M^{-\frac{l+1}{2}}\beta_1^{-\frac{1}{2}}c_0^{\frac{m}{2}}.
\end{aligned}\tag{3.59}$$

By applying the same estimation as [19, Lemma 4.1 (1)] to (3.58) and using (3.4), (3.7), (3.21), (3.22), (3.30), (3.35), (3.44), (3.45), (3.48), (3.50), (3.54), (3.59) for $l'\in\{l+1,l+2,\cdots,N_h\}$ we observe that

$$\begin{aligned}
&\left|\frac{h}{N(\beta_1)}F_0^l(\beta_1)-\frac{h}{N(\beta_2)}F_0^l(\beta_2)\right|\\
&\le\left|\frac{h}{N(\beta_1)}F_0^{l+1}(\beta_1)-\frac{h}{N(\beta_2)}F_0^{l+1}(\beta_2)\right|
\end{aligned}\tag{3.60}$$

$$
\begin{aligned}
&+\left|\frac{h}{N(\beta_1)}T_0^{l+1}(\beta_1)-\frac{h}{N(\beta_2)}T_0^{l+1}(\beta_2)\right|\\
&+\sum_{m=2}^{2N_v}(M^{-l-1}+M^{l+1-N_h})c_0^{\frac{m}{2}}\\
&\qquad\cdot(|F_m^{l+1}(\beta_1)-F_m^{l+1}(\beta_2)|_0+|\hat{F}_m^{l+1}(\beta_1)-\hat{F}_m^{l+1}(\beta_2)|_0)\\
&+\sum_{m=2}^{2N_v}M^{-\frac{l+1}{2}}\beta_1^{-\frac{1}{2}}c_0^{\frac{m}{2}}(\|\hat{F}_m^{l+1}(\beta_2)\|_{0,0}+\|F_m^{l+1}(\beta_2)\|_{0,0})\\
&+c\sum_{m=2}^{2N_v}2^m c_0^{\frac{m}{2}}\Bigg(|\hat{F}_m^{l+1}(\beta_1)-\hat{F}_m^{l+1}(\beta_2)|_0\\
&\qquad+\beta_1^{-\frac{1}{2}}\sum_{a=1}^{2}\sum_{t=0}^{1}\|\hat{F}_m^{l+1}(\beta_a)\|_{0,t}\\
&\qquad+|T_m^{l+1}(\beta_1)-T_m^{l+1}(\beta_2)|_0\\
&\qquad+\beta_1^{-\frac{1}{2}}\sum_{a=1}^{2}\sum_{t=0}^{1}\|T_m^{l+1}(\beta_a)\|_{0,t}\Bigg)\\
&+c\sum_{m=2N_v+2}^{N(\beta_2)}2^m c_0^{\frac{m}{2}}\Bigg(|J_m^{l+1}(\beta_1)-J_m^{l+1}(\beta_2)|_0\\
&\qquad+\beta_1^{-\frac{1}{2}}\sum_{a=1}^{2}\sum_{t=0}^{1}\|J_m^{l+1}(\beta_a)\|_{0,t}\Bigg)\\
\le&\left|\frac{h}{N(\beta_1)}F_0^{l+1}(\beta_1)-\frac{h}{N(\beta_2)}F_0^{l+1}(\beta_2)\right|\\
&+c^{N_v}\beta_1^{-\frac{1}{2}}\alpha^{-2}(M^{-\frac{l+1}{2}}+M^{-\frac{l+1}{N_v-1}}+M^{l+1-N_h})\\
\le&\, c^{N_v}\beta_1^{-\frac{1}{2}}\alpha^{-2}\sum_{j=l}^{N_h-1}(M^{-\frac{j+1}{2}}+M^{-\frac{j+1}{N_v-1}}+M^{j+1-N_h})\\
\le&\, c^{N_v}\beta_1^{-\frac{1}{2}}\alpha^{-2}.
\end{aligned}
$$

By putting (3.48), (3.50), (3.51), (3.55), (3.56), (3.57), (3.60) together we obtain that

$$
(3.61)\quad \left|\frac{h}{N(\beta_1)}F_0^{l}(\beta_1)-\frac{h}{N(\beta_2)}F_0^{l}(\beta_2)\right|+\left|\frac{h}{N(\beta_1)}T_0^{l}(\beta_1)-\frac{h}{N(\beta_2)}T_0^{l}(\beta_2)\right|
$$

$$\le c^{N_v}\alpha^{-2}\beta_1^{-\frac{1}{2}},$$

(3.62)
$$\sum_{m=2}^{2N_v} c_0^{\frac{m}{2}}\alpha^m(|F_m^l(\beta_1)-F_m^l(\beta_2)|_0+|T_m^l(\beta_1)-T_m^l(\beta_2)|_0) \le c^{N_v}\alpha^{-2}\beta_1^{-\frac{1}{2}},$$

(3.63)
$$M^{-\frac{N_v}{N_v-1}l}\sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}}\alpha^m M^{\frac{l}{2N_v-2}m} \cdot(|F_m^l(\beta_1)-F_m^l(\beta_2)|_0+|T_m^l(\beta_1)-T_m^l(\beta_2)|_0) \le (c^{N_v}\alpha^{-2}+cM\alpha^{-2}+c^{N_v}M^{-\frac{1}{N_v-1}})\beta_1^{-\frac{1}{2}}.$$

Recall that in the derivation of the above inequalities we assumed the conditions $\alpha \ge c$, $M \ge c^{N_v^2}$ with a constant $c \in \mathbb{R}_{>0}$ independent of any parameter including c'. If we start by the conditions (3.18) and (3.19) with a sufficiently large constant c', then the conditions $\alpha \ge c$, $M \ge c^{N_v^2}$ are satisfied, the right-hand side of (3.61) is less than $\beta_1^{-1/2}\alpha^{-1}$ and the right-hand sides of (3.62), (3.63) are less than $\beta_1^{-1/2}$. Therefore, we obtain (3.43), (3.44), (3.45) for l on the assumptions (3.18), (3.19) with a generic constant c', which does not depend on any parameter. The induction with $l \in \{0,1,\cdots,N_h\}$ now concludes the proof. □

4. The Infrared Integration

In this section we perform the multi-scale analysis around the singular point of the covariance in momentum space, namely the infrared analysis. The output of the Matsubara ultra-violet integration is substituted into the infrared integration as the initial data. So the infrared integration is the second step of the whole multi-scale integration process. Conservation of symmetries is essential to validate the iteration of the integration. We have to keep track of the preserved symmetries as well as analyticity and scale-dependent bound properties of Grassmann polynomials during the iteration. For this purpose it is convenient to organize sets of Grassmann polynomials having the relevant properties and define maps between these sets resembling the real renormalization group maps in advance. We plan to do so in the first subsection. In the second subsection we will complete the proof of Theorem 1.6 by making use of the tools developed in the preceding subsection. We should remark that in principle one can reach our main result by combining the materials prepared so far in this paper with calculations parallel to those presented in [19, Section 7]. Apart from proving the theorem itself, this section is aimed at providing a more organized construction of the infrared integration than the previous version [19, Section 7] so that the readers can confirm the validity of the infrared integration more clearly.

Throughout this section we assume that

$$M \geq 2, \quad h \geq e^{4d}, \quad L \geq \beta,$$

unless stated otherwise.

4.1. General lemmas

Let $n \in \mathbb{N}$ and let D be a bounded domain of $\mathbb{C}^n$ satisfying that $\overline{\mathbf{z}} \in \overline{D}$ for any $\mathbf{z} \in \overline{D}$, where $\overline{D}$ denotes the closure of D. Set

$$\begin{aligned}
&C\left(\overline{D}; \bigwedge \mathcal{V}\right) \\
&:= \left\{ J \in \mathrm{Map}\left(\overline{D}, \bigwedge \mathcal{V}\right) \ \middle|\ \mathbf{U} \mapsto J(\mathbf{U})(\psi) \text{ is continuous in } \overline{D} \right\}, \\
&C^{\omega}\left(D; \bigwedge \mathcal{V}\right) \\
&:= \left\{ J \in \mathrm{Map}\left(D, \bigwedge \mathcal{V}\right) \ \middle|\ \mathbf{U} \mapsto J(\mathbf{U})(\psi) \text{ is analytic in } D \right\}.
\end{aligned}$$

See [19, Subsection 2.2] for the meaning of continuity and analyticity of Grassmann polynomials. We are going to define a subset of $C(\overline{D}; \bigwedge \mathcal{V}) \cap C^{\omega}(D; \bigwedge \mathcal{V})$ to which Grassmann polynomials dealt in our infrared analysis belong. To describe symmetric properties of Grassmann polynomials, let us fix some notational conventions. Let S be a bijective map from I to I and Q be a map from I to $\mathbb{R}$. The maps $S_m : I^m \to I^m$, $Q_m : I^m \to \mathbb{R}$ ($m \in \mathbb{N}$) are defined by

$$\begin{aligned}
S_m(X_1, X_2, \cdots, X_m) &:= (S(X_1), S(X_2), \cdots, S(X_m)), \\
Q_m(X_1, X_2, \cdots, X_m) &:= \sum_{j=1}^{m} Q(X_j).
\end{aligned}$$

For $f(\psi)=\sum_{m=0}^{N}(\frac{1}{h})^m\sum_{\mathbf{X}\in I^m}f_m(\mathbf{X})\psi_{\mathbf{X}}\in\bigwedge\mathcal{V}$, define $f(\mathcal{R}\psi),\overline{f}(\psi)\in\bigwedge\mathcal{V}$ by

$$\begin{aligned}f(\mathcal{R}\psi)&:=\sum_{m=0}^{N}\left(\frac{1}{h}\right)^m\sum_{\mathbf{X}\in I^m}f_m(\mathbf{X})e^{iQ_m(S_m(\mathbf{X}))}\psi_{S_m(\mathbf{X})},\\ \overline{f}(\psi)&:=\sum_{m=0}^{N}\left(\frac{1}{h}\right)^m\sum_{\mathbf{X}\in I^m}\overline{f_m(\mathbf{X})}\psi_{\mathbf{X}}.\end{aligned}$$

In fact these notational rules have been introduced in [19, Subsection 3.3]. In addition, for $\mathbf{x}\in\mathbb{Z}^d$ we let $r_L(\mathbf{x})$ denote a site of $\Gamma(L)$ satisfying $\mathbf{x}=r_L(\mathbf{x})$ in $(\mathbb{Z}/L\mathbb{Z})^d$.

Now, for parameters c_0, $\alpha\in\mathbb{R}_{\ge 1}$, $M\in\mathbb{R}_{\ge 2}$, $l\in\mathbb{Z}_{\le 0}$ we define the subset $\mathcal{S}(D,c_0,\alpha,M)(l)$ of $C(\overline{D};\bigwedge\mathcal{V})\cap C^{\omega}(D;\bigwedge\mathcal{V})$ as follows. $J\in C(\overline{D};\bigwedge\mathcal{V})$ $\cap C^{\omega}(D;\bigwedge\mathcal{V})$ belongs to $\mathcal{S}(D,c_0,\alpha,M)(l)$ if and only if J satisfies the following properties.

(i)

$$\frac{h}{N}|J_0(\mathbf{U})|\le M^{(d+\frac{3}{2})l}\alpha^{-1},\tag{4.1}$$

$$M^{-(d+\frac{3}{2})l+tl}\sum_{m=2}^{N}c_0^{\frac{m}{2}}\alpha^m M^{\frac{d}{2}lm}\|J_m(\mathbf{U})\|_{l,t}\le 1,\ (\forall\mathbf{U}\in\overline{D},t\in\{0,1\}).\tag{4.2}$$

(ii)

$$J(\mathbf{U})(\psi)=J(\mathbf{U})(\mathcal{R}\psi),\ (\forall\mathbf{U}\in\overline{D}),$$

for each $S:I\to I$ and $Q:I\to\mathbb{R}$ defined as follows.

$$\begin{aligned}&S((\rho,\mathbf{x},\sigma,x,\theta)):=(\rho,\mathbf{x},\sigma,x,\theta),\\ &Q((\rho,\mathbf{x},\sigma,x,\theta)):=\frac{\pi}{2}\theta,\ (\forall(\rho,\mathbf{x},\sigma,x,\theta)\in I).\end{aligned}\tag{4.3}$$

$$\begin{aligned}&S((\rho,\mathbf{x},\sigma,x,\theta)):=(\rho,\mathbf{x},\sigma,x,\theta),\\ &Q((\rho,\mathbf{x},\sigma,x,\theta)):=\pi 1_{\sigma=\uparrow},\ (\forall(\rho,\mathbf{x},\sigma,x,\theta)\in I).\end{aligned}\tag{4.4}$$

$$S((\rho,\mathbf{x},\sigma,x,\theta)):=(\rho,\mathbf{x},-\sigma,x,\theta),\tag{4.5}$$

$$Q((\rho,\mathbf{x},\sigma,x,\theta)) := 0,\ (\forall(\rho,\mathbf{x},\sigma,x,\theta)\in I).$$

$$\begin{aligned}(4.6)\qquad &S((\rho,\mathbf{x},\sigma,x,\theta)) := (\rho, r_L(\mathbf{x}+\mathbf{z}),\sigma, r_\beta(x+s),\theta),\\ &Q((\rho,\mathbf{x},\sigma,x,\theta)) := \pi n_\beta(r_\beta(x-s)+s),\ (\forall(\rho,\mathbf{x},\sigma,x,\theta)\in I),\end{aligned}$$

where $\mathbf{z}\in\mathbb{Z}^d$ and $s\in(1/h)\mathbb{Z}$ are arbitrarily taken and fixed.

$$\begin{aligned}(4.7)\qquad &S((\rho,\mathbf{x},\sigma,x,\theta)) := (\rho, r_L(-\mathbf{x}-b(\rho)),\sigma,x,\theta),\\ &Q((\rho,\mathbf{x},\sigma,x,\theta)) := \theta\langle\mathbf{x},(2\pi/L)\boldsymbol{\varepsilon}^L\rangle + \theta\langle b(\rho),(\pi/L)\boldsymbol{\varepsilon}^L\rangle,\\ &(\forall(\rho,\mathbf{x},\sigma,x,\theta)\in I).\end{aligned}$$

(iii)

$$J(\mathbf{U})(\psi) = \overline{J(\overline{\mathbf{U}})}(\mathcal{R}\psi),\ (\forall \mathbf{U}\in\overline{D}),$$

for each $S: I\to I$ and $Q: I\to\mathbb{R}$ defined as follows.

$$\begin{aligned}(4.8)\qquad &S((\rho,\mathbf{x},\sigma,x,\theta)) := (\rho,\mathbf{x},\sigma,r_\beta(-x),-\theta),\\ &Q((\rho,\mathbf{x},\sigma,x,\theta)) := \pi(1_{\theta=1}+1_{x\neq 0}),\ (\forall(\rho,\mathbf{x},\sigma,x,\theta)\in I).\end{aligned}$$

$$\begin{aligned}(4.9)\qquad &S((\rho,\mathbf{x},\sigma,x,\theta)) := (\rho,\mathbf{x},\sigma,x,-\theta),\\ &Q((\rho,\mathbf{x},\sigma,x,\theta)) := \langle b(\rho),\boldsymbol{\pi}\rangle,\ (\forall(\rho,\mathbf{x},\sigma,x,\theta)\in I).\end{aligned}$$

Moreover, on the assumption (3.3) we define the subset $\hat{\mathcal{S}}(D,c_0,\alpha,M)(l)$ of $\mathcal{S}(D,c_0,\alpha,M)(l)(\beta_1)\times\mathcal{S}(D,c_0,\alpha,M)(l)(\beta_2)$ as follows. $(J(\beta_1),J(\beta_2))$ $\in\mathcal{S}(D,c_0,\alpha,M)(l)(\beta_1)\times\mathcal{S}(D,c_0,\alpha,M)(l)(\beta_2)$ belongs to $\hat{\mathcal{S}}(D,c_0,\alpha,M)(l)$ if and only if

$$(4.10)\qquad \left|\frac{h}{N(\beta_1)}J_0(\beta_1)(\mathbf{U}) - \frac{h}{N(\beta_2)}J_0(\beta_2)(\mathbf{U})\right| \le \beta_1^{-\frac{1}{2}}M^{(d+\frac{1}{2})l}\alpha^{-1},$$

$$\begin{aligned}(4.11)\qquad &M^{-(d+\frac{1}{2})l}\sum_{m=2}^{N(\beta_2)} c_0^{\frac{m}{2}}\alpha^m M^{\frac{d}{2}lm}|J_m(\beta_1)(\mathbf{U}) - J_m(\beta_2)(\mathbf{U})|_l\\ &\le \beta_1^{-\frac{1}{2}},\ (\forall\mathbf{U}\in\overline{D}).\end{aligned}$$

We will later define a set designed to contain kernels of quadratic Grassmann polynomials belonging to $\mathcal{S}(D,c_0,\alpha,M)(l)$. Since one criterion to be an element of the set involves cut-off functions for the infrared integration, let us define the cut-off functions at this stage. Set

$$\begin{aligned} f_{\mathbf{t}} &:= \frac{1}{4}\min_{p\in\{1,2,\cdots,d\}} t_p^2\Bigg(1-\frac{1}{2}\max_{m\in\{1,2,\cdots,d\}}\Bigg(\sum_{j=1}^{m-1}|1+e^{i\theta_{j,m}}| \\ &\qquad\qquad + \sum_{j=m+1}^{d}|1+e^{i\theta_{m,j}}|\Bigg)\Bigg), \\ M_{IR} &:= \frac{\sqrt{6}}{\pi}\left(\frac{\pi^2}{3}M_{UV}^2+d\right)^{\frac{1}{2}}, \\ N_\beta &:= \min\left\{\left\lfloor\frac{\log\left(\frac{\pi}{\beta}\left(\frac{\pi}{\sqrt{3}}M_{IR}\right)^{-1}\right)}{\log M}\right\rfloor, 0\right\}. \end{aligned}$$

By the assumption (3.1), $f_{\mathbf{t}} \le 1/4$. Using the function ϕ introduced in Subsection 3.1, we define the functions $\chi_l : \mathbb{R}^{d+1} \to \mathbb{R}$ ($l \in \mathbb{Z}$) by

$$\begin{aligned} &\chi_l(\omega,\mathbf{k}) \\ &:= \phi(M_{UV}^{-2}\omega^2)\left(\phi\left(M_{IR}^{-2}M^{-2(l+1)}\left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^{d}|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)\right)\right. \\ &\qquad\qquad \left.-\phi\left(M_{IR}^{-2}M^{-2l}\left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^{d}|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)\right)\right). \end{aligned}$$

We can check that

$$\begin{aligned} &\phi\left(M_{IR}^{-2}M^{-2}\left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^{d}|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)\right)=1, \\ &(\forall\omega\in\mathbb{R}\text{ with }\phi(M_{UV}^{-2}\omega^2)\neq 0,\mathbf{k}\in\mathbb{R}^d), \\ &\phi\left(M_{IR}^{-2}M^{-2N_\beta}\left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^{d}|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)\right)=0, \\ &(\forall\omega\in\mathbb{R}\text{ with }|\omega|\ge\pi/\beta,\mathbf{k}\in\mathbb{R}^d). \end{aligned} \tag{4.12}$$

These equalities imply that

$$\sum_{l=0}^{N_\beta}\chi_l(\omega,\mathbf{k})=\phi(M_{UV}^{-2}\omega^2),\ (\forall\omega\in\mathcal{M},\mathbf{k}\in\mathbb{R}^d).$$

The support property of χ_l is described as follows.

$$\chi_l(\omega,\mathbf{k})\begin{cases}=0, & \text{if } \left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^d|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)^{\frac{1}{2}}\\ & \quad\le\frac{\pi}{\sqrt{6}}M_{IR}M^l,\\ \in[0,1], & \text{if } \frac{\pi}{\sqrt{6}}M_{IR}M^l\\ & \quad<\left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^d|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)^{\frac{1}{2}}\\ & \quad<\frac{\pi}{\sqrt{3}}M_{IR}M^{l+1},\\ =0, & \text{if } \left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^d|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)^{\frac{1}{2}}\\ & \quad\ge\frac{\pi}{\sqrt{3}}M_{IR}M^{l+1}.\end{cases}\tag{4.13}$$

We define the functions $\chi_{\le l}:\mathbb{R}^{d+1}\to\mathbb{R}$ ($l\in\mathbb{Z}$ with $l\ge N_\beta$), $\hat{\chi}_{\le m}:\mathbb{R}^{d+1}\to\mathbb{R}$ ($m\in\mathbb{Z}$) by

$$\begin{aligned}&\chi_{\le l}(\omega,\mathbf{k}):=\sum_{j=l}^{N_\beta}\chi_j(\omega,\mathbf{k}),\\ &\hat{\chi}_{\le m}(\omega,\mathbf{k})\\ &\ :=\phi(M_{UV}^{-2}\omega^2)\phi\left(M_{IR}^{-2}M^{-2(m+1)}\left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^d|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)\right),\\ &(\forall(\omega,\mathbf{k})\in\mathbb{R}^{d+1}).\end{aligned}$$

Here let us list properties of these cut-off functions for later use. For simplicity we write $\partial/\partial k_0$ in place of the differential operator $\partial/\partial\omega$ in the following. Note that the condition $L\ge\beta$ is necessary in the proof of the item (5) of the next lemma.

LEMMA 4.1.

(1) *Assume that* $0 < \beta_1 \le \beta_2$. *Then,*

$$\chi_{\le l}(\beta_1)(\omega, \mathbf{k}) = \chi_{\le l}(\beta_2)(\omega, \mathbf{k}) = \hat{\chi}_{\le l}(\omega, \mathbf{k}),$$
$$(\forall (\omega, \mathbf{k}) \in \mathbb{R}^{d+1} \text{ with } |\omega| \ge \pi/\beta_1,\ l \in \mathbb{Z} \text{ with } l \ge N_{\beta_1}).$$

(2)

$$\hat{\chi}_{\le l}(\omega, \mathbf{k}) = 0,\ (\forall (\omega, \mathbf{k}) \in \mathbb{R}^{d+1} \text{ with } |\omega| \ge \pi h,\ l \in \mathbb{Z}).$$

(3) *There exists a constant* $c_\chi \in \mathbb{R}_{>0}$ *independent of any parameter such that*

$$\left| \left(\frac{\partial}{\partial k_j} \right)^n \hat{\chi}_{\le l}(\omega, \mathbf{k}) \right|, \ \left| \left(\frac{\partial}{\partial k_j} \right)^n \chi_l(\omega, \mathbf{k}) \right| \le (c_\chi \mathrm{w}(l)^{-1})^n (n!)^2,$$
$$(\forall (\omega, \mathbf{k}) \in \mathbb{R}^{d+1}, n \in \mathbb{N} \cup \{0\}, j \in \{0, 1, \cdots, d\}, l \in \{0, -1, \cdots, N_\beta\}).$$

(4) *If there exists* $(\omega, \mathbf{k}) \in \mathbb{R}^{d+1}$ *with* $|\omega| \ge \pi/\beta$ *such that* $\hat{\chi}_{\le 0}(\omega, \mathbf{k}) \ne 0$, *then*

$$\frac{1}{\beta} \le M_{IR} M^{N_\beta + 1}.$$

(5) *Assume that* $1/\beta \le M_{IR} M^{N_\beta + 1}$. *Then, there exists a constant* $c(M, d) \in \mathbb{R}_{>0}$ *depending only on* M, d *such that the following inequalities hold for any* $l \in \mathbb{Z}$ *with* $l \ge N_\beta$, $(\omega', \mathbf{k}') \in \mathbb{R}^{d+1}$.

$$\frac{1}{\beta L^d} \sum_{(\omega, \mathbf{k}) \in \mathcal{M} \times \Gamma(L)^*} 1_{\hat{\chi}_{\le l}(\omega + \omega', \mathbf{k} + \mathbf{k}') \ne 0} \le c(M, d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d+1)l},$$
$$\int_{-\infty}^{\infty} d\omega \frac{1}{L^d} \sum_{\mathbf{k} \in \Gamma(L)^*} 1_{\hat{\chi}_{\le l}(\omega, \mathbf{k} + \mathbf{k}') \ne 0} \le c(M, d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d+1)l},$$
$$\frac{1}{L^d} \sum_{\mathbf{k} \in \Gamma(L)^*} 1_{\hat{\chi}_{\le l}(\omega', \mathbf{k} + \mathbf{k}') \ne 0} \le c(M, d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{dl}.$$

PROOF. (1): The claim follows from (4.12).

(2): The assumption $h \ge e^{4d}$ and the support property of $\phi(M_{UV}^{-2} \omega^2)$ imply that result.

(3): The proof for this claim is essentially same as the proof for [19, Lemma 7.4]. Here we especially need to use the fact $f_{\mathbf{t}} \le 1$.

(4): This was essentially proved in [19, Lemma 7.5].

(5): We can derive the claimed inequalities from the support property of $\hat{\chi}_{\le l}$ and the assumptions $1/\beta \le M_{IR} M^{N_\beta+1}$, $L \ge \beta$. □

Set

$$
\begin{aligned}
&C^\infty(\mathbb{R}^{d+1}; \mathrm{Mat}(2^d, \mathbb{C}))\\
&:= \{f : \mathbb{R}^{d+1} \to \mathrm{Mat}(2^d, \mathbb{C}) \mid f(\cdot)(\rho, \eta) \in C^\infty(\mathbb{R}^{d+1}; \mathbb{C})\ (\forall \rho, \eta \in \mathcal{B})\}.
\end{aligned}
$$

Here we define the subset $\mathcal{K}(D, \alpha, M)(l)$ of $\mathrm{Map}(\overline{D}, C^\infty(\mathbb{R}^{d+1}; \mathrm{Mat}(2^d, \mathbb{C})))$ which is designed to contain kernels of relevant quadratic Grassmann polynomials. Let $l \in \mathbb{Z}_{\le 0}$ and c_χ be the constant appearing in Lemma 4.1 (3). $W \in \mathrm{Map}(\overline{D}, C^\infty(\mathbb{R}^{d+1}; \mathrm{Mat}(2^d, \mathbb{C})))$ belongs to $\mathcal{K}(D, \alpha, M)(l)$ if and only if W satisfies the following conditions.

(i) $\mathbf{U} \mapsto W(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta)$ is continuous in $\overline{D}$, analytic in D for any $(\omega, \mathbf{k}) \in \mathbb{R}^{d+1}$, $\rho, \eta \in \mathcal{B}$.

(ii)
$$
\begin{aligned}
&W(\mathbf{U})(\omega, \mathbf{k}) = W(\mathbf{U})(\omega, \mathbf{p}),\\
&(\forall \mathbf{U} \in \overline{D}, \omega \in \mathbb{R}, \mathbf{k}, \mathbf{p} \in \mathbb{R}^d \text{ with } \mathbf{k} = \mathbf{p} \text{ in } (\mathbb{R}/2\pi\mathbb{Z})^d).
\end{aligned}
$$

(iii)
$$
\begin{aligned}
&W(\mathbf{U})(\omega, \mathbf{k}) = W(\overline{\mathbf{U}})(-\omega, \mathbf{k})^*,\\
&(\forall \mathbf{U} \in \overline{D}, (\omega, \mathbf{k}) \in \mathcal{M} \times ((2\pi/L)\mathbb{Z})^d).
\end{aligned}
$$

(iv)
$$
\begin{aligned}
&U_d(\boldsymbol{\pi}) W(\mathbf{U})(\omega, \mathbf{k}) U_d(\boldsymbol{\pi})^* = -W(\overline{\mathbf{U}})(\omega, \mathbf{k})^*,\\
&(\forall \mathbf{U} \in \overline{D}, (\omega, \mathbf{k}) \in \mathcal{M} \times ((2\pi/L)\mathbb{Z})^d).
\end{aligned}
$$

(v)
$$
\begin{aligned}
&U_d\left(\frac{\pi}{L}\boldsymbol{\varepsilon}^L\right) U_d(\mathbf{k}) W(\mathbf{U}) \left(\omega, -\mathbf{k} - \frac{2\pi}{L}\boldsymbol{\varepsilon}^L\right) U_d(\mathbf{k})^* U_d\left(\frac{\pi}{L}\boldsymbol{\varepsilon}^L\right)^*\\
&= W(\mathbf{U})(\omega, \mathbf{k}),\ (\forall \mathbf{U} \in \overline{D}, (\omega, \mathbf{k}) \in \mathcal{M} \times ((2\pi/L)\mathbb{Z})^d).
\end{aligned}
$$

(vi)

$$\prod_{j=0}^{d}\left(\sum_{n_j=0}^{\infty}\left(\frac{1}{2c_{\chi}+\pi}\right)^{n_j}\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\right)\left|\prod_{p=0}^{d}\left(\frac{\partial}{\partial k_p}\right)^{n_p}W(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)\right| \cdot 1_{\sum_{q=0}^{d}n_q>0} \le \alpha^{-2}M^{l},\ (\forall \mathbf{U}\in\overline{D},(\omega,\mathbf{k})\in\mathbb{R}^{d+1},\rho,\eta\in\mathcal{B}). \tag{4.14}$$

(vii)

$$|1_{\hat{\chi}_{\le j}(\omega,\mathbf{k})\neq 0}W(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)|\le\alpha^{-2}M^{j},\quad (\forall \mathbf{U}\in\overline{D},(\omega,\mathbf{k})\in\mathbb{R}^{d+1},\rho,\eta\in\mathcal{B},j\in\mathbb{Z}\text{ with }j\ge N_{\beta}). \tag{4.15}$$

On the assumption (3.3) we define the subset $\hat{\mathcal{K}}(D,\alpha,M)(l)$ of $\mathcal{K}(D,\alpha,M)(l)(\beta_1)\times\mathcal{K}(D,\alpha,M)(l)(\beta_2)$ as follows. $(W(\beta_1),W(\beta_2))\in\mathcal{K}(D,\alpha,M)(l)(\beta_1)\times\mathcal{K}(D,\alpha,M)(l)(\beta_2)$ belongs to $\hat{\mathcal{K}}(D,\alpha,M)(l)$ if and only if

$$\prod_{j=0}^{d}\left(\sum_{n_j=0}^{\infty}\left(\frac{1}{2c_{\chi}+\pi^2}\right)^{n_j}\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\right) \cdot\left|\prod_{p=0}^{d}\left(\frac{\partial}{\partial k_p}\right)^{n_p}(W(\beta_1)(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)-W(\beta_2)(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta))\right| \le\beta_1^{-\frac{1}{2}}\alpha^{-2},\ (\forall \mathbf{U}\in\overline{D},(\omega,\mathbf{k})\in\mathbb{R}^{d+1},\rho,\eta\in\mathcal{B}). \tag{4.16}$$

Let us make an inequality which will enable us to substitute an element of $\mathcal{K}(D,\alpha,M)(l)$ into the denominator of the free covariance.

Lemma 4.2. *There exists a constant $c(d)\in\mathbb{R}_{>0}$ depending only on d such that if $\alpha\ge c(d)$, the following inequality holds for any $W\in\mathcal{K}(D,\alpha,M)(l)$.*

$$\|(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-W(\mathbf{U})(\omega,\mathbf{k}))^{-1}\|_{2^d\times 2^d}\le M^{-l'},$$
$$(\forall l'\in\mathbb{Z}\text{ with }l'\ge N_{\beta},(\omega,\mathbf{k})\in\mathbb{R}^{d+1}\text{ satisfying }\chi_{l'}(\omega,\mathbf{k})\neq 0,\mathbf{U}\in\overline{D}).$$

PROOF. By Lemma 2.3 (5) and (4.13),

$$\begin{aligned}\|(i\omega I_{2^d}-\mathcal{E}(\mathbf{k}))^{-1}\|_{2^d\times 2^d} &\le \left(\omega^2+f_{\mathbf{t}}\sum_{j=1}^{d}|1+e^{i\frac{\pi}{L}\varepsilon_j^L+ik_j}|^2\right)^{-\frac{1}{2}}\\ &\le \frac{\sqrt{6}}{\pi}M_{IR}^{-1}M^{-l'}.\end{aligned}$$

Thus, by (4.15),

$$\begin{aligned}&\|(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-W(\mathbf{U})(\omega,\mathbf{k}))^{-1}\|_{2^d\times 2^d}\\ &\le \|(i\omega I_{2^d}-\mathcal{E}(\mathbf{k}))^{-1}\|_{2^d\times 2^d}\sum_{n=0}^{\infty}\|(i\omega I_{2^d}-\mathcal{E}(\mathbf{k}))^{-1}W(\mathbf{U})(\omega,\mathbf{k})\|_{2^d\times 2^d}^n\\ &\le \frac{\sqrt{6}}{\pi}M_{IR}^{-1}M^{-l'}\sum_{n=0}^{\infty}(c(d)\alpha^{-2})^n\le M^{-l'}.\end{aligned}$$

In order to derive the last inequality, we used the condition $\alpha\ge c(d)$. □

At every step of the iterative IR integration we receive a Grassmann polynomial from the preceding IR integration and substitute the kernel of its quadratic term into the covariance. Our aim here is to construct lemmas which justify this process. Let us define the maps $r'_\beta:[0,\beta)\to[-\beta/2,\beta/2)$, $s_L:[0,L)\to[-L/2,L/2)$, $r'_L:[0,L)^d\to[-L/2,L/2)^d$ by

$$r'_\beta(x):=\begin{cases} x & \text{if } x\in\left[0,\frac{\beta}{2}\right),\\ x-\beta & \text{if } x\in\left[\frac{\beta}{2},\beta\right),\end{cases}\qquad s_L(x):=\begin{cases} x & \text{if } x\in\left[0,\frac{L}{2}\right),\\ x-L & \text{if } x\in\left[\frac{L}{2},L\right),\end{cases}$$
$$r'_L(\mathbf{x}):=(s_L(x_1),s_L(x_2),\cdots,s_L(x_d)).$$

Let $l\in\{0,-1,\cdots,N_\beta\}$ and $(J^0,J^{-1},\cdots,J^l)\in\prod_{j=0}^{l}\mathcal{S}(D,c_0,\alpha,M)(j)$. For $\mathbf{U}\in\overline{D}$, $(\omega,\mathbf{k})\in\mathbb{R}^{d+1}$, $\rho,\eta\in\mathcal{B}$, set

$$\begin{aligned}(4.17)\qquad &W^j(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)\\ &:=\frac{2}{h}\sum_{(\mathbf{x},s)\in\Gamma(L)\times[0,\beta)_h}e^{-i\langle\mathbf{k},r'_L(\mathbf{x})\rangle-i\omega r'_\beta(s)}(-1)^{n_\beta(r'_\beta(s))}\\ &\qquad\cdot J_2^j(\mathbf{U})((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1)),\end{aligned}$$

$$(j = 0, -1, \cdots, l),$$

$$E_l(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta) \tag{4.18}$$

$$:= 1_{\frac{1}{\beta} \le M_{IR} M^{N_\beta + 1}} \sum_{j=0}^{l} \hat{\chi}_{\le j}(\omega, \mathbf{k}) W^j(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta).$$

In fact J^j mimics the output of the infrared integration at one scale. The kernel of its quadratic part is characterized as in (4.17) and substituted into the covariance. The covariance at scale l contains a collection of the kernels of the form (4.18). We are going to prove that $E_l \in \mathcal{K}(D, \alpha, M)(l)$ and $(E_l(\beta_1), E_l(\beta_2)) \in \hat{\mathcal{K}}(D, \alpha, M)(l)$ on the assumption (3.3), which is important information for the validity of the process. We need the next lemma.

Lemma 4.3.

(1)

$$\prod_{j=0}^{d} \left(\sum_{n_j=0}^{\infty} \left(\frac{2}{\pi}\right)^{n_j} \frac{\mathrm{w}(l)^{n_j}}{(2n_j)!} \right) \left| \prod_{p=0}^{d} \left(\frac{\partial}{\partial k_p}\right)^{n_p} W^l(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta) \right|$$
$$\le 2 \| J_2^l(\mathbf{U}) \|_{l,0}, \quad (\forall \mathbf{U} \in \overline{D}, (\omega, \mathbf{k}) \in \mathbb{R}^{d+1}, \rho, \eta \in \mathcal{B}).$$

(2) *Assume that* (3.3) *holds. Then,*

$$\prod_{j=0}^{d} \left(\sum_{n_j=0}^{\infty} \left(\frac{2}{\pi^2}\right)^{n_j} \frac{\mathrm{w}(l)^{n_j}}{(2n_j)!} \right)$$
$$\cdot \left| \prod_{p=0}^{d} \left(\frac{\partial}{\partial k_p}\right)^{n_p} (W^l(\beta_1)(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta) - W^l(\beta_2)(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta)) \right|$$
$$\le 2 | J_2^l(\beta_1)(\mathbf{U}) - J_2^l(\beta_2)(\mathbf{U}) |_l + \frac{4\pi}{\beta_1} \sum_{a=1}^{2} \| J_2^l(\beta_a)(\mathbf{U}) \|_{l,1},$$
$$(\forall \mathbf{U} \in \overline{D}, (\omega, \mathbf{k}) \in \mathbb{R}^{d+1}, \rho, \eta \in \mathcal{B}).$$

Proof. (1): Note that

$$\prod_{j=0}^{d}\left(\left(\frac{2}{\pi}\right)^{n_j}\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\right)\left|\prod_{p=0}^{d}\left(\frac{\partial}{\partial k_p}\right)^{n_p}W^l(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)\right|$$
$$\le \frac{2}{h}\sum_{(\mathbf{x},s)\in\Gamma(L)\times[0,\beta)_h}\prod_{j=0}^{d}\left(\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}d_j((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1))^{n_j}\right)$$
$$\cdot|J_2^l(\mathbf{U})((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1))|.$$

This inequality leads to the result.

(2):

$$\prod_{j=0}^{d}\left(\left(\frac{2}{\pi^2}\right)^{n_j}\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\right)$$
$$\cdot\left|\prod_{p=0}^{d}\left(\frac{\partial}{\partial k_p}\right)^{n_p}(W^l(\beta_1)(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)-W^l(\beta_2)(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta))\right|$$
$$\le\frac{2}{h}\sum_{(\mathbf{x},s)\in\Gamma(L)\times[-\beta_1/4,\beta_1/4)_h}$$
$$\cdot\prod_{j=0}^{d}\left(\frac{1}{(2n_j)!}\left(\frac{\mathrm{w}(l)}{\pi}\right)^{n_j}\hat{d}_j((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1))^{n_j}\right)$$
$$\cdot|J_2^l(\beta_1)(\mathbf{U})(R_{\beta_1}((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1)))$$
$$-J_2^l(\beta_2)(\mathbf{U})(R_{\beta_2}((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1)))|$$
$$+\frac{2}{h}\sum_{a=1}^{2}\sum_{(\mathbf{x},s)\in\Gamma(L)\times[0,\beta_a)_h}\left(\frac{2}{\pi}\right)^{n_0}$$
$$\cdot\left(1_{s\in[\frac{\beta_1}{4},\frac{\beta_a}{2})}\frac{|s|^{n_0}}{(\frac{\beta_a}{2\pi}|e^{i\frac{2\pi}{\beta_a}s}-1|)^{n_0+1}}+1_{s\in[\frac{\beta_a}{2},\beta_a-\frac{\beta_1}{4})}\frac{|s-\beta_a|^{n_0}}{(\frac{\beta_a}{2\pi}|e^{i\frac{2\pi}{\beta_a}s}-1|)^{n_0+1}}\right)$$
$$\cdot|d_0(\beta_a)((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1))|$$
$$\cdot\prod_{j=0}^{d}\left(\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}d_j(\beta_a)((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1))^{n_j}\right)$$
$$\cdot|J_2^l(\beta_a)(\mathbf{U})((\rho,\mathbf{x},\uparrow,s,-1),(\eta,\mathbf{0},\uparrow,0,1))|.$$

The result follows from the above inequality. □

In the following c_χ is the constant appearing in Lemma 4.1 (3).

LEMMA 4.4. *There exists a constant $c(d,M,c_w,c_\chi)\in\mathbb{R}_{>0}$ depending only on d,M,c_w,c_χ such that if $c_0\ge c(d,M,c_w,c_\chi)f_{\mathbf{t}}^{-\frac{1}{2}}$, the following statements hold.*

(1)

$$E_l\in\mathcal{K}(D,\alpha,M)(l).$$

(2) *Assume that* (3.3) *holds,* $l\in\{0,-1,\cdots,N_{\beta_1}\}$ *and* $(J^j(\beta_1),J^j(\beta_2))\in\hat{\mathcal{S}}(D,c_0,\alpha,M)(j)$ $(j=0,-1,\cdots,l)$*. Then,*

$$(E_l(\beta_1),E_l(\beta_2))\in\hat{\mathcal{K}}(D,\alpha,M)(l).$$

PROOF. (1): It suffices to consider the case that $1/\beta\le M_{IR}M^{N_\beta+1}$. The continuity and analyticity with $\mathbf{U}$ is clear. Let us prove the invariant properties. The periodicity claimed in (ii) follows from the definition. Since

$$\hat{\chi}_{\le j}(\omega,\mathbf{k})=\hat{\chi}_{\le j}(-\omega,\mathbf{k})=\hat{\chi}_{\le j}\left(\omega,-\mathbf{k}-\frac{2\pi}{L}\boldsymbol{\varepsilon}^L\right),\quad(\forall(\omega,\mathbf{k})\in\mathbb{R}^{d+1}),$$

it is sufficient to confirm the invariances of W^j $(j\in\{0,-1,\cdots,l\})$ to prove the invariances of E_l. The proof for the invariance of W^j claimed in (iii), (iv), (v) is parallel to the proof for [19, Lemma 7.6 (2), (7.25), (7.26), (7.27)] respectively. Here we only provide the sketch of the proof. The invariance in (iii) is proved by combining the anti-symmetry of $J_2^j(\mathbf{U})(\cdot)$, the invariance $J_2^j(\mathbf{U})(\psi)=J_2^j(\mathbf{U})(\mathcal{R}\psi)$ for $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.6) and the invariance $J_2^j(\mathbf{U})(\psi)=\overline{J_2^j(\overline{\mathbf{U}})(\mathcal{R}\psi)}$ for $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.8). The invariance in (iv) follows from the anti-symmetry of $J_2^j(\mathbf{U})(\cdot)$, the invariance $J_2^j(\mathbf{U})(\psi)=J_2^j(\mathbf{U})(\mathcal{R}\psi)$ for $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.6) and the invariance $J_2^j(\mathbf{U})(\psi)=\overline{J_2^j(\overline{\mathbf{U}})(\mathcal{R}\psi)}$ for $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.9). The invariance in (v) is due to the invariance $J_2^j(\mathbf{U})(\psi)=J_2^j(\mathbf{U})(\mathcal{R}\psi)$ for $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.6) and (4.7).

Next let us show the bound property (vi). Take $n_0, n_1, \cdots, n_d \in \mathbb{N} \cup \{0\}$ satisfying $\sum_{j=0}^{d} n_j > 0$. Using (4.2), Lemma 4.1 (3) and Lemma 4.3 (1), we can derive that for any $(\omega, \mathbf{k}) \in \mathbb{R}^{d+1}$,

$$\begin{aligned}
(4.19)\quad & \left|\prod_{j=0}^{d}\left(\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\left(\frac{\partial}{\partial k_j}\right)^{n_j}\right)E_l(\omega,\mathbf{k})(\rho,\eta)\right| \\
& \le M^l \sum_{p=0}^{l} M^{-p} \prod_{j=0}^{d}\left(\frac{\mathrm{w}(p)^{n_j}}{(2n_j)!}\right)\prod_{q=0}^{d}\left(\sum_{m_q=0}^{n_q}\begin{pmatrix} n_q \\ m_q \end{pmatrix}\right) \\
& \quad \cdot \left|\prod_{r=0}^{d}\left(\frac{\partial}{\partial k_r}\right)^{m_r}\hat{\chi}_{\le p}(\omega,\mathbf{k})\right|\left|\prod_{s=0}^{d}\left(\frac{\partial}{\partial k_s}\right)^{n_s-m_s} W^p(\omega,\mathbf{k})(\rho,\eta)\right| \\
& \le M^l \sum_{p=0}^{l} M^{-p} \prod_{j=0}^{d}\left(\frac{\mathrm{w}(p)^{n_j}}{(2n_j)!}\right)\prod_{q=0}^{d}\left(\sum_{m_q=0}^{n_q}\begin{pmatrix} n_q \\ m_q \end{pmatrix}\right) \\
& \quad \cdot \left|\prod_{r=0}^{d}(c_\chi \mathrm{w}(p)^{-1})^{m_r}(m_r!)^2\right| \\
& \quad \cdot \left|\prod_{s=0}^{d}\left(\frac{\pi}{2}\mathrm{w}(p)^{-1}\right)^{n_s-m_s}(2(n_s-m_s))!\right| 2\|J_2^p\|_{p,0} \\
& \le 2M^l \sum_{p=0}^{l} M^{-p}\|J_2^p\|_{p,0}\prod_{j=0}^{d}\left(\sum_{m_j=0}^{n_j}\begin{pmatrix} n_j \\ m_j \end{pmatrix} c_\chi^{m_j}\left(\frac{\pi}{2}\right)^{n_j-m_j}\right) \\
& \le \frac{2}{1-M^{-\frac{1}{2}}}\left(c_\chi+\frac{\pi}{2}\right)^{\sum_{j=0}^{d} n_j} c_0^{-1}\alpha^{-2}M^l,
\end{aligned}$$

which implies that

$$\begin{aligned}
& \prod_{j=0}^{d}\left(\sum_{n_j=0}^{\infty}\left(\frac{1}{2c_\chi+\pi}\right)^{n_j}\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\right)\left|\prod_{p=0}^{d}\left(\frac{\partial}{\partial k_p}\right)^{n_p}E_l(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)\right| \\
& \quad \cdot 1_{\sum_{q=0}^{d} n_q>0} \\
& \le \frac{2^{d+1}}{1-M^{-\frac{1}{2}}}c_0^{-1}\alpha^{-2}M^l \le \frac{2^{d+1}}{1-2^{-\frac{1}{2}}}c_0^{-1}\alpha^{-2}M^l, \\
& (\forall \mathbf{U} \in \overline{D}, (\omega,\mathbf{k}) \in \mathbb{R}^{d+1}, \rho,\eta \in \mathcal{B}),
\end{aligned}$$

where we used the condition $M \ge 2$. Thus, if $c_0 \ge 2^{d+2}/(1-2^{-\frac{1}{2}})$, E_l satisfies the inequality in (vi).

It remains to prove the bound property (vii). Take $\rho, \eta \in \mathcal{B}$. If $e^{i\langle b(\rho)-b(\eta), \boldsymbol{\pi}\rangle} = 1$, the invariances (iii), (iv) imply that

$$\begin{aligned}
&E_l(\mathbf{U})(\omega, \mathbf{k})(\rho, \eta) = -E_l(\mathbf{U})(-\omega, \mathbf{k})(\rho, \eta),\\
&(\forall \mathbf{U} \in \overline{D}, (\omega, \mathbf{k}) \in \mathcal{M} \times ((2\pi/L)\mathbb{Z})^d).
\end{aligned}$$

Thus, by (4.19),

$$\begin{aligned}
|E_l(\omega, \mathbf{k})(\rho, \eta)| &\le \frac{1}{2}\left|E_l(\omega, \mathbf{k})(\rho, \eta) - E_l\left(\frac{\pi}{\beta}, \mathbf{k}\right)(\rho, \eta)\right| \\
&\quad + \frac{1}{2}\left|E_l(\omega, \mathbf{k})(\rho, \eta) - E_l\left(-\frac{\pi}{\beta}, \mathbf{k}\right)(\rho, \eta)\right| \\
&\le c(d, M, c_\chi, c_w)\left(|\omega| + \frac{1}{\beta}\right) c_0^{-1}\alpha^{-2}, \\
&(\forall (\omega, \mathbf{k}) \in \mathcal{M} \times ((2\pi/L)\mathbb{Z})^d),
\end{aligned} \tag{4.20}$$

where $c(d, M, c_\chi, c_w) \in \mathbb{R}_{>0}$ is a constant depending only on d, M, c_χ, c_w. If $e^{i\langle b(\rho)-b(\eta), \boldsymbol{\pi}\rangle} = -1$, the invariance (v) and the periodicity (ii) yield that

$$\begin{aligned}
&E_l(\mathbf{U})(\omega, \boldsymbol{\pi})(\rho, \eta) = -e^{i\langle b(\rho)-b(\eta), \frac{\pi}{L}\boldsymbol{\varepsilon}^L\rangle} E_l(\mathbf{U})\left(\omega, \boldsymbol{\pi} - \frac{2\pi}{L}\boldsymbol{\varepsilon}^L\right)(\rho, \eta),\\
&(\forall \mathbf{U} \in \overline{D}, \omega \in \mathcal{M}).
\end{aligned}$$

Thus, by using the bound (4.2), Lemma 4.3 (1) and (4.19) we have that

$$\begin{aligned}
|E_l(\omega, \mathbf{k})(\rho, \eta)| &\le \frac{1}{2}|E_l(\omega, \mathbf{k})(\rho, \eta) - E_l(\omega, \boldsymbol{\pi})(\rho, \eta)| \\
&\quad + \frac{1}{2}\left|E_l(\omega, \mathbf{k})(\rho, \eta) - E_l\left(\omega, \boldsymbol{\pi} - \frac{2\pi}{L}\boldsymbol{\varepsilon}^L\right)(\rho, \eta)\right| \\
&\quad + \frac{c(d)}{L}\left|E_l\left(\omega, \boldsymbol{\pi} - \frac{2\pi}{L}\boldsymbol{\varepsilon}^L\right)(\rho, \eta)\right| \\
&\le c(d, M, c_\chi, c_w)\left(\sum_{j=1}^{d}|k_j - \pi| + \frac{1}{L}\right) c_0^{-1}\alpha^{-2}, \\
&(\forall (\omega, \mathbf{k}) \in \mathcal{M} \times ((2\pi/L)\mathbb{Z})^d).
\end{aligned} \tag{4.21}$$

By coupling (4.20) with (4.21) we obtain

$$|E_l(\omega,\mathbf{k})(\rho,\eta)| \le c(d,M,c_\chi,c_w)\left(|\omega|+\sum_{j=1}^{d}|k_j-\pi|+\frac{1}{\beta}+\frac{1}{L}\right)c_0^{-1}\alpha^{-2},$$
$$(\forall(\omega,\mathbf{k})\in\mathcal{M}\times((2\pi/L)\mathbb{Z})^d,\rho,\eta\in\mathcal{B}).$$

For any $(\omega,\mathbf{k}) \in \mathbb{R}^{d+1}$ there exists $(\hat{\omega},\hat{\mathbf{k}}) \in \mathcal{M}\times((2\pi/L)\mathbb{Z})^d$ such that $|\omega-\hat{\omega}| \le \pi/\beta$, $\|\mathbf{k}-\hat{\mathbf{k}}\|_{\mathbb{R}^d} \le \sqrt{d}\pi/L$. By taking into account this fact, we can deduce from the above inequality for $(\hat{\omega},\hat{\mathbf{k}})$ and (4.19) that

$$|E_l(\omega,\mathbf{k})(\rho,\eta)| \le c(d,M,c_\chi,c_w)\left(|\omega|+\sum_{j=1}^{d}|k_j-\pi|+\frac{1}{\beta}+\frac{1}{L}\right)c_0^{-1}\alpha^{-2},$$
$$(\forall(\omega,\mathbf{k})\in\mathbb{R}^{d+1},\rho,\eta\in\mathcal{B}).$$

Then, by using the periodicity of $E_l(\omega,\mathbf{k})$ with $\mathbf{k}$, the support property of $\hat{\chi}_{\le j}$, the assumption $1/L \le 1/\beta \le M_{IR}M^{N_\beta+1}$ and the fact $f_{\mathbf{t}} \le 1$ we reach the inequality that

$$\begin{aligned}&|E_l(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)| \le c(d,M,c_\chi,c_w)f_{\mathbf{t}}^{-\frac{1}{2}}c_0^{-1}\alpha^{-2}M^j,\\&(\forall\mathbf{U}\in\overline{D}, j\in\mathbb{Z}\text{ with } j\ge N_\beta,\\&\quad(\omega,\mathbf{k})\in\mathbb{R}^{d+1}\text{ satisfying }\hat{\chi}_{\le j}(\omega,\mathbf{k})\neq 0,\rho,\eta\in\mathcal{B}).\end{aligned}$$

Thus, E_l satisfies the property (vii) under the assumption that $c_0 \ge c(d,M,c_\chi,c_w)f_{\mathbf{t}}^{-\frac{1}{2}}$.

(2): Take $n_0,n_1,\cdots,n_d \in \mathbb{N}\cup\{0\}$. By (4.2), (4.11), Lemma 4.1 (3), Lemma 4.3 (2) and the assumption $M \ge 2$,

$$\begin{aligned}&\left|\prod_{j=0}^{d}\left(\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\left(\frac{\partial}{\partial k_j}\right)^{n_j}\right)(E_l(\beta_1)(\omega,\mathbf{k})(\rho,\eta)-E_l(\beta_2)(\omega,\mathbf{k})(\rho,\eta))\right|\\&\le\sum_{p=0}^{l}\prod_{j=0}^{d}\left(\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\sum_{m_j=0}^{n_j}\left(\begin{array}{c}n_j\\m_j\end{array}\right)\right)\left|\prod_{q=0}^{d}\left(\frac{\partial}{\partial k_q}\right)^{m_q}\hat{\chi}_{\le p}(\omega,\mathbf{k})\right|\end{aligned}$$

$$\cdot\left|\prod_{r=0}^{d}\left(\frac{\partial}{\partial k_r}\right)^{n_r-m_r}(W^p(\beta_1)(\omega,\mathbf{k})(\rho,\eta)-W^p(\beta_2)(\omega,\mathbf{k})(\rho,\eta))\right|$$
$$\le\sum_{p=0}^{l}\prod_{j=0}^{d}\left(\sum_{m_j=0}^{n_j}\left(\begin{array}{c}n_j\\m_j\end{array}\right)c_\chi^{m_j}\left(\frac{\pi^2}{2}\right)^{n_j-m_j}\right)$$
$$\cdot\left(2|J_2^p(\beta_1)-J_2^p(\beta_2)|_p+\frac{4\pi}{\beta_1}\sum_{a=1}^{2}\|J_2^p(\beta_a)\|_{p,1}\right)$$
$$\le\left(c_\chi+\frac{\pi^2}{2}\right)^{\sum_{j=0}^{d}n_j}\sum_{p=0}^{l}(2\beta_1^{-\frac{1}{2}}c_0^{-1}\alpha^{-2}M^{\frac{p}{2}}+4\pi\beta_1^{-1}c_0^{-1}\alpha^{-2}M^{\frac{p}{2}})$$
$$\le\left(c_\chi+\frac{\pi^2}{2}\right)^{\sum_{j=0}^{d}n_j}\frac{2+4\pi}{1-2^{-\frac{1}{2}}}c_0^{-1}\alpha^{-2}\beta_1^{-\frac{1}{2}}.$$

This inequality implies that

$$\prod_{j=0}^{d}\left(\sum_{n_j=0}^{\infty}\left(\frac{1}{2c_\chi+\pi^2}\right)^{n_j}\frac{\mathrm{w}(l)^{n_j}}{(2n_j)!}\right)$$
$$\cdot\left|\prod_{p=0}^{d}\left(\frac{\partial}{\partial k_j}\right)^{n_p}(E_l(\beta_1)(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)-E_l(\beta_2)(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta))\right|$$
$$\le\frac{2^{d+1}(2+4\pi)}{1-2^{-\frac{1}{2}}}c_0^{-1}\alpha^{-2}\beta_1^{-\frac{1}{2}},\quad(\forall\mathbf{U}\in\overline{D},\rho,\eta\in\mathcal{B},(\omega,\mathbf{k})\in\mathbb{R}^{d+1}).$$

Thus, the claim holds true if $c_0\ge 2^{d+1}(2+4\pi)/(1-2^{-\frac{1}{2}})$. □

In the next lemma we summarize properties of a function of $\mathbf{U}\in\overline{D}$ which resembles the final output of the infrared integration. In the following $C^\omega(D;\mathbb{C})$ denotes the set of analytic functions in D.

LEMMA 4.5. *Assume that $l\in\{0,-1,\cdots,N_\beta\}$, $G_l\in\mathcal{K}(D,\alpha,M)(l)$, $G_{l+1}\in\mathcal{K}(D,\alpha,M)(l+1)$ if $l\le -1$, $G_{l+1}=0$ if $l=0$. Moreover, assume that*

$$G_l(\mathbf{U})(\omega,\mathbf{k})-G_{l+1}(\mathbf{U})(\omega,\mathbf{k})=O,$$
$$(\forall\mathbf{U}\in\overline{D},(\omega,\mathbf{k})\in\mathbb{R}^{d+1}\ \textit{with}\ \hat{\chi}_{\le l}(\omega,\mathbf{k})=0).$$

Define the function $H_l : \overline{D} \to \mathbb{C}$ *by*

$$
\begin{aligned}
&H_l(\mathbf{U})\\
&:= \frac{1}{\beta L^d} \sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*} \log(\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\mathbf{U})(\omega,\mathbf{k}))^{-1}\\
&\qquad\qquad \cdot (G_l(\mathbf{U})(\omega,\mathbf{k}) - G_{l+1}(\mathbf{U})(\omega,\mathbf{k})))).
\end{aligned}
$$

Then, there exist a constant $c(d) \in \mathbb{R}_{>0}$ *depending only on* d *and a constant* $c(d, M, c_w, c_\chi) \in \mathbb{R}_{>0}$ *depending only on* d, M, c_w, c_χ *such that the following statements hold true if* $\alpha \geq c(d)$*.*

(1)
$$H_l \in C(\overline{D}; \mathbb{C}) \cap C^\omega(D; \mathbb{C}).$$

(2)
$$|H_l(\mathbf{U})| \leq c(d, M, c_w, c_\chi) f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d+1)l} \alpha^{-2}, \quad (\forall \mathbf{U} \in \overline{D}).$$

(3) *In addition, assume that* (3.3) *holds,* $l \in \{0, -1, \cdots, N_{\beta_1}\}$, $(G_l(\beta_1), G_l(\beta_2)) \in \hat{\mathcal{K}}(D, \alpha, M)(l)$ *and* $(G_{l+1}(\beta_1), G_{l+1}(\beta_2)) \in \hat{\mathcal{K}}(D, \alpha, M)(l + 1)$ *if* $l \leq -1$*. Then,*

$$
\begin{aligned}
&|H_l(\beta_1)(\mathbf{U}) - H_l(\beta_2)(\mathbf{U})|\\
&\leq c(d, M, c_w, c_\chi) \beta_1^{-\frac{1}{2}} f_{\mathbf{t}}^{-\frac{d}{2}} M^{dl} \alpha^{-2}, \quad (\forall \mathbf{U} \in \overline{D}).
\end{aligned}
$$

PROOF. (1), (2): Take $j \in \{l, l-1, \cdots, N_\beta\}$. It follows from Lemma 4.2 and (4.15) that for any $(\omega, \mathbf{k}) \in \mathbb{R}^{d+1}$ satisfying $\chi_j(\omega, \mathbf{k}) \neq 0$,

$$\|(i\omega I_{2^d} - \mathcal{E}(\omega, \mathbf{k}) - G_{l+1}(\omega, \mathbf{k}))^{-1}\|_{2^d\times 2^d} \leq M^{-j}, \tag{4.22}$$

$$\|G_l(\omega, \mathbf{k}) - G_{l+1}(\omega, \mathbf{k})\|_{2^d\times 2^d} \leq c(d)\alpha^{-2} M^j, \tag{4.23}$$

on the assumption that α is larger than a positive constant depending only on d. By using the above inequalities and Lemma 4.1 (1),(4),(5) we see that

$$|H_l(\mathbf{U})| \leq \frac{1}{\beta L^d} \sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*} \sum_{j=l}^{N_\beta} 1_{\chi_j(\omega,\mathbf{k})\neq 0} \tag{4.24}$$

$$\cdot \sum_{n=1}^{\infty} |\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\mathbf{U})(\omega,\mathbf{k}))^{-1} \cdot (G_l(\mathbf{U})(\omega,\mathbf{k}) - G_{l+1}(\mathbf{U})(\omega,\mathbf{k}))) - 1|^n$$
$$\le \frac{1}{\beta L^d} \sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*} \sum_{j=l}^{N_\beta} 1_{\chi_j(\omega,\mathbf{k})\neq 0} \sum_{n=1}^{\infty} (c(d)\alpha^{-2})^n$$
$$\le c(d,M) f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d+1)l} \alpha^{-2}, \quad (\forall \mathbf{U} \in \overline{D}).$$

This implies (2). Take $j \in \{l, l-1, \cdots, N_\beta\}$ and $(\omega,\mathbf{k}) \in \mathbb{R}^{d+1}$ satisfying $\chi_j(\omega,\mathbf{k}) \neq 0$. Since

$$(4.25) \qquad (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\mathbf{U})(\omega,\mathbf{k}))^{-1} = \sum_{n=0}^{\infty} ((i\omega I_{2^d} - \mathcal{E}(\mathbf{k}))^{-1} G_{l+1}(\mathbf{U})(\omega,\mathbf{k}))^n (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}))^{-1}$$

and this series converges uniformly with $\mathbf{U}$,

$$(i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\cdot)(\omega,\mathbf{k}))^{-1}(\rho,\eta) \in C(\overline{D};\mathbb{C}) \cap C^\omega(D;\mathbb{C}), \quad (\forall \rho,\eta \in \mathcal{B}).$$

Thus,

$$\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\cdot)(\omega,\mathbf{k}))^{-1}(G_l(\cdot)(\omega,\mathbf{k}) - G_{l+1}(\cdot)(\omega,\mathbf{k}))) \in C(\overline{D};\mathbb{C}) \cap C^\omega(D;\mathbb{C}).$$

Moreover, an estimation similar to (4.24) implies that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\mathbf{U})(\omega,\mathbf{k}))^{-1} \cdot (G_l(\mathbf{U})(\omega,\mathbf{k}) - G_{l+1}(\mathbf{U})(\omega,\mathbf{k}))) - 1)^n$$

converges uniformly with $\mathbf{U}$ and thus

$$\log(\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\cdot)(\omega,\mathbf{k}))^{-1} \cdot (G_l(\cdot)(\omega,\mathbf{k}) - G_{l+1}(\cdot)(\omega,\mathbf{k})))) \in C(\overline{D};\mathbb{C}) \cap C^\omega(D;\mathbb{C}).$$

Therefore, the claim (1) holds true.

(3): Let us prepare a couple of necessary inequalities. By (4.14),

$$\left\|\frac{\partial}{\partial\omega}(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_a)(\mathbf{U})(\omega,\mathbf{k}))\right\|_{2^d\times 2^d}\le c(d,M,c_w,c_\chi), \tag{4.26}$$

$$\begin{aligned}&\left\|\frac{\partial}{\partial\omega}(G_l(\beta_a)(\mathbf{U})(\omega,\mathbf{k})-G_{l+1}(\beta_a)(\mathbf{U})(\omega,\mathbf{k}))\right\|_{2^d\times 2^d}\\&\le c(d,M,c_w,c_\chi)\alpha^{-2},\\&(\forall \mathbf{U}\in\overline{D},(\omega,\mathbf{k})\in\mathbb{R}^{d+1},a\in\{1,2\}).\end{aligned} \tag{4.27}$$

Then, define the functions $\hat{H}_l(\beta_a):\overline{D}\to\mathbb{C}$ $(a=1,2)$ by

$$\begin{aligned}\hat{H}_l(\beta_a)(\mathbf{U}):=&\frac{1}{2\pi L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\left(\int_{-\pi h}^{-\frac{\pi}{\beta_a}}d\omega+\int_{\frac{\pi}{\beta_a}}^{\pi h}d\omega\right)\\&\cdot\log(\det(I_{2^d}-(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_a)(\mathbf{U})(\omega,\mathbf{k}))^{-1}\\&\qquad\cdot(G_l(\beta_a)(\mathbf{U})(\omega,\mathbf{k})-G_{l+1}(\beta_a)(\mathbf{U})(\omega,\mathbf{k})))).\end{aligned}$$

By using Lemma 4.1 (1),(2),(4),(5), (4.22), (4.23), (4.26) and (4.27) we deduce that

$$\begin{aligned}&|H_l(\beta_a)(\mathbf{U})-\hat{H}_l(\beta_a)(\mathbf{U})|\\&\le\frac{1}{2\pi L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\sum_{m=0}^{\frac{\beta_a h}{2}-1}\\&\cdot\left(\int_{\frac{\pi}{\beta_a}+\frac{2\pi}{\beta_a}m}^{\frac{\pi}{\beta_a}+\frac{2\pi}{\beta_a}(m+1)}d\omega\int_{\frac{\pi}{\beta_a}+\frac{2\pi}{\beta_a}m}^{\omega}d\eta\right.\\&\qquad\left.+\int_{-\frac{\pi}{\beta_a}-\frac{2\pi}{\beta_a}(m+1)}^{-\frac{\pi}{\beta_a}-\frac{2\pi}{\beta_a}m}d\omega\int_{\omega}^{-\frac{\pi}{\beta_a}-\frac{2\pi}{\beta_a}m}d\eta\right)\\&\cdot\Bigg|\frac{\partial}{\partial\eta}\log(\det(I_{2^d}-(i\eta I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_a)(\eta,\mathbf{k}))^{-1}\\&\qquad\cdot(G_l(\beta_a)(\eta,\mathbf{k})-G_{l+1}(\beta_a)(\eta,\mathbf{k}))))\Bigg|\\&\le\frac{1}{\beta_1 L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\sum_{j=l}^{N_{\beta_a}}\left(\int_{\frac{\pi}{\beta_a}}^{\pi h}d\omega+\int_{-\pi h}^{-\frac{\pi}{\beta_a}}d\omega\right)1_{\chi_j(\omega,\mathbf{k})\neq 0}\end{aligned} \tag{4.28}$$

$$
\begin{aligned}
&\cdot\left|\frac{\partial}{\partial\omega}\log(\det(I_{2^d}-(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k}))^{-1}\right.\\
&\qquad\qquad\qquad\left.\cdot(G_l(\beta_a)(\omega,\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k}))))\right|\\
&\le\frac{1}{\beta_1L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\sum_{j=l}^{N_{\beta_a}}\left(\int_{\frac{\pi}{\beta_a}}^{\pi h}d\omega+\int_{-\pi h}^{-\frac{\pi}{\beta_a}}d\omega\right)1_{\chi_j(\omega,\mathbf{k})\neq 0}\\
&\quad\cdot c(d)\Big\|\frac{\partial}{\partial\omega}((i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k}))^{-1}\\
&\qquad\qquad\cdot(G_l(\beta_a)(\omega,\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k})))\Big\|_{2^d\times 2^d}\\
&\le\frac{1}{\beta_1L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\sum_{j=l}^{N_{\beta_a}}\left(\int_{\frac{\pi}{\beta_a}}^{\pi h}d\omega+\int_{-\pi h}^{-\frac{\pi}{\beta_a}}d\omega\right)1_{\chi_j(\omega,\mathbf{k})\neq 0}\\
&\quad\cdot\Big(c(d)\|(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k}))^{-1}\|_{2^d\times 2^d}^2\\
&\qquad\cdot\Big\|\frac{\partial}{\partial\omega}(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k}))\Big\|_{2^d\times 2^d}\\
&\qquad\cdot\|G_l(\beta_a)(\omega,\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k})\|_{2^d\times 2^d}\\
&\quad+c(d)\|(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k}))^{-1}\|_{2^d\times 2^d}\\
&\qquad\cdot\Big\|\frac{\partial}{\partial\omega}(G_l(\beta_a)(\omega,\mathbf{k})-G_{l+1}(\beta_a)(\omega,\mathbf{k}))\Big\|_{2^d\times 2^d}\Big)\\
&\le c(d,M,c_w,c_\chi)\beta_1^{-1}f_{\mathbf{t}}^{-\frac{d}{2}}\alpha^{-2}\sum_{j=l}^{N_{\beta_a}}M^{dj}\\
&\le c(d,M,c_w,c_\chi)\beta_1^{-1}f_{\mathbf{t}}^{-\frac{d}{2}}\alpha^{-2}M^{dl}.
\end{aligned}
$$

Take $j\in\{l,l-1,\cdots,N_{\beta_1}\}$ and $(\omega,\mathbf{k})\in\mathbb{R}^{d+1}$ with $\chi_j(\omega,\mathbf{k})\neq 0$. Note that for any $a,b\in\mathbb{C}\backslash\mathbb{R}_{\le 0}$ with $|a-1|\le 1/2$, $|b-1|\le 1/2$, $|\log a-\log b|\le 2|a-b|$. Using this inequality, (4.16), (4.22), (4.23) as well as the assumption $\alpha\ge c(d)$, we can justify the following calculation.

$$
\begin{aligned}
&|\log(\det(I_{2^d}-(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_1)(\omega,\mathbf{k}))^{-1}\\
&\qquad\qquad\cdot(G_l(\beta_1)(\omega,\mathbf{k})-G_{l+1}(\beta_1)(\omega,\mathbf{k}))))
\end{aligned}
$$

$$
\begin{aligned}
&- \log(\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))^{-1} \\
&\qquad\qquad \cdot (G_l(\beta_2)(\omega,\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))))| \\
&\le 2|\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_1)(\omega,\mathbf{k}))^{-1} \\
&\qquad\qquad \cdot (G_l(\beta_1)(\omega,\mathbf{k}) - G_{l+1}(\beta_1)(\omega,\mathbf{k}))) \\
&\quad - \det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))^{-1} \\
&\qquad\qquad \cdot (G_l(\beta_2)(\omega,\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k})))| \\
&\le c(d)\|(i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_1)(\omega,\mathbf{k}))^{-1} \\
&\qquad \cdot (G_l(\beta_1)(\omega,\mathbf{k}) - G_{l+1}(\beta_1)(\omega,\mathbf{k})) \\
&\qquad - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))^{-1} \\
&\qquad\quad \cdot (G_l(\beta_2)(\omega,\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))\|_{2^d\times 2^d} \\
&\le c(d)\|(i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_1)(\omega,\mathbf{k}))^{-1}\|_{2^d\times 2^d} \\
&\quad \cdot \|(i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))^{-1}\|_{2^d\times 2^d} \\
&\quad \cdot \|G_{l+1}(\beta_1)(\omega,\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k})\|_{2^d\times 2^d} \\
&\quad \cdot \|G_l(\beta_1)(\omega,\mathbf{k}) - G_{l+1}(\beta_1)(\omega,\mathbf{k})\|_{2^d\times 2^d} \\
&\quad + c(d)\|(i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))^{-1}\|_{2^d\times 2^d} \\
&\qquad \cdot (\|G_l(\beta_1)(\omega,\mathbf{k}) - G_l(\beta_2)(\omega,\mathbf{k})\|_{2^d\times 2^d} \\
&\qquad\quad + \|G_{l+1}(\beta_1)(\omega,\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k})\|_{2^d\times 2^d}) \\
&\le c(d) M^{-j} \beta_1^{-\frac{1}{2}} \alpha^{-2}.
\end{aligned}
$$

It follows from this inequality, Lemma 4.1 (1),(4),(5), (4.22) and (4.23) that

$$
\begin{aligned}
(4.29)\quad & |\hat{H}_l(\beta_1)(\mathbf{U}) - \hat{H}_l(\beta_2)(\mathbf{U})| \\
&\le \frac{1}{2\pi L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\left(\int_{-\pi h}^{-\frac{\pi}{\beta_1}} d\omega + \int_{\frac{\pi}{\beta_1}}^{\pi h} d\omega\right)\sum_{j=l}^{N_{\beta_1}} 1_{\chi_j(\omega,\mathbf{k})\neq 0} \\
&\quad \cdot |\log(\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_1)(\omega,\mathbf{k}))^{-1} \\
&\qquad\qquad \cdot (G_l(\beta_1)(\omega,\mathbf{k}) - G_{l+1}(\beta_1)(\omega,\mathbf{k})))) \\
&\qquad - \log(\det(I_{2^d} - (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))^{-1} \\
&\qquad\qquad \cdot (G_l(\beta_2)(\omega,\mathbf{k}) - G_{l+1}(\beta_2)(\omega,\mathbf{k}))))| \\
&\quad + \frac{1}{2\pi L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\left(\int_{-\frac{\pi}{\beta_1}}^{-\frac{\pi}{\beta_2}} d\omega + \int_{\frac{\pi}{\beta_2}}^{\frac{\pi}{\beta_1}} d\omega\right)\sum_{j=l}^{N_{\beta_2}} 1_{\chi_j(\omega,\mathbf{k})\neq 0}
\end{aligned}
$$

$$\begin{aligned}
&\cdot|\log(\det(I_{2^d}-(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_{l+1}(\beta_2)(\omega,\mathbf{k}))^{-1}\\
&\qquad\qquad\cdot(G_l(\beta_2)(\omega,\mathbf{k})-G_{l+1}(\beta_2)(\omega,\mathbf{k}))))|\\
&\le\frac{1}{2\pi L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\left(\int_{-\pi h}^{-\frac{\pi}{\beta_1}}d\omega+\int_{\frac{\pi}{\beta_1}}^{\pi h}d\omega\right)\\
&\quad\cdot\sum_{j=l}^{N_{\beta_1}}1_{\chi_j(\omega,\mathbf{k})\neq 0}c(d)M^{-j}\beta_1^{-\frac{1}{2}}\alpha^{-2}\\
&\quad+\frac{1}{2\pi L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\left(\int_{-\frac{\pi}{\beta_1}}^{-\frac{\pi}{\beta_2}}d\omega+\int_{\frac{\pi}{\beta_2}}^{\frac{\pi}{\beta_1}}d\omega\right)\sum_{j=l}^{N_{\beta_2}}1_{\chi_j(\omega,\mathbf{k})\neq 0}c(d)\alpha^{-2}\\
&\le c(d,M)\beta_1^{-\frac{1}{2}}f_{\mathbf{t}}^{-\frac{d}{2}}\alpha^{-2}M^{dl}.
\end{aligned}$$

By coupling (4.28) with (4.29) we obtain the claimed inequality. □

Here we introduce sets of covariances. In the next subsection we will see that the actual covariances in the infrared integration belong to these sets. For $l\in\mathbb{Z}_{\le 0}$ we define the subset $\mathcal{R}(D,c_0,M)(l)$ of $\mathrm{Map}(\overline{D},\mathrm{Map}(I_0^2,\mathbb{C}))$ as follows. $C\in\mathrm{Map}(\overline{D},\mathrm{Map}(I_0^2,\mathbb{C}))$ belongs to $\mathcal{R}(D,c_0,M)(l)$ if and only if the following statements hold.

(i) $C(\cdot)(\mathbf{X})\in C(\overline{D};\mathbb{C})\cap C^{\omega}(D;\mathbb{C})$, $(\forall\mathbf{X}\in I_0^2)$.

(ii)

$$\begin{aligned}
&|\det(\langle\mathbf{p}_i,\mathbf{q}_j\rangle_{\mathbb{C}^m}C(\mathbf{U})(X_i,Y_j))_{1\le i,j\le n}|\le(c_0M^{dl})^n,\\
&(\forall m,n\in\mathbb{N},\mathbf{p}_i,\mathbf{q}_i\in\mathbb{C}^m\text{ with }\|\mathbf{p}_i\|_{\mathbb{C}^m},\|\mathbf{q}_i\|_{\mathbb{C}^m}\le 1,\\
&\ X_i,Y_i\in I_0\ (i=1,2,\cdots,n),\mathbf{U}\in\overline{D}).
\end{aligned}\tag{4.30}$$

(iii)

$$\|\widetilde{C}(\mathbf{U})\|_{l-1,t}\le c_0M^{-l-tl},\quad(\forall t\in\{0,1\},\mathbf{U}\in\overline{D}),\tag{4.31}$$

where $\widetilde{C}(\mathbf{U}):I^2\to\mathbb{C}$ is the anti-symmetric extension of $C(\mathbf{U})$ defined as in (3.9).

(iv)

$$\widetilde{C}(\mathbf{U})(\mathbf{X}) = e^{iQ_2(S_2(\mathbf{X}))}\widetilde{C}(\mathbf{U})(S_2(\mathbf{X})),\ (\forall \mathbf{X} \in I^2, \mathbf{U} \in \overline{D}),$$

for each $S : I \to I$, $Q : I \to \mathbb{R}$ defined in (4.3), (4.4), (4.5), (4.6), (4.7).

(v)

$$\widetilde{C}(\mathbf{U})(\mathbf{X}) = e^{-iQ_2(S_2(\mathbf{X}))}\overline{\widetilde{C}(\overline{\mathbf{U}})(S_2(\mathbf{X}))},\ (\forall \mathbf{X} \in I^2, \mathbf{U} \in \overline{D}),$$

for each $S : I \to I$, $Q : I \to \mathbb{R}$ defined in (4.8), (4.9).

It will be important to measure the difference between the covariances defined at different temperatures. For this purpose we introduce the subset $\hat{\mathcal{R}}(D, c_0, M)(l)$ of $\mathcal{R}(\beta_1)(D, c_0, M)(l) \times \mathcal{R}(\beta_2)(D, c_0, M)(l)$ on the assumption (3.3) as follows. $(C(\beta_1), C(\beta_2)) \in \mathcal{R}(\beta_1)(D, c_0, M)(l) \times \mathcal{R}(\beta_2)(D, c_0, M)(l)$ belongs to $\hat{\mathcal{R}}(D, c_0, M)(l)$ if and only if the following statements hold true.

(i)

$$\begin{aligned}(4.32)\qquad &|\det(\langle \mathbf{p}_i, \mathbf{q}_j\rangle_{\mathbb{C}^m} C(\beta_1)(\mathbf{U})(R_{\beta_1}(X_i, Y_j)))_{1\le i,j\le n}\\ &\quad - \det(\langle \mathbf{p}_i, \mathbf{q}_j\rangle_{\mathbb{C}^m} C(\beta_2)(\mathbf{U})(R_{\beta_2}(X_i, Y_j)))_{1\le i,j\le n}|\\ &\le \beta_1^{-\frac{1}{2}} M^{-l}(c_0 M^{dl})^n,\\ &(\forall m, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^m \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^m}, \|\mathbf{q}_i\|_{\mathbb{C}^m} \le 1,\\ &\quad X_i, Y_i \in \hat{I}_0\ (i = 1, 2, \cdots, n), \mathbf{U} \in \overline{D}).\end{aligned}$$

(ii)

$$(4.33)\qquad |\widetilde{C}(\beta_1)(\mathbf{U}) - \widetilde{C}(\beta_2)(\mathbf{U})|_{l-1} \le \beta_1^{-\frac{1}{2}} c_0 M^{-2l},\ (\forall \mathbf{U} \in \overline{D}).$$

Take any $l \in \{0, -1, \cdots, N_\beta\}$ and $G_l \in \mathcal{K}(D, \alpha, M)(l)$. The same estimation as in Lemma 4.2 ensures that we can define $C_l \in \mathrm{Map}(\overline{D}, \mathrm{Map}(I_0^2, \mathbb{C}))$ by

$$\begin{aligned}(4.34)\quad C_l(\mathbf{U})(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y) :=& \frac{\delta_{\sigma,\tau}}{\beta L^d} \sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*} e^{i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega}\chi_l(\omega, \mathbf{k})\\ &\cdot (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_l(\mathbf{U})(\omega, \mathbf{k}))^{-1}(\rho, \eta).\end{aligned}$$

In fact C_l is intended to be a generalization of the actual covariance appearing in the infrared integration process which we perform in the next subsection. Let us summarize properties of C_l.

LEMMA 4.6. *Assume that*

$$M \geq 8(d+1)^2(c_\chi + (1+\sqrt{2})^2(8c_\chi + 4\pi)). \tag{4.35}$$

Then, there exist a constant $c(d, M, c_w, c_\chi) \in \mathbb{R}_{>0}$ depending only on d, M, c_w, c_χ and a constant $c(d) \in \mathbb{R}_{>0}$ depending only on d such that the following statements hold if $c_0 \geq c(d, M, c_w, c_\chi) f_{\mathbf{t}}^{-\frac{d}{2}}$ and $\alpha \geq c(d)$.

(1)

$$C_l \in \mathcal{R}(D, c_0, M)(l).$$

(2) *Assume in addition that* (3.3) *holds,* $l \in \{0, -1, \cdots, N_{\beta_1}\}$ *and* $(G_l(\beta_1), G_l(\beta_2)) \in \hat{\mathcal{K}}(D, \alpha, M)(l)$. *Then,*

$$(C_l(\beta_1), C_l(\beta_2)) \in \hat{\mathcal{R}}(D, c_0, M)(l).$$

REMARK 4.7. To guarantee that C_l, $(C_l(\beta_1), C_l(\beta_2))$ satisfy (4.31), (4.33) respectively, we use the condition (4.35). The bound properties (4.31), (4.33) are crucially important for changing the measurement with the scale l to that with the next scale $l-1$ at every step of the infrared integration. We prefer to make explicit a sufficient condition for M to justify the crux of our RG regime. Also, recall that the only condition of M apart from the basic condition $M \geq 2$ so far is $M \geq c^{N_v^2}$ for some generic constant $c \in \mathbb{R}_{>0}$ in (3.18). The inequality (4.35) is the second nontrivial condition imposed on M.

The next lemma will be useful in the proof of Lemma 4.6.

LEMMA 4.8. *Let $A_j, B, C \in \mathbb{R}_{\geq 0}$ $(j = 0, 1, \cdots, d)$, $D \in \mathbb{R}_{>0}$ and assume that $A_j^n B \leq C D^n (n!)^2$, $(\forall j \in \{0, 1, \cdots, d\}, n \in \mathbb{N} \cup \{0\})$. Then,*

$$B \leq 4C e^{-\sum_{j=0}^{d}\left(\frac{A_j}{(d+1)^2 D}\right)^{1/2}}.$$

Proof. By assumption,

$$\frac{1}{n!}\left(\frac{A_j}{4D}\right)^{\frac{n}{2}}B^{\frac{1}{2}}\le C^{\frac{1}{2}}\left(\frac{1}{2}\right)^n,\quad (\forall j\in\{0,1,\cdots,d\},n\in\mathbb{N}\cup\{0\}).$$

By summing both sides over $n\in\mathbb{N}\cup\{0\}$ and squaring them we obtain

$$e^{\left(\frac{A_j}{D}\right)^{1/2}}B\le 4C,\quad (\forall j\in\{0,1,\cdots,d\}),$$

which leads to the result. □

Proof of Lemma 4.6. (1): The expansion of the integrand of C_l as in (4.25) converges uniformly with respect to $\mathbf{U}\in\overline{D}$. This implies that C_l satisfies the property (i) of $\mathcal{R}(D,c_0,M)(l)$. Let us check that C_l satisfies the invariant properties. The invariance with $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.3), (4.4), (4.5) is clear. The invariance with S, Q defined in (4.6) straightforwardly follows from the definitions. We can refer to the proof of the same invariance in [19, Lemma 7.13 (3)]. For $(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in I_0$, $\mathbf{U}\in\overline{D}$,

$$\begin{aligned}
&C_l(\mathbf{U})((\rho,r_L(-\mathbf{x}-b(\rho)),\sigma,x),(\eta,r_L(-\mathbf{y}-b(\eta)),\tau,y))\\
&\quad\cdot e^{i\langle-\mathbf{x}-b(\rho),\frac{2\pi}{L}\boldsymbol{\varepsilon}^L\rangle+i\langle b(\rho),\frac{\pi}{L}\boldsymbol{\varepsilon}^L\rangle}e^{-i\langle-\mathbf{y}-b(\eta),\frac{2\pi}{L}\boldsymbol{\varepsilon}^L\rangle-i\langle b(\eta),\frac{\pi}{L}\boldsymbol{\varepsilon}^L\rangle}\\
&=\frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle+i(x-y)\omega}\chi_l\left(\omega,-\mathbf{k}-\frac{2\pi}{L}\boldsymbol{\varepsilon}^L\right)U_d\left(\frac{\pi}{L}\boldsymbol{\varepsilon}^L\right)U_d(\mathbf{k})\\
&\quad\cdot\left(i\omega I_{2^d}-\mathcal{E}\left(-\mathbf{k}-\frac{2\pi}{L}\boldsymbol{\varepsilon}^L\right)-G_l(\mathbf{U})\left(\omega,-\mathbf{k}-\frac{2\pi}{L}\boldsymbol{\varepsilon}^L\right)\right)^{-1}\\
&\quad\cdot U_d(\mathbf{k})^*U_d\left(\frac{\pi}{L}\boldsymbol{\varepsilon}^L\right)^*(\rho,\eta)\\
&=C_l(\mathbf{U})((\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)),
\end{aligned}$$

where we used the facts $\chi_l(\omega,-\mathbf{k}-(2\pi/L)\boldsymbol{\varepsilon}^L)=\chi_l(\omega,\mathbf{k})$, $\mathcal{E}(\mathbf{k})=E(-\boldsymbol{\varepsilon}^L,-\boldsymbol{\theta})(\mathbf{k})$, Lemma 2.3 (2) and the invariance (v) of $\mathcal{K}(D,\alpha,M)(l)$. The above equality implies the invariance with $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.7). Thus, we have checked that all the invariances in the item (iv) of $\mathcal{R}(D,c_0,M)(l)$ hold.

An argument based on the invariances $\mathcal{E}(\mathbf{k})=\mathcal{E}(\mathbf{k})^*$, $G_l(\mathbf{U})(\omega,\mathbf{k})=G_l(\overline{\mathbf{U}})(-\omega,\mathbf{k})^*$ ($\forall\mathbf{U}\in\overline{D},(\omega,\mathbf{k})\in\mathcal{M}\times((2\pi/L)\mathbb{Z})^d$), parallel to the proof of

[19, Lemma 7.13 (4)] shows the invariance

$$\widetilde{C_l}(\mathbf{U})(\mathbf{X}) = e^{-iQ_2(S_2(\mathbf{X}))}\overline{\widetilde{C_l}(\overline{\mathbf{U}})(S_2(\mathbf{X}))}$$

with S, Q defined in (4.8).

For $(\rho, \mathbf{x}, \sigma, x), (\eta, \mathbf{y}, \tau, y) \in I_0$,

$$\begin{aligned}
&e^{i\langle b(\rho),\boldsymbol{\pi}\rangle + i\langle b(\eta),\boldsymbol{\pi}\rangle}\overline{C_l(\overline{\mathbf{U}})(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)}\\
&= \frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*} e^{i\langle \mathbf{y}-\mathbf{x},\mathbf{k}\rangle + i(y-x)\omega}\chi_l(\omega,\mathbf{k})\\
&\quad\cdot e^{i\langle b(\rho),\boldsymbol{\pi}\rangle + i\langle b(\eta),\boldsymbol{\pi}\rangle}(-i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_l(\overline{\mathbf{U}})(\omega,\mathbf{k})^*)^{-1}(\eta,\rho)\\
&= -C_l(\mathbf{U})(\eta\mathbf{y}\tau y, \rho\mathbf{x}\sigma x),
\end{aligned}$$

where we used Lemma 2.3 (1) and the invariance in (iv) of $\mathcal{K}(D,\alpha,M)(l)$. This equality leads to the invariance with $S : I \to I$, $Q : I \to \mathbb{R}$ defined in (4.9). Thus, C_l satisfies the invariances stated in the item (v) of $\mathcal{R}(D, c_0, M)(l)$.

By combining Lemma 4.1 (4),(5), Lemma 4.2 with the standard application of Gram's inequality we can show that

$$\begin{aligned}
&|\det(\langle \mathbf{p}_i, \mathbf{q}_j\rangle_{\mathbb{C}^m} C_l(\mathbf{U})(X_i, Y_j))_{1\le i,j\le n}| \le (c(d,M)f_{\mathbf{t}}^{-\frac{d}{2}}M^{dl})^n,\\
&(\forall m, n \in \mathbb{N}, \mathbf{p}_i, \mathbf{q}_i \in \mathbb{C}^m \text{ with } \|\mathbf{p}_i\|_{\mathbb{C}^m}, \|\mathbf{q}_i\|_{\mathbb{C}^m} \le 1,\\
&\quad X_i, Y_i \in I_0\ (i = 1, 2, \cdots, n), \mathbf{U} \in \overline{D}).
\end{aligned} \tag{4.36}$$

This means that C_l satisfies the determinant bound (4.30) for any $c_0 \ge c(d,M)f_{\mathbf{t}}^{-\frac{d}{2}}$.

It remains to prove (4.31). Take $j \in \{0, 1, \cdots, d\}$. By Lemma 2.3 (4), (4.14), the facts that $\|A\|_{2^d\times 2^d} \le 2^d \max_{\rho,\eta\in\mathcal{B}} |A(\rho,\eta)|$ $(\forall A \in \mathrm{Mat}(2^d, \mathbb{C}))$, $(2n)! \le 2^{2n}(n!)^2$ $(\forall n \in \mathbb{N})$, $\mathrm{w}(0) \le 1$ and the condition $\alpha^2 \ge 2^d$ we have

$$\begin{aligned}
&\left\|\left(\frac{\partial}{\partial k_j}\right)^n (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_l(\omega,\mathbf{k}))\right\|_{2^d\times 2^d}\\
&\le 1 + 2^d\alpha^{-2}M^l(2c_\chi + \pi)^n \mathrm{w}(l)^{-n}(2n)!\\
&\le 2M^l(8c_\chi + 4\pi)^n\mathrm{w}(l)^{-n}(n!)^2, \quad (\forall n \in \mathbb{N}_{\ge 1}, (\omega,\mathbf{k}) \in \mathbb{R}^{d+1}).
\end{aligned} \tag{4.37}$$

Take any $(\omega,\mathbf{k})\in\mathbb{R}^{d+1}$ satisfying $\chi_l(\omega,\mathbf{k})\neq 0$. By (4.37) and Lemma 4.2 we can apply [19, Lemma C.3 (2)] with $s=M^{-l}$, $q=2M^l$, $r=(8c_\chi+4\pi)\mathrm{w}(l)^{-1}$, $t=2$ to deduce that

$$\begin{aligned}(4.38)\quad &\left\|\left(\frac{\partial}{\partial k_j}\right)^n(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_l(\omega,\mathbf{k}))^{-1}\right\|_{2^d\times 2^d}\\ &\le\frac{M^{-2l}\cdot 2M^l}{(1+(M^{-l}\cdot 2M^l)^{\frac{1}{2}})^2}\\ &\quad\cdot\big((8c_\chi+4\pi)\mathrm{w}(l)^{-1}(1+(M^{-l}\cdot 2M^l)^{\frac{1}{2}})^2\big)^n(n!)^2\\ &\le M^{-l}\big((1+\sqrt{2})^2(8c_\chi+4\pi)\mathrm{w}(l)^{-1}\big)^n(n!)^2,\quad(\forall n\in\mathbb{N}\cup\{0\}).\end{aligned}$$

Moreover, by Lemma 4.1 (3),

$$\begin{aligned}(4.39)\quad &\left\|\left(\frac{\partial}{\partial k_j}\right)^n\chi_l(\omega,\mathbf{k})(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_l(\omega,\mathbf{k}))^{-1}\right\|_{2^d\times 2^d}\\ &\le M^{-l}\big((c_\chi+(1+\sqrt{2})^2(8c_\chi+4\pi))\mathrm{w}(l)^{-1}\big)^n(n!)^2,\\ &(\forall n\in\mathbb{N}\cup\{0\},(\omega,\mathbf{k})\in\mathbb{R}^{d+1}).\end{aligned}$$

By using the above inequality and Lemma 4.1 (4),(5) we can estimate as follows.

$$\begin{aligned}&\left\|\left(\frac{\beta}{2\pi}\right)^n(e^{-i(x-y)\frac{2\pi}{\beta}}-1)^nC_l(\cdot\mathbf{x}\sigma x,\cdot\mathbf{y}\tau y)\right\|_{2^d\times 2^d}\\ &\le\frac{1}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*}1_{\frac{1}{\beta}\le M_{IR}M^{N_\beta+1}}\prod_{j=1}^n\left(\frac{\beta}{2\pi}\int_0^{\frac{2\pi}{\beta}}d\omega_j\right)\\ &\quad\cdot\left\|\left(\frac{\partial}{\partial\omega'}\right)^n\chi_l(\omega',\mathbf{k})(i\omega' I_{2^d}-\mathcal{E}(\mathbf{k})-G_l(\omega',\mathbf{k}))^{-1}\right\|_{2^d\times 2^d}\Bigg|_{\omega'=\omega+\sum_{j=1}^n\omega_j}\\ &\le M^{-l}\big((c_\chi+(1+\sqrt{2})^2(8c_\chi+4\pi))\mathrm{w}(l)^{-1}\big)^n(n!)^2\\ &\quad\cdot\prod_{j=1}^n\left(\frac{\beta}{2\pi}\int_0^{\frac{2\pi}{\beta}}dw_j\right)\frac{1}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*}1_{\frac{1}{\beta}\le M_{IR}M^{N_\beta+1}}1_{\chi_l(\omega+\sum_{j=1}^n\omega_j,\mathbf{k})\neq 0}\\ &\le c(M,d)f_{\mathbf{t}}^{-\frac{d}{2}}M^{dl}\big((c_\chi+(1+\sqrt{2})^2(8c_\chi+4\pi))\mathrm{w}(l)^{-1}\big)^n(n!)^2.\end{aligned}$$

By repeating the same procedure as above we have that

$$|d_j(\mathbf{X})^n\widetilde{C_l}(\mathbf{X})|$$

$$\leq c(M,d)f_{\mathbf{t}}^{-\frac{d}{2}}M^{dl}\big((c_{\chi}+(1+\sqrt{2})^2(8c_{\chi}+4\pi))\mathrm{w}(l)^{-1}\big)^n(n!)^2,$$
$$(\forall j\in\{0,1,\cdots,d\},n\in\mathbb{N}\cup\{0\},\mathbf{X}\in I^2).$$

Here we can apply Lemma 4.8 to derive that

$$|\widetilde{C_l}(\mathbf{X})|\leq c(M,d)f_{\mathbf{t}}^{-\frac{d}{2}}M^{dl}e^{-\sum_{j=0}^{d}\left(\frac{\mathrm{w}(l)d_j(\mathbf{X})}{(d+1)^2(c_{\chi}+(1+\sqrt{2})^2(8c_{\chi}+4\pi))}\right)^{1/2}}, \quad (\forall \mathbf{X}\in I^2). \tag{4.40}$$

Moreover, on the assumption (4.35),

$$|\widetilde{C_l}(\mathbf{X})|\leq c(M,d)f_{\mathbf{t}}^{-\frac{d}{2}}M^{dl}e^{-2\sqrt{2}\sum_{j=0}^{d}(\mathrm{w}(l-1)d_j(\mathbf{X}))^{1/2}},\quad (\forall \mathbf{X}\in I^2).$$

which implies that

$$\|\widetilde{C_l}(\mathbf{U})\|_{l-1,t}\leq c(M,d,c_w)f_{\mathbf{t}}^{-\frac{d}{2}}M^{-l-tl},\quad (\forall t\in\{0,1\},\mathbf{U}\in\overline{D}).$$

Thus, if $c_0\geq c(M,d,c_w)f_{\mathbf{t}}^{-\frac{d}{2}}$, the covariance C_l satisfies the inequality (4.31).

(2): First note that the assumption $\beta_a\geq 1$ implies that $1/\beta_a\leq M_{IR}M^{N_{\beta_a}+1}$ $(a=1,2)$ and thus the results of Lemma 4.1 (5) for β_1, β_2 are available. For $l\in\{0,-1,\cdots,N_{\beta_a}\}$, $a\in\{1,2\}$, define $C_{ont,l}(\beta_a)\in\mathrm{Map}(\overline{D},\mathrm{Map}(\hat{I}_0^2,\mathbb{C}))$ by

$$\begin{aligned}&C_{ont,l}(\beta_a)(\mathbf{U})(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)\\&:=(-1)^{n_{\beta_a}(x)+n_{\beta_a}(y)}\frac{\delta_{\sigma,\tau}}{2\pi L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\int_{-\pi h}^{\pi h}d\omega e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle+i(x-y)\omega}\chi_l(\omega,\mathbf{k})\\&\quad\cdot(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-G_l(\beta_a)(\mathbf{U})(\omega,\mathbf{k}))^{-1}(\rho,\eta).\end{aligned}$$

By taking into account Lemma 4.1 (2) we can justify the following transformation. For any $(\mathbf{x},\sigma,x),(\mathbf{y},\tau,y)\in\Gamma(L)\times\{\uparrow,\downarrow\}\times[-\beta_1/4,\beta_1/4)_h$,

$$\begin{aligned}&C_{ont,l}(\beta_a)(\cdot\mathbf{x}\sigma x,\cdot\mathbf{y}\tau y)-C_l(\beta_a)(\cdot\mathbf{x}\sigma r_{\beta_a}(x),\cdot\mathbf{y}\tau r_{\beta_a}(y))\\&=(-1)^{n_{\beta_a}(x)+n_{\beta_a}(y)}\frac{\delta_{\sigma,\tau}}{2\pi L^d}\sum_{\mathbf{k}\in\Gamma(L)^*}e^{i\langle\mathbf{x}-\mathbf{y},\mathbf{k}\rangle}\sum_{m=0}^{\beta_a h}\int_{-\pi h-\frac{\pi}{\beta_a}+\frac{2\pi}{\beta_a}m}^{-\pi h-\frac{\pi}{\beta_a}+\frac{2\pi}{\beta_a}(m+1)}d\omega\end{aligned}$$

$$\cdot \int_{-\pi h - \frac{\pi}{\beta_a} + \frac{2\pi}{\beta_a} m}^{\omega} du \frac{\partial}{\partial u}(e^{i(x-y)u}\chi_l(u,\mathbf{k})(iuI_{2^d} - \mathcal{E}(\mathbf{k}) - G_l(\beta_a)(u,\mathbf{k}))^{-1}).$$

Then, by Lemma 4.1 (5) and (4.39),

$$\begin{aligned}(4.41)\quad &\|C_{ont,l}(\beta_a)(\cdot\mathbf{x}\sigma x, \cdot\mathbf{y}\tau y) - C_l(\beta_a)(\cdot\mathbf{x}\sigma r_{\beta_a}(x), \cdot\mathbf{y}\tau r_{\beta_a}(y))\|_{2^d\times 2^d}\\ &\le c(d,M,c_w,c_\chi)\beta_1^{-1} f_{\mathbf{t}}^{-\frac{d}{2}} M^{dl}(|x-y| + M^{-l}).\end{aligned}$$

Calculation parallel to that leading to (4.40) yields that

$$\begin{aligned}(4.42)\quad &|\widetilde{C_{ont,l}}(\beta_a)(\mathbf{X})|\\ &\le c(M,d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{dl} e^{-\sum_{j=0}^{d}\left(\frac{\mathrm{w}(l)\hat{d}_j(\mathbf{X})}{(d+1)^2(c_\chi + (1+\sqrt{2})^2(8c_\chi+4\pi))}\right)^{1/2}},\\ &(\forall \mathbf{X} \in \hat{I}^2).\end{aligned}$$

By using the inequality $d_j(R_{\beta_a}(\mathbf{X})) \ge (2/\pi)\hat{d}_j(\mathbf{X})$ $(\forall \mathbf{X} \in \hat{I}^2)$ we can derive from (4.40) that

$$\begin{aligned}(4.43)\quad &|\widetilde{C_l}(\beta_a)(R_{\beta_a}(\mathbf{X}))|\\ &\le c(M,d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{dl} e^{-\sum_{j=0}^{d}\left(\frac{2\mathrm{w}(l)\hat{d}_j(\mathbf{X})}{\pi(d+1)^2(c_\chi + (1+\sqrt{2})^2(8c_\chi+4\pi))}\right)^{1/2}},\\ &(\forall \mathbf{X} \in \hat{I}^2).\end{aligned}$$

By putting (4.41), (4.42), (4.43) together,

$$\begin{aligned}(4.44)\quad &|\widetilde{C_{ont,l}}(\beta_a)(\mathbf{X}) - \widetilde{C_l}(\beta_a)(R_{\beta_a}(\mathbf{X}))|\\ &\le c(d,M,c_w,c_\chi)\beta_1^{-\frac{1}{2}} f_{\mathbf{t}}^{-\frac{d}{2}} M^{dl}(\hat{d}_0(\mathbf{X})^{\frac{1}{2}} + M^{-\frac{l}{2}})\\ &\quad \cdot e^{-\sum_{j=0}^{d}\left(\frac{\mathrm{w}(l)\hat{d}_j(\mathbf{X})}{2\pi(d+1)^2(c_\chi + (1+\sqrt{2})^2(8c_\chi+4\pi))}\right)^{1/2}}\\ &\le c(d,M,c_w,c_\chi)\beta_1^{-\frac{1}{2}} f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d-\frac{1}{2})l}\\ &\quad \cdot e^{-\sum_{j=0}^{d}\left(\frac{\mathrm{w}(l)\hat{d}_j(\mathbf{X})}{4\pi(d+1)^2(c_\chi + (1+\sqrt{2})^2(8c_\chi+4\pi))}\right)^{1/2}},\\ &(\forall \mathbf{X} \in \hat{I}^2, a \in \{1,2\}).\end{aligned}$$

We need to establish a decay bound on $C_{ont,l}(\beta_1) - C_{ont,l}(\beta_2)$. Remark that

$$C_{ont,l}(\beta_1)(\cdot\mathbf{x}\sigma x, \cdot\mathbf{y}\tau y) - C_{ont,l}(\beta_2)(\cdot\mathbf{x}\sigma x, \cdot\mathbf{y}\tau y)$$

$$
\begin{aligned}
&= (-1)^{1_{x<0}+1_{y<0}} \frac{\delta_{\sigma,\tau}}{2\pi L^d} \sum_{\mathbf{k}\in\Gamma(L)^*} \int_{-\pi h}^{\pi h} d\omega e^{i\langle \mathbf{x}-\mathbf{y},\mathbf{k}\rangle + i(x-y)\omega} \chi_l(\omega,\mathbf{k})\\
&\cdot (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_l(\beta_1)(\omega,\mathbf{k}))^{-1}(G_l(\beta_1)(\omega,\mathbf{k}) - G_l(\beta_2)(\omega,\mathbf{k}))\\
&\cdot (i\omega I_{2^d} - \mathcal{E}(\mathbf{k}) - G_l(\beta_2)(\omega,\mathbf{k}))^{-1}.
\end{aligned}
$$

Then, by Lemma 4.1 (5), (4.16), (4.38), (4.39) and the fact $(n!)^2 \le (2n)! \le 2^{2n}(n!)^2$ we have that for $j \in \{0,1,\cdots,d\}$, $\mathbf{X} \in \hat{I}^2$,

$$
\begin{aligned}
&\hat{d}_j(\mathbf{X})^n |\widetilde{C_{ont,l}}(\beta_1)(\mathbf{X}) - \widetilde{C_{ont,l}}(\beta_2)(\mathbf{X})|\\
&\le c(M,d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d-1)l} \beta_1^{-\frac{1}{2}} \alpha^{-2}\\
&\quad\cdot \sum_{m_1=0}^{n} \binom{n}{m_1} \big((c_\chi + (1+\sqrt{2})^2(8c_\chi + 4\pi))\mathrm{w}(l)^{-1}\big)^{m_1} (m_1!)^2\\
&\quad\cdot \sum_{m_2=0}^{n-m_1} \binom{n-m_1}{m_2} ((2c_\chi + \pi^2)\mathrm{w}(l)^{-1})^{m_2}(2m_2)!\\
&\quad\cdot ((1+\sqrt{2})^2(8c_\chi + 4\pi)\mathrm{w}(l)^{-1})^{n-m_1-m_2}((n-m_1-m_2)!)^2\\
&\le c(M,d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d-1)l} \beta_1^{-\frac{1}{2}} \alpha^{-2}(2n)!\\
&\quad\cdot \big((c_\chi + (1+\sqrt{2})^2(8c_\chi+4\pi))\mathrm{w}(l)^{-1} + (2c_\chi + \pi^2)\mathrm{w}(l)^{-1}\\
&\qquad + (1+\sqrt{2})^2(8c_\chi + 4\pi)\mathrm{w}(l)^{-1}\big)^n\\
&\le c(M,d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d-1)l} \beta_1^{-\frac{1}{2}} \alpha^{-2}(n!)^2\\
&\quad\cdot (4\pi(c_\chi + (1+\sqrt{2})^2(8c_\chi+4\pi))\mathrm{w}(l)^{-1})^n,
\end{aligned}
$$

which combined with Lemma 4.8 implies that

$$
\begin{aligned}
&|\widetilde{C_{ont,l}}(\beta_1)(\mathbf{X}) - \widetilde{C_{ont,l}}(\beta_2)(\mathbf{X})|\\
&\le c(M,d) f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d-1)l} \beta_1^{-\frac{1}{2}} \alpha^{-2}\\
&\quad\cdot e^{-\sum_{j=0}^{d}\left(\frac{\mathrm{w}(l)\hat{d}_j(\mathbf{X})}{4\pi(d+1)^2(c_\chi + (1+\sqrt{2})^2(8c_\chi+4\pi))}\right)^{1/2}}.
\end{aligned}
\tag{4.45}
$$

On the assumption (4.35), the inequalities (4.44), (4.45) yield that

$$
|\widetilde{C_l}(\beta_1)(\mathbf{U})(R_{\beta_1}(\mathbf{X})) - \widetilde{C_l}(\beta_2)(\mathbf{U})(R_{\beta_2}(\mathbf{X}))| \tag{4.46}
$$

$$\le c(d,M,c_w,c_\chi)\beta_1^{-\frac{1}{2}}f_{\mathbf{t}}^{-\frac{d}{2}}M^{(d-1)l}e^{-\sqrt{2}\sum_{j=0}^{d}\left(\frac{1}{\pi}\mathrm{w}(l-1)\hat{d}_j(\mathbf{X})\right)^{1/2}},$$
$$(\forall \mathbf{X}\in \hat{I}^2, \mathbf{U}\in\overline{D}),$$

and thus,

$$|\widetilde{C}_l(\beta_1)(\mathbf{U})-\widetilde{C}_l(\beta_2)(\mathbf{U})|_{l-1}\le c(d,M,c_w,c_\chi)f_{\mathbf{t}}^{-\frac{d}{2}}\beta_1^{-\frac{1}{2}}M^{-2l},\quad (\forall \mathbf{U}\in\overline{D}).$$

Therefore, the inequality (4.33) holds for $c_0\ge c(d,M,c_w,c_\chi)f_{\mathbf{t}}^{-\frac{d}{2}}$. By using (4.36), (4.46) and applying the Cauchy-Binet formula as in the proof of Lemma 3.1 (4) we can prove that (4.32) holds for $c_0\ge c(d,M,c_w,c_\chi)\ f_{\mathbf{t}}^{-\frac{d}{2}}$. □

We conclude this subsection by describing the recursive structure of the infrared integration in terms of the scale-dependent sets of Grassmann polynomials and covariances introduced so far. The proof of the following lemma is essentially based on the general results [19, Lemma 3.9, Proposition 5.6, Proposition 5.9]. See [19, Subsection 2.2] for the meaning of uniform convergence of a sequence of Grassmann polynomials.

Lemma 4.9. *There exists a constant $c\in\mathbb{R}_{>0}$ independent of any parameter such that if*

$$M^{d-\frac{3}{2}}\ge c,\quad \alpha\ge cM^{d+\frac{3}{2}},\tag{4.47}$$

the following statements hold true.

(1) *If $l\in\mathbb{Z}_{<0}$,*

$$J^{l+1}\in\mathcal{S}(D,c_0,\alpha,M)(l+1),\quad C_{l+1}\in\mathcal{R}(D,c_0,M)(l+1),$$

then,

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d}{dz}\right)^n\Big|_{z=0}\log\left(\int e^{z\sum_{m=4}^{N}J_m^{l+1}(\mathbf{U})(\psi+\psi^1)}d\mu_{C_{l+1}(\mathbf{U})}(\psi^1)\right)$$

uniformly converges with $\mathbf{U}\in\overline{D}$. Let J^l denote it. Then,

$$J^l\in\mathcal{S}(D,c_0,\alpha,M)(l).$$

(2) *In addition, assume that* (3.3) *holds and*

$$(J^{l+1}(\beta_1), J^{l+1}(\beta_2)) \in \hat{\mathcal{S}}(D, c_0, \alpha, M)(l+1),$$
$$(C_{l+1}(\beta_1), C_{l+1}(\beta_2)) \in \hat{\mathcal{R}}(D, c_0, M)(l+1).$$

Then,

$$(J^l(\beta_1), J^l(\beta_2)) \in \hat{\mathcal{S}}(D, c_0, \alpha, M)(l).$$

Proof. (1): Let us define F^l, $T^{l,(n)} \in \mathrm{Map}(\overline{D}, \bigwedge \mathcal{V})$ $(n \in \mathbb{N}_{\ge 2})$ by

$$F^l(\mathbf{U})(\psi) := \frac{d}{dz}\Big|_{z=0} \log\left(\int e^{z\sum_{m=4}^{N} J_m^{l+1}(\mathbf{U})(\psi+\psi^1)} d\mu_{C_{l+1}(\mathbf{U})}(\psi^1)\right),$$
$$T^{l,(n)}(\mathbf{U})(\psi)$$
$$:= \frac{1}{n!}\left(\frac{d}{dz}\right)^n\Big|_{z=0} \log\left(\int e^{z\sum_{m=4}^{N} J_m^{l+1}(\mathbf{U})(\psi+\psi^1)} d\mu_{C_{l+1}(\mathbf{U})}(\psi^1)\right).$$

It is implied by [19, Proposition 5.6] with $a_1 = d$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1/2$ that on the assumption (4.47)

$$\frac{h}{N}\left(|F_0^l(\mathbf{U})| + \sum_{n=2}^{\infty} |T_0^{l,(n)}(\mathbf{U})|\right) \le M^{(d+\frac{3}{2})l}\alpha^{-1},$$
$$M^{-(d+\frac{3}{2})l+tl} \sum_{m=2}^{N} c_0^{\frac{m}{2}} \alpha^m M^{\frac{d}{2}lm}\left(\|F_m^l(\mathbf{U})\|_{l,t} + \sum_{n=2}^{\infty} \|T_m^{l,(n)}(\mathbf{U})\|_{l,t}\right) \le 1,$$
$$(\forall \mathbf{U} \in \overline{D},\ t \in \{0,1\}).$$

Moreover, it is clear from the derivation of the inequalities "(5.63)", "(5.66)" in the proof of [19, Proposition 5.6] that

$$\sum_{n=2}^{\infty} \sup_{\mathbf{U}\in\overline{D}} \left(|T_0^{l,(n)}(\mathbf{U})| + \sum_{m=2}^{N} \|T_m^{l,(n)}(\mathbf{U})\|_{l,0}\right) < \infty.$$

Thus, $F^l + \sum_{n=2}^{\infty} T^{l,(n)}$ uniformly converges with respect to $\mathbf{U} \in \overline{D}$. By the definition of the free integration and the tree formula (see, e.g. [27, Theorem 3]), $F^l, T^{l,(n)}$ $(n \in \mathbb{N}_{\ge 2})$ consist of finite sums and products of J^{l+1}, C_{l+1}. Thus, $F^l, T^{l,(n)} \in C(\overline{D}; \bigwedge \mathcal{V}) \cap C^\omega(D; \bigwedge \mathcal{V})$ $(\forall n \in \mathbb{N}_{\ge 2})$. Therefore, the uniform convergent property ensures that $J^l \in C(\overline{D}; \bigwedge \mathcal{V}) \cap$

$C^{\omega}(D;\bigwedge\mathcal{V})$. The above inequalities imply the bound properties (4.1), (4.2). We can apply [19, Lemma 3.9] to prove that J^l inherits the invariant properties claimed in the items (ii), (iii) of $\mathcal{S}(D,c_0,\alpha,M)(l)$ from J^{l+1} and C_{l+1}. Therefore, $J^l\in\mathcal{S}(D,c_0,\alpha,M)(l)$.

(2): On the assumption (4.47) the claim follows from [19, Proposition 5.9] with $a_1=d$, $a_2=1$, $a_3=1$, $a_4=1/2$. □

4.2. Completion of the infrared integration

Here we implement the infrared integration scheme to prove Theorem 1.6. Most of the necessary tools for justifying the multi-scale integration have already been prepared in the preceding sections. By putting together these lemmas and a lemma separately made in Appendix C we will reach the proof of Theorem 1.6. First of all let us describe properties of the output of the Matsubara UV integration in terms of the sets $\mathcal{S}(D,c_0,\alpha,M)(0)$, $\hat{\mathcal{S}}(D,c_0,\alpha,M)(0)$. In the following $C_{>0}^{\delta}:I_0^2\to\mathbb{C}$ ($\delta=+,-$) are the covariances defined in (2.8), (2.9) with the cut-off function $\phi(M_{UV}^{-2}h^2|1-e^{i\omega/h}|^2)$ in place of $\chi(h|1-e^{i\omega/h}|)$.

LEMMA 4.10. *There exist a constant $c\in\mathbb{R}_{>0}$ independent of any parameter and a constant $c(M,d)\in\mathbb{R}_{\ge 1}$ depending only on M, d such that if* (3.18) *holds with c, the following statements hold for any $c_0\ge c(M,d)$ and $r\in\mathbb{R}_{>0}$ satisfying*

$$re^{d\mathrm{w}(0)^{1/2}}\sum_{m=1}^{N_v}c_0^m\alpha^{2m}v_m(\mathrm{w}(0))\le\frac{1}{2}. \tag{4.48}$$

(1) *There exist $r(\beta,L)\in\mathbb{R}_{>0}$ dependent on β, L, independent of h and $J^0\in\mathcal{S}(D(r)^{n_v},c_0,\alpha,M)(0)$ such that*

$$\begin{aligned}J^0(\mathbf{U})(\psi)=\frac{1}{2}\sum_{\delta\in\{+,-\}}&\log\left(\int e^{-V^{\delta}(\mathbf{U})(\psi+\psi^1)}d\mu_{C_{>0}^{\delta}}(\psi^1)\right)\\&+\beta V_0^L(\mathbf{U}),\\(\forall\mathbf{U}\in\overline{D(r(\beta,L))}^{n_v}).&\end{aligned} \tag{4.49}$$

(2) *On the assumption* (3.3),

$$(J^0(\beta_1),J^0(\beta_2))\in\hat{\mathcal{S}}(D(r)^{n_v},c_0,\alpha,M)(0).$$

Proof. There exists a constant $c(M,d)\in\mathbb{R}_{\ge 1}$ depending only on M, d such that the conclusions of Lemma 3.1 hold for any $c_0\ge c(M,d)$. Fix such c_0. Let $F^{N_h,\delta}(\psi), T^{N_h,\delta}(\psi), J^{N_h,\delta}(\psi), F^{l,\delta}(\psi), T^{l,\delta,(n)}(\psi), J^{l,\delta}(\psi)$ ($\in \bigwedge\mathcal{V}$) ($\delta\in\{+,-\}, l\in\{0,1,\cdots,N_h-1\}, n\in\mathbb{N}_{\ge 2}$) be defined as in the beginning of Subsection 3.2. Here we explicitly show the dependency on the parameter δ, while we concealed it in Subsection 3.2 and Subsection 3.3. Then, set $J^0(\psi):=(J^{0,+}(\psi)+J^{0,-}(\psi))/2$. By Lemma 3.4 and Lemma 3.7 there exists a constant $c\in\mathbb{R}_{>0}$ independent of any parameter such that if (3.18) holds with c, the following bounds hold with any $r\in\mathbb{R}_{>0}$ satisfying (4.48).

$$\begin{aligned}&\frac{h}{N}|J_0^0(\mathbf{U})|\le\alpha^{-1},\\ &\sum_{m=2}^{N}c_0^{\frac{m}{2}}\alpha^m\|J_m^0(\mathbf{U})\|_{0,t}\le 1,\quad (\forall t\in\{0,1\},\mathbf{U}\in\overline{D(r)}^{n_v}).\end{aligned}$$

On the assumption (3.3),

$$\begin{aligned}&\left|\frac{h}{N(\beta_1)}J_0^0(\beta_1)(\mathbf{U})-\frac{h}{N(\beta_2)}J_0^0(\beta_2)(\mathbf{U})\right|\le\beta_1^{-\frac{1}{2}}\alpha^{-1},\\ &\sum_{m=2}^{N(\beta_2)}c_0^{\frac{m}{2}}\alpha^m|J_m^0(\beta_1)(\mathbf{U})-J_m^0(\beta_2)(\mathbf{U})|_0\le\beta_1^{-\frac{1}{2}},\quad (\forall\mathbf{U}\in\overline{D(r)}^{n_v}).\end{aligned}$$

Moreover, by Lemma 3.4

$$\begin{aligned}(4.50)\quad &\sum_{n=2}^{\infty}\sup_{\mathbf{U}\in\overline{D(r)}^{n_v}}\sum_{\delta\in\{+,-\}}\left(|T_0^{l,\delta,(n)}(\mathbf{U})|+\sum_{m=2}^{N}\|T_m^{l,\delta,(n)}(\mathbf{U})\|_{0,0}\right)<\infty,\\ &(\forall l\in\{0,1,\cdots,N_h-1\}).\end{aligned}$$

Let us prove that

$$(4.51)\qquad J^{l,\delta}\in C\left(\overline{D(r)}^{n_v};\bigwedge\mathcal{V}\right)\cap C^{\omega}\left(D(r)^{n_v};\bigwedge\mathcal{V}\right)\quad (\forall\delta\in\{+,-\})$$

for any $l\in\{0,1,\cdots,N_h\}$. It is clear from the definition that (4.51) holds for $l=N_h$. Assume that (4.51) holds for $l+1$. Then, by definition $F^{l,\delta}, T^{l,\delta,(n)}\in C(\overline{D(r)}^{n_v};\bigwedge\mathcal{V})\cap C^{\omega}(D(r)^{n_v};\bigwedge\mathcal{V})$ ($\forall\delta\in\{+,-\}, n\in\mathbb{N}_{\ge 2}$). The bound property (4.50) implies that $\sum_{n=2}^{\infty}T^{l,\delta,(n)}$ uniformly converges. Thus (4.51) holds for l. By induction, (4.51) holds for any $l \in \{0,1,\cdots,N_h\}$. We especially have that $J^0 \in C(\overline{D(r)}^{n_v};\bigwedge\mathcal{V})\cap C^{\omega}(D(r)^{n_v};\bigwedge\mathcal{V})$.

By the same argument as in the proof of [19, Proposition 6.4 (3)] we can conclude that there exists $r(\beta,L)\in\mathbb{R}_{>0}$ depending on β, L and independent of h such that

$$J^0(\mathbf{U})(\psi)=\frac{1}{2}\sum_{\delta\in\{+,-\}}\log\left(\int e^{-V^{\delta}(\mathbf{U})(\psi+\psi^1)+\beta V_0^L(\mathbf{U})}d\mu_{C^{\delta}_{>0}}(\psi^1)\right),\quad (\forall \mathbf{U}\in\overline{D(r(\beta,L))}^{n_v}). \tag{4.52}$$

We can see from the properties of $V^{\delta}(\mathbf{U})(\psi)$, $V_0^L(\mathbf{U})$ and the definition of logarithm of Grassmann polynomial (see, e.g. [19, Subsection 2.2]) that the right-hand side of (4.52) is equal to that of (4.49) if $\max_{j\in\{1,2,\cdots,n_v\}}|U_j|$ is sufficiently small. The inequality (2.15) implies that there exists $r(\beta,L)'\in\mathbb{R}_{>0}$ dependent on β,L, independent of h such that the right-hand side of (4.49) is analytic with $\mathbf{U}$ in $D(r(\beta,L)')^{n_v}$. Thus, by using the identity theorem and continuity and taking $r(\beta,L)$ smaller independently of h if necessary we obtain the equality (4.49) for $\mathbf{U}\in\overline{D(r(\beta,L))}^{n_v}$.

It remains to check that J^0 satisfies the invariant properties. Recall the definitions (2.5), (2.11). The invariance

$$-V^{\delta}(\mathbf{U})(\mathcal{R}\psi)+\beta V_0^L(\mathbf{U})=-V^{\delta}(\mathbf{U})(\psi)+\beta V_0^L(\mathbf{U}),\quad (\forall \mathbf{U}\in\overline{D(r)}^{n_v},\delta\in\{+,-\}) \tag{4.53}$$

for $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.3), (4.4), (4.5), (4.6) follows from the definition of V^{δ}, (1.6), (1.7), (1.8) and (1.9) respectively. The properties (1.8), (1.10) and (1.11) imply (4.53) for S,Q defined in (4.7) as well. Moreover, the property (1.12) ensures that

$$-\overline{V^{\delta}(\overline{\mathbf{U}})}(\mathcal{R}\psi)+\beta\overline{V_0^L(\overline{\mathbf{U}})}=-V^{\delta}(\mathbf{U})(\psi)+\beta V_0^L(\mathbf{U}),\quad (\forall \mathbf{U}\in\overline{D(r)}^{n_v},\delta\in\{+,-\}) \tag{4.54}$$

for $S:I\to I$, $Q:I\to\mathbb{R}$ defined in (4.8). It also follows from the definition of $V^{\delta}(\psi)$, (1.11) and (1.12) that

$$-\overline{V^{\delta}(\overline{\mathbf{U}})}(\mathcal{R}\psi)+\beta\overline{V_0^L(\overline{\mathbf{U}})}=-V^{-\delta}(\mathbf{U})(\psi)+\beta V_0^L(\mathbf{U}),\quad (\forall \mathbf{U}\in\overline{D(r)}^{n_v},\delta\in\{+,-\}) \tag{4.55}$$

for $S : I \to I$, $Q : I \to \mathbb{R}$ defined in (4.9).

Next let us confirm some invariances involving the covariances $C_{>0}^{+}$, $C_{>0}^{-}$. Let $S : I \to I$, $Q : I \to \mathbb{R}$ be one of those defined in (4.3), (4.4), (4.5), (4.6), (4.7). It is the same procedure as in the proof of Lemma 4.6 (1) to prove that

$$e^{iQ_2(S_2(\mathbf{X}))}\widetilde{C_{>0}^{\delta}}(S_2(\mathbf{X})) = \widetilde{C_{>0}^{\delta}}(\mathbf{X}), \quad (\forall \mathbf{X} \in I^2, \delta \in \{+,-\}).$$

Moreover, we can see that for any $\mathbf{X} \in I^m$, $\delta \in \{+,-\}$,

$$\begin{aligned}
(4.56) \qquad \int \psi_{\mathbf{X}} d\mu_{C_{>0}^{\delta}}(\psi) &= e^{-\sum_{\mathbf{Y}\in I^2} \widetilde{C_{>0}^{\delta}}(\mathbf{Y})\frac{\partial}{\partial \psi_{\mathbf{Y}}}} \psi_{\mathbf{X}}\Big|_{\psi=0} \\
&= e^{-\sum_{\mathbf{Y}\in I^2} e^{-iQ_2(S_2(\mathbf{Y}))}\widetilde{C_{>0}^{\delta}}(\mathbf{Y})\frac{\partial}{\partial \psi_{S_2(\mathbf{Y})}}} \\
&\quad \cdot e^{iQ_m(S_m(\mathbf{X}))}\psi_{S_m(\mathbf{X})}\Big|_{\psi=0} \\
&= e^{-\sum_{\mathbf{Y}\in I^2} \widetilde{C_{>0}^{\delta}}(\mathbf{Y})\frac{\partial}{\partial \psi_{\mathbf{Y}}}} (\mathcal{R}\psi)_{\mathbf{X}}\Big|_{\psi=0} \\
&= \int (\mathcal{R}\psi)_{\mathbf{X}} d\mu_{C_{>0}^{\delta}}(\psi).
\end{aligned}$$

Let $S : I \to I$, $Q : I \to \mathbb{R}$ be defined in (4.8). By repeating the argument parallel to the proof of [19, Lemma 7.13 (4)] we obtain that

$$e^{-iQ_2(S_2(\mathbf{X}))}\overline{\widetilde{C_{>0}^{\delta}}(S_2(\mathbf{X}))} = \widetilde{C_{>0}^{\delta}}(\mathbf{X}), \quad (\forall \mathbf{X} \in I^2, \delta \in \{+,-\}).$$

Moreover, for any $\mathbf{X} \in I^m$, $\delta \in \{+,-\}$,

$$\begin{aligned}
\int \psi_{\mathbf{X}} d\mu_{C_{>0}^{\delta}}(\psi) &= e^{-\sum_{\mathbf{Y}\in I^2} e^{iQ_2(S_2(\mathbf{Y}))}\widetilde{C_{>0}^{\delta}}(\mathbf{Y})\frac{\partial}{\partial \psi_{S_2(\mathbf{Y})}}} e^{-iQ_m(S_m(\mathbf{X}))}\psi_{S_m(\mathbf{X})}\Big|_{\psi=0} \\
&= e^{-\sum_{\mathbf{Y}\in I^2} \overline{\widetilde{C_{>0}^{\delta}}(\mathbf{Y})}\frac{\partial}{\partial \psi_{\mathbf{Y}}}} e^{-iQ_m(S_m(\mathbf{X}))}\psi_{S_m(\mathbf{X})}\Big|_{\psi=0} \\
&= \int e^{-iQ_m(S_m(\mathbf{X}))}\psi_{S_m(\mathbf{X})} d\mu_{\overline{C_{>0}^{\delta}}}(\psi),
\end{aligned}$$

or

$$(4.57) \qquad \int \psi_{\mathbf{X}} d\mu_{\overline{C_{>0}^{\delta}}}(\psi) = \int (\mathcal{R}\psi)_{\mathbf{X}} d\mu_{C_{>0}^{\delta}}(\psi).$$

For S, Q defined in (4.9) and $(\rho,\mathbf{x},\sigma,x,\theta)$, $(\eta,\mathbf{y},\tau,y,\xi)\in I$,

$$\begin{aligned}(4.58)\quad & e^{-iQ_2(S_2(\rho\mathbf{x}\sigma x\theta,\eta\mathbf{y}\tau y\xi))}\overline{\widetilde{C_{>0}^{\delta}}(S_2(\rho\mathbf{x}\sigma x\theta,\eta\mathbf{y}\tau y\xi))}\\ &= -\frac{1}{2}e^{i\langle b(\rho),\boldsymbol{\pi}\rangle+i\langle b(\eta),\boldsymbol{\pi}\rangle}\\ &\quad\cdot(1_{(\theta,\xi)=(1,-1)}\overline{C_{>0}^{\delta}(\eta\mathbf{y}\tau y,\rho\mathbf{x}\sigma x)}-1_{(\theta,\xi)=(-1,1)}\overline{C_{>0}^{\delta}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)}).\end{aligned}$$

Moreover, by Lemma 2.3 (1),

$$\begin{aligned}& e^{i\langle b(\rho),\boldsymbol{\pi}\rangle+i\langle b(\eta),\boldsymbol{\pi}\rangle}\overline{C_{>0}^{+}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)}\\ &= \frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*}e^{i\langle\mathbf{y}-\mathbf{x},\mathbf{k}\rangle+i(y-x)\omega}(1-\chi_{h,0}(\omega))\\ &\quad\cdot e^{i\langle b(\rho),\boldsymbol{\pi}\rangle+i\langle b(\eta),\boldsymbol{\pi}\rangle}h^{-1}(I_{2^d}-e^{i\frac{\omega}{h}I_{2^d}+\mathcal{E}(\mathbf{k})})^{-1}(\eta,\rho)\\ &= \frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*}e^{i\langle\mathbf{y}-\mathbf{x},\mathbf{k}\rangle+i(y-x)\omega}(1-\chi_{h,0}(\omega))h^{-1}\\ &\quad\cdot U_d(\boldsymbol{\pi})(I_{2^d}-e^{i\frac{\omega}{h}I_{2^d}+\mathcal{E}(\mathbf{k})})^{-1}U_d(\boldsymbol{\pi})^*(\eta,\rho)\\ &= -C_{>0}^{-}(\eta\mathbf{y}\tau y,\rho\mathbf{x}\sigma x).\end{aligned}$$

This implies that for $\delta\in\{+,-\}$,

$$e^{i\langle b(\rho),\boldsymbol{\pi}\rangle+i\langle b(\eta),\boldsymbol{\pi}\rangle}\overline{C_{>0}^{\delta}(\rho\mathbf{x}\sigma x,\eta\mathbf{y}\tau y)}=-C_{>0}^{-\delta}(\eta\mathbf{y}\tau y,\rho\mathbf{x}\sigma x).$$

By substituting this equality into (4.58) we obtain

$$e^{-iQ_2(S_2(\mathbf{X}))}\overline{\widetilde{C_{>0}^{\delta}}(S_2(\mathbf{X}))}=\widetilde{C_{>0}^{-\delta}}(\mathbf{X}),\quad(\forall\mathbf{X}\in I^2,\delta\in\{+,-\}).$$

Furthermore, based on this equality, transformations parallel to the derivation of (4.57) yield

$$(4.59)\quad \int\psi_{\mathbf{X}}d\mu_{\overline{C_{>0}^{\delta}}}(\psi)=\int(\mathcal{R}\psi)_{\mathbf{X}}d\mu_{C_{>0}^{-\delta}}(\psi),\quad(\forall\mathbf{X}\in I^m,\delta\in\{+,-\}).$$

Fix $\mathbf{U}\in\overline{D(r(\beta,L))}^{n_v}$. Let S, Q be one of those defined in (4.3), (4.4), (4.5), (4.6), (4.7). By (4.52), (4.53) and (4.56),

$$J^0(\mathbf{U})(\mathcal{R}\psi)=\frac{1}{2}\sum_{\delta\in\{+,-\}}\log\left(\int e^{-V^{\delta}(\mathbf{U})(\mathcal{R}\psi+\mathcal{R}\psi^1)+\beta V_0^L(\mathbf{U})}d\mu_{C_{>0}^{\delta}}(\psi^1)\right)$$

$$= J^0(\mathbf{U})(\psi).$$

Let S, Q be defined by (4.8). By (4.52), (4.54) and (4.57),

$$\begin{aligned}\overline{J^0(\overline{\mathbf{U}})}(\mathcal{R}\psi) &= \frac{1}{2}\sum_{\delta\in\{+,-\}}\log\left(\int e^{-\overline{V^\delta(\overline{\mathbf{U}})}(\mathcal{R}\psi+\mathcal{R}\psi^1)+\beta\overline{V_0^L(\overline{\mathbf{U}})}}d\mu_{C^\delta_{>0}}(\psi^1)\right)\\ &= J^0(\mathbf{U})(\psi).\end{aligned}$$

Finally, let S, Q be defined by (4.9). By (4.52), (4.55) and (4.59),

$$\begin{aligned}\overline{J^0(\overline{\mathbf{U}})}(\mathcal{R}\psi) &= \frac{1}{2}\sum_{\delta\in\{+,-\}}\log\left(\int e^{-\overline{V^\delta(\overline{\mathbf{U}})}(\mathcal{R}\psi+\mathcal{R}\psi^1)+\beta\overline{V_0^L(\overline{\mathbf{U}})}}d\mu_{C^{-\delta}_{>0}}(\psi^1)\right)\\ &= J^0(\mathbf{U})(\psi).\end{aligned}$$

Since $\mathbf{U}\mapsto J^0(\mathbf{U})(\psi)$, $\mathbf{U}\mapsto J^0(\mathbf{U})(\mathcal{R}\psi)$, $\mathbf{U}\mapsto \overline{J^0(\overline{\mathbf{U}})}(\mathcal{R}\psi)$ are continuous in $\overline{D(r)}^{n_v}$ and analytic in $D(r)^{n_v}$, the identity theorem and the continuity ensure the claimed invariances for any $\mathbf{U}\in\overline{D(r)}^{n_v}$. $\square$

Note that we can choose a constant $c(d)\in\mathbb{R}_{>0}$ depending only on d, a constant $c(d,c_\chi,N_v)\in\mathbb{R}_{>0}$ depending only on d, c_χ, N_v so that if

$$h\ge e^{4d},\quad L\ge\beta,\quad M\ge c(d,c_\chi,N_v),\quad \alpha\ge c(d)M^{d+\frac{3}{2}},$$

all the assumptions imposed on h, L, β, α, M in Lemma 4.4, Lemma 4.5, Lemma 4.6, Lemma 4.9, Lemma 4.10 are satisfied. Then, there exists a constant $c(d,M,c_w,c_\chi)\in\mathbb{R}_{>0}$ depending only on d, M, c_w, c_χ such that the conclusions of Lemma 4.4, Lemma 4.6, Lemma 4.10 hold for $c_0 := c(d,M,c_w,c_\chi)f_{\mathbf{t}}^{-\frac{d}{2}}$. With this c_0, set

$$r_{max} := \frac{1}{2}\left(e^{d\mathrm{w}(0)^{1/2}}\sum_{m=1}^{N_v}c_0^m\alpha^{2m}v_m(\mathrm{w}(0))\right)^{-1}.$$

To make clear, let us sum up these assumptions. From now we assume that

$$\begin{aligned}(4.60)\qquad & h\ge e^{4d},\quad L\ge\beta,\quad M\ge c(d,c_\chi,N_v),\quad \alpha\ge c(d)M^{d+\frac{3}{2}},\\ & c_0 = c(d,M,c_w,c_\chi)f_{\mathbf{t}}^{-\frac{d}{2}},\end{aligned}$$

$$r_{max} = \frac{1}{2}\left(e^{d\mathrm{w}(0)^{1/2}}\sum_{m=1}^{N_v} c_0^m\alpha^{2m}v_m(\mathrm{w}(0))\right)^{-1}.$$

Recall that in Lemma 2.12 we derived the Grassmann integral formulation

$$\log\left(\int e^{\frac{1}{2}(R^+(\psi)+R^-(\psi))}d\mu_{C^\infty_{\le 0}}(\psi)\right)$$

assuming that the coupling constants are sufficiently small, where

$$R^\delta(\psi) = \log\left(\int e^{-V^\delta(\psi+\psi^1)}d\mu_{C^\delta_{>0}}(\psi^1)\right)\quad (\delta = +,-).$$

Here we consider the cut-off functions $\chi(h|1-e^{i\omega/h}|)$, $\chi(|\omega|)$ inside $C^\delta_{>0}$, $C^\infty_{\le 0}$ as $\phi(M_{UV}^{-2}h^2|1-e^{i\omega/h}|^2)$, $\phi(M_{UV}^{-2}\omega^2)$ respectively. The next lemma shows how we can analytically continue this Grassmann integral formulation by means of the iterative infrared multi-scale integration or the renormalization group method.

LEMMA 4.11. *The following statements hold true.*

(1) *There exist*

$$\begin{aligned}&J_0^l \in \mathcal{S}(D(r_{max})^{n_v}, c_0, \alpha, M)(l)\cap C(\overline{D(r_{max})}^{n_v};\mathbb{C})\\ &\quad (l = 0,-1,\cdots,N_\beta-1),\\ &E_l \in \mathcal{K}(D(r_{max})^{n_v},\alpha,M)(l)\quad (l = 0,-1,\cdots,N_\beta)\end{aligned}$$

and $r(\beta,L)\in\mathbb{R}_{>0}$ depending on β, L, independent of h such that

$$\begin{aligned}&E_l(\mathbf{U})(\omega,\mathbf{k}) - E_{l+1}(\mathbf{U})(\omega,\mathbf{k}) = O,\\ &(\forall l\in\{0,-1,\cdots,N_\beta\}, \mathbf{U}\in\overline{D(r_{max})}^{n_v},\\ &\quad (\omega,\mathbf{k})\in\mathbb{R}^{d+1}\ \textit{with}\ \hat{\chi}_{\le l}(\omega,\mathbf{k}) = 0),\end{aligned}\tag{4.61}$$

$$\begin{aligned}&-\frac{1}{\beta L^d}\log\left(\int e^{\frac{1}{2}(R^+(\mathbf{U})(\psi)+R^-(\mathbf{U})(\psi))}d\mu_{C^\infty_{\le 0}}(\psi)\right)\\ &= -\frac{1}{\beta L^d}\sum_{l=0}^{N_\beta-1}J_0^l(\mathbf{U})\end{aligned}\tag{4.62}$$

$$
\begin{aligned}
&-\sum_{l=0}^{N_\beta}\frac{2}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}\times\Gamma(L)^*}\\
&\quad\cdot\log(\det(I_{2^d}-(i\omega I_{2^d}-\mathcal{E}(\mathbf{k})-E_{l+1}(\mathbf{U})(\omega,\mathbf{k}))^{-1}\\
&\qquad\qquad\cdot(E_l(\mathbf{U})(\omega,\mathbf{k})-E_{l+1}(\mathbf{U})(\omega,\mathbf{k}))))\\
&+\frac{1}{L^d}V_0^L(\mathbf{U}),\quad(\forall\mathbf{U}\in\overline{D(r(\beta,L))}^{n_v}),
\end{aligned}
$$

where we set $E_1 := 0$.

(2) *In addition, assume* (3.3). *Then,* $J_0^l(\beta_a)$, $E_l(\beta_a)$ *introduced in* (1) *for* $a = 1, 2$ *satisfy that*

$$
\begin{aligned}
&(J_0^l(\beta_1),J_0^l(\beta_2))\\
&\in\hat{\mathcal{S}}(D(r_{max})^{n_v},c_0,\alpha,M)(l)\cap(C(\overline{D(r_{max})}^{n_v};\mathbb{C})\times C(\overline{D(r_{max})}^{n_v};\mathbb{C})),\\
&(\forall l\in\{0,-1,\cdots,N_{\beta_1}-1\}),\\
&(E_l(\beta_1),E_l(\beta_2))\in\hat{\mathcal{K}}(D(r_{max})^{n_v},\alpha,M)(l),\quad(\forall l\in\{0,-1,\cdots,N_{\beta_1}\}).
\end{aligned}
$$

Proof. (1): By Lemma 4.10 (1) there exist $J^0 \in \mathcal{S}(D(r_{max})^{n_v}, c_0, \alpha, M)(0)$ and $r(\beta, L) \in \mathbb{R}_{>0}$ depending on β, L, independent of h such that (4.49) holds. Let $l \in \{0, -1, \cdots, N_\beta\}$ and assume that we have $(J^0, J^{-1}, \cdots, J^l) \in \prod_{j=0}^{l} \mathcal{S}(D(r_{max})^{n_v}, c_0, \alpha, M)(j)$. Define $W^j, E_l \in \mathrm{Map}(\overline{D(r_{max})}^{n_v}, \mathrm{Map}(\mathbb{R}^{d+1}, \mathrm{Mat}(2^d, \mathbb{C})))$ $(j = 0, -1, \cdots, l)$ by (4.17), (4.18) respectively. By Lemma 4.4 (1), $E_l \in \mathcal{K}(D(r_{max})^{n_v}, \alpha, M)(l)$. Define $C_l \in \mathrm{Map}(\overline{D(r_{max})}^{n_v}, \mathrm{Map}(I_0^2, \mathbb{C}))$ by (4.34) with E_l in place of G_l. We can apply Lemma 4.6 (1) to conclude that $C_l \in \mathcal{R}(D(r_{max})^{n_v}, c_0, M)(l)$. Define $J^{l-1} \in \mathrm{Map}(\overline{D(r_{max})}^{n_v}, \bigwedge \mathcal{V})$ by

$$
\begin{aligned}
&J^{l-1}(\mathbf{U})(\psi)\\
&:=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d}{dz}\right)^n\Big|_{z=0}\log\left(\int e^{z\sum_{m=4}^{N}J_m^l(\mathbf{U})(\psi+\psi^1)}d\mu_{C_l(\mathbf{U})}(\psi^1)\right).
\end{aligned}
$$

By Lemma 4.9 (1), $J^{l-1} \in \mathcal{S}(D(r_{max})^{n_v}, c_0, \alpha, M)(l-1)$. Thus, we have inductively created $J^l \in \mathcal{S}(D(r_{max})^{n_v}, c_0, \alpha, M)(l)$ $(l = 0, -1, \cdots, N_\beta - 1)$, $E_l \in \mathcal{K}(D(r_{max})^{n_v}, \alpha, M)(l)$ $(l = 0, -1, \cdots, N_\beta)$. It is clear from the definition (4.18) that E_l satisfies (4.61). Note that by (4.49) and taking $r(\beta, L)$ smaller independently of h if necessary the left-hand side of (4.62) is equal to

$$-\frac{1}{\beta L^d}\log\left(\int e^{J^0(\mathbf{U})(\psi)}d\mu_{C^{\infty}_{\le 0}}(\psi)\right)+\frac{1}{L^d}V_0^L(\mathbf{U})$$

for any $\mathbf{U}\in\overline{D(r(\beta,L))}^{n_v}$. Then, we can expand the first term in the same way as in the proof of [19, Lemma 7.18 (3)] by taking $r(\beta, L)$ smaller if necessary again and obtain the equality (4.62).

(2): By Lemma 4.10 (2), $(J^0(\beta_1), J^0(\beta_2)) \in \hat{\mathcal{S}}(D(r_{max})^{n_v}, c_0, \alpha, M)(0)$. Assume that $l \in \{0, -1, \cdots, N_{\beta_1}\}$ and $(J^j(\beta_1), J^j(\beta_2)) \in \hat{\mathcal{S}}(D(r_{max})^{n_v}, c_0, \alpha, M)(j)$ for all $j \in \{0, -1, \cdots, l\}$. By Lemma 4.4 (2), $(E_l(\beta_1), E_l(\beta_2)) \in \hat{\mathcal{K}}(D(r_{max})^{n_v}, \alpha, M)(l)$. Then, by Lemma 4.6 (2), $(C_l(\beta_1), C_l(\beta_2)) \in \hat{\mathcal{R}}(D(r_{max})^{n_v}, c_0, M)(l)$. Then, by Lemma 4.9 (2) $(J^{l-1}(\beta_1), J^{l-1}(\beta_2)) \in \hat{\mathcal{S}}(D(r_{max})^{n_v}, c_0, \alpha, M)(l-1)$. The induction with l ensures that the result holds true. □

Remark 4.12. In fact the derivation of (4.62) well describes how to update the covariance and integrate the Grassmann polynomial by using the updated covariance at every step of the IR integration. Despite its conceptional importance, here we only refer to the corresponding part of [19, Lemma 7.18 (3)] without reproducing it, since this paper is intended to be a continuation of [19]. However, we should remark that we need to use the relation

$$\begin{aligned}J_2^l(\mathbf{U})(\psi) = \frac{1}{h^2}&\sum_{(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in I_0}\frac{\delta_{\sigma,\tau}}{\beta L^d}\sum_{(\omega,\mathbf{k})\in\mathcal{M}_h\times\Gamma(L)^*}e^{i\langle\mathbf{k},\mathbf{x}-\mathbf{y}\rangle+i\omega(x-y)}\\&\cdot W^l(\mathbf{U})(\omega,\mathbf{k})(\rho,\eta)\psi_{\rho\mathbf{x}\sigma x}\overline{\psi}_{\eta\mathbf{y}\tau y},\\(\mathbf{U}\in\overline{D(r_{max})}^{n_v})&\end{aligned}$$

to update the covariance by substituting W^l. To derive this equality, we use the invariance $J_2^l(\mathbf{U})(\psi) = J_2^l(\mathbf{U})(\mathcal{R}\psi)$ with $S: I \to I$, $Q: I \to \mathbb{R}$ defined in (4.3), (4.4), (4.5), (4.6), embodied in $\mathcal{S}(D(r_{max})^{n_v}, c_0, \alpha, M)(l)$. See [19, Lemma 7.6 (1)] for the derivation of the same relation.

Define $J_{end} \in \mathrm{Map}(\overline{D(r_{max})}^{n_v}, \mathbb{C})$ by the right-hand side of (4.62). Analyticity and convergent properties of the free energy density follow from the properties of J_{end}. Let us summarize them in the next two lemmas.

LEMMA 4.13. *There exists a constant $c'(d,M,c_w,c_\chi) \in \mathbb{R}_{>0}$ depending only on d, M, c_w, c_χ such that the following statements hold true.*

(1)

$$J_{end} \in C(\overline{D(r_{max})}^{n_v};\mathbb{C}) \cap C^\omega(D(r_{max})^{n_v};\mathbb{C}).$$

(2)

$$|J_{end}(\mathbf{U})| \le c'(d,M,c_w,c_\chi) f_{\mathbf{t}}^{-\frac{d}{2}} \alpha^{-1} + v_0 r_{max}, \quad (\forall \mathbf{U} \in \overline{D(r_{max})}^{n_v}).$$

(3) *In addition, assume* (3.3). *Then,*

$$|J_{end}(\beta_1)(\mathbf{U}) - J_{end}(\beta_2)(\mathbf{U})| \le c'(d,M,c_w,c_\chi)\beta_1^{-\frac{1}{2}} f_{\mathbf{t}}^{-\frac{d}{2}} \alpha^{-1},$$
$$(\forall \mathbf{U} \in \overline{D(r_{max})}^{n_v}).$$

REMARK 4.14. Since we have fixed the (d,M,c_w,c_χ)-dependent constant $c(d,M,c_w,c_\chi)$ in (4.60), we use the different notation $c'(d,M,c_w,c_\chi)$ to express a positive constant depending only on d,M,c_w,c_χ.

PROOF OF LEMMA 4.13. Since (4.61) holds, Lemma 4.5 (1) implies the claim (1). Moreover, by (4.1) and Lemma 4.5 (2),

$$\begin{aligned}
&|J_{end}(\mathbf{U})| \\
&\le c(d) \sum_{l=0}^{N_\beta - 1} M^{(d+\frac{3}{2})l} \alpha^{-1} + c'(d,M,c_w,c_\chi) \sum_{l=0}^{N_\beta} f_{\mathbf{t}}^{-\frac{d}{2}} M^{(d+1)l} \alpha^{-2} + v_0 r_{max} \\
&\le c'(d,M,c_w,c_\chi) f_{\mathbf{t}}^{-\frac{d}{2}} \alpha^{-1} + v_0 r_{max}, \quad (\forall \mathbf{U} \in \overline{D(r_{max})}^{n_v}).
\end{aligned}$$

Thus, the claim (2) is true.

To prove the claim (3), assume (3.3). By (4.1), (4.10) and Lemma 4.5 (2),(3),

$$\begin{aligned}
&|J_{end}(\beta_1)(\mathbf{U}) - J_{end}(\beta_2)(\mathbf{U})| \\
&\le \left| \frac{1}{\beta_1 L^d} \sum_{l=0}^{N_{\beta_1}-1} J_0^l(\beta_1)(\mathbf{U}) - \frac{1}{\beta_2 L^d} \sum_{l=0}^{N_{\beta_1}-1} J_0^l(\beta_2)(\mathbf{U}) \right|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\beta_2 L^d}\sum_{l=N_{\beta_1}-2}^{N_{\beta_2}}|J_0^l(\beta_2)(\mathbf{U})|\\
&+c'(d,M,c_w,c_\chi)\beta_1^{-\frac{1}{2}}f_{\mathbf{t}}^{-\frac{d}{2}}\sum_{l=0}^{N_{\beta_1}}M^{dl}\alpha^{-2}\\
&+c'(d,M,c_w,c_\chi)f_{\mathbf{t}}^{-\frac{d}{2}}\sum_{l=N_{\beta_1}-1}^{N_{\beta_2}}M^{(d+1)l}\alpha^{-2}\\
\le\ & c'(d,M,c_w,c_\chi)\beta_1^{-\frac{1}{2}}f_{\mathbf{t}}^{-\frac{d}{2}}\alpha^{-1},\quad (\forall \mathbf{U}\in\overline{D(r_{max})}^{n_v}).\ \square
\end{aligned}
$$

Lemma 4.15. *Let K be a non-empty compact set of $\mathbb{C}^{n_v}$ satisfying $K\subset D(r_{max})^{n_v}$. Then, the following statements hold.*

(1) *For any $\beta\in\mathbb{R}_{>0}$, $L\in\mathbb{N}$ with $L\ge\beta$, $J_{end}(\beta,L,h)$ converges in $C(K;\mathbb{C})$ as $h\to\infty(h\in(2/\beta)\mathbb{N})$.*

(2) *Set $J(\beta,L):=\lim_{h\to\infty,h\in(2/\beta)\mathbb{N}}J_{end}(\beta,L,h)$. Then, for any $\beta\in\mathbb{R}_{>0}$, $J(\beta,L)$ converges in $C(K;\mathbb{C})$ as $L\to\infty(L\in\mathbb{N})$.*

(3) *Set $J(\beta):=\lim_{L\to\infty,L\in\mathbb{N}}J(\beta,L)$. Then, $J(\beta)$ converges in $C(K;\mathbb{C})$ as $\beta\to\infty(\beta\in\mathbb{N})$.*

Proof. Though the proof is parallel to the proof of [19, Lemma 7.20], we present it for completeness. Take $r_0\in(0,r_{max})$ and $\varepsilon\in(0,r_0)$. Since $J_{end}\in C^\omega(D(r_{max})^{n_v};\mathbb{C})$ by Lemma 4.13 (1), we have that for any $\mathbf{U}\in\overline{D(r_0-\varepsilon)}^{n_v}$,

$$
J_{end}(\beta,L,h)(\mathbf{U})=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n J_{end}(\beta,L,h)(z\mathbf{U})\Big|_{z=0}. \tag{4.63}
$$

By Lemma 4.13 (2),

$$
\sup_{\mathbf{U}\in\overline{D(r_0-\varepsilon)}^{n_v}}\left|\frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n J_{end}(\beta,L,h)(z\mathbf{U})\Big|_{z=0}\right| \tag{4.64}
$$

$$
\begin{aligned}
&= \sup_{\mathbf{U}\in\overline{D(r_0-\varepsilon)}^{n_v}} \left| \frac{1}{2\pi i}\oint_{|z|=r_0/(r_0-\varepsilon)} dz \frac{J_{end}(\beta,L,h)(z\mathbf{U})}{z^{n+1}} \right| \\
&\le \left(\frac{r_0-\varepsilon}{r_0}\right)^n (c'(d,M,c_w,c_\chi) f_{\mathbf{t}}^{-\frac{d}{2}}\alpha^{-1} + v_0 r_{max}), \\
&(\forall n \in \mathbb{N}\cup\{0\}).
\end{aligned}
$$

By Lemma 2.12 (1) and Lemma 4.11 (1) there exist h-independent constants h_0, $c_1 \in \mathbb{R}_{>0}$ such that for any $h \in (2/\beta)\mathbb{N}$ with $h \ge h_0$ and $\mathbf{U} \in \overline{D(r_0-\varepsilon)}^{n_v}$,

$$
\begin{aligned}
(4.65)\qquad & \frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n J_{end}(\beta,L,h)(z\mathbf{U})\Big|_{z=0} \\
&= \frac{1}{2\pi i}\oint_{|z|=c_1} dz \frac{1}{z^{n+1}} \\
&\quad \cdot \Bigg(-\frac{1}{\beta L^d}\log\left(\int e^{\frac{1}{2}(R^+(z\mathbf{U})(\psi)+R^-(z\mathbf{U})(\psi))} d\mu_{C_{\le 0}^{\infty}}(\psi)\right) \\
&\qquad + \frac{1}{\beta L^d}\log\left(\int e^{-V(z\mathbf{U})(\psi)} d\mu_C(\psi)\right)\Bigg) \\
&\quad - \frac{1}{\beta L^d n!}\left(\frac{\partial}{\partial z}\right)^n \log\left(\int e^{-zV(\mathbf{U})(\psi)} d\mu_C(\psi)\right)\Bigg|_{z=0}.
\end{aligned}
$$

Here we used that $V(\mathbf{U})(\psi)$ is linear with $\mathbf{U}$. By Lemma 2.12 (2) the first term of the right-hand side of (4.65) uniformly converges to 0 with respect to $\mathbf{U} \in \overline{D(r_0-\varepsilon)}^{n_v}$ as $h \to \infty$. By Lemma C.3 proved in Appendix C the second term of the right-hand side of (4.65) uniformly converges with $\mathbf{U} \in \overline{D(r_0-\varepsilon)}^{n_v}$ as we send $h \to \infty$ and then send $L \to \infty$. Therefore,

$$
\begin{aligned}
&\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} \frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n J_{end}(\beta,L,h)(z\cdot)\Big|_{z=0}, \\
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} \frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n J_{end}(\beta,L,h)(z\cdot)\Big|_{z=0}
\end{aligned}
$$

converge in $C(\overline{D(r_0-\varepsilon)}^{n_v};\mathbb{C})$. Since the right-hand side of (4.64) is summable over $n \in \mathbb{N}\cup\{0\}$, we can apply the dominated convergence theorem in $l^1(\mathbb{N}\cup\{0\}; C(\overline{D(r_0-\varepsilon)}^{n_v};\mathbb{C}))$ to the expansion (4.63) and conclude that

$$\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} J_{end}(\beta,L,h),\quad \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} J_{end}(\beta,L,h)$$

converge in $C(\overline{D(r_0-\varepsilon)}^{n_v};\mathbb{C})$. Set

$$J(\beta):=\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} J_{end}(\beta,L,h).$$

By taking the limits in the inequality obtained in Lemma 4.13 (3) we see that $(J(\beta))_{\beta\in\mathbb{N}}$ is a Cauchy sequence in $C(\overline{D(r_0-\varepsilon)}^{n_v};\mathbb{C})$. Thus, $\lim_{\beta\to\infty,\beta\in\mathbb{N}} J(\beta)$ converges in this Banach space. For any compact set K of $\mathbb{C}^{n_v}$ satisfying $K\subset D(r_{max})^{n_v}$ we can choose $r_0\in(0,r_{max})$ and $\varepsilon\in(0,r_0)$ so that $K\subset\overline{D(r_0-\varepsilon)}^{n_v}$. Thus, the claimed convergence results in $C(K;\mathbb{C})$ follow from the above arguments. □

Before proceeding to the proof of Theorem 1.6 we state a couple of necessary lemmas, which are close to [19, Lemma E.2, Lemma E.3]. In the proofs of these lemmas $\|\cdot\|_{\mathfrak{B}(F_f)}$ denotes the operator norm for linear operators on $F_f(L^2(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}))$.

LEMMA 4.16. *For any* $\beta\in\mathbb{R}_{\ge 1}$,

$$\begin{aligned}&\left|\frac{1}{\beta L^d}\log\left(\frac{\mathrm{Tr}\, e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H_0}}\right)-\frac{1}{\lfloor\beta\rfloor L^d}\log\left(\frac{\mathrm{Tr}\, e^{-\lfloor\beta\rfloor H}}{\mathrm{Tr}\, e^{-\lfloor\beta\rfloor H_0}}\right)\right|\\
&\le\int_{\lfloor\beta\rfloor}^{\beta}d\gamma\frac{1}{\gamma^2L^d}\left|\log\left(\frac{\mathrm{Tr}\, e^{-\gamma H}}{\mathrm{Tr}\, e^{-\gamma H_0}}\right)\right|\\
&\quad+\Bigg(v_0\max_{l\in\{1,2,\cdots,n_v\}}|U_l|+2^{d+1}\sum_{j=1}^{N_v}v_j(0)\max_{l\in\{1,2,\cdots,n_v\}}|U_l|\\
&\qquad+2^{d+2}\sup_{\mathbf{k}\in\mathbb{R}^d}\|E(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{k})\|_{2^d\times 2^d}\Bigg)\log\left(\frac{\beta}{\lfloor\beta\rfloor}\right).\end{aligned}$$

PROOF. Since $\|\psi_X^*\|_{\mathfrak{B}(F_f)}=\|\psi_X\|_{\mathfrak{B}(F_f)}=1$ $(\forall X\in\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\})$,

$$\|V\|_{\mathfrak{B}(F_f)}\le L^dv_0\max_{l\in\{1,2,\cdots,n_v\}}|U_l|+2^{d+1}L^d\sum_{j=1}^{N_v}v_j(0)\max_{l\in\{1,2,\cdots,n_v\}}|U_l|.$$

By using this inequality in place of the inequality "(E.3)" and straightforwardly following the proof of [19, Lemma E.2], we can derive the claimed inequality. □

We may consider $\mathbf{U}$ inside the operator H as complex variables.

LEMMA 4.17. *For any $r \in \mathbb{R}_{>0}$ there exists a domain O of $\mathbb{C}$ such that $(-r, r) \subset O$ and $\log(\mathrm{Tr}\, e^{-\beta H})$ is analytic with respect to $\mathbf{U}$ in O^{n_v}.*

PROOF. Take any $r \in \mathbb{R}_{>0}$, $\mathbf{a} \in [-1, 1]^{n_v}$, $\mathbf{U} \in [-r, r]^{n_v}$ and $\delta \in [0, 1]$. Note that

$$\begin{aligned}
&|\mathrm{Tr}\, e^{-\beta(H_0+V(\mathbf{U}+i\delta\mathbf{a}))} - \mathrm{Tr}\, e^{-\beta(H_0+V(\mathbf{U}))}| \\
&\le \int_0^\delta d\varepsilon \left| \frac{d}{d\varepsilon} \mathrm{Tr}\, e^{-\beta(H_0+V(\mathbf{U})+i\varepsilon V(\mathbf{a}))} \right| \\
&\le \delta\beta 2^{2^{d+1}L^d} \|V(\mathbf{a})\|_{\mathfrak{B}(F_f)} e^{\beta(\|H_0\|_{\mathfrak{B}(F_f)}+\|V(\mathbf{U})\|_{\mathfrak{B}(F_f)}+\|V(\mathbf{a})\|_{\mathfrak{B}(F_f)})},
\end{aligned}$$

where we used the linearity of V with respect to the coupling constants. Thus,

$$\begin{aligned}
&\mathrm{Re}\,\mathrm{Tr}\, e^{-\beta(H_0+V(\mathbf{U}+i\delta\mathbf{a}))} \\
&\ge \mathrm{Tr}\, e^{-\beta(H_0+V(\mathbf{U}))} \\
&\quad - \delta\beta 2^{2^{d+1}L^d} \|V(\mathbf{a})\|_{\mathfrak{B}(F_f)} e^{\beta(\|H_0\|_{\mathfrak{B}(F_f)}+\|V(\mathbf{U})\|_{\mathfrak{B}(F_f)}+\|V(\mathbf{a})\|_{\mathfrak{B}(F_f)})} \\
&\ge e^{-\beta L^d r v_0} \\
&\quad - \delta\beta 2^{2^{d+1}L^d} \sup_{\substack{\mathbf{z}\in[-r,r]^{n_v} \\ \mathbf{b}\in[-1,1]^{n_v}}} \left(\|V(\mathbf{b})\|_{\mathfrak{B}(F_f)} e^{\beta(\|H_0\|_{\mathfrak{B}(F_f)}+\|V(\mathbf{z})\|_{\mathfrak{B}(F_f)}+\|V(\mathbf{b})\|_{\mathfrak{B}(F_f)})} \right) \\
&> 0, \quad (\forall \mathbf{U} \in [-r, r]^{n_v}, \mathbf{a} \in [-1, 1]^{n_v}),
\end{aligned}$$

if δ is sufficiently small. Therefore, there exists $\delta \in \mathbb{R}_{>0}$ such that $\log(\mathrm{Tr}\, e^{-\beta H})$ is analytic with respect to $\mathbf{U}$ in the domain $\{x + iy \mid x \in (-r, r), y \in (-\delta, \delta)\}^{n_v}$. □

Here we can give the proof of Theorem 1.6.

PROOF OF THEOREM 1.6. Assume that the condition (4.60) holds. By Lemma 4.15 there exist

$$J(\beta, L), J(\beta), J \in C(\overline{D(r_{max}/2)}^{n_v}; \mathbb{C}) \cap C^\omega(D(r_{max}/2)^{n_v}; \mathbb{C})$$

such that

$$\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} J_{end}(\beta,L,h)=J(\beta,L),\quad (\forall\beta\in\mathbb{R}_{>0},L\in\mathbb{N}\text{ with }L\ge\beta), \tag{4.66}$$

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} J(\beta,L)=J(\beta),\quad (\forall\beta\in\mathbb{R}_{>0}), \tag{4.67}$$

$$\lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{N}}} J(\beta)=J \tag{4.68}$$

in $C(\overline{D(r_{max}/2)}^{n_v};\mathbb{C})$. By Lemma 2.7 (2), Lemma 2.12 (2), Lemma 4.11 (1) and (4.66) there exists a constant $c_1\in\mathbb{R}_{>0}$ independent of h such that for any $\mathbf{U}\in\overline{D(c_1)}^{n_v}\cap\mathbb{R}^{n_v}$,

$$\begin{aligned}
&J(\beta,L)(\mathbf{U}) \\
&=\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}}\Bigg(-\frac{1}{\beta L^d}\log\left(\int e^{\frac{1}{2}(R^+(\mathbf{U})(\psi)+R^-(\mathbf{U})(\psi))}d\mu_{C^\infty_{\le 0}}(\psi)\right)\\
&\qquad\qquad+\frac{1}{\beta L^d}\log\left(\int e^{-V(\mathbf{U})(\psi)}d\mu_C(\psi)\right)\Bigg)\\
&\quad-\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}}\frac{1}{\beta L^d}\log\left(\int e^{-V(\mathbf{U})(\psi)}d\mu_C(\psi)\right)\\
&=-\frac{1}{\beta L^d}\log\left(\frac{\operatorname{Tr}e^{-\beta H}}{\operatorname{Tr}e^{-\beta H_0}}\right).
\end{aligned} \tag{4.69}$$

By Lemma 4.17 there exists a domain $O\subset\mathbb{C}^{n_v}$ such that $\overline{D(r_{max}/2)}^{n_v}\cap\mathbb{R}^{n_v}\subset O$ and the right-hand side of (4.69) is analytic with $\mathbf{U}$ in O. Thus, by the identity theorem and continuity the equality (4.69) holds for any $\mathbf{U}\in\overline{D(r_{max}/2)}^{n_v}\cap\mathbb{R}^{n_v}$, $\beta\in\mathbb{R}_{>0}$, $L\in\mathbb{N}$ with $L\ge\beta$. Then, by Lemma 2.3 (3), Lemma 4.13 (2) and Lemma 4.16,

$$\begin{aligned}
&\left|\frac{1}{\beta L^d}\log\left(\frac{\operatorname{Tr}e^{-\beta H}}{\operatorname{Tr}e^{-\beta H_0}}\right)-\frac{1}{\lfloor\beta\rfloor L^d}\log\left(\frac{\operatorname{Tr}e^{-\lfloor\beta\rfloor H}}{\operatorname{Tr}e^{-\lfloor\beta\rfloor H_0}}\right)\right|\\
&\le\int_{\lfloor\beta\rfloor}^{\beta}d\gamma\frac{1}{\gamma}\left(c'(d,M,c_w,c_\chi)f_{\mathbf{t}}^{-\frac{d}{2}}\alpha^{-1}+\frac{v_0}{2}r_{max}\right)\\
&\quad+\left(\frac{v_0}{2}r_{max}+2^d\sum_{j=1}^{N_v}v_j(0)r_{max}+2^{d+3}d\right)\log\left(\frac{\beta}{\lfloor\beta\rfloor}\right)
\end{aligned} \tag{4.70}$$

$$\le \left(c'(d,M,c_w,c_\chi)f_{\mathbf{t}}^{-\frac{d}{2}}\alpha^{-1}+v_0r_{max}+2^d\sum_{j=1}^{N_v}v_j(0)r_{max}+2^{d+3}d\right)$$
$$\cdot\log\left(\frac{\beta}{\lfloor\beta\rfloor}\right),$$

for any $\mathbf{U}\in\overline{D(r_{max}/2)}^{n_v}\cap\mathbb{R}^{n_v}$, $\beta\in\mathbb{R}_{\ge 1}$, $L\in\mathbb{N}$ with $L\ge\beta$.

Let $\alpha_\rho^L(\mathbf{k})$ $(\rho\in\mathcal{B})$ denote the eigen values of $E(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{k})$ for $\mathbf{k}\in\Gamma(L)^*$. Then, by [19, Lemma E.1],

$$-\frac{1}{\beta(2L)^d}\log(\mathrm{Tr}\,e^{-\beta H_0})=-\frac{2}{\beta(2L)^d}\sum_{\rho\in\mathcal{B}}\sum_{\mathbf{k}\in\Gamma(L)^*}\log(1+e^{-\beta\alpha_\rho^L(\mathbf{k})})$$
$$=-\frac{2}{\beta(2L)^d}\sum_{\mathbf{k}\in\Gamma(L)^*}\log\det(I_{2^d}+e^{-\beta E(\boldsymbol{\varepsilon}^L,\theta)(\mathbf{k})}).$$

We can deduce from the definition that $\lim_{L\to\infty,L\in\mathbb{N}}E(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{k})(\rho,\eta)$ converges for any $\mathbf{k}\in\mathbb{R}^d$, $\rho,\eta\in\mathcal{B}$ and if we set $E(\boldsymbol{\theta})(\mathbf{k}):=\lim_{L\to\infty,L\in\mathbb{N}}E(\boldsymbol{\varepsilon}^L,\boldsymbol{\theta})(\mathbf{k})$, $E(\boldsymbol{\theta})(\mathbf{k})^*=E(\boldsymbol{\theta})(\mathbf{k})$ $(\forall\mathbf{k}\in\mathbb{R}^d)$. Let $\alpha_\rho(\mathbf{k})$ $(\rho\in\mathcal{B})$ be the eigen values of $E(\boldsymbol{\theta})(\mathbf{k})$ for $\mathbf{k}\in\mathbb{R}^d$. Then, by Lemma 2.3 (3), $|\alpha_\rho(\mathbf{k})|\le 2d$ $(\forall\rho\in\mathcal{B},\mathbf{k}\in\mathbb{R}^d)$. By considering these facts we can apply the dominated convergence theorem to prove that

$$\begin{aligned}(4.71)\quad&\lim_{\substack{L\to\infty\\L\in\mathbb{N}}}\left(-\frac{1}{\beta(2L)^d}\log(\mathrm{Tr}\,e^{-\beta H_0})\right)\\&=-\frac{2}{\beta(4\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\log\det(I_{2^d}+e^{-\beta E(\theta)(\mathbf{k})})\\&=-\frac{2}{\beta(4\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\sum_{\rho\in\mathcal{B}}\log(1+e^{-\beta\alpha_\rho(\mathbf{k})}),\\&\lim_{\substack{\beta\to\infty\\\beta\in\mathbb{R}_{>0}}}\lim_{\substack{L\to\infty\\L\in\mathbb{N}}}\left(-\frac{1}{\beta(2L)^d}\log(\mathrm{Tr}\,e^{-\beta H_0})\right)\\&=\frac{2}{(4\pi)^d}\int_{[0,2\pi]^d}d\mathbf{k}\sum_{\rho\in\mathcal{B}}1_{\alpha_\rho(\mathbf{k})<0}\alpha_\rho(\mathbf{k}).\end{aligned}$$

Let us define $F(\beta,L)$, $F(\beta)$, $F\in C(\overline{D(r_{max}/2)}^{n_v};\mathbb{C})\cap C^\omega(D(r_{max}/2)^{n_v};$

$\mathbb{C}$) ($\beta \in \mathbb{R}_{>0}, L \in \mathbb{N}$ with $L \ge \beta$) by

$$\begin{aligned}
&F(\beta, L) := 2^{-d} J(\beta, L) - \frac{1}{\beta(2L)^d} \log(\mathrm{Tr}\, e^{-\beta H_0}),\\
&F(\beta) := \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} F(\beta, L),\\
&F := \lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{N}}} F(\beta).
\end{aligned}$$

By (4.67), (4.68), (4.70) and the fact that (4.69) holds for any $\mathbf{U} \in \overline{D(r_{max}/2)}^{n_v} \cap \mathbb{R}^{n_v}$, $\beta \in \mathbb{R}_{\ge 1}$, $L \in \mathbb{N}$ with $L \ge \beta$ we see that

$$\begin{aligned}
&|F(\beta)(\mathbf{U}) - F(\mathbf{U})|\\
&\le 2^{-d}|J(\beta)(\mathbf{U}) - J(\lfloor\beta\rfloor)(\mathbf{U})| + 2^{-d}|J(\lfloor\beta\rfloor)(\mathbf{U}) - J(\mathbf{U})|\\
&\quad + \left| \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{1}{\beta(2L)^d} \log(\mathrm{Tr}\, e^{-\beta H_0}) - \lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{R}_{>0}}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{1}{\beta(2L)^d} \log(\mathrm{Tr}\, e^{-\beta H_0}) \right|\\
&\le \left(c'(M, d, c_w, c_\chi) 2^{-d} f_{\mathbf{t}}^{-\frac{d}{2}} \alpha^{-1} + 2^{-d} v_0 r_{max} + \sum_{j=1}^{N_v} v_j(0) r_{max} + 2^3 d \right)\\
&\quad \cdot \log\left(\frac{\beta}{\lfloor\beta\rfloor} \right)\\
&\quad + 2^{-d}|J(\lfloor\beta\rfloor)(\mathbf{U}) - J(\mathbf{U})|\\
&\quad + \left| \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{1}{\beta(2L)^d} \log(\mathrm{Tr}\, e^{-\beta H_0}) - \lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{R}_{>0}}} \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \frac{1}{\beta(2L)^d} \log(\mathrm{Tr}\, e^{-\beta H_0}) \right|,\\
&(\forall \mathbf{U} \in \overline{D(r_{max}/2)}^{n_v} \cap \mathbb{R}^{n_v}, \beta \in \mathbb{R}_{\ge 1}).
\end{aligned}$$

Then, (4.68) and (4.71) imply that $\lim_{\beta\to\infty,\beta\in\mathbb{R}_{>0}} F(\beta) = F$ in $C(\overline{D(r_{max}/2)}^{n_v} \cap \mathbb{R}^{n_v}; \mathbb{C})$. By the same basic argument as the final part of [19, Proof of Theorem 1.1, Section 7] we can deduce from the above convergence property that $\lim_{\beta\to\infty,\beta\in\mathbb{R}_{>0}} F(\beta) = F$ in $C(\overline{D(r_{max}/2)}^{n_v}; \mathbb{C})$.

To conclude the proof of the theorem under the assumption (3.1), we may conceal the dependency on the artificial parameters α, M, c_w, c_χ in (4.60). Then, we can read from the conditions (4.60) and $4 f_{\mathbf{t}} \le 1$ that there exists a constant $c(d, N_v) \in \mathbb{R}_{>0}$ depending only on d, N_v such that

$$\left(\sum_{m=1}^{N_v} c(d,N_v)^m v_m(c(d,N_v))\right)^{-1} (4f_{\mathbf{t}})^{\frac{dN_v}{2}} \le \frac{r_{max}}{2}.$$

The left-hand side of the above inequality is equal to R set in Theorem 1.6 if (3.1) holds. By recalling Lemma 2.5 we see that the above arguments have proved the theorem under the assumption (3.1).

Let us show that the theorem in the general case follows from the theorem proved under (3.1). Let us temporarily write $R_{\mathbf{t}}$ in place of R. Set $t_{max} := \max_{j\in\{1,2,\cdots,d\}} t_j$. By the theorem for the Hamiltonian $\frac{1}{t_{max}}\mathsf{H}_0 + \mathsf{V}$, there exist $F(\beta, L)$, $F(\beta)$, $F \in C(\overline{D(R_{\mathbf{t}/t_{max}})}^{n_v};\mathbb{C}) \cap C^{\omega}(D(R_{\mathbf{t}/t_{max}})^{n_v};\mathbb{C})$ such that

$$\begin{aligned}
&F(\beta,L)(\mathbf{U}) = -\frac{1}{\beta(2L)^d}\log(\mathrm{Tr}\, e^{-\beta(\frac{1}{t_{max}}\mathsf{H}_0+\mathsf{V})}),\\
&(\forall \mathbf{U} \in \overline{D(R_{\mathbf{t}/t_{max}})}^{n_v} \cap \mathbb{R}^{n_v}, \beta \in \mathbb{R}_{>0}, L \in \mathbb{N} \text{ with } L \ge \beta),\\
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} F(\beta,L) = F(\beta) \text{ in } C(\overline{D(R_{\mathbf{t}/t_{max}})}^{n_v};\mathbb{C}),\\
&\lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{R}_{>0}}} F(\beta) = F \text{ in } C(\overline{D(R_{\mathbf{t}/t_{max}})}^{n_v};\mathbb{C}).
\end{aligned}$$

Then, by the linearity of $V_m^L(\mathbf{U})(\cdot)$ with $\mathbf{U}$,

$$\begin{aligned}
&F(t_{max}\beta, L)\left(\frac{1}{t_{max}}\mathbf{U}\right) = -\frac{1}{t_{max}\beta(2L)^d}\log(\mathrm{Tr}\, e^{-\beta\mathsf{H}}),\\
&(\forall \mathbf{U} \in \overline{(t_{max}D(R_{\mathbf{t}/t_{max}}))}^{n_v} \cap \mathbb{R}^{n_v}, \beta \in \mathbb{R}_{>0}, L \in \mathbb{N} \text{ with } L \ge t_{max}\beta).
\end{aligned}$$

Since $t_{max}D(R_{\mathbf{t}/t_{max}}) = D(R_{\mathbf{t}})$,

$$t_{max}F(t_{max}\beta, L)\left(\frac{1}{t_{max}}\cdot\right) \in C(\overline{D(R_{\mathbf{t}})}^{n_v};\mathbb{C}) \cap C^{\omega}(D(R_{\mathbf{t}})^{n_v};\mathbb{C})$$

and

$$\begin{aligned}
&\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} t_{max}F(t_{max}\beta, L)\left(\frac{1}{t_{max}}\cdot\right) = t_{max}F(t_{max}\beta)\left(\frac{1}{t_{max}}\cdot\right),\\
&\lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{R}_{>0}}} t_{max}F(t_{max}\beta)\left(\frac{1}{t_{max}}\cdot\right) = t_{max}F\left(\frac{1}{t_{max}}\cdot\right) \text{ in } C(\overline{D(R_{\mathbf{t}})}^{n_v};\mathbb{C}).
\end{aligned}$$

Thus, the theorem has been proved. □

Appendix A. Reordering in a Non-Commutative $\mathbb{C}$-Algebra

Here we prove a lemma which is used in the proof of Lemma 1.1. Though the actual problem involves the Fermionic creation/annihilation operators, we set up the problem in a non-commutative $\mathbb{C}$-algebra for simplicity. Let $n \in \mathbb{N}$. Let A be a $\mathbb{C}$-algebra spanned by products of the elements $a_1, a_2, \cdots, a_n$, $a_1^*, a_2^*, \cdots, a_n^*$ satisfying the relation

$$\begin{aligned}(A.1)\qquad & a_j^* a_k + a_k a_j^* = \delta_{j,k}, \quad a_j a_k + a_k a_j = 0, \\ & a_j^* a_k^* + a_k^* a_j^* = 0. \quad (\forall j, k \in \{1, 2, \cdots, n\}).\end{aligned}$$

Set $S := \{1, 2, \cdots, n\}$. We call a function $f_m : S^m \times S^m \to \mathbb{C}$ bi-anti-symmetric if

$$\begin{aligned}& f_m((x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(m)}), (y_{\tau(1)}, y_{\tau(2)}, \cdots, y_{\tau(m)})) \\ & = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) f_m((x_1, x_2, \cdots, x_m), (y_1, y_2, \cdots, y_m)), \\ & (\forall \sigma, \tau \in \mathbb{S}_m, (x_1, x_2, \cdots, x_m), (y_1, y_2, \cdots, y_m) \in S^m).\end{aligned}$$

For $\mathbf{X} = (x_1, x_2, \cdots, x_m) \in S^m$ let $a_{\mathbf{X}}$, $a_{\mathbf{X}}^*$ denote $a_{x_1} a_{x_2} \cdots a_{x_m}$, $a_{x_1}^* a_{x_2}^* \cdots a_{x_m}^*$ respectively. Moreover, let $\widetilde{\mathbf{X}}$ denote $(x_m, x_{m-1}, \cdots, x_1)$.

LEMMA A.1. *For any bi-anti-symmetric function $f_m : S^m \times S^m \to \mathbb{C}$,*

$$\begin{aligned}& \sum_{\mathbf{X}, \mathbf{Y} \in S^m} f_m(\mathbf{X}, \mathbf{Y}) a_{\mathbf{X}}^* a_{\mathbf{Y}} \\ & = \sum_{l=0}^{m} \sum_{\substack{\mathbf{X}, \mathbf{Y} \in S^{m-l} \\ \mathbf{Z} \in S^l}} (-1)^{m-l} \begin{pmatrix} m \\ l \end{pmatrix}^2 l! f_m((\mathbf{X}, \mathbf{Z}), (\widetilde{\mathbf{Z}}, \mathbf{Y})) a_{\mathbf{Y}} a_{\mathbf{X}}^*.\end{aligned}$$

PROOF. We prove the claim by induction with m. The equality for $m = 1$ follows from the relation (A.1). Assume that the claim is true for some $m \in \mathbb{N}$. Let $f_{m+1} : S^{m+1} \times S^{m+1} \to \mathbb{C}$ be a bi-anti-symmetric function. By the hypothesis of induction,

$$\sum_{\mathbf{X}, \mathbf{Y} \in S^{m+1}} f_{m+1}(\mathbf{X}, \mathbf{Y}) a_{\mathbf{X}}^* a_{\mathbf{Y}}$$

$$= \sum_{x,y\in S}\sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m-l}\\ \mathbf{Z}\in S^l}}(-1)^{m-l}\begin{pmatrix} m\\ l\end{pmatrix}^2 l! f_{m+1}((x,\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y},y))$$
$$\cdot a_x^* a_{\mathbf{Y}} a_{\mathbf{X}}^* a_y.$$

Moreover, by (A.1) and the bi-anti-symmetric property of f_{m+1},

$$\sum_{\mathbf{X},\mathbf{Y}\in S^{m+1}} f_{m+1}(\mathbf{X},\mathbf{Y})a_{\mathbf{X}}^* a_{\mathbf{Y}}$$
$$= \sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l}\\ \mathbf{Z}\in S^l}}(-1)^{m-l}\begin{pmatrix} m\\ l\end{pmatrix}^2 l! f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))$$
$$\cdot a_{x_1}^* a_{y_1}\cdots a_{y_{m-l}} a_{x_2}^*\cdots a_{x_{m-l+1}}^* a_{y_{m-l+1}}$$
$$= \sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m-l}\\ \mathbf{Z}\in S^{l+1}}}\begin{pmatrix} m\\ l\end{pmatrix}^2 l! f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{y_1}\cdots a_{y_{m-l-1}} a_{\mathbf{X}}^* a_{y_{m-l}}$$
$$+ \sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l}\\ \mathbf{Z}\in S^l}}(-1)^{m-l+1}\begin{pmatrix} m\\ l\end{pmatrix}^2 l! f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))$$
$$\cdot a_{y_1} a_{x_1}^* a_{y_2}\cdots a_{y_{m-l}} a_{x_2}^*\cdots a_{x_{m-l+1}}^* a_{y_{m-l+1}}$$
$$= 2\sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m-l}\\ \mathbf{Z}\in S^{l+1}}}\begin{pmatrix} m\\ l\end{pmatrix}^2 l! f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{y_1}\cdots a_{y_{m-l-1}} a_{\mathbf{X}}^* a_{y_{m-l}}$$
$$+ \sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l}\\ \mathbf{Z}\in S^l}}(-1)^{m-l+2}\begin{pmatrix} m\\ l\end{pmatrix}^2 l! f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))$$
$$\cdot a_{y_1} a_{y_2} a_{x_1}^* a_{y_3}\cdots a_{y_{m-l}} a_{x_2}^*\cdots a_{x_{m-l+1}}^* a_{y_{m-l+1}}$$
$$= \sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m-l}\\ \mathbf{Z}\in S^{l+1}}}\begin{pmatrix} m\\ l\end{pmatrix}^2 l!(m-l) f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))$$
$$\cdot a_{y_1}\cdots a_{y_{m-l-1}} a_{\mathbf{X}}^* a_{y_{m-l}}$$

$$
\begin{aligned}
&+\sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l}\\ \mathbf{Z}\in S^{l}}}\left(\begin{array}{c} m\\ l\end{array}\right)^2 l! f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{y_1}\cdots a_{y_{m-l}}a_{\mathbf{X}}^*a_{y_{m-l+1}}\\
&=\sum_{l=0}^{m+1}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l}\\ \mathbf{Z}\in S^{l}}}\left(1_{l\ge 1}\left(\begin{array}{c} m\\ l-1\end{array}\right)^2(l-1)!(m+1-l)+1_{l\le m}\left(\begin{array}{c} m\\ l\end{array}\right)^2 l!\right)\\
&\qquad\cdot f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{y_1}\cdots a_{y_{m-l}}a_{\mathbf{X}}^*a_{y_{m-l+1}}.
\end{aligned}
$$

Set

$$
D(l,m):=1_{l\ge 1}\left(\begin{array}{c} m\\ l-1\end{array}\right)^2(l-1)!(m+1-l)+1_{l\le m}\left(\begin{array}{c} m\\ l\end{array}\right)^2 l!.
$$

Then, by considering that $D(m+1,m)=0$,

$$
\begin{aligned}
&\sum_{\mathbf{X},\mathbf{Y}\in S^{m+1}} f_{m+1}(\mathbf{X},\mathbf{Y})a_{\mathbf{X}}^*a_{\mathbf{Y}}\\
&=\sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l}\\ \mathbf{Z}\in S^{l}}} D(l,m)f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{y_1}\cdots a_{y_{m-l}}a_{\mathbf{X}}^*a_{y_{m-l+1}}\\
&=\sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m-l}\\ \mathbf{Z}\in S^{l+1}}}(-1)^{m-l}D(l,m)f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{\mathbf{Y}}a_{\mathbf{X}}^*\\
&\quad+\sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l}\\ \mathbf{Z}\in S^{l}}}(-1)D(l,m)f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))\\
&\qquad\cdot a_{y_1}\cdots a_{y_{m-l}}a_{x_1}^*\cdots a_{x_{m-l-1}}^*a_{x_{m-l}}^*a_{y_{m-l+1}}a_{x_{m-l+1}}^*\\
&=\sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m-l}\\ \mathbf{Z}\in S^{l+1}}}(-1)^{m-l}(m-l+1)D(l,m)f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{\mathbf{Y}}a_{\mathbf{X}}^*\\
&\quad+\sum_{l=0}^{m}\sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l}\\ \mathbf{Z}\in S^{l}}}(-1)^{m-l+1}D(l,m)f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{\mathbf{Y}}a_{\mathbf{X}}^*
\end{aligned}
$$

$$= \sum_{l=0}^{m+1} \sum_{\substack{\mathbf{X},\mathbf{Y}\in S^{m+1-l} \\ \mathbf{Z}\in S^l}} (1_{l\geq 1}(-1)^{m+1-l}(m+2-l)D(l-1,m) \\ + (-1)^{m+1-l}D(l,m))f_{m+1}((\mathbf{X},\mathbf{Z}),(\widetilde{\mathbf{Z}},\mathbf{Y}))a_{\mathbf{Y}}a_{\mathbf{X}}^*.$$

By calculation we can derive that

$$1_{l\geq 1}(-1)^{m+1-l}(m+2-l)D(l-1,m)+(-1)^{m+1-l}D(l,m) \\ = (-1)^{m+1-l}\left(\begin{array}{c} m+1 \\ l \end{array}\right)^2 l!,$$

which implies the claimed equality for $m+1$. The induction with m concludes the proof. □

Appendix B. The Flux Phase Problem on a Periodic Hyper-Cubic Lattice

In order to deduce Corollary 1.7 from Theorem 1.6, we need to know when the free energy density is minimum in the flux phase problem on the hyper-cubic lattice $\Gamma(2L)$ with the periodic boundary condition. A sufficient condition was essentially proved by Lieb in [20]. It was also claimed by Macris and Nachtergaele in [22]. In [19, Appendix A] we restated Lieb's theorem on a periodic square lattice with supplementary arguments which were not explicit in the letter [20]. In order to assist the readers in deriving Corollary 1.7 from Theorem 1.6, here we restate Lieb's theorem on the flux phase problem on the periodic hyper-cubic lattice with explanations of how to extend the arguments in [19, Appendix A] into the d-dimensional case. Not to confuse the problem, we should make clear the dependency between the original article [20], the preceding section [19, Appendix A] and this section. In this section we admit the basic lemmas [19, Lemma A.2, Lemma A.4] and the contents of the proof of [19, Theorem A.5] which was based on the original key lemma [20, Lemma]. For those who know how to apply the reflection positivity lemma [20, Lemma] well, there is no need to follow the proof of Theorem B.4 below. However, we should remark that Lemma B.3 claimed below itself is necessary to prove not only Theorem B.4 but also Theorem 1.6. In fact Lemma B.3 is referred in the proof of Lemma 2.5 in Section 2.

First let us extend [19, Lemma A.2] into the d-dimensional case. Assume that phases φ_1, $\varphi_2 : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ satisfy (1.1) and

$$\begin{aligned}&\varphi_1(\mathbf{x}+\mathbf{e}_j,\mathbf{x})+\varphi_1(\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j)\\&\quad+\varphi_1(\mathbf{x}+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k)+\varphi_1(\mathbf{x},\mathbf{x}+\mathbf{e}_k)\\&=\varphi_2(\mathbf{x}+\mathbf{e}_j,\mathbf{x})+\varphi_2(\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j)\\&\quad+\varphi_2(\mathbf{x}+\mathbf{e}_k,\mathbf{x}+\mathbf{e}_j+\mathbf{e}_k)+\varphi_2(\mathbf{x},\mathbf{x}+\mathbf{e}_k)\quad(\text{mod }2\pi),\\&\sum_{m=0}^{2L-1}\varphi_1(\mathbf{x}+(m+1)\mathbf{e}_j,\mathbf{x}+m\mathbf{e}_j)=\sum_{m=0}^{2L-1}\varphi_2(\mathbf{x}+(m+1)\mathbf{e}_j,\mathbf{x}+m\mathbf{e}_j)\\&(\text{mod }2\pi),\quad(\forall\mathbf{x}\in\mathbb{Z}^d,j,k\in\{1,2,\cdots,d\}).\end{aligned}$$

For $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$ we simply write $(\varphi_1 - \varphi_2)(\mathbf{x},\mathbf{y})$ in place of $\varphi_1(\mathbf{x},\mathbf{y}) - \varphi_2(\mathbf{x},\mathbf{y})$.

LEMMA B.1. *Assume that $n \ge 2$, $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n \in \Gamma(2L)$ and for any $j \in \{1,2,\cdots,n\}$ there exists $p \in \{1,2,\cdots,d\}$ such that $\mathbf{x}_j - \mathbf{x}_{j+1} = \mathbf{e}_p$ or $-\mathbf{e}_p$ in $(\mathbb{Z}/2L\mathbb{Z})^d$, where $\mathbf{x}_{n+1} := \mathbf{x}_1$. Then,*

$$\sum_{j=1}^{n}(\varphi_1-\varphi_2)(\mathbf{x}_{j+1},\mathbf{x}_j)=0\quad(\textit{mod }2\pi).\tag{B.1}$$

PROOF. It follows from [19, Lemma A.2] that if there are $p,q \in \{1,2,\cdots,d\}$ such that for any $j \in \{1,2,\cdots,n\}$ $\mathbf{x}_j - \mathbf{x}_{j+1}$ is equal to one of $\mathbf{e}_p$, $-\mathbf{e}_p$, $\mathbf{e}_q$, $-\mathbf{e}_q$ in $(\mathbb{Z}/2L\mathbb{Z})^d$, then (B.1) holds. Let us call this property $(\star)$.

As hypothesis of induction, let us assume that there exists $l \in \{1,2,\cdots,d-1\}$ such that if for any $j \in \{1,2,\cdots,n\}$ $\mathbf{x}_j - \mathbf{x}_{j+1}$ is equal to one of $\mathbf{e}_1$, $-\mathbf{e}_1$, $\mathbf{e}_2$, $-\mathbf{e}_2$, $\cdots$, $\mathbf{e}_l$, $-\mathbf{e}_l$ in $(\mathbb{Z}/2L\mathbb{Z})^d$, then (B.1) holds. Let $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n \in \Gamma(2L)$ satisfy that for any $j \in \{1,2,\cdots,n\}$, $\mathbf{x}_j - \mathbf{x}_{j+1}$ is equal to one of $\mathbf{e}_1$, $-\mathbf{e}_1$, $\mathbf{e}_2$, $-\mathbf{e}_2$, $\cdots$, $\mathbf{e}_{l+1}$, $-\mathbf{e}_{l+1}$ in $(\mathbb{Z}/2L\mathbb{Z})^d$. Let us prove (B.1) for $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_n$. If $\mathbf{x}_j - \mathbf{x}_{j+1}$ is equal to $\mathbf{e}_{l+1}$ or $-\mathbf{e}_{l+1}$ in $(\mathbb{Z}/2L\mathbb{Z})^d$ for any $j \in \{1,2,\cdots,n\}$, then (B.1) follows from $(\star)$. Assume that there exist $k_1,k_2,\cdots,k_m \in \{1,2,\cdots,n\}$ such that

$$\begin{aligned}&k_1<k_2<\cdots<k_m,\\&\mathbf{x}_{k_p}-\mathbf{x}_{k_p+1}\neq\mathbf{e}_{l+1},-\mathbf{e}_{l+1}\text{ in }(\mathbb{Z}/2L\mathbb{Z})^d\quad(\forall p\in\{1,2,\cdots,m\}),\end{aligned}$$

$$\mathbf{x}_j - \mathbf{x}_{j+1} = \mathbf{e}_{l+1} \text{ or } -\mathbf{e}_{l+1} \text{ in } (\mathbb{Z}/2L\mathbb{Z})^d$$
$$(\forall j \in \{1,2,\cdots,n\}\backslash\{k_1,k_2,\cdots,k_m\}).$$

If $m = 1$, again (B.1) follows from $(\star)$. Assume that $m \ge 2$. Define the map $P : \Gamma(2L) \to \Gamma(2L)$ by

$$P(\mathbf{x}) := (\mathbf{x}(1),\cdots,\mathbf{x}(l),\mathbf{x}_1(l+1),\mathbf{x}(l+2),\cdots,\mathbf{x}(d)).$$

For any $j \in \{1,2,\cdots,m-1\}$ we can choose $\mathbf{y}_{j,1},\mathbf{y}_{j,2},\cdots,\mathbf{y}_{j,q_j} \in \Gamma(2L)$ so that $\mathbf{y}_{j,1} = \mathbf{x}_{k_j+1}$, $\mathbf{y}_{j,q_j} = P(\mathbf{x}_{k_j+1})$, $\mathbf{y}_{j,p} - \mathbf{y}_{j,p+1} = \mathbf{e}_{l+1}$ or $-\mathbf{e}_{l+1}$ in $(\mathbb{Z}/2L\mathbb{Z})^d$ $(\forall p \in \{1,2,\cdots,q_j-1\})$. By $(\star)$,

$$\text{(B.2)}\quad (\varphi_1-\varphi_2)(P(\mathbf{x}_{k_1+1}),\mathbf{x}_1) = \sum_{r=1}^{k_1}(\varphi_1-\varphi_2)(\mathbf{x}_{r+1},\mathbf{x}_r) + \sum_{p=1}^{q_1-1}(\varphi_1-\varphi_2)(\mathbf{y}_{1,p+1},\mathbf{y}_{1,p}) \pmod{2\pi}.$$

Moreover, by $(\star)$, for any $j \in \{1,2,\cdots,m-2\}$,

$$\text{(B.3)}\quad \begin{aligned}&(\varphi_1-\varphi_2)(P(\mathbf{x}_{k_{j+1}+1}),P(\mathbf{x}_{k_j+1}))\\ &= -\sum_{p=1}^{q_j-1}(\varphi_1-\varphi_2)(\mathbf{y}_{j,p+1},\mathbf{y}_{j,p}) + \sum_{r=k_j+1}^{k_{j+1}}(\varphi_1-\varphi_2)(\mathbf{x}_{r+1},\mathbf{x}_r)\\ &\quad + \sum_{p=1}^{q_{j+1}-1}(\varphi_1-\varphi_2)(\mathbf{y}_{j+1,p+1},\mathbf{y}_{j+1,p}) \pmod{2\pi},\end{aligned}$$

$$\text{(B.4)}\quad \begin{aligned}&(\varphi_1-\varphi_2)(\mathbf{x}_{n+1},P(\mathbf{x}_{k_{m-1}+1}))\\ &= -\sum_{p=1}^{q_{m-1}-1}(\varphi_1-\varphi_2)(\mathbf{y}_{m-1,p+1},\mathbf{y}_{m-1,p})\\ &\quad + \sum_{r=k_{m-1}+1}^{n}(\varphi_1-\varphi_2)(\mathbf{x}_{r+1},\mathbf{x}_r) \pmod{2\pi}.\end{aligned}$$

By adding (B.2), (B.3), (B.4) together,

$$\text{(B.5)}\quad (\varphi_1-\varphi_2)(P(\mathbf{x}_{k_1+1}),\mathbf{x}_1) + \sum_{j=1}^{m-2}(\varphi_1-\varphi_2)(P(\mathbf{x}_{k_{j+1}+1}),P(\mathbf{x}_{k_j+1}))$$

$$
\begin{aligned}
&+ (\varphi_1 - \varphi_2)(\mathbf{x}_{n+1}, P(\mathbf{x}_{k_{m-1}+1})) \\
&= \sum_{r=1}^{n} (\varphi_1 - \varphi_2)(\mathbf{x}_{r+1}, \mathbf{x}_r) \pmod{2\pi}.
\end{aligned}
$$

By the hypothesis of induction the left-hand side of (B.5) is 0 (mod 2π) and thus (B.1) holds.

The induction with $l \in \{1, 2, \cdots, d\}$ concludes the proof. □

The next lemma is the d-dimensional version of [19, Lemma A.3]. However, the content is essentially same as [21, Lemma 2.1].

LEMMA B.2 ([21, Lemma 2.1]). *There exists a function* $\theta : \Gamma(2L) \to \mathbb{R}$ *such that for any* $\mathbf{x}, \mathbf{y} \in \Gamma(2L)$ *satisfying that* $\mathbf{x} - \mathbf{y}$ *is equal to one of* $\mathbf{e}_1$, $-\mathbf{e}_1$, $\mathbf{e}_2$, $-\mathbf{e}_2$, $\cdots$, $\mathbf{e}_d$, $-\mathbf{e}_d$ *in* $(\mathbb{Z}/2L\mathbb{Z})^d$,

$$
\varphi_1(\mathbf{x}, \mathbf{y}) = \varphi_2(\mathbf{x}, \mathbf{y}) + \theta(\mathbf{x}) - \theta(\mathbf{y}) \quad (\mathit{mod}\ 2\pi).
$$

PROOF. Define $\theta : \Gamma(2L) \to \mathbb{R}$ by

$$
\begin{aligned}
&\theta((x_1, x_2, \cdots, x_d)) \\
&:= 1_{x_1 \ge 1} \sum_{j=0}^{x_1 - 1} (\varphi_1 - \varphi_2)((j+1, 0, \cdots, 0), (j, 0, \cdots, 0)) \\
&\quad + 1_{x_2 \ge 1} \sum_{j=0}^{x_2 - 1} (\varphi_1 - \varphi_2)((x_1, j+1, 0, \cdots, 0), (x_1, j, 0, \cdots, 0)) + \cdots \\
&\quad + 1_{x_d \ge 1} \sum_{j=0}^{x_d - 1} (\varphi_1 - \varphi_2)((x_1, \cdots, x_{d-1}, j+1), (x_1, \cdots, x_{d-1}, j)).
\end{aligned}
$$

Then, Lemma B.1 implies that for any $\mathbf{x}$, $\mathbf{y} \in \Gamma(2L)$ satisfying that $\mathbf{x} - \mathbf{y}$ is equal to one of $\mathbf{e}_1$, $-\mathbf{e}_1$, $\mathbf{e}_2$, $-\mathbf{e}_2$, $\cdots$, $\mathbf{e}_d$, $-\mathbf{e}_d$ in $(\mathbb{Z}/2L\mathbb{Z})^d$,

$$
\theta(\mathbf{x}) + (\varphi_1 - \varphi_2)(\mathbf{y}, \mathbf{x}) - \theta(\mathbf{y}) = 0 \pmod{2\pi}. \ \square
$$

With a phase $\varphi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ satisfying (1.1) we define the free Hamiltonian $\mathsf{H}_0(\varphi)$ by (1.20) and set $\mathsf{H}(\varphi) = \mathsf{H}_0(\varphi) + \mathsf{V}$ with the generalized interaction V defined in (1.15).

Lemma B.3.

$$\operatorname{Tr} e^{-\beta \mathsf{H}(\varphi_1)} = \operatorname{Tr} e^{-\beta \mathsf{H}(\varphi_2)}.$$

Proof. By using the function θ introduced in Lemma B.2 and following the proof of [19, Lemma A.4] we can construct the unitary transform B on $F_f(L^2(\Gamma(2L) \times \{\uparrow, \downarrow\}))$ so that $B\mathsf{H}(\varphi_2)B^* = \mathsf{H}(\varphi_1)$. Here we need the invariance (1.11) to ensure that $B\mathsf{V}B^* = \mathsf{V}$. This implies the result. □

Here we can state the sufficient condition to be a minimizer of the flux phase problem. In the following we restrict the interaction V to have the reflection positive form (1.19).

Theorem B.4 ([20]). *Assume that the phase θ_L satisfies* (1.1), (1.2) *with $\theta_{j,k} = \pi$ for any $j,k \in \{1, 2, \cdots, d\}$ with $j < k$ and* (1.3) *with $\varepsilon_l^L = 1_{L \in 2\mathbb{Z}}$ for any $l \in \{1, 2, \cdots, d\}$. Then,*

$$\begin{aligned}
&-\frac{1}{\beta} \log(\operatorname{Tr} e^{-\beta \mathsf{H}(\theta_L)}) \\
&= \min\left\{ -\frac{1}{\beta} \log(\operatorname{Tr} e^{-\beta \mathsf{H}(\varphi)}) \ \Big| \ \varphi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R} \text{ satisfying } (1.1) \right\}.
\end{aligned}$$

Proof. By Lemma B.3 it is sufficient to prove the existence of a phase with the claimed properties minimizing the free energy. For any $j, k \in \{1, 2, \cdots, d\}$, $\mathbf{x} \in \mathbb{Z}^d$, $s \in \mathbb{Z}$ and $\eta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, set

$$\begin{aligned}
f_{j,k}(\eta)(\mathbf{x}) &:= \eta(\mathbf{x} + \mathbf{e}_j, \mathbf{x}) + \eta(\mathbf{x} + \mathbf{e}_j + \mathbf{e}_k, \mathbf{x} + \mathbf{e}_j) \\
&\quad + \eta(\mathbf{x} + \mathbf{e}_k, \mathbf{x} + \mathbf{e}_j + \mathbf{e}_k) + \eta(\mathbf{x}, \mathbf{x} + \mathbf{e}_k), \\
f_j(\eta)(\mathbf{x}) &:= \sum_{m=0}^{2L-1} \eta(\mathbf{x} + (m+1)\mathbf{e}_j, \mathbf{x} + m\mathbf{e}_j), \\
H_j(s) &:= \left\{ \left(y_1, \cdots, y_{j-1}, s + \frac{1}{2}, y_{j+1}, \cdots, y_d \right) \in \mathbb{R}^d \right.
\end{aligned}$$

$$\Big| \, y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_d \in \mathbb{R} \Big\}.$$

Since the interaction V is assumed to satisfy the positivity convention, we can apply the reflection positivity lemma [20, Lemma] with respect to the cutting hyper-plane $H_j(s)$. Recall that in the proof of [19, Theorem A.5] first we did the reflection with the horizontal line $\{(x, 1/2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ and secondly we did the reflections with the vertical lines $\{(s + 1/2, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$ $(s = 0, 1, \cdots, L-1)$. The argument involving the reflections with the hyper-planes $H_2(0)$, $H_1(s)$ $(s = 0, 1, \cdots, L-1)$, parallel to the proof of [19, Theorem A.5], proves that there exists a phase φ satisfying (1.1),

$$f_{1,2}(\varphi)(\mathbf{x}) = \pi, \quad f_1(\varphi)(\mathbf{x}) = f_2(\varphi)(\mathbf{x}) = 1_{L \in 2\mathbb{N}}\pi \pmod{2\pi}, \ (\forall \mathbf{x} \in \mathbb{Z}^d)$$

and minimizing the free energy. This concludes the proof in the case $d = 2$. Let us consider the case that $d \geq 3$. As hypothesis of induction, assume that $l \in \{2, 3, \cdots, d-1\}$ and there exists a phase φ satisfying (1.1),

$$\begin{aligned} &f_{j,k}(\varphi)(\mathbf{x}) = \pi, \quad f_m(\varphi)(\mathbf{x}) = 1_{L \in 2\mathbb{N}}\pi \pmod{2\pi}, \\ &(\forall \mathbf{x} \in \mathbb{Z}^d, j, k, m \in \{1, 2, \cdots, l\} \text{ with } j < k) \end{aligned} \tag{B.6}$$

and minimizing the free energy. For $s \in \{0, 1, \cdots, L-1\}$ let us define the map $\mathrm{Ref}_s : \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$\mathrm{Ref}_s(\mathbf{x}) := (x_1, \cdots, x_l, 2s + 1 - x_{l+1}, x_{l+2}, \cdots, x_d).$$

Then, define the transform R_s on $\mathrm{Map}(\mathbb{Z}^d \times \mathbb{Z}^d, \mathbb{R})$ by

$$\begin{aligned} &R_s(\eta)(\mathbf{x}, \mathbf{y}) \\ &:= \begin{cases} \eta(\mathrm{Ref}_s(\mathbf{x}), \mathrm{Ref}_s(\mathbf{y})) + \pi & \text{if } \exists j, k \in \{s+1, s+2, \cdots, s+L\} \text{ s.t.} \\ & \mathbf{x}(l+1) = j, \ \mathbf{y}(l+1) = k \pmod{2L}, \\ \eta(\mathbf{x}, \mathbf{y}) & \text{otherwise}, \end{cases} \\ &(\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d). \end{aligned}$$

Also, for any function $\theta : \mathbb{Z}^d \to \mathbb{R}$ satisfying that $\theta(\mathbf{x}) = \theta(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ with $\mathbf{x} = \mathbf{y}$ in $(\mathbb{Z}/2L\mathbb{Z})^d$ we define the transform G_θ on $\mathrm{Map}(\mathbb{Z}^d \times \mathbb{Z}^d, \mathbb{R})$ by

$$G_\theta(\eta)(\mathbf{x}, \mathbf{y}) := \eta(\mathbf{x}, \mathbf{y}) + \theta(\mathbf{x}) - \theta(\mathbf{y}), \quad (\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d).$$

Note that if $\eta \in \mathrm{Map}(\mathbb{Z}^d \times \mathbb{Z}^d, \mathbb{R})$ satisfies (1.1) and (B.6), so do $R_s(\eta)$, $G_\theta(\eta)$. We reform $\mathrm{Tr}\, e^{-\beta \mathsf{H}(\varphi)}$ by repeating the reflection with respect to the hyper-planes $H_{l+1}(s)$ $(s = 0, 1, \cdots, L-1)$ and the gauge transformations. This procedure is parallel to the part of the proof of [19, Theorem A.5] demonstrating the reflections with respect to the vertical lines $\{(s+1/2, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$ $(s = 0, 1, \cdots, L-1)$. Here we consider the $l+1$-th coordinate, the k-th coordinate $(k \in \{1, 2, \cdots, l\})$ as the first coordinate, the second coordinate respectively in the part of the proof of [19, Theorem A.5] after the first reflection with the horizontal line $\{(x, 1/2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. By replacing $\mathbf{e}_1$, $\mathbf{e}_2$ by $\mathbf{e}_{l+1}$, $\mathbf{e}_k$ respectively there we can see that there exists a phase $\varphi' \in \mathrm{Map}(\mathbb{Z}^d \times \mathbb{Z}^d, \mathbb{R})$ satisfying (1.1),

$$
\begin{aligned}
& f_{k,l+1}(\varphi')(\mathbf{x}) = \pi, \quad f_{l+1}(\varphi')(\mathbf{x}) = 1_{L \in 2\mathbb{N}} \pi \pmod{2\pi}, \\
& (\forall \mathbf{x} \in \mathbb{Z}^d, k \in \{1, 2, \cdots, l\})
\end{aligned}
$$

and minimizing the free energy. In fact the phase φ' is derived by repeatedly applying the transforms R_s, G_θ with $s \in \{0, 1, \cdots, L-1\}$ and some periodic functions $\theta : \mathbb{Z}^d \to \mathbb{R}$ to the phase φ given by the induction hypothesis. As remarked above, the phase φ' still satisfies (B.6). The induction with l concludes the existence of a phase with the claimed properties. $\square$

Appendix C. Lemmas for the Time-Continuum, Infinite-Volume Limit

Here we prove that each term of the Taylor expansion of the free energy density with respect to the amplitude of the interaction converges in the time-continuum, infinite-volume limit. This fact is used to prove that the free energy itself converges in these limits in Subsection 4.2. Basic ideas of this section are not essentially different from those in [17, Appendix B], [18, Appendix D], [19, Appendix D]. Since we introduced a class of interactions, which are different from the interactions in the preceding papers and some properties of our interactions are necessary to prove the fact of concern, we should again demonstrate the major part of the proof.

For $n \in \mathbb{N} \cup \{0\}$ set

$$
a_n(L, h)(\mathbf{U}) := -\frac{1}{\beta L^d n!} \left(\frac{\partial}{\partial z} \right)^n \log \left(\int e^{-zV(\mathbf{U})(\psi)} d\mu_C(\psi) \right) \Big|_{z=0},
$$

where the Grassmann Gaussian integral is same as that considered in Lemma 2.7. Our aim here is to prove the uniform convergence property of $a_n(L,h)(\mathbf{U})$ with the coupling $\mathbf{U}$ as $h, L \to \infty$. The covariance $C : (\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\times[0,\beta))^2 \to \mathbb{C}$ was originally defined in (2.6). We can periodically extend the domain of C into $(\mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta))^2$. Then, by taking into account Lemma 2.3 (3),(4) we can see that the same procedure as in the derivation of the inequalities [19, (D.3), (D.4), Appendix D] yields the following results.

LEMMA C.1. *There exists a constant $c(\beta,d,(t_j)_{1\le j\le d}) \in \mathbb{R}_{>0}$ depending only on β, d, $(t_j)_{1\le j\le d}$ such that the following inequalities hold.*

$$\text{(C.1)}\qquad |C(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)| \le \frac{c(\beta,d,(t_j)_{1\le j\le d})}{1+\sum_{j=1}^{d}\left|\frac{L}{2\pi}\left(e^{i\frac{2\pi}{L}\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle}-1\right)\right|^{d+1}},$$
$$(\forall(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in\mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta)),$$

$$\text{(C.2)}\qquad |C(\rho\mathbf{x}\sigma x, \eta\mathbf{y}\tau y)| \le \frac{c(\beta,d,(t_j)_{1\le j\le d})}{1+\left(\frac{2}{\pi}\right)^{d+1}\sum_{j=1}^{d}|\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle|^{d+1}}$$
$$(\forall(\rho,\mathbf{x},\sigma,x),(\eta,\mathbf{y},\tau,y)\in\mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta)$$
$$\textit{with } |\langle\mathbf{x}-\mathbf{y},\mathbf{e}_j\rangle|\le L/2\ (\forall j\in\{1,2,\cdots,d\})).$$

For conciseness we set

$$\begin{aligned}
&J := \mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\times\{1,-1\},\\
&J_c := \mathcal{B}\times\left[-\frac{L}{2},\frac{L}{2}\right)\times\{\uparrow,\downarrow\}\times\{1,-1\},\\
&J_0 := \mathcal{B}\times\{\mathbf{0}\}\times\{\uparrow,\downarrow\}\times\{1,-1\},\quad J_\infty := \mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times\{1,-1\}.
\end{aligned}$$

Using the original kernels V_m^L $(m=0,1,\cdots,N_v)$ of the interaction, we define $V_0^0\in\mathrm{Map}(\mathbb{C}^{n_v},\mathbb{C})$, $V_{2m}^0\in\mathrm{Map}(\mathbb{C}^{n_v},\mathrm{Map}(J_\infty^{2m},\mathbb{C}))$ $(m=1,2,\cdots,N_v)$ by

$$\begin{aligned}
&V_0^0(\mathbf{U}) := V_0^L(\mathbf{U}),\\
&V_{2m}^0(\mathbf{U})((\rho_1,\mathbf{x}_1,\sigma_1,\theta_1),\cdots,(\rho_{2m},\mathbf{x}_{2m},\sigma_{2m},\theta_{2m}))\\
&:= \frac{1}{(2m)!}\sum_{\xi\in\mathbb{S}_{2m}}\mathrm{sgn}(\xi)1_{(\theta_{\xi(1)},\cdots,\theta_{\xi(m)})=(1,\cdots,1),(\theta_{\xi(m+1)},\cdots,\theta_{\xi(2m)})=(-1,\cdots,-1)}
\end{aligned}$$

$$\cdot V_m^L(\mathbf{U})(((2\mathbf{x}_{\xi(1)}+b(\rho_{\xi(1)}),\sigma_{\xi(1)}),\cdots,(2\mathbf{x}_{\xi(m)}+b(\rho_{\xi(m)}),\sigma_{\xi(m)})),$$
$$((2\mathbf{x}_{\xi(m+1)}+b(\rho_{\xi(m+1)}),\sigma_{\xi(m+1)}),\cdots,(2\mathbf{x}_{\xi(2m)}+b(\rho_{\xi(2m)}),\sigma_{\xi(2m)}))).$$

The next lemma summarizes some properties of V_0^0, V_{2m}^0 which we will use later.

Lemma C.2. *For any $r\in\mathbb{R}_{>0}$, $m\in\{1,2,\cdots,N_v\}$ the following statements hold.*

$$\text{(C.3)}\qquad \sup_{\mathbf{U}\in\overline{D(r)}^{n_v}}|V_{2m}^0(\mathbf{U})(\rho_1\mathbf{0}\sigma_1\theta_1,\rho_2\mathbf{x}_2\sigma_2\theta_2,\cdots,\rho_{2m}\mathbf{x}_{2m}\sigma_{2m}\theta_{2m})|$$
$$\le re^d v_m(1)e^{-\frac{1}{2m-1}\sum_{p=2}^{2m}\sum_{j=1}^{d}(\frac{2}{\pi}|\langle\mathbf{x}_p,\mathbf{e}_j\rangle|)^{1/2}},$$
$$(\forall(\rho_1,\sigma_1,\theta_1)\in\mathcal{B}\times\{\uparrow,\downarrow\}\times\{1,-1\},$$
$$(\rho_j,\mathbf{x}_j,\sigma_j,\theta_j)\in J_\infty\cap J_c\ (j=2,3,\cdots,2m)).$$

$\frac{1}{L^d}V_0^0, V_{2m}^0(\mathbf{X})$ *converge in* $C(\overline{D(r)}^{n_v};\mathbb{C})$ *as* $L\to\infty(L\in\mathbb{N})$ *for any* $\mathbf{X}\in J_\infty^{2m}$.

Proof. Take any $\mathbf{U}\in\overline{D(r)}^{n_v}$, $(\rho_j,\mathbf{x}_j,\sigma_j,\theta_j)\in J_\infty\cap J_c$ $(j=1,2,\cdots,2m)$, $p\in\{2,3,\cdots,2m\}$. Note that

$$e^{\sum_{j=1}^{d}(\frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle\mathbf{x}_1-\mathbf{x}_p,\mathbf{e}_j\rangle}-1|)^{1/2}}\le e^{\sum_{j=1}^{d}(\frac{L}{\pi}|e^{i\frac{\pi}{L}\langle 2\mathbf{x}_1+b(\rho_1)-2\mathbf{x}_p-b(\rho_p),\mathbf{e}_j\rangle}-1|)^{1/2}+d}.$$

By this inequality, the linearity with $\mathbf{U}$, (1.5), (1.12) and (1.14),

$$e^{\sum_{j=1}^{d}(\frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle\mathbf{x}_1-\mathbf{x}_p,\mathbf{e}_j\rangle}-1|)^{1/2}}|V_{2m}^0(\mathbf{U})(\rho_1\mathbf{x}_1\sigma_1\theta_1,\cdots,\rho_{2m}\mathbf{x}_{2m}\sigma_{2m}\theta_{2m})|$$
$$\le re^d v_m(1).$$

Thus,

$$\sup_{\mathbf{U}\in\overline{D(r)}^{n_v}}|V_{2m}^0(\mathbf{U})(\rho_1\mathbf{x}_1\sigma_1\theta_1,\cdots,\rho_{2m}\mathbf{x}_{2m}\sigma_{2m}\theta_{2m})|$$
$$\le re^d v_m(1)e^{-\frac{1}{2m-1}\sum_{p=2}^{2m}\sum_{j=1}^{d}(\frac{L}{2\pi}|e^{i\frac{2\pi}{L}\langle\mathbf{x}_1-\mathbf{x}_p,\mathbf{e}_j\rangle}-1|)^{1/2}}.$$

The above inequality implies (C.3). The claimed convergence properties follow from that $\mathbf{U}\mapsto V_0^L(\mathbf{U})$, $\mathbf{U}\mapsto V_m^L(\mathbf{U})(\mathbf{X})$ are linear and $\frac{1}{L^d}\frac{\partial}{\partial U_j}V_0^L(\mathbf{U})$, $\frac{\partial}{\partial U_j}V_m^L(\mathbf{U})(\mathbf{X})$ converge as $L\to\infty$. □

For convenience in the proof of the next lemma we introduce some more notations. Define the transform P_0 on J_∞^m by

$$\begin{aligned}&P_0(((\rho_1,\mathbf{x}_1,\sigma_1,\theta_1),\cdots,(\rho_m,\mathbf{x}_m,\sigma_m,\theta_m)))\\&=((\rho_1,\mathbf{0},\sigma_1,\theta_1),\cdots,(\rho_m,\mathbf{0},\sigma_m,\theta_m)).\end{aligned}$$

Define the map P_s from J_∞ to $\mathbb{Z}^d$ by $P_s((\rho,\mathbf{x},\sigma,\theta)) := \mathbf{x}$. Moreover, define the map P_L from J_∞^m to J^m by

$$\begin{aligned}&P_L(((\rho_1,\mathbf{x}_1,\sigma_1,\theta_1),\cdots,(\rho_m,\mathbf{x}_m,\sigma_m,\theta_m)))\\&=((\rho_1,\mathbf{x}_1',\sigma_1,\theta_1),\cdots,(\rho_m,\mathbf{x}_m',\sigma_m,\theta_m)),\end{aligned}$$

where $\mathbf{x}_j' \in \Gamma(L)$ and $\mathbf{x}_j = \mathbf{x}_j'$ in $(\mathbb{Z}/L\mathbb{Z})^d$ $(j = 1, 2, \cdots, m)$. We also define a map from $(\mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta)\times\{1,-1\})^m$ to $(\mathcal{B}\times\Gamma(L)\times\{\uparrow,\downarrow\}\times[0,\beta)\times\{1,-1\})^m$ in the same way as above and let P_L denote the map, though this is abuse of notation. For any $\mathbf{X} = ((\rho_1,\mathbf{x}_1,\sigma_1,\theta_1),\cdots,(\rho_m,\mathbf{x}_m,\sigma_m,\theta_m)) \in J_\infty^m$, $s \in [0,\beta)$ we define $(\mathbf{X}|s) \in (\mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta)\times\{1,-1\})^m$ by

$$(\mathbf{X}|s) := ((\rho_1,\mathbf{x}_1,\sigma_1,s,\theta_1),\cdots,(\rho_m,\mathbf{x}_m,\sigma_m,s,\theta_m)).$$

Furthermore, for any $\mathbf{X} = ((\rho_1,\mathbf{x}_1,\sigma_1,s_1,\theta_1),\cdots,(\rho_m,\mathbf{x}_m,\sigma_m,s_m,\theta_m)) \in (\mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta)\times\{1,-1\})^m$ and $\mathbf{x}\in\mathbb{Z}^d$ we define $\mathbf{X}+\mathbf{x} \in (\mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta)\times\{1,-1\})^m$ by

$$\mathbf{X}+\mathbf{x} := ((\rho_1,\mathbf{x}_1+\mathbf{x},\sigma_1,s_1,\theta_1),\cdots,(\rho_m,\mathbf{x}_m+\mathbf{x},\sigma_m,s_m,\theta_m)).$$

For any $\mathbf{Y} \in J_\infty^m$ and $\mathbf{x}\in\mathbb{Z}^d$ we also define $\mathbf{Y}+\mathbf{x} \in J_\infty^m$ in the same way. We use the same notational rules for different power m for simplicity. We should make clear that the notations introduced above are used only in the rest of this section, not used anywhere else in this paper.

Here let us note that

$$V(\psi) = \beta V_0^0 + \frac{1}{h}\sum_{s\in[0,\beta)_h}\sum_{m=1}^{N_v}\sum_{\mathbf{X}\in J^{2m}} V_{2m}^0(\mathbf{X})\psi_{(\mathbf{X}|s)}.$$

LEMMA C.3. *For any $r \in \mathbb{R}_{>0}$, $n \in \mathbb{N}\cup\{0\}$ the following statements hold true.*

(1) *$a_n(L,h)$ converges in $C(\overline{D(r)}^{n_v};\mathbb{C})$ as $h\to\infty (h\in(2/\beta)\mathbb{N})$.*

(2) *Set $a_n(L) := \lim_{h\to\infty,h\in(2/\beta)\mathbb{N}} a_n(L,h)$. Then, $a_n(L)$ converges in $C(\overline{D(r)}^{n_v};\mathbb{C})$ as $L\to\infty (L\in\mathbb{N})$.*

Proof. Since $a_0(L,h)=0$, the claims are trivial for $n=0$. First let us prove the claims for $n=1$. By the translation invariance (1.9) and the periodicity (1.8),

$$\begin{aligned}
a_1(L,h) &= \frac{1}{\beta L^d}\int V(\psi)d\mu_C(\psi)\\
&= \frac{1}{L^d}V_0^0 + \frac{1}{L^d}\sum_{m=1}^{N_v}\sum_{\mathbf{X}\in J^{2m}} V_{2m}^0(\mathbf{X})\int \psi_{(\mathbf{X}|0)}d\mu_C(\psi)\\
&= \frac{1}{L^d}V_0^0 + \frac{1}{L^d}\sum_{m=1}^{N_v}\sum_{\mathbf{X}\in J^{2m}} V_{2m}^0(\mathbf{X}-P_s(X_1))\\
&\qquad\qquad \cdot \int \psi_{P_L((\mathbf{X}|0)-P_s(X_1))}d\mu_C(\psi)\\
&= \frac{1}{L^d}V_0^0 + \sum_{m=1}^{N_v}\sum_{X\in J_0}\sum_{\mathbf{X}\in J^{2m-1}} V_{2m}^0(X,\mathbf{X})\int \psi_{((X,\mathbf{X})|0)}d\mu_C(\psi)\\
&= \frac{1}{L^d}V_0^0 + \sum_{m=1}^{N_v}\sum_{X\in J_0}\sum_{\mathbf{X}\in J_\infty^{2m-1}} 1_{\mathbf{X}\in J_c^{2m-1}} V_{2m}^0(X,\mathbf{X})\\
&\qquad\qquad \cdot \int \psi_{P_L(((X,\mathbf{X})|0))}d\mu_C(\psi).
\end{aligned}$$

Set

$$F_m'(\mathbf{U})(X,\mathbf{X}) := 1_{\mathbf{X}\in J_c^{2m-1}} V_{2m}^0(\mathbf{U})(X,\mathbf{X})\int \psi_{P_L(((X,\mathbf{X})|0))}d\mu_C(\psi).$$

Note that F_m' is independent of h. Then, it follows from (C.1), the convergence property of C as $L\to\infty$ and Lemma C.2 that

$$\begin{aligned}
&\sup_{\mathbf{U}\in\overline{D(r)}^{n_v}} |F_m'(\mathbf{U})(X,\mathbf{X})|\\
&\le m!c(\beta,d,(t_j)_{1\le j\le d})^m r e^d v_m(1) e^{-\frac{1}{2m-1}\sum_{p=1}^{2m-1}\sum_{j=1}^{d}(\frac{2}{\pi}|\langle P_s(X_p),\mathbf{e}_j\rangle|)^{1/2}}
\end{aligned}$$

and $\lim_{L\to\infty,L\in\mathbb{N}} F'_m(\cdot)(X,\mathbf{X})$ converges in $C(\overline{D(r)}^{n_v};\mathbb{C})$ for any $m\in\{1,2,\cdots,N_v\}$, $X\in J_0$, $\mathbf{X}\in J_\infty^{2m-1}$. Therefore, by the dominated convergence theorem in $L^1(J_0\times J_\infty^{2m-1},C(\overline{D(r)}^{n_v};\mathbb{C}))$ and the convergence property of $(1/L^d)V_0^0$ we see that $a_1(L,h)$ has the claimed convergence properties.

Let $n\ge 2$. Here we need to recall the tree formula for $a_n(L,h)$. We adopt a version of the tree formula [27, Theorem 3], which states that

$$\begin{aligned}a_n(L,h)=\frac{(-1)^{n+1}}{n!\beta L^d}\sum_{T\in\mathbb{T}_n}\prod_{\{p,q\}\in T}(\Delta_{p,q}(C)+\Delta_{q,p}(C))\\ \cdot ope(T,C)\prod_{j=1}^n V(\psi^j)\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}},\end{aligned}$$

where $\mathbb{T}_n$ is the set of all trees over the vertices $\{1,2,\cdots,n\}$,

$$\Delta_{r,s}(C):=-\sum_{X,Y\in I_0}C(X,Y)\frac{\partial}{\partial\overline{\psi}_X^r}\frac{\partial}{\partial\psi_Y^s},\quad(\forall r,s\in\{1,2,\cdots,n\})$$

with the Grassmann left derivatives $\partial/\partial\overline{\psi}_X^r$, $\partial/\partial\psi_X^r$, and

$$ope(T,C):=\int_{[0,1]^{n-1}}d\mathbf{s}\sum_{\xi\in\mathbb{S}_n(T)}\varphi(T,\xi,\mathbf{s})e^{\sum_{r,s=1}^n M(T,\xi,\mathbf{s})(r,s)\Delta_{r,s}(C)},$$

with a T-dependent subset $\mathbb{S}_n(T)$ of $\mathbb{S}_n$, a (T,ξ)-dependent function $\varphi(T,\xi,\cdot)\in C([0,1]^{n-1};\mathbb{R}_{\ge 0})$ satisfying

$$\int_{[0,1]^{n-1}}d\mathbf{s}\sum_{\xi\in\mathbb{S}_n(T)}\varphi(T,\xi,\mathbf{s})=1,\quad(\forall T\in\mathbb{T}_n),\tag{C.4}$$

and a (T,ξ)-dependent matrix-valued function $M(T,\xi,\cdot)\in C([0,1]^{n-1};\mathrm{Mat}(n,\mathbb{R}))$ satisfying

$$\begin{aligned}&|M(T,\xi,\mathbf{s})(r,s)|\le 1,\\ &(\forall T\in\mathbb{T}_n,\xi\in\mathbb{S}_n(T),\mathbf{s}\in[0,1]^{n-1},r,s\in\{1,2,\cdots,n\}).\end{aligned}\tag{C.5}$$

The important bound property of the operator $ope(T,C)$ is that

$$\left|ope(T,C)\psi_{\mathbf{X}_1}^1\psi_{\mathbf{X}_2}^2\cdots\psi_{\mathbf{X}_n}^n\Big|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\right|\tag{C.6}$$

$$\leq \left\lfloor \frac{1}{2}\sum_{k=1}^{n} m_k \right\rfloor ! c(\beta, d, (t_j)_{1\leq j\leq d})^{\frac{1}{2}\sum_{k=1}^{n} m_k},$$
$$(\forall m_j \in \mathbb{N}\cup\{0\}, \mathbf{X}_j \in I^{m_j}\ (j = 1, 2, \cdots, n)),$$

which follows from (C.1), (C.4) and (C.5). The proof of [16, Lemma 4.5] essentially shows how to derive (C.6).

Define the anti-symmetric function $\widetilde{C} : (\mathcal{B} \times \mathbb{Z}^d \times \{\uparrow,\downarrow\} \times [0,\beta) \times \{1,-1\})^2 \to \mathbb{C}$ by

$$\widetilde{C}((X,\theta),(Y,\xi)) := \frac{1}{2}(1_{(\theta,\xi)=(1,-1)}C(X,Y) - 1_{(\theta,\xi)=(-1,1)}C(Y,X)),$$
$$(\forall X, Y \in \mathcal{B} \times \mathbb{Z}^d \times \{\uparrow,\downarrow\} \times [0,\beta), \theta, \xi \in \{1,-1\}).$$

Then, we have that

$$\Delta_{p,q}(C) + \Delta_{q,p}(C) = -2\sum_{\mathbf{X}\in I^2} \widetilde{C}(\mathbf{X})\frac{\partial}{\partial\psi^p_{X_1}}\frac{\partial}{\partial\psi^q_{X_2}}.$$

The term $a_n(L,h)$ can be expanded as follows.

$$\begin{aligned}
&a_n(L,h)\\
&= \frac{2^{n-1}}{n!\beta}\sum_{T\in\mathbb{T}_n}\prod_{j=1}^{n}\left(\sum_{m_j=2}^{2N_v}\right)\frac{1}{L^d}ope(T,C)\prod_{\{p,q\}\in T}\left(\sum_{\mathbf{Y}\in I^2}\widetilde{C}(\mathbf{Y})\frac{\partial}{\partial\psi^p_{Y_1}}\frac{\partial}{\partial\psi^q_{Y_2}}\right)\\
&\quad\cdot\prod_{k=1}^{n}\left(\left(\frac{1}{h}\right)^{m_k}\sum_{\mathbf{X}_k\in I^{m_k}}V_{m_k}(\mathbf{X}_k)\psi^k_{\mathbf{X}_k}\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}.
\end{aligned}$$

For $T \in \mathbb{T}_n$ and $j \in \{1, 2, \cdots, n\}$ let $d_j(T)$ denote the degree of the vertex j in T. Fix $m_j \in \{2, 4, \cdots, 2N_v\}$ $(j = 1, 2, \cdots, n)$. If $d_j(T)$ is larger than m_j for some j, the derivatives along the lines of T erase the Grassmann polynomials completely and thus such a tree does not contribute to the result. Take any $T \in \mathbb{T}_n$ satisfying $d_j(T) \leq m_j$ $(\forall j \in \{1, 2, \cdots, n\})$. Then, set

$$a'_n(L,h) := \frac{1}{L^d}ope(T,C)\prod_{\{p,q\}\in T}\left(\sum_{\mathbf{Y}\in I^2}\widetilde{C}(\mathbf{Y})\frac{\partial}{\partial\psi^p_{Y_1}}\frac{\partial}{\partial\psi^q_{Y_2}}\right)$$

$$\cdot \prod_{k=1}^{n}\left(\left(\frac{1}{h}\right)^{m_k}\sum_{\mathbf{X}_k\in I^{m_k}}V_{m_k}(\mathbf{X}_k)\psi^k_{\mathbf{X}_k}\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}.$$

It suffices to prove the convergence properties of $a'_n(L,h)$ instead of $a_n(L,h)$. By changing the numbering if necessary, we may assume that if $\{p,q\}\in T$ and $p<q$ the length of the shortest path between 1 and p in T is shorter than that between 1 and q. Then, we can define the map $f:\{2,3,\cdots,n\}\to\{1,2,\cdots,n-1\}$ by $f(q):=p$ with $p\in\{1,2,\cdots,q-1\}$ satisfying $\{p,q\}\in T$.

To shorten formulas, we use the notational convention that for integers $l,l+1,\cdots,l+m$ and objects $w_l,w_{l+1},\cdots,w_{l+m}$,

$$\prod_{\substack{j=l\\ order}}^{l+m} w_j,\quad \prod_{\substack{j=l+m\\ order}}^{l} w_j$$

denote

$$w_l w_{l+1}\cdots w_{l+m},\quad w_{l+m}w_{l+m-1}\cdots w_l$$

respectively. Also, it will be convenient to write $X\subset\mathbf{Y}$ for $X\in J$, $\mathbf{Y}\in J^n$ if there exists $j\in\{1,2,\cdots,n\}$ such that $X=Y_j$. By using the notations introduced so far, anti-symmetry, translation invariance and periodicity from part to part we can transform as follows.

$$\begin{aligned}
&a'_n(L,h)\\
&=\frac{1}{L^d}ope(T,C)\prod_{j=1}^{n}\left(\frac{1}{h}\sum_{s_j\in[0,\beta)_h}\right)\\
&\quad\cdot\prod_{\substack{q=n\\ order}}^{2}\left(\sum_{Y,Z_q\in J}\widetilde{C}((Y|s_{f(q)}),(Z_q|s_q))\frac{\partial}{\partial\psi^{f(q)}_{(Y|s_{f(q)})}}\right)\sum_{\mathbf{X}_1\in J^{m_1}}V^0_{m_1}(\mathbf{X}_1)\psi^1_{(\mathbf{X}_1|s_1)}\\
&\quad\cdot\prod_{\substack{k=2\\ order}}^{n}\left(m_k\sum_{\mathbf{X}_k\in J^{m_k-1}}V^0_{m_k}(Z_k,\mathbf{X}_k)\psi^k_{(\mathbf{X}_k|s_k)}\right)\Bigg|_{\substack{\psi^j=0\\(\forall j\in\{1,2,\cdots,n\})}}\\
&=\frac{1}{L^d}\prod_{j=1}^{n}\left(\frac{1}{h}\sum_{s_j\in[0,\beta)_h}\right)\sum_{\substack{X_0\in J\\ \mathbf{X}_1\in J^{m_1-1}}}V^0_{m_1}(X_0,\mathbf{X}_1+P_s(X_0))
\end{aligned}$$

$$\cdot \prod_{q=2}^{n} \Bigg(m_q \sum_{\substack{Y_q, Z_q \in J \\ \mathbf{X}_q \in J^{m_q - 1}}} \widetilde{C}((Y_q|s_{f(q)}) + P_s(X_0), (Z_q|s_q) + P_s(X_0))$$

$$\cdot V^0_{m_q}(Z_q + P_s(X_0), \mathbf{X}_q + P_s(X_0)) \Bigg)$$

$$\cdot ope(T, C) \prod_{\substack{k=n \\ order}}^{2} \left(\frac{\partial}{\partial \psi^{f(k)}_{P_L((Y_k + P_s(X_0)|s_{f(k)}))}} \right)$$

$$\cdot \psi^1_{(X_0|s_1)} \psi^1_{P_L((\mathbf{X}_1 + P_s(X_0)|s_1))} \prod_{\substack{l=2 \\ order}}^{n} \psi^l_{P_L((\mathbf{X}_l + P_s(X_0)|s_l))} \Bigg|_{\substack{\psi^j = 0 \\ (\forall j \in \{1,2,\cdots,n\})}}$$

$$= \prod_{j=1}^{n} \left(\frac{1}{h} \sum_{s_j \in [0,\beta)_h} \right) \sum_{\substack{X_0 \in J_0 \\ \mathbf{X}_1 \in J^{m_1 - 1}}} V^0_{m_1}(X_0, \mathbf{X}_1)$$

$$\cdot \prod_{q=2}^{n} \Bigg(m_q \sum_{\substack{Y_q, Z_q \in J \\ \mathbf{X}_q \in J^{m_q - 1}}} \widetilde{C}((Y_q|s_{f(q)}), (Z_q|s_q)) V^0_{m_q}(Z_q, \mathbf{X}_q) \Bigg)$$

$$\cdot ope(T, C) \prod_{\substack{k=n \\ order}}^{2} \left(\frac{\partial}{\partial \psi^{f(k)}_{(Y_k|s_{f(k)})}} \right) \psi^1_{(X_0|s_1)} \psi^1_{(\mathbf{X}_1|s_1)} \prod_{\substack{l=2 \\ order}}^{n} \psi^l_{(\mathbf{X}_l|s_l)} \Bigg|_{\substack{\psi^j = 0 \\ (\forall j \in \{1,2,\cdots,n\})}}$$

$$= \prod_{j=1}^{n} \left(\frac{1}{h} \sum_{s_j \in [0,\beta)_h} \right) \sum_{\substack{X_0 \in J_0 \\ \mathbf{X}_1 \in J^{m_1 - 1}}} V^0_{m_1}(X_0, \mathbf{X}_1)$$

$$\cdot \prod_{q=2}^{n} \Bigg(m_q \sum_{\substack{Y_q, Z_q \in J \\ \mathbf{X}_q \in J^{m_q - 1}}} \widetilde{C}((Y_q|s_{f(q)}), (Z_q|s_q)) V^0_{m_q}(Z_q, \mathbf{X}_q + P_s(Z_q)) \Bigg)$$

$$\cdot ope(T, C) \prod_{\substack{k=n \\ order}}^{2} \left(\frac{\partial}{\partial \psi^{f(k)}_{(Y_k|s_{f(k)})}} \right)$$

$$\cdot\, \psi^1_{(X_0|s_1)}\psi^1_{(\mathbf{X}_1|s_1)} \prod_{\substack{l=2\\ order}}^{n} \psi^l_{P_L((\mathbf{X}_l+P_s(Z_l)|s_l))}\Bigg|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n\})}}$$

$$= \prod_{j=1}^{n}\left(\frac{1}{h}\sum_{s_j\in[0,\beta)_h}\right)\sum_{\substack{X_0\in J_0\\ \mathbf{X}_1\in J^{m_1-1}}} V^0_{m_1}(X_0,\mathbf{X}_1)$$

$$\cdot \prod_{q=2}^{n}\left(m_q \sum_{\substack{Y_q,Z_q\in J\\ \mathbf{X}_q\in J^{m_q-1}}} \widetilde{C}((Y_q|s_{f(q)}),(Z_q|s_q))V^0_{m_q}(P_0(Z_q),\mathbf{X}_q)\right)$$

$$\cdot\, ope(T,C)\prod_{\substack{k=n\\ order}}^{2}\left(\frac{\partial}{\partial\psi^{f(k)}_{(Y_k|s_{f(k)})}}\right)$$

$$\cdot\, \psi^1_{(X_0|s_1)}\psi^1_{(\mathbf{X}_1|s_1)} \prod_{\substack{l=2\\ order}}^{n} \psi^l_{P_L((\mathbf{X}_l+P_s(Z_l)|s_l))}\Bigg|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n\})}}$$

$$= \prod_{j=1}^{n}\left(\frac{1}{h}\sum_{s_j\in[0,\beta)_h}\right)\sum_{\substack{X_0\in J_0\\ \mathbf{X}_1\in J^{m_1-1}}} V^0_{m_1}(X_0,\mathbf{X}_1)$$

$$\cdot \prod_{q=2}^{n}\left(m_q \sum_{\substack{Y_q,Z_q\in J\\ \mathbf{X}_q\in J^{m_q-1}}} \widetilde{C}((Y_q|s_{f(q)}),(Z_q+P_s(Y_q)|s_q))V^0_{m_q}(P_0(Z_q),\mathbf{X}_q)\right)$$

$$\cdot\, ope(T,C)\prod_{\substack{k=n\\ order}}^{2}\left(\frac{\partial}{\partial\psi^{f(k)}_{(Y_k|s_{f(k)})}}\right)$$

$$\cdot\, \psi^1_{(X_0|s_1)}\psi^1_{(\mathbf{X}_1|s_1)} \prod_{\substack{l=2\\ order}}^{n} \psi^l_{P_L((\mathbf{X}_l+P_s(Z_l)+P_s(Y_l)|s_l))}\Bigg|_{\substack{\psi^j=0\\ (\forall j\in\{1,2,\cdots,n\})}}$$

$$= \prod_{j=1}^{n}\left(\frac{1}{h}\sum_{s_j\in[0,\beta)_h}\right)\sum_{\substack{X_0\in J_0\\ \mathbf{X}_1\in J^{m_1-1}}} V^0_{m_1}(X_0,\mathbf{X}_1)$$

$$\cdot \prod_{q=2}^{n} \left(\sum_{\substack{Y_q \in J_0, Z_q \in J \\ \mathbf{X}_q \in J^{m_q - 1}}} \widetilde{C}((Y_q|s_{f(q)}), (Z_q|s_q)) V^0_{m_q}(P_0(Z_q), \mathbf{X}_q) \right)$$
$$\cdot F((s_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n),$$

where F is the function on

$$[0,\beta)_h^n \times J_0 \times J_\infty^{m_1 - 1} \times \prod_{j=2}^{n} J_\infty^{m_j - 1} \times J_0^{n-1} \times J_\infty^{n-1}$$

defined by

$$F((s_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n)$$
$$:= \prod_{q=2}^{n} \left(m_q \sum_{\mathbf{y}_q \in \Gamma(L)} \right) ope(T, C) \prod_{\substack{k=n \\ order}}^{2} \left(\frac{\partial}{\partial \psi^{f(k)}_{(Y_k + \mathbf{y}_k | s_{f(k)})}} \right)$$
$$\cdot \psi^1_{(X_0|s_1)} \psi^1_{P_L((\mathbf{X}_1|s_1))} \prod_{\substack{l=2 \\ order}}^{n} \psi^l_{P_L((\mathbf{X}_l + P_s(Z_l) + \mathbf{y}_l | s_l))} \Bigg|_{\substack{\psi^j = 0 \\ (\forall j \in \{1,2,\cdots,n\})}}.$$

Note that

$$F((s_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n)$$
$$= \prod_{j=2}^{n} m_j \prod_{r \in f^{-1}(1)} \left(\sum_{\mathbf{y}_r \in \Gamma(L)} 1_{Y_r + \mathbf{y}_r \subset P_L((X_0, \mathbf{X}_1))} \right)$$
$$\cdot \prod_{\substack{q=2 \\ order}}^{n-1} \left(\prod_{p \in f^{-1}(q)} \left(\sum_{\mathbf{y}_p \in \Gamma(L)} 1_{Y_p + \mathbf{y}_p \subset P_L(\mathbf{X}_q + P_s(Z_q) + \mathbf{y}_q)} \right) \right)$$
$$\cdot ope(T, C) \prod_{\substack{k=n \\ order}}^{2} \left(\frac{\partial}{\partial \psi^{f(k)}_{(Y_k + \mathbf{y}_k | s_{f(k)})}} \right)$$
$$\cdot \psi^1_{(X_0|s_1)} \psi^1_{P_L((\mathbf{X}_1|s_1))} \prod_{\substack{l=2 \\ order}}^{n} \psi^l_{P_L((\mathbf{X}_l + P_s(Z_l) + \mathbf{y}_l | s_l))} \Bigg|_{\substack{\psi^j = 0 \\ (\forall j \in \{1,2,\cdots,n\})}},$$

and thus by (C.6),

(C.7) $\quad |F((s_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n)|$

$$\le \prod_{j=2}^{n} m_j \prod_{r\in f^{-1}(1)} m_1 \prod_{q=2}^{n-1}\prod_{p\in f^{-1}(q)} (m_q-1)\left(\frac{1}{2}\sum_{l=1}^{n} m_l - n+1\right)!$$
$$\cdot c(\beta,d,(t_j)_{1\le j\le d})^{\frac{1}{2}\sum_{l=1}^{n} m_l-n+1}$$
$$\le \prod_{j=1}^{n} m_j^{d_T(j)}\left(\frac{1}{2}\sum_{l=1}^{n} m_l-n+1\right)! c(\beta,d,(t_j)_{1\le j\le d})^{\frac{1}{2}\sum_{l=1}^{n} m_l-n+1},$$
$$(\forall((s_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n) \in [0,\beta)_h^n \times \mathbf{J}_\infty).$$

Here we set

$$\mathbf{J}_\infty := J_0 \times J_\infty^{m_1-1} \times \prod_{j=2}^{n} J_\infty^{m_j-1} \times J_0^{n-1} \times J_\infty^{n-1}.$$

For any $s \in [0,\beta)$ we let $\hat{s}$ denote an element of $[0,\beta)_h$ satisfying $s \in [\hat{s}, \hat{s}+1/h)$. By periodicity we can rewrite $a_n'(L,h)$ as follows.

$$a_n'(L,h) = \prod_{j=1}^{n}\left(\int_0^\beta ds_j\right) \sum_{\substack{X_0\in J_0\\ \mathbf{X}_1\in J_\infty^{m_1-1}}} V_{m_1}^0(X_0,\mathbf{X}_1)$$
$$\cdot \prod_{q=2}^{n}\left(\sum_{\substack{Y_q\in J_0, Z_q\in J_\infty\\ \mathbf{X}_q\in J_\infty^{m_q-1}}} \widetilde{C}((Y_q|\hat{s}_{f(q)}),(Z_q|\hat{s}_q)) V_{m_q}^0(P_0(Z_q),\mathbf{X}_q)\right)$$
$$\cdot 1_{((\mathbf{X}_j)_{j=1}^n,(Z_j)_{j=2}^n)\in \prod_{j=1}^n J_c^{m_j-1}\times J_c^{n-1}}$$
$$\cdot F((\hat{s}_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n).$$

By (C.2), (C.3) and (C.7),

$$\text{(C.8)}\quad \sup_{\mathbf{U}\in\overline{D(r)}^{nv}} \Bigg| V_{m_1}^0(\mathbf{U})(X_0,\mathbf{X}_1)$$
$$\cdot \prod_{q=2}^{n}(\widetilde{C}((Y_q|\hat{s}_{f(q)}),(Z_q|\hat{s}_q)) V_{m_q}^0(\mathbf{U})(P_0(Z_q),\mathbf{X}_q))$$
$$\cdot 1_{((\mathbf{X}_j)_{j=1}^n,(Z_j)_{j=2}^n)\in \prod_{j=1}^n J_c^{m_j-1}\times J_c^{n-1}}$$
$$\cdot F((\hat{s}_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n)\Bigg|$$

$$\le \left(\frac{1}{2}\sum_{l=1}^{n} m_l - n + 1\right)!$$
$$\cdot \prod_{q=1}^{n}\left(\frac{re^d v_{m_q/2}(1)c(\beta,d,(t_j)_{1\le j\le d})^{m_q/2} m_q^{d_T(q)}}{1+1_{q\neq 1}(\frac{2}{\pi})^{d+1}\sum_{j=1}^{d}|\langle P_s(Z_q),\mathbf{e}_j\rangle|^{d+1}}\right.$$
$$\left.\cdot e^{-\frac{1}{m_q-1}\sum_{l=1}^{m_q-1}\sum_{j=1}^{d}(\frac{2}{\pi}|\langle P_s(X_{q,l}),\mathbf{e}_j\rangle|)^{1/2}}\right),$$
$$(\forall((s_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n) \in [0,\beta)^n \times \mathbf{J}_\infty).$$

The right-hand side of (C.8) is integrable with respect to

$$((s_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n)$$

over $[0,\beta)^n \times \mathbf{J}_\infty$.

Since F becomes a finite sum of products of the covariance C after applying all the Grassmann derivatives to the monomial, the domain of F can be naturally extended into $[0,\beta)^n \times \mathbf{J}_\infty$. Moreover, we can see that the function $F : [0,\beta)^n \times \mathbf{J}_\infty \to \mathbb{C}$ is independent of h. Since $(s,t) \mapsto \widetilde{C}((X|s),(Y|t))$ is continuous a.e. in $[0,\beta)^2$ for any $X, Y \in J_\infty$, so is $\mathbf{s} \mapsto F(\mathbf{s}, Z)$ a.e. in $[0,\beta)^n$ for any $Z \in \mathbf{J}_\infty$. Thus, for any $X, Y \in J_\infty$, $Z \in \mathbf{J}_\infty$,

$$\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} \widetilde{C}((X|\hat{s}),(Y|\hat{t})) = \widetilde{C}((X|s),(Y|t)) \text{ a.e. } (s,t)\in[0,\beta)^2,$$
$$\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} F((\hat{s}_j)_{j=1}^n, Z) = F((s_j)_{j=1}^n, Z) \text{ a.e. } (s_j)_{j=1}^n \in [0,\beta)^n.$$

Furthermore, by using the fact that $\lim_{L\to\infty, L\in\mathbb{N}} C(\mathbf{X})$ converges for any $\mathbf{X} \in (\mathcal{B}\times\mathbb{Z}^d\times\{\uparrow,\downarrow\}\times[0,\beta))^2$ we can check that $\lim_{L\to\infty,L\in\mathbb{N}} F(\mathbf{s}, Z)$ converges for any $(\mathbf{s}, Z) \in [0,\beta)^n \times \mathbf{J}_\infty$.

Now we can apply the dominated convergence theorem in $L^1([0,\beta)^n \times \mathbf{J}_\infty, C(\overline{D(r)}^{n_v};\mathbb{C}))$ to prove that $a_n'(L,h)$ converges in $C(\overline{D(r)}^{n_v};\mathbb{C})$ as $h \to \infty (h \in (2/\beta)\mathbb{N})$ and

(C.9)
$$\lim_{\substack{h\to\infty\\ h\in(2/\beta)\mathbb{N}}} a_n'(L,h)$$
$$= \prod_{j=1}^{n}\left(\int_0^\beta ds_j\right) \sum_{\substack{X_0\in J_0\\ \mathbf{X}_1\in J_\infty^{m_1-1}}} V_{m_1}^0(X_0,\mathbf{X}_1)$$

$$\cdot \prod_{q=2}^{n}\left(\sum_{\substack{Y_q\in J_0, Z_q\in J_\infty\\ \mathbf{X}_q\in J_\infty^{m_q-1}}}\widetilde{C}((Y_q|s_{f(q)}),(Z_q|s_q))V_{m_q}^0(P_0(Z_q),\mathbf{X}_q)\right)$$
$$\cdot 1_{((\mathbf{X}_j)_{j=1}^n,(Z_j)_{j=2}^n)\in\prod_{j=1}^n J_c^{m_j-1}\times J_c^{n-1}}$$
$$\cdot F((s_j)_{j=1}^n, X_0, \mathbf{X}_1, (\mathbf{X}_j)_{j=2}^n, (Y_j)_{j=2}^n, (Z_j)_{j=2}^n).$$

Set $a_n'(L) := \lim_{h\to\infty, h\in(2/\beta)\mathbb{N}} a_n'(L,h)$. By sending $h\to\infty$ we obtain the inequality (C.8) with a.e. $(s_j)_{j=1}^n \in [0,\beta)^n$ in place of $(\hat{s}_j)_{j=1}^n$ in the left-hand side. Then, by the convergence property of V_{2m}^0 proved in Lemma C.2, the convergence properties of F and $\widetilde{C}$ in the limit $L\to\infty$ and that

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} 1_{((\mathbf{X}_j)_{j=1}^n,(Z_j)_{j=2}^n)\in\prod_{j=1}^n J_c^{m_j-1}\times J_c^{n-1}} = 1,$$
$$\left(\forall ((\mathbf{X}_j)_{j=1}^n, (Z_j)_{j=2}^n) \in \prod_{j=1}^n J_\infty^{m_j-1}\times J_\infty^{n-1}\right),$$

we can again apply the dominated convergence theorem in $L^1([0,\beta)^n \times \mathbf{J}_\infty, C(\overline{D(r)}^{n_v};\mathbb{C}))$ to deduce from (C.9) that $a_n'(L)$ converges in $C(\overline{D(r)}^{n_v};\mathbb{C})$ as $L\to\infty (L\in\mathbb{N})$. □

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Supplementary List of Notations

Parameters and constants

Notation	Description	Reference
$\theta_{j,k}$ $(1\le j<k\le d)$	flux per plaquette	Subsection 1.2
ε_j^L $(j=1,2,\cdots,d)$	flux per large circles around periodic lattice	Subsection 1.2
t_j $(j=1,2,\cdots,d)$	hopping amplitude	Subsection 1.2
n_v	number of coupling constants	Subsection 1.2
N_v	maximal degree of interacting part of Hamiltonian	Subsection 1.2

$v_0, v_m(c)$ $(m = 1, \cdots, N_v)$	integral of kernels of interaction	beginning of Subsection 1.3
$\boldsymbol{\pi}$	$(\pi, \pi, \cdots, \pi) \in \mathbb{R}^d$	Subsection 2.1
$\boldsymbol{\varepsilon}^L$	$(\varepsilon_j^L)_{1\le j\le d}$	Subsection 2.1
$\boldsymbol{\theta}$	$(\theta_{j,k})_{1\le j<k\le d}$	Subsection 2.1
N	$2^{d+2}L^d\beta h$, cardinality of the index set I	Subsection 2.2
M_{UV}	$\frac{2\sqrt{6}}{\pi}(2d+1)$	Subsection 3.1
$\mathrm{w}(0)$	$c_w(d+1)^{-2}M^{-2}$	Subsection 3.2
$f_{\mathbf{t}}$	parameter depending only on $(t_j)_{1\le j\le d}$, $(\theta_{j,k})_{1\le j<k\le d}$	Subsection 4.1
c_χ	constant appearing in an upper bound on derivatives of cut-off functions	Lemma 4.1 (3)

Sets and spaces

Notation	Description	Reference
$\Gamma(L)$	$\{0, 1, \cdots, L-1\}^d$	Subsection 1.2
$\mathrm{Map}(A, B)$	set of maps from A to B	Subsection 1.2
$D(c)$	$\{z \in \mathbb{C} \mid \lvert z\rvert < c\}$	Subsection 1.4
$C(\overline{D}; \mathbb{C})$	set of continuous functions on $\overline{D}$	Subsection 1.4
$\mathrm{Mat}(n, \mathbb{C})$	set of $n \times n$ complex matrices	Subsection 2.1
$\mathcal{B}$	$\{1, 2, 3, \cdots, 2^d\}$	Subsection 2.1
$\Gamma(L)^*$	$\{0, \frac{2\pi}{L}, \cdots, \frac{2\pi}{L}(L-1)\}^d$	Subsection 2.1
$C(\overline{D}; \bigwedge \mathcal{V})$	set of Grassmann polynomials continuous with $\mathbf{z} \in \overline{D}$	Subsection 4.1
$C^\omega(D; \bigwedge \mathcal{V})$	set of Grassmann polynomials analytic with $\mathbf{z} \in D$	Subsection 4.1
$\mathcal{S}(D, c_0, \alpha, M)\ (l)$	subset of $C(\overline{D}; \bigwedge \mathcal{V}) \cap C^\omega(D; \bigwedge \mathcal{V})$	Subsection 4.1
$\hat{\mathcal{S}}(D, c_0, \alpha, M)\ (l)$	subset of $\mathcal{S}(D, c_0, \alpha, M)(l)(\beta_1)\times \mathcal{S}(D, c_0, \alpha, M)(l)(\beta_2)$	Subsection 4.1
$\mathcal{K}(D, \alpha, M)(l)$	subset of $\mathrm{Map}(\overline{D}, C^\infty(\mathbb{R}^{d+1}; \mathrm{Mat}(2^d, \mathbb{C})))$	Subsection 4.1

$\hat{\mathcal{K}}(D,\alpha,M)(l)$	subset of $\mathcal{K}(D,\alpha,M)(l)(\beta_1) \times \mathcal{K}(D,\alpha,M)(l)(\beta_2)$	Subsection 4.1
$C^{\omega}(D;\mathbb{C})$	set of analytic functions in D	Subsection 4.1
$\mathcal{R}(D,c_0,M)(l)$	subset of $\mathrm{Map}(\overline{D},\mathrm{Map}(I_0^2,\mathbb{C}))$	Subsection 4.1
$\hat{\mathcal{R}}(D,c_0,M)(l)$	subset of $\mathcal{R}(D,c_0,M)(l)(\beta_1) \times \mathcal{R}(D,c_0,M)(l)(\beta_2)$	Subsection 4.1

Functions and maps

Notation	Description	Reference
H	1-band Hamiltonian	Subsection 1.2
H_0	kinetic part of H	Subsection 1.2
V	interacting part of H	Subsection 1.2
b	bijection from $\mathcal{B}$ to $\{0,1\}^d$	Subsection 2.1
$U_d((\xi)_{1\le j\le d})$	$2^d \times 2^d$ diagonal unitary matrix	Subsection 2.1
ν	bijection from $\mathcal{B}\times\Gamma(L)$ to $\Gamma(2L)$	Subsection 2.1
H	2^d-band Hamiltonian	Subsection 2.1
H_0	kinetic part of H	Subsection 2.1
V	interacting part of H	Subsection 2.1
$\mathcal{E}$	$2^d \times 2^d$-matrix-valued function	Subsection 2.2

Other notations

Notation	Description	Reference
$\mathbf{e}_j$ $(j=1,2,\cdots,d)$	standard basis of $\mathbb{R}^d$	Subsection 1.2
$\|\cdot\|_{n\times n}$	operator norm for $n\times n$-matrices	Subsection 2.1

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Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Tokyo 153-8914, Japan
E-mail: kashima@ms.u-tokyo.ac.jp