Twist maps on quantized coordinate algebras

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Abstract. Quantum unipotent subgroups and quantum unipotent cells are quantum analogues of the coordinate algebras of unipotent subgroups and unipotent cells of Kac-Moody groups, respectively. Here unipotent subgroups are affine algebraic groups and unipotent cells are affine algebraic varieties. Those quantized coordinate algebras have many interesting structures and reflect structures of other mathematical objects via categorifications. Typical examples are their dual canonical bases and quantum cluster algebra structures.

In this thesis, we develop the theory of quantum analogues of twist maps on those quantized coordinate algebras. In particular, we study their compatibility with specific bases and quantum cluster algebra structures of those quantized coordinate algebras. Here the twist maps indicate the Fomin-Zelevinsky twist maps (henceforth the FZ-twist maps) between unipotent subgroups and the Berenstein-Fomin-Zelevinsky twist automorphisms (henceforth the BFZ-twist automorphisms) on unipotent cells.

The quantum analogues of the FZ-twist maps were constructed by Lenagan-Yakimov. We show that these quantum FZ-twist maps are restricted to bijections between the dual canonical bases of quantum unipotent subgroups, and preserve quantum analogues of certain unipotent generalized minors, in particular, specific determinantal identities, called quantum T-systems.

The quantum analogues of the BFZ-twist automorphisms were obtained by Berenstein-Rupel when the Weyl group elements corresponding to quantum unipotent cells are squares of Coxeter elements. In this thesis, we construct the quantum analogues of the BFZ-twist automorphisms on arbitrary quantum unipotent cells in a different method. Our approach relies on the relations between the structures of quantum unipotent subgroups, quantum unipotent cells and non-unipotent quantized coordinate algebras. We define appropriately the dual canonical bases of the quantum unipotent cells and show that these quantum BFZ-twist automorphisms are restricted to permutations on the dual canonical bases. Moreover we prove that the quantum BFZ-twist automorphisms are categorified by representations of preprojective algebras following Geiß-Leclerc-Schröer's theory when the corresponding Lie algebra is symmetric. As a corollary, we show the compatibility between quantum BFZ-twist automorphisms and quantum cluster monomials. At last, the Chamber Ansatz formulae for quantum unipotent cells are obtained by means of the quantum BFZ-twist automorphisms. These formulae tell that our quantum BFZ-twist automorphisms are generalizations of Berenstein-Rupel's ones.

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Contents

Introduction

Organization of the thesis

This thesis is divided into three chapters. In Chapter I, we fix our notations and review known properties of our main targets, quantized enveloping algebras, quantized coordinate algebras and canonical bases. The definition of quantum cluster algebras is also reviewed. In Chapter II and III, we develop the theory of twist maps in the quantum settings from the basis-theoretic viewpoint. In particular, we focus on their compatibility with dual canonical bases and quantum cluster monomials. In Chapter II, we deal with quantum analogues of Fomin-Zelevinsky twist maps, introduced by Lenagan-Yakimov. In Section II.1, we show that quantum analogues of Fomin-Zelevinsky twist maps are restricted to bijections between dual canonical bases of quantum unipotent subgroups. In Section II.2, we treat quantum analogues of generalized minors on unipotent groups, called unipotent quantum minors. They are typical example of dual canonical basis elements and some of them are quantum cluster monomials of quantum unipotent subgroups. We prove that the image of some unipotent quantum minors under quantum analogues of Fomin-Zelevinsky twist maps are also unipotent quantum minors. In Chapter III, we deal with quantum analogues of Berenstein-Fomin-Zelevinsky twist automorphisms. In Section III.1, we construct quantum analogues of Berenstein-Fomin-Zelevinsky twist automorphisms and show that they are restricted to permutations on dual canonical bases of quantum unipotent cells. In Section III.2, we prove that quantum analogues of Berenstein-Fomin-Zelevinsky twist automorphisms are categorified by representations of preprojective algebras following Geiß-Leclerc-Schröer's theory when the corresponding Lie algebra is symmetric. As a corollary, we show the compatibility between quantum analogues of Berenstein-Fomin-Zelevinsky twist automorphisms and quantum cluster monomials. In Section III.3, the "Chamber Ansatz formulae" for quantum unipotent cells are established by means of quantum analogues of Berenstein-Fomin-Zelevinsky twist automorphisms.

Backgrounds and Main results

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ its triangular decomposition, G the connected simply-connected complex algebraic group with the Lie algebra $\mathfrak{g}$, and $N_\pm$, H the closed subgroups of G with the Lie algebras $\mathfrak{n}_\pm$, $\mathfrak{h}$, respectively. Set $B_- := N_- H$, $B_+ := HN_+$, which are called Borel subgroups. The Weyl group $N_G(H)/H$ is denoted by W. (Here we take $\mathfrak{g}$ as a semisimple Lie algebra for simplicity, however all results are valid in arbitrary symmetrizable Kac-Moody settings unless otherwise specified.)

For $w \in W$, the algebraic subgroup $N_-(w) := N_- \cap \dot{w}N_+\dot{w}^{-1}$ is called *a unipotent subgroup*, and the affine algebraic variety $N_-^w := N_- \cap B_+\dot{w}B_+$ is called *a unipotent cell*. Here $\dot{w}$ denotes an arbitrary lift of w to $N_G(H)$. We deal with the quantum analogues $\mathbf{A}_q[N_-(w)]$ and $\mathbf{A}_q[N_-^w]$ of the coordinate algebras $\mathbb{C}[N_-(w)]$ and $\mathbb{C}[N_-^w]$, respectively. These algebras $\mathbf{A}_q[N_-(w)]$ and $\mathbf{A}_q[N_-^w]$ are introduced by De Concini, Kac and Procesi [**10**], [**11**]. The algebras $\mathbf{A}_q[N_-(w)]$ and $\mathbf{A}_q[N_-^w]$ are called *the quantum unipotent subgroup* and *the quantum unipotent cell*, respectively.

An important research topic concerning those quantized coordinate algebras is the theory of their specific bases. Our main interests are *dual canonical bases(= upper global bases)* in the sense of Lusztig [**40, 41, 44**] and Kashiwara [**29**]. A guiding principle of algebraic and combinatorial research on such remarkable bases is a *quantum cluster algebra structure*. It provides the special elements, called *quantum cluster monomials*, in the given algebra which are constructed by the successive procedure, called *mutation*. See Section I.2 for the precise definition of the quantum cluster algebras.

The dual canonical bases and the quantum cluster algebra structures reflect structures of other mathematical objects; In [**40**] (and his subsequent work [**41, 44**]), Lusztig has originally constructed the canonical bases by using *perverse sheaves* on affine spaces arising from the representation theory of quivers. In his construction, the canonical basis elements correspond to simple perverse sheaves. The multiplication structure of the dual canonical basis elements is a shadow of the images of those simple perverse sheaves under "the restriction functor". In [**29**], Kashiwara has independently constructed the canonical bases (more precisely, he called them *global crystal bases*, and subsequently their coincidence with canonical bases is proved), and shown that they have some *combinatorial* structures, called *Kashiwara crystals*. When $\mathfrak{g}$ is symmetric, the algebra $\mathbf{A}_q[N_-]$ is isomorphic to the deformed Grothendieck ring of an appropriate monoidal subcategory of finite dimensional representations of *quantum affine algebras*, and the dual canonical basis corresponds to the basis coming from simple objects [**23**]. The quantum cluster algebra structures on $\mathbf{A}_q[N_-(w)]$ and $\mathbf{A}_q[N_-^w]$ are categorified by representations of *the preprojective algebras* [**20**], and we deal with this kind of categorification in this thesis. There is also a monoidal categorification of quantum cluster algebra structures on $\mathbf{A}_q[N_-(w)]$ and $\mathbf{A}_q[N_-^w]$ through *quiver Hecke algebras* [**27**], [**28**].

In this thesis, we develop the theory of quantum analogues of "*twist maps*" from the basis-theoretic viewpoint. Ideally (and, in consequence, actually), they should be nontrivial (anti-)algebra isomorphisms of the quantized coordinate algebras. Hence they *preserve* multiplication structure. If these quantum twist maps preserve specific bases, especially dual canonical bases, then such results may suggest the new "symmetries" of various mathematical objects. The main aim of this paper is to establish the foundation of this direction: (1) to show the compatibility between dual canonical bases and the "known" quantum twist maps (2) to *construct* quantum analogues of twist maps and show their compatibility with dual canonical bases and quantum cluster monomials. Moreover, as an application of (2), we deduce quantum analogues of *the Chamber Ansatz formulae*, which we will explain below. The results in Chapter II correspond to (1) and the results in Chapter III correspond to (2).

Here we present explicit forms of (non-quantum) twist maps whose quantum analogues are discussed in this thesis. Set $G_0 := N_- H N_+$ and let $g = [g]_-[g]_0[g]_+$ be the corresponding (unique) decomposition for $g \in G_0$.

Definition (Twist maps [**4**], [**6**], [**14**]). Let $w \in W$. There exists a biregular automorphism $\eta_w \colon N_-^w \to N_-^w$ given by

$$y \mapsto [y^T \dot{w}]_-,$$

here y^T is a transpose of y in G and $\dot{w}$ is an arbitrary lift of w to $N_G(H)$. This is called *the BFZ-twist automorphism.* Moreover, there exists a biregular isomorphism $\tau_w \colon N_-(w^{-1}) \to N_-(w)$ given by

$$y \mapsto \overline{w}(y^\vee)^{-1}\overline{w}^{-1},$$

here $\vee$ is a involutive group automorphisms of G which interchanges positive and negative Chevalley generators (cf. Definition I.1.5), and $\overline{w}$ is a specific lift of w to $N_G(H)$. This is called *the FZ-twist map* (with respect to the y-coordinate in the sense of [**14**]).

These are introduced in order to solve the "*factorization problems*" which arose from research on totally positive elements in G. Indeed, we consider the quantum analogue of factorization problems for unipotent cells in Section III.3 and explain them below. These twist maps induce the $\mathbb{C}$-algebra isomorphisms $\eta_w^* \colon \mathbb{C}[N_-^w] \to \mathbb{C}[N_-^w]$ and $\tau_w^* \colon \mathbb{C}[N_-(w)] \to \mathbb{C}[N_-(w^{-1})]$ respectively. It is known that these isomorphisms are compatible with some specific basis elements of $\mathbb{C}[N_-(w)]$ and $\mathbb{C}[N_-^w]$, for instance, [**14**, Lemma 2.25], [**19**, Theorem 6]. We pursue such phenomena in quantum settings, focusing on especially dual canonical bases. Note that dual canonical bases are originally defined in quantum settings. Chapter II discusses quantum analogues of FZ-twist maps, and Chapter III deals with quantum analogues of BFZ-twist automorphisms.

Before explaining the details of our main results, we briefly confirm our setup and prepare notations from representation theory. The quantized coordinate algebras above are defined from the Drinfeld-Jimbo quantized enveloping algebra $\mathbf{U}_q := \mathbf{U}_q(\mathfrak{g})$ associated with $\mathfrak{g}$, which is an algebra over the rational function field $\mathbb{Q}(q)$ in one variable q. The algebra $\mathbf{U}_q$ is a Hopf algebra which is a quantum analogue of the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$. The coordinate algebra $\mathbb{C}[N_-]$ is isomorphic to the graded dual $(\mathbf{U}(\mathfrak{n}_-))^*_{\mathrm{gr}}$ of the enveloping algebra of $\mathfrak{n}_-$. Note that the algebra structure of $(\mathbf{U}(\mathfrak{n}_-))^*_{\mathrm{gr}}$ comes from the coalgebra structure of $\mathbf{U}(\mathfrak{n}_-)$. Hence $\mathbf{A}_q[N_-]$ is defined as a graded dual $(\mathbf{U}_q^-)^*_{\mathrm{gr}}$ of the negative half $\mathbf{U}_q^-$ of $\mathbf{U}_q$. Here the algebra structure of $(\mathbf{U}_q^-)^*_{\mathrm{gr}}$ is defined as a dual of the twisted coalgebra structure of $\mathbf{U}_q^-$. In quantum settings, $\mathbf{A}_q[N_-]$ is actually isomorphic to $\mathbf{U}_q^-$ as $\mathbb{Q}(q)$-algebras by the existence of an appropriate nondegenerate pairing. The quantum unipotent subgroups $\mathbf{A}_q[N_-(w)]$ are defined as certain subalgebras of $\mathbf{A}_q[N_-]$, and the quantum unipotent cells $\mathbf{A}_q[N_-^w]$ are defined as some localizations of quotient algebras of $\mathbf{A}_q[N_-]$.

Let P_+ be the set of dominant integral weights. For $\lambda \in P_+$, $V(\lambda)$ denotes the integrable highest weight $\mathbf{U}_q$-module with highest weight λ. Fix a highest weight vector u_λ and let $u_{w\lambda} \in V(\lambda)$ be the (unique) canonical basis element of weight $w\lambda$. Then there exists a

$\mathbb{Q}(q)$-bilinear form $(\ ,\ )^{\varphi}_{\lambda}\colon V(\lambda)\times V(\lambda)\to\mathbb{Q}(q)$ such that $(u_\lambda,u_\lambda)^{\varphi}_{\lambda}=1$ and $(x.u,u')^{\varphi}_{\lambda}=(u,\varphi(x).u')^{\varphi}_{\lambda}$ for $x\in\mathbf{U}_q$, $u,u'\in V(\lambda)$. Here φ is the $\mathbb{Q}(q)$-anti-algebra involution which is a quantum analogue of the transpose in G (see Definition I.1.5). Now two vectors $u,u'\in V(\lambda)$ determine the element $D_{u,u'}$ of $(\mathbf{U}_q^-)^*_{\mathrm{gr}}=\mathbf{A}_q[N_-]\simeq\mathbf{U}_q^-$ given by $n\mapsto(u,n.u')^{\varphi}_{\lambda}$ for $n\in\mathbf{U}_q^-$. We write $D_{u_{w\lambda},u_{w'\lambda}}$ as $D_{w\lambda,w'\lambda}$ for $w,w'\in W$ and $\lambda\in P_+$. The elements of this form are called *unipotent quantum minors*, which are quantum analogues of generalized minors on the unipotent group N_-. The unipotent quantum minors are important and manageable examples of the dual canonical basis elements.

Quantum FZ-twist maps: The quantum analogue $\tau_{w,q}$ of the FZ-twist map τ_w^* is introduced by Lenagan-Yakimov [**39**]. This quantum FZ-twist map $\tau_{w,q}$ is constructed as a composition of well-known algebra automorphisms and anti-automorphisms on the quantized enveloping algebras, more precisely, $\tau_{w,q}:=*\circ T_{w^{-1}}\circ S\circ\vee\circ*\colon\mathbf{A}_q[N_-(w)]\to\mathbf{A}_q[N_-(w^{-1})]$; here let T_w be Lusztig's braid group symmetry, S the antipode, $*$ the $*$-involution and by $\vee$ the involution analogous to the one in the definition of twist maps (see Definition I.1.5 and I.1.28). Note that $\mathbf{A}_q[N_-]=(\mathbf{U}_q^-)^*_{\mathrm{gr}}$ is isomorphic to $\mathbf{U}_q^-$ as $\mathbb{Q}(q)$-algebras In particular, $\tau_{w,q}$ is a $\mathbb{Q}(q)$-anti-algebra involution. (We use the $*$-involution for the technical reason.) We first show the following:

THEOREM (Theorem II.1.10). *The quantum FZ-twist map $\tau_{w,q}$ is restricted to a bijection from the dual canonical basis of $\mathbf{A}_q[N_-(w)]$ to that of $\mathbf{A}_q[N_-(w^{-1})]$.*

The quantum unipotent subgroup $\mathbf{A}_q[N_-(w)]$ has another specific basis called the (dual) Poincaré-Birkhoff-Witt type basis. It is known that the transition matrices between the dual canonical basis and the dual Poincaré-Birkhoff-Witt type basis are unitriangular with respect to "the left lexicographic order". As a corollary of the theorem above, we proved that this unitriangularity also holds under "the right lexicographic order" (Corollary II.1.11). If $\mathfrak{g}$ is not of finite type, this reverse unitriangularity is a new symmetry.

Next we show the compatibility between quantum FZ-twist maps and unipotent quantum minors. The following statement is a quantum analogue of [**14**, Lemma 2.25] and a slight refinement of [**39**, The equality (6.7)].

THEOREM (Theorem II.2.8, Corollary II.2.14). *Let $w_1,w_2\in W$. Suppose that w_1 and w_2 are less than or equal to w with respect to the weak right Bruhat order (see Proposition II.2.7). Then we have $D_{w_2\lambda,w_1\lambda}\in\mathbf{A}_q[N_-(w)]$, $D_{w^{-1}w_1\lambda,w^{-1}w_2\lambda}\in\mathbf{A}_q[N_-(w^{-1})]$, and*

$$\tau_{w,q}(D_{w_2\lambda,w_1\lambda})=D_{w^{-1}w_1\lambda,w^{-1}w_2\lambda}.$$

In particular, $\tau_{w,q}$ maps the quantum analogues of specific determinantal identities, called the quantum T-system, *in $\mathbf{A}_q[N_-(w)]$ to those in $\mathbf{A}_q[N_-(w^{-1})]$.*

Note that the quantum T-systems are specific mutation sequences consisting of unipotent quantum minors from the viewpoint of quantum cluster algebras.

Quantum BFZ-twist automorphisms: Berenstein-Rupel constructed a quantum analogue of the BFZ-twist automorphism on $\mathbf{A}_q[N_-^w]$ in the case that w is a square of Coxeter elements, by using quantum cluster algebra structures [**5**, Theorem 2.10]. In this thesis, we construct a quantum analogue $\eta_{w,q}$ of the BFZ-twist automorphism η_w^* on an *arbitrary* quantum unipotent cell $\mathbf{A}_q[N_-^w]$ without referring quantum cluster algebra structures. Our construction depends on the relations between the structure of quantum unipotent subgroups, quantum unipotent cells and a quantum analogue of the coordinate algebra of G. We present the explicit form of our quantum BFZ-twist automorphisms here. Since $\mathbf{A}_q[N_-^w]$ is some localization of a quotient algebra of $\mathbf{A}_q[N_-]$, there exists the element $[D_{u,u'}] \in \mathbf{A}_q[N_-^w]$ derived from $D_{u,u'} \in \mathbf{A}_q[N_-]$ for every $u, u' \in V(\lambda)$, $\lambda \in P_+$:

THEOREM (Theorem III.1.42). *Let $w \in W$. There exists a $\mathbb{Q}(q)$-algebra automorphism $\eta_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w]$ given by*

$$[D_{u,u_\lambda}] \mapsto q^{-(\lambda, \mathrm{wt}\, u - \lambda)}[D_{w\lambda,\lambda}]^{-1}[D_{u_{w\lambda},u}], \qquad [D_{w\lambda,\lambda}]^{-1} \mapsto q^{(\lambda, w\lambda-\lambda)}[D_{w\lambda,\lambda}]$$

for a weight vector $u \in V(\lambda)$ and $\lambda \in P_+$. Here $\mathrm{wt}\, u$ denotes the weight of u and $(\ ,\ )$ is the standard bilinear form on $\mathfrak{h}^*$.

We call $\eta_{w,q}$ *the quantum BFZ-twist automorphism* on $\mathbf{A}_q[N_-^w]$. Actually, it is shown in [**37**, Corollary 5.5] that this automorphism $\eta_{w,q}$ coincides with the BFZ-twist automorphism $\eta_w^* \colon \mathbb{C}[N_-^w] \to \mathbb{C}[N_-^w]$ when we consider an appropriate specialization to $q = 1$. Our aim is to study the compatibility between quantum BFZ-twist automorphisms and dual canonical bases. We define naturally the dual canonical basis $\widetilde{\mathbf{B}}^{\mathrm{up},w}$ of $\mathbf{A}_q[N_-^w]$ (Definition III.1.35) and prove the following.

THEOREM (Theorem III.1.42). *The quantum BFZ-twist automorphism $\eta_{w,q}$ is restricted to a permutation on $\widetilde{\mathbf{B}}^{\mathrm{up},w}$.*

Since this is a *permutation*, we can consider the iterated application of quantum BFZ-twist automorphisms, and the "*orbit*" of dual canonical basis elements. In fact, we prove that, if w is a longest element w_0 of W, then $\eta_{w_0,q}$ has "6-periodicity" (Theorem III.1.45). The necessary and sufficient condition for such periodicity is unclear. When $\mathfrak{g}$ is symmetric, the periodicity is categorified as the "periodicity" of (relative) syzygy functors on representations of preprojective algebras via *Geiß-Leclerc-Schröer's additive categorification*, which is our next topic. In fact, "6-periodicity" is a well-known property in such context.

We next show an additive categorification of quantum BFZ-twist automorphisms in the sense of Geiß-Leclerc-Schröer. Here we assume that $\mathfrak{g}$ is symmetric. In particular, we show that quantum BFZ-twist automorphisms preserve quantum cluster monomials.

Geiß-Leclerc-Schröer have categorified the (non-quantum) BFZ-twist automorphisms [**19**] and quantum cluster algebra structures on quantum unipotent subgroups (and quantum unipotent cells) [**20**] by using representations of the preprojective algebra Π associated with $\mathfrak{g}$. They used specific full subcategories $\mathcal{C}_w$, $w \in W$ of Π-modules which are introduced by Buan-Iyama-Reiten-Scott [**8**] and independently by Geiß-Leclerc-Schröer [**17**] for specific w. Lusztig's construction [**43**] of $\mathbb{C}[N_-] = (\mathbf{U}(\mathfrak{n}_-))^*_{\mathrm{gr}}$ associates each $X \in \mathcal{C}_w$ with a regular function $[\varphi_X] \in \mathbb{C}[N_-^w]$. There exists a quantum analogue $Y_R \in \mathbf{A}_q[N_-^w]$ of $[\varphi_R] \in \mathbb{C}[N_-^w]$ if $R \in \mathcal{C}_w$ is a *reachable* Π-module in the sense of [**18, 20**]. In terms of quantum cluster algebras, the elements Y_R correspond to the quantum cluster monomials. If R is projective-injective in $\mathcal{C}_w$ (in fect, $\mathcal{C}_w$ is a Frobenius category), then Y_R is invertible in $\mathbf{A}_q[N_-^w]$. For $X \in \mathcal{C}_w$, $I(X)$ denotes the injective hull of X in $\mathcal{C}_w$ and $\Omega_w^{-1}(X)$ denotes the cokernel of the corresponding embedding $X \to I(X)$. Then we prove the following theorem:

THEOREM (Theorem III.2.20). *Let $w \in W$. Then for every reachable Π-module $R \in \mathcal{C}_w$, we have*

$$\eta_{w,q}(Y_R) \simeq Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}.$$

Here $\simeq$ stands for the coincidence up to some powers of q.

This result is a quantum analogue of Geiß-Leclerc-Schröer's result in [**19**, Theorem 6] (Proposition III.2.19) and we actually use their result essentially in our proof. This is regarded as an additive categorification of the quantum BFZ-twist automorphisms. An important corollary is the following (Corollary III.2.21): for a reachable Π-module $R \in \mathcal{C}_w$, Y_R is a dual canonical basis element if and only if $Y_{\Omega_w^{-1}(R)}$ is so. Therefore the property that a quantum cluster monomial belongs to dual canonical basis is preserved in an "*orbit*" of syzygy functors. Actually, Kang-Kashiwara-Kim-Oh have shown that all quantum cluster monomials belong to the dual canonical bases by using a categorification via representations of quiver Hecke algebras [**27**], [**28**]. Hence we have already known that every Y_R is a dual canonical basis element. However, the understanding of the orbits of syzygy functors may provide another approach to this strong result. Indeed, there is now no proof of it through the additive categorification. It would be interesting to determine the dual canonical basis elements obtained from the corollary above and, for example, quantum T-systems.

At last, we consider quantum analogues of the "factorization problems" for unipotent cells as mentioned above. These problems are considered by Berenstein, Fomin and Zelevinsky in [**4**], [**6**] in order to study the totally positive elements in Schubert cells. Here we explain them in non-quantum settings. Let $\{\alpha_i \text{ (resp. } h_i) \mid i \in I\}$ be the set of simple roots (resp. simple coroots), $\{s_i \mid i \in I\} \subset W$ the set of simple reflections, $\{\varpi_i \mid i \in I\} \subset P_+$ the set of fundamental weights, that is, $\langle h_i, \varpi_j \rangle = \delta_{ij}$ for $i, j \in I$. Denote by f_i a root vector of $\mathfrak{g}$ corresponding to $-\alpha_i$, by $y_i \colon \mathbb{C} \to N_-, t \mapsto \exp(tf_i)$ the 1-parameter subgroup corresponding to f_i. For $w \in W$, denote by $\ell(w)$ the length of w and by $I(w) \subset I^{\ell(w)}$ the set of the reduced words of w. For $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$, there exists a map $y_{\boldsymbol{i}} \colon (\mathbb{C}^\times)^\ell \to N_-^w$ given by

$$(t_1, \dots, t_\ell) \mapsto \exp(t_1 f_{i_1}) \cdots \exp(t_\ell f_{i_\ell}).$$

Then it is known that $y_{\boldsymbol{i}}$ is a birational map. The problem on finding an explicit description of the inverse birational map $y_{\boldsymbol{i}}^{-1}$ is called *the factorization problem.* By the way, if $n \in N_-^w$ is in the image of $y_{\boldsymbol{i}}$ and $y_{\boldsymbol{i}}^{-1}(n) \in \mathbb{R}_{>0}^\ell$, then n is called a totally positive element in N_-^w. This problem is also formulated in terms of coordinate algebras: the map $y_{\boldsymbol{i}}$ induces an embedding of algebras

$$y_{\boldsymbol{i}}^* \colon \mathbb{C}[N_-^w] \to \mathbb{C}[t_1^{\pm 1}, \dots, t_\ell^{\pm 1}].$$

The problem is to describe explicitly each t_k ($k = 1, \dots, \ell$) as a rational function on N_-^w. Berenstein, Fomin and Zelevinsky solved this problem by using generalized minors and the BFZ-twist automorphisms [**4**], [**6**]. Indeed, this problem is the original motivation for the BFZ-twist automorphisms. The resulting formulae are called *the Chamber Ansatz* *formulae* [**4**, Theorem 1.4], [**6**, Theorem 1.4]. We present the exact quantum analogue of them below. We already have the quantum analogues of the coordinate algebra $\mathbb{C}[N_-^w]$, generalized minors and the BFZ-twist automorphism η_w^*. Actually, there also exists a quantum analogue of the embedding $y_{\boldsymbol{i}}^*\colon \mathbb{C}[N_-^w] \to \mathbb{C}[t_1^{\pm 1}, \dots, t_\ell^{\pm 1}]$, which is known as a Feigin homomorphism. This is an injective $\mathbb{Q}(q)$-algebra homomorphism $\Phi_{\boldsymbol{i}}\colon \mathbf{A}_q[N_-^w] \to \mathcal{L}_{\boldsymbol{i}}$, where $\mathcal{L}_{\boldsymbol{i}}$ is a "non-commutative" Laurent polynomial algebra(=a quantum torus) in ℓ-variables $t_1, \cdots t_\ell$. By using these materials, we obtain the following exact quantum analogues of the Chamber Ansatz formulae.

Theorem (Theorem III.3.6, Corollary III.3.9). *Let $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. For $j = 1, \dots, \ell$, set $w_{\leq j} = s_{i_1} \cdots s_{i_j}$. Then, for $k = 1, \dots, \ell$, we have*

$$(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})([D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}])(=: D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}) \simeq t_1^{-d_1} t_2^{-d_2} \cdots t_k^{-d_k},$$

where $d_j := \langle w_{\leq j} h_{i_j}, w_{\leq k}\varpi_{i_k}\rangle$, $j = 1, \dots, k$. Here $\simeq$ stands for the coincidence up to some powers of q. These formulae are equivalent to the following:

$$t_k \simeq (D'^{(\boldsymbol{i})}_{w_{\leq k-1}\varpi_{i_k}, \varpi_{i_k}})^{-1} (D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}})^{-1} \prod_{j \in I \setminus \{i_k\}} (D'^{(\boldsymbol{i})}_{w_{\leq k}\varpi_j, \varpi_j})^{-a_{j,i_k}},$$

here $a_{ij} := \langle h_i, \alpha_j \rangle$ for $i, j \in I$. Note that the right-hand side is determined up to powers of q.

This is a generalization of Berenstein-Rupel's result [**5**, Corollary 1.2]. By this theorem, we can say that the quantum BFZ-automorhisms in this thesis are generalization of Berenstein-Rupel's quantum twist automorphisms [**5**, Theorem 2.10], which has been constructed in the case that w is a square of a Coxeter element. Hence the quantum BFZ-automorhisms $\eta_{w,q}$ are the ones predicted in [**5**, Conjecture 2.12 (c)]. Moreover, their compatibility with dual canonical bases corresponds to [**5**, Conjecture 2.17 (a)]. We should remark that they treat quantum unipotent cells as subalgebras of quantum Laurent polynomial algebras via Feigin homomorphisms and construct quantum BFZ-twist automorphisms by using quantum cluster algebra structures defined in quantum Laurent polynomial algebras. We hope that this result provides the new interesting tools for the study of quantum cluster algebra structure of quantum unipotent subgroups and quantum unipotent cells. For example, by Feigin homomorphisms and the Chamber Ansatz formulae, we can obtain an expression of an element of $\mathbf{A}_q[N_-^w]$ as a Laurent polynomial in variables $\{\eta_{w,q}^{-1}([D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}]) \mid k = 1, \dots, \ell\}$.

Further questions

In this thesis, we consider quantum analogues of Chamber Ansatz formulae only for unipotent cells. There are the Chamber Ansatz formulae for double Bruhat cells [**14**], and they are the original motivation for FZ-twist maps. The quantum analogue of their Chamber Ansatz formulae may be interesting for the theory of quantum cluster algebra structures on quantum double Bruhat cells. By the way, the quantum cluster algebra structures on quantum double Bruhat cells are obtained in [**22**] and they consider quantum analogues of FZ-twist maps for double Bruhat cells.

It would be also interesting to explain the existence of the quantum FZ-twist maps via categorifications.

Since the quantum BFZ-twist automorphism $\eta_{w,q}$ is an *automorphism*, we can apply $\eta_{w,q}$ on $\mathbf{A}_q[N_-^w]$ repeatedly. Moreover we should remark that the image of a unipotent quantum minor under the quantum BFZ-twist automorphism is not necessarily a unipotent quantum minor. Hence, roughly speaking, we can obtain the "difficult" dual canonical basis elements from the "easy" dual canonical basis elements by iterated application of quantum BFZ-twist automorphisms. Therefore it would be interesting to investigate how "many" dual canonical basis elements are obtained from unipotent quantum minors and their appropriate monomials by this procedure. Theorem III.1.45 is considered as a result related with this investigation.

It is also unclear whether quantum BFZ-twist automorphisms are categorified by using finite dimensional representations of quantum affine algebras or quiver Hecke algebras. In particular, it is unknown that quantum BFZ-twist automorphisms preserve the basis coming from the simple modules of quiver Hecke algebras.

The Chamber Ansatz formulae for quantum unipotent cells state the monomiality of $(\Phi_{\boldsymbol{i}} \circ \eta_{q,w}^{-1})([D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}])$. This is a non-trivial point, and it would be interesting to understand this phenomenon via categorifications. Actually, in non-quantum settings, Geiß-Leclerc-Schröer have obtained an explanation by using their additive categorification [**19**].

CHAPTER I

Preliminaries

In this chapter, we fix our basic notations and summarize known properties of quantized enveloping algebras, quantized coordinate algebras, canonical bases and quantum cluster algebras. In Section I.1, we review the several objects related with quantized enveloping algebras, which are our main subjects. In Section I.2, we prepare the fundamental notions concerning quantum cluster algebras, which are algebraic and combinatorial frameworks for the study of canonical bases.

General notation

The following are general notations in this thesis.

(1) For a vector space V over a field k, set $V^* := \mathrm{Hom}_k(V, k)$. Denote by $\langle\ ,\ \rangle \colon V^* \times V \to k$, $(f, u) \mapsto \langle f, u\rangle$ the canonical pairing.
(2) For a k-algebra $\mathscr{A}$, we set $[a_1, a_2] := a_1a_2 - a_2a_1$ for $a_1, a_2 \in \mathscr{A}$. *An Ore set $\mathscr{M}$ of $\mathscr{A}$* stands for a left and right Ore set consisting of non-zero divisors. Denote by $\mathscr{A}[\mathscr{M}^{-1}]$ the algebra of fractions with respect to the Ore set $\mathscr{M}$. In this case, $\mathscr{A}$ is naturally a subalgebra of $\mathscr{A}[\mathscr{M}^{-1}]$. See [**21**, Chapter 6] for more details.
(3) *An $\mathscr{A}$-module V* means a left $\mathscr{A}$-module. The action of $\mathscr{A}$ on V is denoted by $a.v$ for $a \in \mathscr{A}$ and $v \in V$. In this case, V^* is regarded as a right $\mathscr{A}$-module by $\langle f.a, v\rangle = \langle f, a.v\rangle$ for $f \in V^*$, $a \in \mathscr{A}$ and $v \in V$.
(4) For two letters i, j, the symbol δ_{ij} stands for the Kronecker delta.

I.1. Quantized enveloping algebras

Quantum analogues of Fomin-Zelevinsky twist maps, which will be treated in Chapter II, are the $\mathbb{Q}(q)$-anti-algebra isomorphisms between quantum unipotent subgroups. In subsection I.1.8, we present the definition of quantum unipotent subgroups. Quantum analogues of Berenstein-Fomin-Zelevinsky twist maps, which will be dealt with in Chapter III, are $\mathbb{Q}(q)$-algebra automorphisms on quantum unipotent cells. In subsection I.1.9, we present the definition of "closed version" of quantum unipotent cells. To obtain the "non-closed version", we consider localizations of these objects, which will be presented in Chapter III.

I.1.1. Lie theoretic setups.

Definition I.1.1. A root datum consists of the following data;

(1) I : a finite index set,
(2) $\mathfrak{h}$: a finite dimensional $\mathbb{Q}$-vector space,

(3) $P \subset \mathfrak{h}^*$: a lattice, called weight lattice,
(4) $P^* = \{h \in \mathfrak{h} \mid \langle h, P\rangle \subset \mathbb{Z}\}$ with the canonical pairing $\langle\ ,\ \rangle : P^* \times P \to \mathbb{Z}$, called the coweight lattice,
(5) $\{\alpha_i\}_{i\in I} \subset P$: a subset, called the set of simple roots,
(6) $\{h_i\}_{i\in I} \subset P^*$: a subset, called the set of simple coroots,
(7) $(\ ,\ ) : P \times P \to \mathbb{Q}$: a $\mathbb{Q}$-valued symmetric $\mathbb{Z}$-bilinear form on P.

satisfying the following conditions:

(a) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ for $i \in I$,
(b) $\langle h_i, \lambda\rangle = 2\,(\alpha_i, \lambda)\,/\,(\alpha_i, \alpha_i)$ for $\lambda \in P$ and $i \in I$,
(c) $A = (a_{ij})_{i,j\in I} := (\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a symmetrizable generalized Cartan matrix, that is $\langle h_i, \alpha_i\rangle = 2$, $\langle h_i, \alpha_j\rangle \in \mathbb{Z}_{\leq 0}$ for $i \neq j$ and, $\langle h_i, \alpha_j\rangle = 0$ is equivalent to $\langle h_j, \alpha_i\rangle = 0$,
(d) $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$, $\{h_i\}_{i\in I} \subset \mathfrak{h}$ are linearly independent subsets.

The $\mathbb{Z}$-submodule $Q = \sum_{i\in I} \mathbb{Z}\alpha_i \subset P$ is called the root lattice, $Q^\vee = \sum_{i\in I} \mathbb{Z}h_i \subset P^*$ is called the coroot lattice. We set $Q_+ = \sum \mathbb{Z}_{\geq 0}\alpha_i \subset Q$ and $Q_- = -Q_+$. For $\alpha = \sum_{i\in I} m_i\alpha_i \in Q$, we set $\mathrm{ht}\,(\alpha) = \sum_{i\in I} m_i \in \mathbb{Z}$.

Let $P_+ := \{\lambda \in P \mid \langle h_i, \lambda\rangle \in \mathbb{Z}_{\geq 0}$ for all $i \in I\}$ and we assume that there exists $\{\varpi_i\}_{i\in I} \subset P_+$ such that $\langle h_i, \varpi_j\rangle = \delta_{ij}$. An element of P_+ is called a dominant integral weight. Set $\rho := \sum_{i\in I} \varpi_i \in P_+$. Then $\langle h_i, \rho\rangle = 1$ and $(\alpha_i, \rho) = (\alpha_i, \alpha_i)/2$ for all $i \in I$.

Fix elements $\{\varpi_i^\vee\}_{i\in I} \subset \mathfrak{h}$ such that $\langle \varpi_i^\vee, \alpha_j\rangle = \delta_{ij}$ for $i, j \in I$. We do not assume that $\{\varpi_i^\vee\}_{i\in I} \subset P^*$.

Definition I.1.2. Let W be the Weyl group associated with the above root datum, that is, the group generated by $\{s_i\}_{i\in I}$ with the defining relations $s_i^2 = e$ for $i \in I$ and $(s_is_j)^{m_{ij}} = e$ for $i, j \in I$, $i \neq j$. Here e is the unit of W, $m_{ij} = 2$ (resp. $3, 4, 6, \infty$) if $a_{ij}a_{ji} = 0$ (resp. $1, 2, 3, \geq 4$), and $w^\infty := e$ for any $w \in W$. We have the group homomorphisms $W \to \mathrm{Aut}\,\mathfrak{h}$ and $W \to \mathrm{Aut}\,\mathfrak{h}^*$ given by

$$s_i\,(h) = h - \langle h, \alpha_i\rangle h_i \qquad\qquad s_i\,(\mu) = \mu - \langle h_i, \mu\rangle\alpha_i$$

for $h \in \mathfrak{h}$ and $\mu \in \mathfrak{h}^*$. For an element w of W, $\ell(w)$ denotes the length of w, that is, the smallest integer ℓ such that there exist $i_1, \ldots, i_\ell \in I$ with $w = s_{i_1} \cdots s_{i_\ell}$. For $w \in W$, set

$$I(w) := \{\boldsymbol{i} = (i_1, \ldots, i_{\ell(w)}) \in I^{\ell(w)} \mid w = s_{i_1} \cdots s_{i_{\ell(w)}}\}.$$

Here we set $I(e) := \{\emptyset\}$. An element of $I(w)$ is called a reduced word of w.

I.1.2. Notations for non-quantum objects. In this thesis, we do not deal with specializations of quantum objects to non-quantum objects. Nevertheless, when describing quantum objects, we use notations of non-quantum objects to clarify non-quantum counterparts. Hence we explain them here. However, since we do not need those objects logically, we drop their precise definitions. See, for example, [**37**] for precise definitions and specializations of quantum objects.

Let $\mathfrak{g} := \mathfrak{g}(A)$ be the Kac-Moody Lie algebra associate with the symmetrizable generalized Cartan matrix A, and G a corresponding Kac-Moody group. Denote by Φ_+ the set of positive roots of $\mathfrak{g}$ and set $\mathfrak{n}_\pm := \bigoplus_{\beta \in \Phi_+} \mathfrak{g}_{\pm\beta}$, $\hat{\mathfrak{n}}_\pm := \prod_{\beta \in \Phi_+} \mathfrak{g}_{\pm\beta}$. Here $\mathfrak{g}_{\pm\beta}$ denotes the root space of $\pm\beta$. Let $N_\pm (\subset G)$ be the pro-unipotent pro-group whose pro-nilpotent pro-Lie algebra is $\hat{\mathfrak{n}}_\pm$. Let $H(\subset G)$ the algebraic torus whose character lattice is P. Write $B_\pm := HN_\pm$. Set

$$\begin{aligned} G_0 &:= N_- H N_+ \\ N_-(w) &:= N_- \cap \dot{w} N_+ \dot{w}^{-1} \\ N_-^w &:= N_- \cap B_+ \dot{w} B_+ \\ X(w) &:= \bigcup_{w' \in W; w' \leq w} B_+ \dot{w}' B_+ / B_+ \end{aligned}$$

for $w \in W$. Here $\dot{w}''$ is an arbitrary lift of $w'' \in W$ to G, and $\leq$ denotes the Bruhat order on W. Then $N_-(w)$ (resp. N_-^w, $X(w)$) is called the *unipotent subgroup* (resp. the *unipotent cell*, the *Schubert variety*) associated with w. In this thesis, we deal with quantum analogues of coordinate algebras of these objects and intersections of some of them.

I.1.3. Quantized enveloping algebras. In this subsection, we present the definitions of quantized enveloping algebras and their variants.

NOTATION I.1.3. Let q be an indeterminate. Set

$$\begin{aligned} &q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}, \ [n] := \frac{q^n - q^{-n}}{q - q^{-1}} \text{ for } n \in \mathbb{Z}, \\ &\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \dfrac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[1]} & \text{if } n \in \mathbb{Z}, k \in \mathbb{Z}_{>0}, \\ 1 & \text{if } n \in \mathbb{Z}, k = 0, \end{cases} \\ &[n]! := [n][n-1]\cdots[1] \text{ for } n \in \mathbb{Z}_{>0}, [0]! := 1. \end{aligned}$$

Note that $[n], \begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{Z}[q^{\pm 1}]$ and $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{[n]!}{[k]![n-k]!}$ if $n \geq k \geq 0$. For a rational function $R \in \mathbb{Q}(q)$, we define R_i as the rational function obtained from X by substituting q by q_i, $i \in I$.

DEFINITION I.1.4. The quantized enveloping algebra $\mathbf{U}_q (:= \mathbf{U}_q(\mathfrak{g}))$ is the unital associative $\mathbb{Q}(q)$-algebra (associated with $(P, I, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I}, (\ ,\ ))$) defined by the generators

$$e_i, f_i \ (i \in I), q^h \ (h \in P^*),$$

and the relations (i)–(iv) below:

(i) $q^0 = 1, \ q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^*$,

(ii) $q^h e_i = q^{\langle h, \alpha_i \rangle} e_i q^h, \ q^h f_i = q^{-\langle h, \alpha_i \rangle} f_i q^h$ for $h \in P^*, i \in I$,

(iii) $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$ where $t_i := q^{\frac{(\alpha_i, \alpha_i)}{2} h_i}$,

(iv) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i x_i^k x_j x_i^{1-a_{ij}-k} = 0$ for $i, j \in I$ with $i \neq j$, and $x = e, f$.

The $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q$ generated by $\{e_i\}_{i\in I}$ (resp. $\{f_i\}_{i\in I}$, $\{q^h\}_{h\in P^*}$, $\{e_i, q^h\}_{i\in I, h\in P^*}$, $\{f_i, q^h\}_{i\in I, h\in P^*}$) will be denoted by $\mathbf{U}_q^+$ (resp. $\mathbf{U}_q^-$, $\mathbf{U}_q^0$, $\mathbf{U}_q^{\geq 0}$, $\mathbf{U}_q^{\leq 0}$). For $\alpha = \sum_{i\in I} m_i\alpha_i \in Q$, $m_i \in \mathbb{Z}$, we set $t_\alpha := \prod_{i\in I} t_i^{m_i}$. In particular, $t_{\alpha_i} = t_i$ for $i \in I$.

For $\alpha \in Q$, write $(\mathbf{U}_q)_\alpha := \{x \in \mathbf{U}_q \mid q^h x q^{-h} = q^{\langle \alpha, h\rangle} x \text{ for all } h \in P^*\}$. The elements of $(\mathbf{U}_q)_\alpha$ are said to be homogeneous. For a homogeneous element $x \in (\mathbf{U}_q)_\alpha$, we set $\operatorname{wt} x = \alpha$. For any subset $X \subset \mathbf{U}_q$ and $\alpha \in Q$, we set $X_\alpha := X \cap (\mathbf{U}_q)_\alpha$.

The algebra $\mathbf{U}_q$ has a Hopf algebra structure. In this thesis, we take the coproduct $\Delta\colon \mathbf{U}_q \to \mathbf{U}_q \otimes \mathbf{U}_q$, the counit $\varepsilon\colon \mathbf{U}_q \to \mathbb{Q}(q)$ and the antipode $S\colon \mathbf{U}_q \to \mathbf{U}_q$ as follows:

$$\begin{aligned}
\Delta(e_i) &= e_i \otimes t_i^{-1} + 1 \otimes e_i, & \varepsilon(e_i) &= 0, & S(e_i) &= -e_i t_i,\\
\Delta(f_i) &= f_i \otimes 1 + t_i \otimes f_i, & \varepsilon(f_i) &= 0, & S(f_i) &= -t_i^{-1} f_i,\\
\Delta\left(q^h\right) &= q^h \otimes q^h, & \varepsilon\left(q^h\right) &= 1, & S\left(q^h\right) &= q^{-h}.
\end{aligned}$$

for $i \in I$, $h \in P^*$.

For $i \in I$, define $\mathbf{U}_{q,i}$ as the Hopf subalgebra of $\mathbf{U}_q$ generated by $\{e_i, f_i, t_i^{\pm 1}\}$. Denote by $\iota_i : \mathbf{U}_{q,i} \hookrightarrow \mathbf{U}_q$ the natural inclusion of a Hopf algebra.

Definition I.1.5. Let $\vee\colon \mathbf{U}_q \to \mathbf{U}_q$ be the $\mathbb{Q}(q)$-algebra, anti-coalgebra involution defined by

$$e_i^\vee = f_i, \qquad f_i^\vee = e_i, \qquad \left(q^h\right)^\vee = q^{-h}.$$

Let $^-\colon \mathbb{Q}(q) \to \mathbb{Q}(q)$, $^-\colon \mathbf{U}_q \to \mathbf{U}_q$ be the $\mathbb{Q}$-algebra involutions defined by

$$\overline{q} = q^{-1}, \qquad \overline{e_i} = e_i, \qquad \overline{f_i} = f_i, \qquad \overline{q^h} = q^{-h}.$$

Let $*, \varphi, \psi\colon \mathbf{U}_q \to \mathbf{U}_q$ be the $\mathbb{Q}(q)$-anti-algebra involutions defined by

$$\begin{aligned}
*(e_i) &= e_i, & *(f_i) &= f_i, & *\left(q^h\right) &= q^{-h},\\
\varphi(e_i) &= f_i, & \varphi(f_i) &= e_i, & \varphi\left(q^h\right) &= q^h\\
\psi(e_i) &= q_i^{-1} t_i^{-1} f_i, & \psi(f_i) &= q_i^{-1} t_i e_i, & \psi\left(q^h\right) &= q^h.
\end{aligned}$$

Remark that ψ is also a $\mathbb{Q}(q)$-coalgebra homomorphism, and $\varphi = \vee \circ * = * \circ \vee$.

Definition I.1.6. For $i \in I$, define the $\mathbb{Q}(q)$-linear maps e_i' and ${}_ie'\colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ by

$$\begin{aligned}
e_i'(xy) &= e_i'(x)\, y + q_i^{\langle h_i, \operatorname{wt} x\rangle} x e_i'(y), & e_i'(f_j) &= \delta_{ij},\\
{}_ie'(xy) &= q_i^{\langle h_i, \operatorname{wt} y\rangle}\, {}_ie'(x)\, y + x\, {}_ie'(y), & {}_ie'(f_j) &= \delta_{ij}
\end{aligned}$$

for homogeneous elements $x, y \in \mathbf{U}_q^-$. For $i \in I$, define the $\mathbb{Q}(q)$-linear maps f_i' and ${}_if'\colon \mathbf{U}_q^+ \to \mathbf{U}_q^+$ by

$$\begin{aligned}
f_i'(xy) &= f_i'(x)\, y + q_i^{-\langle h_i, \operatorname{wt} x\rangle} x f_i'(y), & f_i'(e_j) &= \delta_{ij},\\
{}_if'(xy) &= q_i^{-\langle h_i, \operatorname{wt} y\rangle}\, {}_if'(x)\, y + x\, {}_if'(y), & {}_if'(e_j) &= \delta_{ij}
\end{aligned}$$

for homogeneous elements $x, y \in \mathbf{U}_q^+$. We have $* \circ {}_ie' \circ *|_{\mathbf{U}_q^-} = e_i'$ and $* \circ {}_if' \circ *|_{\mathbf{U}_q^+} = f_i'$. Moreover $\vee \circ f_i' \circ \vee|_{\mathbf{U}_q^-} = e_i'$ and $\vee \circ {}_if' \circ \vee|_{\mathbf{U}_q^+} = {}_ie'$.

In this thesis, we use the following variant $\check{\mathbf{U}}_q$ of the quantized enveloping algebra $\mathbf{U}_q$.

Definition I.1.7. A variant $\check{\mathbf{U}}_q$ of the quantized enveloping algebra $\mathbf{U}_q$ is the unital associative $\mathbb{Q}(q)$-algebra defined by the generators

$$e_i, f_i\ (i \in I), q^{\mu}\ (\mu \in P),$$

and the relations (i)–(iv) below:

(i) $q^0 = 1,\ q^{\mu}q^{\mu'} = q^{\mu+\mu'}$ for $\mu, \mu' \in P$,

(ii) $q^{\mu}e_i = q^{(\mu,\alpha_i)}e_i q^{\mu},\ q^{\mu}f_i = q^{-(\mu,\alpha_i)}f_i q^{\mu}$ for $\mu \in P, i \in I$,

(iii) $[e_i, f_j] = \delta_{ij}\dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$ where $t_i := q^{\alpha_i}$ (abuse of notation),

(iv) $\displaystyle\sum_{k=0}^{1-a_{ij}}(-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i x_i^k x_j x_i^{1-a_{ij}-k} = 0$ for $i, j \in I$ with $i \neq j$, and $x = e, f$.

The $\mathbb{Q}(q)$-algebra $\check{\mathbf{U}}_q$ has a Hopf algebra structure given by the same formulae as $\mathbf{U}_q$. The notions, notations and maps defined in Definition I.1.4 and I.1.5 are immediately translated into those for $\check{\mathbf{U}}_q$. Note that $\check{\mathbf{U}}_q^{\pm}$ can be identified with $\mathbf{U}_q^{\pm}$ respectively in an obvious way.

Notation I.1.8. Set $x_i^{(n)} := x_i^n/[n]_i! \in \mathbf{U}_q$ for $i \in I$, $n \in \mathbb{Z}_{\geq 0}$ and $x = e, f$.

I.1.4. Drinfeld pairings and Lusztig pairings. Some nondegenerate bilinear forms play a role of bridges between quantized enveloping algebras and their dual objects. The maps in Definition I.1.16 are important for our study of quantized coordinate algebras.

Proposition I.1.9 ([**12**], [**47**]). *There uniquely exists a $\mathbb{Q}(q)$-bilinear map $(\ ,\ )_D \colon \check{\mathbf{U}}_q^{\geq 0} \times \mathbf{U}_q^{\leq 0} \to \mathbb{Q}(q)$ such that*

(i) $(\Delta(x), y_1 \otimes y_2)_D = (x, y_1 y_2)_D$ *for* $x \in \check{\mathbf{U}}_q^{\geq 0}, y_1, y_2 \in \mathbf{U}_q^{\leq 0}$,

(ii) $(x_2 \otimes x_1, \Delta(y))_D = (x_1 x_2, y)_D$ *for* $x_1, x_2 \in \check{\mathbf{U}}_q^{\geq 0}, y \in \mathbf{U}_q^{\leq 0}$,

(iii) $(e_i, q^h)_D = (q^{\lambda}, f_i)_D = 0$ *for* $i \in I$ *and* $h \in P^*,\ \lambda \in P$,

(iv) $(q^{\lambda}, q^h)_D = q^{-\langle \lambda, h\rangle}$ *for* $\lambda \in P, h \in P^*$,

(v) $(e_i, f_j)_D = -\delta_{ij}\dfrac{1}{q_i - q_i^{-1}}$ *for* $i, j \in I$.

This bilinear form is called the Drinfeld pairing. It has the following properties:

(1) For $\alpha, \beta \in Q_+$, $(\ ,\ )_D\,|_{(\check{\mathbf{U}}_q^{\geq 0})_{\alpha} \times (\mathbf{U}_q^{\leq 0})_{-\beta}} = 0$ unless $\alpha = \beta$.

(2) $(\ ,\ )_D|_{(\mathbf{U}_q^{+})_{\alpha} \times (\mathbf{U}_q^{-})_{-\alpha}}$ is nondegenerate.

(3) $(q^{\lambda}x, q^h y)_D = q^{-\langle \lambda, h\rangle}(x, y)_D$ for $\lambda \in P,\ h \in P^*$ and $x \in \mathbf{U}_q^{+}, y \in \mathbf{U}_q^{-}$.

Definition I.1.10. Define the $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_L \colon \mathbf{U}_q^{-} \times \mathbf{U}_q^{-} \to \mathbb{Q}(q)$ by $(x, y)_L := (\psi(x), y)_D$ for $x, y \in \mathbf{U}_q^{-}$. Then this bilinear form satisfies

$$(1,1)_L = 1, \qquad (f_i x, y)_L = \frac{1}{1 - q_i^2}(x, e_i'(y))_L, \qquad (x f_i, y)_L = \frac{1}{1 - q_i^2}(x, {}_i e'(y))_L.$$

This is a symmetric bilinear form, called the Lusztig pairing. The bilinear form $(\ ,\ )_L$ is the unique symmetric $\mathbb{Q}(q)$-bilinear form satisfying the properties above. Moreover, $(\ ,\ )_L$ is nondegenerate and has the following property:

$$(*(x), *(y))_L = (x, y)_L \tag{I.1}$$

for all $x, y \in \mathbf{U}_q^-$.

Define the $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_L^+ \colon \mathbf{U}_q^+ \times \mathbf{U}_q^+ \to \mathbb{Q}(q)$ by $(x, y)_L^+ := (x, \psi(y))_D$ for $x, y \in \mathbf{U}_q^+$. Then this bilinear form satisfies

$$(1,1)_L^+ = 1, \qquad (e_i x, y)_L^+ = \frac{1}{1-q_i^2}(x, f_i'(y))_L^+, \qquad (xe_i, y)_L^+ = \frac{1}{1-q_i^2}(x, {}_if'(y))_L^+.$$

The forms $(\ ,\ )_L$ and $(\ ,\ )_L^+$ are related as follows:

$$(x, y)_L = (x^\vee, y^\vee)_L^+ \tag{I.2}$$

for all $x, y \in \mathbf{U}_q^-$.

The following dual bar involution is useful for the study of dual canonical bases. We also prepare the twisted dual bar involution, which is compatible with the algebra structure of $\mathbf{U}_q^-$.

Definition I.1.11. For a homogeneous $x \in \mathbf{U}_q^-$, we define $\sigma(x) = \sigma_L(x) \in \mathbf{U}_q^-$ by the property that

$$(\sigma(x), y)_L = \overline{(x, \overline{y})_L}$$

for an arbitrary $y \in \mathbf{U}_q^-$. By the nondegeneracy of $(\ ,\ )_L$, the element $\sigma(x)$ is well-defined. This map $\sigma \colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ is called the dual bar involution.

The following proposition can be proved in the same manner as [**34**, Proposition 3.2].

Proposition I.1.12. *For a homogeneous element $x \in \mathbf{U}_q^-$, we have*

$$\sigma(x) = (-1)^{\mathrm{ht}(\mathrm{wt}\, x)} q^{(\mathrm{wt}\, x, \mathrm{wt}\, x)/2 - (\mathrm{wt}\, x, \rho)} (^- \circ *)(x).$$

In particular, for homogeneous elements $x, y \in \mathbf{U}_q^-$, we have

$$\sigma(xy) = q^{(\mathrm{wt}\, x, \mathrm{wt}\, y)} \sigma(y)\sigma(x).$$

Definition I.1.13. Define a $\mathbb{Q}(q)$-linear isomorphism $c_{\mathrm{tw}} \colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ by

$$x \mapsto q^{(\mathrm{wt}\, x, \mathrm{wt}\, x)/2 - (\mathrm{wt}\, x, \rho)} x$$

for every homogeneous element $x \in \mathbf{U}_q^-$. Set $\sigma' := c_{\mathrm{tw}}^{-1} \circ \sigma \colon \mathbf{U}_q^- \to \mathbf{U}_q^-$. We call σ' the twisted dual bar involution. By Proposition I.1.12, $\sigma'(x) = (-1)^{\mathrm{ht}(\mathrm{wt}\, x)} (^- \circ *)(x)$ for every homogeneous element $x \in \mathbf{U}_q^-$. In particular, σ' is a $\mathbb{Q}$-anti-algebra involution.

Remark I.1.14. Let $x \in \mathbf{U}_q^-$ be a homogeneous element. Then,

$$\sigma(x) = x \text{ if and only if } \sigma'(x) = q^{-(\mathrm{wt}\, x, \mathrm{wt}\, x)/2 + (\mathrm{wt}\, x, \rho)} x.$$

We prepare a convenient lemma here. See also Definition I.1.16.

LEMMA I.1.15. *For $\mu \in P$, $h \in P^*$, $y_1, y_2 \in \mathbf{U}_q^-$ and $x_1, x_2 \in \mathbf{U}_q^+$, we have*

$$(\psi(y_1 q^\mu), y_2 q^h)_D = q^{-\langle \mu, h\rangle}(y_1, y_2)_L, \qquad (x_1 q^\mu, \psi(x_2 q^h))_D = q^{-\langle \mu, h\rangle}(x_1, x_2)_L^+.$$

Proof. We have

$$\begin{aligned}(\psi(y_1 q^\mu), y_2 q^h)_D &= (q^\mu \psi(y_1), y_2 q^h)_D \\ &= (\psi(y_1) \otimes q^\mu, \Delta(y_2)\Delta(q^h))_D \\ &= (\psi(y_1) \otimes q^\mu, y_2 q^h \otimes q^h)_D \\ &= q^{-\langle \mu, h\rangle}(\psi(y_1), y_2 q^h)_D \\ &= q^{-\langle \mu, h\rangle}(\psi(y_1), y_2 q^h)_D \\ &= q^{-\langle \mu, h\rangle}((\psi \otimes \psi)(\Delta(y_1)), y_2 \otimes q^h)_D \\ &= q^{-\langle \mu, h\rangle}((\psi \otimes \psi)(y_1 \otimes 1), y_2 \otimes q^h)_D = q^{-\langle \mu, h\rangle}(y_1, y_2)_L.\end{aligned}$$

The second equality is proved in the same manner. □

The following maps connect the algebra structures of (half) quantized enveloping algebras with those of their dual spaces. Note that the dual space of coalgebras have algebra structures dual to their coalgebra structures.

DEFINITION I.1.16. Define the following linear maps:

$$\begin{aligned}\Phi \colon \check{\mathbf{U}}_q^{\leq 0} &\to (\mathbf{U}_q^{\leq 0})^*,\, y_1 \mapsto (y_2 \mapsto (\psi(y_1), y_2)_D)\,, \\ \Phi^+ \colon \check{\mathbf{U}}_q^{\geq 0} &\to (\mathbf{U}_q^{\geq 0})^*,\, x_1 \mapsto (x_2 \mapsto (x_1, \psi(x_2))_D)\,.\end{aligned}$$

By the properties of the Drinfeld pairing, Φ is an injective algebra homomorphism and Φ^+ is an injective anti-algebra homomorphism.

I.1.5. Quantized coordinate algebras. We use quantized coordinate algebras associated with $\mathbf{U}_q$ and their subalgebras when constructing quantum analogues of the Berenstein-Fomin-Zelevinsky automorphisms in Chapter III. We begin with preparation of the terminologies for representations of $\mathbf{U}_q$.

DEFINITION I.1.17. Let V be a left (resp. right) $\mathbf{U}_q$-module. For $\mu \in P$, we set

$$V_\mu := \{u \in V \mid q^h.u = q^{\langle h, \mu\rangle} u \text{ (resp. } u.q^h = q^{\langle h, \mu\rangle} u) \text{ for all } h \in P^*\}.$$

This is called the weight space of V of weight μ, and for $u \in V_\mu$, we write $\operatorname{wt} u := \mu$. For a $\mathbf{U}_q$-module $V = \bigoplus_{\mu \in P} V_\mu$ with weight space decomposition, its graded dual $\bigoplus_{\mu \in P} V_\mu^*$ is denoted by $V^\star$. Recall that $V^\star$ is a right $\mathbf{U}_q$-module.

DEFINITION I.1.18. A left (resp. right) $\mathbf{U}_q$-module V is said to be integrable if

- V has the weight space decomposition $V = \bigoplus_{\mu \in P} V_\mu$, and
- for every $i \in I$, the actions of e_i and f_i on V are locally nilpotent, or equivalently $\dim_{\mathbb{Q}(q)} \mathbf{U}_{q,i}.u < \infty$ (resp. $\dim_{\mathbb{Q}(q)} u.\mathbf{U}_{q,i} < \infty$) for all $u \in V$.

Let $\lambda \in P_+$. The integrable irreducible $\mathbf{U}_q$-module with highest (resp. lowest) weight λ (resp. $-\lambda$) is denoted by $V(\lambda)$ (resp. $V(-\lambda)$). We fix its highest (resp. lowest) weight vector u_λ (resp. $u_{-\lambda}$). For $w \in W$, define the elements $u_{\pm w\lambda} \in V(\pm\lambda)$ by

$$u_{w\lambda} = f_{i_1}^{(\langle h_{i_1}, s_{i_2} \cdots s_{i_\ell}\lambda\rangle)} \cdots f_{i_{\ell-1}}^{(\langle h_{i_{\ell-1}}, s_{i_\ell}\lambda\rangle)} f_{i_\ell}^{(\langle h_{i_\ell}, \lambda\rangle)}.u_\lambda$$
$$u_{-w\lambda} = e_{i_1}^{(\langle h_{i_1}, s_{i_2} \cdots s_{i_\ell}\lambda\rangle)} \cdots e_{i_{\ell-1}}^{(\langle h_{i_{\ell-1}}, s_{i_\ell}\lambda\rangle)} e_{i_\ell}^{(\langle h_{i_\ell}, \lambda\rangle)}.u_{-\lambda}$$

for $(i_1, \dots, i_\ell) \in I(w)$. It is known that these elements do not depend on the choice of $(i_1, \dots, i_\ell) \in I(w)$ and $w \in W$. See, for example, [**44**, Proposition 39.3.7]. These vectors $u_{\pm w\lambda}$ are called the extremal weight vectors of weight $\pm w\lambda$.

Definition I.1.19. Let $\lambda \in P_+ \cup (-P_+)$. Then there exists a unique nondegenerate symmetric $\mathbb{Q}(q)$-bilinear form $(\ ,\ )_\lambda^\varphi$ on $V(\lambda)$ such that

$$(u_\lambda, u_\lambda)_\lambda^\varphi = 1 \qquad\qquad (x.u, u')_\lambda^\varphi = (u, \varphi(x).u')_\lambda^\varphi$$

for $u, u' \in V(\lambda)$ and $x \in \mathbf{U}_q$. For $u \in V(\lambda)$, we set $u^* := (u' \mapsto (u, u')_\lambda^\varphi) \in V(\lambda)^*$. Note that $(u_{w\lambda}, u_{w\lambda})_\lambda^\varphi = 1$ for all $w \in W$. Set $f_{w\lambda} := u_{w\lambda}^* \in V(\lambda)$. Note that $V(\lambda)^\star = \{u^* \mid u \in V(\lambda)\}$.

Definition I.1.20. Let V be a $\mathbf{U}_q$-module. For $f \in V^*$ and $u \in V$, define the element $c_{f,u}^V \in \mathbf{U}_q^*$ by

$$x \mapsto \langle f, x.u\rangle$$

for $x \in \mathbf{U}_q$. An element of this form is called a (quantum) matrix coefficient. For $\lambda \in P$, a matrix coefficient $c_{f,u}^{V(\lambda)}$ will be briefly denoted by $c_{f,u}^\lambda$. Moreover, for $w, w' \in W$ and $\lambda \in P_+$, we write

$$c_{w\lambda, w'\lambda}^\lambda := c_{f_{w\lambda}, u_{w'\lambda}}^\lambda.$$

Definition I.1.21. The quantized coordinate algebra $\mathbf{R}_q := \mathbf{R}_q(\mathfrak{g})$ associated with $\mathbf{U}_q$ is the $\mathbb{Q}(q)$-vector subspace of $\mathbf{U}_q^*$ spanned by the elements

$$\left\{c_{f,u}^\lambda \mid f \in V(\lambda)^\star, u \in V(\lambda) \text{ and } \lambda \in P_+\right\}.$$

Then $\mathbf{R}_q$ is a subalgebra of $\mathbf{U}_q^*$, and isomorphic to $\bigoplus_{\lambda \in P_+} V(\lambda)^\star \otimes V(\lambda)$ as a $\mathbf{U}_q$-bimodule [**33**, Chapter 7]. This isomorphism is known as the Peter-Weyl isomorphism. Here the $\mathbf{U}_q$-bimodule structure on $\mathbf{U}_q^*$ (and $\mathbf{R}_q$) is given by $\langle (x.F.y), z\rangle = \langle F, yzx\rangle$ for $F \in \mathbf{U}_q^*$ and $x, y, z \in \mathbf{U}_q$. Recall that the algebra structure of $\mathbf{U}_q^*$ is induced from the coalgebra structure of $\mathbf{U}_q$.

Example I.1.22. In type A case, the quantized coordinate algebra $\mathbf{R}_q(\mathfrak{sl}_n)$ is isomorphic to the unital associative $\mathbb{Q}(q)$-algebra with the generators $\{c_{ij}\}_{i,j=1,\dots,n}$ and the follwing defining relations:

(i) $c_{ij}c_{ik} = qc_{ik}c_{ij}$ if $j < k$,
(ii) $c_{ij}c_{kj} = qc_{kj}c_{ij}$ if $i < k$,
(iii) $[c_{ij}, c_{k\ell}] = 0$ if $i < k, j > \ell$,
(iv) $c_{ij}c_{k\ell} - c_{k\ell}c_{ij} = (q - q^{-1})c_{i\ell}c_{kj}$ if $i < k, j < \ell$,
(v) $\sum_{\tau \in \mathfrak{S}_n} (-q)^{\ell(\tau)} c_{1\tau(1)}c_{2\tau(2)} \cdots c_{n\tau(n)} = 1$, here $\mathfrak{S}_n$ is the symmetric group of degree n.

In this case, we can identify the index set I of simple roots with $\{1, \ldots, n-1\}$ in a natural way, and then $V(\varpi_1)$ is n-dimensional module. Write $u_k := f_{k-1} \ldots f_1.u_{\varpi_1} (\neq 0)$ for $k = 1, \ldots, n$. Then the c_{ij} $(i, j = 1, \ldots, n)$ above corresponds to the matrix coefficient $c^{\varpi_i}_{u_i,u_j}$ in the Peter-Weyl type presentation. The relation (v) is nothing but the quantum analogue of the condition that determinants are equal to 1. Hence $\mathbf{R}_q(\mathfrak{sl}_n)$ is a quantum analogue of the coordinate algebra of SL_n.

Here we define some subalgebras of the quantized coordinate algebra $\mathbf{R}_q$. See, for instance, [**26**, Chapter 9, 10], [**48**, Chapter 3].

Definition I.1.23. Let $w, w' \in W$ and $\lambda \in P_+$. Set

$$\mathbf{R}^{w(+)}_q(\lambda) := \{c^{\lambda}_{f,u_{w\lambda}} \mid f \in V(\lambda)^{\star}\} \qquad \mathbf{R}^{w(+)}_q := \sum_{\lambda' \in P_+} \mathbf{R}^{w(+)}_q(\lambda') \subset \mathbf{R}_q,$$

$$\mathbf{Q}^{w(+)}_{w'}(\lambda) := \{c^{\lambda}_{f,u_{w\lambda}} \mid f \in V(\lambda)^{\star},\ \langle f, \mathbf{U}^+_q.u_{w'\lambda}\rangle = 0\} \quad \mathbf{Q}^{w(+)}_{w'} := \sum_{\lambda' \in P_+} \mathbf{Q}^{w(+)}_{w'}(\lambda') \subset \mathbf{R}_q.$$

When $w = e$, we write $\mathbf{R}^{e(+)}_q$ (resp. $\mathbf{Q}^{e(+)}_{w'}$) as $\mathbf{R}^+_q$ (resp. $\mathbf{Q}^+_{w'}$). It is easy to show that, for all $w \in W$, $\mathbf{R}^{w(+)}_q$ is a subalgebra of $\mathbf{R}_q$, and isomorphic to $\mathbf{R}^+_q$ as $\mathbb{Q}(q)$-algebras via $c^{\lambda}_{f,u_{w\lambda}} \mapsto c^{\lambda}_{f,u_{\lambda}}$. Moreover, for $w', w \in W$, $\mathbf{Q}^{w(+)}_{w'}$ is a two-sided ideal of $\mathbf{R}^{w(+)}_q$, and the previous isomorphism induces the isomorphism from $\mathbf{R}^{w(+)}_q/\mathbf{Q}^{w(+)}_{w'}$ to $\mathbf{R}^+_q/\mathbf{Q}^+_{w'}$.

Definition I.1.24. The restriction map $R_{\geq 0}\colon \mathbf{U}^*_q \to (\mathbf{U}^{\geq 0}_q)^*$ (resp. $R_{\leq 0}\colon \mathbf{U}^*_q \to (\mathbf{U}^{\leq 0}_q)^*$) induces the $\mathbb{Q}(q)$-algebra homomorphism $R_{\geq 0}\colon \mathbf{R}_q \to (\mathbf{U}^{\geq 0}_q)^*$ (resp. $R_{\leq 0}\colon \mathbf{R}_q \to (\mathbf{U}^{\leq 0}_q)^*$).

We prepare a lemma for the definition of the map Ψ_- in Definition I.1.26. This is the argument in [**26**, Proposition 9.2.11].

Lemma I.1.25. *Let $F \in \mathbf{U}^*_q$ be an element such that $F.q^h = q^{\langle\lambda,h\rangle}F$ and $q^h.F = q^{\langle\mu,h\rangle}F$ for all $h \in P^*$ and some $\lambda, \mu \in P$. Then we have*

$$R_{\leq 0}(F) \in \Phi(\check{\mathbf{U}}^{\leq 0}_q).$$

In particular, $R_{\leq 0}(\mathbf{R}_q) \subset \Phi(\check{\mathbf{U}}^{\leq 0}_q)$.

Proof. For a homogeneous element $x \in \mathbf{U}^-_q$, we have $\langle F, x\rangle = 0$ unless $\mathrm{wt}\, x = \lambda - \mu$. Since $(\ ,\ )_L \mid_{(\mathbf{U}^-_q)_{\lambda-\mu} \times (\mathbf{U}^-_q)_{\lambda-\mu}}$ is nondegenerate, there uniquely exists $x_F \in (\mathbf{U}^-_q)_{\lambda-\mu}$ such that $(x_F, x)_L = \langle F, x\rangle$ for all $x \in (\mathbf{U}^-_q)_{\lambda-\mu}$. Then, by Lemma I.1.15,

$$\langle R_{\leq 0}(F), xq^h\rangle = q^{\langle\mu,h\rangle}\langle F, x\rangle = q^{\langle\mu,h\rangle}(x_F, x)_L = \langle \Phi(x_F q^{-\mu}), xq^h\rangle$$

for $h \in P^*$ and $x \in \mathbf{U}^-_q$. This proves the lemma. □

Definition I.1.26. By Lemma I.1.25, we can define an injective $\mathbb{Q}(q)$-algebra homomorphism

$$\Psi_- := \Phi^{-1} \mid_{R_{\leq 0}(\mathbf{R}_q)}\colon R_{\leq 0}(\mathbf{R}_q) \to \check{\mathbf{U}}^{\leq 0}_q.$$

I.1.6. Lusztig's braid group symmetries. In this subsection, we present the definition of braid group actions on integrable modules and quantized enveloping algebras. We also review the fundamental properties of them. All statements in this subsections are found, for example, in [**44**], [**46**].

DEFINITION I.1.27. Let $V = \bigoplus_{\mu \in P} V_\mu$ be an integrable $\mathbf{U}_q$-module. We can define a $\mathbb{Q}(q)$-linear automorphism $T_i \colon V \to V$ for $i \in I$ by

$$T_i(u) := \sum_{-a+b-c=\langle h_i,\mu\rangle} (-1)^b q_i^{-ac+b} e_i^{(a)} f_i^{(b)} e_i^{(c)}.u$$

for $u \in V_\mu$ and $\mu \in P$. Its inverse map is given by

$$T_i^{-1}(u) = \sum_{a-b+c=\langle h_i,\mu\rangle} (-1)^b q_i^{ac-b} f_i^{(a)} e_i^{(b)} f_i^{(c)}.u$$

for $u \in V_\mu$ and $\mu \in P$.

DEFINITION I.1.28. We can define a $\mathbb{Q}(q)$-algebra automorphism $T_i \colon \mathbf{U}_q \to \mathbf{U}_q$ for $i \in I$ by the following formulae:

$$T_i\left(q^h\right) = q^{s_i(h)},$$

$$T_i(e_j) = \begin{cases} -f_i t_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{-r} e_i^{(s)} e_j e_i^{(r)} & \text{for } j \neq i, \end{cases}$$

$$T_i(f_j) = \begin{cases} -t_i^{-1} e_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{r} f_i^{(r)} f_j f_i^{(s)} & \text{for } j \neq i. \end{cases}$$

Its inverse map is given by

$$T_i^{-1}\left(q^h\right) = q^{s_i(h)},$$

$$T_i^{-1}(e_j) = \begin{cases} -t_i^{-1} f_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{-r} e_i^{(r)} e_j e_i^{(s)} & \text{for } j \neq i, \end{cases}$$

$$T_i^{-1}(f_j) = \begin{cases} -e_i t_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{r} f_i^{(s)} f_j f_i^{(r)} & \text{for } j \neq i. \end{cases}$$

The following are fundamental properties of T_i.

PROPOSITION I.1.29. *Let V be an integrable $\mathbf{U}_q$-module.*

(1) *For $i \in I$, $T_i(x.u) = T_i(x).T_i(u)$ for $u \in V$ and $x \in \mathbf{U}_q$.*
(2) *For $w \in W$, the composition maps $T_w := T_{i_1} \cdots T_{i_\ell} \colon V \to V$, $\mathbf{U}_q \to \mathbf{U}_q$ do not depend on the choice of $(i_1, \ldots, i_\ell) \in I(w)$.*
(3) *For $\mu \in P$ and $w \in W$, T_w maps V_μ to $V_{w\mu}$.*

(4) *For* $i \in I$, $T_i \circ \vee \circ {}^- = \vee \circ {}^- \circ T_i^{-1}$ *on* $\mathbf{U}_q$.

PROPOSITION I.1.30. *Let* V *be an integrable* $\mathbf{U}_q$*-module and* $i \in I$. *Then, for* $u \in V_\mu \cap \mathrm{Ker}(e_i.)$ *and* $u' \in V_{\mu'} \cap \mathrm{Ker}(f_i.)$, *we have*

$$T_i^{-1}(u) = f_i^{(\langle h_i, \mu \rangle)}.u \qquad T_i(u') = e_i^{(-\langle h_i, \mu' \rangle)}.u'.$$

In particular, for $\lambda \in P_+$ *and* $w \in W$, *we have*

$$u_{w\lambda} = (T_{w^{-1}})^{-1}(u_\lambda) \qquad u_{-w\lambda} = T_w(u_{-\lambda}).$$

We have the following invariance of the bilinear form $(\ , \ )_L$ under the braid group symmetry T_i.

PROPOSITION I.1.31. (1) *For* $i \in I$, *we have* $\mathrm{Ker}\, e'_i = \mathbf{U}_q^- \cap T_i \mathbf{U}_q^-$ *and* $\mathrm{Ker}\, {}_ie' = \mathbf{U}_q^- \cap T_i^{-1} \mathbf{U}_q^-$.

(2) *For* $i \in I$ *and* $x, y \in \mathrm{Ker}\, e'_i$, *we have* $(x, y)_L = \left(T_i^{-1}(x), T_i^{-1}(y)\right)_L$.

I.1.7. Canonical/Dual canonical bases. We review basic properties of canonical/dual canonical bases of quantized enveloping algebras and highest weight integrable modules. See, for example, [**32**] for the fundamental results on crystal bases and canonical bases. We refer to [**30**] for the definition of the category of Kashiwara crystals $(\mathscr{B}; \mathrm{wt}, \{\varepsilon_i\}_{i\in I}, \{\varphi_i\}_{i\in I}, \{\tilde{e}_i\}_{i\in I}, \{\tilde{f}_i\}_{i\in I})$ associated with $(P, \{\alpha_i\}_{i\in I}, \{h_i\}_{i\in I})$.

Denote by $\mathscr{B}(\infty)$ (resp. $\mathscr{B}(\lambda), \lambda \in P_+$) the crystal associated with $\mathbf{U}_q^-$ (resp. $V(\lambda)$). The unique element of $\mathscr{B}(\infty)$ with weight 0 is denoted by $\tilde{u}_\infty$, and the unique element of $\mathscr{B}(\lambda)$ with weight $w\lambda$ is denoted by $u_{w\lambda}$ for $\lambda \in P_+$ and $w \in W$, by abuse of notation.

Set $\mathcal{A} := \mathbb{Q}[q^{\pm 1}]$. Denote by $\mathbf{U}_\mathcal{A}^-$ the $\mathcal{A}$-subalgebra of $\mathbf{U}_q^-$ generated by the elements $\{f_i^{(n)}\}_{i\in I, n\in\mathbb{Z}_{\geq 0}}$. Lusztig [**40, 41, 44**] and Kashiwara [**29**] have constructed the specific $\mathbb{Q}(q)$-basis $\mathbf{B}^{\mathrm{low}}$ (resp. $\mathbf{B}^{\mathrm{low}}(\lambda)$, $\lambda \in P_+$) of $\mathbf{U}_q^-$ (resp. $V(\lambda)$), called the canonical basis (or the lower global basis), which is also an $\mathcal{A}$-basis of $\mathbf{U}_\mathcal{A}^-$ (resp. $V_\mathcal{A}(\lambda) := \mathbf{U}_\mathcal{A}^-.u_\lambda$). Moreover the elements of $\mathbf{B}^{\mathrm{low}}$ (resp. $\mathbf{B}^{\mathrm{low}}(\lambda)$) are parametrized by the Kashiwara crystal $\mathscr{B}(\infty)$ (resp. $\mathscr{B}(\lambda)$). We write $\mathbf{B}^{\mathrm{low}} = \{G(\tilde{b})\}_{\tilde{b}\in\mathscr{B}(\infty)}$ and $\mathbf{B}^{\mathrm{low}}(\lambda) = \{g(b)\}_{b\in\mathscr{B}(\lambda)}$.

NOTATION I.1.32. Let $i \in I$ and $\lambda \in P_+$. For $b \in \mathscr{B}(\lambda)$ and $\tilde{b} \in \mathscr{B}(\infty)$, write

$$\tilde{e}_i^{\max} b := \tilde{e}_i^{\varepsilon_i(b)} b \neq 0 \qquad \tilde{f}_i^{\max} b := \tilde{f}_i^{\varphi_i(b)} b \neq 0 \qquad \tilde{e}_i^{\max} \tilde{b} := \tilde{e}_i^{\varepsilon_i(\tilde{b})} \tilde{b} \neq 0.$$

Then $\tilde{e}_i(\tilde{e}_i^{\max} b) = 0$, $\tilde{f}_i(\tilde{f}_i^{\max} b) = 0$ and $\tilde{e}_i(\tilde{e}_i^{\max} \tilde{b}) = 0$.

DEFINITION I.1.33. Denote by $\mathbf{B}^{\mathrm{up}}$ (resp. $\mathbf{B}^{\mathrm{up}}(\lambda)$, $\lambda \in P_+$) the basis of $\mathbf{U}_q^-$ (resp. $V(\lambda)$) dual to $\mathbf{B}^{\mathrm{low}}$ (resp. $\mathbf{B}^{\mathrm{low}}(\lambda)$) with respect to the bilinear form $(\ , \ )_L$ (resp. $(\ , \ )_\lambda^\varphi$), that is, $\mathbf{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(\tilde{b})\}_{\tilde{b}\in\mathscr{B}(\infty)}$ (resp. $\mathbf{B}^{\mathrm{up}}(\lambda) = \{g^{\mathrm{up}}(b)\}_{b\in\mathscr{B}(\lambda)}$) such that

$$(G(\tilde{b}), G^{\mathrm{up}}(\tilde{b}'))_L = \delta_{\tilde{b},\tilde{b}'} \qquad (\text{resp. } (g(b), g^{\mathrm{up}}(b'))_\lambda^\varphi = \delta_{b,b'})$$

for any $\tilde{b}, \tilde{b}' \in \mathscr{B}(\infty)$ (resp. $b, b' \in \mathscr{B}(\lambda)$).

Definition I.1.34. For $\lambda \in P_+$, define a $\mathbf{U}_q^-$-module surjective homomorphism $\pi_\lambda \colon \mathbf{U}_q^- \to V(\lambda)$ by

$$\pi_\lambda(y) = y.u_\lambda.$$

Proposition I.1.35 ([**29**, Theorem 5, Lemma 7.3.2]). *Let $\lambda \in P_+$. There exists a surjective map $\pi_\lambda \colon \mathscr{B}(\infty) \to \mathscr{B}(\lambda) \coprod \{0\}$ such that*

$$\pi_\lambda(G(\tilde{b})) = g(\pi_\lambda(\tilde{b}))$$

for $\tilde{b} \in \mathscr{B}(\infty)$, here $g(0) = 0$. Moreover π_λ induces a bijection $\pi_\lambda^{-1}(\mathscr{B}(\lambda)) \to \mathscr{B}(\lambda)$.

Definition I.1.36. Let $\lambda \in P_+$. Define $j_\lambda \colon V(\lambda) \hookrightarrow \mathbf{U}_q^-$ as the dual homomorphism of π_λ given by the nondegenerate bilinear forms $(\ , \ )_\lambda^\varphi : V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ and $(\ , \ )_L : \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$, that is

$$(j_\lambda(v), y)_L = (v, \pi_\lambda(y))_\lambda^\varphi = (v, y.u_\lambda)_\lambda^\varphi.$$

Proposition I.1.37 ([**29**, Theorem 5]). *There is an injective map $\jmath_\lambda \colon \mathscr{B}(\lambda) \hookrightarrow \mathscr{B}(\infty)$ such that*

$$(g^{\mathrm{up}}(b), G(\tilde{b}').u_\lambda)_\lambda^\varphi = \delta_{\tilde{b}', \jmath_\lambda(b)}$$

for any $b \in \mathscr{B}(\lambda)$ and $\tilde{b}' \in \mathscr{B}(\infty)$. That is, we have $j_\lambda(g^{\mathrm{up}}(b)) = G^{\mathrm{up}}(\jmath_\lambda(b))$.

Remark I.1.38. Let $\lambda \in P_+$. Then,

- $\mathrm{wt}\, \jmath_\lambda(b) = \mathrm{wt}\, b - \lambda$ for $b \in \mathscr{B}(\lambda)$, and
- $\jmath_\lambda(\pi_\lambda(b)) = b$ for $b \in \pi_\lambda^{-1}(\mathscr{B}(\lambda))$.

Proposition I.1.39 ([**29**, Lemma 7.3.4],). *For all $\tilde{b} \in \mathscr{B}(\infty)$, we have*

$$\overline{G(\tilde{b})} = G(\tilde{b})$$

Note that this implies

$$\sigma(G^{\mathrm{up}}(\tilde{b})) = G^{\mathrm{up}}(\tilde{b}).$$

Proposition I.1.40 ([**30**, Theorem 2.1.1]). *There exist bijections $* \colon \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ such that*

$$*G(\tilde{b}) = G(*\tilde{b})$$

for $\tilde{b} \in \mathscr{B}(\infty)$. Note that this implies

$$*G^{\mathrm{up}}(\tilde{b}) = G^{\mathrm{up}}(*\tilde{b}).$$

See the equality (I.1).

Definition I.1.41. The bijections $*$ give new crystal structures on $\mathscr{B}(\infty)$, defined by the maps

$$\mathrm{wt}^* := \mathrm{wt} \circ * = \mathrm{wt}, \ \varepsilon_i^* := \varepsilon_i \circ *, \ \varphi_i^* := \varphi_i \circ *, \ \tilde{e}_i^* := * \circ \tilde{e}_i \circ *, \ \tilde{f}_i^* := * \circ \tilde{f}_i \circ *.$$

Note that $\mathrm{wt}^* = \mathrm{wt}$ for $\mathscr{B}(\infty)$. For $\tilde{b} \in \mathscr{B}(\infty)$, set $(\tilde{e}_i^*)^{\max}\tilde{b} := (\tilde{e}_i^*)^{\varepsilon_i^*(\tilde{b})}\tilde{b} \neq 0$. Then $\tilde{e}_i^*((\tilde{e}_i^*)^{\max}\tilde{b}) = 0$.

PROPOSITION I.1.42 ([**32**, Proposition 8.2]). *Let $\lambda \in P_+$. Then we have*

$$\jmath_\lambda(\mathscr{B}(\lambda)) = \{\tilde{b} \in \mathscr{B}(\infty) \mid \varepsilon_i^*(\tilde{b}) \le \langle h_i, \lambda \rangle \text{ for all } i \in I\}.$$

PROPOSITION I.1.43 ([**33**, Lemma 5.1.1]). *For $i \in I$, $\lambda \in P_+$, $b \in \mathscr{B}(\lambda)$ and $\tilde{b} \in \mathscr{B}(\infty)$, we have*

$$\begin{aligned}
e_i^{(\varepsilon_i(b))}.g^{\mathrm{up}}(b) &= g^{\mathrm{up}}(\tilde{e}_i^{\max} b) & e_i^{(k)}.g^{\mathrm{up}}(b) &= 0 \text{ if } k > \varepsilon_i(b),\\
f_i^{(\varphi_i(b))}.g^{\mathrm{up}}(b) &= g^{\mathrm{up}}(\tilde{f}_i^{\max} b) & f_i^{(k)}.g^{\mathrm{up}}(b) &= 0 \text{ if } k > \varphi_i(b),\\
(e_i')^{(\varepsilon_i(\tilde{b}))} G^{\mathrm{up}}(\tilde{b}) &= (1-q_i^2)^{(\varepsilon_i(\tilde{b}))} G^{\mathrm{up}}(\tilde{e}_i^{\max} \tilde{b}) & (e_i')^{(k)} G^{\mathrm{up}}(\tilde{b}) &= 0 \text{ if } k > \varepsilon_i(\tilde{b}),\\
({}_ie')^{(\varepsilon_i^*(\tilde{b}))} G^{\mathrm{up}}(\tilde{b}) &= (1-q_i^2)^{(\varepsilon_i^*(\tilde{b}))} G^{\mathrm{up}}((\tilde{e}_i^*)^{\max} \tilde{b}) & ({}_ie')^{(k)} G^{\mathrm{up}}(\tilde{b}) &= 0 \text{ if } k > \varepsilon_i^*(\tilde{b}).
\end{aligned}$$

Here $(e_i')^{(n)} := (e_i')^n/[n]_i!$ and $({}_ie')^{(n)} := ({}_ie')^n/[n]_i!$ for $n \in \mathbb{Z}_{\ge 0}$.

I.1.8. Quantum unipotent subgroups. A quantum unipotent subgroup is a quantum analogue of the coordinate algebra $\mathbb{C}[N_-(w)]$ of a unipotent subgroup $N_-(w)$. See, for example, [**37**] for the specialization to $q = 1$. These algebras are introduced by De Concini-Kac-Procesi [**10**]. The quantum unipotent subgroup has the dual canonical basis and (dual) Poincaré-Birkhoff-Witt type bases.

DEFINITION I.1.44. (1) For $w \in W$, we set $\mathbf{U}_q^-(w) = \mathbf{U}_q^- \cap T_w\left(\mathbf{U}_q^{\ge 0}\right)$. These subalgebras of $\mathbf{U}_q^-$ are called quantum nilpotent subalgebras.

(2) Let $w \in W$ and $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$. For $\boldsymbol{c} = (c_1, \cdots, c_\ell) \in \mathbb{Z}_{\ge 0}^\ell$, we set

$$\begin{aligned}
F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i}) &:= f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots \left(T_{i_1} \cdots T_{i_{\ell-1}}\right)(f_{i_\ell}^{(c_\ell)}),\\
F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) &:= F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i}) / \left(F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i}), F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i})\right)_L.
\end{aligned}$$

PROPOSITION I.1.45 ([**2**, Proposition 2.3], [**10**, Proposition 2.2] [**44**, Proposition 38.2.3]). (1) *$F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i}) \in \mathbf{U}_q^-(w)$ for $\boldsymbol{c} \in \mathbb{Z}_{\ge 0}^\ell$ and $\{F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\ge 0}^\ell}$ forms a basis of $\mathbf{U}_q^- \cap T_w\left(\mathbf{U}_q^{\ge 0}\right)$.*

(2) *$\{F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\ge 0}^\ell}$ is an orthogonal basis of $\mathbf{U}_q^-(w)$, more precisely, we have*

$$\left(F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i}), F^{\mathrm{low}}(\boldsymbol{c}', \boldsymbol{i})\right)_L = \delta_{\boldsymbol{c}, \boldsymbol{c}'} \prod_{k=1}^{\ell} \prod_{j=1}^{c_k} (1 - q_{i_k}^{2j})^{-1}. \tag{I.5}$$

In particular, $\{F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\ge 0}^\ell}$ is also a basis of $\mathbf{U}_q^-(w)$. The basis $\{F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\ge 0}^\ell}$ is called the (lower) Poincaré-Birkhoff-Witt type basis (henceforth the PBW-type basis) associated with $\boldsymbol{i} \in I(w)$, and the basis $\{F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\ge 0}^\ell}$ is called the dual (or upper) Poincaré-Birkhoff-Witt type basis (henceforth the dual PBW-type basis).

DEFINITION I.1.46. For $w \in W$, we set

$$\begin{aligned}
\mathbf{U}_q^+(w) &:= \left(\mathbf{U}_q^-(w)\right)^\vee,\\
\mathbf{A}_q[N_-(w)] &:= *\left(\mathbf{U}_q^-(w)\right).
\end{aligned}$$

We call $\mathbf{A}_q[N_-(w)]$ a quantum unipotent subgroup. The quantum unipotent subgroup has a Q_--graded algebra structure induced from that of $\mathbf{U}_q^-$. Note that $\varphi(\mathbf{A}_q[N_-(w)]) = \mathbf{U}_q^+(w)$. Set

$$E^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i}) := F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i})^\vee \qquad E^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) := F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})^\vee$$
$$F^{\mathrm{low}}_{-1}(\boldsymbol{c}, \boldsymbol{i}) := *(F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i})) \qquad F^{\mathrm{up}}_{-1}(\boldsymbol{c}, \boldsymbol{i}) := *(F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}))$$

for $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$ and $\boldsymbol{i} \in I(w)$. Then $\{E^{\mathrm{low(resp.\ up)}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell(w)}}$ is a basis of $\mathbf{U}_q^+(w)$, and $\{F_{-1}^{\mathrm{low(resp.\ up)}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell(w)}}$ is a basis of $\mathbf{A}_q[N_-(w)]$. Moreover

$$(F^{\mathrm{low}}_{-1}(\boldsymbol{c}, \boldsymbol{i}), F^{\mathrm{up}}_{-1}(\boldsymbol{c}', \boldsymbol{i}))_L = (E^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i}), E^{\mathrm{up}}(\boldsymbol{c}', \boldsymbol{i}))_L^+ = \delta_{\boldsymbol{c},\boldsymbol{c}'}.$$

Proposition I.1.47 ([**34**, Theorem 4.25, Theorem 4.29]). *Let $w \in W$ and $\boldsymbol{i} \in I(w)$.*
(1) *$\mathbf{U}_q^-(w) \cap \mathbf{B}^{\mathrm{up}}$ is a basis of $\mathbf{U}_q^-(w)$.*
(2) *every element $G^{\mathrm{up}}(b)$ of $\mathbf{U}_q^-(w) \cap \mathbf{B}^{\mathrm{up}}$ satisfies the following conditions:*

(DCB1) *$\sigma(G^{\mathrm{up}}(b)) = G^{\mathrm{up}}(b)$, and*
(DCB2) *$G^{\mathrm{up}}(b) = F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) + \sum_{\boldsymbol{c}'<\boldsymbol{c}} d^{\boldsymbol{i}}_{\boldsymbol{c},\boldsymbol{c}'} F^{\mathrm{up}}(\boldsymbol{c}', \boldsymbol{i})$ with $d^{\boldsymbol{i}}_{\boldsymbol{c},\boldsymbol{c}'} \in q\mathbb{Z}[q]$ for a unique $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$.*

Here $<$ denotes the left lexicographic order on $\mathbb{Z}_{\geq 0}^{\ell(w)}$, that is, we write $(c'_1, \ldots, c'_{\ell(w)}) < (c_1, \ldots, c_{\ell(w)})$ if and only if there exists $k \in \{1, \ldots, \ell(w)\}$ such that $c'_1 = c_1, \ldots, c'_{k-1} = c_{k-1}$ and $c'_k < c_k$.

Definition I.1.48. Proposition I.1.47 (2) says that $F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})$ determines a unique dual canonical basis element $G^{\mathrm{up}}(b)$ in $\mathbf{U}_q^-(w)$. We write the corresponding element of $\mathscr{B}(\infty)$ as $b(\boldsymbol{c}, \boldsymbol{i})$. Then $\mathbf{U}_q^-(w) \cap \mathbf{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(b(\boldsymbol{c}, \boldsymbol{i}))\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell(w)}}$. Write $\mathscr{B}(\mathbf{U}_q^-(w)) := \{b(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell(w)}}$. Set $b_{-1}(\boldsymbol{c}, \boldsymbol{i}) := *(b(\boldsymbol{c}, \boldsymbol{i}))$. Then $\mathbf{A}_q[N_-(w)] \cap \mathbf{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(b_{-1}(\boldsymbol{c}, \boldsymbol{i}))\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell(w)}}$. Recall that $\mathcal{A} := \mathbb{Q}[q^{\pm 1}]$. Set

$$\mathbf{A}_{\mathcal{A}}[N_-(w)] := \{x \in \mathbf{A}_q[N_-(w)] \mid (x, \mathbf{U}_{\mathcal{A}}^-)_L \in \mathcal{A}\} = \sum_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell(w)}} \mathcal{A} G^{\mathrm{up}}(b_{-1}(\boldsymbol{c}, \boldsymbol{i})).$$

Then $\mathbf{A}_{\mathcal{A}}[N_-(w)]$ is an $\mathcal{A}$-subalgebra of $\mathbf{A}_q[N_-(w)]$.

Remark I.1.49. In fact, the element $G^{\mathrm{up}}(b(\boldsymbol{c}, \boldsymbol{i}))$ is characterized by the property (DCB1) in Proposition I.1.47 and the following property:

(DCB2)′ $G^{\mathrm{up}}(b) - F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) \in \sum_{\boldsymbol{c}'\in\mathbb{Z}_{\geq 0}^{\ell(w)}} q\mathbb{Z}[q] F^{\mathrm{up}}(\boldsymbol{c}', \boldsymbol{i})$.

Remark I.1.50. For $k = 1, \ldots, \ell(w)$, we set $\boldsymbol{c}_k := (0, \ldots, 0, \overset{\overset{k}{\vee}}{1}, 0, \ldots, 0)$. Then we have $G^{\mathrm{up}}(b(\boldsymbol{c}_k, \boldsymbol{i})) = F^{\mathrm{up}}(\boldsymbol{c}_k, \boldsymbol{i})$.

Remark I.1.51. The unitriangular property in Proposition I.1.47 (2) is equivalent to the following unitriangular property:

$$F^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i})=\sum\nolimits_{\boldsymbol{c}'\in\mathbb{Z}_{\geq 0}^{\ell(w)}}\left[F^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i}):G^{\mathrm{up}}(b(\boldsymbol{c}',\boldsymbol{i}))\right]G^{\mathrm{up}}(b(\boldsymbol{c}',\boldsymbol{i}))\ \text{with}$$

$$\left[F^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i}):G^{\mathrm{up}}(b(\boldsymbol{c}',\boldsymbol{i}))\right]\begin{cases}\in\delta_{\boldsymbol{c}',\boldsymbol{c}}+q\mathbb{Z}[q] & \text{if } \boldsymbol{c}'\leq\boldsymbol{c}\\ =0 & \text{otherwise.}\end{cases}$$

In fact, these unitriangular properties also hold when we consider the right lexicographic order on $\mathbb{Z}_{\geq 0}^{\ell(w)}$. See Corollary II.1.11.

I.1.9. Quantum closed unipotent cells. A quantum closed unipotent cell is a quantum analogue of the coordinate algebra $\mathbb{C}[N_-\cap X(w)]$ of a closed unipotent cell $N_-\cap X(w)$. Here we identify N_- with its image under the natural projection $G\to G/B_+$. See, for example, [**37**] for the specialization to $q=1$. The quantum closed unipotent cells are essentially introduced by De Concini-Procesi [**11**]. They also have the dual canonical bases.

Quantum closed unipotent cells are actually "related" to quantum unipotent subgroups. See Proposition III.1.41. The definition of quantum unipotent cells are presented in Definition III.1.29.

Proposition I.1.52 ([**30**, Proposition 3.2.3, 3.2.5]). *For $\lambda\in P_+$, $w\in W$ and $\boldsymbol{i}=(i_1,\cdots,i_\ell)\in I(w)$, we set*

$$\mathscr{B}_w(\lambda):=\left\{\widetilde{f}_{i_1}^{a_1}\cdots\widetilde{f}_{i_\ell}^{a_\ell}u_\lambda\mid\boldsymbol{a}=(a_1,\cdots,a_\ell)\in\mathbb{Z}_{\geq 0}^{\ell}\right\}\setminus\{0\}\subset\mathscr{B}(\lambda).$$

Then we have

$$V_w(\lambda):=\mathbf{U}_q^{+}.u_{w\lambda}=\sum_{b\in\mathscr{B}_w(\lambda)}\mathbb{Q}(q)\,g(b).$$

This $\mathbf{U}_q^{\geq 0}$-module $V_w(\lambda)$ is called a Demazure module.

(2) *For $w\in W$ and $\boldsymbol{i}=(i_1,\cdots,i_\ell)\in I(w)$, we set*

$$\mathscr{B}_w(\infty)=\left\{\widetilde{f}_{i_1}^{a_1}\cdots\widetilde{f}_{i_\ell}^{a_\ell}\tilde{u}_\infty\mid\boldsymbol{a}=(a_1,\cdots,a_\ell)\in\mathbb{Z}_{\geq 0}^{\ell}\right\}$$

and $\mathbf{U}_w^{-}:=\sum_{a_1,\dots,a_\ell\in\mathbb{Z}_{\geq 0}}\mathbb{Q}(q)\,f_{i_1}^{a_1}\cdots f_{i_\ell}^{a_\ell}$. Then we have

$$\mathbf{U}_w^{-}=\sum_{\tilde{b}\in\mathscr{B}_w(\infty)}\mathbb{Q}(q)\,G(\tilde{b}).$$

For more details on Demazure modules and their crystal bases, see Kashiwara [**30**].

Remark I.1.53. We have

$$\bigcup_{\lambda\in P_+}\overline{\jmath}_\lambda(\mathscr{B}_w(\lambda))=\mathscr{B}_w(\infty).$$

See also Theorem III.1.9.

DEFINITION I.1.54. Let $w \in W$. Set

$$\left(\mathbf{U}_w^-\right)^\perp := \{x \in \mathbf{U}_q^- \mid (x, \mathbf{U}_w^-)_L = 0\}.$$

Then, by the equality $\Delta\left(\mathbf{U}_w^-\right) \subset \mathbf{U}_w^-\mathbf{U}_q^0 \otimes \mathbf{U}_w^-$ and properties of Drinfeld and Lusztig pairings, $(\mathbf{U}_w^-)^\perp$ is a two-sided ideal of $\mathbf{U}_q^-$. Hence we obtain a $\mathbb{Q}(q)$-algebra $\mathbf{U}_q^- / \left(\mathbf{U}_w^-\right)^\perp$, denoted by $\mathbf{A}_q[N_- \cap X(w)]$ and called the quantum closed unipotent cell. The quantum closed unipotent cell has a Q_--graded algebra structure induced from that of $\mathbf{U}_q^-$. Note that

$$\left(\mathbf{U}_w^-\right)^\perp = \sum_{\tilde{b} \in \mathscr{B}(\infty) \setminus \mathscr{B}_w(\infty)} \mathbb{Q}\left(q\right) G^{\mathrm{up}}(\tilde{b}).$$

Describe the canonical projection $\mathbf{U}_q^- \to \mathbf{A}_q[N_- \cap X(w)]$ as $x \mapsto [x]$. The element $[x]$ clearly depends on w, however, we omit to write w because it will cause no confusion below.

I.2. Quantum cluster algebras

In this section, We review the definitions of skew-symmetric quantum cluster algebras. Roughly speaking, quantum cluster algebras are subalgebras of quantum tori with infinitely many generators and relations, which are obtained by some inductive procedure, called mutation. The main references are [**7**] and [**20**]. Quantum cluster algebras are frameworks of the algebraic and combinatorial research on quantum objects. In Section III.2, we review an additive categorification of the quantum cluster algebras due to Geiß-Leclerc-Schröer. In this case, the resulting quantum cluster algebras are isomorphic to quantum unipotent subgroups, and we will consider quantum Berenstein-Fomin-Zelevinsky twist automorphisms in their settings.

NOTATION I.2.1. For $m, m' \in \mathbb{Z}_{\geq 0}$ with $m \leq m'$, set $[m, m'] := \{k \in \mathbb{Z} \mid m \leq k \leq m'\}$.

DEFINITION I.2.2. Let ℓ be a positive integer such that $n \leq \ell$. Let $\Lambda = (\lambda_{ij})_{i,j \in [1,\ell]}$ be a skew-symmetric integer matrix. This skew-symmetric integer matrix Λ determines a skew-symmetric $\mathbb{Z}$-bilinear form $\mathbb{Z}^\ell \times \mathbb{Z}^\ell \to \mathbb{Z}$ by $\Lambda(\boldsymbol{e}_i, \boldsymbol{e}_j) = \lambda_{ij}$ for $i, j \in [1, \ell]$, denoted also by Λ. Here $\{\boldsymbol{e}_i \mid i \in [1, \ell]\}$ denotes the standard basis of $\mathbb{Z}^\ell$. *The based quantum torus* $\mathcal{T}(= \mathcal{T}(\Lambda))$ associated with Λ is the $\mathbb{Q}[q^{\pm 1/2}]$-algebra defined as follows; as a $\mathbb{Q}[q^{\pm 1/2}]$-module $\mathcal{T}$ is free and has a $\mathbb{Q}[q^{\pm 1/2}]$-basis $\{X^{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathbb{Z}^\ell\}$. The multiplication is defined by

$$X^{\boldsymbol{a}} X^{\boldsymbol{b}} = q^{\Lambda(a,b)/2} X^{\boldsymbol{a}+\boldsymbol{b}}$$

for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^\ell$. Then

- $\mathcal{T}$ is an associative algebra,
- $X^{\boldsymbol{a}} X^{\boldsymbol{b}} = q^{\Lambda(\boldsymbol{a},\boldsymbol{b})} X^{\boldsymbol{b}} X^{\boldsymbol{a}}$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^\ell$,
- $X^{\boldsymbol{0}} = 1$ and $(X^{\boldsymbol{a}})^{-1} = X^{-\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathbb{Z}^\ell$.

Hence the based quantum torus $\mathcal{T}$ is a quantum analogue of Laurent polynomial algebras in ℓ-variables(=the coordinate algebra of the ℓ-dimensional algebraic torus). More precisely, $\mathbb{Q} \otimes_{\mathbb{Q}[q^{\pm 1/2}]} \mathcal{T}$ is naturally isomorphic to $\mathbb{Q}[X_1^{\pm 1}, \dots, X_\ell^{\pm 1}]$, here $\mathbb{Q}$ is a $\mathbb{Q}[q^{\pm 1/2}]$-module via $q^{\pm 1/2} \mapsto 1$. The based quantum torus $\mathcal{T}$ is contained in the skew-field of fractions $\mathcal{F}(= \mathcal{F}(\Lambda))$ [**7**, Appendix A]. Note that $\mathcal{F}$ is a $\mathbb{Q}(q^{1/2})$-algebra. Write $X_i := X^{\boldsymbol{e}_i}$ for $i \in [1, \ell]$.

Next we define an important operation, called *mutation*. Let $\widetilde{B} = (b_{ij})_{i\in[1,\ell], j\in[1,\ell-n]}$ be an $\ell \times (\ell - n)$ integer matrix. Its submatrix $B = (b_{ij})_{i,j\in[1,\ell-n]}$ of $\widetilde{B}$ is called *the principal part of* $\widetilde{B}$. The pair $(\Lambda, \widetilde{B})$ is said to be *compatible* if, for $i \in [1, \ell]$ and $j \in [1, \ell - n]$,

$$\sum_{k=1}^{\ell} b_{kj}\lambda_{ki} = \delta_{ij} d_j \text{ for some } d_i \in \mathbb{Z}_{>0}.$$

Note that, when $(\Lambda, \widetilde{B})$ is compatible, $\widetilde{B}$ has full rank $\ell - n$ and the principal part $B = (b_{ij})_{i,j\in[1,\ell-n]}$ is skew-symmetrizable [**7**, Proposition 3.3]. We will assume that B is skew-symmetric.

For $k \in [1, \ell - n]$, define $E^{(k)} = (e_{ij})_{i,j\in[1,\ell]}$ and $F^{(k)} = (f_{ij})_{i,j\in[1,\ell-n]}$ as follows;

$$e_{ij} = \begin{cases} \delta_{i,j} & \text{if } j \neq k, \\ -1 & \text{if } i = j = k, \\ \max(0, -b_{ik}) & \text{if } i \neq j = k, \end{cases} \qquad f_{ij} = \begin{cases} \delta_{i,j} & \text{if } i \neq k, \\ -1 & \text{if } i = j = k, \\ \max(0, b_{kj}) & \text{if } i = k \neq j. \end{cases}$$

Set

$$\mu_k(\Lambda) = (E^{(k)})^T \Lambda E^{(k)} \qquad \mu_k(\widetilde{B}) = E^{(k)} \widetilde{B} F^{(k)}.$$

Then $\mu_k(\Lambda, \widetilde{B}) := (\mu_k(\widetilde{B}), \mu_k(\Lambda))$ is again compatible [**7**, Proposition 3.4]. It is said that $\mu_k(\Lambda, \widetilde{B})$ is obtained from $(\Lambda, \widetilde{B})$ by *the mutation in direction* k. Note that $\mu_k(\mu_k(\Lambda, \widetilde{B})) = (\Lambda, \widetilde{B})$.

The pair $\mathscr{S} = (\{X_i\}_{i\in[1,\ell]}, \widetilde{B}, \Lambda)$ is called *a quantum seed in* $\mathcal{F}$, and $\{X_i\}_{i\in[1,\ell]}$ is called *the quantum cluster of* $\mathscr{S}$. For $k \in [1, \ell - n]$, define $\mu_k(\{X_i\}_{i\in[1,\ell]}) = \{X'_i\}_{i\in[1,\ell]} \subset \mathcal{F} \setminus \{0\}$ by

(M1) $X'_i = X_i$ if $i \neq k$,
(M2) $X'_k = X^{-\boldsymbol{e}_k - \sum_{j;-b_{jk}>0} b_{jk}\boldsymbol{e}_j} + X^{-\boldsymbol{e}_k + \sum_{j;b_{jk}>0} b_{jk}\boldsymbol{e}_j}$.

Then there is an injective $\mathbb{Q}[q^{\pm 1/2}]$-algebra homomorphisms $\mathcal{T}(\mu_k(\Lambda)) \to \mathcal{F}(\Lambda)$ given by $X_i^{\pm 1} \mapsto (X'_i)^{\pm 1}$ ($i \in [1, \ell]$). Moreover there exist a basis $\{\boldsymbol{c}_i\}_{i\in[1,\ell]}$ of $\mathbb{Z}^{\ell}$ and a $\mathbb{Q}(q^{1/2})$-algebra automorphism $\vartheta\colon \mathcal{F}(\Lambda) \to \mathcal{F}(\Lambda)$ such that $\vartheta(X^{\boldsymbol{c}_i}) = X'_i$ for $i \in [1, \ell]$ [**7**, Proposition 4.7]. Hence the map above is extended to the isomorphism $\mathcal{F}(\mu_k(\Lambda)) \to \mathcal{F}(\Lambda)$. Through this isomorphism, we identify $\mathcal{F}(\mu_k(\Lambda))$ with $\mathcal{F}(\Lambda)$, and henceforth always write $\mathcal{F}$ for this skew-field. Write

$$\mu_k(\mathscr{S}) := (\mu_k(\{X_i\}_{i\in[1,\ell]}), \mu_k(\widetilde{B}), \mu_k(\Lambda))$$

and this is called *a quantum seed obtained from the mutation of* $\mathscr{S}$ *in direction* k. By the argument above, we can consider the iterated mutations in arbitrary various directions $k \in [1, \ell - n]$. Note that $\mu_k(\mu_k(\mathscr{S}')) = \mathscr{S}'$ for any quantum seed $\mathscr{S}'$ and $k \in [1, \ell - n]$. The subset $\{X_i \mid i \in [\ell - n + 1, \ell]\}$, called the set of *frozen variables*, is contained in the quantum cluster of an arbitrary seed obtained by iterated mutations of $\mathscr{S}$.

The quantum cluster algebra $\mathscr{A}_{q^{\pm 1/2}}(\mathscr{S})$ is defined as the $\mathbb{Q}[q^{\pm 1/2}]$-subalgebra of $\mathcal{F}$ generated by the union of the quantum clusters of all quantum seeds obtained by iterated mutations of $\mathscr{S}$. An element $M \in \mathscr{A}_{q^{\pm 1/2}}(\mathscr{S})$ is called *a quantum cluster monomial* if there exists a quantum cluster $\{X'_i = (X')^{\boldsymbol{e}_i}\}_{i \in [1,\ell]}$ of a quantum seed obtained by iterated mutations of $\mathscr{S}$ such that $M = (X')^{\boldsymbol{a}}$ for some $\boldsymbol{a} \in \mathbb{Z}^{\ell}_{\geq 0}$.

The following property is known as *the Laurent phenomenon.*

Proposition I.2.3 ([**7**, Corollary 5.2]). *The quantum cluster algebra* $\mathcal{A}_{q^{\pm 1/2}}(\mathscr{S})$ *is contained in the based quantum torus generated by the quantum cluster of an arbitrary quantum seed obtained by iterated mutations of* $\mathscr{S}$*.*

In fact, the $\mathbb{Q}$-subalgebra $\mathbb{Q} \otimes_{\mathbb{Q}[q^{\pm 1/2}]} \mathcal{A}_{q^{\pm 1/2}}(\mathscr{S})$ of $\mathbb{Q} \otimes_{\mathbb{Q}[q^{\pm 1/2}]} \mathcal{T} \simeq \mathbb{Q}[X_1^{\pm 1}, \dots, X_\ell^{\pm 1}]$ is called *a cluster algebra.* This is an algebra associated with the data $(\{X_i\}_{i \in [1,\ell]}, \widetilde{B})$. In other words, Λ is a datum of "noncommutativity".

CHAPTER II

Quantum Fomin-Zelevinsky twist maps

In this chapter, we deal with quantum analogues of Fomin-Zelevinsky twist maps (henceforth quantum FZ-twist maps). See Introduction for their definitions in non-quantum settings. Quantum FZ-twist maps are introduced by Lenagan-Yakimov [**39**]. They are $\mathbb{Q}(q)$-anti-algebra isomorphisms between quantum unipotent subgroups. In Section II.1, we show that quantum FZ-twist maps are restricted to bijections between the dual canonical bases of quantum unipotent subgroups. As a corollary, we obtain the unitriangular property between dual canonical bases and dual PBW-type bases under the "reverse" lexicographic order. This is a new symmetry when $\mathfrak{g}$ is not of finite type. In Section II.2, we show that quantum FZ-twist maps induce bijections between certain unipotent quantum minors. This result is a quantum analogue of [**14**, Lemma 2.25]. In particular, quantum FZ-twist maps preserve the specific quantum determinantal identities, called quantum T-systems.

II.1. Quantum Fomin-Zelevinsky twist maps and dual canonical bases

We define quantum FZ-twist maps following Lenagan-Yakimov [**39**]. In this thesis, we refer to their restriction to quantum unipotent subgroups as quantum FZ-twist maps (Definition II.1.8). We show the compatibility between quantum FZ-twist maps and dual canonical bases of quantum unipotent subgroups (Theorem II.1.10). When $\mathfrak{g}$ is of finite type, the symmetries of quantum FZ-twist maps are related with the symmetries coming from $*$-involution. We remark this point in the last part of this section.

Definition II.1.1 ([**39**, Section 6.1]). For $w \in W$, we consider the $\mathbb{Q}(q)$-algebra anti-automorphisms Θ_w and Θ_w^* of $\mathbf{U}_q$ defined by

$$\Theta_w := T_w \circ S \circ \vee \qquad\qquad \Theta_w^* := * \circ T_w \circ S \circ \vee \circ *.$$

For a homogeneous element $x \in \mathbf{U}_q$, we have $\mathrm{wt}\,(\Theta_w(x)) = \mathrm{wt}\,(\Theta_w^*(x)) = -w\,\mathrm{wt}\,(x)$.

The following lemma follows from the straightforward check on the generators of $\mathbf{U}_q$.

Lemma II.1.2. *For $i \in I$, we have $T_i \circ S \circ \vee = S \circ \vee \circ T_i^{-1}$.*

By this lemma and $(S \circ \vee)^2 = \mathrm{id}$, we have $(\Theta_w)^{-1} = \Theta_{w^{-1}}$ and $(\Theta_w^*)^{-1} = \Theta_{w^{-1}}^*$.

Remark II.1.3. In Definition II.1.8, we define quantum FZ-twist maps by using the "$*$-versions" Θ_w^* in order to match them with our definition of quantum unipotent subgroups. However, in the proof of the following statements, we use the "simplified-versions" Θ_w for abbreviation.

PROPOSITION II.1.4. *For $w \in W$ and $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$, we have*

$$\Theta_{w^{-1}}\left(T_{i_1} \cdots T_{i_{k-1}}(f_{i_k})\right) = T_{i_\ell} \cdots T_{i_{k+1}}(f_{i_k}) \text{ for } k = 1, \ldots, \ell.$$

Proof. It can be easily checked that

$$(T_i \circ S \circ \vee)(f_i) = f_i.$$

Hence by Lemma II.1.2 we have

$$\begin{aligned}\Theta_{w^{-1}}\left(T_{i_1} \cdots T_{i_{k-1}}(f_{i_k})\right) &= (T_{i_\ell} \cdots T_{i_1} \circ S \circ \vee)\left(T_{i_1} \cdots T_{i_{k-1}}(f_{i_k})\right) \\ &= (T_{i_\ell} \cdots T_{i_k} \circ S \circ \vee)(f_{i_k}) \\ &= \left(T_{i_\ell} \cdots T_{i_{k+1}}\right)(f_{i_k}).\end{aligned}$$

□

DEFINITION II.1.5. (1) For $\boldsymbol{i} = (i_1, \cdots, i_\ell) \in I(w)$, we set $\boldsymbol{i}^{\mathrm{rev}} = (i_\ell, \ldots, i_1) \in I(w^{-1})$. (2) For $\boldsymbol{c} = (c_1, \ldots, c_\ell) \in \mathbb{Z}^{\ell}_{\geq 0}$, we set $\boldsymbol{c}^{\mathrm{rev}} := (c_\ell, \ldots, c_1) \in \mathbb{Z}^{\ell}_{\geq 0}$.

PROPOSITION II.1.6. *For $w \in W$, $\boldsymbol{i} \in I(w)$ and $\boldsymbol{c} \in \mathbb{Z}^{\ell}_{\geq 0}$, we have*

$$\Theta_{w^{-1}}\left(F^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})\right) = F^{\mathrm{up}}(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}), \qquad \Theta^{*}_{w^{-1}}\left(F^{\mathrm{up}}_{-1}(\boldsymbol{c}, \boldsymbol{i})\right) = F^{\mathrm{up}}_{-1}(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}).$$

Proof. The latter follows from the former. By the equality (I.5), we have

$$\left(F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i}), F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i})\right)_L = \left(F^{\mathrm{low}}(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}), F^{\mathrm{low}}(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}})\right)_L.$$

Hence it suffices to show that $\Theta_{w^{-1}}\left(F^{\mathrm{low}}(\boldsymbol{c}, \boldsymbol{i})\right) = F^{\mathrm{low}}(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}})$. This follows immediately from Proposition II.1.4. □

By Proposition II.1.6, $\Theta_{w^{-1}}$ (resp. $\Theta^{*}_{w^{-1}}$) is also regarded as a $\mathbb{Q}(q)$-algebra anti-isomorphism from $\mathbf{U}_q^-(w)$ (resp. $\mathbf{A}_q[N_-(w)]$) to $\mathbf{U}_q^-(w^{-1})$ (resp. $\mathbf{A}_q[N_-(w^{-1})]$).

LEMMA II.1.7. *Let $w \in W$. For $x, x' \in \mathbf{U}_q^-(w)$ and $y, y' \in \mathbf{A}_q[N_-(w)]$, we have*

$$(x, x')_L = (\Theta_{w^{-1}}(x), \Theta_{w^{-1}}(x'))_L, \qquad (y, y')_L = (\Theta^{*}_{w^{-1}}(y), \Theta^{*}_{w^{-1}}(y'))_L.$$

Proof. This follows immediately from Proposition II.1.6 and the equality (I.5). □

DEFINITION II.1.8. Let $w \in W$. set

$$\tau_{w,q} := \Theta^{*}_{w^{-1}}\,|_{\mathbf{A}_q[N_-(w)]} \colon \mathbf{A}_q[N_-(w)] \to \mathbf{A}_q[N_-(w^{-1})].$$

We call this $\mathbb{Q}(q)$-anti-algebra isomorphism $\tau_{w,q}$ a quantum FZ-twist map. By Lemma II.1.7, we have

$$(y, \Theta^{*}_{w}(y'))_L = (\tau_{w,q}(y), y')_L$$

for $y \in \mathbf{A}_q[N_-(w)]$ and $y' \in \mathbf{A}_q[N_-(w^{-1})]$.

PROPOSITION II.1.9. *Let $w \in W$. For $x \in \mathbf{U}_q^-(w)$ and $y \in \mathbf{A}_q[N_-(w)]$, we have*

$$(\Theta_{w^{-1}} \circ \sigma)(x) = (\sigma \circ \Theta_{w^{-1}})(x) \qquad (\tau_{w,q} \circ \sigma)(y) = (\sigma \circ \tau_{w,q})(y).$$

Proof. The latter follows from the former and the equality $* \circ \sigma = \sigma \circ *$ on $\mathbf{U}_q^-$, which is derived from Proposition I.1.12. We may assume that x is homogeneous. On generators, by Remark I.1.50, we have

$$\begin{aligned}\left(\Theta_{w^{-1}} \circ \sigma\right)\left(F^{\mathrm{up}}\left(\boldsymbol{c}_k, \boldsymbol{i}\right)\right) &= \Theta_{w^{-1}}\left(F^{\mathrm{up}}\left(\boldsymbol{c}_k, \boldsymbol{i}\right)\right) \\ &= F^{\mathrm{up}}\left(\boldsymbol{c}_k^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right) = \sigma\left(F^{\mathrm{up}}\left(\boldsymbol{c}_k^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right)\right) \\ &= \left(\sigma \circ \Theta_{w^{-1}}\right)\left(F^{\mathrm{up}}\left(\boldsymbol{c}_k, \boldsymbol{i}\right)\right).\end{aligned}$$

Assume that the desired equality holds for homogeneous elements $x', x'' \in \mathbf{U}_q^-(w)$. Then, by Proposition I.1.12, we have

$$\begin{aligned}\left(\Theta_{w^{-1}} \circ \sigma\right)\left(x'x''\right) &= q^{(\mathrm{wt}(x'),\mathrm{wt}(x''))}\Theta_{w^{-1}}\left(\sigma\left(x''\right)\sigma\left(x'\right)\right) \\ &= q^{(\mathrm{wt}(x'),\mathrm{wt}(x''))}\Theta_{w^{-1}}\left(\sigma\left(x'\right)\right)\Theta_{w^{-1}}\left(\sigma\left(x''\right)\right) \\ &= q^{\left(-w^{-1}\mathrm{wt}(x'),-w^{-1}\mathrm{wt}(x'')\right)}\sigma\left(\Theta_{w^{-1}}\left(x'\right)\right)\sigma\left(\Theta_{w^{-1}}\left(x''\right)\right) \\ &= \sigma\left(\Theta_{w^{-1}}\left(x''\right)\Theta_{w^{-1}}\left(x'\right)\right) \\ &= \left(\sigma \circ \Theta_{w^{-1}}\right)\left(x'x''\right).\end{aligned}$$

Hence we obtained the assertion. □

Now we prove the compatibility between quantum FZ-twist maps and dual canonical bases. Recall Definition I.1.48.

Theorem II.1.10. *Let $w \in W$ and $\boldsymbol{i} \in I(w)$. For $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, we have*

$$\Theta_{w^{-1}}\left(G^{\mathrm{up}}\left(b\left(\boldsymbol{c}, \boldsymbol{i}\right)\right)\right) = G^{\mathrm{up}}\left(b\left(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right)\right) \quad \tau_{w,q}\left(G^{\mathrm{up}}\left(b_{-1}\left(\boldsymbol{c}, \boldsymbol{i}\right)\right)\right) = G^{\mathrm{up}}\left(b_{-1}\left(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right)\right).$$

In particular, $\tau_{w,q}$ (resp. $\Theta_{w^{-1}}$) induces a bijection between the dual canonical basis of $\mathbf{A}_q[N_-(w)]$ and that of $\mathbf{A}_q[N_-(w^{-1})]$ (resp. the dual canonical basis of $\mathbf{U}_q^-(w)$ and that of $\mathbf{U}_q^-(w^{-1})$).

Proof. The former implies the latter. We have already checked that $\Theta_{w^{-1}}\left(G^{\mathrm{up}}\left(b\left(\boldsymbol{c}, \boldsymbol{i}\right)\right)\right) \in \mathbf{U}_q^-(w^{-1})$. Hence by Remark I.1.49 we only have to show that

$$\sigma\left(\Theta_{w^{-1}}\left(G^{\mathrm{up}}\left(b\left(\boldsymbol{c}, \boldsymbol{i}\right)\right)\right)\right) = \Theta_{w^{-1}}\left(G^{\mathrm{up}}\left(b\left(\boldsymbol{c}, \boldsymbol{i}\right)\right)\right),$$

$$\Theta_{w^{-1}}\left(G^{\mathrm{up}}\left(b\left(\boldsymbol{c}, \boldsymbol{i}\right)\right)\right) - F^{\mathrm{up}}\left(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right) \in \sum_{\boldsymbol{c}' \in \mathbb{Z}_{\geq 0}^{\ell(w)}} q\mathbb{Z}[q]F^{\mathrm{up}}\left(\boldsymbol{c}', \boldsymbol{i}^{\mathrm{rev}}\right).$$

The latter follows from Proposition I.1.47 and Proposition II.1.6. The former follows from Proposition I.1.39 and II.1.9. □

Recall the notation in Remark I.1.51. By applying $\Theta_{w^{-1}}$ to the expansion of the dual PBW basis into the dual canonical basis in Remark I.1.51, we obtain the following corollary. This symmetry is new when $\mathfrak{g}$ is not finite dimensional. See also Remark II.1.17.

Corollary II.1.11. *Let $w \in W$ and $\boldsymbol{i} \in I(w)$. For $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, we have*

$$\left[F^{\mathrm{up}}\left(\boldsymbol{c}, \boldsymbol{i}\right) : G^{\mathrm{up}}\left(b\left(\boldsymbol{c}', \boldsymbol{i}\right)\right)\right] = \left[F^{\mathrm{up}}\left(\boldsymbol{c}^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right) : G^{\mathrm{up}}\left(b\left(\left(\boldsymbol{c}'\right)^{\mathrm{rev}}, \boldsymbol{i}^{\mathrm{rev}}\right)\right)\right]. \tag{II.1}$$

In particular, we can write the expansion as follows:

$$F^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i}) = G^{\mathrm{up}}(b(\boldsymbol{c},\boldsymbol{i})) + \sum_{\boldsymbol{c}'<\boldsymbol{c},\, \boldsymbol{c}'<_{\mathrm{r}}\boldsymbol{c}} [F^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i}) : G^{\mathrm{up}}(b(\boldsymbol{c}',\boldsymbol{i}))]\, G^{\mathrm{up}}(b(\boldsymbol{c}',\boldsymbol{i})),$$

here $<_{\mathrm{r}}$ *denotes the right lexicographic order on* $\mathbb{Z}_{\geq 0}^{\ell(w)}$*, which is deterined by the condition that* $\boldsymbol{c}' <_{\mathrm{r}} \boldsymbol{c}$ *if and only if* $(\boldsymbol{c}')^{\mathrm{rev}} < \boldsymbol{c}^{\mathrm{rev}}$.

In the rest of this section, we assume that $\mathfrak{g}$ is a finite dimensional complex simple Lie algebra. Let $w_0 \in W$ be the longest element of W. There is a unique Dynkin diagram automorphism θ with $-w_0(\alpha_i) = \alpha_{\theta(i)}$ for all $i \in I$. For a reduced word $\boldsymbol{i} = (i_1, \cdots, i_N) \in I(w_0)$, the sequence $(i_2, \cdots, i_N, \theta(i_1))$ is also a reduced word of w_0.

Definition II.1.12. We define a $\mathbb{Q}(q)$-algebra automorphism on $\mathbf{U}_q(\mathfrak{g})$ defined by

$$\theta(e_i) = e_{\theta(i)} \qquad \theta(f_i) = f_{\theta(i)} \qquad \theta\left(q^h\right) = q^{-w_0(h)}.$$

Proposition II.1.13 ([**25**, Proposition 8.20], [**45**, Proposition 3.2]). *For* $w(\alpha_i) = \alpha_j \in \Pi$*, we have*

$$T_w(x_i) = x_j$$

where $x = e, f$.

Proposition II.1.14. *We have*

$$\theta \circ *(= * \circ \theta) = \Theta_{w_0} = \tau_{w_0,q}.$$

Proof. By Proposition II.1.13, we have

$$T_{w_0}(e_i) = -f_{\theta(i)}t_{\theta(i)}, \qquad T_{w_0}(f_i) = -t_{\theta(i)}^{-1}e_{\theta(i)}, \qquad T_{w_0}\left(q^h\right) = q^{w_0(h)}.$$

Hence the proposition follows from the straightforward check on the generators of $\mathbf{U}_q$. □

By Proposition II.1.14, we obtain the following corollary.

Corollary II.1.15. *For* $\boldsymbol{i} = (i_1, \cdots, i_N) \in I(w_0)$ *and* $\boldsymbol{c} = (c_1, \cdots, c_N) \in \mathbb{Z}_{\geq 0}^N$*, we have*

$$(\theta \circ *)(G^{\mathrm{up}}(b(\boldsymbol{c},\boldsymbol{i}))) = G^{\mathrm{up}}(b(\boldsymbol{c}^{\mathrm{rev}},\boldsymbol{i}^{\mathrm{rev}})).$$

Moreover by Lemma II.1.7 we obtain the following corollary.

Corollary II.1.16. *Let* $\boldsymbol{i} = (i_1, \cdots, i_N) \in I(w_0)$ *and* $\boldsymbol{c} = (c_1, \cdots, c_N) \in \mathbb{Z}_{\geq 0}^N$.
(1) *We have*

$$(\theta \circ *)(G(b(\boldsymbol{c},\boldsymbol{i}))) = G(b(\boldsymbol{c}^{\mathrm{rev}},\boldsymbol{i}^{\mathrm{rev}})).$$

(2) *Write* $G(b(\boldsymbol{c},\boldsymbol{i})) = \sum_{\boldsymbol{c}'} \left(G(b(\boldsymbol{c},\boldsymbol{i})) : F^{\mathrm{low}}(\boldsymbol{c}',\boldsymbol{i})\right) F^{\mathrm{low}}(\boldsymbol{c}',\boldsymbol{i})$. *Then we have*

$$\left(G(b(\boldsymbol{c},\boldsymbol{i})) : F^{\mathrm{low}}(\boldsymbol{c}',\boldsymbol{i})\right) = \left(G(b(\boldsymbol{c}^{\mathrm{rev}},\boldsymbol{i}^{\mathrm{rev}})) : F^{\mathrm{low}}\left((\boldsymbol{c}')^{\mathrm{rev}},\boldsymbol{i}^{\mathrm{rev}}\right)\right).$$

In particular, we have

$$G(b(\boldsymbol{c},\boldsymbol{i})) = F^{\mathrm{low}}(\boldsymbol{c},\boldsymbol{i}) + \sum_{\boldsymbol{c}<\boldsymbol{c}',\boldsymbol{c}<_{\mathrm{r}}\boldsymbol{c}'} \left(G(b(\boldsymbol{c},\boldsymbol{i})) : F^{\mathrm{low}}(\boldsymbol{c}',\boldsymbol{i})\right) F^{\mathrm{low}}(\boldsymbol{c}',\boldsymbol{i}).$$

REMARK II.1.17. We have to remark that Corollary II.1.15 was already proved by Lusztig [**40**, 2.11]. In fact, if $\mathfrak{g}$ is of finite type, we can also show the equality (II.1) in Corollary II.1.11 without using quantum FZ-twist maps, by the results in [**40**, 2.11] together with [**46**, Proposition 3.4.7, Corollary 3.4.8], [**42**, Theorem 1.2]. Note that $\Theta_w = (T_{w_0 w^{-1}})^{-1} \circ \theta \circ *$ for all $w \in W$ if $\mathfrak{g}$ is of finite type.

II.2. Quantum Fomin-Zelevinsky twist maps and unipotent quantum minors

We again assume that $\mathfrak{g}$ is an arbitrary Kac-Moody Lie algebra. Unipotent quantum minors are typical and manageable elements of dual canonical bases. In this section, we show that the images of some unipotent quantum minors under quantum FZ-twist maps are also described by unipotent quantum minors (Theorem II.2.8). In particular, quantum FZ-twist maps preserve quantum analogue of specific determinantal identities, called quantum T-system (Corollary II.2.14).

DEFINITION II.2.1. For $\lambda \in P_+ \cup (-P_+)$ and $w, w' \in W$, define an element $D_{w\lambda, w'\lambda} \in \mathbf{U}_q^-$ by the following property:

$$(D_{w\lambda, w'\lambda}, x)_L = (u_{w\lambda}, x.u_{w'\lambda})_\lambda^\varphi$$

for $x \in \mathbf{U}_q^-$. By the nondegeneracy of the bilinear form $(\ , \ )_L$, this element is uniquely determined. An element of this form is called a unipotent quantum minor. Note that, if $D_{w\lambda, w'\lambda} \neq 0$, then $\mathrm{wt}\,(D_{w\lambda, w'\lambda}) = w\lambda - w'\lambda$. See [**34**, Section 6].

PROPOSITION II.2.2 ([**30**, Proposition 4.1]). *The unipotent quantum minors belong to* $\mathbf{B}^{\mathrm{up}}$.

The unipotent quantum minors associated with lowest weight modules are related with those associated with highest weight modules via $*$-involution.

PROPOSITION II.2.3. *For $\lambda \in P_+$ and $w, w' \in W$, we have*

$$*D_{-w\lambda, -w'\lambda} = D_{w'\lambda, w\lambda}.$$

Proof. For all $x \in \mathbf{U}_q^-$, we have

$$\begin{aligned}(*D_{-w\lambda, -w'\lambda}, x)_L &= (D_{-w\lambda, -w'\lambda}, *x)_L \\ &= (u_{-w\lambda}, *(x).u_{-w'\lambda})_{-\lambda}^\varphi = (u_{-w'\lambda}, (x)^\vee.u_{-w\lambda})_{-\lambda}^\varphi.\end{aligned}$$

We can consider a new $\mathbf{U}_q$-module $V(-\lambda)^\vee$ which has the same underlying vector space as $V(-\lambda)$ and is endowed with the action $\bullet$ of $\mathbf{U}_q$ given by $x \bullet u = (x)^\vee.u$ for $x \in \mathbf{U}_q$ and $u \in V(-\lambda)^\vee$. Then there exists a $\mathbf{U}_q$-module isomorphism $\Upsilon : V(\lambda) \to V(-\lambda)^\vee$ given by $u_\lambda \mapsto u_{-\lambda}$. Moreover $\Phi(u_{v\lambda}) = u_{-v\lambda}$ for all $v \in W$. Indeed, for $(i_1, \dots, i_\ell) \in I(v)$,

$$\begin{aligned}\Upsilon(u_{v\lambda}) &= \Upsilon\left(f_{i_1}^{(\langle h_{i_1}, s_{i_2}\cdots s_{i_\ell}\lambda\rangle)} \cdots f_{i_{\ell-1}}^{(\langle h_{i_{\ell-1}}, s_{i_\ell}\lambda\rangle)} f_{i_\ell}^{(\langle h_{i_\ell}, \lambda\rangle)}.u_\lambda\right) \\ &= e_{i_1}^{(\langle h_{i_1}, s_{i_2}\cdots s_{i_\ell}\lambda\rangle)} \cdots e_{i_{\ell-1}}^{(\langle h_{i_{\ell-1}}, s_{i_\ell}\lambda\rangle)} e_{i_\ell}^{(\langle h_{i_\ell}, \lambda\rangle)}.u_{-\lambda} = u_{-v\lambda}.\end{aligned}$$

Hence

$$\begin{aligned}(*D_{-w\lambda,-w'\lambda},x)_L &= (u_{-w'\lambda},(x)^\vee.u_{-w\lambda})^\varphi_{-\lambda}\\ &= (u_{-w'\lambda},(\Upsilon\circ\Upsilon^{-1})(x\bullet u_{-w\lambda}))^\varphi_{-\lambda}\\ &= (u_{-w'\lambda},\Upsilon(x.u_{w\lambda}))^\varphi_{-\lambda}\\ &= (u_{w'\lambda},x.u_{w\lambda})^\varphi_{\lambda} = (D_{w'\lambda,w\lambda},x)_L\end{aligned}$$

for all $x\in\mathbf{U}_q^-$. This proves the proposition. □

We consider the unipotent quantum minors which belong to $\mathbf{U}_q^-(w)$.

PROPOSITION II.2.4 ([**35**, Proposition 3.4]). *For $w\in W$ and $\boldsymbol{i}=(i_1,\dots,i_\ell)\in I(w)$, we have*

$$\mathbf{U}_q^-\cap T_w\left(\mathbf{U}_q^-\right)=\mathbf{U}_q^-\cap T_{i_1}\left(\mathbf{U}_q^-\right)\cap T_{i_1}T_{i_2}\left(\mathbf{U}_q^-\right)\cap\cdots\cap T_{i_1}T_{i_2}\cdots T_{i_\ell}\left(\mathbf{U}_q^-\right).$$

PROPOSITION II.2.5 ([**35**, Theorem 1.1]). *Let $w\in W$. Then the multiplication map induces the $\mathbb{Q}(q)$-linear isomorphism:*

$$\mathbf{U}_q^-(w)\otimes\left(\mathbf{U}_q^-\cap T_w\mathbf{U}_q^-\right)\xrightarrow{\sim}\mathbf{U}_q^-.$$

LEMMA II.2.6. *For $w\in W$, set $\mathbf{U}_q^-(w)^\perp:=\{x\in\mathbf{U}_q^-\mid(x,\mathbf{U}_q^-(w))_L=0\}$. Then the multiplication map induces the $\mathbb{Q}(q)$-linear isomorphism:*

$$\mathbf{U}_q^-(w)\otimes\left(\mathbf{U}_q^-\cap T_w\mathbf{U}_q^-\cap\mathrm{Ker}\varepsilon\right)\xrightarrow{\sim}\mathbf{U}_q^-(w)^\perp.$$

Recall that ε is the counit of $\mathbf{U}_q$. In particular, $\mathbf{U}_q^-(w)^\perp$ is a left ideal of $\mathbf{U}_q^-$.

Proof. By Proposition II.2.5, we have a decomposition $\mathbf{U}_q^-=\mathbf{U}_q^-(w)\oplus\mathbf{U}_q^-(w)(\mathbf{U}_q^-\cap T_w(\mathbf{U}_q^-)\cap\mathrm{Ker}\,\varepsilon)$ of a $\mathbb{Q}(q)$-vector space. By the way, we also have $\mathbf{U}_q^-=\mathbf{U}_q^-(w)\oplus\mathbf{U}_q^-(w)^\perp$. Hence it suffices to prove the following inclusion:

$$\mathbf{U}_q^-(w)(\mathbf{U}_q^-\cap T_w(\mathbf{U}_q^-)\cap\mathrm{Ker}\,\varepsilon)\subset\mathbf{U}_q^-(w)^\perp.$$

It is shown by using Proposition I.1.31 and Proposition II.2.4 repeatedly. □

PROPOSITION II.2.7. *Let $\lambda\in P_+$ and $w_1,w_2,w\in W$. Suppose that w_2 is less than or equal to w with respect to the weak right Bruhat order, that is, $\ell(w)=\ell(w_2)+\ell(w_2^{-1}w)$. Then*

$$D_{-w_1\lambda,-w_2\lambda}\in\mathbf{U}_q^-(w)\qquad\qquad D_{w_2\lambda,w_1\lambda}\in\mathbf{A}_q[N_-(w)].$$

Proof. By Proposition II.2.3, the latter is equivalent to the former. Since $\mathbf{U}_q(w)=\{x\in\mathbf{U}_q^-\mid(x,\mathbf{U}_q^-(w)^\perp)_L=0\}$, it suffices to show that

$$(u_{-w_1\lambda},\mathbf{U}_q^-(w)^\perp.u_{-w_2\lambda})^\varphi_{-\lambda}=0.$$

For every homogeneous element $x\in\mathbf{U}_q^-\cap T_w(\mathbf{U}_q^-)\cap\mathrm{Ker}\,\varepsilon$, we have $w_2^{-1}\,\mathrm{wt}\,x\in Q_-$ by Proposition II.2.4. Here note that there exists $\boldsymbol{i}=(i_1,i_2,\dots,i_\ell)\in I(w)$ such that $(i_1,i_2,\dots,i_{\ell(w_2)})\in I(w_2)$. Therefore, by Proposition I.1.29 and I.1.30,

$$x.u_{-w_2\lambda}=T_{w_2}\left((T_{w_2})^{-1}(x).u_{-\lambda}\right)=0.$$

Hence Lemma II.2.6 implies the assertion. □

The following is a quantum analogue of [**14**, Lemma 2.25].

THEOREM II.2.8. *Let $\lambda \in P_+$ and $w_1, w_2 \in W$. Suppose that w_1 and w_2 are less than or equal to w with respect to the weak right Bruhat order. Then we have*

$$\Theta_{w^{-1}}(D_{-w_1\lambda,-w_2\lambda}) = D_{-w^{-1}w_2\lambda,-w^{-1}w_1\lambda} \qquad \tau_{w,q}(D_{w_2\lambda,w_1\lambda}) = D_{w^{-1}w_1\lambda,w^{-1}w_2\lambda}.$$

Proof. The latter follows from the former by Proposition II.2.3. By Proposition II.2.7, we have $D_{-w_1\lambda,-w_2\lambda} \in \mathbf{U}_q^-(w)$. Therefore we have $\Theta_{w^{-1}}(D_{-w_1\lambda,-w_2\lambda}) \in \mathbf{U}_q^-(w^{-1})$. By Lemma II.1.7, for $x \in \mathbf{U}_q^-(w^{-1})$,

$$\begin{aligned}(\Theta_{w^{-1}}(D_{-w_1\lambda,-w_2\lambda}), x)_L &= (D_{-w_1\lambda,-w_2\lambda}, \Theta_w(x))_L \\ &= (u_{-w_1\lambda}, \Theta_w(x).u_{-w_2\lambda})^{\varphi}_{-\lambda} \\ &= (u_{-w_2\lambda}, (\varphi\circ\Theta_w)(x).u_{-w_1\lambda})^{\varphi}_{-\lambda}.\end{aligned}$$

Now $\varphi\circ\Theta_w$ is a $\mathbb{Q}(q)$-algebra automorphism of $\mathbf{U}_q$. Hence we can consider a new $\mathbf{U}_q$-module $V'(-\lambda)$ which has the same underlying vector space as $V(-\lambda)$ and is endowed with the action $\star$ of $\mathbf{U}_q$ given by $x \star u = (\varphi\circ\Theta_w)(x).u$ for $x \in \mathbf{U}_q$ and $u \in V'(-\lambda)$.

Then there exists a $\mathbf{U}_q$-module isomorphism $V(-\lambda) \to V'(-\lambda)$ given by $u_{-\lambda} \mapsto u_{-w\lambda}$. Note that $(\varphi\circ\Theta_w)(q^h) = q^{w(h)}$ for $h \in P^*$. Hence the vector $u_{-w_i\lambda} \in V'(-\lambda)$ is a vector of weight $-w^{-1}w_i\lambda$ $(i = 1, 2)$. Moreover it is well-known that the weight space of $V(-\lambda)$ of weight μ is 1-dimensional for all $\mu \in -W\lambda$. Therefore as in the proof of Proposition II.2.3 we have

$$\begin{aligned}(u_{-w_2\lambda}, (\varphi\circ\Theta_w)(x).u_{-w_1\lambda})^{\varphi}_{-\lambda} &= \zeta\,(u_{-w^{-1}w_2\lambda}, x.u_{-w^{-1}w_1\lambda})^{\varphi}_{-\lambda} \\ &= \zeta\,(D_{-w^{-1}w_2\lambda,-w^{-1}w_1\lambda}, x)_L\end{aligned}$$

for some $\zeta \in \mathbb{Q}(q)^{\times}$ and all $x \in \mathbf{U}_q^-(w^{-1})$. By our assumption, $w^{-1}w_1$ is less than or equal to w^{-1} with respect to the weak right Bruhat order. Therefore $D_{-w^{-1}w_2\lambda,-w^{-1}w_1\lambda} \in \mathbf{U}_q^-(w^{-1})$ by Proposition II.2.7. Hence $\Theta_{w^{-1}}(D_{-w_1\lambda,-w_2\lambda}) = \zeta D_{-w^{-1}w_2\lambda,-w^{-1}w_1\lambda}$.

On the other hand, by Theorem II.1.10, $\Theta_{w^{-1}}(D_{-w_1\lambda,-w_2\lambda}) \in \mathbf{B}^{\mathrm{up}} \cap \mathbf{U}_q^-(w^{-1})$. Therefore, by Proposition II.2.2, $\zeta = 1$ and $\Theta_{w^{-1}}(D_{-w_1\lambda,-w_2\lambda}) = D_{-w^{-1}w_2\lambda,-w^{-1}w_1\lambda}$. □

As a corollary of Theorem II.2.8, we show the compatibility between quantum FZ-twist maps and quantum analogues of specific determinantal identities, called quantum T-systems. From the view point of the theory of quantum cluster algebras, quantum T-system is the specific mutation sequences of the quantum cluster algebra $\mathbf{A}_q[N_-(w)]$ [**20**].

NOTATION II.2.9. When we fix $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$, we write

$$\begin{aligned}k^+ &:= \min(\{\ell+1\} \cup \{k+1 \le j \le \ell \mid i_j = i_k\}), \\ k^- &:= \max(\{0\} \cup \{1 \le j \le k-1 \mid i_j = i_k\}), \\ k^-(i) &:= \max(\{0\} \cup \{1 \le j \le k-1 \mid i_j = i\}), \\ k^{\max} &:= \max\{1 \le j \le \ell \mid i_j = i_k\}, \\ k^{\min} &:= \min\{1 \le j \le \ell \mid i_j = i_k\}.\end{aligned}$$

for $k = 1, \dots, \ell$ and $i \in I$.

Definition II.2.10. Let $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. For $0 \le b \le d \le \ell$ and $j \in I$, we set

$$D(d, b; j)(= D^{\boldsymbol{i}}(d, b; j)) := D_{\mu(d,j),\mu(b,j)},$$

here $\mu(b, j)(= \mu^{\boldsymbol{i}}(b, j)) := s_{i_1} \cdots s_{i_b} \varpi_j$. By Proposition II.2.7, this is an element of $\mathbf{A}_q[N_-(w)]$. Moreover, when $i_b = i_d = j$, we write $D(d, b)(= D^{\boldsymbol{i}}(d, b)) := D(d, b; j)$. Note that $D(d, 0) = D_{s_{i_1} \cdots s_{i_d} \varpi_{i_d}, \varpi_{i_d}}$ for $1 \le d \le \ell$ and $D(b, b) := 1$ for $b = 0, \dots, \ell$. Then, for $0 \le b \le d \le \ell$ and $j \in I$, we have $D(d, b; j) = D(d^-(j), b^-(j))$.

Recall that $a_{ji} := \langle h_j, \alpha_i \rangle$ for $i, j \in I$.

Proposition II.2.11 ([**20**, Proposition 5.5]). *Let $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. Fix an arbitrary total order on I. Suppose that the integers b, d satisfy that $1 \le b < d \le \ell$ and $i_b = i_d = i$. Then we have*

$$q^A D(d, b) D(d^-, b^-) = q_i^{-1} q^B D(d, b^-) D(d^-, b) + q^C \overrightarrow{\prod_{j \in I \setminus \{i\}}} D(d^-(j), b^-(j))^{-a_{ji}} \tag{II.2}$$

$$= q_i^{-1} q^{B'} D(d^-, b) D(d, b^-) + q^C \overrightarrow{\prod_{j \in I \setminus \{i\}}} D(d^-(j), b^-(j))^{-a_{ji}}, \tag{II.3}$$

here

$$A = (\mu(b, i), \mu(b^-, i) - \mu(d^-, i)),$$
$$B = (\mu(b^-, i), \mu(b, i) - \mu(d^-, i)), \qquad B' = (\mu(b, i), \mu(b^-, i) - \mu(d, i)),$$

$$C = \sum_{j \in I \setminus \{i\}} \binom{-a_{ji}}{2} (\mu(b, j), \mu(b, j) - \mu(d, j)) + \sum_{j,k \in I \setminus \{i\}; j < k} a_{ji} a_{ki} (\mu(b, j), \mu(b, k) - \mu(d, k)),$$

and $\overrightarrow{\prod}$ denotes a product with respect to the increasing order from left to right. This system of equalities is called the quantum T-system in $\mathbf{A}_q[N_-(w)]$.

Remark II.2.12. Note that our convention is different from the one in [**20**], and Geiß-Leclerc-Schröer always assume that $\mathfrak{g}$ is symmetric. Nevertheless, we can prove the equality above in the same manner as in [**20**]. See also Remark III.2.11 below.

Example II.2.13. We consider the case that $\mathfrak{g} = \mathfrak{sl}_3$, $I = \{1, 2\}$. Note that $N_-(w_0) = N_-$ in this case and this is the group of unipotent lower triangular 3×3 matrices. The following is a basic determinantal identities in the non-quantum settings:

$$D_{3,2} D_{2,1} = D_{3,1} + D_{23,12} \tag{II.4}$$

Here D_{J_1,J_2} denotes the regular function on N_- which assigns to a matrix its minor with row-set J_1 and column-set J_2. This is nothing but the classical counterpart of the equality in Proposition II.2.11 with $w = w_0 = s_1s_2s_1$, $\boldsymbol{i} = (1, 2, 1)$ and $i = 1$, $b = 1$, $d = 3$. In fact, the unipotent quantum minors appearing in this equality are the quantum analogues of the ones in (II.4). By using the same notation, the equality (II.2) (which is equivalent to (II.3)) is described as follows:

$$D_{3,2}D_{2,1} = q^{-1}D_{3,1} + D_{23,12}.$$

Corollary II.2.14. *The quantum FZ-twist map $\tau_{w,q}$ maps the quantum T-system in $\mathbf{A}_q[N_-(w)]$ to the one in $\mathbf{A}_q[N_-(w^{-1})]$.*

Proof. Fix $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. Let b, d the integers such that $1 \le b < d \le \ell$ and $i_b = i_d = i$. For $a = 1, \dots, \ell$, set $a_{\mathrm{r}} := \ell - a + 1$. For simplicity of notation, we write $a_{\mathrm{r}}^- := (a_{\mathrm{r}})^-$ and $a_{\mathrm{r}}^-(j) := (a_{\mathrm{r}})^-(j)$ for $a = 1, \dots, \ell$. Note that $w^{-1}\mu^{\boldsymbol{i}}(a', j) = \mu^{\boldsymbol{i}^{\mathrm{rev}}}(\ell - a', j)$ for $a' = 0, \dots, \ell$ and $j \in I$. In particular, for $a = b, d$, we have $w^{-1}\mu^{\boldsymbol{i}}(a, i) = \mu^{\boldsymbol{i}^{\mathrm{rev}}}(a_{\mathrm{r}}^-, i)$, $w^{-1}\mu^{\boldsymbol{i}}(a^-, i) = \mu^{\boldsymbol{i}^{\mathrm{rev}}}(a_{\mathrm{r}}, i)$ and $w^{-1}\mu^{\boldsymbol{i}}(a, j) = \mu^{\boldsymbol{i}^{\mathrm{rev}}}(a_{\mathrm{r}}, j)$ if $j \neq i$. Hence, by applying $\tau_{w,q}$ to both sides of (II.2) and using Theorem II.2.8, we obtain

$$\text{(II.5)} \qquad \begin{aligned} &q^A D^{\boldsymbol{i}^{\mathrm{rev}}}(b_{\mathrm{r}}, d_{\mathrm{r}}) D^{\boldsymbol{i}^{\mathrm{rev}}}(b_{\mathrm{r}}^-, d_{\mathrm{r}}^-) \\ &= q_i^{-1} q^B D^{\boldsymbol{i}^{\mathrm{rev}}}(b_{\mathrm{r}}^-, d_{\mathrm{r}}) D^{\boldsymbol{i}^{\mathrm{rev}}}(b_{\mathrm{r}}, d_{\mathrm{r}}^-) + q^C \overrightarrow{\prod_{j \in I^{\mathrm{rev}} \setminus \{i\}}} D^{\boldsymbol{i}^{\mathrm{rev}}}(b_{\mathrm{r}}^-(j), d_{\mathrm{r}}^-(j))^{-a_{ji}}, \end{aligned}$$

here A, B and C are the same as in Proposition II.2.11 and I^{rev} denotes the index set I with the reverse total order. By the way,

$$(\mu^{\boldsymbol{i}}(b, i), \mu^{\boldsymbol{i}}(b^-, i)) = (s_i\varpi_i, \varpi_i) = (\mu^{\boldsymbol{i}}(d, i), \mu^{\boldsymbol{i}}(d^-, i)),$$
$$(\mu^{\boldsymbol{i}}(a', j), \mu^{\boldsymbol{i}}(a', k)) = (\varpi_j, \varpi_k) \text{ for all } a' = 0, \dots, \ell \text{ and } j, k \in I.$$

Therefore we have

$$\begin{aligned} A = (\mu^{\boldsymbol{i}}(b, i), \mu^{\boldsymbol{i}}(b^-, i) - \mu^{\boldsymbol{i}}(d^-, i)) &= (\mu^{\boldsymbol{i}}(d^-, i), \mu^{\boldsymbol{i}}(d, i) - \mu^{\boldsymbol{i}}(b, i)) \\ &= (\mu^{\boldsymbol{i}^{\mathrm{rev}}}(d_{\mathrm{r}}, i), \mu^{\boldsymbol{i}^{\mathrm{rev}}}(d_{\mathrm{r}}^-, i) - \mu^{\boldsymbol{i}^{\mathrm{rev}}}(b_{\mathrm{r}}^-, i)), \\ B = (\mu^{\boldsymbol{i}}(b^-, i), \mu^{\boldsymbol{i}}(b, i) - \mu^{\boldsymbol{i}}(d^-, i)) &= (\mu^{\boldsymbol{i}}(d^-, i), \mu^{\boldsymbol{i}}(d, i) - \mu^{\boldsymbol{i}}(b^-, i)) \\ &= (\mu^{\boldsymbol{i}^{\mathrm{rev}}}(d_{\mathrm{r}}, i), \mu^{\boldsymbol{i}^{\mathrm{rev}}}(d_{\mathrm{r}}^-, i) - \mu^{\boldsymbol{i}}(b_{\mathrm{r}}, i)), \end{aligned}$$

$$\begin{aligned} C &= \sum_{j \in I \setminus \{i\}} \binom{-a_{ji}}{2} (\mu^{\boldsymbol{i}}(b, j), \mu^{\boldsymbol{i}}(b, j) - \mu^{\boldsymbol{i}}(d, j)) \\ &\quad + \sum_{j,k \in I \setminus \{i\}; j<k} a_{ji}a_{ki}(\mu^{\boldsymbol{i}}(b, j), \mu^{\boldsymbol{i}}(b, k) - \mu^{\boldsymbol{i}}(d, k)) \\ &= \sum_{j \in I^{\mathrm{rev}} \setminus \{i\}} \binom{-a_{ji}}{2} (\mu^{\boldsymbol{i}^{\mathrm{rev}}}(d_{\mathrm{r}}, j), \mu^{\boldsymbol{i}^{\mathrm{rev}}}(d_{\mathrm{r}}, j) - \mu^{\boldsymbol{i}^{\mathrm{rev}}}(b_{\mathrm{r}}, j)) \\ &\quad + \sum_{j,k \in I^{\mathrm{rev}} \setminus \{i\}; k<j} a_{ki}a_{ji}(\mu^{\boldsymbol{i}^{\mathrm{rev}}}(d_{\mathrm{r}}, k), \mu^{\boldsymbol{i}^{\mathrm{rev}}}(d_{\mathrm{r}}, j) - \mu^{\boldsymbol{i}^{\mathrm{rev}}}(b_{\mathrm{r}}, j)). \end{aligned}$$

Therefore the equality (II.5) belongs to the quantum T-system in $\mathbf{A}_q[N_-(w^{-1})]$. □

CHAPTER III

Quantum Berenstein-Fomin-Zelevinsky twist automorphisms

In this chapter, we consider quantum analogues of Berenstein-Fomin-Zelevinsky twist automorphisms (henceforth quantum BFZ-twist automorphisms). See Introduction for their definitions in non-quantum settings. In Section III.1, we construct quantum BFZ-twist automorphisms, and show that they preserve dual canonical bases of quantum unipotent cells, which are defined also in Section III.1. Our quantum BFZ-twist automorphisms on arbitrary quantum unipotent cells are generalizations of the quantum BFZ-twist automorphisms in [**5**, Theorem 2.10] and correspond to those in [**5**, Conjecture 2.12 (c)]. However our approach to the construction is different from their way. Berenstein-Rupel used a quantum cluster algebra structure on a quantum unipotent cell $\mathbf{A}_q[N_-^w]$ and mainly dealt with the case that w is a square of a Coxeter element. Our method relies on the relation between the structures of quantum unipotent cells $\mathbf{A}_q[N_-^w]$ and those of quantized coordinate algebras $\mathbf{R}_q$. The compatibility between quantum BFZ-twist automorphisms and dual canonical bases corresponds [**5**, Conjecture 2.17 (a)]. In Section III.2, we obtain an additive categorification of quantum BFZ-twist automorphisms by using Geiß-Leclerc-Schröer's theory when $\mathfrak{g}$ is symmetric. As a corollary, we show the compatibility between quantum BFZ-twist automorphisms and quantum cluster monomials. In Section III.3, we prove the Chamber Ansatz formulae for arbitrary quantum unipotent cells by using quantum BFZ-twist automorphisms. This is a generalization of [**5**, Corllary 1.2].

III.1. Quantum Berenstein-Fomin-Zelevinsky twist automorphisms

In this section, we construct quantum BFZ-twist automorphisms on quantum unipotent cells (Theorem III.1.42). Quantum BFZ-twist automorphisms are $\mathbb{Q}(q)$-algebra automorphisms on quantum unipotent cells. Quantum unipotent cells are localizations of quantum closed unipotent cells, but our construction of quantum BFZ-twist automorphisms requires the quantized coordinate algebras $\mathbf{R}_q$ associated with $\mathbf{U}_q$. We also define dual canonical bases of quantum unipotent cells (Definition III.1.35). Then quantum BFZ-twist automorphisms are restricted to permutations on these dual canonical bases. This compatibility is essentially useful when we consider an additive categorification of quantum BFZ-twist automorphisms in Section III.2.

III.1.1. Unipotent quantum matrix coefficients. We introduce quantum analogues of matrix coefficients on unipotent groups N_- and variants $j_{w\lambda}^{\vee}$ of j_λ in Proposition I.1.37. They are useful for describing quantum BFZ-twist automorphisms (Theorem III.1.30, III.1.42).

DEFINITION III.1.1. For $\lambda \in P_+$ and $u, u' \in V(\lambda)$, define the element $D_{u,u'} \in \mathbf{U}_q^-$ by

$$(D_{u,u'}, x)_L = (u, x.u')_\lambda^\varphi$$

for all $x \in \mathbf{U}_q^-$. We call an element of this form a unipotent quantum matrix coefficient. Note that $\mathrm{wt}\,(D_{u,u'}) = \mathrm{wt}\, u - \mathrm{wt}\, u'$ for weight vectors $u, u' \in V(\lambda)$ and $D_{u_{w\lambda}, u_{w'\lambda}} = D_{w\lambda, w'\lambda}$ for $w, w' \in W$. Recall Definition II.2.1.

DEFINITION III.1.2. Let $\lambda \in P_+$. Define a surjective $\mathbb{Q}(q)$-linear map $\pi_{w\lambda}^\vee \colon \mathbf{U}_q^- \to V_w(\lambda)$ by

$$\pi_{w\lambda}^\vee(y) = y^\vee.u_{w\lambda}.$$

PROPOSITION III.1.3 ([**44**, Proposition 25.2.6],[**31**, 8.2.2 (iii), (iv)]). *Let $\lambda \in P_+$ and $w \in W$. Then there exists a surjective map $\pi_{w\lambda}^\vee \colon \mathscr{B}(\infty) \to \mathscr{B}_w(\lambda) \coprod \{0\}$ such that*

$$\pi_{w\lambda}^\vee(G(b)) = g(\pi_{w\lambda}^\vee(b))$$

for $b \in \mathscr{B}(\infty)$, here $g(0) = 0$. Moreover, $\pi_{w\lambda}^\vee$ induces a bijection $(\pi_{w\lambda}^\vee)^{-1}(\mathscr{B}_w(\lambda)) \to \mathscr{B}_w(\lambda)$.

DEFINITION III.1.4. Let $\lambda \in P_+$ and $w \in W$. Set $V_w(\lambda)^\perp := \{u \in V(\lambda) \mid (u, V_w(\lambda))_\lambda^\varphi = 0\}$. Define $j_{w\lambda}^\vee \colon V(\lambda)/V_w(\lambda)^\perp \hookrightarrow \mathbf{U}_q^-$ as the dual homomorphism of $\pi_{w\lambda}^\vee$ given by the nondegenerate bilinear forms $(\ ,\ )_\lambda^\varphi : V(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$ and $(\ ,\ )_L : \mathbf{U}_q^- \times \mathbf{U}_q^- \to \mathbb{Q}(q)$, that is,

$$(j_{w\lambda}^\vee(u), y)_L = (u, \pi_{w\lambda}^\vee(y))_\lambda^\varphi (= (u, y^\vee.u_{w\lambda})_\lambda^\varphi = (\varphi(y^\vee).u, u_{w\lambda})_\lambda^\varphi).$$

In the following, the map $V(\lambda) \to \mathbf{U}_q^-$ given by $u \mapsto j_{w\lambda}^\vee(p_w(u))$ is also denoted by $j_{w\lambda}^\vee$, here p_w is the canonical projection $V(\lambda) \to V(\lambda)/V_w(\lambda)^\perp$.

The following proposition immediately follows from Proposition III.1.3.

PROPOSITION III.1.5. *Let $\lambda \in P_+$ and $w \in W$. Then there is an injective map $\jmath_{w\lambda}^\vee \colon \mathscr{B}_w(\lambda) \hookrightarrow \mathscr{B}(\infty)$ such that*

$$(g^{\mathrm{up}}(b), G(\tilde{b}')^\vee.u_{w\lambda})_\lambda^\varphi = \delta_{\tilde{b}', \jmath_{w\lambda}^\vee(b)}$$

for any $b \in \mathscr{B}_w(\lambda)$ and $\tilde{b}' \in \mathscr{B}(\infty)$. That is, we have $j_{w\lambda}^\vee(g^{\mathrm{up}}(b)) = G^{\mathrm{up}}(\jmath_{w\lambda}^\vee(b))$. .

REMARK III.1.6. Let $\lambda \in P_+$ and $w \in W$. Then,

- $\mathrm{wt}\, \jmath_{w\lambda}^\vee(b) = -\mathrm{wt}\, b + w\lambda$ for $b \in \mathscr{B}_w(\lambda)$, and
- $\jmath_{w\lambda}^\vee(\pi_{w\lambda}^\vee(\tilde{b})) = \tilde{b}$ for $\tilde{b} \in (\pi_{w\lambda}^\vee)^{-1}(\mathscr{B}_w(\lambda))$.

PROPOSITION III.1.7. *Let $\lambda \in P_+$ and $w \in W$. Then the following hold:*

(1) $D_{g^{\mathrm{up}}(b), u_\lambda} = G^{\mathrm{up}}(\jmath_\lambda(b))$ *for all $b \in \mathscr{B}(\lambda)$,*
(2) $D_{u_{w\lambda}, g^{\mathrm{up}}(b)} = G^{\mathrm{up}}(*\jmath_{w\lambda}^\vee(b))$ *for all $b \in \mathscr{B}_w(\lambda)$, and*
(3) $D_{u_{w\lambda}, g^{\mathrm{up}}(b)} = 0$ *for all $b \in \mathscr{B}(\lambda) \setminus \mathscr{B}_w(\lambda)$.*

Proof. The equality (1) follows immediately by Proposition I.1.37. For $y \in \mathbf{U}_q^-$, we have

$$
\begin{aligned}
(D_{u_{w\lambda},g^{\mathrm{up}}(b)}, y)_L &= (u_{w\lambda}, y.g^{\mathrm{up}}(b))^{\varphi}_{\lambda} \\
&= (g^{\mathrm{up}}(b), (*(y))^{\vee}.u_{w\lambda})^{\varphi}_{\lambda} \\
&= (G^{\mathrm{up}}(\bar{\jmath}^{\vee}_{w\lambda}(b)), *(y))_L \\
&= (G^{\mathrm{up}}(*\bar{\jmath}^{\vee}_{w\lambda}(b)), y)_L.
\end{aligned}
$$

This completes the proof of (2). The assertion (3) follows from the similar calculation and Proposition I.1.52. ☐

Proposition III.1.8 ([**34**, Corollary 6.4]). *Let $w \in W$ and $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$. For $i \in I$, define $\boldsymbol{n}^{(i)} = (n_1^{(i)}, \ldots, n_\ell^{(i)}) \in \mathbb{Z}^{\ell}_{\geq 0}$ by*

$$
n_k^{(i)} = \begin{cases} 1 & \text{if } i_k = i, \\ 0 & \text{otherwise.} \end{cases}
$$

For $\lambda \in P_+$, set $\boldsymbol{n}^{\lambda} := \sum_{i\in I} \langle \lambda, h_i \rangle \boldsymbol{n}^{(i)}$. Then we have

$$
D_{w\lambda,\lambda} = G^{\mathrm{up}}(b_{-1}(\boldsymbol{n}^{\lambda}, \boldsymbol{i})).
$$

III.1.2. Kumar-Peterson identity. We investigate the map $\bar{\jmath}^{\vee}_{w\lambda}$ a little more. Kumar and Peterson studied the identity which expresses the "characters" of the coordinate ring $\mathbb{C}[X_w \cap U_v]$ of the intersection $X(w) \cap U_v$ of Schubert varieties $X(w)$ and "v-translates of the open cell U_v" as the limit of a family of "twisted" characters of Demazure modules in general Kac-Moody Lie algebras, see Kumar [**38**, Theorem 12.1.3] for details. The following bijection can be considered as a crystalized Kumar-Peterson identity for the special case $v = w$.

Theorem III.1.9. *We have*

$$
\bigcup_{\lambda \in P_+} \bar{\jmath}^{\vee}_{w\lambda}(\mathscr{B}_w(\lambda)) = \mathscr{B}\left(\mathbf{U}_q^-(w)\right).
$$

The rest of this subsection is devoted to the proof of Theorem III.1.9.

Lemma III.1.10. *For $y \in \mathbf{U}_q^-(w)^{\perp}$, we have $y^{\vee}.u_{w\lambda} = 0$ for all $\lambda \in P_+$.*

Proof. By Lemma II.2.5, we write $y = \sum y_{(1)} y_{(2)}$ with $y_{(1)} \in \mathbf{U}_q^-(w)$ and homogeneous elements $y_{(2)} \in \mathbf{U}_q^- \cap T_w \mathbf{U}_q^- \cap \operatorname{Ker} \varepsilon$. Then, by Proposition I.1.29 and I.1.30, we have

$$
y^{\vee}.u_{w\lambda} = (T_{w^{-1}})^{-1} \left(\sum T_{w^{-1}}\left(y^{\vee}_{(1)}\right) T_{w^{-1}}\left(y^{\vee}_{(2)}\right).u_{\lambda} \right) = 0
$$

because $\mathrm{wt}(T_{w^{-1}}(y^{\vee}_{(2)})) \in Q_+ \setminus \{0\}$. ☐

Proposition III.1.11. *We have*

$$
\bigcup_{\lambda \in P_+} \bar{\jmath}^{\vee}_{w\lambda}(\mathscr{B}_w(\lambda)) \subset \mathscr{B}\left(\mathbf{U}_q^-(w)\right).
$$

Proof. Let $\pi(w) : \mathbf{U}_q^- \to \mathbf{U}_q^-$ be the orthogonal projection with respect to $\mathbf{U}_q^- = \mathbf{U}_q^-(w) \oplus \mathbf{U}_q^-(w)^\perp$. Since $\mathbf{U}_q^-(w)^\perp \cap \mathbf{B}^{\mathrm{low}}$ is a basis of $\mathbf{U}_q^-(w)^\perp$ by Proposition I.1.47, we have $\pi(w)(G(\tilde{b})) \neq 0$ if and only if $\tilde{b} \in \mathscr{B}\left(\mathbf{U}_q^-(w)\right)$ for $\tilde{b} \in \mathscr{B}(\infty)$. Let $\tilde{b} \in \bigcup_{\lambda \in P_+} \overline{\jmath}_{w\lambda}^\vee(\mathscr{B}_w(\lambda))$. Then there exists $\lambda \in P_+$ such that $G(\tilde{b})^\vee . u_{w\lambda} \neq 0$. By Proposition III.1.10, we have

$$G(\tilde{b})^\vee . u_{w\lambda} = (\pi(w)(G(\tilde{b})))^\vee . u_{w\lambda}.$$

In particular, we have $\pi(w)(G(\tilde{b})) \neq 0$. This completes the proof. □

We prove the opposite inclusion.

Proposition III.1.12. *We have*

$$\mathscr{B}\left(\mathbf{U}_q^-(w)\right) \subset \bigcup_{\lambda \in P_+} \overline{\jmath}_{w\lambda}^\vee(\mathscr{B}_w(\lambda)).$$

Proof. Let $\tilde{b} \in \mathscr{B}\left(\mathbf{U}_q^-(w)\right)$, that is $0 \neq \pi(w)(G(\tilde{b})) \in \mathbf{U}_q^-(w)$. (See the proof of Proposition III.1.11.) By Proposition III.1.3 and Remark III.1.6, it suffices to show that $G(\tilde{b})^\vee . u_{w\lambda} = (\pi(w)(G(\tilde{b})))^\vee . u_{w\lambda} \neq 0$ for some $\lambda \in P_+$. Note that $(\pi(w)(G(\tilde{b})))^\vee . u_{w\lambda} \neq 0$ is equivalent to $\overline{(\pi(w)(G(\tilde{b})))}^\vee . u_{w\lambda} \neq 0$.

By the way, it follows from Proposition I.1.29 that

$$\overline{y}^\vee . u_{w\lambda} = (T_{w^{-1}})^{-1}\left((T_{w^{-1}} \circ \vee \circ {}^-)(y) . u_\lambda\right) = (T_{w^{-1}})^{-1}\left(\left(\vee \circ {}^- \circ T_w^{-1}\right)(y) . u_\lambda\right).$$

Since $y_0 := \pi(w)(G(\tilde{b})) \in \mathbf{U}_q^- \cap T_w \mathbf{U}_q^{\geq 0}$, we have $(\vee \circ {}^- \circ T_w^{-1})(y_0) \in \mathbf{U}_q^{\leq 0}$. It is well-known that, for $\xi \in Q_-$, there exists an element $\lambda \in P_+$ such that the projection $\left(\mathbf{U}_q^-\right)_\xi \to V(\lambda)_{\xi+\lambda}$ given by $y \mapsto y.u_\lambda$ is an isomorphism of vector space. Hence it can be shown that there exists $\lambda \in P_+$ such that $(\vee \circ {}^- \circ T_w^{-1})(y_0).u_\lambda \neq 0$. □

III.1.3. Other descriptions of quantum unipotent subgroups and quantum unipotent cells. In this subsection, we describe the algebras, quantum unipotent subgroups and quantum unipotent cells, by using the quantized coordinate algebra $\mathbf{R}_q$. The following descriptions are essentially shown in [**26**, 9.1.7], [**49**, Theorem 3.7]. However, we state them again emphasizing the terms of dual canonical bases. Actually, we can now prove each statement immediately.

Notation III.1.13. Let $v, w \in W$. By abuse of notation, we describe the canonical projection $\mathbf{R}_q^{w(+)} \to \mathbf{R}_q^{w(+)}/\mathbf{Q}_v^{w(+)}$ as $c \mapsto [c]$.

Definition III.1.14. Let $\lambda \in P_+$. Set

$$\mathbf{U}_q^-(\lambda) := j_\lambda(V(\lambda)) = \sum\nolimits_{b \in \mathscr{B}(\lambda)} \mathbb{Q}(q) G^{\mathrm{up}}(\overline{\jmath}_\lambda(b)).$$

Recall Definition I.1.24 and I.1.26. The following propositions follow from Lemma I.1.15 and Proposition III.1.7.

Proposition III.1.15. *The $\mathbb{Q}(q)$-algebra homomorphism $\mathbf{R}_q^+ \to \check{\mathbf{U}}_q^{\leq 0}$, $F \mapsto (\Psi_- \circ R_{\leq 0})(F)$ induces the $\mathbb{Q}(q)$-algebra isomorphism $\mathcal{I}\colon \mathbf{R}_q^+ \to \sum_{\lambda \in P_+} \mathbf{U}_q^-(\lambda) q^{-\lambda}$.*

Proposition III.1.16. *For $\lambda \in P_+$ and $b \in \mathscr{B}(\lambda)$, we have*

$$\mathcal{I}\left(c^{\lambda}_{g^{\mathrm{up}}(b)^*, u_\lambda}\right) = G^{\mathrm{up}}\left(\overline{\jmath}_\lambda(b)\right) q^{-\lambda} = D_{g^{\mathrm{up}}(b), u_\lambda} q^{-\lambda}.$$

In particular, for $w \in W$, we have

$$\mathcal{I}\left(\mathbf{Q}_w^+(\lambda)\right) = \sum\nolimits_{b \in \mathscr{B}(\lambda) \setminus \mathscr{B}_w(\lambda)} \mathbb{Q}(q) G^{\mathrm{up}}\left(\overline{\jmath}_\lambda(b)\right) q^{-\lambda}.$$

Definition III.1.17. An element z of $\mathbf{R}_q^+$ (resp. $\mathbf{R}_q^+/\mathbf{Q}_w^+$) is said to be q-central if, for every weight vectors $f \in V(\lambda)^\star$ and $\lambda \in P_+$, there exists $l \in \mathbb{Z}$ such that

$$z c^{\lambda}_{f,u_\lambda} = q^l c^{\lambda}_{f,u_\lambda} z \quad (\text{resp. } z[c^{\lambda}_{f,u_\lambda}] = q^l [c^{\lambda}_{f,u_\lambda}] z).$$

Corollary III.1.18. *The set $\mathcal{S} = \{c^{\lambda}_{\lambda,\lambda}\}_{\lambda \in P_+}$ is an Ore set in $\mathbf{R}_q^+$ consisting of q-central elements. In particular, $[\mathcal{S}] := \{[c^{\lambda}_{\lambda,\lambda}]\}_{\lambda \in P_+}$ is an Ore set in $\mathbf{R}_q^+/\mathbf{Q}_w^+$ consisting of q-central elements.*

By Corollary III.1.18, we can consider the algebra $(\mathbf{R}_q^+/\mathbf{Q}_w^+)[[\mathcal{S}]^{-1}]$. Proposition III.1.15 and III.1.16 together with Remark I.1.53 immediately imply the following proposition. This gives the description of $\mathbf{A}_q[N_- \cap X(w)]$ in terms of the quantized coordinate algebra $\mathbf{R}_q$. This kind of description appears in [**26**, 9.1.7].

Proposition III.1.19. *Let $w \in W$. Set $\mathbf{A}_q[N_- \cap X(w)]^{\mathrm{ex}} := \check{\mathbf{U}}_q^{\leq 0}/(\mathbf{U}_w^-)^\perp \check{\mathbf{U}}_q^0$. Note that $(\mathbf{U}_w^-)^\perp \check{\mathbf{U}}_q^0$ is a two-sided ideal of $\check{\mathbf{U}}_q^{\leq 0}$. Then the $\mathbb{Q}(q)$-algebra isomorphism $\mathcal{I}$ induces the $\mathbb{Q}(q)$-algebra isomorphism*

$$\mathcal{I}_w\colon \left(\mathbf{R}_q^+/\mathbf{Q}_w^+\right)[[\mathcal{S}]^{-1}] \to \mathbf{A}_q[N_- \cap X(w)]^{\mathrm{ex}}.$$

Moreover the $\mathbb{Q}(q)$-algebra $\sum_{\lambda \in P_+} \left(\mathbf{R}_q^+(\lambda)/\mathbf{Q}_w^+\right)[c^{\lambda}_{\lambda,\lambda}]^{-1} (\subset \left(\mathbf{R}_q^+/\mathbf{Q}_w^+\right)[[\mathcal{S}]^{-1}])$ is isomorphic to $\mathbf{A}_q[N_- \cap X(w)]$.

Next we consider the algebra $\mathbf{R}_q^{w(+)}/\mathbf{Q}_w^{w(+)}$, which is isomorphic to $\mathbf{R}_q^+/\mathbf{Q}_w^+$. See Definition I.1.23.

Definition III.1.20. Let $w \in W$ and $\lambda \in P_+$. Set

$$\mathbf{U}_q^+(w,\lambda) := \left(j^\vee_{w\lambda}\left(V(\lambda)/V_w(\lambda)^\perp\right)\right)^\vee = \sum\nolimits_{b \in \mathscr{B}_w(\lambda)} \mathbb{Q}(q) G^{\mathrm{up}}\left(\overline{\jmath}^\vee_{w\lambda}(b)\right)^\vee.$$

The following proposition follows again from the nondegeneracy of the Drinfeld pairing, the equality (I.2), Lemma I.1.15, Proposition I.1.40 and Proposition III.1.7.

Proposition III.1.21. *Let $w \in W$. The restriction map $R_{\geq 0}\colon \mathbf{U}_q^* \to (\mathbf{U}_q^{\geq 0})^*$ induces the algebra homomorphism $R^w_{\geq 0}\colon \mathbf{R}_q^{w(+)} \to (\mathbf{U}_q^{\geq 0})^*$, and it satisfies $\mathrm{Ker}(R^w_{\geq 0}) = \mathbf{Q}_w^{w(+)}$ and $\mathrm{Im}\, R^w_{\geq 0} \subset \mathrm{Im}\, \Phi^+$. Hence $R^w_{\geq 0}$ induces the $\mathbb{Q}(q)$-algebra isomorphism $\overline{R}^w_{\geq 0}\colon \mathbf{R}_q^{w(+)}/\mathbf{Q}_w^{w(+)} \to \mathrm{Im}\, R^w_{\geq 0}$. Moreover we have an well-defined anti-algebra isomorphism $\mathcal{I}_w^+\colon \mathbf{R}_q^+/\mathbf{Q}_w^+ \to$*

$\sum_{\lambda\in P_+}\mathbf{U}_q^+(w,\lambda)q^{-w\lambda}$ *given by* $[c^{\lambda}_{f,u_\lambda}]\mapsto\left((\Phi^+)^{-1}\circ\overline{R}^{w}_{\geq 0}\right)\left([c^{\lambda}_{f,u_{w\lambda}}]\right)$ *for* $f\in V(\lambda)^{\star}$, $\lambda\in P_+$. *We have*

$$\mathcal{I}^+_w\left([c^{\lambda}_{g^{\mathrm{up}}(b)^*,u_\lambda}]\right)=G^{\mathrm{up}}\left(\jmath^{\vee}_{w\lambda}(b)\right)^{\vee}q^{-w\lambda}=\varphi(D_{u_{w\lambda},g^{\mathrm{up}}(b)})q^{-w\lambda}$$

for $b\in\mathscr{B}_w(\lambda)$.

Corollary III.1.22. *The set* $[{}_w\mathcal{S}]:=\{[c^{\lambda}_{w\lambda,\lambda}]\}_{\lambda\in P_+}$ *is an Ore set in* $\mathbf{R}_q^+/\mathbf{Q}_w^+$ *consisting of q-central elements.*

Remark III.1.23. The description in Proposition III.1.21 implies that the algebra $\mathbf{R}_q^+/\mathbf{Q}_w^+$ has no zero divisors.

By Corollary III.1.22, we can consider the $\mathbb{Q}(q)$-algebra $(\mathbf{R}_q^+/\mathbf{Q}_w^+)[[{}_w\mathcal{S}]^{-1}]$. Proposition III.1.21 immediately implies the following proposition. This gives the description of $\mathbf{A}_q[N_-(w)]$ in terms of the quantized coordinate algebra $\mathbf{R}_q$. This kind of description appears in [**49**, Theorem 3.7].

Proposition III.1.24. *Let* $w\in W$. *Then* $\mathcal{I}^+_w$ *induces the anti-algebra isomorphism*

$$\mathcal{I}^+_w\colon\left(\mathbf{R}_q^+/\mathbf{Q}_w^+\right)[[{}_w\mathcal{S}]^{-1}]\to\mathbf{U}_q^+(w)\check{\mathbf{U}}_q^0.$$

Moreover the $\mathbb{Q}(q)$-*algebra* $\sum_{\lambda\in P_+}\left(\mathbf{R}_q^+(\lambda)/\mathbf{Q}_w^+\right)[c^{\lambda}_{w\lambda,\lambda}]^{-1}(\subset\left(\mathbf{R}_q^+/\mathbf{Q}_w^+\right)[[{}_w\mathcal{S}]^{-1}])$ *is anti-isomorphic to* $\mathbf{U}_q^+(w)$, *and is isomorphic to* $\mathbf{A}_q[N_-(w)]$ *via* φ.

Proof. It suffices to show that $\sum_{\lambda\in P_+}\mathbf{U}_q^+(w,\lambda)=\mathbf{U}_q^+(w)$. This follows from Theorem III.1.9. □

III.1.4. Quantum BFZ-twist isomorphisms and dual canonical bases. In this subsection, we consider two kinds of localized algebras, $\mathbf{A}_q[N_-(w)\cap\dot{w}G_0]$ and $\mathbf{A}_q[N_-^w]$. The former is a localization of the quantum unipotent subgroup $\mathbf{A}_q[N_-(w)]$ and the latter is a localization of $\mathbf{A}_q[N_-\cap X(w)]$ (Definition III.1.29). The latter is called the quantum unipotent cell. The aim of this section is construct a quantum analogue of the BFZ-twist automorphism on $\mathbf{A}_q[N_-^w]$. In preparation for it, we construct a $\mathbb{Q}(q)$-algebra isomorphism from $\mathbf{A}_q[N_-^w]$ to $\mathbf{A}_q[N_-(w)\cap\dot{w}G_0]$ in a “twisted” way (Theorem III.1.30). This is a quantum analogue of $\tilde{\eta}_w\mid_{N_-(w)\cap\dot{w}G_0}(=:\gamma_w)$ in [**18**, Subsection 8.2] (see also [**18**, Proposition 8.4 (iv)]). Actually, the construction of this isomorphism is an essential step for our construction of a quantum BFZ-twist automorphism. Moreover we naturally define the dual canonical bases of $\mathbf{A}_q[N_-(w)\cap\dot{w}G_0]$ and $\mathbf{A}_q[N_-^w]$ (Definition III.1.35). At last, we show that the isomorphism above induces a bijection between these dual canonical bases.

Notation III.1.25. Let $\mathbf{V}$ be a $\mathbb{Z}[q^{\pm1}]$-module. For a subset $\mathcal{M}\subset\mathbf{V}$, write $q^{\mathbb{Z}}\mathcal{M}:=\{q^km\mid m\in\mathcal{M},k\in\mathbb{Z}\}$.

The following lemma easily follows from Corollary III.1.18 and III.1.22. See also [**21**, Proposition 6.3]. This localization is important in the proof of Theorem III.1.30.

Lemma III.1.26. *Let* $w\in W$. *Then the set* $[{}_w\tilde{\mathcal{S}}]:=q^{\mathbb{Z}}\{[c^{\lambda}_{w\lambda,\lambda}c^{\lambda'}_{\lambda',\lambda'}]\}_{\lambda,\lambda'\in P_+}$ *is an Ore set in* $\mathbf{R}_q^+/\mathbf{Q}_w^+$ *consisting of q-central elements.*

Moreover the maps $(\mathbf{R}_q^+/\mathbf{Q}_w^+)[[\mathcal{S}]^{-1}] \to (\mathbf{R}_q^+/\mathbf{Q}_w^+)[[{}_w\tilde{\mathcal{S}}]^{-1}]$, $[c^{\lambda}_{f,u_\lambda}][c^{\lambda'}_{\lambda',\lambda'}]^{-1} \mapsto [c^{\lambda}_{f,u_\lambda}][c^{\lambda'}_{\lambda',\lambda'}]^{-1}$ *and* $(\mathbf{R}_q^+/\mathbf{Q}_w^+)[[{}_w\mathcal{S}]^{-1}] \to (\mathbf{R}_q^+/\mathbf{Q}_w^+)[[{}_w\tilde{\mathcal{S}}]^{-1}]$, $[c^{\lambda}_{f,u_\lambda}][c^{\lambda'}_{w\lambda',\lambda'}]^{-1} \mapsto [c^{\lambda}_{f,u_\lambda}][c^{\lambda'}_{w\lambda',\lambda'}]^{-1}$ *are injective* $\mathbb{Q}(q)$*-algebra homomorphisms.*

We prove the following proposition (Proposition III.1.27) and theorem (Theorem III.1.30) simultaneously.

PROPOSITION III.1.27. *Let* $w \in W$ *and set* ${}_w\mathcal{D} := q^{\mathbb{Z}}\{D_{w\lambda,\lambda}\}_{\lambda \in P_+}$. *Then the sets* ${}_w\mathcal{D}$ *and* $[{}_w\mathcal{D}]$ *are Ore sets of* $\mathbf{A}_q[N_-(w)]$ *and* $\mathbf{A}_q[N_- \cap X(w)]$ *respectively consisting of* q*-central elements. More explicitly, for* $\lambda, \lambda' \in P_+$ *and homogeneous elements* $x \in \mathbf{A}_q[N_-(w)]$, $y \in \mathbf{A}_q[N_- \cap X(w)]$, *we have*

$$q^{-(\lambda, w\lambda'-\lambda')} D_{w\lambda,\lambda} D_{w\lambda',\lambda'} = D_{w(\lambda+\lambda'),\lambda+\lambda'}$$

$$D_{w\lambda,\lambda} x = q^{(\lambda+w\lambda, \mathrm{wt}\, x)} x D_{w\lambda,\lambda} \text{ in } \mathbf{A}_q[N_-(w)], \text{ and}$$

$$[D_{w\lambda,\lambda}][y] = q^{(\lambda+w\lambda, \mathrm{wt}\, y)} [y][D_{w\lambda,\lambda}] \text{ in } \mathbf{A}_q[N_- \cap X(w)].$$

REMARK III.1.28. In fact, Proposition III.1.27 is a known fact. See, for example, [**34**, Proposition 6.11, Corollary 6.18].

DEFINITION III.1.29. By Proposition III.1.27, we can consider the localizations;

$$\mathbf{A}_q[N_-(w) \cap \dot{w}G_0] := \mathbf{A}_q[N_-(w)][{}_w\mathcal{D}^{-1}],$$

$$\mathbf{A}_q[N_-^w] := \mathbf{A}_q[N_- \cap X(w)][[{}_w\mathcal{D}]^{-1}].$$

Those algebras have Q-graded algebra structures in an obvious way. The algebra $\mathbf{A}_q[N_-^w]$ is called a quantum unipotent cell.

The algebras $\mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$ and $\mathbf{A}_q[N_-^w]$ are isomorphic as follows. These isomorphisms are "almost" the desired quantum BFZ-twist automorphisms. In fact, these are quantum analogues of the maps $\tilde{\eta}_w$ in [**18**, Subsection 8.2] (see also [**18**, Proposition 8.4 (iv)]). See also subsection III.1.5.

THEOREM III.1.30. *There exists an isomorphism of* $\mathbb{Q}(q)$*-algebras*

$$\gamma_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$$

given by

$$[D_{u,u_\lambda}] \mapsto q^{-(\lambda, \mathrm{wt}\, u-\lambda)} D_{w\lambda,\lambda}^{-1} D_{u_{w\lambda},u}, \qquad [D_{w\lambda,\lambda}]^{-1} \mapsto q^{(\lambda, w\lambda-\lambda)} D_{w\lambda,\lambda}$$

for a weight vector $u \in V(\lambda)$ *and* $\lambda \in P_+$.

DEFINITION III.1.31. We call $\gamma_{w,q}$ a quantum twist isomorphism.

PROOF OF PROPOSITION III.1.27 AND THEOREM III.1.30. By Proposition III.1.19 (see also Proposition III.1.16), we have the algebra isomorphism

$$\mathbf{A}_q[N_- \cap X(w)] \xrightarrow{\mathcal{I}_w^{-1}} \sum_{\lambda \in P_+} \left(\mathbf{R}_q^+(\lambda)/\mathbf{Q}_w^+\right) [c^{\lambda}_{\lambda,\lambda}]^{-1} \tag{III.1}$$

given by

$$[D_{u,u_\lambda}] \mapsto [c^\lambda_{u^*,u_\lambda}][c^\lambda_{\lambda,\lambda}]^{-1} \tag{III.2}$$

for $\lambda \in P_+$ and $u \in V(\lambda)$. In particular, $\mathcal{I}_w^{-1}([D_{w\lambda,\lambda}]) = [c^\lambda_{w\lambda,\lambda}][c^\lambda_{\lambda,\lambda}]^{-1}$.

By Lemma III.1.26, $\sum_{\lambda\in P_+}\left(\mathbf{R}_q^+(\lambda)/\mathbf{Q}_w^+\right)[c^\lambda_{\lambda,\lambda}]^{-1}$ is naturally regarded as a subalgebra of $(\mathbf{R}_q^+/\mathbf{Q}_w^+)[[{}_w\tilde{\mathcal{S}}]^{-1}]$, and in the latter algebra, the set $q^{\mathbb{Z}}\{[c^\lambda_{w\lambda,\lambda}][c^\lambda_{\lambda,\lambda}]^{-1}\}_{\lambda\in P_+}$ is a multiplicative set consisting of invertible q-central elements. Hence $[{}_w\mathcal{D}]$ is an Ore set of $\mathbf{A}_q[N_-\cap X(w)]$ consisting of q-central elements, and the algebra isomorphism (III.1) is extended to the algebra isomorphism

$$\mathcal{J}_1\colon \mathbf{A}_q[N_-^w] \to \sum_{\substack{\lambda,\lambda',\lambda''\in P_+\\ \lambda=\lambda'+\lambda''}}\left(\mathbf{R}_q^+(\lambda)/\mathbf{Q}_w^+\right)[c^{\lambda'}_{\lambda',\lambda'}c^{\lambda''}_{w\lambda'',\lambda''}]^{-1}. \tag{III.3}$$

On the other hand, by Proposition III.1.24 (see also Proposition III.1.21), we have an algebra isomorphism

$$\sum_{\lambda\in P_+}\left(\mathbf{R}_q^+(\lambda)/\mathbf{Q}_w^+\right)[c^\lambda_{w\lambda,\lambda}]^{-1} \xrightarrow{\varphi\circ\mathcal{I}_w^+} \mathbf{A}_q[N_-(w)], \tag{III.4}$$

given by

$$[c^\lambda_{w\lambda,\lambda}]^{-1}[c^\lambda_{u^*,u_\lambda}] \mapsto D_{u_{w\lambda},u} \tag{III.5}$$

for $\lambda\in P_+$ and $u \in V(\lambda)$. In particular, $(\varphi\circ\mathcal{I}_w^+)([c^\lambda_{w\lambda,\lambda}]^{-1}[c^\lambda_{\lambda,\lambda}]) = D_{w\lambda,\lambda}$.

As above, the set $q^{\mathbb{Z}}\{[c^\lambda_{w\lambda,\lambda}]^{-1}[c^\lambda_{\lambda,\lambda}]\}_{\lambda\in P_+}$ is a multiplicative set consisting of invertible q-central elements of $(\mathbf{R}_q^+/\mathbf{Q}_w^+)[[{}_w\tilde{\mathcal{S}}]^{-1}]$. Hence ${}_w\mathcal{D}$ is an Ore set of $\mathbf{A}_q[N_-(w)]$ consisting of q-central elements, and the algebra isomorphism (III.4) is extended to the algebra isomorphism

$$\mathcal{J}_2\colon \sum_{\substack{\lambda,\lambda',\lambda''\in P_+\\ \lambda=\lambda'+\lambda''}}\left(\mathbf{R}_q^+(\lambda)/\mathbf{Q}_w^+\right)[c^{\lambda'}_{\lambda',\lambda'}c^{\lambda''}_{w\lambda'',\lambda''}]^{-1} \to \mathbf{A}_q[N_-(w)\cap \dot{w}G_0]. \tag{III.6}$$

By the way, we obtained Proposition III.1.27. The calculation of explicit q-commutation is left to the reader.

By (III.3) and (III.6), we obtain the $\mathbb{Q}(q)$-algebra isomorphism

$$\gamma_{w,q} := \mathcal{J}_2\circ\mathcal{J}_1\colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-(w)\cap\dot{w}G_0].$$

Moreover, for $\lambda\in P_+$ and a weight vector $u\in V(\lambda)$, we have

$$\begin{aligned}\gamma_{w,q}(D_{u,u_\lambda}) &= \mathcal{J}_2([c^\lambda_{u^*,u_\lambda}][c^\lambda_{\lambda,\lambda}]^{-1}) \text{ by (III.2)},\\ &= \mathcal{J}_2(q^{-(\lambda,\mathrm{wt}\,u-\lambda)}[c^\lambda_{\lambda,\lambda}]^{-1}[c^\lambda_{u^*,u_\lambda}])\\ &= \mathcal{J}_2(q^{-(\lambda,\mathrm{wt}\,u-\lambda)}[c^\lambda_{\lambda,\lambda}]^{-1}[c^\lambda_{w\lambda,\lambda}][c^\lambda_{w\lambda,\lambda}]^{-1}[c^\lambda_{u^*,u_\lambda}])\\ &= q^{-(\lambda,\mathrm{wt}\,u-\lambda)}D^{-1}_{w\lambda,\lambda}D_{u_{w\lambda},u} \text{ by (III.5)}.\end{aligned}$$

Moreover,

$$\begin{aligned}1 &= \gamma_{w,q}([D_{w\lambda,\lambda}][D_{w\lambda,\lambda}]^{-1}) \\ &= q^{-(\lambda,w\lambda-\lambda)}D_{w\lambda,\lambda}^{-1}\gamma_{w,q}([D_{w\lambda,\lambda}]^{-1}).\end{aligned}$$

Hence,

$$\gamma_{w,q}([D_{w\lambda,\lambda}]^{-1}) = q^{(\lambda,w\lambda-\lambda)}D_{w\lambda,\lambda}.$$

This completes the proof of the theorem. □

Next we define the dual canonical bases of $\mathbf{A}_q[N_-^w]$ and $\mathbf{A}_q[N_-(w)\cap \dot{w}G_0]$.

Proposition III.1.32 ([**34**, Theorem 6.24, Theorem 6.25]). *Let $w \in W$.*

(1) *For $\lambda \in P_+$ and $b \in \mathscr{B}_w(\infty)$, there exists $b' \in \mathscr{B}_w(\infty)$ such that*

$$q^{-(\lambda,\mathrm{wt}\, b)}[D_{w\lambda,\lambda}][G^{\mathrm{up}}(b)] = [G^{\mathrm{up}}(b')].$$

(2) *For $\lambda \in P_+$, $\boldsymbol{i} \in I(w)$ and $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, we have*

$$q^{-(\lambda,\mathrm{wt}\, b_{-1}(\boldsymbol{c},\boldsymbol{i}))}D_{w\lambda,\lambda}G^{\mathrm{up}}(b_{-1}(\boldsymbol{c},\boldsymbol{i})) = G^{\mathrm{up}}(b_{-1}(\boldsymbol{c}+\boldsymbol{n}^{\lambda},\boldsymbol{i})),$$

where $\boldsymbol{n}^{\lambda}$ is defined as in Proposition III.1.8.

Remark III.1.33. Proposition I.1.12 and III.1.32 also imply the equalities in Proposition III.1.27.

Proposition III.1.34. *Let $w \in W$ and $\boldsymbol{i} \in I(w)$. Then the following hold:*

(1) *The subset*

$$\{q^{(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(b)] \mid \lambda \in P_+, b \in \mathscr{B}_w(\infty)\}$$

of $\mathbf{A}_q[N_-^w]$ forms a basis of $\mathbf{A}_q[N_-^w]$.

(2) *The subset*

$$\{q^{(\lambda,\mathrm{wt}\, b_{-1}(\boldsymbol{c},\boldsymbol{i})+\lambda-w\lambda)}D_{w\lambda,\lambda}^{-1}G^{\mathrm{up}}(b_{-1}(\boldsymbol{c},\boldsymbol{i})) \mid \lambda \in P_+, \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell(w)}\}$$

of $\mathbf{A}_q[N_-(w)\cap \dot{w}G_0]$ forms a basis of $\mathbf{A}_q[N_-(w)\cap \dot{w}G_0]$.

Proof. We prove only (1). The assertion (2) is proved in the same manner. The given subset obviously spans the $\mathbb{Q}(q)$-vector space $\mathbf{A}_q[N_-^w]$. Hence it remains to show that this set is a linearly independent set. For $(\lambda,b),(\lambda',b') \in P_+ \times \mathscr{B}_w(\infty)$, write $(\lambda,b) \sim (\lambda',b')$ if and only if $q^{(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(b)] = q^{(\lambda',\mathrm{wt}\, b'+\lambda'-w\lambda')}[D_{w\lambda',\lambda'}]^{-1}[G^{\mathrm{up}}(b')]$. The relation $\sim$ is clearly an equivalence relation, and we take a complete set F of coset representatives of $(P_+ \times \mathscr{B}_w(\infty))/\sim$. Suppose that there exists a finite subset $F' \subset F$ and $a_{\lambda,b} \in \mathbb{Q}(b)$ ($(\lambda,b) \in F'$) such that $\sum_{(\lambda,b)\in F'} q^{(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}a_{\lambda,b}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(b)] = 0$. There exists $\lambda_0 \in P_+$ such that $\lambda_0-\lambda \in P_+$ for all $\lambda \in P_+$ such that $(\lambda,b) \in F'$ for some $b \in \mathscr{B}_w(\infty)$. Now the equality $\sum_{(\lambda,b)\in F'} q^{(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}a_{\lambda,b}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(b)] = 0$ is equivalent to the equality

$$[D_{w\lambda_0,\lambda_0}]\left(\sum_{(\lambda,b)\in F'} q^{(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}a_{\lambda,b}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(b)]\right) = 0. \tag{III.7}$$

By Proposition III.1.27 and Proposition III.1.32, for $(\lambda, b) \in F'$, we have

$$\begin{aligned}[D_{w\lambda_0,\lambda_0}]\left(q^{(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(b)]\right) &= q^{-(\lambda_0-\lambda,w\lambda-\lambda)+(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}[D_{w(\lambda_0-\lambda),(\lambda_0-\lambda)}][G^{\mathrm{up}}(b)] \\ &= q^{(\lambda_0,\mathrm{wt}\, b+\lambda-w\lambda)}[G^{\mathrm{up}}(b^{(\lambda_0-\lambda)})]\end{aligned}$$

for some $b^{(\lambda_0-\lambda)} \in \mathscr{B}_w(\infty)$. Note that $\mathrm{wt}\, b+\lambda-w\lambda = \mathrm{wt}\, b^{(\lambda_0-\lambda)} - \mathrm{wt}\, D_{w\lambda_0,\lambda_0}$. Therefore if $b^{(\lambda_0-\lambda)} = (b')^{(\lambda_0-\lambda')}$ for $(\lambda,b),(\lambda',b') \in F'$ then we have the equality

$$[D_{w\lambda_0,\lambda_0}]\left(q^{(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(b)]\right) = [D_{w\lambda_0,\lambda_0}]\left(q^{(\lambda',\mathrm{wt}\, b'+\lambda'-w\lambda')}[D_{w\lambda',\lambda'}]^{-1}[G^{\mathrm{up}}(b')]\right),$$

hence $(\lambda, b) = (\lambda', b')$. Thus (III.7) implies $a_{\lambda,b} = 0$ for all $(\lambda, b) \in F'$. This completes the proof. □

Definition III.1.35. Let $w \in W$. We call

$$\widetilde{\mathbf{B}}^{\mathrm{up},w} := \{q^{(\lambda,\mathrm{wt}\, b+\lambda-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(b)] \mid \lambda \in P_+, b \in \mathscr{B}_w(\infty)\}, \text{ and}$$
$$\widetilde{\mathbf{B}}^{\mathrm{up}}(w) := \{q^{(\lambda,\mathrm{wt}\, b_{-1}(\boldsymbol{c},\boldsymbol{i})+\lambda-w\lambda)}D_{w\lambda,\lambda}^{-1}G^{\mathrm{up}}(b_{-1}(\boldsymbol{c},\boldsymbol{i})) \mid \lambda \in P_+, \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell(w)}\}$$

the dual canonical bases of $\mathbf{A}_q[N_-^w]$ and $\mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$, respectively.

For $\lambda \in P$, there exist $\lambda_1, \lambda_2 \in P_+$ such that $\lambda = -\lambda_1 + \lambda_2$. Set

$$D_{w,\lambda} := q^{(\lambda_1,w\lambda-\lambda)}D_{w\lambda_1,\lambda_1}^{-1}D_{w\lambda_2,\lambda_2} \in \widetilde{\mathbf{B}}^{\mathrm{up}}(w).$$

Then $D_{w,\lambda}$ does not depend on the choice of $\lambda_1, \lambda_2 \in P_+$ by Proposition III.1.34. Note that $\mathrm{wt}\, D_{w,\lambda} = w\lambda - \lambda$.

The following is straightforwardly proved by Proposition III.1.27.

Proposition III.1.36. *Let $w \in W$ and $\lambda, \lambda' \in P_+$. Then the following hold:*

(1) $D_{w,\lambda} = q^{(\lambda,w\lambda_1-\lambda_1)}D_{w\lambda_2,\lambda_2}D_{w\lambda_1,\lambda_1}^{-1}$ *for $\lambda_1, \lambda_2 \in P_+$ with $\lambda = -\lambda_1 + \lambda_2$.*
(2) $D_{w,\lambda}D_{w,\lambda'} = q^{(\lambda,w\lambda'-\lambda')}D_{w,\lambda+\lambda'}$. *In particular, $D_{w,\lambda}^{-1} = q^{(\lambda,w\lambda-\lambda)}D_{w,-\lambda}$.*
(3) $D_{w,\lambda}x = q^{(\lambda+w\lambda,\mathrm{wt}\, x)}xD_{w,\lambda}$ *for $\lambda \in P_+$ and a homogeneous element $\mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$.*

Remark III.1.37. By using Proposition III.1.32 (2), we can parametrize explicitly the elements of $\widetilde{\mathbf{B}}^{\mathrm{up}}(w)$. Fix $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$. An element $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}$ is said to have gaps if $\min\{c_k \mid i_k = i\} = 0$ for all $i \in I$. Then, by Propositions III.1.32 (2) and III.1.34 (2), we obtain the non-overlapping parametrization of the elements of $\widetilde{\mathbf{B}}^{\mathrm{up}}(w)$ as follows:

$$\widetilde{\mathbf{B}}^{\mathrm{up}}(w) = \{q^{-(\lambda,\mathrm{wt}\, b_{-1}(\boldsymbol{c},\boldsymbol{i}))}D_{w,\lambda}G^{\mathrm{up}}(b_{-1}(\boldsymbol{c},\boldsymbol{i})) \mid \lambda \in P, \boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell} \text{ has gaps}\}.$$

By Proposition III.1.32, the property of $b_{-1}(\boldsymbol{c},\boldsymbol{i})$ that $b_{-1}(\boldsymbol{c},\boldsymbol{i})$ has gaps does not depend on the choice of $\boldsymbol{i} \in I(w)$.

We define the dual bar involutions on $\mathbf{A}_q[N_-^w]$ and $\mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$, which are useful when we study the dual canonical bases.

Proposition III.1.38. *The following hold:*

(1) *The twisted dual bar involution* σ' *induces* $\mathbb{Q}$*-anti-algebra involutions* $\mathbf{A}_q[N_- \cap X(w)] \to \mathbf{A}_q[N_- \cap X(w)]$ *and* $\mathbf{A}_q[N_-(w)] \to \mathbf{A}_q[N_-(w)]$*. See Definition I.1.13 for the definition of* σ'*. Moreover these maps are extended to* $\mathbb{Q}$*-anti-algebra involutions* $\sigma' \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w]$ *and* $\sigma' \colon \mathbf{A}_q[N_-(w) \cap \dot{w}G_0] \to \mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$*.*

(2) *Define a* $\mathbb{Q}(q)$*-linear isomorphism* $c_{\mathrm{tw}} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w]$ *(resp.* $\mathbf{A}_q[N_-(w) \cap \dot{w}G_0] \to \mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$*) by*

$$x \mapsto q^{(\mathrm{wt}\, x, \mathrm{wt}\, x)/2 - (\mathrm{wt}\, x, \rho)} x$$

for every homogeneous element $x \in \mathbf{A}_q[N_-^w]$ *(resp.* $x \in \mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$*). Set* $\sigma := c_{\mathrm{tw}} \circ \sigma'$*. Then for homogeneous elements* $x, y \in \mathbf{A}_q[N_-^w]$ *(resp.* $\mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$*) we have*

$$\sigma(xy) = q^{(\mathrm{wt}\, x, \mathrm{wt}\, y)} \sigma(y)\sigma(x). \tag{III.8}$$

Moreover the elements of the dual canonical basis $\widetilde{\mathbf{B}}^{\mathrm{up},w}$ *and* $\widetilde{\mathbf{B}}^{\mathrm{up}}(w)$ *are fixed by* σ*.*

Definition III.1.39. The $\mathbb{Q}$-linear isomorphisms σ and $\sigma' \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w]$, $\mathbf{A}_q[N_-(w) \cap \dot{w}G_0] \to \mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$ defined in Proposition III.1.38 will be also called the dual bar involution and the twisted dual bar involution, respectively.

Proof of Proposition III.1.38. Recall that $\sigma'(G^{\mathrm{up}}(b)) = q^{-(\mathrm{wt}\, b, \mathrm{wt}\, b)/2 + (\mathrm{wt}\, b, \rho)} G^{\mathrm{up}}(b)$ for all $b \in \mathscr{B}(\infty)$. See Remark I.1.14. Hence (1) follows from the compatibility of the algebras $\mathbf{A}_q[N_- \cap X(w)]$, $\mathbf{A}_q[N_-(w)]$ and the dual canonical basis (Definition I.1.48, Definition I.1.54), and the universality of localization [**21**, Proposition 6.3]. A direct calculation immediately shows the equality III.8. For $\lambda \in P_+$, we have

$$\begin{aligned} 1 &= \sigma(D_{w\lambda,\lambda} D_{w\lambda,\lambda}^{-1}) \\ &= q^{-(w\lambda - \lambda, w\lambda - \lambda)} \sigma(D_{w\lambda,\lambda}^{-1}) \sigma(D_{w\lambda,\lambda}) \\ &= q^{2(\lambda, w\lambda - \lambda)} \sigma(D_{w\lambda,\lambda}^{-1}) D_{w\lambda,\lambda} \end{aligned}$$

in $\mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$. Hence

$$\sigma(D_{w\lambda,\lambda}^{-1}) = q^{-2(\lambda, w\lambda - \lambda)} D_{w\lambda,\lambda}^{-1}.$$

Let $b \in \mathscr{B}_w(\infty)$. Then, by Proposition III.1.27 and the equality above, we have

$$\begin{aligned} &\sigma(q^{(\lambda, \mathrm{wt}\, b + \lambda - w\lambda)} [D_{w\lambda,\lambda}]^{-1} [G^{\mathrm{up}}(b)]) \\ &= q^{-(\lambda, \mathrm{wt}\, b + \lambda - w\lambda) + (\lambda - w\lambda, \mathrm{wt}\, b)} \sigma([G^{\mathrm{up}}(b)]) \sigma([D_{w\lambda,\lambda}]^{-1}) \\ &= q^{-(\lambda, \mathrm{wt}\, b + \lambda - w\lambda) + (\lambda - w\lambda, \mathrm{wt}\, b) - 2(\lambda, w\lambda - \lambda)} [G^{\mathrm{up}}(b)] [D_{w\lambda,\lambda}]^{-1} \\ &= q^{-(\lambda, \mathrm{wt}\, b + \lambda - w\lambda) + (\lambda - w\lambda, \mathrm{wt}\, b) - 2(\lambda, w\lambda - \lambda) + (\lambda + w\lambda, \mathrm{wt}\, b)} [D_{w\lambda,\lambda}]^{-1} [G^{\mathrm{up}}(b)] \\ &= q^{(\lambda, \mathrm{wt}\, b + \lambda - w\lambda)} [D_{w\lambda,\lambda}]^{-1} [G^{\mathrm{up}}(b)]. \end{aligned}$$

This proves the dual bar invariance property for $\widetilde{\mathbf{B}}^{\mathrm{up},w}$. The assertion for $\tilde{\mathbf{B}}^{\mathrm{up}}(w)$ is proved in the same manner. $\square$

The quantum twist isomorphism $\gamma_{w,q}$ is compatible with the dual canonical bases:

THEOREM III.1.40. *Let $w \in W$. Then the quantum twist isomorphism $\gamma_{w,q}\colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-(w) \cap \dot{w}G_0]$ is restricted to the bijection $\widetilde{\mathbf{B}}^{\mathrm{up},w} \to \widetilde{\mathbf{B}}^{\mathrm{up}}(w)$ given by*

$$q^{(\lambda, \mathrm{wt}(\overline{\jmath}_{\lambda'}(b))+\lambda-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(\overline{\jmath}_{\lambda'}(b))] \mapsto q^{-(\lambda-\lambda', \mathrm{wt}(*\overline{\jmath}^{\vee}_{w\lambda'}(b)))} D_{w,\lambda-\lambda'} G^{\mathrm{up}}(*\overline{\jmath}^{\vee}_{w\lambda'}(b))$$

for $\lambda, \lambda' \in P_+, b \in \mathscr{B}_w(\lambda')$. In particular, $\gamma_{w,q}([D_{w,\lambda}]) = D_{w,-\lambda}$ for $\lambda \in P$, and $\gamma_{w,q} \circ \sigma = \sigma \circ \gamma_{w,q}$.

Proof. By Proposition III.1.7, for $\lambda, \lambda' \in P_+$ and $b \in \mathscr{B}_w(\lambda')$, we have

$$\begin{aligned}
&\gamma_{w,q}(q^{(\lambda, \mathrm{wt}(\overline{\jmath}_{\lambda'}(b))+\lambda-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[G^{\mathrm{up}}(\overline{\jmath}_{\lambda'}(b))]) \\
&= \gamma_{w,q}(q^{(\lambda, \mathrm{wt}\, b-\lambda'+\lambda-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[D_{g^{\mathrm{up}}(b),u_{\lambda'}}]) \\
&= q^{(\lambda, \mathrm{wt}\, b-\lambda'+\lambda-w\lambda)}(q^{(\lambda, w\lambda-\lambda)}D_{w\lambda,\lambda})(q^{-(\lambda', \mathrm{wt}\, b-\lambda')}D^{-1}_{w\lambda',\lambda'}D_{u_{w\lambda'},g^{\mathrm{up}}(b)}) \\
&= q^{-(\lambda-\lambda', \mathrm{wt}(*\overline{\jmath}^{\vee}_{w\lambda'}(b)))}D_{w,\lambda-\lambda'}G^{\mathrm{up}}(*\overline{\jmath}^{\vee}_{w\lambda'}(b)).
\end{aligned}$$

This completes the proof. □

III.1.5. Quantum BFZ-twist automorphisms. We introduce quantum analogues of BFZ-twist automorphisms on quantum unipotent cells (Theorem III.1.42). Since they are automorphisms, we can consider the iterated application of them. In this subsection, we also show the "periodicity" of quantum BFZ-twist automorphisms corresponding to a finite dimensional Lie algebra $\mathfrak{g}$ and the longest element w_0 of W.

The following is known as the (generalized) De Concini-Procesi isomorphism.

PROPOSITION III.1.41 ([**34**, Theorem 5.13], [**37**, Theorem 3.17]). *Let $w \in W$. Define*

$$\iota_w\colon \mathbf{A}_q[N_-(w)] \to \mathbf{A}_q[N_- \cap X(w)],\ x \mapsto [x]$$

as a $\mathbb{Q}(q)$-algebra homomorphisms induced from the canonical projection $\mathbf{U}_q^- \to \mathbf{A}_q[N_- \cap X(w)]$. Recall Definition I.1.46 and I.1.54.

Then ι_w is injective, or equivalently, $*(\mathscr{B}(\mathbf{U}_q^-(w))) \subset \mathscr{B}_w(\infty)$. Moreover ι_w induces an isomorphism;

$$\iota_w\colon \mathbf{A}_q[N_-(w) \cap \dot{w}G_0] \xrightarrow{\sim} \mathbf{A}_q[N_-^w].$$

By Proposition III.1.41, we now obtain quantum BFZ-twist automorphisms on quantum unipotent cells. These are generalizations of the quantum BFZ-twist automorphisms in [**5**, Theorem 2.10] and correspond to those in [**5**, Conjecture 2.12 (c)]. Their compatibility between quantum BFZ-twist automorphisms and dual canonical bases corresponds [**5**, Conjecture 2.17 (a)]. Actually, Berenstein-Rupel dealt with the case that w is a square of a Coxeter element and state their results and conjectures through quantum cluster algebras rather than quantum unipotent cells $\mathbf{A}_q[N_-^w]$. Remark that our method "directly" treat the structures of quantum unipotent cells $\mathbf{A}_q[N_-^w]$ and those of quantized coordinate algebras $\mathbf{R}_q$. See also III.3.8.

THEOREM III.1.42. *Let $w \in W$. Then there exists a $\mathbb{Q}(q)$-algebra automorphism;*

$$\eta_{w,q} := \iota_w \circ \gamma_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w]$$

given by

$$[D_{u,u_\lambda}] \mapsto q^{-(\lambda, \mathrm{wt}\, u - \lambda)}[D_{w\lambda,\lambda}]^{-1}[D_{u_{w\lambda},u}], \qquad [D_{w\lambda,\lambda}]^{-1} \mapsto q^{(\lambda, w\lambda-\lambda)}[D_{w\lambda,\lambda}]$$

for a weight vector $u \in V(\lambda)$ and $\lambda \in P_+$. In particular, $\mathrm{wt}\, \eta_{w,q}([x]) = -\mathrm{wt}[x]$ for a homogeneous element $[x] \in \mathbf{A}_q[N_-^w]$.

Moreover $\eta_{w,q}$ is restricted to a permutation on the dual canonical basis $\widetilde{\mathbf{B}}^{\mathrm{up},w}$. In particular, $\eta_{w,q}$ commutes with the dual bar involution σ. We have $\eta_{w,q}([D_{w,\lambda}]) = [D_{w,-\lambda}]$ for $\lambda \in P_+$.

DEFINITION III.1.43. We call the $\mathbb{Q}(q)$-algebra automorphism $\eta_{w,q} \colon \mathbf{A}_q[N_-^w] \to \mathbf{A}_q[N_-^w]$ the quantum BFZ-twist automorphism on the quantum unipotent cell $\mathbf{A}_q[N_-^w]$.

Indeed, quantum BFZ-twist automorphisms coincide with BFZ-twist automorphisms when we consider the appropriate specialization to $q = 1$. See [**37**, Corollary 5.5].

REMARK III.1.44. In order to apply quantum BFZ-twist automorphisms to a dual canonical basis element $[G^{\mathrm{up}}(\tilde{b})]$, $\tilde{b} \in \mathscr{B}(\lambda)$, we have to find $\lambda \in P_+$ and $b \in \mathscr{B}(\lambda)$ such that $G^{\mathrm{up}}(\tilde{b}) = D_{g^{\mathrm{up}}(b),u_\lambda} = G^{\mathrm{up}}(\bar{\jmath}_\lambda(b))$. By Proposition I.1.42, we can take λ as $\lambda_{\tilde{b}} := \sum_{i \in I} \varepsilon_i^*(\tilde{b})\varpi_i$. Note that $\lambda_{\tilde{b}}$ is "minimal" in an appropriate sense.

Since the map $\eta_{q,w}$ is an automorphism, we can apply it repeatedly. In the rest of this subsection, we show the "6-periodicity" of the specific quantum BFZ-twist automorphisms. Assume that $\mathfrak{g}$ is a finite dimensional Lie algebra, and let w_0 be the longest element of W.

THEOREM III.1.45. *For a homogeneous element $x \in \mathbf{A}_q[N_-^{w_0}]$, we have*

$$\eta_{w_0,q}^6(x) = q^{(\mathrm{wt}\, x + w_0 \,\mathrm{wt}\, x, \mathrm{wt}\, x)} D_{w_0, -\mathrm{wt}\, x - w_0 \,\mathrm{wt}\, x}\, x.$$

REMARK III.1.46.

(III.9) When the action of w_0 on P is given by $\mu \mapsto -\mu$,

the theorem above states that $\eta_{w_0,q}^6 = \mathrm{id}$. Hence $\eta_{w_0,q}$ is "really" periodic. If $\mathfrak{g}$ is simple, then the condition (III.9) is satisfied in the case that $\mathfrak{g}$ is of type B_n, C_n, D_{2n} for $n \in \mathbb{Z}_{>0}$ and E_7, E_8, F_4, G_2. See [**24**, Section 3.7].

When $\mathfrak{g}$ is symmetric, this periodicity is also explained by Geiß-Leclerc-Schröer's additive categorification of BFZ-twist automorphisms (see Section III.2). The periodicity corresponds to the well-known 6-periodicity of syzygy functors [**1**], [**13**], that is, the property that $(\Omega_{w_0}^{-1})^6(M) \simeq M$ for an indecomposable non-projective-injective module M of Π in the notation of Section III.2.

We can consider the similar periodicity problems for every $w \in W$. It would be interesting to find the necessary and sufficient condition of $w \in W$ for periodicity. Since quantum BFZ-twist automorphisms are restricted to permutations on dual canonical bases, the periodicity of a quantum BFZ-twist automorphism $\eta_{w,q}$ is equaivalect to the periodicity of a (non-quantum) BFZ-twist automorphism η_w.

LEMMA III.1.47. *Let* $\lambda \in P_+$. *Take* $u, u' \in V(\lambda)$ *such that* $D_{u,u'} = G^{\mathrm{up}}(\tilde{b})$ *for some* $\tilde{b} \in \mathscr{B}(\infty)$. *Then, for* $i \in I$,

$$\varepsilon_i(\tilde{b}) = \max\{k \in \mathbb{Z}_{\geq 0} \mid D_{e_i^k.u,u'} \neq 0\} \qquad \varepsilon_i^*(\tilde{b}) = \max\{k \in \mathbb{Z}_{\geq 0} \mid D_{u,f_i^k.u'} \neq 0\}.$$

In particular,

$$\varepsilon_i(\bar{\jmath}_\lambda(b)) = \varepsilon_i(b) \qquad \varepsilon_i(\bar{\jmath}^\vee_{w_0\lambda}(b)) = \varphi_i(b)(= \varepsilon_i(b) + \langle h_i, \mathrm{wt}\, b\rangle).$$

Proof. By Proposition I.1.43,

$$\varepsilon_i(\tilde{b}) = \max\{k \in \mathbb{Z}_{\geq 0} \mid (e_i')^k(D_{u,u'}) \neq 0\}. \tag{III.10}$$

For $k \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbf{U}_q^-$, we have

$$\begin{aligned} ((e_i')^k(D_{u,u'}), x)_L &= (1 - q_i^2)^k (D_{u,u'}, f_i^k x)_L \\ &= (1 - q_i^2)^k (u, f_i^k x.u')_\lambda^\varphi \\ &= (1 - q_i^2)^k (e_i^k.u, x.u')_\lambda^\varphi = (1 - q_i^2)^k (D_{e_i^k.u,u'}, x)_L. \end{aligned}$$

Hence $(e_i')^k(D_{u,u'}) = (1 - q_i^2)^k D_{e_i^k.u,u'}$. Combining this equality with (III.10), we obtain the first equality. The second equality is proved in the same manner. The last two equalities are deduced from Proposition I.1.43 and III.1.7. □

PROOF OF THEOREM III.1.45. It is easily seen that we need only check the case that $x \in \mathbf{U}_q^-$. For $i \in I$, we have $D_{s_i\varpi_i,\varpi_i} = (1 - q_i^2) f_i$. We first consider the images of $D_{s_i\varpi_i,\varpi_i}$, $i \in I$ under iterated application of $\eta_{w_0,q}$. If $I = \{i\}$, that is, $\mathfrak{g} = \mathfrak{sl}_2$, the quantum unipotent cell $\mathbf{A}_q[N_-^{w_0}]$ is generated by $D^{\pm 1}_{s_i\varpi_i,\varpi_i} (= D^{\pm 1}_{w_0\varpi_i,\varpi_i})$. In this case, $\eta^2_{w_0,q}(D_{s_i\varpi_i,\varpi_i}) = D_{s_i\varpi_i,\varpi_i}$. Hence $\eta^2_{w_0,q} = \mathrm{id}$, in particular, the theorem holds. Henceforth, we consider the case that $\mathfrak{g}$ does not have ideals of Lie algebras which are isomorphic to $\mathfrak{sl}_2$. We have

$$\eta_{w_0,q}(D_{s_i\varpi_i,\varpi_i}) \simeq D^{-1}_{w_0\varpi_i,\varpi_i} D_{w_0\varpi_i,s_i\varpi_i}.$$

Here $\simeq$ stands for the coincidence up to some powers of q. Now, by Proposition III.1.7, $D_{w_0\varpi_i,s_i\varpi_i} = G^{\mathrm{up}}(*\bar{\jmath}^\vee_{w_0\varpi_i}(u_{s_i\varpi_i}))$. By Lemma III.1.47,

$$\varepsilon_j^*(*\bar{\jmath}^\vee_{w_0\varpi_i}(u_{s_i\varpi_i})) = \varepsilon_j(\bar{\jmath}^\vee_{w_0\varpi_i}(u_{s_i\varpi_i})) = \varphi_j(u_{s_i\varpi_i}) = \begin{cases} -a_{ji} & \text{if } j \neq i, \\ 0 & \text{if } j = i. \end{cases}$$

Therefore $\sum_{j\in I} \varepsilon_j^*(*\bar{\jmath}^\vee_{w_0\varpi_i}(u_{s_i\varpi_i}))\varpi_j = \varpi_i + s_i\varpi_i (=: \lambda_1)$. Recall Remark III.1.44. Then there exists $b_1 \in \mathscr{B}(\lambda_1)$ such that $D_{w_0\varpi_i,s_i\varpi_i} = D_{g^{\mathrm{up}}(b_1),u_{\lambda_1}}$, that is, $\bar{\jmath}_{\lambda_1}(b_1) = *\bar{\jmath}^\vee_{w_0\varpi_i}(u_{s_i\varpi_i})$. Then

$$\eta^2_{w_0,q}(D_{s_i\varpi_i,\varpi_i}) \simeq D_{w_0\varpi_i,\varpi_i} D^{-1}_{w_0\lambda_1,\lambda_1} D_{u_{w_0\lambda_1},g^{\mathrm{up}}(b_1)}.$$

As above, $D_{w_0\lambda_1,g^{\mathrm{up}}(b_1)} = G^{\mathrm{up}}(*\bar{\jmath}^\vee_{w_0\lambda_1}(b_1))$, and by Lemma III.1.47,

$$\begin{aligned} \varepsilon_j^*(*\bar{\jmath}^\vee_{w_0\lambda_1}(b_1)) &= \varepsilon_j(\bar{\jmath}^\vee_{w_0\lambda_1}(b_1)) \\ &= \varepsilon_j(b_1) + \langle h_j, \mathrm{wt}\, b_1\rangle \\ &= \varepsilon_j(\bar{\jmath}_{\lambda_1}(b_1)) + \langle h_j, w_0\varpi_i - s_i\varpi_i + \lambda_1\rangle \\ &= \varepsilon_j(*\bar{\jmath}^\vee_{w_0\varpi_i}(u_{s_i\varpi_i})) + \langle h_j, w_0\varpi_i + \varpi_i\rangle. \end{aligned}$$

By Proposition III.1.7 and Lemma III.1.47,

$$\varepsilon_j(*\bar{\jmath}^{\vee}_{w_0\varpi_i}(u_{s_i\varpi_i})) = \max\{k \in \mathbb{Z}_{\geq 0} \mid D_{e_j^k.u_{w_0\varpi_i},u_{s_i\varpi_i}} \neq 0\}.$$

By the way, recall the map θ on I defined just before Definition II.1.12. Then $w_0\varpi_i = -\varpi_{\theta(i)}$ and $s_{\theta(i)}w_0\varpi_i = w_0 s_i\varpi_i$. When $\mathfrak{g}$ does not have ideals of Lie algebras which are isomorphic to $\mathfrak{sl}_2$, we have $D_{w_0 s_i\varpi_i, s_i\varpi_i} \neq 0$. Therefore $\varepsilon_j(*\bar{\jmath}^{\vee}_{w_0\varpi_i}(u_{s_i\varpi_i})) = \delta_{j,\theta(i)}$. Hence

$$\varepsilon_j^*(*\bar{\jmath}^{\vee}_{w_0\lambda_1}(b_1)) = \delta_{j,\theta(i)} - \delta_{j,\theta(i)} + \delta_{ij} = \delta_{ij}.$$

Therefore $\sum_{j\in I}\varepsilon_j^*(*\bar{\jmath}^{\vee}_{w_0\lambda_1}(b_1))\varpi_j = \varpi_i$. Then there exists $b_2 \in \mathscr{B}(\varpi_i)$ such that $D_{w_0\lambda_1, g^{\mathrm{up}}(b_1)} = D_{g^{\mathrm{up}}(b_2),u_{\varpi_i}}$. Then

$$\begin{aligned}\eta^3_{w_0,q}(D_{s_i\varpi_i,\varpi_i}) &\simeq D^{-1}_{w_0\varpi_i,\varpi_i} D_{w_0\lambda_1,\lambda_1} D^{-1}_{w_0\varpi_i,\varpi_i} D_{u_{w_0\varpi_i},g^{\mathrm{up}}(b_2)}\\ &\simeq D_{w_0,-\alpha_i} D_{u_{w_0\varpi_i},g^{\mathrm{up}}(b_2)}.\end{aligned}$$

Here $\mathrm{wt}\, D_{u_{w_0\varpi_i},g^{\mathrm{up}}(b_2)} = w_0\varpi_i - \mathrm{wt}\, b_2 = w_0\varpi_i - (w_0\lambda_1 - \mathrm{wt}\, b_1 + \varpi_i) = w_0\varpi_i - (w_0\lambda_1 - (w_0\varpi_i - s_i\varpi_i + \lambda_1) + \varpi_i) = -\alpha_{\theta(i)}$. Hence $D_{u_{w_0\varpi_i},g^{\mathrm{up}}(b_2)} = D_{s_{\theta(i)}\varpi_{\theta(i)},\varpi_{\theta(i)}}$ because both hand-sides are unique elements of the dual canonical basis of weight $-\alpha_{\theta(i)}$. Therefore,

$$\eta^6_{w_0,q}(D_{s_i\varpi_i,\varpi_i}) \simeq D_{w_0,\alpha_i-\alpha_{\theta(i)}} D_{s_i\varpi_i,\varpi_i}.$$

Moreover, by Theorem III.1.42, $\eta^6_{w_0,q}(D_{s_i\varpi_i,\varpi_i})$ is an element of dual canonical basis, in particular, dual bar-invariant. Therefore,

$$\eta^6_{w_0,q}(D_{s_i\varpi_i,\varpi_i}) = q^{(\alpha_i-\alpha_{\theta(i)},\alpha_i)} D_{w_0,\alpha_i-\alpha_{\theta(i)}} D_{s_i\varpi_i,\varpi_i}.$$

By this result and Proposition III.1.27, III.1.36, for $i_1,\dots,i_\ell \in I$, we have

$$\begin{aligned}&\eta^6_{w_0,q}(D_{s_{i_1}\varpi_{i_1},\varpi_{i_1}}\cdots D_{s_{i_\ell}\varpi_{i_\ell},\varpi_{i_\ell}})\\ &= q^{\sum_{k=1}^{\ell}(\alpha_{i_k}-\alpha_{\theta(i_k)},\alpha_{i_k})} D_{w_0,\alpha_{i_1}-\alpha_{\theta(i_1)}} D_{s_{i_1}\varpi_{i_1},\varpi_{i_1}} \cdots D_{w_0,\alpha_{i_\ell}-\alpha_{\theta(i_\ell)}} D_{s_{i_\ell}\varpi_{i_\ell},\varpi_{i_\ell}}\\ &= q^{(\sum_{k=1}^{\ell}\alpha_{i_k}-\sum_{k=1}^{\ell}\alpha_{\theta(i_k)},\sum_{k=1}^{\ell}\alpha_{i_k})} D_{w_0,\sum_{k=1}^{\ell}\alpha_{i_k}-\sum_{k=1}^{\ell}\alpha_{\theta(i_k)}} D_{s_{i_1}\varpi_{i_1},\varpi_{i_1}}\cdots D_{s_{i_\ell}\varpi_{i_\ell},\varpi_{i_\ell}}.\end{aligned}$$

This proves the theorem. □

III.2. Geiß-Leclerc-Schröer type categorification

In this section, we consider an additive categorification of the quantum BFZ-twist automorphisms in the sense of Geiß-Leclerc-Schröer. When $\mathfrak{g}$ is symmetric, Geiß-Leclerc-Schröer [**19**] obtained a categorification of the (non-quantum) BFZ-twist automorphisms (Proposition III.2.19). They used subcategories $\mathcal{C}_w$, introduced by Buan-Iyama-Reiten-Scott [**8**] and independently by Geiß-Leclerc-Schröer [**17**] for specific w, of the module category of the preprojective algebra Π corresponding to the Dynkin diagram for $\mathfrak{g}$. Geiß-Leclerc-Schröer [**20**] have also shown that the quantum unipotent subgroup $\mathbf{A}_q[N_-(w)]$ is isomorphic to a certain quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathcal{C}_w)$, which is determined by data of $\mathcal{C}_w$ (Proposition III.2.14). Combining these results, we obtain a categorification of the twist automorphism $\eta_{w,q}$ (Theorem III.2.20). This results state the compatibility between quantum BFZ-twist automorphisms and quantum cluster monomials. See also Corollary III.2.21.

In this section, we always assume that $\mathfrak{g}$ is symmetric. We may assume that $(\alpha_i, \alpha_i) = 2$ for all $i \in I$ and $I = \{1, 2, \dots, n\} = [1, n]$. Note that $q_i = q$ for all $i \in I$. Recall also Section I.2.

We first review Geiß-Leclerc-Schröer's theory with concision in subsection III.2.1. The main references are [**8, 15, 16, 18, 19, 20**]. However our convention is different from Geiß-Leclerc-Schröer's one. See Remark III.2.11. The main result in this section is stated in subsection III.2.2.

III.2.1. Additive categorification of quantum cluster cluster structures on quantum unipotent subgroups and quantum unipotent cells.

Definition III.2.1. *A finite quiver* $\mathbf{Q} = (\mathbf{Q}_0, \mathbf{Q}_1, s, t)$ is a datum such that

- $\mathbf{Q}_0$ is a finite set, called the set of *vertices*,
- $\mathbf{Q}_1$ is a finite set, called the set of *arrows*,
- $s, t \colon \mathbf{Q}_1 \to \mathbf{Q}_0$ are maps, and it is said that $a \in \mathbf{Q}_1$ *starts in a vertex* $s(a)$ and *terminates in a vertex* $t(a)$.

For $i, j \in \mathbf{Q}_0$,

Here we take a quiver $\mathbf{Q}$ such that $\mathbf{Q}_0 = I$, $s(a) \neq t(a)$ for all $a \in \mathbf{Q}_1$ and $a_{ij} (:= \langle h_i, \alpha_j \rangle) = -\#\{a \in \mathbf{Q}_1 \mid s(a) = i, t(a) = j\} - \#\{a \in \mathbf{Q}_1 \mid s(a) = j, t(a) = i\}$. Such a quiver $\mathbf{Q}$ is called *a finite quiver without edge loops which corresponds to the symmetric generalized Cartan matrix A*.

Let $\overline{\mathbf{Q}} = (\mathbf{Q}_0, \overline{\mathbf{Q}}_1 := \mathbf{Q}_1 \coprod \mathbf{Q}_1^*, s, t)$ be *the double quiver of* $\mathbf{Q}$, which is obtained from $\mathbf{Q}$ by adding to each arrow $a \in \mathbf{Q}_1$ an arrow $a^* \in \mathbf{Q}_1$ such that $s(a^*) = t(a)$ and $t(a^*) = s(a)$. Set

$$\Pi := \mathbb{C}\overline{\mathbf{Q}} / (\sum_{a \in \mathbf{Q}_1} (a^* a - a a^*)),$$

Here $\mathbb{C}\overline{\mathbf{Q}}$ is *a path algebra of* $\overline{\mathbf{Q}}$, which is the $\mathbb{C}$-algebra with the generators $\mathbf{e}_i$ $(i \in \mathbf{Q}_0 = I)$, a' $(a' \in \overline{\mathbf{Q}}_1)$ and the relations:

(i) $\mathbf{e}_i \mathbf{e}_j = \delta_{ij} \mathbf{e}_i$ for $i, j \in \mathbf{Q}_0$,
(ii) $a' \mathbf{e}_{s(a')} = \mathbf{e}_{t(a')} a' = a'$ for $a' \in \overline{\mathbf{Q}}_1$,

and $(\sum_{a \in \mathbf{Q}_1} (a^* a - a a^*))$ stands for the two-sided ideal generated by $\sum_{a \in \mathbf{Q}_1} (a^* a - a a^*)$. This is called *the preprojective algebra associated with* $\mathbf{Q}$.

For a finite dimensional Π-module X, write $\underline{\dim}\, X := -\sum_{i \in I} (\dim_{\mathbb{C}} \mathbf{e}_i.X) \alpha_i \in Q_-$. Remark that we do not regard $\underline{\dim}\, X$ as an element of Q_+. A finite dimensional Π-module X is said to be *nilpotent* if there exists $N \in \mathbb{Z}_{\geq 0}$ such that $a_1 \cdots a_N.X = 0$ for any sequence $(a_1, \dots, a_N) \in \overline{\mathbf{Q}}_1^N$ with $s(a_j) = t(a_{j+1})$, $j = 1, \dots, N-1$.

Let $\boldsymbol{d} = (d_j)_{j \in I} \in \mathbb{Z}_{\geq 0}^I$. Set $\operatorname{rep}(\overline{\mathbf{Q}}, \boldsymbol{d}) := \prod_{a' \in \overline{\mathbf{Q}}_1} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s(a')}}, \mathbb{C}^{d_{t(a')}})$ and define an affine variety $\operatorname{rep}(\Pi, \boldsymbol{d})$ by

$$\operatorname{rep}(\Pi, \boldsymbol{d}) := \{(f_{a'})_{a' \in \overline{\mathbf{Q}}_1} \in \operatorname{rep}(\overline{\mathbf{Q}}, \boldsymbol{d}) \mid \textstyle\sum_{a \in \mathbf{Q}_1; s(a) = i} f_{a^*} f_a = \sum_{a \in \mathbf{Q}_1; t(a) = i} f_a f_{a^*} \text{ for all } i \in I\}.$$

An element $(f_{a'})_{a'\in\overline{\mathbf{Q}}_1} \in \operatorname{rep}(\Pi, \boldsymbol{d})$ naturally determines a representation X of Π such that $\underline{\dim} X := -\sum_{i\in I} d_i\alpha_i$. Define an affine variety $\Lambda_{\boldsymbol{d}}$ by

$$\Lambda_{\boldsymbol{d}} := \{(f_{a'})_{a'\in\overline{\mathbf{Q}}_1} \in \operatorname{rep}(\Pi, \boldsymbol{d}) \mid (f_{a'})_{a'\in\overline{\mathbf{Q}}_1} \text{ corresponds to a nilpotent } \Pi\text{-module}\}.$$

The varieties $\{\Lambda_{\boldsymbol{d}} \mid \boldsymbol{d} \in \mathbb{Z}^n_{\geq 0}\}$ are called *nilpotent varieties.* Then $\mathrm{GL}_{\boldsymbol{d}} := \prod_{i\in I} \mathrm{GL}_{d_i}(\mathbb{C})$ acts on $\operatorname{rep}(\Pi, \boldsymbol{d})$ and $\Lambda_{\boldsymbol{d}}$ by $(g_i)_{i\in I}.(f_{a'})_{a'\in\overline{\mathbf{Q}}_1} = (g_{t(a')}f_{a'}g^{-1}_{s(a')})_{a'\in\overline{\mathbf{Q}}_1}$. Then each $\mathrm{GL}_{\boldsymbol{d}}$-orbit of an element of $\operatorname{rep}(\Pi, \boldsymbol{d})$ naturally corresponds to an isomorphism class of Π-modules.

A function $f\colon \Lambda_{\boldsymbol{d}} \to \mathbb{C}$ is called *constructible* if $\operatorname{Im} f$ is a finite set and $f^{-1}(z)$ is a constructible subset (namely, a finite union of locally closed subsets) of $\Lambda_{\boldsymbol{d}}$ for all $z \in \mathbb{C}$. Denote the set of constructible functions $f\colon \Lambda_{\boldsymbol{d}} \to \mathbb{C}$ by $\widetilde{\mathcal{M}}(\Lambda_{\boldsymbol{d}})$. Let $\widetilde{\mathcal{M}}(\Lambda_{\boldsymbol{d}})^{\mathrm{GL}_{\boldsymbol{d}}}$ be the subspace of $\widetilde{\mathcal{M}}(\Lambda_{\boldsymbol{d}})$ consisting of the constructible functions which are constant on the $\mathrm{GL}_{\boldsymbol{d}}$-orbits in $\Lambda_{\boldsymbol{d}}$. For $i \in I$, define $\boldsymbol{e}^{(i)} = (\boldsymbol{e}^{(i)}_j)_{j\in I} \in \mathbb{Z}^I_{\geq 0}$ by $\boldsymbol{e}^{(i)}_j = \delta_{ij}$. Then $\Lambda_{\boldsymbol{e}^{(i)}}$ consists of a point. The corresponding simple Π-module will be denoted by S_i. Then $\widetilde{\mathcal{M}}(\Lambda_{\boldsymbol{e}^{(i)}}) = \mathbb{C}\mathbf{1}_i$, where $\mathbf{1}_i(S_i) = 1$.

Set

$$\widetilde{\mathcal{M}} := \bigoplus_{\boldsymbol{d}\in\mathbb{Z}^I_{\geq 0}} \widetilde{\mathcal{M}}(\Lambda_{\boldsymbol{d}})^{\mathrm{GL}_{\boldsymbol{d}}}.$$

Let $\boldsymbol{d}', \boldsymbol{d}'' \in \mathbb{Z}^I_{\geq 0}$. For $f' \in \widetilde{\mathcal{M}}(\Lambda_{\boldsymbol{d}'})^{\mathrm{GL}_{\boldsymbol{d}'}}$ and $f'' \in \widetilde{\mathcal{M}}(\Lambda_{\boldsymbol{d}''})^{\mathrm{GL}_{\boldsymbol{d}''}}$, define $f' * f''\colon \Lambda_{\boldsymbol{d}'+\boldsymbol{d}''} \to \mathbb{C}$ by

$$(f' * f'')(X) := \sum_{z\in\mathbb{C}} z\chi_c(\{U \mid U \text{ is a submodule of } X, f'(X/U)f''(U) = z\})$$

for $X \in \Lambda_{\boldsymbol{d}'+\boldsymbol{d}''}$. Here χ_c means topological Euler characteristic with respect to cohomology with compact support. This operation makes $\widetilde{\mathcal{M}}$ into an associative $\mathbb{C}$-algebra. Let $\mathcal{M}$ be a $\mathbb{C}$-subalgebra of $\widetilde{\mathcal{M}}$ generated by $\{\mathbf{1}_i\}_{i\in I}$. Lusztig has shown that the algebra $\mathcal{M}$ is isomorphic to the universal enveloping algebra $\mathbf{U}(\mathfrak{n}^-)$ of $\mathfrak{n}^-$:

Proposition III.2.2 ([**43**]). *There exists an isomorphism of* $\mathbb{C}$*-algebras* $\mathbf{U}(\mathfrak{n}^-) \to \mathcal{M}$ *given by* $f_i \mapsto \mathbf{1}_i$*, here* f_i *denotes a root vector of* $\mathfrak{g}$ *corresponding to* $-\alpha_i$ *(abuse of notation).*

By the way, a nilpotent Λ-module X determines a well-defined linear map $\varphi_X\colon \mathcal{M} \to \mathbb{C}$ given by $f \mapsto f(X)$. Through the isomorphism $\mathbf{U}(\mathfrak{n}^-) \simeq \mathcal{M}$ above, we regard φ_X as an element of the graded dual $\mathbf{U}(\mathfrak{n}^-)^*_{\mathrm{gr}}$ of $\mathbf{U}(\mathfrak{n}^-)$, which can be identified with the coordinate algebra $\mathbb{C}[N_-]$ of N_-.

Proposition III.2.3 ([**9**, Lemma 1]). *For any finite dimensional* Π*-module* X, Y*, we have*

$$(\underline{\dim} X, \underline{\dim} Y) = \dim_{\mathbb{C}} \operatorname{Hom}_\Pi(X, Y) + \dim_{\mathbb{C}} \operatorname{Hom}_\Pi(Y, X) - \dim_{\mathbb{C}} \operatorname{Ext}^1_\Pi(X, Y).$$

The following property of φ_X is obtained by Geiß-Leclerc-Schröer.

Proposition III.2.4 ([**15, 16**]). *Let* X, Y *be nilpotent* Π*-modules. The following hold:*

(1) $\varphi_X\varphi_Y = \varphi_{X\oplus Y}$.

(2) *Suppose that* $\dim_{\mathbb{C}} \mathrm{Ext}^1_{\Pi}(X, Y) = 1$. *Write non-split short exact sequences as*

$$0 \to X \to Z_1 \to Y \to 0 \qquad\qquad 0 \to Y \to Z_2 \to X \to 0.$$

Then we have $\varphi_X \varphi_Y = \varphi_{Z_1} + \varphi_{Z_2}$.

Remark III.2.5. Note that, for any finite dimensional Π-modules X, Y, we have

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\Pi}(X, Y) = \dim_{\mathbb{C}} \mathrm{Ext}^1_{\Pi}(Y, X)$$

by Proposition III.2.3.

Definition III.2.6. For a Π-module X and $i \in I$, define $\mathrm{soc}_i(X) \subset X$ by the sum of all submodules of X isomorphic to S_i. For a sequence $(i_1, \ldots, i_k) \in I^k$ ($k \in \mathbb{Z}_{>0}$), there exists a unique chain

$$X \supset X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_k = 0$$

of submodules of X such that $X_{j-1}/X_j \simeq \mathrm{soc}_{i_j}(X/X_j)$ for $j = 1, \ldots, k$. Set $\mathrm{soc}_{(i_1,\ldots,i_k)}(X) := X_0$. For $i \in I$, denote by $\hat{I}_i$ the indecomposable injective Π-module with socle S_i. Let $w \in W$ and $\boldsymbol{i} = (i_1, \ldots, i_\ell) \in I(w)$. For $k = 1, \ldots, \ell$, set

$$V_{\boldsymbol{i},k} := \mathrm{soc}_{(i_1,\ldots,i_k)}(\hat{I}_{i_k}).$$

Set $V_{\boldsymbol{i}} := \bigoplus_{k=1,\ldots,\ell} V_{\boldsymbol{i},k}$. Define $\mathcal{C}_w$ as a full subcategory of the category of Π-modules consisting of all Π-modules X such that there exist $t \in \mathbb{Z}_{>0}$ and a surjective homomorphism $V_{\boldsymbol{i}}^{\oplus t} \to X$. Then it is known that $\mathcal{C}_w$ does not depend on the choice of $\boldsymbol{i} \in I(w)$. Note that all objects of $\mathcal{C}_w$ are nilpotent Π-modules. An object $C \in \mathcal{C}_w$ is called $\mathcal{C}_w$-*projective* (resp. $\mathcal{C}_w$-*injective*) if $\mathrm{Ext}^1_{\Pi}(C, X) = 0$ (resp. $\mathrm{Ext}^1_{\Pi}(X, C) = 0$) for all $X \in \mathcal{C}_w$. The category $\mathcal{C}_w$ is closed under extension and Frobenius. In particular, an object $X \in \mathcal{C}_w$ is $\mathcal{C}_w$-projective if and only if it is $\mathcal{C}_w$-injective. An object T of $\mathcal{C}_w$ is called $\mathcal{C}_w$-*maximal rigid* if $\mathrm{Ext}^1_{\Pi}(T \oplus X, X) = 0$ with $X \in \mathcal{C}_w$ implies that X is isomorphic to a direct summand of a direct sum of T. In fact, $V_{\boldsymbol{i}}$ is a basic $\mathcal{C}_w$-maximal rigid module. Recall that a Π-module M is called *basic* if it is written as a direct sum of pairwise non-isomorphic indecomposable modules. See [**8**] for more details, and [**18**, Subsection 2.4] for more detailed summaries.

Let T be a basic $\mathcal{C}_w$-maximal rigid module $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. We always number indecomposable modules as $T_{\ell-n+i}$ is a $\mathcal{C}_w$-projective-injective module with socle S_i for $i \in I$. Note that this labelling is different from the labelling $V_{\boldsymbol{i}} = \bigoplus_{k \in [1,\ell]} V_{\boldsymbol{i},k}$. Let Γ_T be *the Gabriel quiver of* $A_T := \mathrm{End}_{\Pi}(T)^{\mathrm{op}}$, that is, the vertex set of Γ_T is $[1, \ell]$ and $d_{ij} := \dim_{\mathbb{C}} \mathrm{Ext}^1_{A_T}(S_{T_i}, S_{T_j})$ arrows from i to j, where S_{T_i} is the head of a (projective) A_T-module $\mathrm{Hom}_{\Pi}(T, T_i)$. Define $\widetilde{B}_T = (b_{ij})_{i \in [1,\ell], j \in [1,\ell-n]}$ by $b_{ij} := d_{ji} - d_{ij}$. The following proposition is an essential results for the additive categorification of cluster algebras.

Proposition III.2.7 ([**8**], [**17**]). *In the setting above, the following hold:*

(1) $\ell = \ell(w)$.
(2) *For any* $k \in [1, \ell - n]$, *there exists a unique indecomposable* Π-*module in* $\mathcal{C}_w$ *such that* $T_k^* \not\simeq T_k$ *and* $(T/T_k) \oplus T_k^*$ *is a basic* $\mathcal{C}_w$-*maximal rigid module. This basic*

$\mathcal{C}_w$-maximal rigid module is denoted by $\mu_{T_k}(T)$ *and celled* the mutation of T in direction T_k.

(3) *For any* $k \in [1, \ell - n]$, $\mu_k(\widetilde{B}_T) = \widetilde{B}_{\mu_{T_k}(T)}$. *Recall Definition I.2.2.*

(4) *For any* $k \in [1, \ell - n]$, *we have* $\dim_{\mathbb{C}} \mathrm{Ext}^1_{\Pi}(T_k, T_k^*) = 1$, *and there exists non-split exact sequences*

$$0 \to T_k \to T_- \to T_k^* \to 0 \qquad\qquad 0 \to T_k^* \to T_+ \to T_k \to 0$$

such that $T_- \simeq \bigoplus_{j; b_{jk} < 0} T_j^{\oplus(-b_{jk})}$ *and* $T_+ \simeq \bigoplus_{j; b_{jk} > 0} T_j^{\oplus b_{jk}}$.

Note that, by Proposition III.2.4 and III.2.7, we have

$$\varphi_{T_k}\varphi_{T_k^*} = \prod_{j; b_{jk}<0} \varphi_{T_j}^{-b_{jk}} + \prod_{j; b_{jk}>0} \varphi_{T_j}. \tag{III.11}$$

This is nothing but *an additive categorification of mutation.* See [**18**, Subsection 2.7] and references therein for more details. An object T of $\mathcal{C}_w$ is said to be *reachable (in $\mathcal{C}_w$)* if T is isomorphic to a direct summand of a direct sum of a basic $\mathcal{C}_w$-maximal rigid module which is obtained from $V_{\boldsymbol{i}}$ by iterated mutations. In fact, the notion of reachable does not depend on the choice of $\boldsymbol{i}$ [**8**, Proposition III.4.3].

Recall Notation II.2.9. For $1 \leq a < b \leq \ell$ with $i_a = i_b$, there exists a natural injective homomorphism $V_{\boldsymbol{i},a^-} \to V_{\boldsymbol{i},b}$ of Π-modules, and the cokernel of this homomorphism is denoted by $M_{\boldsymbol{i}}[b,a]$. Here we set $V_{\boldsymbol{i},0} := 0$. In particular, $M_{\boldsymbol{i}}[b, b^{\min}]$ is isomorphic to $V_{\boldsymbol{i},b}$. Geiß-Leclerc-Schröer shows that $M_{\boldsymbol{i}}[b,a]$ is reachable for all $1 \leq a < b \leq \ell$ with $i_a = i_b$ [**18**, Section 13].

Remark III.2.8. Let T be a basic reachable $\mathcal{C}_w$-maximal rigid module, and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. By Proposition III.2.3, for any $i, j \in [1, \ell]$, we have

$$(\underline{\dim}\, T_i, \underline{\dim}\, T_j) = \dim_{\mathbb{C}} \mathrm{Hom}_{\Pi}(T_i, T_j) + \dim_{\mathbb{C}} \mathrm{Hom}_{\Pi}(T_j, T_i).$$

Definition III.2.9. We use the notation in Definition III.2.6. Geiß-Leclerc-Schröer construct a quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathcal{C}_w)$ associated with $\mathcal{C}_w$. We may assume that all elements of $I = [1, n]$ appears in the sequence $\boldsymbol{i}$.

Let T be a basic $\mathcal{C}_w$-maximal rigid module and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. Define $\Lambda_T := (\lambda_{ij})_{i,j \in [1,\ell]}$ by

$$\lambda_{ij} := \dim_{\mathbb{C}} \mathrm{Hom}_{\Pi}(T_i, T_j) - \dim_{\mathbb{C}} \mathrm{Hom}_{\Pi}(T_j, T_i).$$

Geiß-Leclerc-Schröer have shown the following properties:

- $(\widetilde{B}_T, \Lambda_T)$ is compatible in the sense of Definition I.2.2 [**20**, Proposition 10.1],
- $\mu_k(\widetilde{B}_T, \Lambda_T) = (\widetilde{B}_{\mu_{T_k}(T)}, \Lambda_{\mu_{T_k}(T)})$ for $k \in [1, \ell - n]$ [**20**, Proposition 10.2],

The quantum cluster algebra $\mathscr{A}_{q^{\pm 1/2}}(\mathcal{C}_w)$ is defined as the quantum cluster algebra with the initial seed $((X_T)_i)_{i \in [1,\ell]}, \widetilde{B}_T, \Lambda_T)$ for a basic reachable $\mathcal{C}_w$-maximal rigid module T. Note that this algebra $\mathscr{A}_{q^{\pm 1/2}}(C_w)$ does not depend on the choice of T. By the properties above, we may write

$$\mu_k(((X_T)_i)_{i\in[1,\ell]}, \widetilde{B}_T, \Lambda_T) = (((X_{\mu_{T_k}(T)})_i)_{i\in[1,\ell]}, \widetilde{B}_{\mu_{T_k}(T)}, \Lambda_{\mu_{T_k}(T)})$$

for $k \in [1, \ell - n]$. Moreover, for $\boldsymbol{a} = (a_1, \dots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, set $X_{\bigoplus_{i\in[1,\ell]} T_i^{\oplus a_1}} := (X_T)^{\boldsymbol{a}}$. Then the quantum cluster monomials of $\mathscr{A}_{q^{\pm 1/2}}(\mathcal{C}_w)$ is indexed by reachable Π-modules in $\mathcal{C}_w$.

Set

$$Y_R := q^{(\underline{\dim} R, \underline{\dim} R)/4} X_R.$$

for every reachable Π-module R in $\mathcal{C}_w$. Recall that $\underline{\dim} R \in Q_-$. Define the rescaled quantum cluster algebra $\mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)$ as an $\mathcal{A}(:= \mathbb{Q}[q^{\pm 1}])$-subalgebra of $\mathscr{A}_{q^{\pm 1/2}}(\mathcal{C}_w)$ generated by $\{Y_R \mid R \text{ is reachable in } \mathcal{C}_w\}$. For any basic reachable $\mathcal{C}_w$-maximal rigid module $T = T_1 \oplus \dots \oplus T_\ell$, the rescaled quantum cluster algebra $\mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)$ is contained in the rescaled based quantum torus $\mathcal{T}_{\mathcal{A},T} := \mathcal{A}[Y_{T_k}^{\pm 1} \mid k \in [1, \ell]] (\subset \mathcal{F})$ [**20**, Lemma 10.4 and Proposition 10.5] (they are cited as (III.12) and Proposition III.2.12 below). Note that, for $(a_1, \dots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, we have

$$Y_R = q^{\alpha(R)} Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell}, \tag{III.12}$$

here we set $R := \bigoplus_{i\in[1,\ell]} T_i^{\oplus a_i}$ and

$$\alpha(R) := \sum_{i\in[1,\ell]} a_i(a_i - 1) \dim_{\mathbb{C}} \mathrm{Hom}_\Pi(T_i, T_i)/2 + \sum_{i<j; i,j\in[1,\ell]} a_i a_j \dim_{\mathbb{C}} \mathrm{Hom}_\Pi(T_j, T_i).$$

Note that $\mathbf{I} := q^{\mathbb{Z}}\{Y_{\bigoplus_{i\in[\ell-n+1,\ell]} T_i^{a_i}} \mid (a_{\ell-n+1}, \dots, a_\ell) \in \mathbb{Z}_{\geq 0}^n\}$ is an Ore set in $\mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)$. Set $\widetilde{\mathscr{A}_{q^{\pm 1}}}(\mathcal{C}_w) := \mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)[\mathbf{I}^{-1}]$, and $\mathscr{A}_{\mathbb{Q}(q)}(\mathcal{C}_w) := \mathbb{Q}(q) \otimes_{\mathcal{A}} \mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)$, $\widetilde{\mathscr{A}_{\mathbb{Q}(q)}}(\mathcal{C}_w) := \mathbb{Q}(q) \otimes_{\mathcal{A}} \widetilde{\mathscr{A}_{q^{\pm 1}}}(\mathcal{C}_w)$.

For $X \in \mathcal{C}_w$, denote by $I(X)$ the injective hull of X in $\mathcal{C}_w$, and by $\Omega_w^{-1}(X)$ the cokernel of the corresponding injective homomorphism $X \to I(X)$. Hence we have an exact sequence

$$0 \to X \to I(X) \to \Omega_w^{-1}(X) \to 0.$$

Proposition III.2.10 ([**18**, Proposition 13.4]). *Let $w \in W$, T a basic reachable $\mathcal{C}_w$-maximal rigid module and $T = T_1 \oplus \dots \oplus T_\ell$ its indecomposable decomposition. Then $T' := \Omega_w^{-1}(T) \oplus \bigoplus_{i\in I} T_{\ell-n+i}$ is also a basic reachable $\mathcal{C}_w$-maximal rigid module; hence there exists a bijection $[1, \ell - n] \to [1, \ell - n], k \mapsto k^*$ such that $T'_{k^*} = \Omega_w^{-1}(T_k)$.*

Let $k \in [1, \ell - n]$ and write $\mu_{T_k}(T) = (T/T_k) \oplus T_k^*$. Then we have

$$\mu_{T'_{k^*}}(T') = (T'/T'_{k^*}) \oplus \Omega_w^{-1}(T_k^*).$$

Remark III.2.11. Let $w \in W$. In this remark, we explain the difference between our convention and Geiß-Leclerc-Schröer's one in [**18**], [**20**], [**19**]. An object $\mathcal{X}$ in Geiß-Leclerc-Schröer's papers is denoted by $\mathcal{X}^{\mathrm{GLS}}$ here.

The category $\mathcal{C}_w$ is the same category as $\mathcal{C}_{w^{-1}}^{\mathrm{GLS}}$. Moreover $N_-(w) = (N(w^{-1})^{\mathrm{GLS}})^T$ and $N_-^w = ((N^{w^{-1}})^{\mathrm{GLS}})^T$, here $(-)^T$ denotes the transpose in the Kac-Moody "group" G (see, for example, [**37**]). We omitted the definition of φ_X for a finite dimensional nilpotent Π-module X, however the algebra $\mathcal{M}$ which is used for its precise definition (see Definition

III.2.1) is the same space as $\mathcal{M}^{\mathrm{GLS}}$ in [**18**, Subsection 2.2] equipped with the opposite convolution product.

Thus there exist algebra isomorphisms $\mathbb{C}[N_-(w)] \to \mathbb{C}[N(w^{-1})^{\mathrm{GLS}}]$ and $\mathbb{C}[N_-^w] \to \mathbb{C}[N^{w^{-1},\mathrm{GLS}}]$ given by $f \to f \circ (-)^T$. Moreover $\varphi_X = \varphi_X^{\mathrm{GLS}} \circ (-)^T$ for all $X \in \mathcal{C}_w = \mathcal{C}_{w^{-1}}^{\mathrm{GLS}}$. See also [**18**, Chapter 6]. (This is the reason why we consider the opposite product on $\mathcal{M}$.)

The quantum nilpotent subalgebra $\mathbf{U}_q(\mathfrak{n}(w^{-1}))^{\mathrm{GLS}}$ in [**20**] is equal to $\mathbf{A}_q[N_-(w)]^\vee$. Geiß-Leclerc-Schröer consider a $\mathbb{Q}(q)$-algebra $\mathbf{A}_q(\mathfrak{n}(w^{-1}))^{\mathrm{GLS}}$, called the quantum coordinate ring, which is defined in $(\mathbf{U}_q^+)^*$ [**20**, (4.6)], and define an algebra isomorphism $\Psi^{\mathrm{GLS}} \colon \mathbf{U}_q(\mathfrak{n}(w^{-1}))^{\mathrm{GLS}} \to \mathbf{A}_q(\mathfrak{n}(w^{-1}))^{\mathrm{GLS}}$ by using a nondegenerate bilinear form $(-,-)^{\mathrm{GLS}}$ [**20**, Proposition 4.1]. Actually, for $x \in (\mathbf{U}_q^+)_\beta$, $y \in (\mathbf{U}_q^+)_{\beta'}$ $(\beta, \beta' \in Q_+)$, we have

$$\begin{aligned}(x,y)^{\mathrm{GLS}} &= \delta_{\beta,\beta'}(1-q^{-2})^{\mathrm{ht}\,\beta}\overline{(\overline{x},\overline{y})_L^+} \\ &= \delta_{\beta,\beta'}(1-q^{-2})^{\mathrm{ht}\,\beta}\overline{(\overline{x^\vee},\overline{y^\vee})_L} \\ &= \delta_{\beta,\beta'}(1-q^{-2})^{\mathrm{ht}\,\beta}(x^\vee,\sigma(\overline{y^\vee}))_L \\ &= q^{(\beta,\beta)/2}(q^{-1}-q)^{\mathrm{ht}\,\beta}(x^\vee,\varphi(y))_L.\end{aligned}$$

The last equality follows from Proposition I.1.12. By the way, there exists a $\mathbb{Q}(q)$-algebra automorphism $m_{\mathrm{norm}} \colon \mathbf{U}_q^- \to \mathbf{U}_q^-$ given by $f_i \mapsto (q^{-1}-q)^{-1} f_i$ for $i \in I$. We now have the following $\mathbb{Q}(q)$-algebra isomorphism;

$$I_{\mathrm{norm}} \colon \mathbf{A}_q[N_-(w)] \xrightarrow{m_{\mathrm{norm}}} \mathbf{A}_q[N_-(w)] \xrightarrow{\vee} \mathbf{U}_q(\mathfrak{n}(w^{-1}))^{\mathrm{GLS}} \xrightarrow{\Psi^{\mathrm{GLS}}} \mathbf{A}_q(\mathfrak{n}(w^{-1}))^{\mathrm{GLS}},$$

which maps $x \in (\mathbf{U}_q^-)_\beta$ $(\beta \in -Q_+)$ to $q^{(\beta,\beta)/2}(x, \varphi(-))_L$. By using this isomorphism, we describe their results. Note that $I_{\mathrm{norm}}(D_{w\lambda,w'\lambda}) = q^{(w\lambda-w'\lambda,w\lambda-w'\lambda)/2} D^{\mathrm{GLS}}_{w'\lambda,w\lambda}$ for $w, w' \in W$ and $\lambda \in P_+$ [**20**, (5.5)].

The definitions of the quantum cluster algebra $\mathscr{A}_{q^{\pm 1/2}}(\mathcal{C}_w) = \mathscr{A}_{q^{\pm 1/2}}(\mathcal{C}_{w^{-1}}^{\mathrm{GLS}})$ are the same. We have $Y_R = q^{(\underline{\dim}\,R,\underline{\dim}\,R)/2} Y_R^{\mathrm{GLS}}$ for every reachable Π-module R [**20**, (10.16)]. Note that $(\underline{\dim}\,R, \underline{\dim}\,R)/2 \in \mathbb{Z}$. Therefore we have $\mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w) = \mathscr{A}_{\mathbb{A}}(\mathcal{C}_{w^{-1}}^{\mathrm{GLS}})^{\mathrm{GLS}}$.

The following propositions describe mutations of quantum clusters and twisted dual bar involutions in $\mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)$. cf. (III.11).

Proposition III.2.12 ([**20**, Proposition 10.5]). *Let T be a basic reachable $\mathcal{C}_w$-maximal rigid module, and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. Fix $k \in [1, \ell - n]$. Write $\tilde{B}_T = (b_{ij})_{i\in[1,\ell], j\in[1,\ell-n]}$ and $\mu_{T_k}(T) = (T/T_k) \oplus T_k^*$. Set $T_+ := \bigoplus_{j; b_{jk}>0} T_j^{\oplus b_{jk}}$ and $T_- := \bigoplus_{j; b_{jk}<0} T_j^{\oplus(-b_{jk})}$. Then we have*

$$Y_{T_k^*} Y_{T_k} = q^{-\dim_{\mathbb{C}} \mathrm{Hom}_\Pi(T_k, T_k^*)}(qY_{T_+} + Y_{T_-}).$$

Proposition III.2.13 ([**20**, Lemma 10.6, Lemma 10.7]). *Let T be a basic reachable $\mathcal{C}_w$-maximal rigid module. Then there exists a unique $\mathbb{Q}$-anti-algebra involution σ'_T on $\mathcal{T}_{\mathcal{A},T}$ such that*

$$q \mapsto q^{-1},\ Y_R \mapsto q^{-(\underline{\dim}\,R,\underline{\dim}\,R)/2 + (\underline{\dim}\,R,\rho)} Y_R$$

for every direct summand R of a direct sum of T. Moreover σ'_T induces $\mathbb{Q}$-anti-algebra-involutions σ' on $\mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)$ and $\widetilde{\mathscr{A}_{q^{\pm 1}}}(\mathcal{C}_w)$, and σ' does not depend on the choice of a basic reachable $\mathcal{C}_w$-maximal rigid module T.

Geiß-Leclerc-Schröer showed that a rescaled quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathcal{C}_w)$ gives an additive categorification of the quantum unipotent subgroup $\mathbf{A}_q[N_-(w)]$ as follows.

Proposition III.2.14 ([**20**, Theorem 12.3]). *Let $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. Then there is an isomorphism of $\mathbb{Q}(q)$-algebras $\kappa\colon \mathbf{A}_q[N_-(w)] \to \mathscr{A}_{\mathbb{Q}(q)}(\mathcal{C}_w)$ given by*

$$D_{s_{i_1}\cdots s_{i_b}\varpi_{i_b}, s_{i_1}\cdots s_{i_{d^-}}\varpi_{i_d}} \mapsto Y_{M[b,d]}$$

for all $1 \leq d < b \leq \ell$ with $i_b = i_d$. Moreover we have $\sigma' \circ \kappa = \kappa \circ \sigma'$. See Definition I.1.13.

By Proposition III.1.41, this result also gives an additive categorification of the quantum unipotent cell $\mathbf{A}_q[N_-^w]$.

Corollary III.2.15. *Let $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. Then there is an isomorphism of $\mathbb{Q}(q)$-algebras $\widetilde{\kappa}\colon \mathbf{A}_q[N_-^w] \to \widetilde{\mathscr{A}_{\mathbb{Q}(q)}}(\mathcal{C}_w)$ given by*

$$[D_{s_{i_1}\cdots s_{i_b}\varpi_{i_b}, s_{i_1}\cdots s_{i_{d^-}}\varpi_{i_d}}] \mapsto Y_{M[b,d]}$$

for all $1 \leq d < b \leq \ell$ with $i_b = i_d$. Moreover we have $\sigma' \circ \widetilde{\kappa} = \widetilde{\kappa} \circ \sigma'$. See Definition III.1.39.

The following is the classical counterpart of the results above due to Geiß-Leclerc-Schröer. Note that we explain it as a “specialization” of the results above but it is actually the preceding result of them.

Proposition III.2.16 ([**18**, Theorem 3.1, Theorem 3.3]). *Let $w \in W$. For every reachable Π-module R in $\mathcal{C}_w$, we have $\varphi_R \in \mathbb{C}[N_-(w)]$, and the correspondence*

$$\varphi_R (\text{resp. } [\varphi_R]) \mapsto 1 \otimes Y_R.$$

gives the $\mathbb{C}$-algebra isomorphism from $\mathbb{C}[N_-(w)]$ (resp. $\mathbb{C}[N_-^w]$) to $\mathbb{C} \otimes_{\mathcal{A}} \mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)$ (resp. $\mathbb{C} \otimes_{\mathcal{A}} \widetilde{\mathscr{A}_{q^{\pm 1}}}(\mathcal{C}_w)$).

Remark III.2.17. The isomorphism in Proposition III.2.16 is the “specialization” of the one in Proposition III.2.14. However Geiß-Leclerc-Schröer did not prove the isomorphism between the standard $\mathcal{A}$-form $\mathbf{A}_{\mathcal{A}}[N_-(w)]$ of $\mathbf{A}_q[N_-(w)]$ and $\mathscr{A}_{q^{\pm 1}}(\mathcal{C}_w)$. See [**20**, Conjecture 12.7].

Definition III.2.18. Let T be a basic reachable $\mathcal{C}_w$-maximal rigid module and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. Then a Q_--grading on $\mathbb{Q}[q^{\pm 1}][Y_k \mid k = 1, \dots, \ell] (\subset \mathcal{T}_{\mathcal{A},T})$ given by $\operatorname{wt} Y_{T_k} = \underline{\dim}\, T_k$ is extended to the Q-grading on $\mathcal{T}_{\mathcal{A},T}$. A homogeneous element $X \in \mathcal{T}_{\mathcal{A},T}$ is said to be dual bar invariant if

$$\sigma'_T(X) = q^{-(\operatorname{wt} X, \operatorname{wt} X)/2 + (\operatorname{wt} X, \rho)} X.$$

When $X \in \mathscr{A}_{\mathbb{Q}(q)}(\mathcal{C}_w)$ (resp. $\widetilde{\mathscr{A}_{\mathbb{Q}(q)}}(\mathcal{C}_w)$), the Q-grading and the definition of dual bar invariance of homogeneous elements are compatible with the corresponding notions in $\mathbf{A}_q[N_-(w)]$ (resp. $\mathbf{A}_q[N_-^w]$) via κ (resp. $\widetilde{\kappa}$). See Remark I.1.14. Note that Y_R is dual bar invariant for any reachable Π-module R.

Geiß-Leclerc-Schröer also obtained an additive categorification of the twist automorphism η_w^* on the coordinate algebra $\mathbb{C}[N_-^w]$ of a unipotent cell N_-^w in non-quantum settings. Here the image of φ_X under the restriction map $\mathbb{C}[N_-] \to \mathbb{C}[N_-^w]$ is denoted by $[\varphi_X]$.

Proposition III.2.19 ([**19**, Theorem 6]). *Let $w \in W$. Then for every $X \in \mathcal{C}_w$ we have*

$$\eta_w^*([\varphi_X]) = \frac{[\varphi_{\Omega_w^{-1}(X)}]}{[\varphi_{I(X)}]}.$$

III.2.2. Quantum twist automorphisms and the quantum cluster algebra structure. Our main result in this section is the following quantum analogue of Proposition III.2.19. Recall Proposition III.2.10.

Theorem III.2.20. *Let $w \in W$, T a basic reachable $\mathcal{C}_w$-maximal rigid module, and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. Through $\widetilde{\kappa}$ in Corollary III.2.15, we regard the quantum twist map $\eta_{w,q}$ as an algebra automorphism on $\widetilde{\mathscr{A}_{\mathbb{Q}(q)}}(\mathcal{C}_w)$. Then, for every reachable Π-module R in $\mathcal{C}_w$, we have*

$$\eta_{w,q}(Y_R) = q^{\sum_{i\in I} \lambda_i \dim_{\mathbb{C}} \mathbf{e}_i.R} Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}. \tag{III.13}$$

here we write $I(R) = \bigoplus_{i\in I} T_{\ell-n+i}^{\oplus\lambda_i}$.

Before proving Theorem III.2.20, we show its corollary.

Corollary III.2.21. *Let R be a reachable Π-module in $\mathcal{C}_w$. Then $\kappa^{-1}(Y_R) \in \mathbf{B}^{\mathrm{up}} \cap \mathbf{A}_q[N_-(w)]$ if and only if $\kappa^{-1}(Y_{\Omega_w^{-1}(R)}) \in \mathbf{B}^{\mathrm{up}} \cap \mathbf{A}_q[N_-(w)]$.*

Proof. By Theorem III.1.42 and III.2.20, $\kappa^{-1}(Y_R) \in \mathbf{B}^{\mathrm{up}} \cap \mathbf{A}_q[N_-(w)]$ if and only if $\widetilde{\kappa}^{-1}(q^{\sum_{i\in I} \lambda_i \dim_{\mathbb{C}} \mathbf{e}_i.R} Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}) \in \widetilde{\mathbf{B}}^{\mathrm{up},w}$. By Theorem III.1.42 and the dual bar invariance of Y_R, the element $q^{\sum_{i\in I} \lambda_i \dim_{\mathbb{C}} \mathbf{e}_i.R} Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}$ is also dual bar invariant. Combining this fact with the definition of $\widetilde{\mathbf{B}}^{\mathrm{up},w} = \iota_w(\widetilde{\mathbf{B}}^{\mathrm{up}}(w))$ and the dual bar invariance of $Y_{\Omega_w^{-1}(R)}$, we have $\widetilde{\kappa}^{-1}(q^{\sum_{i\in I} \lambda_i \dim_{\mathbb{C}} \mathbf{e}_i.R} Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}) \in \widetilde{\mathbf{B}}^{\mathrm{up},w}$ if and only if $\kappa^{-1}(Y_{\Omega_w^{-1}(R)}) \in \mathbf{B}^{\mathrm{up}} \cap \mathbf{A}_q[N_-(w)]$. □

Remark III.2.22. Kang-Kashiwara-Kim-Oh [**27, 28**] have shown that all (rescaled) quantum cluster monomials belong to $\mathbf{B}^{\mathrm{up}}$ by using the categorification via representations of quiver Hecke algebras. Hence we have already known that Y_R is an element of $\mathbf{B}^{\mathrm{up}}$ for an arbitrary reachable Π-module in $\mathcal{C}_w$. However there is now no proof of this strong result through the additive categorification above. Therefore it would be interesting to determine the quantum monomials in $\mathbf{B}^{\mathrm{up}}$ which are obtained from Corollary III.2.21 and, for example, $(Y_{V_{\boldsymbol{i}}})^{\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$ and $\boldsymbol{i} \in I(w)$. Actually, it is easy to show that $(Y_{V_{\boldsymbol{i}}})^{\boldsymbol{a}} \in \mathbf{B}^{\mathrm{up}}$ by Proposition III.1.32. For iterated application of quantum BFZ-twist automorphisms, see also subsection III.1.5. Moreover it is unclear whether a quantum BFZ-twist automorphism $\eta_{w,q}$ is categorified by using finite dimensional representations of quiver Hecke algebras. In particular, it is unclear whether quantum BFZ-twist automorphisms preserve the basis coming from the simple modules of quiver Hecke algebras.

The rest of this subsection is devoted to the proof of Theorem III.2.20. In this proof, we essentially use Geiß-Leclerc-Schröer's theory.

LEMMA III.2.23. *Let T be a basic reachable $\mathcal{C}_w$-maximal rigid module and $T = T_1 \oplus \cdots \oplus T_\ell$ its indecomposable decomposition. Take $(a_1, \dots, a_\ell) \in \mathbb{Z}^\ell$. Then there exists a unique integer m such that $q^m Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell}$ is dual bar invariant in $\mathcal{T}_{\mathcal{A},T}$.*

Proof. We have

$$\begin{aligned}\sigma'_T(q^m Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell}) &= q^{-m} \sigma'_T(Y_{T_\ell})^{a_\ell} \cdots \sigma'_T(Y_{T_1})^{a_1} \\ &= q^{-m+\sum_{i\in[1,\ell]} a_i(-(\underline{\dim}\, T_i, \underline{\dim}\, T_i)/2 + (\underline{\dim}\, T_i, \rho))} Y_{T_\ell}^{a_\ell} \cdots Y_{T_1}^{a_1} \\ &= q^{-m+\sum_{i\in[1,\ell]} a_i(-(\underline{\dim}\, T_i, \underline{\dim}\, T_i)/2 + (\underline{\dim}\, T_i, \rho)) - \sum_{i<j} a_i a_j \lambda_{ij}} Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell}.\end{aligned}$$

Here we write $\Lambda_T = (\lambda_{ij})_{i,j\in[1,\ell]}$. Therefore $q^m Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell}$ is dual bar invariant if and only if

$$\begin{aligned}&m - \sum_{i\in[1,\ell]} a_i^2 (\underline{\dim}\, T_i, \underline{\dim}\, T_i)/2 - \sum_{i<j} a_i a_j (\underline{\dim}\, T_i, \underline{\dim}\, T_j) + \sum_{i\in[1,\ell]} a_i (\underline{\dim}\, T_i, \rho) \\ &= -m + \sum_{i\in[1,\ell]} a_i(-(\underline{\dim}\, T_i, \underline{\dim}\, T_i)/2 + (\underline{\dim}\, T_i, \rho)) - \sum_{i<j} a_i a_j \lambda_{ij}.\end{aligned}$$

By Remark III.2.8, this is equivalent to

$$2m = \sum_{i\in[1,\ell]} a_i(a_i - 1)(\underline{\dim}\, T_i, \underline{\dim}\, T_i)/2 + 2\sum_{i<j} a_i a_j \dim_{\mathbb{C}} \operatorname{Hom}_{\Pi}(T_j, T_i).$$

The right-hand side is an element of $2\mathbb{Z}$. Therefore we can take an integer $m \in \mathbb{Z}$ uniquely which satisfies this equality. □

REMARK III.2.24. For $(a_1, \dots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, the dual bar invariant element in $q^{\mathbb{Z}}\{Y_{T_1}^{a_1} \cdots Y_{T_\ell}^{a_\ell}\}$ is nothing but $Y_{\bigoplus_{i\in[1,\ell]} T_i^{\oplus a_i}}$. See Definition III.2.9.

LEMMA III.2.25. *With the notation in Theorem III.2.20, $q^{\sum_{i\in I} \lambda_i \dim_{\mathbb{C}} \mathbf{e}_i . R} Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)}$ is dual bar invariant.*

Proof. By Proposition III.2.14,

$$\kappa^{-1}(Y_{I(R)}) = D_{w\lambda,\lambda},$$

here $\lambda := \sum_{j\in I} \lambda_j \varpi_j$. Hence, by Proposition III.1.27, we have

$$\begin{aligned}\kappa^{-1}(Y_{I(R)} Y_{\Omega_w^{-1}(R)}) &= D_{w\lambda,\lambda} \kappa^{-1}(Y_{\Omega_w^{-1}(R)}) \\ &= q^{(\lambda + w\lambda, \underline{\dim}\, \Omega_w^{-1}(R))} \kappa^{-1}(Y_{\Omega_w^{-1}(R)}) D_{w\lambda,\lambda} \\ &= q^{(\lambda + w\lambda, \underline{\dim}\, \Omega_w^{-1}(R))} \kappa^{-1}(Y_{\Omega_w^{-1}(R)} Y_{I(R)}).\end{aligned}$$

By the way, $\underline{\dim}\, \Omega_w^{-1}(R) = \underline{\dim}\, I(R) - \underline{\dim}\, R = w\lambda - \lambda - \underline{\dim}\, R$. Hence $(\lambda + w\lambda, \underline{\dim}\, \Omega_w^{-1}(R)) = -(\lambda + w\lambda, \underline{\dim}\, R)$. Therefore

$$Y_{I(R)}^{-1} Y_{\Omega_w^{-1}(R)} = q^{(\lambda + w\lambda, \underline{\dim}\, R)} Y_{\Omega_w^{-1}(R)} Y_{I(R)}^{-1}$$

Note that $\sum_{i\in I}\lambda_i \dim_{\mathbb{C}} \mathbf{e}_i.R = -(\lambda, \underline{\dim} R)$. We have

$$\begin{aligned}
&q^{(\underline{\dim}\,\Omega_w^{-1}(R)-\underline{\dim}\,I(R),\underline{\dim}\,\Omega_w^{-1}(R)-\underline{\dim}\,I(R))/2-(\underline{\dim}\,\Omega_w^{-1}(R)-\underline{\dim}\,I(R),\rho)}\sigma'_T(q^{-(\lambda,\underline{\dim}\,R)}Y_{I(R)}^{-1}Y_{\Omega_w^{-1}(R)})\\
&= q^{(\underline{\dim}\,R,\underline{\dim}\,R)/2+(\underline{\dim}\,R,\rho)}\sigma'_T(q^{(w\lambda,\underline{\dim}\,R)}Y_{\Omega_w^{-1}(R)}Y_{I(R)}^{-1})\\
&= q^{(\underline{\dim}\,R,\underline{\dim}\,R)/2+(\underline{\dim}\,R,\rho)-(w\lambda,\underline{\dim}\,R)}\sigma'_T(Y_{I(R)}^{-1})\sigma'_T(Y_{\Omega_w^{-1}(R)})\\
&= q^{(\underline{\dim}\,R,\underline{\dim}\,R)/2-(\underline{\dim}\,\Omega_w^{-1}(R),\underline{\dim}\,\Omega_w^{-1}(R))/2+(\underline{\dim}\,I(R),\underline{\dim}\,I(R))/2-(w\lambda,\underline{\dim}\,R)}Y_{I(R)}^{-1}Y_{\Omega_w^{-1}(R)}\\
&= q^{(\underline{\dim}\,I(R),\underline{\dim}\,R)-(w\lambda,\underline{\dim}\,R)}Y_{I(R)}^{-1}Y_{\Omega_w^{-1}(R)}\\
&= q^{-(\lambda,\underline{\dim}\,R)}Y_{I(R)}^{-1}Y_{\Omega_w^{-1}(R)}.
\end{aligned}$$

This competes the proof. □

Lemma III.2.26. *Let T be a basic reachable $\mathcal{C}_w$-maximal rigid module and $T = T_1\oplus\cdots\oplus T_\ell$ its indecomposable decomposition. Then the equality* (III.13) *with $R = T_k$ holds for all $k = 1,\dots,\ell$ if and only if the one with $R = T_1^{\oplus a_1}\oplus\cdots\oplus T_\ell^{\oplus a_\ell}$ holds for all $(a_1,\dots,a_\ell)\in\mathbb{Z}_{\geq 0}^\ell$.*

Proof. The latter obviously implies the former. Suppose that the equality (III.13) holds for $R = T_k$, $k = 1,\dots,\ell$. Write

$$\eta_{w,q}(Y_{T_k}) = q^{m_k}Y_{I(T_k)}^{-1}Y_{\Omega_w^{-1}(T_k)},\ m_k\in\mathbb{Z},$$

for $k = 1,\dots,\ell$. Set $R = T_1^{\oplus a_1}\oplus\cdots\oplus T_\ell^{\oplus a_\ell}$ for $(a_1,\dots,a_\ell)\in\mathbb{Z}_{\geq 0}^\ell$. Note that $I(R) = I(T_1)^{\oplus a_1}\oplus\cdots\oplus I(T_\ell)^{\oplus a_\ell}$ and $\Omega_w^{-1}(R) = \Omega_w^{-1}(T_1)^{\oplus a_1}\oplus\cdots\oplus\Omega_w^{-1}(T_\ell)^{\oplus a_\ell}$. (Actually $I(T_{\ell-n+i}) = T_{\ell-n+i}$ and $\Omega_w^{-1}(T_{\ell-n+i}) = 0$ for $i\in I$.) There exist unique $A_1, A_2, A_3\in\mathbb{Z}$ such that the following hold:

$$\begin{aligned}
\eta_{w,q}(Y_R) &= q^{A_1}\eta_{w,q}(Y_{T_1}^{a_1}\cdots Y_{T_\ell}^{a_\ell})\\
&= q^{A_1}(q^{m_1}Y_{I(T_1)}^{-1}Y_{\Omega_w^{-1}(T_1)})^{a_1}\cdots(q^{m_\ell}Y_{I(T_\ell)}^{-1}Y_{\Omega_w^{-1}(T_\ell)})^{a_\ell}\\
&= q^{A_2}(Y_{I(T_1)}^{a_1}\cdots Y_{I(T_\ell)}^{a_\ell})^{-1}Y_{\Omega_w^{-1}(T_1)}^{a_1}\cdots Y_{\Omega_w^{-1}(T_\ell)}^{a_\ell}\\
&= q^{A_3}Y_{I(R)}^{-1}Y_{\Omega_w^{-1}(R)}.
\end{aligned}$$

Moreover $\eta_{w,q}(Y_R)$ is dual bar invariant because of the dual bar invariance of Y_R and Theorem III.1.42. Hence, by Lemma III.2.23 and Lemma III.2.25, the equality (III.13) also holds for R. □

Proof of Theorem III.2.20. Recall that we always assume that $T_{\ell-n+i}$ is a $\mathcal{C}_w$-projective-injective module with socle S_i for all $i\in I = [1,n]$, in particular, the isomorphism class of $T_{\ell-n+i}$ does not depend on the choice of T. From now on, we identify $\widetilde{\mathscr{A}_{\mathbb{Q}(q)}}(\mathcal{C}_w)$ with $\mathbf{A}_q[N_-^w]$ via $\widetilde{\kappa}$. First we consider the case that R in the statement of Theorem III.2.20 is equal to $T_{\ell-n+i}$ for $i\in I$. Then

$$\begin{aligned}
\eta_{w,q}(Y_{T_{\ell-n+i}}) &= \eta_{w,q}([D_{w\varpi_i,\varpi_i}])\\
&= q^{-(\varpi_i,w\varpi_i-\varpi_i)}[D_{w\varpi_i,\varpi_i}]^{-1}\\
&= q^{\dim_{\mathbb{C}}\mathbf{e}_i.T_{\ell-n+i}}Y_{T_{\ell-n+i}}^{-1},
\end{aligned}$$

which is the desired equality in this case since $I(T_{\ell-n+i}) = T_{\ell-n+i}$ and $\Omega_w^{-1}(T_{\ell-n+i}) = 0$. Next we consider the case that $R = V_{\boldsymbol{i},k}$ for some $\boldsymbol{i} \in I(w)$ and $k \in [1,\ell]$ with $k^+ \neq \ell+1$. Then $I(V_{\boldsymbol{i},k}) = V_{\boldsymbol{i},k^{\max}}$ and $\Omega_w^{-1}(V_{\boldsymbol{i},k}) = M_{\boldsymbol{i}}[k^{\max}, k^+]$. Therefore we have

$$\begin{aligned}\eta_{w,q}(Y_{V_{\boldsymbol{i},k}}) &= \eta_{w,q}(D_{s_{i_1}\cdots s_{i_k}\varpi_{i_k},\varpi_{i_k}}) \\ &= q^{-(\varpi_{i_k}, s_{i_1}\cdots s_{i_k}\varpi_{i_k}-\varpi_{i_k})} D^{-1}_{w\varpi_{i_k},\varpi_{i_k}} D_{uw\varpi_{i_k}, s_{i_1}\cdots s_{i_k}\varpi_{i_k}} \\ &= q^{-(\varpi_{i_k}, \underline{\dim}\, V_{\boldsymbol{i},k})} Y^{-1}_{V_{\boldsymbol{i},k^{\max}}} Y_{M_{\boldsymbol{i}}[k^{\max},k^+]} \\ &= q^{\dim_{\mathbb{C}} \mathbf{e}_i . V_{\boldsymbol{i},k}} Y^{-1}_{I(V_{\boldsymbol{i},k})} Y_{\Omega_w^{-1}(V_{\boldsymbol{i},k})}.\end{aligned}$$

Suppose that the equality (III.13) hold for $R = T_1^{\oplus a_1} \oplus \cdots \oplus T_\ell^{\oplus a_\ell}$, where $T = T_1 \oplus \cdots \oplus T_\ell$ is a basic reachable $\mathcal{C}_w$-maximal rigid module. Fix $k \in [1, \ell-n]$. Write $\mu_{T_k}(T) = (T/T_k) \oplus T_k^*$ and $I(T_k^*) = \bigoplus_{i\in I} T_{\ell-n+i}^{\oplus \lambda_i}$. By Lemma III.2.26, it remains to prove the following equality;

$$\eta_{w,q}(Y_{T_k^*}) = q^{\sum_{i\in I} \lambda_i \dim_{\mathbb{C}} \mathbf{e}_i . T_k^*} Y^{-1}_{I(T_k^*)} Y_{\Omega_w^{-1}(T_k^*)}. \tag{III.14}$$

Write $\tilde{B}_T = (b_{ij})_{i\in[1,\ell], j\in[1,\ell-n]}$. Set $T_+ := \bigoplus_{j; b_{jk}>0} T_j^{\oplus b_{jk}}$ and $T_- := \bigoplus_{j; b_{jk}<0} T_j^{\oplus(-b_{jk})}$. By (III.11) and Proposition III.2.19, we have

$$\eta_w^*([\varphi_{T_k}][\varphi_{T_k^*}]) = \eta_w^*([\varphi_{T_+}] + [\varphi_{T_-}]) = \frac{[\varphi_{\Omega_w^{-1}(T_+)}]}{[\varphi_{I(T_+)}]} + \frac{[\varphi_{\Omega_w^{-1}(T_-)}]}{[\varphi_{I(T_-)}]},$$

and

$$\eta_w^*([\varphi_{T_k}][\varphi_{T_k^*}]) = \frac{[\varphi_{\Omega_w^{-1}(T_k)}]}{[\varphi_{I(T_k)}]} \cdot \frac{[\varphi_{\Omega_w^{-1}(T_k^*)}]}{[\varphi_{I(T_k^*)}]}.$$

Therefore

$$[\varphi_{\Omega_w^{-1}(T_k)}][\varphi_{\Omega_w^{-1}(T_k^*)}] = [\varphi_{I(T_k\oplus T_k^*)}] \left(\frac{[\varphi_{\Omega_w^{-1}(T_+)}]}{[\varphi_{I(T_+)}]} + \frac{[\varphi_{\Omega_w^{-1}(T_-)}]}{[\varphi_{I(T_-)}]} \right). \tag{III.15}$$

By Proposition III.2.10, $T' := \Omega_w^{-1}(T) \oplus \bigoplus_{i\in I} T_{\ell-n+i}$ is a basic reachable $\mathcal{C}_w$-maximal rigid module; hence there exists a bijection $[1,\ell-n] \to [1,\ell-n], j \mapsto j^*$ such that $T'_{j^*} = \Omega_w^{-1}(T_j)$. Moreover we have

$$\mu_{T'_{k^*}}(T') = (T'/T'_{k^*}) \oplus \Omega_w^{-1}(T_k^*).$$

Write $\tilde{B}_{T'} = (b'_{ij})_{i\in[1,\ell], j\in[1,\ell-n]}$ and $(T'_{k^*})^* := \Omega_w^{-1}(T_k^*)$. Set $T'_+ := \bigoplus_{j; b'_{j^*k^*}>0} (T'_{j^*})^{\oplus b'_{j^*k^*}}$ and $T'_- := \bigoplus_{j; b'_{j^*k^*}<0} (T'_{j^*})^{\oplus(-b'_{j^*k^*})}$. Then, by (III.11) and (III.15), we have

$$[\varphi_{I(T_k\oplus T_k^*)}] \left(\frac{[\varphi_{\Omega_w^{-1}(T_+)}]}{[\varphi_{I(T_+)}]} + \frac{[\varphi_{\Omega_w^{-1}(T_-)}]}{[\varphi_{I(T_-)}]} \right) = [\varphi_{T'_+}] + [\varphi_{T'_-}]. \tag{III.16}$$

We now recall our assumption that the equality (III.13) hold for $R = T_1^{\oplus a_1} \oplus \cdots \oplus T_\ell^{\oplus a_\ell}$. By Proposition III.2.12 and our assumption, there exist unique $A_1, A'_1, A_2, A'_2, A_3 \in \mathbb{Z}$ such that

$$\begin{aligned}\eta_{w,q}(Y_{T_k}Y_{T_k^*}) &= \eta_{w,q}(q^{A_1}Y_{T_+}+q^{A_2}Y_{T_-})\\ &= q^{A_1'}Y_{I(T_+)}^{-1}Y_{\Omega_w^{-1}(T_+)}+q^{A_2'}Y_{I(T_-)}^{-1}Y_{\Omega_w^{-1}(T_-)},\end{aligned}$$

and

$$\eta_{w,q}(Y_{T_k}Y_{T_k^*}) = q^{A_3}Y_{I(T_k)}^{-1}Y_{T'_{k^*}}\eta_{w,q}(Y_{T_k^*}).$$

Therefore, by (III.16), there exist unique $A, A_1'', A_2'' \in \mathbb{Z}$ such that

$$\begin{aligned}\eta_{w,q}(Y_{T_k^*}) &= q^{A}Y_{T'_{k^*}}^{-1}Y_{I(T_k^*)}^{-1}Y_{I(T_k\oplus T_k^*)}(q^{A_1'}Y_{I(T_+)}^{-1}Y_{\Omega_w^{-1}(T_+)}+q^{A_2'}Y_{I(T_-)}^{-1}Y_{\Omega_w^{-1}(T_-)})\\ &= Y_{I(T_k^*)}^{-1}Y_{T'_{k^*}}^{-1}\left(q^{A_1''}Y_{T'_+}+q^{A_2''}Y_{T'_-}\right).\end{aligned}$$

Note that all rescaled quantum cluster monomials appearing in the rightmost side are elements of the standard basis of the based quantum torus $\mathcal{T}_{\mathcal{A},T'}$. By Theorem III.1.42 and Proposition III.2.13, $\eta_{w,q}(Y_{T_k^*})$ is dual bar invariant. Hence $q^{A_1''}Y_{I(T_k^*)}^{-1}Y_{T'_{k^*}}^{-1}Y_{T'_+}$ and $q^{A_2''}Y_{I(T_k^*)}^{-1}Y_{T'_{k^*}}^{-1}Y_{T'_-}$ are dual bar invariant elements of $\mathcal{T}_{\mathcal{A},T'}$. By Lemma III.2.23, A_1'' and A_2'' are uniquely determined by this property. On the other hand, by Proposition III.2.12, $q^{\sum_{i\in I}\lambda_i \dim_{\mathbb{C}} \mathbf{e}_i.T_k^*}Y_{I(T_k^*)}^{-1}Y_{(T'_{k^*})^*}$ is of the following form as an element of $\mathcal{T}_{q^{\pm1},T'}$;

$$Y_{I(T_k^*)}^{-1}Y_{T'_{k^*}}^{-1}\left(q^{M_1}Y_{T'_+}+q^{M_2}Y_{T'_-}\right),\ M_1, M_2 \in \mathbb{Z}.$$

Moreover, by Lemma III.2.25, $q^{\sum_{i\in I}\lambda_i \dim_{\mathbb{C}} \mathbf{e}_i.T_k^*}Y_{I(T_k^*)}^{-1}Y_{(T'_{k^*})^*} = q^{\sum_{i\in I}\lambda_i \dim_{\mathbb{C}} \mathbf{e}_i.T_k^*}Y_{I(T_k^*)}^{-1}Y_{\Omega_w^{-1}(T_k^*)}$ is dual bar invariant. Hence, by the argument above, $M_1 = A_1'$ and $M_2 = A_2''$. Therefore we obtain the equality III.14, which completes the proof. □

III.3. Quantum Chamber Ansatz

We again consider an arbitrary symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. In this section, we prove quantum analogues of the Chamber Ansatz formulae for unipotent cells (Corollary III.3.9) by using the quantum BFZ-twist automorphisms constructed in Section III.1. The quantum analogues of birational homomorphisms between algebraic tori and unipotent cells are known as Feigin homomorphisms. By Feigin homomorphisms, we can realize quantum unipotent cells in q-Laurent polynomial algebras. Quantum Chamber Ansatz formulae provide explicit description of the variables of the q-Laurent polynomial algebras in terms of elements of quantum unipotent cells.

Definition III.3.1. Let $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I^\ell$. The q-polynomial algebra (resp. the q-Laurent polynomial algebra) $\mathcal{P}_{\boldsymbol{i}}$ (resp. $\mathcal{L}_{\boldsymbol{i}}$) is the unital associative $\mathbb{Q}(q)$-algebra generated by $t_1, \dots, t_\ell$ (resp. $t_1^{\pm1}, \dots, t_\ell^{\pm1}$) subject to the relations;

$$\begin{aligned}&t_jt_k = q^{(\alpha_{i_j},\alpha_{i_k})}t_kt_j \text{ for } 1 \le j < k \le \ell,\\ &t_kt_k^{-1} = t_k^{-1}t_k = 1 \text{ for } 1 \le k \le \ell.\end{aligned}$$

Set $\mathcal{U}_{\boldsymbol{i}}^- := \prod_{\alpha\in Q_+}\mathcal{P}_{\boldsymbol{i}}\otimes_{\mathbb{Q}(q)}(\mathbf{U}_q^-)_{-\alpha}$. We write an element $(p_{(-\alpha)}\otimes x_{(-\alpha)})_{\alpha\in Q_+}$ $(p_{(-\alpha)}\in\mathcal{P}_{\boldsymbol{i}}, x_{(-\alpha)}\in(\mathbf{U}_q^-)_{-\alpha})$ of $\mathcal{U}_{\boldsymbol{i}}^-$ as $\sum_{\alpha\in Q_+}p_{(-\alpha)}x_{(-\alpha)}$. The vector space $\mathcal{U}_{\boldsymbol{i}}^-$ has the $\mathbb{Q}(q)$-algebra structure given by

$$\left(\sum_{\alpha\in Q_+}p_{(-\alpha)}x_{(-\alpha)}\right)\left(\sum_{\alpha\in Q_+}p'_{(-\alpha)}x'_{(-\alpha)}\right)=\sum_{\alpha\in Q_+}\left(\sum_{\substack{\beta,\beta'\in Q_+\\ \beta+\beta'=\alpha}}p_{(-\beta)}p'_{(-\beta')}x_{(-\beta)}x'_{(-\beta')}\right)$$

for $p_{(-\alpha)},p'_{(-\alpha)}\in\mathcal{P}_{\boldsymbol{i}}$, $x_{(-\alpha)},x'_{(-\alpha)}\in(\mathbf{U}_q^-)_{-\alpha}$. Set

$$y_{\boldsymbol{i}}:=\exp_{q_{i_1}}(t_1f_{i_1})\cdots\exp_{q_{i_\ell}}(t_\ell f_{i_\ell}).$$

where

$$\exp_{q_{i_k}}(t_kf_{i_k}):=\sum_{m\in\mathbb{Z}_{\geq 0}}q_{i_k}^{m(m-1)/2}t_k^mf_{i_k}^{(m)}\in\mathcal{U}_{\boldsymbol{i}}^-$$

for $1\leq k\leq\ell$. Then we can define the $\mathbb{Q}(q)$-linear map $\Phi_{\boldsymbol{i}}\colon\mathbf{U}_q^-\to\mathcal{P}_{\boldsymbol{i}}$ by

$$x\mapsto(x,y_{\boldsymbol{i}})_L:=\sum_{\boldsymbol{a}=(a_1,\ldots,a_\ell)\in\mathbb{Z}_{\geq 0}^\ell}q_{\boldsymbol{i}}(\boldsymbol{a})(x,f_{i_1}^{(a_1)}\cdots f_{i_\ell}^{(a_\ell)})_Lt_1^{a_1}\cdots t_\ell^{a_\ell}$$

where

$$q_{\boldsymbol{i}}(\boldsymbol{a}):=\prod_{k=1}^{\ell}q_{i_k}^{a_k(a_k-1)/2}.$$

Note that the all but finitely many summands in the right-hand side are zero. The map $\Phi_{\boldsymbol{i}}$ is called a Feigin homomorphism.

Proposition III.3.2 ([**3**]). (1) *For* $\boldsymbol{i}\in I^\ell$, *the map* $\Phi_{\boldsymbol{i}}$ *is a* $\mathbb{Q}(q)$*-algebra homomorphism.*

(2) *For* $w\in W$ *and* $\boldsymbol{i}\in I(w)$, *we have* $\operatorname{Ker}\Phi_{\boldsymbol{i}}=(\mathbf{U}_w^-)^\perp$.

(3) *For* $w\in W$, $\boldsymbol{i}=(i_1,\ldots,i_\ell)\in I(w)$ *and* $\lambda\in P_+$, *we have*

$$\Phi_{\boldsymbol{i}}(D_{w\lambda,\lambda})=q_{\boldsymbol{i}}(\boldsymbol{a})t_1^{a_1}\cdots t_\ell^{a_\ell}$$

where $\boldsymbol{a}=(a_1,\ldots,a_\ell)$ *with* $a_k:=\langle h_{i_k},s_{i_{k+1}}\cdots s_{i_\ell}\lambda\rangle$.

Remark III.3.3. For any $\boldsymbol{i}=(i_1,\ldots,i_\ell)\in I^\ell$, we have $\Phi_{\boldsymbol{i}}((1-q_i^2)f_i)=\sum_{k;i_k=i}t_k$.

Definition III.3.4. Let $w\in W$ and $\boldsymbol{i}\in I(w)$. By Proposition III.3.2 and the universality of localization, we have the embedding of an algebra $\mathbf{A}_q[N_-^w]\to\mathcal{L}_{\boldsymbol{i}}$, also denoted by $\Phi_{\boldsymbol{i}}$.

Definition III.3.5. Let $w\in W$ and suppose that its reduced word $\boldsymbol{i}=(i_1,\ldots,i_\ell)\in I(w)$ is fixed. Write $w_{\leq k}:=s_{i_1}\cdots s_{i_k}$ and $w_{k\leq}:=s_{i_k}\cdots s_{i_\ell}$ for $k=1,\ldots,\ell$.

In the following theorem, we need the inverse of quantum BFZ-twist automorphisms. By Theorem III.1.42,

$$\eta_{w,q}^{-1}([D_{u_{w\lambda},u}])=q^{(\lambda,\operatorname{wt}u-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[D_{u,u_\lambda}]\qquad\text{(III.17)}$$

for a weight vector $u \in V(\lambda)$ and $\lambda \in P_+$.

THEOREM III.3.6. *Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $k = 1, \dots, \ell$. Then we have*

$$(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})([D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}]) = \left(\prod_{j=1}^{k} q_{i_j}^{d_j(d_j+1)/2}\right) t_1^{-d_1} t_2^{-d_2} \cdots t_k^{-d_k},$$

where $d_j := \langle w_{\leq j} h_{i_j}, w_{\leq k}\varpi_{i_k}\rangle$, $j = 1, \dots, k$.

REMARK III.3.7. Note that, by Proposition II.2.7, $D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}} \in \mathbf{A}_q[N_-(w)]$.

REMARK III.3.8. Theorem III.3.6 is a generalization of [**5**, Corollary 1.2], where they treat the case that w is a square of a Coxeter element. Moreover, by Theorem III.3.6, we can say that the quantum BFZ-twist automorphisms $\eta_{w,q}$ is a generalization of Berenstein-Rupel's quantum BFZ-twist automorphisms [**5**, Theorem 2.10]. This result corresponds to [**5**, Conjecture 2.12 (c)]. Therefore Theorem III.1.42 corresponds to [**5**, Conjecture 2.17 (a)]. However we do not deal with their upper quantum cluster algebras.

Proof. If $w = e$, there is nothing to prove. From now on, we assume that the length ℓ of w is greater than 0. The proof is by induction on k. Let $k = 1$. Take $\lambda \in P_+$ such that $\langle h_{i_1}, w\lambda\rangle < 0$. Then it is easily seen that

$$D_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}} = [\langle h_{i_1}, w_{2\leq}\lambda\rangle]_{i_1}^{-1} D_{u_{w\lambda},\, e_{i_1}.u_{w\lambda}}.$$

Hence, by (III.17),

$$(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})([D_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}}]) = q_{i_1}^{\langle h_{i_1}, \lambda\rangle} [\langle h_{i_1}, w_{2\leq}\lambda\rangle]_{i_1}^{-1} \Phi_{\boldsymbol{i}}\left([D_{w\lambda,\lambda}]^{-1}[D_{e_{i_1}.u_{w\lambda}, u_\lambda}]\right).$$

By Proposition III.3.2 (3), we have

$$\begin{aligned}
\Phi_{\boldsymbol{i}}\left([D_{w\lambda,\lambda}]^{-1}\right) &= q_{\boldsymbol{i}}(\boldsymbol{c})^{-1} t_\ell^{-c_\ell} \cdots t_1^{-c_1},\\
\Phi_{\boldsymbol{i}}\left([D_{e_{i_1}.u_{w\lambda}, u_\lambda}]\right) &= q_{\boldsymbol{i}}(\boldsymbol{c} - (1, 0, \dots, 0))[c_1]_{i_1} t_1^{c_1 - 1} t_2^{c_2} \cdots t_\ell^{c_\ell},
\end{aligned}$$

where $\boldsymbol{c} = (c_1, \dots, c_\ell)$ with $c_j := \langle h_{i_j}, w_{j+1\leq}\lambda\rangle$. Combining the above equalities, we obtain

$$\begin{aligned}
(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})([D_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}}]) &= q_{i_1}^{\langle h_{i_1}, \lambda - w_{2\leq}\lambda - \sum_{j=2}^{\ell} c_j \alpha_{i_j}\rangle + 1} t_1^{-1}\\
&= q_{i_1} t_1^{-1}.
\end{aligned}$$

This proves the assertion in the case $k = 1$.

Assume that $k > 1$. By Proposition III.1.7 and Theorem III.1.9, we can take $\lambda \in P_+$ and $b \in \mathscr{B}_w(\lambda)$ such that $D_{u_{w\lambda}, g^{\mathrm{up}}(b)} = D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}}$.

CLAIM 1. $D_{u_{w\lambda}, g^{\mathrm{up}}(\tilde{f}_{i_1}^{\max} b)} = D_{w_{\leq k}\varpi_{i_k}, s_{i_1}\varpi_{i_k}}$. *Here $\tilde{f}_{i_1}^{\max} b := \tilde{f}_{i_1}^{\varphi_{i_1}(b)} b = \tilde{f}_{i_1}^{\delta_{i_1, i_k}} b$.*

PROOF OF CLAIM 1. Let $\delta := \delta_{i_1, i_k}$. Since $u_{s_{i_1}\varpi_{i_k}} = f_{i_1}^{\delta}.u_{\varpi_{i_k}}$, we have

$$D_{u_{w\lambda}, f_{i_1}^{(p)}.g^{\mathrm{up}}(b)} = \begin{cases} D_{w_{\leq k}\varpi_{i_k}, s_{i_1}\varpi_{i_k}} \neq 0 & \text{if } p = \delta,\\ 0 & \text{if } p > \delta.\end{cases}$$

On the other hand, by Proposition I.1.43,

$$f_{i_1}^{(p)}.g^{\mathrm{up}}(b) = \begin{cases} g^{\mathrm{up}}(\tilde{f}_{i_1}^{\max}b) & \text{if } p = \varphi_{i_1}(b), \\ 0 & \text{if } p > \varphi_{i_1}(b), \end{cases}$$

and $\tilde{f}_{i_1}^{\max}b \in \mathscr{B}_w(\lambda)$ by Proposition I.1.52. Hence,

$$D_{u_{w\lambda}, f_{i_1}^{(p)}.g^{\mathrm{up}}(b)} = \begin{cases} D_{u_{w\lambda}, g^{\mathrm{up}}(\tilde{f}_{i_1}^{\max}b)} \neq 0 & \text{if } p = \varphi_{i_1}(b), \\ 0 & \text{if } p > \varphi_{i_1}(b). \end{cases}$$

Combining the above arguments, we obtain $\varphi_{i_1}(b) = \delta$ and $D_{w_{\leq k}\varpi_{i_k}, s_{i_1}\varpi_{i_k}} = D_{u_{w\lambda}, g^{\mathrm{up}}(\tilde{f}_{i_1}^{\max}b)}$. □

We write $b_2 := \tilde{e}_{i_1}^{\max}b$.

Claim 2. *We have*

$$D_{u_{w\lambda}, g^{\mathrm{up}}(b_2)} = q_{i_1}^{(X-1-2\langle h_{i_1}, w_{\leq k}\varpi_{i_k}\rangle)X/2} D_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}}^X D_{w_{\leq k}\varpi_{i_k}, \varpi_{i_k}},$$

where $X := -\langle h_{i_1}, w\lambda - w_{\leq k}\varpi_{i_k}\rangle$.

Proof of Claim 2. By [**44**, Corollary 3.1.8], for $p \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbf{U}_q^-$, we have

$$xe_{i_1}^{(p)} = \sum_{p'+p''+p'''=p} A(p', p'', p''') t_{i_1}^{-p'''} e_{i_1}^{(p'')} ({}_{i_1}e')^{p'} (e'_{i_1})^{p'''}(x) t_{i_1}^{p'},$$

where

$$A(p', p'', p''') := (-q_{i_1})^{p'''} q_{i_1}^{p'p''+p'p'''+p''p'''+p'^2} \frac{1}{(1-q_{i_1}^2)^{p'}[p']_{i_1}!} \frac{1}{(1-q_{i_1}^2)^{p'''}[p''']_{i_1}!}.$$

Therefore, for $x \in \mathbf{U}_q^-$, we have

$$\begin{aligned}
&(D_{u_{w\lambda}, e_{i_1}^{(p)}.g^{\mathrm{up}}(b)}, x)_L \\
&= (u_{w\lambda}, x e_{i_1}^{(p)}.g^{\mathrm{up}}(b))_\lambda^\varphi \\
&= \sum_{p'+p''+p'''=p} A(p',p'',p''')(u_{w\lambda}, t_{i_1}^{-p'''} e_{i_1}^{(p'')} ({}_{i_1}e')^{p'} (e'_{i_1})^{p'''}(x) t_{i_1}^{p'}.g^{\mathrm{up}}(b))_\lambda^\varphi \\
&= \sum_{p'+p''=p} A(p',0,p'')(u_{w\lambda}, t_{i_1}^{-p''} ({}_{i_1}e')^{p'} (e'_{i_1})^{p''}(x) t_{i_1}^{p'}.g^{\mathrm{up}}(b))_\lambda^\varphi \\
&= \sum_{p'+p''=p} A(p',0,p'') q_{i_1}^{p'\langle h_{i_1}, \mathrm{wt}\, b\rangle - p''\langle h_{i_1}, w\lambda\rangle} (u_{w\lambda}, ({}_{i_1}e')^{p'} (e'_{i_1})^{p''}(x).g^{\mathrm{up}}(b))_\lambda^\varphi \\
&= \sum_{p'+p''=p} A(p',0,p'') q_{i_1}^{p'\langle h_{i_1}, w\lambda + \varpi_{i_k} - w_{\le k}\varpi_{i_k}\rangle - p''\langle h_{i_1}, w\lambda\rangle} (D_{w_{\le k}\varpi_{i_k}, \varpi_{i_k}}, ({}_{i_1}e')^{p'} (e'_{i_1})^{p''}(x))_L \\
&= \sum_{p'+p''=p} A(p',0,p'') q_{i_1}^{p'\langle h_{i_1}, w\lambda + \varpi_{i_k} - w_{\le k}\varpi_{i_k}\rangle - p''\langle h_{i_1}, w\lambda\rangle} (D^{p''}_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}} D_{w_{\le k}\varpi_{i_k}, \varpi_{i_k}} D^{p'}_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}}, x)_L \\
&= \sum_{p'+p''=p} A(p',0,p'') q_{i_1}^{p'\langle h_{i_1}, w\lambda - 2w_{\le k}\varpi_{i_k}\rangle - p''\langle h_{i_1}, w\lambda\rangle} (D^{p}_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}} D_{w_{\le k}\varpi_{i_k}, \varpi_{i_k}}, x)_L.
\end{aligned}$$

Note that the last equality follows from Proposition III.1.27. Therefore we have

$$D_{u_{w\lambda}, e_{i_1}^{(p)}.g^{\mathrm{up}}(b)} = \sum_{p'+p''=p} A(p',0,p'') q_{i_1}^{p'\langle h_{i_1}, w\lambda - 2w_{\le k}\varpi_{i_k}\rangle - p''\langle h_{i_1}, w\lambda\rangle} D^{p}_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}} D_{w_{\le k}\varpi_{i_k}, \varpi_{i_k}}.$$

In particular, since $g^{\mathrm{up}}(b_2) = e_{i_1}^{(\varepsilon_{i_1}(b))} g^{\mathrm{up}}(b) = e_{i_1}^{(-\langle h_{i_1}, w\lambda - w_{\le k}\varpi_{i_k}\rangle)} g^{\mathrm{up}}(b)$ by Claim 1, we have

(III.18)
$$D_{u_{w\lambda}, g^{\mathrm{up}}(b_2)} = \frac{q_{i_1}^{-\langle h_{i_1}, w_{\le k}\varpi_{i_k}\rangle X}}{(1-q_{i_1}^2)^X} \left(\sum_{p'+p''=X} (-q_{i_1})^{p''} q_{i_1}^{p''X} \frac{1}{[p']_{i_1}![p'']_{i_1}!} \right) D^{X}_{s_{i_1}\varpi_{i_1}, \varpi_{i_1}} D_{w_{\le k}\varpi_{i_k}, \varpi_{i_k}}.$$

Recall that $X = -\langle h_{i_1}, w\lambda - w_{\le k}\varpi_{i_k}\rangle$. By the way, the following equality is well-known. See for instance [**44**, 1.3.1].

$$\sum_{t=0}^{a} q^{t(a-1)} \frac{[a]!}{[t]![a-t]!} z^t = \prod_{j=0}^{a-1} (1 + q^{2j} z)$$

for $a \in \mathbb{Z}_{\ge 0}$. Substituting q by q_{i_1}, a by X and z by $-q_{i_1}^2$, we have

$$\sum_{t=0}^{X} (-q_{i_1})^t q_{i_1}^{tX} \frac{[X]_{i_1}!}{[t]_{i_1}![X-t]_{i_1}!} = \prod_{j=1}^{X} (1 - q_{i_1}^{2j}).$$

Combining this equality with (III.18), we obtain

$$\begin{aligned}D_{u_{w\lambda},g^{\mathrm{up}}(b_2)} &= \frac{q_{i_1}^{-\langle h_{i_1},w_{\leq k}\varpi_{i_k}\rangle X}\prod_{j=1}^{X}(1-q_{i_1}^{2j})}{(1-q_{i_1}^2)^X[X]_{i_1}!}D^X_{s_{i_1}\varpi_{i_1},\varpi_{i_1}}D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}\\ &= q_{i_1}^{(X-1-2\langle h_{i_1},w_{\leq k}\varpi_{i_k}\rangle)X/2}D^X_{s_{i_1}\varpi_{i_1},\varpi_{i_1}}D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}.\end{aligned}$$

□

By Claim 2 and $(\Phi_{\boldsymbol{i}}\circ\eta_{w,q}^{-1})(D_{s_{i_1}\varpi_{i_1},\varpi_{i_1}}) = q_{i_1}t_1^{-1}$, we have

$$\begin{aligned}&(\Phi_{\boldsymbol{i}}\circ\eta_{w,q}^{-1})([D_{u_{w\lambda},g^{\mathrm{up}}(b_2)}])\\ &= q_{i_1}^{(c_1+\langle s_{i_1}h_{i_1},w_{\leq k}\varpi_{i_k}\rangle-1)X/2}(\Phi_{\boldsymbol{i}}\circ\eta_{w,q}^{-1})([D_{s_{i_1}\varpi_{i_1},\varpi_{i_1}}])^X(\Phi_{\boldsymbol{i}}\circ\eta_{w,q}^{-1})([D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}])\\ &= q_{i_1}^{(c_1+\langle s_{i_1}h_{i_1},w_{\leq k}\varpi_{i_k}\rangle+1)X/2}t_1^{-X}(\Phi_{\boldsymbol{i}}\circ\eta_{w,q}^{-1})([D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}]).\end{aligned}\tag{III.19}$$

Since our aim is to calculate $(\Phi_{\boldsymbol{i}}\circ\eta_{w,q}^{-1})([D_{w_{\leq k}\varpi_{i_k},\varpi_{i_k}}])$, we describe $(\Phi_{\boldsymbol{i}}\circ\eta_{w,q}^{-1})([D_{u_{w\lambda},g^{\mathrm{up}}(b_2)}])$ in a different way. Now we have

$$\begin{aligned}&\eta_{w,q}^{-1}([D_{u_{w\lambda},g^{\mathrm{up}}(b_2)}])\\ &= q^{(\lambda,\mathrm{wt}\,b_2-w\lambda)}[D_{w\lambda,\lambda}]^{-1}[D_{g^{\mathrm{up}}(b_2),u_\lambda}] = q^{(\lambda,\varpi_{i_k}-w_{\leq k}\varpi_{i_k}+X\alpha_{i_1})}[D_{w\lambda,\lambda}]^{-1}[D_{g^{\mathrm{up}}(b_2),u_\lambda}].\end{aligned}$$

Moreover,

$$\begin{aligned}&\Phi_{\boldsymbol{i}}([D_{g^{\mathrm{up}}(b_2),u_\lambda}])\\ &= \sum_{\boldsymbol{a}=(a_1,\dots,a_\ell)\in\mathbb{Z}_{\geq 0}^\ell} q_{\boldsymbol{i}}(\boldsymbol{a})(g^{\mathrm{up}}(b_2), f_{i_1}^{(a_1)}\cdots f_{i_\ell}^{(a_\ell)}.u_\lambda)^\varphi_\lambda t_1^{a_1}\cdots t_\ell^{a_\ell}\\ &= \sum_{(a_2,\dots,a_\ell)\in\mathbb{Z}_{\geq 0}^{\ell-1}} q_{\boldsymbol{i}}((0,a_2,\dots,a_\ell))(g^{\mathrm{up}}(b_2), f_{i_2}^{(a_2)}\cdots f_{i_\ell}^{(a_\ell)}.u_\lambda)^\varphi_\lambda t_2^{a_2}\cdots t_\ell^{a_\ell}.\end{aligned}\tag{III.20}$$

The last equality holds because $e_{i_1}.g^{\mathrm{up}}(b_2) = 0$. Here we prepare one more claim.

Claim 3. *Set* $\mu_2 := w_{2\leq}\lambda$. *Then* $D_{u_{\mu_2},g^{\mathrm{up}}(b_2)} = D_{s_{i_1}w_{\leq k}\varpi_{i_k},\varpi_{i_k}}$.

PROOF OF CLAIM 3. By Proposition I.1.29, I.1.30, I.1.31 and Claim 1, for $x \in \mathbf{U}_q^-$, we have

$$\begin{aligned}
&(D_{s_{i_1} w_{\le k}\varpi_{i_k},\varpi_{i_k}}, x)_L \\
&= (u_{s_{i_1} w_{\le k}\varpi_{i_k}}, x.u_{\varpi_{i_k}})^{\varphi}_{\varpi_{i_k}} \\
&= \begin{cases} (u_{s_{i_1} w_{\le k}\varpi_{i_k}}, x.u_{\varpi_{i_k}})^{\varphi}_{\varpi_{i_k}} & \text{if } x \in \mathbf{U}_q^- \cap T_{i_1}(\mathbf{U}_q^-) = \operatorname{Ker} e'_{i_1}, \\ 0 & \text{if } x \in f_{i_1}\mathbf{U}_q^- = (\operatorname{Ker} e'_{i_1})^{\perp}, \end{cases} \\
&= \begin{cases} (u_{w_{\le k}\varpi_{i_k}}, T_{i_1}^{-1}(x).u_{s_{i_1}\varpi_{i_k}})^{\varphi}_{\varpi_{i_k}} & \text{if } x \in \mathbf{U}_q^- \cap T_{i_1}(\mathbf{U}_q^-), \\ 0 & \text{if } x \in f_{i_1}\mathbf{U}_q^- = (\operatorname{Ker} e'_{i_1})^{\perp}, \end{cases} \\
&= \begin{cases} (u_{w\lambda}, T_{i_1}^{-1}(x).g^{\mathrm{up}}(\tilde{f}_{i_1}^{\max} b))^{\varphi}_{\lambda} & \text{if } x \in \mathbf{U}_q^- \cap T_{i_1}(\mathbf{U}_q^-), \\ 0 & \text{if } x \in f_{i_1}\mathbf{U}_q^- = (\operatorname{Ker} e'_{i_1})^{\perp}, \end{cases} \\
&= (u_{\mu_2}, x.g^{\mathrm{up}}(b_2))^{\varphi}_{\lambda} \\
&= (D_{u_{\mu_2}, g^{\mathrm{up}}(b_2)}, x)_L.
\end{aligned}$$

This completes the proof. □

Set $\boldsymbol{i}_{2\le} := (i_2, \dots, i_\ell)$ and identify $\mathcal{L}_{\boldsymbol{i}_{2\le}}$ with the subalgebra of $\mathcal{L}_{\boldsymbol{i}}$ generated by $t_2^{\pm 1}, \dots, t_\ell^{\pm 1}$. Write

$$C_2 := \prod_{j=2}^{k} q_{i_j}^{\langle s_{i_1} w_{\le j} h_{i_j}, s_{i_1} w_{\le k}\varpi_{i_k}\rangle (\langle s_{i_1} w_{\le j} h_{i_j}, s_{i_1} w_{\le k}\varpi_{i_k}\rangle + 1)/2} = \prod_{j=2}^{k} q_{i_j}^{d_j(d_j+1)/2}.$$

By our induction assumption, Proposition III.3.2 (3) and Claim 3, we have

$$\begin{aligned}
&C_2 t_2^{-\langle w_{\le 2} h_{i_2}, w_{\le k}\varpi_{i_k}\rangle} \cdots t_k^{-\langle w_{\le k} h_{i_k}, w_{\le k}\varpi_{i_k}\rangle} \left(=: C_2 \overrightarrow{\prod_{j=2,\dots,k}} t_j^{-\langle w_{\le j} h_{i_j}, w_{\le k}\varpi_{i_k}\rangle} \right) \\
&= (\Phi_{\boldsymbol{i}_{2\le}} \circ \eta^{-1}_{w_{2\le},q})([D_{s_{i_1} w_{\le k}\varpi_{i_k},\varpi_{i_k}}]) \\
&= (\Phi_{\boldsymbol{i}_{2\le}} \circ \eta^{-1}_{w_{2\le},q})([D_{u_{\mu_2}, g^{\mathrm{up}}(b_2)}]) \\
&= \Phi_{\boldsymbol{i}_{2\le}}(q^{(\lambda, \varpi_{i_k} - w_{\le k}\varpi_{i_k} + \langle h_{i_1}, w_{\le k}\varpi_{i_k}\rangle \alpha_{i_1})} [D_{w_{2\le}\lambda,\lambda}]^{-1} [D_{g^{\mathrm{up}}(b_2), u_\lambda}]) \\
&= q^{(\lambda, \varpi_{i_k} - w_{\le k}\varpi_{i_k} + \langle h_{i_1}, w_{\le k}\varpi_{i_k}\rangle \alpha_{i_1})} q_{\boldsymbol{i}_{2\le}}(\boldsymbol{c}')^{-1} t_\ell^{-c_\ell} \cdots t_2^{-c_2} \\
&\quad \times \sum_{\boldsymbol{a}' = (a_2, \dots, a_\ell) \in \mathbb{Z}_{\ge 0}^{\ell-1}} q_{\boldsymbol{i}_{2\le}}(\boldsymbol{a}') (g^{\mathrm{up}}(b_2), f_{i_2}^{(a_2)} \cdots f_{i_\ell}^{(a_\ell)}.u_\lambda)^{\varphi}_{\lambda} t_2^{a_2} \cdots t_\ell^{a_\ell},
\end{aligned}$$

where $\boldsymbol{c}' = (c_2, \dots, c_\ell)$ with $c_j := \langle h_{i_j}, w_{j+1\le}\lambda\rangle$. Therefore,

$$\begin{aligned}
&\sum_{\boldsymbol{a}' = (a_2, \dots, a_\ell) \in \mathbb{Z}_{\ge 0}^{\ell-1}} q_{\boldsymbol{i}_{2\le}}(\boldsymbol{a}') (g^{\mathrm{up}}(b_2), f_{i_2}^{(a_2)} \cdots f_{i_\ell}^{(a_\ell)}.u_\lambda)^{\varphi}_{\lambda} t_2^{a_2} \cdots t_\ell^{a_\ell} \\
&= C_2 q^{-(\lambda, \varpi_{i_k} - w_{\le k}\varpi_{i_k} + \langle h_{i_1}, w_{\le k}\varpi_{i_k}\rangle \alpha_{i_1})} q_{\boldsymbol{i}_{2\le}}(\boldsymbol{c}') t_2^{c_2} \cdots t_\ell^{c_\ell} \overrightarrow{\prod_{j=2,\dots,k}} t_j^{-\langle w_{\le j} h_{i_j}, w_{\le k}\varpi_{i_k}\rangle}.
\end{aligned} \tag{III.21}$$

Combining (III.20) and (III.21), we obtain the following equality ($\boldsymbol{c} = (c_1, \cdots c_\ell), c_1 := \langle h_{i_1}, w_{2\le}\lambda\rangle$):

$$\begin{aligned}(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})([D_{u_{w\lambda}, g^{\mathrm{up}}(b_2)}]) &= q^{(\lambda, \varpi_{i_k} - w_{\le k}\varpi_{i_k} + X\alpha_{i_1})}\Phi_{\boldsymbol{i}}([D_{w\lambda,\lambda}]^{-1}[D_{g^{\mathrm{up}}(b_2), u_\lambda}]) \\ &= C_2 q_{i_1}^{-\langle h_{i_1},\lambda\rangle\langle h_{i_1}, w\lambda\rangle} q_{\boldsymbol{i}}(\boldsymbol{c})^{-1} q_{\boldsymbol{i}_{2\le}}(\boldsymbol{c}') t_\ell^{-c_\ell}\dots t_1^{-c_1} t_2^{c_2}\cdots t_\ell^{c_\ell} \overrightarrow{\prod_{j=2,\dots,k}} t_j^{-\langle w_{\le j}h_{i_j}, w_{\le k}\varpi_{i_k}\rangle} \\ &= C_2 q_{i_1}^{-\langle h_{i_1},\lambda\rangle\langle h_{i_1}, w\lambda\rangle - c_1(c_1-1)/2 - \sum_{j=2}^{\ell}\langle h_{i_1}, c_j\alpha_{i_j}\rangle} t_1^{-c_1} \overrightarrow{\prod_{j=2,\dots,k}} t_j^{-\langle w_{\le j}h_{i_j}, w_{\le k}\varpi_{i_k}\rangle} \\ &= C_2 q_{i_1}^{c_1(c_1+1)/2} t_1^{-c_1} \overrightarrow{\prod_{j=2,\dots,k}} t_j^{-\langle w_{\le j}h_{i_j}, w_{\le k}\varpi_{i_k}\rangle}.\end{aligned} \tag{III.22}$$

Recall that $X = -\langle h_{i_1}, w\lambda - w_{\le k}\varpi_{i_k}\rangle = c_1 - \langle s_{i_1}h_{i_1}, w_{\le k}\varpi_{i_k}\rangle$. By (III.22) and (III.19), we obtain

$$\begin{aligned}&(\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})([D_{w_{\le k}\varpi_{i_k}, \varpi_{i_k}}]) \\ &= C_2 q_{i_1}^{-(c_1 + \langle s_{i_1}h_{i_1}, w_{\le k}\varpi_{i_k}\rangle + 1)(c_1 - \langle s_{i_1}h_{i_1}, w_{\le k}\varpi_{i_k}\rangle)/2 + c_1(c_1+1)/2} t_1^{-\langle s_{i_1}h_{i_1}, w_{\le k}\varpi_{i_k}\rangle} \overrightarrow{\prod_{j=2,\dots,k}} t_j^{-\langle w_{\le j}h_{i_j}, w_{\le k}\varpi_{i_k}\rangle} \\ &= C_2 q_{i_1}^{\langle s_{i_1}h_{i_1}, w_{\le k}\varpi_{i_k}\rangle(\langle s_{i_1}h_{i_1}, w_{\le k}\varpi_{i_k}\rangle + 1)/2} t_1^{-\langle s_{i_1}h_{i_1}, w_{\le k}\varpi_{i_k}\rangle} \overrightarrow{\prod_{j=2,\dots,k}} t_j^{-\langle w_{\le j}h_{i_j}, w_{\le k}\varpi_{i_k}\rangle}.\end{aligned}$$

This completes the proof. □

The following is a direct corollary of Theorem III.3.6. These equalities are exact quantum analogues of the Chamber Ansatz formulae for unipotent cells [**4**, Theorem 1.4], [**6**, Theorem 1.4]. See also the proof of [**6**, Theorem 4.3].

COROLLARY III.3.9. *Let $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$. For $j = 1, \dots, \ell$, set*

$$D'^{(\boldsymbol{i})}_{w_{\le j}\varpi_{i_j}, \varpi_{i_j}} := (\Phi_{\boldsymbol{i}} \circ \eta_{w,q}^{-1})([D_{w_{\le j}\varpi_{i_j}, \varpi_{i_j}}]).$$

By Theorem III.3.6, these elements are Laurent monomials in $\mathcal{L}_{\boldsymbol{i}}$. Then, for $k = 1, \dots, \ell$,

$$t_k \simeq (D'^{(\boldsymbol{i})}_{w_{\le k-1}\varpi_{i_k}, \varpi_{i_k}})^{-1} (D'^{(\boldsymbol{i})}_{w_{\le k}\varpi_{i_k}, \varpi_{i_k}})^{-1} \prod_{j \in I\setminus\{i_k\}} (D'^{(\boldsymbol{i})}_{w_{\le k}\varpi_j, \varpi_j})^{-a_{j,i_k}},$$

here $\simeq$ means the coincidence up to some powers of q. Note that the right-hand side is determined up to powers of q.

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