A cohomological study of the existence problem of compact Clifford-Klein forms

その他のタイトル	コンパクトClifford-Klein形の存在問題のコホモロジー的研究
学位授与年月日	2017-03-23
URL	http://doi.org/10.15083/00076099

博士論文

論文題目　A cohomological study of the existence problem of compact Clifford–Klein forms
(コンパクト Clifford–Klein 形の存在問題のコホモロジー的研究)

氏　　名　森田 陽介

A cohomological study of the existence problem of compact Clifford–Klein forms

Yosuke Morita

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my supervisor Toshiyuki Kobayashi. It was invaluable to me, especially when I felt stuck, that he always trusted and encouraged me. He always listened patiently my ideas—often vague and naive—and gave me insightful advices. My way of doing mathematics is without doubt deeply influenced by him, and I hope I continue learning from him. In retrospect, it was not until I gave talks in his undergraduate seminar that I realized, in the true sense, how important (and difficult!) it is to *really understand* theorems, definitions or proofs.

It is my pleasure to express my sincere thanks to the professors of the Graduate School of Mathematical Sciences of the University of Tokyo, especially Mikio Furuta, Masahiko Kanai, Hideko Sekiguchi and Taro Yoshino, for their lectures, advices and supports. I am also happy to thank colleagues in the University of Tokyo, notably the participants of Professor Kobayashi's graduate seminar and the members (and the regular guests) of Room 457. I have benefitted in particular from conservations about Clifford–Klein forms and related subjects with Takayuki Okuda, Yoshiki Oshima, Yuichiro Tanaka, Severin Barmeier, Masatoshi Kitagawa, Yosuke Kubota, Koichi Tojo, Yoshinori Tanimura and Kazuki Kannaka.

My special thanks go to Fanny Kassel and Nicolas Tholozan. Discussions with them were very inspiring to me. I would like to express my gratitude to Hee Oh for her generous hospitality and mathematical advices during my stay at Yale University in September–October 2016. I would like to thank also to Ilia Smilga for helpful discussion during that stay. I am grateful to Shin Nayatani and Koji Fujiwara for their interest in my work and for their support.

Finally, I wish to thank my mother Naoko and my father Hirofumi; I could not have done anything without their support.

This work was supported by JSPS KAKENHI Grant Number 14J08233 and the Program for Leading Graduate Schools, MEXT, Japan.

Contents

Chapter 1

Introduction

1.1 The existence problems of compact Clifford–Klein forms and compact manifolds locally modelled on homogeneous spaces

Let G be a Lie group and H its closed subgroup. A manifold is said to be *locally modelled on the homogeneous space* G/H if it has an atlas with values in G/H whose transition functions are given by the left translations by elements of G. One standard way to construct a manifold locally modelled on G/H is to take a quotient $\Gamma\backslash G/H$ of the homogeneous space G/H by the action of a *discontinuous group* Γ. Here, a discrete subgroup Γ of G is called a discontinuous group for G/H if the the projection $G/H \to \Gamma\backslash G/H$ forms a principal (flat) Γ-bundle, or equivalently, the Γ-action on G/H is proper (= properly discontinuous) and free. We then call $\Gamma\backslash G/H$ a *Clifford–Klein form*. Various geometric structures are described in terms of local models and Clifford–Klein forms.

Example 1.1.1. *A pseudo-Riemannian manifold of signature* (p, q) *(with* $p \geqslant 2$ *for simplicity) with constant positive sectional curvature is nothing but a manifold locally modelled on a homogeneous space* $\mathrm{O}(p+1, q)/\mathrm{O}(p, q)$*. The geodesically complete ones exactly correspond to the compact Clifford–Klein forms of* $\mathrm{O}(p+1, q)/\mathrm{O}(p, q)$*.*

Suppose that the G-action on G/H is effective and preserves some Riemannian metric, or slightly more generally, that H is compact. Then, every discrete subgroup Γ of G acts properly on G/H, and therefore the quotient $\Gamma\backslash G/H$ always carries a natural orbifold (or Satake's V-manifold) structure. If, in addition, G is linear and Γ is finitely generated, there always exists a finite-index subgroup of Γ that acts freely on G/H by Selberg's lemma [52, Lem. 8]. Thus, the study of Clifford–Klein forms of G/H is almost equivalent to that of discrete subgroups of G. In contrast, if H is noncompact, a discrete subgroup of G does not necessarily act properly on G/H, and the study of Clifford–Klein forms is much more difficult. Systematic study in this general setting was initiated by T. Kobayashi in the late 1980s ([20], [22], [23]). In particular, the following problem has attracted considerable attention:

Problem 1.1.2. *Given a homogeneous space G/H, determine if there is a compact Clifford–Klein form of G/H. More generally, determine if there is a compact manifold locally modelled on G/H.*

Remark 1.1.3. When G/H is a homogeneous space of reductive type, it is conjectured that every compact manifold locally modelled on G/H is in fact a Clifford–Klein form.

Problem 1.1.2 has been studied by various methods derived from diverse fields in mathematics (see excellent surveys [26], [27] by Kobayashi himself on this problem and also surveys [28], [29], [31], [34], [38]). Geometrically, one of the most interesting cases is when G/H is a semisimple symmetric space. In that case, there are two available approaches:

(I) A criterion for properness in terms of Cartan projection of G ([20], [23], [2]).

(II) Comparison of relative Lie algebra cohomology and de Rham cohomology ([30], [20], [3]).

Note that a dynamical approach ([61], [13], [35], [36]) and an approach from unitary representation theory ([37], [53]) are not applicable to the case of semisimple symmetric spaces, whereas they give some results in the nonsymmetric case (cf. [27, §4.8]). Although these various approaches have been developed in last three decades, Problem 1.1.2 is still unsolved for many homogeneous spaces, including many semisimple symmetric spaces. For instance, it is not known whether there exists a compact Clifford–Klein form of $\mathrm{O}(4,3)/\mathrm{O}(4,2)$ (or a compact manifold locally modelled on $\mathrm{O}(4,3)/\mathrm{O}(4,2)$).

In this thesis, we study Problem 1.1.2 from the approach (II). This method is based on the following elementary observation: if M is a manifold locally modelled on a homogeneous space G/H (with H connected for simplicity), we can define a differential graded algebra homomorphism

$$\eta : (\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \simeq \Omega(G/H)^G \to \Omega(M)$$

by patching G-invariant differential forms on open sets of G/H by left translations, and therefore we have a homomorphism

$$\eta : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(M; \mathbb{R})$$

from relative Lie algebra cohomology to de Rham cohomology. The relative Lie algebra cohomology $H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$ is determined by the model homogeneous space G/H and does not depend on the topology of M. By contrast, the de Rham cohomology $H^\bullet(M;\mathbb{R})$ is determined by the topology of M and does not depend on its locally homogeneous geometric structure. We can extract some topological information of M from the homomorphism η that relate these two cohomologies. T. Kobayashi and K. Ono [30], [20] were the first to apply this homomorphism η to the study of Problem 1.1.2. They compared the Euler characteristic in relative Lie algebra cohomology with the one in de Rham cohomology, and deduced necessary conditions for the existence of compact Clifford–Klein forms. F. Labourie pointed out that their result [30, Cor. 5] is also valid for a manifold locally modelled on homogeneous spaces ([26, Notes 3.13]). Y. Benoist and F. Labourie [3] obtained another necessary condition by investigating the invariant symplectic forms on homogeneous spaces.

1.2 Main results of this thesis

In the first half of Chapter 2, we prove the following theorem:

Theorem 1.2.1 (Theorem 2.1.2)**.** *Let G be a Lie group and H its closed subgroup with finitely many connected components. Put $N = \dim G - \dim H$.*

(1) *If $(\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \neq 0$ and $H^N(\mathfrak{g},\mathfrak{h};\mathbb{R}) = 0$, then there is no compact manifold locally modelled on G/H.*

(2) *Take a maximal compact subgroup K_H of H. Let*

$$i : H^N(\mathfrak{g},\mathfrak{h};\mathbb{R}) \to H^N(\mathfrak{g},\mathfrak{k}_H;\mathbb{R})$$

be the homomorphism induced by the inclusion $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \hookrightarrow (\Lambda(\mathfrak{g}/\mathfrak{k}_H)^*)^{\mathfrak{k}_H}$. If i is not injective, there is no compact manifold locally modelled on G/H.

We recover the necessary conditions of Kobayashi–Ono and Benoist–Labourie from Theorem 1.2.1 (see Proposition 2.6.1 and Remark 3.1.4).

From the second half of Chapter 2 to Chapter 4, we give various examples of homogeneous spaces to which Theorem 1.2.1 is applicable. First, in the second half of Chapter 2, we consider the case of homogeneous spaces of nonreductive type. We obtain, for instance, the following:

Example 1.2.2 (see Example 2.7.6 and Remark 2.7.5)**.** *Let G be a real linear solvable algebraic group and $F \in \mathfrak{g}^*$. Then, there does not exist a compact manifold locally modelled on the coadjoint orbit $G/\operatorname{Stab}(F)$ unless $G/\operatorname{Stab}(F)$ is zero-dimensional.*

In Chapters 3–4, we study the case of homogeneous spaces of reductive type. In Chapter 3, we give a necessary and sufficient condition for the applicability of Theorem 1.2.1 (2):

Theorem 1.2.3 (Theorem 3.1.3)**.** *Let G/H be a homogeneous space of reductive type. We write $P_{\mathfrak{g}^*}$ and $P_{\mathfrak{h}^*}$ for the space of primitive elements in $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ and $(\Lambda\mathfrak{h}^*)^{\mathfrak{h}}$, respectively. Then, the homomorphism $i : H^\bullet(\mathfrak{g},\mathfrak{h};\mathbb{R}) \to H^\bullet(\mathfrak{g},\mathfrak{k}_H;\mathbb{R})$ is not injective if and only if the linear map* $\operatorname{rest} : (P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ *induced from the restriction map $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}} \to (\Lambda\mathfrak{h}^*)^{\mathfrak{h}}$ is not surjective, where $(\,\cdot\,)^{-\theta}$ denotes the (-1)-eigenspace for θ.*

Remark 1.2.4. Theorem 1.2.1 (1) is not applicable to any homogeneous space of reductive type.

The proof of Theorem 1.2.3 uses an isomorphism constructed by H. Cartan, C. Chevalley, J.-L. Koszul and A. Weil [11] between relative Lie algebra cohomology $H^\bullet(\mathfrak{g},\mathfrak{h};\mathbb{R})$ and the cohomology $H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}})$ of a pure Sullivan algebra defined from a transgression in the Weil algebra.

As a direct application of Theorems 1.2.1 and 1.2.3, we prove the following conjecture proposed by T. Kobayashi in 1989:

Conjecture 1.2.5 ([21, Conj. 6.4])**.** *A homogeneous space G/H of reductive type does not admit a compact Clifford–Klein form if* $\operatorname{rank} G - \operatorname{rank} K < \operatorname{rank} H - \operatorname{rank} K_H$.

Remark 1.2.6. We show this conjecture not only for Clifford–Klein forms but, more generally, for manifolds locally modelled on homogeneous spaces. However, we do not know whether this is an essential generalization (see Remark 1.1.3).

In Chapter 4, we classify the semisimple symmetric spaces to which Theorem 1.2.1 (2) is applicable:

Theorem 1.2.7 (Theorem 4.1.2)**.** *Let $(\mathfrak{g},\mathfrak{h})$ be a semisimple symmetric pair. Then the following two conditions are equivalent:*

(A) *The homomorphism $i : H^\bullet(\mathfrak{g},\mathfrak{h};\mathbb{R}) \to H^\bullet(\mathfrak{g},\mathfrak{k}_H;\mathbb{R})$ induced from the inclusion map $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \hookrightarrow (\Lambda(\mathfrak{g}/\mathfrak{k}_H)^*)^{\mathfrak{k}_H}$ is injective.*

(B) *The pair $(\mathfrak{g},\mathfrak{h})$ is isomorphic (up to possibly outer automorphisms) to a direct sum of the following irreducible symmetric pairs* (B-1)–(B-5)*.*

- (B-1) $(\mathfrak{l},\mathfrak{l})$ *($\mathfrak{l}$: simple Lie algebra).*
- (B-2) $(\mathfrak{l}\oplus\mathfrak{l},\Delta\mathfrak{l})$ *($\mathfrak{l}$: simple Lie algebra).*
- (B-3) $(\mathfrak{l}_{\mathbb{C}},\mathfrak{l})$ *($\mathfrak{l}_{\mathbb{C}}$: complex simple Lie algebra, $\mathfrak{l}$: real form of $\mathfrak{l}_{\mathbb{C}}$).*
- (B-4) *A pair $(\mathfrak{g}',\mathfrak{h}')$ such that* $\operatorname{rank}\mathfrak{h}' = \operatorname{rank}\mathfrak{k}_{H'}$, *where $\mathfrak{k}_{H'}$ is a maximal compact subalgebra of $\mathfrak{h}'$.*
- (B-5)
  - $(\mathfrak{sl}(2n+1,\mathbb{C}),\mathfrak{so}(2n+1,\mathbb{C}))$ $(n \geqslant 1)$,
  - $(\mathfrak{sl}(2n,\mathbb{C}),\mathfrak{sp}(n,\mathbb{C}))$ $(n \geqslant 2)$,
  - $(\mathfrak{so}(2n,\mathbb{C}),\mathfrak{so}(2n-1,\mathbb{C}))$ $(n \geqslant 4)$,

- $(\mathfrak{e}_{6,\mathbb{C}}, \mathfrak{f}_{4,\mathbb{C}})$.

The proof of Theorem 1.2.7 uses Theorem 1.2.3 and Berger's classification of the irreducible symmetric pairs [4].

From this classification, we obtain some new examples of semisimple symmetric spaces that do not admit compact Clifford–Klein forms:

Example 1.2.8. *There does not exist a compact manifold locally modelled on* $\mathrm{SL}(p+q,\mathbb{R})/\mathrm{SO}_0(p,q)$ *($p, q \geqslant 1$, p, q: odd).*

See Table 4.1 for other examples and Remark 1.2.13 for comparison with previous results. We give some nonsymmetric and nonlinear examples in Chapter 4 too.

Finally, in Chapter 5, we prove the following refinement of Theorem 1.2.1 (2):

Theorem 1.2.9 (Theorem 5.1.1)**.** *Let G be a connected linear Lie group and H its connected closed subgroup. Assume that $H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \neq 0$ ($N = \dim G - \dim H$). Let K_H be a maximal compact subgroup of H and T_H a maximal torus of K_H. Let $I^\bullet = \bigoplus_{n\in\mathbb{N}} I^n$ be the graded ideal of $H^\bullet(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})$ generated by*

$$\bigoplus_{C,p} \mathrm{im}(i : H^p(\mathfrak{g}, \mathfrak{c}; \mathbb{R}) \to H^p(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})),$$

where the direct sum runs all connected compact subgroups C of G containing T_H and all $p > N + \dim K_H - \dim C$. If

$$\mathrm{im}(i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})) \subset I^N$$

holds, G/H does not admit a compact Clifford–Klein form.

Remark 1.2.10. We do not know whether Theorem 5.1.1 applies to the case of manifolds locally modelled on G/H.

The key to the proof of Theorem 1.2.9 is to combine the homomorphism η from relative Lie algebra cohomology to de Rham cohomology with the upper-bound estimate for cohomological dimensions of discontinuous groups.

We give a systematic way to find semisimple symmetric spaces G/H that apply Theorem 1.2.9 using the theorem of Cartan–Chevalley–Koszul–Weil [11] and ε-families of semisimple symmetric pairs, introduced by T. Oshima and J. Sekiguchi [50] (see Propositions 5.4.1 and 5.5.3).

Example 1.2.11. *If $p, q \geqslant 1$ and q is odd, then $\mathrm{O}(p+1,q)/\mathrm{O}(p,q)$ does not admit a compact Clifford–Klein form. In other words, every complete pseudo-Riemannian manifold of signature (p,q) with positive constant sectional curvature is noncompact if $p, q \geqslant 1$ and q is odd.*

Example 1.2.12. *If* $p, q \geqslant 1$*, then* $\mathrm{SL}(p+q,\mathbb{R})/\mathrm{SO}_0(p,q)$*,* $\mathrm{SL}(p+q,\mathbb{C})/\mathrm{SU}(p,q)$ *and* $\mathrm{SL}(p+q,\mathbb{H})/\mathrm{Sp}(p,q)$ *do not admit compact Clifford–Klein forms.*

See Subsection 5.1.3 for other examples.

Remark 1.2.13. We mention some results on the existence problem of compact Clifford–Klein forms related to the above examples:

- Example 1.2.11 generalizes results of [33] (for $p, q \geqslant 1$, p, q: odd) and [2] (for $(p, q) = (2n, 2n+1)$). It is also known that compact Clifford–Klein forms do not exist for $p \geqslant q \geqslant 1$ (Calabi–Markus phenomenon [10], [59], [60], [20]). On the other hand, compact Clifford–Klein forms exist for $(p, q) = (1, 2n)$, $(3, 4n)$ and $(7, 8)$ ([33], [26]).

- Example 1.2.12 generalizes Example 1.2.8 (for p, q: odd, over $\mathbb{R}$). It also generalizes results of [23] (for $p = q$, over $\mathbb{R}$ or $\mathbb{C}$) and [2] (for $p = q$ or $q + 1$, over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$).

Some results in this thesis, including the proof of Conjecture 1.2.5, Examples 1.2.11 and 1.2.12 are also obtained by N. Tholozan [56] at about the same time (see Remarks 3.1.5 and 5.1.7).

The results of Chapter 2 are contained in [43]. We mention that Theorem 1.2.1 (2) was first proved in [41] for the case of Clifford–Klein forms under the assumption that G/H is of reductive type. The results of Chapter 4 were announced in [42]. Many of the examples in Chapter 4 were first found in [41]. The results of Chapter 5 are contained in [44].

Chapter 2

Homogeneous spaces of nonreductive type locally modelling no compact manifold

2.1 Introduction

Let G/H be a homogeneous space. A manifold is called locally modelled on G/H if it is covered by open sets that are diffeomorphic to open sets of G/H and their coordinate changes are given by left translations by elements of G. A typical example is a double coset space $\Gamma\backslash G/H$, where Γ is a discrete subgroup of G acting properly and freely on G/H. In this case Γ is called a discontinuous group for G/H and $\Gamma\backslash G/H$ is called a Clifford–Klein form. A manifold locally modelled on a homogeneous space is a fundamental object of the study of "geometry" in the sense of Klein's Erlangen program. Thus, one of the central questions in geometry is to understand topological features of manifolds locally modelled on a given homogeneous space.

We study the following problem proposed by T. Kobayashi:

Problem 2.1.1 ([20])**.** *When does a homogeneous space model some compact manifold? When does a homogeneous space admit a compact Clifford–Klein form?*

Various methods have been applied to study this problem (See surveys [26], [31], [34] and references therein). One is a cohomological method, that is, to investigate "locally invariant" differential forms on a manifold locally modelled on a homogeneous space and their cohomology classes. This method was initiated by Kobayashi–Ono [30] and then extended by Kobayashi [20] and Benoist–Labourie [3].

In this chapter, we give a new cohomological obstruction to the existence of compact manifolds locally modelled on homogeneous spaces and find it useful even when G is not reductive. Note that, for a nonreductive Lie group G, less is known about Problem 2.1.1 in particular because we cannot use the properness criterion of Benoist [2] and Kobayashi [25] anymore.

We use lowercase German letters for the Lie algebras of Lie groups denoted by uppercase Roman letters. For example, the Lie algebras of G, K_H and $\mathrm{Stab}(X)$ are $\mathfrak{g}$, $\mathfrak{k}_H$ and $\mathfrak{stab}(X)$, respectively. Then, our main result is stated as follows:

Theorem 2.1.2. *Let G be a Lie group and H its closed subgroup with finitely many connected components. Put $N = \dim G - \dim H$.*

(1) *If $(\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \neq 0$ and $H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) = 0$, then there is no compact manifold locally modelled on G/H.*

(2) *Take a maximal compact subgroup K_H of H. Let*

$$i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$$

be the homomorphism induced by the inclusion $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \hookrightarrow (\Lambda(\mathfrak{g}/\mathfrak{k}_H)^*)^{\mathfrak{k}_H}$. If i is not injective, there is no compact manifold locally modelled on G/H.

Some applications of this theorem to homogeneous spaces of nonreductive type are given in Sections 2.7–2.8. The case of homogeneous spaces of reductive type is studied in Chapters 3–4.

The idea of Theorem 2.1.2 (1) is already implicit in [3]. We shall give its proof for the sake of completeness. Theorem 2.1.2 (2) is first proved in [41] under the assumption that G/H is a homogeneous space of reductive type. The key to the proof is to combine the homomorphism $\eta : H^p(\mathfrak{g}, H; \mathbb{R}) \to H^p(M; \mathbb{R})$ (see Section 2.2) with an observation that a fibre bundle with contractible fibre induces an isomorphism between the cohomologies of the total space and the base space. We here generalize it to the nonreductive case by separating the Poincaré duaity argument from the other parts of the proof (cf. Corollary 2.5.1).

Theorem 2.1.2 generalizes some earlier results in [30], [20] and [3] (see Section 2.6 and Remark 3.1.4).

2.2 Preliminaries

In this section, we review the definition of the homomorphism $\eta : H^p(\mathfrak{g}, H; \mathbb{R}) \to H^p(M; \mathbb{R})$, which plays a foundational role in the cohomological study of Problem 2.1.1.

Let X be a real analytic manifold with an action of a Lie group G. Recall that a (G, X)-structure on a manifold M is a collection of $(U_i)_{i \in I}$, $(\phi_i)_{i \in I}$, $(g_{ij})_{i,j \in I}$, where $(U_i)_{i \in I}$ is an open covering of M, ϕ_i is a diffeomorphism from U_i to some open set of X, and $g_{ij} : U_i \cap U_j \to G$ is a locally constant map satisfying

$$g_{ij}(p)\phi_j(p) = \phi_i(p) \quad (p \in U_i \cap U_j).$$

We assume the cocycle condition for the transition functions $(g_{ij})_{i,j\in I}$:

$$g_{ii}(p) = 1 \quad (p \in U_i), \qquad g_{ij}(p)g_{jk}(p)g_{ki}(p) = 1 \quad (p \in U_i \cap U_j \cap U_k).$$

It is automatically satisfied if X is connected and G acts on X effectively. We mainly consider the case when G acts transitively on X, namely, $X = G/H$ for some closed subgroup H of G. A manifold equipped with a $(G, G/H)$-structure is also called a manifold locally modelled on G/H.

Let M be a manifold equipped with a (G, X)-structure $(U_i)_{i\in I}$, $(\phi_i)_{i\in I}$, $(g_{ij})_{i,j\in I}$. Let $\pi : E \to X$ be a G-equivariant fibre bundle on X with typical fibre F. Patching $(\phi_i^* E)_{i\in I}$ by $(g_{ij})_{i,j\in I}$, we get a fibre bundle $\pi_M : E_M \to M$ with the same typical fibre F. We call it the locally G-equivariant bundle over M corresponding to E. By definition E_M naturally equips a (G, E)-structure. We can define

$$\eta : \Gamma(X; E)^G \to \Gamma(M; E_M)$$

also by patching construction. In particular, if $X = G/H$ and $E = \Lambda^p T^* X$, this is written as

$$\eta : (\Lambda^p(\mathfrak{g}/\mathfrak{h})^*)^H \to \Omega^p(M).$$

Here, we naturally identified $\Omega^p(G/H)^G$ with $(\Lambda^p(\mathfrak{g}/\mathfrak{h})^*)^H$. Taking cohomology, we get a homomorphism

$$\eta : H^p(\mathfrak{g}, H; \mathbb{R}) \to H^p(M; \mathbb{R})$$

(see e.g. [15, §1.3], [40, §2.2] for the definition of relative Lie algebra cohomology $H^p(\mathfrak{g}, H; \mathbb{R})$). Such a homomorphism η appears explicitly or implicitly in many branches of geometry and representation theory, e.g. the Matsushima–Murakami formula [39], characteristic classes of foliations [8], a generalization of Hirzebruch's proportionality principle [30] and the existence problem of a compact manifold locally modelled on homogeneous spaces [30], [20], [3].

2.3 Proof of Theorem 2.1.2

Lemma 2.3.1. *Let G be a Lie group and H its closed subgroup with finitely many connected components. We write H_0 for the identity component of H. If there is no compact manifold locally modelled on G/H_0, neither is on G/H.*

Proof of Lemma 2.3.1. This is well-known at least for Clifford–Klein forms. Suppose there is a compact manifold M locally modelled on G/H. Consider the locally G-equivariant fibre bundle $\pi_M : M_0 \to M$ corresponding to $\pi : G/H_0 \to G/H$. Then the total space M_0 is locally modelled on G/H_0 and compact. □

Thus we may assume H to be connected without loss of generality. Now, it is enough to see:

Proposition 2.3.2. *Let G be a Lie group and H its closed subgroup. Put $N = \dim G - \dim H$.*

(1) *If $(\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^H \neq 0$ and $H^N(\mathfrak{g}, H; \mathbb{R}) = 0$, then there is no compact manifold locally modelled on G/H.*

(2) *Suppose that H has finitely many connected components. Take a maximal compact subgroup K_H of H. If the homomorphism*

$$i : H^N(\mathfrak{g}, H; \mathbb{R}) \to H^N(\mathfrak{g}, K_H; \mathbb{R})$$

is not injective, then there is no compact manifold locally modelled on G/H.

Remark 2.3.3. Proposition 2.3.2 (1) holds true even if H has infinitely many connected components.

Proof of Proposition 2.3.2. (1). Suppose, on the contrary, that there is a compact manifold M locally modelled on G/H. Take a nonzero element Φ of $(\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^H$; it is identified with a G-invariant volume form on G/H. Hence $\eta(\Phi) \in \Omega^N(M)$ is a volume form on M by construction of η, and $[\eta(\Phi)] \neq 0$ in $H^N(M; \mathbb{R})$ by compactness of M. On the other hand, $[\Phi] = 0$ in $H^N(\mathfrak{g}, H; \mathbb{R})$ by assumption, and $[\eta(\Phi)] = 0$ in $H^N(M; \mathbb{R})$. This is contradiction.

(2). Let M be a compact manifold locally modelled on G/H. Let $\pi_M : E_M \to M$ be the locally G-equivariant fibre bundle on M corresponding to $\pi : G/K_H \to G/H$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^N(\mathfrak{g}, H; \mathbb{R}) & \xrightarrow{i} & H^N(\mathfrak{g}, K_H; \mathbb{R}) \\ {\scriptstyle\eta}\downarrow & & {\scriptstyle\eta}\downarrow \\ H^N(M; \mathbb{R}) & \xrightarrow{\pi_M^*} & H^N(E_M; \mathbb{R}). \end{array}$$

We saw in the proof of (1) that the homomorphism $\eta : H^N(\mathfrak{g}, H; \mathbb{R}) \to H^N(M; \mathbb{R})$ is injective. The typical fibre H/K_H of the fibre bundle $\pi_M : E_M \to M$ is contractible by the Cartan–Malcev–Iwasawa–Mostow theorem (cf. [18, Ch. XV, Th. 3.1]), thus $\pi_M^* : H^N(M; \mathbb{R}) \to H^N(E_M; \mathbb{R})$ is an isomorphism. These yield the injectivity of $i : H^N(\mathfrak{g}, H; \mathbb{R}) \to H^N(\mathfrak{g}, K_H; \mathbb{R})$. □

2.4 Equivalent form of Theorem 2.1.2 (1)

It is sometimes useful to rewrite Theorem 2.1.2 (1) as follows:

Proposition 2.4.1. *Let G be a Lie group and H its closed subgroup with finitely many connected components. Let $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ denote the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. If the $\mathfrak{h}$-action on $\mathfrak{g}/\mathfrak{h}$ is trace-free (i.e.* $\operatorname{tr}(\operatorname{ad}_{\mathfrak{g}/\mathfrak{h}}(X)) = 0$ *for all $X \in \mathfrak{h}$) and the $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$-action on $\mathfrak{g}/\mathfrak{h}$ is not trace-free, then there is no compact manifold locally modelled on G/H.*

Proof. This is a direct consequence of Theorem 2.1.2 (1) and the lemma below. □

Lemma 2.4.2. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ its subalgebra. Put $N = \dim \mathfrak{g} - \dim \mathfrak{h}$.*

(1) *The $\mathfrak{h}$-action on $\mathfrak{g}/\mathfrak{h}$ is trace-free if and only if $(\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \neq 0$.*

(2) *The $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$-action on $\mathfrak{g}/\mathfrak{h}$ is trace-free if and only if $H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \neq 0$.*

Proof. (1). This follows immediately from the definition of the $\mathfrak{h}$-action on $\Lambda^N(\mathfrak{g}/\mathfrak{h})^*$.

(2). Let ι denote the interior product and $\mathcal{L}$ the $\mathfrak{g}$-action on $\Lambda\mathfrak{g}^*$. Assume that $(\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \neq 0$ and fix a nonzero element Φ of $(\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$. We wish to determine when

$$d : (\Lambda^{N-1}(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \to (\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$$

is a zero map. Every element of $\Lambda^{N-1}(\mathfrak{g}/\mathfrak{h})^*$ is written in the form $\iota(Y)\Phi$ ($Y \in \mathfrak{g}$), and the choice of such Y is unique up to $\mathfrak{h}$. For $X \in \mathfrak{h}$,

$$\mathcal{L}(X)\iota(Y)\Phi = \iota(Y)\mathcal{L}(X)\Phi - \iota([X,Y])\Phi = \iota([X,Y])\Phi.$$

It is equal to zero if and only if $[X,Y] \in \mathfrak{h}$. Thus $\iota(Y)\Phi$ is $\mathfrak{h}$-invariant if and only if $Y \in \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. Now,

$$d\iota(Y)\Phi = \mathcal{L}(Y)\Phi - \iota(Y)d\Phi = \mathcal{L}(Y)\Phi = -\operatorname{tr}(\operatorname{ad}_{\mathfrak{g}/\mathfrak{h}}(Y))\Phi.$$

Hence $d = 0$ on $(\Lambda^{N-1}(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$ if and only if the $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$-action on $\mathfrak{g}/\mathfrak{h}$ is trace-free. □

2.5 The lower-degree parts

Under a suitable assumption, Theorem 2.1.2 (2) is extended to the lower-degree parts of cohomology:

Corollary 2.5.1. *Let G be a unimodular Lie group and H a closed subgroup of G that is reductive in G. Take a maximal compact subgroup K_H of H. If the homomorphism $i : H^p(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^p(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ is not injective for some $p \in \mathbb{N}$, then there is no compact manifold locally modelled on G/H.*

Remark 2.5.2. In this chapter, we say that a Lie group G is unimodular if the adjoint action of $\mathfrak{g}$ on itself is trace-free. If G is connected, it is equivalent to the existence of bi-invaraint Haar measure on G.

Proof of Corollary 2.5.1. Put $N = \dim G - \dim H$. By Theorem 2.1.2 (2), it suffices to see that, if $i : H^p(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^p(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ is injective for $p = N$, it is also injective for $0 \leqslant p \leqslant N - 1$. This follows from the standard Poincaré duality argument. Take any nonzero cohomology class $\alpha \in H^p(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$. By the Poincaré duality [32, Th. 12.1], we can pick $\beta \in H^{N-p}(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$ such that $\alpha \wedge \beta \neq 0$ in $H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$. Then $\eta(\alpha \wedge \beta) \neq 0$ by assumption, which yields $\eta(\alpha) \neq 0$. □

Remark 2.5.3. If G is a unimodular Lie group and H is a closed subgroup of G that is reductive in G, Theorem 2.1.2 (1) is not applicable to G/H because $H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \neq 0$ ($N = \dim G - \dim H$).

2.6 Relation with earlier results

We recover a result of Benoist–Labourie [3] from Theorem 2.1.2, though our proof relies on the crucial parts of [3].

Proposition 2.6.1 ([3, Th. 1])**.** *Let G be a connected semisimple Lie group and H its unimodular subgroup with finitely many connected components. If the centre $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$ contains a nonzero hyperbolic element, then there is no compact manifold locally modelled on G/H.*

Proof. We may assume H to be connected by Lemma 2.3.1. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form. In [3], it is shown that our assumptions yield the existence of $X \in \mathfrak{g}$ such that:

- X is a nonzero hyperbolic element.
- $H \subset \mathrm{Stab}(X)$.
- Let $\omega = dX$. Let $N = \dim(G/H)$ and $2m = \dim(G/\mathrm{Stab}(X))$. If we take $\mu \in (\Lambda^{N-2m}(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$ so that $\mu \wedge \omega^m \neq 0$, then $d(\mu \wedge \omega^{m-1}) = 0$.

Here, $\mathrm{Stab}(X) \subset G$ is the stabilizer of X in G. Remark that $\omega = dX$ is an element of $(\Lambda^2(\mathfrak{g}/\mathfrak{stab}(X))^*)^{\mathrm{Stab}(X)}$ ($\subset (\Lambda^2(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$) and satisfies $\omega^m \neq 0$ ($2m = \dim(G/\mathrm{Stab}(X))$).

If $[\mu \wedge \omega^m]_{\mathfrak{g},\mathfrak{h}} = 0$ in $H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$, then the proposition follows from Theorem 2.1.2 (1). Thus we assume $[\mu \wedge \omega^m]_{\mathfrak{g},\mathfrak{h}} \neq 0$. Since every element of $\mathfrak{k}_H$ commutes with X and is elliptic, $X \in ((\mathfrak{g}/\mathfrak{k}_H)^*)^{\mathfrak{k}_H}$. Hence $[\mu \wedge \omega^m]_{\mathfrak{g},\mathfrak{k}_H} = [d(X \wedge \mu \wedge \omega^{m-1})]_{\mathfrak{g},\mathfrak{k}_H} = 0$ in $H^N(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$. Apply Theorem 2.1.2 (2). □

We shall see later that results by Kobayashi and Ono ([30, Cor. 5], [20, Prop. 4.10]) are recovered from Theorem 2.1.2 (2), too (see Remark 3.1.4).

2.7 Examples (1): nonreductive Lie groups

In the rest of this chapter, we shall give some applications of Theorem 2.1.2. In this section, we study the case when G is nonreductive.

Example 2.7.1. *Let G be a simply connected nonunimodular Lie group and*

$$G = S \ltimes R \quad (S\text{: semisimple, } R\text{: solvable})$$

be its Levi decomposition. Take any closed unimodular subgroup H of S with finitely many connected components. Then there is no compact manifold locally modelled on G/H.

In fact, we can show a slightly more general result:

Example 2.7.2. *Let G be a nonunimodular Lie group. Let G' be a closed subgroup of G such that $\mathfrak{g}'$ is reductive in $\mathfrak{g}$ and the adjoint action of $\mathfrak{z}(\mathfrak{g}')$ on $\mathfrak{g}$ is trace-free. Here $\mathfrak{z}(\mathfrak{g}')$ denotes the centre of $\mathfrak{g}'$. Let H be any closed unimodular subgroup of G' with finitely many connected components. Then there is no compact manifold locally modelled on G/H.*

Proof of Example 2.7.2. By Proposition 2.4.1, it suffices to check that:

(i) The $\mathfrak{h}$-action on $\mathfrak{g}/\mathfrak{h}$ is trace-free.

(ii) The $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$-action on $\mathfrak{g}/\mathfrak{h}$ is not trace-free.

We will show the stronger results:

(i′) The $\mathfrak{g}'$-action on $\mathfrak{g}$ is trace-free.

(ii′) The $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}')$-action on $\mathfrak{g}$ is not trace-free.

Here $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}')$ denotes the centralizer of $\mathfrak{g}'$ in $\mathfrak{g}$.

Let us prove (i′). Since $\mathfrak{g}'$ is reductive, we have a direct sum decomposition $\mathfrak{g}' = \mathfrak{z}(\mathfrak{g}') \oplus [\mathfrak{g}', \mathfrak{g}']$. By our assumption, $\mathfrak{z}(\mathfrak{g}')$ acts trace-freely on $\mathfrak{g}$. Also, $[\mathfrak{g}', \mathfrak{g}']$ acts trace-freely on $\mathfrak{g}$ since it is a semisimple Lie algebra.

Now let us prove (ii′). Let

$$\mathfrak{g}_1 = \{X \in \mathfrak{g} : \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X)) = 0\}.$$

Since $\mathfrak{g}'$ is reductive in $\mathfrak{g}$, we can pick a $\mathfrak{g}'$-invariant subspace $\mathfrak{g}_2$ complementary to $\mathfrak{g}_1$ in $\mathfrak{g}$. Note that $\mathfrak{g}_2 \neq \{0\}$ and $\operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X)) \neq 0$ for any nonzero element X of $\mathfrak{g}_2$. We have $[\mathfrak{g}', \mathfrak{g}_2] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_1$, while $[\mathfrak{g}', \mathfrak{g}_2] \subset \mathfrak{g}_2$ by $\mathfrak{g}'$-invariance of $\mathfrak{g}_2$. This means $\mathfrak{g}_2 \subset \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}')$. From these (ii′) follows. □

Next we consider coadjoint orbits. Let G be a Lie group and $F \in \mathfrak{g}^*$. The coadjoint orbit $G.F \subset \mathfrak{g}^*$ of F is G-diffeomorphic to $G/\operatorname{Stab}(F)$, where

$\mathrm{Stab}(F) = \{g \in G : g.F = F\}$ is the stabilizer of F in G. Let $\omega = dF$, in other words,

$$\omega(X,Y) = -\langle F, [X,Y]\rangle \quad (X, Y \in \mathfrak{g}).$$

Then ω is an element of $(\Lambda^2(\mathfrak{g}/\mathfrak{stab}(F))^*)^{\mathrm{Stab}(F)}$ satisfying $d\omega = 0$ and $\omega^m \neq 0$, where $2m = \dim(G/\mathrm{Stab}(F))$. Under the identification $(\Lambda^2(\mathfrak{g}/\mathfrak{stab}(F))^*)^{\mathrm{Stab}(F)} \simeq \Omega^2(G/\mathrm{Stab}(F))^G$, ω corresponds to the Kirillov–Kostant–Souriau symplectic form. Applying Theorem 2.1.2 to this setting, we obtain:

Example 2.7.3. *Let G be a Lie group and $F \in \mathfrak{g}^*$. Assume that* $\dim(G/\mathrm{Stab}(F)) > 0$ *and* $\mathrm{Stab}(F)$ *has finitely many connected components. If $F|_{\mathfrak{k}_{\mathrm{Stab}(F)} \cap [\mathfrak{g},\mathfrak{g}]} = 0$, then there is no compact manifold locally modelled on $G/\mathrm{Stab}(F)$.*

Remark 2.7.4. The condition $\dim(G/\mathrm{Stab}(F)) > 0$ holds if and only if $F|_{[\mathfrak{g},\mathfrak{g}]} \neq 0$.

Remark 2.7.5. If G is a real linear algebraic group, the number of the connected components of $\mathrm{Stab}(F)$ (in the Euclidean topology) is always finite by Whitney's theorem [58, Th. 3]. For a nonalgebraic Lie group G, it may be infinite. An easy example is:

$$G = (\text{universal covering of } \mathrm{SL}(2,\mathbb{R})), \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g} \simeq \mathfrak{g}^*.$$

Here we identified $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form.

Proof of Example 2.7.3. Put $2m = \dim(G/\mathrm{Stab}(F))$. Recall that ω^m is a nonzero element of $(\Lambda^{2m}(\mathfrak{g}/\mathfrak{stab}(F))^*)^{\mathfrak{stab}(F)}$. By Theorem 2.1.2 (1), we only need to consider the case when $[\omega^m]_{\mathfrak{g},\mathfrak{stab}(F)} \neq 0$. Thus, by Theorem 2.1.2 (2), it suffices to prove that $[\omega^m]_{\mathfrak{g},\mathfrak{k}_{\mathrm{Stab}(F)}} = 0$. Since

$$\ker(d : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*) = (\mathfrak{g}^*)^{\mathfrak{g}} = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*,$$

our assumption $F|_{\mathfrak{k}_{\mathrm{Stab}(F)} \cap [\mathfrak{g},\mathfrak{g}]} = 0$ may be rewritten as:

$$F + F' \in ((\mathfrak{g}/\mathfrak{k}_{\mathrm{Stab}(F)})^*)^{\mathfrak{k}_{\mathrm{Stab}(F)}} \quad \text{for some } F' \in \ker(d : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*).$$

We obtain

$$[\omega^m]_{\mathfrak{g},\mathfrak{k}_{\mathrm{Stab}(F)}} = [d((F+F') \wedge \omega^{m-1})]_{\mathfrak{g},\mathfrak{k}_{\mathrm{Stab}(F)}} = 0 \quad \text{in } H^{2m}(\mathfrak{g}, \mathfrak{k}_{\mathrm{Stab}(F)}; \mathbb{R})$$

as required. $\square$

When G is a linear solvable Lie group, Example 2.7.3 gives the following result:

Example 2.7.6. *Let G be a linear solvable Lie group and $F \in \mathfrak{g}^*$. Assume that* $\dim(G/\operatorname{Stab}(F)) > 0$ *and* $\operatorname{Stab}(F)$ *has finitely many connected components. Then there is no compact manifold locally modelled on* $G/\operatorname{Stab}(F)$.

Remark 2.7.7. In Example 2.7.6, if G is simply connected, then $G/\operatorname{Stab}(F)$ admits an infinite discontinuous group ([24, Th. 2.2]).

Remark 2.7.8. In Example 2.7.6, the linearity of G is crucial. Consider the nonlinear nilpotent Lie group

$$G := \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} / \left\{ \begin{pmatrix} 1 & 0 & n \\ & 1 & 0 \\ & & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Its 2-dimensional coadjoint orbits have connected stabilizers, but admit compact Clifford–Klein forms.

Proof of Example 2.7.6. Let G_0 be the identity component of G and $[G_0, G_0]$ be its commutator subgroup. Then $[G_0, G_0]$ is closed in G and it does not contain a compact subgroup other than $\{1\}$ [18, Ch. XVIII, Th. 3.2]. In particular $K_{\operatorname{Stab}(F)} \cap [G_0, G_0] = \{1\}$ and hence $\mathfrak{k}_{\operatorname{Stab}(F)} \cap [\mathfrak{g}, \mathfrak{g}] = 0$. Thus, we can apply Example 2.7.3. □

2.8 Examples (2): reductive Lie groups

In this section, we study the case when G is reductive and H is not reductive in G.

Example 2.8.1. *Let G be a reductive Lie group and $P = MAN$ be a proper parabolic subgroup of G. Then there is no compact manifold locally modelled on G/N.*

Proof. Since $\mathfrak{g}$ and $\mathfrak{n}$ are unimodular, the $\mathfrak{n}$-action on $\mathfrak{g}/\mathfrak{n}$ is trace-free. On the other hand, $\mathfrak{a}$ normalizes $\mathfrak{n}$ and contains an element X such that $\operatorname{tr}_{\mathfrak{n}}(X) \neq 0$. Since $\mathfrak{g}$ is unimodular, such X also satisfies $\operatorname{tr}_{\mathfrak{g}/\mathfrak{n}}(X) \neq 0$. Thus, we can apply Proposition 2.4.1. □

Example 2.8.2. *Let G be a real linear semisimple algebraic group and $X \in \mathfrak{g}$. Let* $\operatorname{Stab}(X) \subset G$ *be the stabilizer of X in G. Let $X = X_e + X_h + X_n$ be the decomposition of X into elliptic, hyperbolic and nilpotent parts. If X is not a semisimple element (i.e. $X_n \neq 0$), then there is no compact manifold locally modelled on* $G/\operatorname{Stab}(X)$.

Remark 2.8.3. The study of Problem 2.1.1 for $G/\operatorname{Stab}(X)$, where G and X are as in Example 2.8.2, was started by [23] and then extended by [3]. We list their results here:

- Assume that X is a semisimple element (i.e. $X_n = 0$). If $\operatorname{Stab}(X) \neq \operatorname{Stab}(X_e)$, namely, if $G/\operatorname{Stab}(X)$ does not carry a G-invariant complex structure, then $G/\operatorname{Stab}(X)$ does not admit a compact Clifford–Klein form ([23, Th. 1.3]).

- If X is a nilpotent element (i.e. $X = X_n$), then there is no compact manifold locally modelled on $G/\operatorname{Stab}(X)$ ([3, Cor. 4]).

- If $X_h \neq 0$, then there is no compact manifold locally modelled on $G/\operatorname{Stab}(X)$ ([3, Cor. 5]).

Combining [3, Cor. 5] and Example 2.8.2, we conclude that, if X is not an elliptic element (i.e. if $X \neq X_e$), then there is no compact manifold locally modelled on $G/\operatorname{Stab}(X)$.

Proof of Example 2.8.2. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form. Let $\omega = dX$. Then ω is an element of $(\Lambda^2(\mathfrak{g}/\mathfrak{stab}(X))^*)^{\operatorname{Stab}(X)}$ satisfying $d\omega = 0$ and $\omega^m \neq 0$ ($2m = \dim(G/\operatorname{Stab}(X))$). By Theorem 2.1.2 (1), we may assume $[\omega^m]_{\mathfrak{g},\mathfrak{stab}(X)} = 0$. Then, by Theorem 2.1.2 (2), it is enough to prove that $[\omega^m]_{\mathfrak{g},\mathfrak{k}_{\operatorname{Stab}(X)}} = 0$.

Put $X_{ss} = X_e + X_h$. Let $\omega_{ss} = dX_{ss}$ and $\omega_n = dX_n$. They are elements of $(\Lambda^2(\mathfrak{g}/\mathfrak{stab}(X))^*)^{\mathfrak{stab}(X)}$ because $Y \in \mathfrak{g}$ commutes with X if and only if it commutes with X_{ss} and X_n. Since every element of $\mathfrak{k}_{\operatorname{Stab}(X)}$ commutes with X_n and is elliptic, X_n is perpendicular to $\mathfrak{k}_{\operatorname{Stab}(X)}$. Therefore, $X_n \in ((\mathfrak{g}/\mathfrak{k}_{\operatorname{Stab}(X)})^*)^{\mathfrak{k}_{\operatorname{Stab}(X)}}$. We have

$$\begin{aligned}
[\omega^m]_{\mathfrak{g},\mathfrak{k}_{\operatorname{Stab}(X)}} &= [\sum_{k=0}^{m} \frac{m!}{k!(m-k)!}\, \omega_{ss}^{m-k} \wedge \omega_n^k]_{\mathfrak{g},\mathfrak{k}_{\operatorname{Stab}(X)}} \\
&= [\omega_{ss}^m + d(X_n \wedge \sum_{k=1}^{m} \frac{m!}{k!(m-k)!}\, \omega_{ss}^{m-k} \wedge \omega_n^{k-1})]_{\mathfrak{g},\mathfrak{k}_{\operatorname{Stab}(X)}} \\
&= [\omega_{ss}^m]_{\mathfrak{g},\mathfrak{k}_{\operatorname{Stab}(X)}} \quad \text{in } H^{2m}(\mathfrak{g}, \mathfrak{k}_{\operatorname{Stab}(X)}; \mathbb{R}).
\end{aligned}$$

Let us prove that $\omega_{ss}^m = 0$. To see this, it suffices to show that $\mathfrak{stab}(X) \subsetneq \mathfrak{stab}(X_{ss})$. Let us assume the contrary: $\mathfrak{stab}(X) = \mathfrak{stab}(X_{ss})$. Take a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g} \otimes \mathbb{C}$ containing X_{ss}. Then we have

$$\mathfrak{j} \subset \mathfrak{stab}(X_{ss}) \otimes \mathbb{C} = \mathfrak{stab}(X) \otimes \mathbb{C} \subset \mathfrak{stab}(X_n) \otimes \mathbb{C}.$$

Since $\mathfrak{j}$ is a maximal abelian subalgebra of $\mathfrak{g} \otimes \mathbb{C}$, we have $X_n \in \mathfrak{j}$. This is impossible because $\mathfrak{j}$ consists of semisimple elements. □

Chapter 3

Proof of Kobayashi's rank conjecture on Clifford–Klein forms

3.1 Introduction

In 1989, T. Kobayashi conjectured the following:

Conjecture 3.1.1 ([21, Conj. 6.4]). *A homogeneous space G/H of reductive type does not admit a compact Clifford–Klein form if* $\operatorname{rank} G - \operatorname{rank} K < \operatorname{rank} H - \operatorname{rank} K_H$.

In this chapter, we prove this conjecture. Recall that a homogeneous space G/H is called of reductive type if G is a linear reductive Lie group with Cartan involution θ and H is a closed subgroup of G with finitely many connected components such that $\theta(H) = H$. We write K and K_H for the corresponding maximal compact subgroups of G and H, namely, $K = G^\theta$ and $K_H = H^\theta$, respectively. A Clifford–Klein form of a homogeneous space G/H is a quotient space $\Gamma\backslash G/H$, where Γ is a discrete subgroup of G acting properly and freely on G/H. It is a typical example of a manifold locally modelled on G/H, i.e. a manifold obtained by patching open sets of G/H by left translations by elements of G. Since the late 1980s, the existence problem of compact Clifford–Klein forms has been studied by various methods (e.g. [20], [23], [61], [2], [37]).

We deduce Conjecture 3.1.1 from the following cohomological obstruction to the existence of compact Clifford–Klein forms proved in Chapter 2:

Fact 3.1.2 (cf. Corollary 2.5.1). *Let G/H be a homogeneous space of reductive type. Let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}_H$ denote the Lie algebras of G, H and K_H, respectively. If the homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ induced from the inclusion $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \hookrightarrow (\Lambda(\mathfrak{g}/\mathfrak{k}_H)^*)^{\mathfrak{k}_H}$ is not injective, then there* *does not exist a compact manifold locally modelled on the homogeneous space G/H (and, in particular, a compact Clifford–Klein form of G/H).*

The main result of this chapter is the following necessary and sufficient condition for injectivity of $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$:

Theorem 3.1.3 (see Theorem 3.4.1)**.** *Let G/H be a homogeneous space of reductive type. We write $P_{\mathfrak{g}^*}$ and $P_{\mathfrak{h}^*}$ for the space of primitive elements in $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ and $(\Lambda\mathfrak{h}^*)^{\mathfrak{h}}$, respectively. Then, the homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ is not injective if and only if the linear map* rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ *induced from the restriction map $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}} \to (\Lambda\mathfrak{h}^*)^{\mathfrak{h}}$ is not surjective, where $(\,\cdot\,)^{-\theta}$ denotes the (-1)-eigenspace for θ.*

Conjecture 3.1.1 follows immediately from Fact 3.1.2 and Theorem 3.1.3. Indeed, it is classically known that $\dim (P_{\mathfrak{g}^*})^{-\theta} = \operatorname{rank} G - \operatorname{rank} K$ and $\dim (P_{\mathfrak{h}^*})^{-\theta} = \operatorname{rank} H - \operatorname{rank} K_H$ (see [16, Ch. X, §7]).

The proof of Theorem 3.1.3 is based on the theory of H. Cartan, C. Chevalley, J.-L. Koszul and A. Weil that gives an isomorphism between the relative Lie algebra cohomology $H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R})$ and the cohomology of a pure Sullivan algebra defined from a transgression for $\mathfrak{g}$ ([11]). By this theory, the proof is reduced to computation of invariant polynomials and a spectral sequence for pure Sullivan algebras.

Remark 3.1.4. Kobayashi and Ono proved Conjecture 3.1.1 in the case of $\operatorname{rank} G = \operatorname{rank} H$, investigating the Euler class of tangent bundle of a compact Clifford–Klein form ([30, Cor. 5], [20, Prop. 4.10]). Fact 3.1.2 can be regarded as an extension of their result to all the Chern–Weil characteristic classes (cf. Theorem 3.4.1 (i) ⇔ (ii) and [41, Prop. 6.1]).

Remark 3.1.5. Tholozan [56, ver. 2] independently announced the proof of Conjecture 3.1.1. His argument can be seen as a refinement of Fact 3.1.2. On the other hand, it seems that his proof cannot be applied to the case of manifolds locally modelled on G/H.

The organization of this chapter is as follows. In Section 3.2, we recall the definition of pure Sullivan algebras and construct a spectral sequence arising from a homomorphism of pure Sullivan algebras. In Section 3.3, we recall the theory of transgressions for a reductive Lie algebra and the Sullivan model for a reductive pair, mostly without proof, and apply the spectral sequence constructed in Section 3.2 to this setting. In Section 3.4, we give the proof of Theorem 3.1.3 using results in Section 3.3.

3.2 Preliminaries on pure Sullivan algebras

Since Theorem 3.1.3 is a purely algebraic statement, we work over an arbitrary field $\mathbb{K}$ of characteristic 0, rather than over $\mathbb{R}$, in the rest of this chapter. Given a graded vector space V, we define a new graded vector space $\widetilde{V}$ by $\widetilde{V} = V[-1]$, i.e. by putting $\widetilde{V}^n = V^{n-1}$ for each $n \in \mathbb{N}$. We write $\widetilde{v}$ for the element of $\widetilde{V}$ corresponding to $v \in V$. Similarly, we write $\widetilde{Q}$ for the element of $S^p\widetilde{V}$ corresponding to $Q \in S^pV$. For $v \in V$, we denote by $\varepsilon(v)$ and $\mu(v)$ the left multiplications by v on ΛV and SV, respectively. For $\alpha \in V^*$, we denote by $\iota(\alpha)$ and $\partial(\alpha)$ the derivations of ΛV and SV uniquely determined by $\iota(\alpha)v = \alpha(v)$ and $\partial(\alpha)v = \alpha(v)$ $(v \in V)$, respectively. We always use the Koszul sign convention, namely, we multiply by $(-1)^{pq}$ when we interchange two objects of homogeneous degrees p and q, respectively.

3.2.1 Pure Sullivan algebras

Let $U = \bigoplus_{n\geqslant 1} U^{2n-1}$ and $V = \bigoplus_{n\geqslant 1} V^{2n-1}$ be finite-dimensional, oddly and positively graded vector spaces. Let $f : S\widetilde{U} \to S\widetilde{V}$ be a graded algebra homomorphism. Define a differential δ_f on a graded algebra $\Lambda U \otimes S\widetilde{V}$ by the formula

$$\delta_f = \sum_i \iota(e^i) \otimes \mu(f(\widetilde{e_i})),$$

where $(e_i)_i$ is a basis of U and $(e^i)_i$ the basis of U^* dual to $(e_i)_i$. It is called the Koszul differential associated with f. In other words, the Koszul differential δ_f is a unique derivation satisfying

$$\delta_f(u \otimes 1) = 1 \otimes f(\widetilde{u}), \quad \delta_f(1 \otimes \widetilde{v}) = 0 \qquad (u \in U,\ v \in V).$$

Thus, δ_f does not depend on the choice of a basis $(e^i)_i$. A differential graded algebra of the form $(\Lambda U \otimes S\widetilde{V},\ -\delta_f)$ is called a pure Sullivan algebra.

Remark 3.2.1. The minus sign in our definition of a pure Sullivan algebra is inserted just for convenience and is not essential. Indeed, $1 \otimes \operatorname{sgn} : (\Lambda U \otimes S\widetilde{V}, -\delta_f) \xrightarrow{\sim} (\Lambda U \otimes S\widetilde{V}, \delta_f)$ is an isomorphism of differential graded algebras, where sgn denotes the automorphism of $S\widetilde{V}$ defined by $\operatorname{sgn}|_{S^p\widetilde{V}} = (-1)^p$.

The Koszul differential on $\Lambda V \otimes S\widetilde{V}$ associated with the identity map $1_{S\widetilde{V}}$ on $S\widetilde{V}$ is denoted by δ_V instead of $\delta_{1_{S\widetilde{V}}}$.

3.2.2 A spectral sequence for pure Sullivan algebras

Let U, V and W be finite-dimensional, oddly and positively graded vector spaces. Let $f : S\widetilde{U} \to S\widetilde{V}$ and $g : S\widetilde{V} \to S\widetilde{W}$ be graded algebra homomorphisms. Then,

$$1 \otimes g : (\Lambda U \otimes S\widetilde{V}, -\delta_f) \to (\Lambda U \otimes S\widetilde{W}, -\delta_{gf})$$

is a differential graded algebra homomorphism.

The Koszul differentials δ_f on $\Lambda U \otimes S\widetilde{V}$, δ_g on $\Lambda V \otimes S\widetilde{W}$, δ_{gf} on $\Lambda U \otimes S\widetilde{W}$ and δ_V on $\Lambda V \otimes S\widetilde{V}$ are naturally extended to the diferentials on $\Lambda U \otimes$

$S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$, which we shall denote by the same symbols. We define a differential graded algebra homomorphism

$$m : (\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_f - \delta_g + \delta_V) \to (\Lambda U \otimes S\widetilde{W}, -\delta_{gf})$$

by

$$m(\phi \otimes \widetilde{Q} \otimes \psi \otimes \widetilde{R}) = 0 \qquad (\phi \in \Lambda U,\ Q \in SV,\ \psi \in \Lambda^+ V,\ R \in SW),$$
$$m(\phi \otimes \widetilde{Q} \otimes 1 \otimes \widetilde{R}) = \phi \otimes g(\widetilde{Q})\widetilde{R} \qquad (\phi \in \Lambda U,\ Q \in SV,\ R \in SW).$$

Proposition 3.2.2. *The homomorphism m is a relative Sullivan model for the homomorphism $1 \otimes g : (\Lambda U \otimes S\widetilde{V}, -\delta_f) \to (\Lambda U \otimes S\widetilde{W}, -\delta_{gf})$, i.e.*

(i) *The diagram*

$$\begin{array}{ccc} (\Lambda U \otimes S\widetilde{V}, -\delta_f) & \xrightarrow{1\otimes g} & (\Lambda U \otimes S\widetilde{W}, -\delta_{gf}) \\ & \searrow^{i} & \uparrow{\scriptstyle m} \\ & & (\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_f - \delta_g + \delta_V) \end{array}$$

commutes, where i is the natural inclusion.

(ii) *It induces an isomorphism in cohomology:*

$$m : H^\bullet(\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_f - \delta_g + \delta_V) \xrightarrow{\sim} H^\bullet(\Lambda U \otimes S\widetilde{W}, -\delta_{gf}).$$

Proposition 3.2.2 should be known to experts, but we give its proof in Subsection 3.2.3 for the sake of completeness.

Let us define a filtration $(F^p)_{p\in\mathbb{N}}$ of the differential graded algebra $(\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_f - \delta_g + \delta_V)$ by

$$F^p = \bigoplus_{k \geqslant p} (\Lambda U \otimes S\widetilde{V})^p \otimes \Lambda V \otimes S\widetilde{W}.$$

The next proposition is easily obtained from routine computations and the identification $m : H^\bullet(\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_f - \delta_g + \delta_V) \xrightarrow{\sim} H^\bullet(\Lambda U \otimes S\widetilde{W}, -\delta_{gf})$.

Proposition 3.2.3. *The spectral sequence $(E_r^{p,q}, d_r)$ associated with the filtration $(F^p)_{p\in\mathbb{N}}$ satisfies the following:*

(1) $E_2^{p,q} = H^p(\Lambda U \otimes S\widetilde{V}, -\delta_f) \otimes H^q(\Lambda V \otimes S\widetilde{W}, -\delta_g)$.

(2) *The spectral sequence $(E_r^{p,q}, d_r)$ converges to $H^{p+q}(\Lambda U \otimes S\widetilde{W}, -\delta_{gf})$.*

(3) *The homomorphism $1 \otimes g : H^p(\Lambda U \otimes S\widetilde{V}, -\delta_f) \to H^p(\Lambda U \otimes S\widetilde{W}, -\delta_{gf})$ is factorized as*

$$H^p(\Lambda U \otimes S\widetilde{V}, -\delta_f) \xrightarrow{\sim} E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(\Lambda U \otimes S\widetilde{W}, -\delta_{gf}).$$

3.2.3 Proof of Proposition 3.2.2

The condition (i) is trivial. Let us verify the condition (ii).

For $(p,q) \in \mathbb{N}^2$, let $\pi_{p,q}$ denote the projection of $\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$ given by

$$\pi_{p,q}(x) = \begin{cases} 0 & \text{on} \quad \Lambda U \otimes S^{p'}\widetilde{V} \otimes \Lambda^{q'} V \otimes S\widetilde{W},\ (p',q') \neq (p,q), \\ 1 & \text{on} \quad \Lambda U \otimes S^{p}\widetilde{V} \otimes \Lambda^{q} V \otimes S\widetilde{W}. \end{cases}$$

We write π instead of $\pi_{0,0}$ when we regard $\pi_{0,0}$ as a map from $\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$ to $\Lambda U \otimes S\widetilde{W}$. Define a linear endomorphism κ of $\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$ by

$$\kappa = \begin{cases} \dfrac{1}{p+q} \displaystyle\sum_j 1 \otimes \partial(\widetilde{f^j}) \otimes \varepsilon(f_j) \otimes 1 & \text{on} \quad \begin{array}{l} \Lambda U \otimes S^{p}\widetilde{V} \otimes \Lambda^{q} V \otimes S\widetilde{W}, \\ (p,q) \neq (0,0), \end{array} \\ 0 & \text{on} \quad \Lambda U \otimes \mathbb{K} \otimes \mathbb{K} \otimes S\widetilde{W}, \end{cases}$$

where $(f_j)_j$ is a basis of V and $(f^j)_j$ the basis of V^* dual to $(f_j)_j$. One can easily show that $\delta_V \kappa + \kappa \delta_V = 1 - \pi_{0,0}$ (see [17, §3.1]). Since

$$(\delta_g \kappa)(\Lambda U \otimes S^{p}\widetilde{V} \otimes \Lambda^{q} V \otimes S\widetilde{W}) \subset \Lambda U \otimes S^{p-1}\widetilde{V} \otimes \Lambda^{q} V \otimes S\widetilde{W},$$

the infinite sum $\sum_{p=0}^{\infty} (\delta_g \kappa)^p$ is well-defined as a linear automorphism of $\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$, whose inverse is $1 - \delta_g \kappa$. Define an endomorphism ϕ of the graded algebra $\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$ by

$$\begin{aligned} \phi(u \otimes 1 \otimes 1 \otimes 1) &= u \otimes 1 \otimes 1 \otimes 1 \\ &\qquad + \kappa \sum_{p=0}^{\infty} (\delta_g \kappa)^p (1 \otimes f(\widetilde{u}) \otimes 1 \otimes 1) && (u \in U), \\ \phi(1 \otimes \widetilde{v} \otimes 1 \otimes 1) &= 1 \otimes \widetilde{v} \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes g(\widetilde{v}) && (v \in V), \\ \phi(1 \otimes 1 \otimes v \otimes 1) &= 1 \otimes 1 \otimes v \otimes 1 && (v \in V), \\ \phi(1 \otimes 1 \otimes 1 \otimes \widetilde{w}) &= 1 \otimes 1 \otimes 1 \otimes \widetilde{w} && (w \in W). \end{aligned}$$

Lemma 3.2.4. (1) $\phi(-\delta_{gf} + \delta_V) = (-\delta_f - \delta_g + \delta_V)\phi$.

(2) *For any* $x \in \Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$*, there exists* $n \in \mathbb{N}$ *such that* $(1-\phi)^n x = 0$.

(3) $m\phi = \pi$.

Proof of Lemma 3.2.4. We identify U, $\widetilde{V}$, V and $\widetilde{W}$ as graded subspaces of $\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$ in the natural way.

(1). Since both sides are derivations of $\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$, it suffices to verify this equality on U, $\widetilde{V}$, V and $\widetilde{W}$. The only nontrivial equality is

$$\phi(-\delta_{gf} + \delta_V)(u \otimes 1 \otimes 1 \otimes 1) = (-\delta_f - \delta_g + \delta_V)\phi(u \otimes 1 \otimes 1 \otimes 1) \qquad (u \in U).$$

The left-hand side is equal to $-1 \otimes 1 \otimes 1 \otimes gf(\widetilde{u})$, while the right-hand side is computed as

$$\begin{aligned}
&(-\delta_f - \delta_g + \delta_V)\phi(u \otimes 1 \otimes 1 \otimes 1)\\
=& -1 \otimes f(\widetilde{u}) \otimes 1 \otimes 1 + (-\delta_g + \delta_V)\kappa \sum_{p=0}^{\infty} (\delta_g\kappa)^p (1 \otimes f(\widetilde{u}) \otimes 1 \otimes 1)\\
=&(-1 + \delta_V\kappa) \sum_{p=0}^{\infty} (\delta_g\kappa)^p (1 \otimes f(\widetilde{u}) \otimes 1 \otimes 1)\\
=& -(\pi_{0,0} + \kappa\delta_V) \sum_{p=0}^{\infty} (\delta_g\kappa)^p (1 \otimes f(\widetilde{u}) \otimes 1 \otimes 1)\\
=& -\pi_{0,0} \sum_{p=0}^{\infty} (\delta_g\kappa)^p (1 \otimes f(\widetilde{u}) \otimes 1 \otimes 1)\\
=& -\sum_{p=0}^{\infty} (\delta_g\kappa)^p \pi_{p,0} (1 \otimes f(\widetilde{u}) \otimes 1 \otimes 1).
\end{aligned}$$

Thus, it is enough to see that

$$(\delta_g\kappa)^p (1 \otimes \widetilde{Q} \otimes 1 \otimes \widetilde{R}) = 1 \otimes 1 \otimes 1 \otimes g(\widetilde{Q})\widetilde{R} \qquad (\widetilde{Q} \in S^p\widetilde{V},\ \widetilde{R} \in S\widetilde{W}) \ (*_p)$$

holds for every $p \in \mathbb{N}$. Obviously $(*_0)$ is true. Let us assume that $(*_{p-1})$ is true for some $p \geqslant 1$. Then

$$\begin{aligned}
(\delta_g\kappa)^p (1 \otimes \widetilde{Q} \otimes 1 \otimes \widetilde{R}) &= \frac{1}{p}(\delta_g\kappa)^{p-1} \sum_j 1 \otimes \partial(\widetilde{f}_j)\widetilde{Q} \otimes 1 \otimes g(\widetilde{f}^j)\widetilde{R}\\
&= 1 \otimes 1 \otimes 1 \otimes g\left(\frac{1}{p}\sum_j \mu(\widetilde{f}^j)\partial(\widetilde{f}_j)\widetilde{Q}\right)\widetilde{R}
\end{aligned}$$

by the induction hypothesis. Since $\sum_j \mu(\widetilde{f}^j)\partial(\widetilde{f}_j) = p$ on $S^p\widetilde{V}$, we have

$$1 \otimes 1 \otimes 1 \otimes g\left(\frac{1}{p}\sum_j \mu(\widetilde{f}^j)\partial(\widetilde{f}_j)\widetilde{Q}\right)\widetilde{R} = 1 \otimes 1 \otimes 1 \otimes g(\widetilde{Q})\widetilde{R}.$$

Hence $(*_p)$ is also true. This completes the proof of Lemma 3.2.4 (1).

(2). Put $A = \{x \in \Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W} : (1-\phi)^n x = 0 \text{ for some } n \in \mathbb{N}\}$. Notice that A is a subalgebra of $\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$. Indeed, the equality

$(1-\phi)(xx') = (1-\phi)(x)x' + \phi(x)(1-\phi)(x')$ implies that, if $(1-\phi)^n x = 0$ and $(1-\phi)^{n'}x' = 0$, then $(1-\phi)^{n+n'-1}(xx') = 0$. Therefore, it suffices to show that $U, \widetilde{V}, V, \widetilde{W} \subset A$. The inclusions $\widetilde{V}, V, \widetilde{W} \subset A$ are obvious. This implies $\mathbb{K} \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W} \subset A$. Now, $U \subset A$ follows from $(1-\phi)(U) \subset \mathbb{K} \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}$.

(3). Since both sides are graded algebra homomorphisms, it suffices to verify this equality on U, $\widetilde{V}$, V and $\widetilde{W}$. The only nontrivial equality is

$$m\phi(u \otimes 1 \otimes 1 \otimes 1) = \pi(u \otimes 1 \otimes 1 \otimes 1) \qquad (u \in U),$$

which follows from $\pi\kappa = 0$. □

By Lemma 3.2.4,

$$\phi : (\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_{gf} + \delta_V) \xrightarrow{\sim} (\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_f - \delta_g + \delta_V)$$

is a differential graded algebra isomorphism, whose inverse is $\sum_{k=0}^{\infty}(1-\phi)^k$, that makes the diagram

$$\begin{array}{ccc} (\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_{gf} + \delta_V) & & \\ \phi \downarrow \wr & \searrow^{\pi} & \\ (\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_f - \delta_g + \delta_V) & \xrightarrow{m} & (\Lambda U \otimes S\widetilde{W}, -\delta_{gf}) \end{array}$$

commute. Now, it suffices to show that the projection

$$\pi : (\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_{gf} + \delta_V) \to (\Lambda U \otimes S\widetilde{W}, -\delta_{gf})$$

induces an isomorphism in cohomology. Let

$$i : (\Lambda U \otimes S\widetilde{W}, -\delta_{gf}) \to (\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_{gf} + \delta_V)$$

denote the natural inclusion. We have $\pi i = 1$ and

$$i\pi = \pi_{0,0} = 1 - \delta_V\kappa - \kappa\delta_V = 1 - (-\delta_{gf} + \delta_V)\kappa - \kappa(-\delta_{gf} + \delta_V).$$

Therefore, $\pi : H^\bullet(\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_{gf} + \delta_V) \to H^\bullet(\Lambda U \otimes S\widetilde{W}, -\delta_{gf})$ is an isomorphism with inverse $i : H^\bullet(\Lambda U \otimes S\widetilde{W}, -\delta_{gf}) \to H^\bullet(\Lambda U \otimes S\widetilde{V} \otimes \Lambda V \otimes S\widetilde{W}, -\delta_{gf} + \delta_V)$. This completes the proof of Proposition 3.2.2. □

3.3 Preliminaries on the relative Lie algebra cohomology of reductive pairs

In this section, we recall the Cartan–Chevalley–Koszul–Weil theory (announced in [11]) on transgressions for a reductive Lie algebra and the Sullivan model for a reductive pair. We mostly omit the proofs. See [16] for details on this subject.

We retain the notation of Section 3.2. We always regard the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ as a graded vector space concentrated in degree 1. Thus $\widetilde{\mathfrak{g}^*}$ is concentrated in degree 2. We write $\mathcal{L}$ for the $\mathfrak{g}$-action on the exterior algebra $\Lambda\mathfrak{g}^*$. Given an automorphism θ of a Lie algebra $\mathfrak{g}$, we denote by the same symbol θ the induced automorphisms of $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$, $(S\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$, etc.

3.3.1 Relative Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ its subalgebra. Let d be a differential on the exterior algebra $\Lambda\mathfrak{g}^*$ given by

$$(d\alpha)(X_1,\ldots,X_{p+1}) = \sum_{1\leqslant i<j\leqslant p+1} \alpha([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{p+1})$$
$$(\alpha\in\Lambda^p\mathfrak{g}^*, X_1,\ldots,X_{p+1}\in\mathfrak{g}).$$

The graded subalgebra

$$(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} = \{\alpha\in\Lambda\mathfrak{g}^* : \iota(X)\alpha = 0,\ \mathcal{L}(X)\alpha = 0 \text{ for all } X\in\mathfrak{h}\}$$

of $\Lambda\mathfrak{g}^*$ is closed under the differential d. The cohomology of the differential graded algebra $((\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d)$ is denoted by $H^\bullet(\mathfrak{g},\mathfrak{h};\mathbb{K})$ and called the relative Lie algebra cohomology of a pair $(\mathfrak{g},\mathfrak{h})$.

3.3.2 The Cartan model of equivariant cohomology and the Chern–Weil homomorphism ([16, Ch. VIII, §4], [17, §§2–5])

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ its subalgebra. Define a differential $d_{\mathfrak{g},\mathfrak{h}}$ on a graded algebra $(\Lambda\mathfrak{g}^*\otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}$ by the formula

$$d_{\mathfrak{g},\mathfrak{h}} = d\otimes 1 - \sum_j \iota(F_j)\otimes\mu(\widetilde{F^j}),$$

where $(F_j)_j$ is a basis of $\mathfrak{h}$ and $(F^j)_j$ the basis of $\mathfrak{h}^*$ dual to $(F_j)_j$. The cohomology of a differential graded algebra $((\Lambda\mathfrak{g}^*\otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}})$ is called the Cartan model of $\mathfrak{h}$-equivariant cohomology of $\Lambda\mathfrak{g}^*$. The natural inclusion

$$w : ((S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, 0) \to ((\Lambda\mathfrak{g}^*\otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}}), \qquad \widetilde{Q}\mapsto 1\otimes\widetilde{Q}$$

induces a homomorphism $w : (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}} \to H^\bullet((\Lambda\mathfrak{g}^*\otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}})$, called the Chern–Weil homomorphism.

One has a natural inclusion of differential graded algebras

$$\epsilon : ((\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d) \to ((\Lambda\mathfrak{g}^*\otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}}), \qquad \alpha\mapsto\alpha\otimes 1.$$

Fact 3.3.1. *When* $\mathfrak{h}$ *has an* $\mathfrak{h}$*-invariant complementary linear subspace* V *in* $\mathfrak{g}$ *(e.g. when* $\mathfrak{h} = \mathfrak{g}$ *or* $\mathfrak{h}$ *is reductive in* $\mathfrak{g}$*), the inclusion* ϵ *induces an isomorphism* $\epsilon : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K}) \xrightarrow{\sim} H^\bullet((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}})$.

The inverse isomorphism is constructed as follows. Let π_V denote the projection $\Lambda\mathfrak{g}^* = \Lambda\mathfrak{h}^* \otimes \Lambda V^* \twoheadrightarrow \Lambda V^*$. Let $\chi : S\widetilde{\mathfrak{h}^*} \to \Lambda V$ be a graded algebra homomorphism induced from a graded linear map

$$\widetilde{\mathfrak{h}^*} \to \Lambda^2 V^*, \qquad \widetilde{g} \mapsto -g([\cdot,\cdot]),$$

where $g \in \mathfrak{h}^*$ is regarded as an element of $\mathfrak{g}^*$ by putting $g|_V = 0$. Then, a graded algebra homomorphism

$$\psi_V : \Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*} \to \Lambda V^* \ (\simeq \Lambda(\mathfrak{g}/\mathfrak{h})^*), \qquad \alpha \otimes \widetilde{Q} \mapsto \pi_V(\alpha) \wedge \chi(\widetilde{Q}).$$

restricts to the differential graded algebra homomorphism

$$\psi_V : ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}}) \to ((\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d).$$

This ψ_V induces the inverse of ϵ in cohomology. We simply write w for the composition

$$(\epsilon^{-1} \circ w =)\, \psi_V \circ w : (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}} \to H^\bullet((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}}) \xrightarrow{\sim} H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$$

and call it the Chern–Weil homomorphism.

3.3.3 The Cartan map ([16, Ch. VI, §2])

Let $\mathfrak{g}$ be a Lie algebra. By Fact 3.3.1, one has

$$H^\bullet((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}, d_{\mathfrak{g},\mathfrak{g}}) \simeq H^p(\mathfrak{g}, \mathfrak{g}; \mathbb{K}) = \begin{cases} \mathbb{K} & (p = 0) \\ 0 & (p \geqslant 1). \end{cases}$$

Thus, for $\widetilde{P} \in ((S\widetilde{\mathfrak{g}^*})^{\mathfrak{g}})^{2k}$ $(= (S^k\widetilde{\mathfrak{g}^*})^{\mathfrak{g}})$ $(k \geqslant 1)$, there uniquely exists an element $\rho_{\mathfrak{g}}(\widetilde{P})$ of $(\Lambda^{2k-1}\mathfrak{g}^*)^{\mathfrak{g}}$ such that $d_{\mathfrak{g},\mathfrak{g}}(\rho_{\mathfrak{g}}(\widetilde{P}) \otimes 1 + \Omega) = -1 \otimes \widetilde{P}$ for some $\Omega \in (\Lambda\mathfrak{g}^* \otimes S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ (the uniqueness follows from $d|_{(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}} = 0$). This defines a linear map $\rho_{\mathfrak{g}} : (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}} \to (\Lambda^+\mathfrak{g}^*)^{\mathfrak{g}}$ of degree -1, called the Cartan map for $\mathfrak{g}$.

3.3.4 Transgressions ([16, Ch. VI, §§3–4])

Let $\mathfrak{g}$ be a reductive Lie algebra. Then $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ is dual to the graded algebra $(\Lambda\mathfrak{g})^{\mathfrak{g}}$ and therefore admits a graded coalgebra structure in a natural way. In fact, it forms a Hopf algebra. Let $P_{\mathfrak{g}^*} = \bigoplus_{k \geqslant 1} P^{2k-1}_{\mathfrak{g}^*}$ denote the space of primitive elements in $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$.

Fact 3.3.2. *The Cartan map* $\rho_{\mathfrak{g}}$ *for a reductive Lie algebra* $\mathfrak{g}$ *satisfies* $\ker \rho_{\mathfrak{g}} = (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}} \cdot (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ *and* $\operatorname{im} \rho_{\mathfrak{g}} = P_{\mathfrak{g}^*}$.

A linear map $\tau_{\mathfrak{g}} : P_{\mathfrak{g}^*} \to (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ of degree 1 satisfying $\rho_{\mathfrak{g}} \circ \tau_{\mathfrak{g}} = 1$ is called a transgression in the Weil algebra of $\mathfrak{g}$. We simply call it a transgression for $\mathfrak{g}$.

Fact 3.3.3. *A transgression $\tau_{\mathfrak{g}}$ for a reductive Lie algebra $\mathfrak{g}$ induces a graded algebra isomorphism $\widetilde{\tau_{\mathfrak{g}}} : \widetilde{SP_{\mathfrak{g}^*}} \xrightarrow{\sim} (S\widetilde{\mathfrak{g}^*})^*$.*

The condition $\rho_{\mathfrak{g}} \circ \tau_{\mathfrak{g}} = 1$ is equivalent to the existence of a graded linear map $\Omega : P_{\mathfrak{g}^*} \to (\Lambda\mathfrak{g}^* \otimes S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ such that $d_{\mathfrak{g},\mathfrak{g}}(\alpha \otimes 1 + \Omega(\alpha)) = -1 \otimes \tau_{\mathfrak{g}}(\alpha)$. There uniquely exists a transgression $\tau_{\mathfrak{g}}$ for $\mathfrak{g}$ such that this graded linear map Ω can be taken so that $(\iota(Z) \otimes 1)(\Omega(\alpha)) = 0$ for any $Z \in (\Lambda^+\mathfrak{g})^{\mathfrak{g}}$ and $\alpha \in P_{\mathfrak{g}^*}$. It is called the distinguished transgression for $\mathfrak{g}$.

3.3.5 Compatibility with automorphisms

It is obvious from the definition of the Cartan map $\rho_{\mathfrak{g}}$ for a Lie algebra $\mathfrak{g}$ that the following diagram commutes for any automorphism θ of $\mathfrak{g}$:

$$\begin{array}{ccc} (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}} & \xrightarrow[\rho_{\mathfrak{g}}]{} & \Lambda^+\mathfrak{g}^* \\ \theta\big\downarrow & & \theta\big\downarrow \\ (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}} & \xrightarrow[\rho_{\mathfrak{g}}]{} & \Lambda^+\mathfrak{g}^*. \end{array}$$

We say that a transgression $\tau_{\mathfrak{g}}$ for a reductive Lie algebra $\mathfrak{g}$ is compatible with an automorphism θ of $\mathfrak{g}$ if the following diagram commutes:

$$\begin{array}{ccc} P_{\mathfrak{g}^*} & \xrightarrow[\tau_{\mathfrak{g}}]{} & (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}} \\ \theta\big\downarrow & & \theta\big\downarrow \\ P_{\mathfrak{g}^*} & \xrightarrow[\tau_{\mathfrak{g}}]{} & (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}. \end{array}$$

It readily follows from its uniqueness that the distinguished transgression is compatible with any automorphism.

3.3.6 The Sullivan model for a reductive pair ([16, Ch. X, §2])

Now, let $(\mathfrak{g}, \mathfrak{h})$ be a reductive pair, i.e. $\mathfrak{g}$ a reductive Lie algebra and $\mathfrak{h}$ its subalgebra that is reductive in $\mathfrak{g}$. Let $\tau_{\mathfrak{g}} : P_{\mathfrak{g}^*} \to (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ be a transgression for $\mathfrak{g}$ and $\widetilde{\tau_{\mathfrak{g}}} : \widetilde{SP_{\mathfrak{g}^*}} \xrightarrow{\sim} (S\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ the induced isomorphism (see Fact 3.3.3). Define a graded algebra homomorphism $\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}} : \widetilde{SP_{\mathfrak{g}^*}} \to (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}$ by $\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}(\widetilde{\Omega}) = \widetilde{\tau_{\mathfrak{g}}}(\widetilde{\Omega})|_{\mathfrak{h}}$. Let us consider the pure Sullivan algebra $(\Lambda P_{\mathfrak{g}^*} \otimes (S\mathfrak{h}^*)^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}})$ associated with $\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}$:

$$\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}(\alpha \otimes 1) = 1 \otimes \tau_{\mathfrak{g}}(\alpha)|_{\mathfrak{h}} \quad (\alpha \in P_{\mathfrak{g}^*}), \qquad \delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}(1 \otimes \widetilde{Q}) = 0 \quad (Q \in (S\mathfrak{h}^*)^{\mathfrak{h}}).$$

Take a graded linear map $\Omega : P_{\mathfrak{g}^*} \to (\Lambda\mathfrak{g}^* \otimes S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ such that $d_{\mathfrak{g},\mathfrak{g}}(\alpha \otimes 1 + \Omega(\alpha)) = -1 \otimes \tau_{\mathfrak{g}}(\alpha)$, which exists by definition of $\tau_{\mathfrak{g}}$. The Chevalley homomorphism

$$\vartheta_\Omega : (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \to ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}})$$

is a differential graded algebra homomorphism defined by

$$\vartheta_\Omega(\alpha \otimes 1) = \alpha \otimes 1 + (1 \otimes \mathrm{rest})(\Omega(\alpha)), \qquad (\alpha \in P_{\mathfrak{g}^*}),$$
$$\vartheta_\Omega(1 \otimes \widetilde{Q}) = 1 \otimes \widetilde{Q} \qquad (Q \in (S\mathfrak{h}^*)^{\mathfrak{h}}),$$

where $\mathrm{rest} : S\widetilde{\mathfrak{g}^*} \to S\widetilde{\mathfrak{h}^*}$ denotes the restriction map.

Fact 3.3.4. *The Chevalley homomorphism ϑ_Ω induces an isomorphism in cohomology:*

$$\vartheta_\Omega : H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \xrightarrow{\sim} H^\bullet((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}}) \ (\simeq H^\bullet(\mathfrak{g},\mathfrak{h};\mathbb{K})).$$

Remark 3.3.5. Fact 3.3.4 means that the Chevalley homomorphism ϑ_Ω (resp. $\psi_V \circ \vartheta_\Omega$, where ψ_V is as in Subsection 3.3.2) is a Sullivan model for a differential graded algebra $((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}})$ (resp. $((\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d)$).

3.3.7 The Chern–Weil homomorphism in the Sullivan model ([16, Ch. X, §2])

We retain the setting of Subsection 3.3.6. Let $w' : (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}} \to H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}})$ be the homomorphism induced from the inclusion

$$w' : ((S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, 0) \to (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}), \qquad \widetilde{Q} \mapsto 1 \otimes \widetilde{Q}.$$

Proposition 3.3.6. *The homomorphism w' is identified with the Chern–Weil homomorphism $w : (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}} \to H^\bullet((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}})$ $(\simeq H^\bullet(\mathfrak{g},\mathfrak{h};\mathbb{K}))$ via ϑ_Ω (or $\epsilon^{-1} \circ \vartheta_\Omega$).*

Indeed, $w = \vartheta_\Omega \circ w' : ((S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, 0) \to ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}})$.

Proposition 3.3.7. $(\ker w =)\ \ker w' = (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}|_{\mathfrak{h}} \cdot (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}$.

This immediately follows from Fact 3.3.3.

3.3.8 The case of reductive symmetric pairs ([16, Ch. X, §7])

Fact 3.3.8. *Suppose $(\mathfrak{g},\mathfrak{h})$ is a reductive symmetric pair, i.e. $\mathfrak{g}$ is a reductive Lie algebra and $\mathfrak{h} = \mathfrak{g}^\theta$ for some involution θ of $\mathfrak{g}$. Then,*

(1) $\dim(P_{\mathfrak{g}^*})^{-\theta} = \operatorname{rank}\mathfrak{g} - \operatorname{rank}\mathfrak{h}$.

(2) *If $\tau_{\mathfrak{g}}$ is a transgression for $\mathfrak{g}$ that is compatible with θ, the following is a graded algebra isomorphism:*

$$\Lambda(P_{\mathfrak{g}^*})^{-\theta} \otimes \operatorname{im} w' \xrightarrow{\sim} H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}), \quad \alpha \otimes [1 \otimes \widetilde{Q}] \mapsto [\alpha \otimes \widetilde{Q}].$$

3.3.9 Induced homomorphisms

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ and $\mathfrak{l}$ a subalgebra of $\mathfrak{h}$. Then the inclusion

$$i : ((\Lambda(\mathfrak{g}/\mathfrak{l})^*)^{\mathfrak{l}}, d) \to ((\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d)$$

and the restriction

$$1 \otimes \mathrm{rest} : ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}}) \to ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, d_{\mathfrak{g},\mathfrak{l}})$$

are differential graded algebra homomorphisms. The following diagram commutes:

$$\begin{array}{ccc} ((\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d) & \xrightarrow{\epsilon} & ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}}) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1\otimes\mathrm{rest}} \\ ((\Lambda(\mathfrak{g}/\mathfrak{l})^*)^{\mathfrak{l}}, d) & \xrightarrow{\epsilon} & ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, d_{\mathfrak{g},\mathfrak{l}}) \end{array}$$

Suppose, in addition, that $(\mathfrak{g},\mathfrak{h})$ and $(\mathfrak{g},\mathfrak{l})$ are reductive pairs. Take a transgression $\tau_{\mathfrak{g}}$ for $\mathfrak{g}$ and a graded linear map $\Omega : P_{\mathfrak{g}^*} \to (\Lambda\mathfrak{g}^* \otimes S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ such that $d_{\mathfrak{g},\mathfrak{g}}(\alpha \otimes 1 + \Omega(\alpha)) = -1 \otimes \tau_{\mathfrak{g}}(\alpha)$. Then,

$$1 \otimes \mathrm{rest} : (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \to (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}})$$

is a differential graded algebra homomorphism, and the diagram

$$\begin{array}{ccc} (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) & \xrightarrow{\vartheta_\Omega} & ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, d_{\mathfrak{g},\mathfrak{h}}) \\ \downarrow{\scriptstyle 1\otimes\mathrm{rest}} & & \downarrow{\scriptstyle 1\otimes\mathrm{rest}} \\ (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}}) & \xrightarrow{\vartheta_\Omega} & ((\Lambda\mathfrak{g}^* \otimes S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, d_{\mathfrak{g},\mathfrak{l}}) \end{array}$$

commutes. In summary,

Proposition 3.3.9. *The homomorphism*

$$1 \otimes \mathrm{rest} : H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \to H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}})$$

is identified with the homomorphism $i : H^\bullet(\mathfrak{g},\mathfrak{h};\mathbb{K}) \to H^\bullet(\mathfrak{g},\mathfrak{l};\mathbb{K})$ *via* $\epsilon^{-1} \circ \vartheta_\Omega$.

3.3.10 A spectral sequence for the Sullivan models of reductive pairs

As in Subsection 3.3.9, let $(\mathfrak{g},\mathfrak{h})$ be a reductive pair and $\mathfrak{l}$ a subalgebra of $\mathfrak{h}$ such that $(\mathfrak{g},\mathfrak{l})$ is a reductive pair. Let $\tau_{\mathfrak{g}}$ and $\tau_{\mathfrak{h}}$ be transgressions for $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We identify $(S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}$ with $\widetilde{SP_{\mathfrak{h}^*}}$ via $\widetilde{\tau_{\mathfrak{h}}}$. We thus denote by $\delta_{P_{\mathfrak{h}^*}}$ the Koszul differential on $\Lambda P_{\mathfrak{h}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}$ defined by

$$\delta_{P_{\mathfrak{h}^*}}(\beta \otimes 1) = \tau_{\mathfrak{h}}(\beta), \quad \delta_{P_{\mathfrak{h}^*}}(1 \otimes \widetilde{Q}) = 0 \qquad (\beta \in P_{\mathfrak{h}^*}, Q \in (S\mathfrak{h}^*)^{\mathfrak{h}}).$$

Let us apply the spectral sequence constructed in Section 3.2 to the differential graded algebra homomorphism

$$1 \otimes \mathrm{rest} : (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \to (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}}).$$

By Proposition 3.2.2, the differential graded algebra homomorphism

$$m : (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}} \otimes \Lambda P_{\mathfrak{h}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}} - \delta_{\widetilde{\tau_{\mathfrak{h},\mathfrak{l}}}} + \delta_{P_{\mathfrak{h}^*}}) \to (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}})$$

defined by

$$m(\alpha \otimes \widetilde{Q} \otimes \beta \otimes \widetilde{R}) = 0 \quad (\alpha \in \Lambda P_{\mathfrak{g}^*},\ Q \in (S\mathfrak{h}^*)^{\mathfrak{h}},\ \beta \in \Lambda^+ P_{\mathfrak{h}^*},\ R \in (S\mathfrak{l}^*)^{\mathfrak{l}}),$$
$$m(\alpha \otimes \widetilde{Q} \otimes 1 \otimes \widetilde{R}) = \alpha \otimes \widetilde{Q}|_{\mathfrak{l}} \cdot \widetilde{R} \qquad (\alpha \in \Lambda P_{\mathfrak{g}^*},\ Q \in (S\mathfrak{h}^*)^{\mathfrak{h}},\ R \in (S\mathfrak{l}^*)^{\mathfrak{l}})$$

is a relative Sullivan model for the differential graded algebra homomorphism $1 \otimes \mathrm{rest} : (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \to (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}})$. Let $(F^p)_{p\in\mathbb{N}}$ be a filtration of the differential graded algebra $(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}} \otimes \Lambda P_{\mathfrak{h}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}} - \delta_{\widetilde{\tau_{\mathfrak{h},\mathfrak{l}}}} + \delta_{P_{\mathfrak{h}^*}})$ defined by $F^p = \bigoplus_{k \geqslant p} (\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}})^p \otimes \Lambda P_{\mathfrak{h}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}$. Applying Proposition 3.2.3 to this setting, we have the following:

Corollary 3.3.10. *Let $(E_r^{p,q}, d_r)$ be the spectral sequence associated with the filtration $(F^p)_{p\in\mathbb{N}}$. Then,*

(1) $E_2^{p,q} = H^p(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \otimes H^q(\Lambda P_{\mathfrak{h}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{h},\mathfrak{l}}}})$.

(2) *The spectral sequence $(E_r^{p,q}, d_r)$ converges to $H^{p+q}(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}})$.*

(3) *The homomorphism*

$$1 \otimes \mathrm{rest} : H^p(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \to H^p(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}})$$

is factorized as

$$H^p(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \xrightarrow{\sim} E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{l}^*})^{\mathfrak{l}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{l}}}}).$$

3.4 Main theorem

We retain the notation of Section 3.3. Now, let us prove the following theorem that is a more detailed version of Theorem 3.1.3:

Theorem 3.4.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive pair and θ an involution of $\mathfrak{g}$ such that $\theta(\mathfrak{h}) = \mathfrak{h}$. Put $\mathfrak{k}_{\mathfrak{h}} = \mathfrak{h}^\theta$. Let $\tau_{\mathfrak{g}} : P_{\mathfrak{g}^*} \to (S\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}$ be a transgression for $\mathfrak{g}$. Let $\tau_{\mathfrak{h}} : P_{\mathfrak{h}^*} \to (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}$ be a transgression for $\mathfrak{h}$ that is compatible with θ. Then, the following conditions are all equivalent:*

(i) *The homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_{\mathfrak{h}}; \mathbb{K})$ is injective.*

(ii) *The homomorphism* $i|_{\operatorname{im} w} : \operatorname{im} w \to H^\bullet(\mathfrak{g}, \mathfrak{k}_\mathfrak{h}; \mathbb{K})$ *is injective, where* $w : (S\widetilde{\mathfrak{h}^*})^\mathfrak{h} \to H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$ *is the Chern–Weil homomorphism.*

(iii) *The homomorphism*

$$1 \otimes \operatorname{rest} : H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^\mathfrak{h}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \to H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{k}^*_\mathfrak{h}})^{\mathfrak{k}_\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{k}_\mathfrak{h}}}})$$

is injective.

(iv) *The homomorphism*

$$(1 \otimes \operatorname{rest})|_{\operatorname{im} w'} : \operatorname{im} w' \to H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{k}^*_\mathfrak{h}})^{\mathfrak{k}_\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{k}_\mathfrak{h}}}})$$

is injective, where $w' : (S\widetilde{\mathfrak{h}^*})^\mathfrak{h} \to H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^\mathfrak{h}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}})$ *is defined by* $w'(\widetilde{Q}) = [1 \otimes \widetilde{Q}]$.

(v) $((S^+\mathfrak{h}^*)^\mathfrak{h})^{-\theta} \subset (S^+\mathfrak{g}^*)^\mathfrak{g}|_\mathfrak{h} \cdot (S\mathfrak{h}^*)^\mathfrak{h}$.

(vi) *The linear map*

$$\begin{aligned}\overline{\operatorname{rest}} : ((S^+\mathfrak{g}^*)^\mathfrak{g}/((S^+\mathfrak{g}^*)^\mathfrak{g} \cdot (S^+\mathfrak{g}^*)^\mathfrak{g}))^{-\theta}\\ \to ((S^+\mathfrak{h}^*)^\mathfrak{h}/((S^+\mathfrak{h}^*)^\mathfrak{h} \cdot (S^+\mathfrak{h}^*)^\mathfrak{h}))^{-\theta}\end{aligned}$$

induced from the restriction map $(S\mathfrak{g}^*)^\mathfrak{g} \to (S\mathfrak{h}^*)^\mathfrak{h}$ *is surjective.*

(vii) *The linear map* rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ *induced from the restriction map* $(\Lambda\mathfrak{g}^*)^\mathfrak{g} \to (\Lambda\mathfrak{h}^*)^\mathfrak{h}$ *is surjective.*

(viii) *The spectral sequence*

$$\begin{aligned}E_2^{p,q} = H^p(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^\mathfrak{h}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \otimes H^q(\Lambda P_{\mathfrak{h}^*} \otimes (S\widetilde{\mathfrak{k}^*_\mathfrak{h}})^{\mathfrak{k}_\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{h},\mathfrak{k}_\mathfrak{h}}}})\\ \Rightarrow H^{p+q}(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{k}^*_\mathfrak{h}})^{\mathfrak{k}_\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{k}_\mathfrak{h}}}})\end{aligned}$$

defined as in Corollary 3.3.10 collapses at the E_2*-term.*

Proof. (i) ⇒ (ii). Trivial.

(iii) ⇒ (iv). Trivial.

(i) ⇔ (iii). This follows from Proposition 3.3.9.

(ii) ⇔ (iv). This follows from Propositions 3.3.6 and 3.3.9.

(iv) ⇒ (v). Take any $Q \in ((S\mathfrak{h}^*)^\mathfrak{h})^{-\theta}$. Then we have $Q|_{\mathfrak{k}_\mathfrak{h}} = 0$. By (iv), $[1 \otimes \widetilde{Q}] = 0$ in $H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^\mathfrak{h}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}})$. This means $Q \in (S^+\mathfrak{g}^*)^\mathfrak{g}|_\mathfrak{h} \cdot (S\mathfrak{h}^*)^\mathfrak{h}$ by Proposition 3.3.7.

(v) ⇒ (vi). Take any $\overline{Q} \in ((S^+\mathfrak{h}^*)^\mathfrak{h}/((S^+\mathfrak{h}^*)^\mathfrak{h} \cdot (S^+\mathfrak{h}^*)^\mathfrak{h}))^{-\theta}$. Let $Q \in (S^+\mathfrak{h}^*)^\mathfrak{h}$ be a representative of $\overline{Q}$. By (v), we can write

$$\frac{Q - \theta(Q)}{2} = P|_\mathfrak{h} + \sum_{i=1}^{r} P_i|_\mathfrak{h} \cdot Q_i \qquad (P, P_i \in (S^+\mathfrak{g}^*)^\mathfrak{g},\ Q_i \in (S^+\mathfrak{h}^*)^\mathfrak{h}).$$

Put $P' = (P - \theta(P))/2$. Then $\overline{P'} \in ((S^+\mathfrak{g}^*)^{\mathfrak{g}}/((S^+\mathfrak{g}^*)^{\mathfrak{g}} \cdot (S^+\mathfrak{g}^*)^{\mathfrak{g}}))^{-\theta}$ and $\overline{P'|_{\mathfrak{h}}} = \overline{Q}$.

(vi) $\Rightarrow$ (v). We shall prove

$$((S^n\mathfrak{h}^*)^{\mathfrak{h}})^{-\theta} \subset (S^+\mathfrak{g}^*)^{\mathfrak{g}}|_{\mathfrak{h}} \cdot (S\mathfrak{h}^*)^{\mathfrak{h}} \qquad (\dagger_n)$$

by induction on n. Assume that $(\dagger_m)$ is true for $m \leqslant n-1$. Let us take any $Q \in ((S^n\mathfrak{h}^*)^{\mathfrak{h}})^{-\theta}$. By (vi), we can write

$$Q = P|_{\mathfrak{h}} + \sum_{i=1}^{r} Q_i \cdot Q_i' \qquad (P \in (S^n\mathfrak{g}^*)^{\mathfrak{g}},\ Q_i \in (S^{m_i}\mathfrak{h}^*)^{\mathfrak{h}},$$
$$Q_i' \in (S^{n-m_i}\mathfrak{h}^*)^{\mathfrak{h}},\ 1 \leqslant m_i \leqslant n-1)$$

Then,

$$Q = \frac{1}{2}(Q - \theta(Q)) = \frac{1}{2}(P - \theta(P))|_{\mathfrak{h}} + \frac{1}{4}\sum_{i=1}^{r}((Q_i - \theta(Q_i))(Q_i' + \theta(Q_i'))$$
$$- (Q_i + \theta(Q_i))(Q_i' - \theta(Q_i'))).$$

We have

$$Q_i - \theta(Q_i),\ Q_i' - \theta(Q_i') \in (S^+\mathfrak{g}^*)^{\mathfrak{g}}|_{\mathfrak{h}} \cdot (S\mathfrak{h}^*)^{\mathfrak{h}}$$

by the induction hypothesis, and therefore $Q \in (S^+\mathfrak{g}^*)^{\mathfrak{g}}|_{\mathfrak{h}} \cdot (S\mathfrak{h}^*)^{\mathfrak{h}}$. Thus $(\dagger_n)$ is also true.

(vi) $\Leftrightarrow$ (vii). This follows from the commutativity of the diagram

$$\begin{array}{ccc} \Big((S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}/((S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}} \cdot (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}})\Big)^{-\theta} & \xrightarrow[\overline{\rho_{\mathfrak{g}}}]{\sim} & (P_{\mathfrak{g}^*})^{-\theta} \\ \downarrow{\scriptstyle \overline{\mathrm{rest}}} & & \downarrow{\scriptstyle \mathrm{rest}} \\ \Big((S^+\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}/((S^+\widetilde{\mathfrak{h}^*})^{\mathfrak{h}} \cdot (S^+\widetilde{\mathfrak{h}^*})^{\mathfrak{h}})\Big)^{-\theta} & \xrightarrow[\overline{\rho_{\mathfrak{h}}}]{\sim} & (P_{\mathfrak{h}^*})^{-\theta}, \end{array}$$

where $\overline{\rho_{\mathfrak{g}}}$ and $\overline{\rho_{\mathfrak{h}}}$ are the linear isomorphisms induced from the Cartan maps.

(v) $\Rightarrow$ (viii). We shall prove $d_r = 0$ $(r \geqslant 2)$ by induction on r. Let us assume that $d_s = 0$ for $2 \leqslant s \leqslant r-1$. Then

$$E_r^{p,q} = E_2^{p,q} = H^p(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}}) \otimes H^q(\Lambda P_{\mathfrak{h}^*} \otimes (S\widetilde{\mathfrak{k}_{\mathfrak{h}}^*})^{\mathfrak{k}_{\mathfrak{h}}}, -\delta_{\widetilde{\tau_{\mathfrak{h},\mathfrak{k}_{\mathfrak{h}}}}}).$$

By Leibniz's rule, to prove $d_r = 0$, it suffices to see that $d_r|_{E_r^{0,q}} = 0$ for all $q \geqslant 0$. Moreover, by Fact 3.3.8 (2) and again by Leibniz's rule, we only need to prove that

- $d_r([1 \otimes 1] \otimes [1 \otimes \widetilde{R}]) = 0$ for any $R \in (S\mathfrak{k}_{\mathfrak{h}}^*)^{\mathfrak{k}_{\mathfrak{h}}}$.
- $d_r([1 \otimes 1] \otimes [\beta \otimes 1]) = 0$ for any $\beta \in (P_{\mathfrak{h}^*})^{-\theta}$.

By construction of the spectral sequence, we have $d_r([1 \otimes 1] \otimes [1 \otimes \widetilde{R}]) = 0$ and

$$d_r([1 \otimes 1] \otimes [\beta \otimes 1]) = \begin{cases} [1 \otimes \tau_{\mathfrak{h}}(\beta)] \otimes [1 \otimes 1] & \text{if } \beta \in (P_{\mathfrak{h}^*}^{r-1})^{-\theta}, \\ 0 & \text{if } \beta \in (P_{\mathfrak{h}^*}^{q})^{-\theta},\ q \neq r-1. \end{cases}$$

Since $\tau_{\mathfrak{h}}$ is taken to be compatible with θ, it follows that $\tau_{\mathfrak{h}}(\beta) \in ((S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}})^{-\theta}$. By (v), we have $\tau_{\mathfrak{h}}(\beta) \in (S^+\widetilde{\mathfrak{g}^*})^{\mathfrak{g}}|_{\mathfrak{h}} \cdot (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}$. This implies that $[1 \otimes \tau_{\mathfrak{h}}(\beta)] = 0$ in $H^\bullet(\Lambda P_{\mathfrak{g}^*} \otimes (S\widetilde{\mathfrak{h}^*})^{\mathfrak{h}}, -\delta_{\widetilde{\tau_{\mathfrak{g},\mathfrak{h}}}})$ by Proposition 3.3.7. We have thus proved $d_r = 0$.

(viii) ⇒ (iii). This follows immediately from Corollary 3.3.10 (3). □

Chapter 4

Semisimple symmetric spaces that do not model any compact manifold

4.1 Introduction

We continue the study of the existence problem of compact manifold locally modelled on homogeneous spaces. Recall that a manifold is said to be locally modelled on a homogeneous space G/H if it is covered by open sets that are diffeomorphic to open sets of G/H and the transition functions are given by elements of G. We always assume that the transition functions satisfy the cocycle condition (see Section 2.2). A basic example of a manifold locally modelled on G/H is a Clifford–Klein form, that is, a quotient space $\Gamma\backslash G/H$, where Γ is a discrete subgroup of G acting properly and freely on G/H.

We have proved the following fact in Chapter 2:

Fact 4.1.1 (cf. Corollary 2.5.1)**.** *Let G/H be a homogeneous space of reductive type and K_H a maximal compact subgroup of H. Let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}_H$ denote the Lie algebras of G, H and K_H, respectively. If the homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ induced from the inclusion map $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \hookrightarrow (\Lambda(\mathfrak{g}/\mathfrak{k}_H)^*)^{\mathfrak{k}_H}$ is not injective, then there does not exist a compact manifold locally modelled on the homogeneous space G/H (and, in particular, a compact Clifford–Klein form of G/H).*

The main purpose of this chapter is to classify the semisimple symmetric spaces G/H (or rather, the semisimple symmetric pairs $(\mathfrak{g}, \mathfrak{h})$) such that the homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ is injective.

Theorem 4.1.2. *Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair. Then the following two conditions are equivalent:*

(A) *The homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ induced from the inclusion map $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \hookrightarrow (\Lambda(\mathfrak{g}/\mathfrak{k}_H)^*)^{\mathfrak{k}_H}$ is injective.*

(B) *The pair* $(\mathfrak{g}, \mathfrak{h})$ *is isomorphic (up to possibly outer automorphisms) to a direct sum of the following irreducible symmetric pairs* (B-1)–(B-5).

(B-1) $(\mathfrak{l}, \mathfrak{l})$ *($\mathfrak{l}$: simple Lie algebra).*

(B-2) $(\mathfrak{l} \oplus \mathfrak{l}, \Delta\mathfrak{l})$ *($\mathfrak{l}$: simple Lie algebra).*

(B-3) $(\mathfrak{l}_{\mathbb{C}}, \mathfrak{l})$ *($\mathfrak{l}_{\mathbb{C}}$: complex simple Lie algebra, $\mathfrak{l}$: real form of $\mathfrak{l}_{\mathbb{C}}$).*

(B-4) *A pair* $(\mathfrak{g}', \mathfrak{h}')$ *such that* $\operatorname{rank} \mathfrak{h}' = \operatorname{rank} \mathfrak{k}_{H'}$*, where* $\mathfrak{k}_{H'}$ *is a maximal compact subalgebra of* $\mathfrak{h}'$*.*

(B-5)
- $(\mathfrak{sl}(2n+1, \mathbb{C}), \mathfrak{so}(2n+1, \mathbb{C}))$ $(n \geqslant 1)$,
- $(\mathfrak{sl}(2n, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C}))$ $(n \geqslant 2)$,
- $(\mathfrak{so}(2n, \mathbb{C}), \mathfrak{so}(2n-1, \mathbb{C}))$ $(n \geqslant 4)$,
- $(\mathfrak{e}_{6,\mathbb{C}}, \mathfrak{f}_{4,\mathbb{C}})$.

By Fact 4.1.1, there exists a compact manifold locally modelled on a semisimple symmetric space G/H only when the corresponding semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$ satisfies (B).

Remark 4.1.3. Every irreducible symmetric space G/H listed in (B-1)–(B-2) admits a compact Clifford–Klein form. On the other hand, many irreducible symmetric spaces listed in (B-3)–(B-5) do not admit compact Clifford–Klein forms, whereas some of them admit (see [2], [20], [23], [26], [31], [47], [56] and Chapter 5). For instance, $\mathrm{SO}(2n+1, \mathbb{C})/\mathrm{SO}(n+1, n)$ $(n \geqslant 1)$, $\mathrm{SO}(p+1, q)/\mathrm{SO}(p, q)$ ($p \geqslant q \geqslant 1$, pq: even) and $\mathrm{SO}(4n+2, \mathbb{C})/\mathrm{SO}(4n+1, \mathbb{C})$ $(n \geqslant 1)$ do not admit compact Clifford–Klein forms, whereas $\mathrm{SO}(8, \mathbb{C})/\mathrm{SO}(1, 7)$, $\mathrm{SO}(2, 2n)/\mathrm{SO}(1, 2n)$ $(n \geqslant 2)$ and $\mathrm{SO}(8, \mathbb{C})/\mathrm{SO}(7, \mathbb{C})$ admit. To the best of the author's knowledge, no known method is applicable to (B-3)–(B-5) for a more general case of compact manifolds locally modelled on G/H.

In Table 4.1, we list all the irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ that do not satisfy the condition (B) among Berger's classification of the irreducible symmetric pairs [4]. By Fact 4.1.1, there does not exist a compact manifold locally modelled on an irreducible symmetric space G/H if the corresponding pair $(\mathfrak{g}, \mathfrak{h})$ is listed in Table 4.1.

	$\mathfrak{g}$	$\mathfrak{h}$	Conditions
$\star$	$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{so}(2n, \mathbb{C})$	$n \geqslant 1$
	$\mathfrak{sl}(p+q, \mathbb{C})$	$\mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(q, \mathbb{C}) \oplus \mathbb{C}$	$p, q \geqslant 1$
$\star\star$	$\mathfrak{sl}(p+q, \mathbb{R})$	$\mathfrak{so}(p, q)$	$p, q \geqslant 1$, p, q: odd
$\star$	$\mathfrak{su}(p, q)$	$\mathfrak{so}(p, q)$	$p, q \geqslant 1$, p, q: odd
	$\mathfrak{su}(n, n)$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$	$n \geqslant 1$
$\star\star$	$\mathfrak{sl}(2n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \sqrt{-1}\mathbb{R}$	$n \geqslant 2$

$\star$	$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sl}(n,\mathbb{C})\oplus\sqrt{-1}\mathbb{R}$	$n\geqslant 2$
	$\mathfrak{sl}(p+q,\mathbb{R})$	$\mathfrak{sl}(p,\mathbb{R})\oplus\mathfrak{sl}(q,\mathbb{R})\oplus\mathbb{R}$	$p,q\geqslant 1$
	$\mathfrak{sl}(p+q,\mathbb{H})$	$\mathfrak{sl}(p,\mathbb{H})\oplus\mathfrak{sl}(q,\mathbb{H})\oplus\mathbb{R}$	$p,q\geqslant 1$
$\star$	$\mathfrak{so}(p+q,\mathbb{C})$	$\mathfrak{so}(p,\mathbb{C})\oplus\mathfrak{so}(q,\mathbb{C})$	$p,q\geqslant 2$, $(p,q)\neq(2,2)$
$\star$	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(2n,\mathbb{C})$	$n\geqslant 1$
	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{C}$	$n\geqslant 3$
$\star\star$	$\mathfrak{so}(n,n)$	$\mathfrak{so}(n,\mathbb{C})$	$n\geqslant 3$
$\star$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n,\mathbb{C})$	$n\geqslant 3$
$\star\star$	$\mathfrak{so}(p+r,q+s)$	$\mathfrak{so}(p,q)\oplus\mathfrak{so}(r,s)$	$p,q\geqslant 1$, p,q: odd, $r,s\geqslant 0$, $(r,s)\neq(0,0)$, $(p,q,r,s)\neq(1,1,1,1)$
	$\mathfrak{so}(n,n)$	$\mathfrak{sl}(n,\mathbb{R})\oplus\mathbb{R}$	$n\geqslant 3$
	$\mathfrak{so}^*(4n)$	$\mathfrak{sl}(n,\mathbb{H})\oplus\mathbb{R}$	$n\geqslant 2$
	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{C}$	$n\geqslant 1$
$\star$	$\mathfrak{sp}(p+q,\mathbb{C})$	$\mathfrak{sp}(p,\mathbb{C})\oplus\mathfrak{sp}(q,\mathbb{C})$	$p,q\geqslant 1$
$\star$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{C})$	$n\geqslant 1$
$\star$	$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbb{C})$	$n\geqslant 1$
	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R})\oplus\mathbb{R}$	$n\geqslant 1$
	$\mathfrak{sp}(n,n)$	$\mathfrak{sl}(n,\mathbb{H})\oplus\mathbb{R}$	$n\geqslant 1$
$\star$	$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{sp}(4,\mathbb{C})$	—
$\star$	$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{sl}(6,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	—
	$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{so}(10,\mathbb{C})\oplus\mathbb{C}$	—
$\star$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}(6,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$	—
$\star\star$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}(3,\mathbb{H})\oplus\mathfrak{su}(2)$	—
$\star$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{sl}(3,\mathbb{H})\oplus\mathfrak{su}(2)$	—
	$\mathfrak{e}_{6(6)}$	$\mathfrak{so}(5,5)\oplus\mathbb{R}$	—
	$\mathfrak{e}_{6(-26)}$	$\mathfrak{so}(9,1)\oplus\mathbb{R}$	—
$\star$	$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{sl}(8,\mathbb{C})$	—
$\star$	$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{so}(12,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	—
	$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{6,\mathbb{C}}\oplus\mathbb{C}$	—
	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(8,\mathbb{R})$	—
	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(4,\mathbb{H})$	—
	$\mathfrak{e}_{7(-25)}$	$\mathfrak{sl}(4,\mathbb{H})$	—
	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(6)}\oplus\mathbb{R}$	—
	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)}\oplus\mathbb{R}$	—
$\star$	$\mathfrak{e}_{8,\mathbb{C}}$	$\mathfrak{so}(16,\mathbb{C})$	—
$\star$	$\mathfrak{e}_{8,\mathbb{C}}$	$\mathfrak{e}_{7,\mathbb{C}}\oplus\mathfrak{sl}(2,\mathbb{C})$	—
$\star$	$\mathfrak{f}_{4,\mathbb{C}}$	$\mathfrak{sp}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	—
$\star$	$\mathfrak{f}_{4,\mathbb{C}}$	$\mathfrak{so}(9,\mathbb{C})$	—
$\star$	$\mathfrak{g}_{2,\mathbb{C}}$	$\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	—

Table 4.1: $(\mathfrak{g},\mathfrak{h})$ not satisfying (B)

In Table 4.1, the signs $\star\star$, $\star$ and $\circ$ signify

$\star\star$: The nonexistence of compact Clifford–Klein forms of G/H seems to be new.

$\star$: The nonexistence of compact Clifford–Klein forms of G/H had been known before [41], but not for the locally modelled case.

$\circ$: The nonexistence of compact manifolds locally modelled on G/H had been known before [41].

Note that we saw in [41, Cor. 1.4] the nonexistence of compact Clifford–Klein forms of $\star\star$ except for the case $(\mathfrak{e}_{6(6)}, \mathfrak{sl}(3, \mathbb{H}) \oplus \mathfrak{su}(2))$.

The proof of Theorem 4.1.2 uses Berger's classification of the irreducible symmetric pairs and a necessary and sufficient condition for injectivity of the homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ obtained in Chapter 3 (Theorem 3.1.3).

The results of this chapter were announced in [42].

Remark 4.1.4. We correct some minor errors in the announcement [42]:

- The pairs $(\mathfrak{sp}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{C}))$ $(n \geqslant 1)$ and $(\mathfrak{sp}(n, n), \mathfrak{sp}(n, \mathbb{C}))$ $(n \geqslant 1)$ should be labelled as $\star$.
- The pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g} = \mathfrak{so}(4, \mathbb{C}), \mathfrak{so}(2, 2)$ or $\mathfrak{so}^*(4)$ should not be listed because they are not irreducible.

4.2 Preliminaries

We say that $(\mathfrak{g}, \mathfrak{h})$ is a (real) reductive pair if $\mathfrak{g}$ is a reductive Lie algebra with Cartan involution θ and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ such that $\theta(\mathfrak{h}) = \mathfrak{h}$. We put $\mathfrak{k} = \mathfrak{g}^\theta$ and $\mathfrak{k}_H = \mathfrak{h}^\theta$. Similarly, we say that a homogeneous space G/H is of reductive type if G is a linear reductive Lie group with Cartan involution θ and H is a closed subgroup of G with finitely many connected components such that $\theta(H) = H$. We put $K = G^\theta$ and $K_H = H^\theta$. Note that K and K_H are maximal compact subgroups of G and H, respectively.

If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h} = \mathfrak{g}^\sigma$ for some involution σ of $\mathfrak{g}$, we call $(\mathfrak{g}, \mathfrak{h})$ a semisimple symmetric pair. Similarly, if G is a connected linear semisimple Lie group and H is an open subgroup of G^σ for some involution σ of G, we call G/H a semisimple symmetric space. In these situations, we can take the Cartan involution θ so that $\theta\sigma = \sigma\theta$. Therefore, every semisimple symmetric pair is a reductive pair, and every semisimple symmetric space is a homogeneous space of reductive type. We say that a semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is an irreducible symmetric pair if $\mathfrak{g}$ is simple or $(\mathfrak{g}, \mathfrak{h})$ is isomorphic (up to possibly outer automorphisms) to $(\mathfrak{l} \oplus \mathfrak{l}, \Delta\mathfrak{l})$ for some simple Lie algebra $\mathfrak{l}$. A semisimple symmetric space is called an irreducible symmetric space if the corresponding semisimple symmetric pair is irreducible. Every semisimple symmetric pair is uniquely decomposed into irreducible ones. The complete classification of the irreducible symmetric pairs (up to possibly outer automorphisms) is obtained by Berger [4].

Now, let us recall from [16] and Section 3.3 the definition and basic properties of $P_{\mathfrak{g}^*}$. Let $\mathfrak{g}$ be a (real or complex) reductive Lie algebra. It is classicaly known that $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ has a natural Hopf algebra structure. We denote by $P_{\mathfrak{g}^*} = \bigoplus_{k\geqslant 1} P^{2k-1}_{\mathfrak{g}^*}$ the space of primitive elements in $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$. An involution θ of the reductive Lie algebra $\mathfrak{g}$ acts on the space $P_{\mathfrak{g}^*}$ of primitive elements in the natural way. Thus we have the eigenspace decomposition: $P_{\mathfrak{g}^*} = (P_{\mathfrak{g}^*})^{\theta} \oplus (P_{\mathfrak{g}^*})^{-\theta}$.

The Cartan map $\rho_{\mathfrak{g}} : (S^k\mathfrak{g}^*)^{\mathfrak{g}} \to (\Lambda^{2k-1}\mathfrak{g})^{\mathfrak{g}}$ $(k \geqslant 1)$ induces a linear isomorphism

$$\overline{\rho_{\mathfrak{g}}} : (S^+\mathfrak{g}^*)^{\mathfrak{g}}/((S^+\mathfrak{g}^*)^{\mathfrak{g}} \cdot (S^+\mathfrak{g}^*)^{\mathfrak{g}}) \xrightarrow{\sim} P_{\mathfrak{g}^*}.$$

The isomorphism $\overline{\rho_{\mathfrak{g}}}$ commutes with any involution θ of $\mathfrak{g}$:

$$\begin{array}{ccc} (S^+\mathfrak{g}^*)^{\mathfrak{g}}/((S^+\mathfrak{g}^*)^{\mathfrak{g}} \cdot (S^+\mathfrak{g}^*)^{\mathfrak{g}}) & \xrightarrow[\overline{\rho_{\mathfrak{g}}}]{\sim} & P_{\mathfrak{g}^*} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ (S^+\mathfrak{g}^*)^{\mathfrak{g}}/((S^+\mathfrak{g}^*)^{\mathfrak{g}} \cdot (S^+\mathfrak{g}^*)^{\mathfrak{g}}) & \xrightarrow[\overline{\rho_{\mathfrak{g}}}]{\sim} & P_{\mathfrak{g}^*}. \end{array}$$

If $(\mathfrak{g}, \mathfrak{h})$ is a reductive pair, the restriction map $\mathrm{rest} : (\Lambda\mathfrak{g}^*)^{\mathfrak{g}} \to (\Lambda\mathfrak{h}^*)^{\mathfrak{h}}$ induces a linear map

$$\mathrm{rest} : P_{\mathfrak{g}^*} \to P_{\mathfrak{h}^*}.$$

Similarly, $\mathrm{rest} : (S\mathfrak{g}^*)^{\mathfrak{g}} \to (S\mathfrak{h}^*)^{\mathfrak{h}}$ induces a linear map

$$\overline{\mathrm{rest}} : (S^+\mathfrak{g}^*)^{\mathfrak{g}}/((S^+\mathfrak{g}^*)^{\mathfrak{g}} \cdot (S^+\mathfrak{g}^*)^{\mathfrak{g}}) \to (S^+\mathfrak{h}^*)^{\mathfrak{h}}/((S^+\mathfrak{h}^*)^{\mathfrak{h}} \cdot (S^+\mathfrak{h}^*)^{\mathfrak{h}}).$$

The following diagram commutes:

$$\begin{array}{ccc} (S^+\mathfrak{g}^*)^{\mathfrak{g}}/((S^+\mathfrak{g}^*)^{\mathfrak{g}} \cdot (S^+\mathfrak{g}^*)^{\mathfrak{g}}) & \xrightarrow[\overline{\rho_{\mathfrak{g}}}]{\sim} & P_{\mathfrak{g}^*} \\ \downarrow{\scriptstyle\overline{\mathrm{rest}}} & & \downarrow{\scriptstyle\mathrm{rest}} \\ (S^+\mathfrak{h}^*)^{\mathfrak{h}}/((S^+\mathfrak{h}^*)^{\mathfrak{h}} \cdot (S^+\mathfrak{h}^*)^{\mathfrak{h}}) & \xrightarrow[\overline{\rho_{\mathfrak{h}}}]{\sim} & P_{\mathfrak{h}^*}. \end{array}$$

For a reductive Lie algebra $\mathfrak{g}$, the algebra $(S\mathfrak{g}^*)^{\mathfrak{g}}$ is a polynomial algebra of r homogeneous elements ($r = \mathrm{rank}\,\mathfrak{g}$). the symmetric algebra of some r-dimensional graded subspace of $(S\mathfrak{g}^*)^{\mathfrak{g}}$. For a complex simple Lie algebra $\mathfrak{g}$, the degrees of algebraically independent generators of $(S\mathfrak{g}^*)^{\mathfrak{g}}$ are as follows (see e.g. [57, p. 144]):

$\mathfrak{g}$	Degrees
$\mathfrak{sl}(n,\mathbb{C})$	$2,3,\ldots,n$
$\mathfrak{so}(2n+1,\mathbb{C})$	$2,4,\ldots,2n$
$\mathfrak{sp}(n,\mathbb{C})$	$2,4,\ldots,2n$
$\mathfrak{so}(2n,\mathbb{C})$	$2,4,\ldots,2n-2,n$
$\mathfrak{e}_{6,\mathbb{C}}$	$2,5,6,8,9,12$
$\mathfrak{e}_{7,\mathbb{C}}$	$2,6,8,10,12,14,18$
$\mathfrak{e}_{8,\mathbb{C}}$	$2,8,12,14,18,20,24,30$
$\mathfrak{f}_{4,\mathbb{C}}$	$2,6,8,12$
$\mathfrak{g}_{2,\mathbb{C}}$	$2,6$

Table 4.2: Degrees of generators of $(S\mathfrak{g}^*)^{\mathfrak{g}}$

Through the Cartan map $\rho_{\mathfrak{g}}$, we immediately obtain the degrees of a basis of $P_{\mathfrak{g}^*}$:

$\mathfrak{g}$	Degrees
$\mathfrak{sl}(n,\mathbb{C})$	$3,5,7,\ldots,2n-1$
$\mathfrak{so}(2n+1,\mathbb{C})$	$3,7,11,\ldots,4n-1$
$\mathfrak{sp}(n,\mathbb{C})$	$3,7,11,\ldots,4n-1$
$\mathfrak{so}(2n,\mathbb{C})$	$3,7,11,\ldots,4n-5,2n-1$
$\mathfrak{e}_{6,\mathbb{C}}$	$3,9,11,15,17,23$
$\mathfrak{e}_{7,\mathbb{C}}$	$3,11,15,19,23,27,35$
$\mathfrak{e}_{8,\mathbb{C}}$	$3,15,23,27,35,39,47,59$
$\mathfrak{f}_{4,\mathbb{C}}$	$3,11,15,23$
$\mathfrak{g}_{2,\mathbb{C}}$	$3,11$

Table 4.3: Degrees of a basis of $P_{\mathfrak{g}^*}$

If $\mathfrak{g}$ is abelian, then the graded algebra $(S\mathfrak{g}^*)^{\mathfrak{g}} = S\mathfrak{g}^*$ is generated by the elements of degree 1, and hence the graded vector space $P_{\mathfrak{g}^*}$ is concentrated in degree 1.

For the classical cases, the structure of graded algebra $(S\mathfrak{g}^*)^{\mathfrak{g}}$ is explicitly described as follows:

Fact 4.2.1 (see e.g. [9, Ch. VIII, §13]). *Let* $f_k \in (S^k(\mathfrak{gl}(n,\mathbb{C})^*))^{\mathfrak{gl}(n,\mathbb{C})}$ $(k = 1,2,\ldots,n)$ *denote invariant polynomials defined by*

$$\det(\lambda I_n - X) = \lambda^n + f_1(X)\lambda^{n-1} + f_2(X)\lambda^{n-2} + \cdots + f_n(X) \quad (X \in \mathfrak{gl}(n,\mathbb{C})).$$

We use the same notation f_k for the restriction of f_k to $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$ or $\mathfrak{sp}(m,\mathbb{C})$ (if $n=2m$). Then,

- *The graded algebra $(S(\mathfrak{sl}(n,\mathbb{C})^*))^{\mathfrak{sl}(n,\mathbb{C})}$ is the polynomial algebra of $(n-1)$ variables $f_2, f_3, \dots, f_n$. We have $f_1 = 0$.*
- *If $n = 2m+1$, the graded algebra $(S(\mathfrak{so}(n,\mathbb{C})^*))^{\mathfrak{so}(n,\mathbb{C})}$ is the polynomial algebra of m variables $f_2, f_4, \dots, f_{2m}$. We have $f_1 = f_3 = \cdots = f_{2m+1} = 0$.*
- *If $n = 2m$, the graded algebra $(S(\mathfrak{sp}(n,\mathbb{C})^*))^{\mathfrak{sp}(n,\mathbb{C})}$ is the polynomial algebra of m variables $f_2, f_4, \dots, f_{2m}$. We have $f_1 = f_3 = \cdots = f_{2m-1} = 0$.*
- *If $n = 2m$, the graded algebra $(S(\mathfrak{so}(n,\mathbb{C})^*))^{\mathfrak{so}(n,\mathbb{C})}$ is the polynomial algebra of m variables $f_2, f_4, \dots, f_{2m-2}, \tilde{f}$, where $\tilde{f} \in (S^m(\mathfrak{so}(n,\mathbb{C})^*))^{\mathfrak{so}(n,\mathbb{C})}$ is the Pfaffian of $n \times n$ skew-symmeric matrices. We have $f_1 = f_3 = \cdots = f_{2m-1} = 0$ and $f_{2m} = \tilde{f}^2$.*

The following fact, proved in Chapter 3, plays a foundational role in our classification:

Fact 4.2.2 (Theorem 3.1.3)**.** *Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive pair with Cartan involution θ. Then, the homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ is injective if and only if the linear map* $\mathrm{rest} : (P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ *is surjective.*

See Theorem 3.4.1 for some other conditions equivalent to injectivity of $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$.

Fact 4.2.3 (cf. [16, Ch. X, §7])**.** *Let $\mathfrak{g}$ be a real reductive Lie algebra with Cartan involution θ. Let $\rho_{\mathfrak{g}} : (S^k\mathfrak{g}^*)^{\mathfrak{g}} \to (\Lambda^{2k-1}\mathfrak{g})^{\mathfrak{g}}$ $(k \geqslant 1)$ be the Cartan map for $\mathfrak{g}$. Then,*

(1) $(P_{\mathfrak{g}^*})^{-\theta} = \{\rho_{\mathfrak{g}}(P) : P \in (S^+\mathfrak{g}^*)^{\mathfrak{g}},\ P|_{\mathfrak{k}} = 0\}$.

(2) $\dim(P_{\mathfrak{g}^*})^{-\theta} = \mathrm{rank}\,\mathfrak{g} - \mathrm{rank}\,\mathfrak{k}$.

Proposition 4.2.4. *For a simple Lie algebra $\mathfrak{g}$ with Cartan involution θ, the degrees of a basis of $(P_{\mathfrak{g}^*})^{-\theta}$ are as follows:*

$\mathfrak{g}$	Degrees
$\mathfrak{sl}(n,\mathbb{R})$	$5, 9, 13, \dots, 4[\frac{n+1}{2}] - 3$
$\mathfrak{sl}(n,\mathbb{H})$	$5, 9, 13, \dots, 4n-3$
$\mathfrak{so}(p,q)$ (p, q: odd)	$p+q-1$
$\mathfrak{e}_{6(6)}$	$9, 17$
$\mathfrak{e}_{6(-26)}$	$9, 17$

$\mathfrak{sl}(n,\mathbb{C})$	$3,5,7,\dots,2n-1$
$\mathfrak{so}(2n+1,\mathbb{C})$	$3,7,11,\dots,4n-1$
$\mathfrak{sp}(n,\mathbb{C})$	$3,7,11,\dots,4n-1$
$\mathfrak{so}(2n,\mathbb{C})$	$3,7,11,\dots,4n-5,2n-1$
$\mathfrak{e}_{6,\mathbb{C}}$	$3,9,11,15,17,23$
$\mathfrak{e}_{7,\mathbb{C}}$	$3,11,15,19,23,27,35$
$\mathfrak{e}_{8,\mathbb{C}}$	$3,15,23,27,35,39,47,59$
$\mathfrak{f}_{4,\mathbb{C}}$	$3,11,15,23$
$\mathfrak{g}_{2,\mathbb{C}}$	$3,11$
Otherwise	—

Table 4.4: Degrees of a basis of $(P_{\mathfrak{g}^*})^{-\theta}$

Proof. Although this proposition seems to be already known in the early 1960s (cf. [55]), we give its proof for the reader's convenience.

If $\operatorname{rank}\mathfrak{g} = \operatorname{rank}\mathfrak{k}$, we have $(P_{\mathfrak{g}^*})^{-\theta} = \{0\}$ by Fact 4.2.3 (2). Thus we assume $\operatorname{rank}\mathfrak{g} \neq \operatorname{rank}\mathfrak{k}$. It suffices to compute the degrees of a basis of the complexification $(P_{\mathfrak{g}^*})^{-\theta} \otimes \mathbb{C} = (P_{\mathfrak{g}_{\mathbb{C}}^*})^{-\theta}$.

Suppose that $\mathfrak{g}$ is a complex simple Lie algebra. We use the notation $\mathfrak{g}_{\mathbb{R}}$ when we regard $\mathfrak{g}$ as a real simple Lie algebra. We have a natural isomorphism $(\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}} \simeq \mathfrak{g} \oplus \mathfrak{g}$. The Cartan involution θ acts on $P_{(\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}^*} \simeq P_{\mathfrak{g}^*} \oplus P_{\mathfrak{g}^*}$ by $(\alpha_1,\alpha_2) \mapsto (\alpha_2,\alpha_1)$ $(\alpha_1,\alpha_2 \in P_{\mathfrak{g}^*})$. Therefore $\dim(P^k_{(\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}^*})^{-\theta} = \dim(P^k_{\mathfrak{g}^*})$ for every $k \in \mathbb{N}$, and Proposition 4.2.4 follows from Table 4.3.

Suppose that $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$. Then $(\mathfrak{g}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}) = (\mathfrak{sl}(n,\mathbb{C}),\mathfrak{so}(n,\mathbb{C}))$. Let $f_k \in (S^k\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}$ $(2 \leqslant k \leqslant n)$ be as in Fact 4.2.1. The restriction of f_k to $\mathfrak{k}_{\mathbb{C}}$ vanishes if and only if $k = 3,5,\dots,2[\frac{n+1}{2}]-1$. Since the images of f_k $(2 \leqslant k \leqslant n)$ under the Cartan map form a basis of $P_{\mathfrak{g}_{\mathbb{C}}^*}$, we obtain from Fact 4.2.3 (1) that the degrees of a basis of $(P_{\mathfrak{g}_{\mathbb{C}}^*})^{-\theta}$ are $5,9,13,\dots,4[\frac{n+1}{2}]-3$.

Suppose that $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{H})$. Then $(\mathfrak{g}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}) = (\mathfrak{sl}(2n,\mathbb{C}),\mathfrak{sp}(n,\mathbb{C}))$. The restriction of $f_k \in (S^k\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}$ $(2 \leqslant k \leqslant 2n)$ to $\mathfrak{k}_{\mathbb{C}}$ vanishes if and only if $k = 3,5,\dots,2n-1$. Thus, the degrees of a basis of $(P_{\mathfrak{g}_{\mathbb{C}}^*})^{-\theta}$ are $5,9,13,\dots,4n-3$.

Suppose that $\mathfrak{g} = \mathfrak{so}(p,q)$ $(p,q$: odd$)$. Then $(\mathfrak{g}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}) = (\mathfrak{so}(p+q,\mathbb{C}),\mathfrak{so}(p,\mathbb{C}) \oplus \mathfrak{so}(q,\mathbb{C}))$. The restriction of $f_{2k} \in (S^{2k}\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}$ $(1 \leqslant k \leqslant \frac{p+q}{2}-1)$ to $\mathfrak{k}_{\mathbb{C}}$ is nonzero for every k, and that of $\tilde{f} \in (S^{(p+q)/2}\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}$ is zero. Thus, $(P_{\mathfrak{g}_{\mathbb{C}}^*})^{-\theta}$ is a 1-dimensional vector space concentrated in degree $p+q-1$.

Suppose that $\mathfrak{g} = \mathfrak{e}_{6(6)}$. Then $(\mathfrak{g}_{\mathbb{C}},\mathfrak{k}_{\mathbb{C}}) = (\mathfrak{e}_{6,\mathbb{C}},\mathfrak{sp}(4,\mathbb{C}))$. Let us fix algebraically independent generators $g_2,g_5,g_6,g_8,g_9,g_{12}$ $(\deg g_k = k)$ of the algebra $(S\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}$ (cf. Table 4.2). Notice from Table 4.2 that $(S^{2k+1}\mathfrak{k}_{\mathbb{C}}^*)^{\mathfrak{k}_{\mathbb{C}}} = 0$. Therefore $g_5|_{\mathfrak{k}_{\mathbb{C}}} = g_9|_{\mathfrak{k}_{\mathbb{C}}} = 0$. By Fact 4.2.3 (1), the images of g_5 and g_9 under the Cartan map are nonzero elements of $(P_{\mathfrak{g}_{\mathbb{C}}^*})^{-\theta}$. Their degrees are 9 and

17, respectively. Since $\dim(P_{\mathfrak{g}^*_{\mathbb{C}}})^{-\theta} = \operatorname{rank} \mathfrak{g}_{\mathbb{C}} - \operatorname{rank} \mathfrak{k}_{\mathbb{C}} = 2$ (Fact 4.2.3 (2)), they form a basis of $(P_{\mathfrak{g}^*_{\mathbb{C}}})^{-\theta}$.

Finally, suppose that $\mathfrak{g} = \mathfrak{e}_{6(-26)}$. Then $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}) = (\mathfrak{e}_{6,\mathbb{C}}, \mathfrak{f}_{4,\mathbb{C}})$. Again, $(S^{2k+1}\mathfrak{k}^*_{\mathbb{C}})^{\mathfrak{k}_{\mathbb{C}}} = 0$ and $\operatorname{rank} \mathfrak{g}_{\mathbb{C}} - \operatorname{rank} \mathfrak{k}_{\mathbb{C}} = 2$. The degrees of $(P_{\mathfrak{g}^*_{\mathbb{C}}})^{-\theta}$ are hence 9 and 17 by the same argument as the case of $\mathfrak{g} = \mathfrak{e}_{6(6)}$. □

4.3 Some results on surjectivity of the restriction map $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$

In view of Fact 4.2.2, our task is to study the restriction map $\operatorname{rest} : (P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$.

As we saw in Chapter 3, we obtain the following necessary condition for surjectivity of the restriction map $\operatorname{rest} : (P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ from Fact 4.2.3 (2):

Proposition 4.3.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive pair with Cartan involution θ. If $\operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{k} < \operatorname{rank} \mathfrak{h} - \operatorname{rank} \mathfrak{k}_H$, the restriction map $\operatorname{rest} : (P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is not surjective.*

Proof. By Fact 4.2.3 (2),

$$\dim(P_{\mathfrak{g}^*})^{-\theta} = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{k} < \operatorname{rank} \mathfrak{h} - \operatorname{rank} \mathfrak{k}_H = \dim(P_{\mathfrak{h}^*})^{-\theta}.$$ □

Corollary 4.3.2 (Kobayashi's rank conjecture; see Chapter 3 and [56, ver. 2, Th. 6])**.** *If a homogeneous space G/H of reductive type satisfies $\operatorname{rank} G - \operatorname{rank} K < \operatorname{rank} H - \operatorname{rank} K_H$, there does not exist a compact manifold locally modelled on G/H.*

Proof. Combine Fact 4.1.1, Fact 4.2.2 and Proposition 4.3.1. □

Similarly, Proposition 4.2.4 gives some necessary conditions for surjectivity of the restriction map $\operatorname{rest} : (P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$. Let $\mathfrak{g}$ be a reductive Lie algebra and write $\mathfrak{g}$ as

$$\mathfrak{g} \simeq \mathbb{R}^{n^+} \oplus \sqrt{-1}\mathbb{R}^{n^-} \oplus \bigoplus_{\substack{\mathfrak{l}\text{: simple} \\ \text{Lie algebra}}} m_{\mathfrak{l}} \cdot \mathfrak{l}.$$

We then put

$$
\begin{aligned}
d_1(\mathfrak{g}) &= n^+, \\
d_2(\mathfrak{g}) &= \sum_{\substack{\mathfrak{l}:\ \text{complex simple} \\ \text{Lie algebra}}} m_{\mathfrak{l}}, \\
d_3(\mathfrak{g}) &= \sum_{k\geqslant 3} m_{\mathfrak{sl}(k,\mathbb{R})} + \sum_{k\geqslant 3} m_{\mathfrak{sl}(k,\mathbb{C})} + \sum_{k\geqslant 2} m_{\mathfrak{sl}(k,\mathbb{H})}, \\
d_4(\mathfrak{g}) &= m_{\mathfrak{so}(7,1)} + m_{\mathfrak{so}(5,3)} + 2m_{\mathfrak{so}(8,\mathbb{C})} \\
&\qquad + \sum_{k\geqslant 4} m_{\mathfrak{sl}(k,\mathbb{C})} + \sum_{k\geqslant 7,\, k\neq 8} m_{\mathfrak{so}(k,\mathbb{C})} + \sum_{k\geqslant 2} m_{\mathfrak{sp}(k,\mathbb{C})}.
\end{aligned}
$$

Remark 4.3.3. When we compute $d_k(\mathfrak{g})$, we have to be careful in the following accidental isomorphisms:

- $d_1(\mathfrak{g})$: $\mathfrak{so}(2,\mathbb{C}) \simeq \mathbb{C} = \mathbb{R} \oplus \sqrt{-1}\mathbb{R}$, $\mathfrak{so}(1,1) \simeq \mathbb{R}$.
- $d_2(\mathfrak{g})$: $\mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C})$,
- $d_3(\mathfrak{g})$: $\mathfrak{so}(6,\mathbb{C}) \simeq \mathfrak{sl}(4,\mathbb{C})$, $\mathfrak{so}(3,3) \simeq \mathfrak{sl}(4,\mathbb{R})$, $\mathfrak{so}(5,1) \simeq \mathfrak{sl}(2,\mathbb{H})$.
- $d_4(\mathfrak{g})$: $\mathfrak{so}(5,\mathbb{C}) \simeq \mathfrak{sp}(2,\mathbb{C})$, $\mathfrak{so}(6,\mathbb{C}) \simeq \mathfrak{sl}(4,\mathbb{C})$.

Proposition 4.3.4. *Let $(\mathfrak{g},\mathfrak{h})$ be a reductive pair with Cartan involution θ. If $d_k(\mathfrak{g}) < d_k(\mathfrak{h})$ for some $1 \leqslant k \leqslant 4$, the restriction map* rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ *is not surjective.*

Proof. It follows from Proposition 4.2.4 that $d_k(\mathfrak{g}) = \dim(P^{2k-1}_{\mathfrak{g}^*})^{-\theta}$ and $d_k(\mathfrak{h}) = \dim(P^{2k-1}_{\mathfrak{h}^*})^{-\theta}$ $(1 \leqslant k \leqslant 4)$. □

Corollary 4.3.5. *If a homogeneous space G/H of reductive type satisfies $d_k(\mathfrak{g}) < d_k(\mathfrak{h})$ for some $1 \leqslant k \leqslant 4$, there does not exist a compact manifold locally modelled on G/H.*

Proof. Combine Fact 4.1.1, Fact 4.2.2 and Proposition 4.3.4. □

Let $(\mathfrak{g},\mathfrak{h})$ be a semisimple symmetric pair defined by an involution σ. Extend σ to the complex linear involution of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$, and set $\mathfrak{g}^c = \mathfrak{g}^\sigma \oplus \sqrt{-1}\mathfrak{g}^{-\sigma} \subset \mathfrak{g}_{\mathbb{C}}$. Then $(\mathfrak{g}^c,\mathfrak{h})$ becomes a semisimple symmetric pair with involution σ. It is called the c-dual of the semisimple symmetric pair $(\mathfrak{g},\mathfrak{h})$. Note that $\mathfrak{g}^{cc} = \mathfrak{g}$. The c-dual of an irreducible symmetric pair is again irreducible.

Proposition 4.3.6. *Let $(\mathfrak{g},\mathfrak{h})$ be a semisimple symmetric pair and $(\mathfrak{g}^c,\mathfrak{h})$ its c-dual. Then, the restriction map* rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ *is surjective if and only if* rest : $(P_{(\mathfrak{g}^c)^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ *is surjective.*

Proof. Recall that $\mathfrak{g}^c$ and $\mathfrak{g}$ have the same complexification $\mathfrak{g}_{\mathbb{C}}$. Therefore, the linear maps

$$\mathrm{rest} : (P_{\mathfrak{g}^*} \otimes \mathbb{C})^{-\theta} \to (P_{\mathfrak{h}^*} \otimes \mathbb{C})^{-\theta}$$

and

$$\mathrm{rest} : (P_{(\mathfrak{g}^c)^*} \otimes \mathbb{C})^{-\theta} \to (P_{\mathfrak{h}^*} \otimes \mathbb{C})^{-\theta}$$

are the same. Since surjectivity is stable under complexification, we have the desired equivalence. □

A reductive pair $(\mathfrak{g}, \mathfrak{h})$ is called complex if $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{h}$ is a complex subalgebra of $\mathfrak{g}$. We remark that the Cartan involution θ of $\mathfrak{g}$ is antilinear in this case. We use the notation $\mathfrak{g}_{\mathbb{R}}$ (resp. $\mathfrak{h}_{\mathbb{R}}$) when we regard $\mathfrak{g}$ (resp. $\mathfrak{h}$) as a real Lie algebra.

Proposition 4.3.7. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a complex reductive pair with Cartan involution* θ*. Then, the following three conditions are equiavlent:*

(1) *The (real) linear map* $\mathrm{rest} : (P_{\mathfrak{g}_{\mathbb{R}}^*})^{-\theta} \to (P_{\mathfrak{h}_{\mathbb{R}}^*})^{-\theta}$ *is surjective.*

(2) *The (complex) linear map* $\mathrm{rest} : P_{\mathfrak{g}^*} \to P_{\mathfrak{h}^*}$ *is surjective.*

(3) *The graded algebra homomorphism* $\mathrm{rest} : (S\mathfrak{g}^*)^{\mathfrak{g}} \to (S\mathfrak{h}^*)^{\mathfrak{h}}$ *is surjective.*

Proof. (1) ⇔ (2). As we saw in the proof of Proposition 4.2.4, the Cartan involution θ acts on $P_{\mathfrak{g}_{\mathbb{R}}^*} \otimes \mathbb{C} \simeq P_{\mathfrak{g}^*} \oplus P_{\mathfrak{g}^*}$ by $(\alpha_1, \alpha_2) \mapsto (\alpha_2, \alpha_1)$ $(\alpha_1, \alpha_2 \in P_{\mathfrak{g}^*})$. Thus $(P_{\mathfrak{g}_{\mathbb{R}}^*})^{-\theta} \otimes \mathbb{C} \simeq \{(\alpha, -\alpha) : \alpha \in P_{\mathfrak{g}^*}\}$, and similarly $(P_{\mathfrak{h}_{\mathbb{R}}^*})^{-\theta} \otimes \mathbb{C} \simeq \{(\beta, -\beta) : \beta \in P_{\mathfrak{g}^*}\}$. By these isomorphisms, the restriction map $\mathrm{rest} : (P_{\mathfrak{g}_{\mathbb{R}}^*})^{-\theta} \to (P_{\mathfrak{h}_{\mathbb{R}}^*})^{-\theta}$ is rewritten as $(\alpha, -\alpha) \mapsto (\alpha|_{\mathfrak{h}}, -\alpha|_{\mathfrak{h}})$ $(\alpha \in P_{\mathfrak{g}^*})$. This is surjective if and only if so is $\mathrm{rest} : P_{\mathfrak{g}^*} \to P_{\mathfrak{h}^*}$.

(2) ⇔ (3). Recall that the restriction map $\mathrm{rest} : (P_{\mathfrak{g}_{\mathbb{R}}^*})^{-\theta} \to (P_{\mathfrak{h}_{\mathbb{R}}^*})^{-\theta}$ is canonically identified with

$$\overline{\mathrm{rest}} : (S^+\mathfrak{g}^*)^{\mathfrak{g}}/((S^+\mathfrak{g}^*)^{\mathfrak{g}} \cdot (S^+\mathfrak{g}^*)^{\mathfrak{g}}) \to (S^+\mathfrak{h}^*)^{\mathfrak{h}}/((S^+\mathfrak{h}^*)^{\mathfrak{h}} \cdot (S^+\mathfrak{h}^*)^{\mathfrak{h}})$$

via the Cartan maps. Since $(S\mathfrak{g}^*)^{\mathfrak{g}}$ and $(S\mathfrak{h}^*)^{\mathfrak{h}}$ are symmetric algebras over graded vector spaces, the above linear map is surjective if and only if $\mathrm{rest} : (S\mathfrak{g}^*)^{\mathfrak{g}} \to (S\mathfrak{h}^*)^{\mathfrak{h}}$ is surjective. □

Proposition 4.3.8. *Let* $(\mathfrak{g}_0, \mathfrak{h}_0)$ *be a reductive pair with Cartan involution* θ_0*. Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be the complexifications of* $\mathfrak{g}_0$ *and* $\mathfrak{h}_0$*, respectively. We denote by* θ *the Cartan involution of* $\mathfrak{g}$ *such that* $\theta|_{\mathfrak{g}} = \theta_0$*. If the restriction map* $\mathrm{rest} : (P_{\mathfrak{g}_0^*})^{-\theta_0} \to (P_{\mathfrak{h}_0^*})^{-\theta_0}$ *is not surjective, neither is* $\mathrm{rest} : (P_{\mathfrak{g}_{\mathbb{R}}^*})^{-\theta} \to (P_{\mathfrak{h}_{\mathbb{R}}^*})^{-\theta}$*.*

Proof. We write $\theta_{\mathbb{C}}$ for the complex linear extension of θ_0 to $\mathfrak{g}$. We note that $\theta_{\mathbb{C}}$ is a complex linear involution on $\mathfrak{g}$, whereas θ is antilinear. If the linear map rest : $(P_{\mathfrak{g}^*_{\mathbb{R}}})^{-\theta} \to (P_{\mathfrak{h}^*_{\mathbb{R}}})^{-\theta}$ is surjective, so is rest : $P_{\mathfrak{g}^*} \to P_{\mathfrak{h}^*}$ by Proposition 4.3.7 (1) $\Rightarrow$ (2). In particular, it is surjective on the (-1)-eigenspaces for $\theta_{\mathbb{C}}$, namely,

$$\text{rest} : (P_{\mathfrak{g}^*_0})^{-\theta_0} \otimes \mathbb{C} = (P_{\mathfrak{g}^*})^{-\theta_{\mathbb{C}}} \to (P_{\mathfrak{h}^*})^{-\theta_{\mathbb{C}}} = (P_{\mathfrak{h}^*_0})^{-\theta_0} \otimes \mathbb{C}$$

is surjective. This is equivalent to saying that rest : $(P_{\mathfrak{g}^*_0})^{-\theta_0} \to (P_{\mathfrak{h}^*_0})^{-\theta_0}$ is surjective. $\square$

4.4 Proof of Theorem 4.1.2

Notice that a direct sum of two semisimple symmetric spaces $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ satisfies the condition (A) (resp. (B)) if and only if both $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ satisfy (A) (resp. (B)). Recall from Fact 4.2.2 that injectivity of the homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ is equivalent to surjectivity of the linear map rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$. Therefore, it is sufficient to prove the following two claims:

- If an irreducible symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is listed in (B-1)–(B-5), the restriction map rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is surjective.
- If an irreducible symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is listed in Table 4.1, the restriction map rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is not surjective.

If $(\mathfrak{g}, \mathfrak{h})$ satisfies (B-1) or (B-2), surjectivity of rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is obvious.

If $(\mathfrak{g}, \mathfrak{h})$ satisfies (B-3), its c-dual $(\mathfrak{g}^c, \mathfrak{h})$ satisfies (B-2), and therefore the restriction rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is surjective by Proposition 4.3.6.

If $(\mathfrak{g}, \mathfrak{h})$ satisfies (B-4), we have $\dim(P_{\mathfrak{h}})^{-\theta} = \operatorname{rank} \mathfrak{h} - \operatorname{rank} \mathfrak{k}_H = 0$ by Fact 4.2.3 (2). Hence rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is trivially surjective.

Suppose that $(\mathfrak{g}, \mathfrak{h})$ satisfies (B-5). By Proposition 4.3.7 (3) $\Rightarrow$ (1), it suffices to see that rest : $(S\mathfrak{g}^*)^{\mathfrak{g}} \to (S\mathfrak{h}^*)^{\mathfrak{h}}$ is surjective. If $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}(2n+1, \mathbb{C}), \mathfrak{so}(2n+1, \mathbb{C}))$, $(\mathfrak{sl}(2n, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C}))$ or $(\mathfrak{so}(2n, \mathbb{C}), \mathfrak{so}(2n-1, \mathbb{C}))$, the surjectivity easily follows from Fact 4.2.1. If $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{e}_{6,\mathbb{C}}, \mathfrak{f}_{4,\mathbb{C}})$, the surjectivity is shown in [55, p. 322].

Let $(\mathfrak{g}, \mathfrak{h})$ be one of the irreducible symmetric pairs listed in Table 4.1. If $(\mathfrak{g}, \mathfrak{h})$ satisfies $d_k(\mathfrak{g}) < d_k(\mathfrak{h})$ for some $1 \leqslant k \leqslant 4$, the restriction map rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is not surjective by Proposition 4.3.4. The remaining cases are:

$\mathfrak{g}$	$\mathfrak{h}$	Conditions
$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{so}(2n, \mathbb{C})$	$n \geqslant 1$

$\mathfrak{sl}(p+q,\mathbb{R})$	$\mathfrak{so}(p,q)$	$p,q \geqslant 1$, p,q: odd
$\mathfrak{su}(p,q)$	$\mathfrak{so}(p,q)$	$p,q \geqslant 1$, p,q: odd
$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(2n,\mathbb{C})$	$n \geqslant 3$
$\mathfrak{so}(p+r,q+s)$	$\mathfrak{so}(p,q)\oplus\mathfrak{so}(r,s)$	$p,q \geqslant 1$, p,q: odd, $r,s \geqslant 0$, $(r,s) \neq (0,0)$, $(p,q,r,s) \neq (1,1,1,1)$

Table 4.5: $(\mathfrak{g},\mathfrak{h})$ not satisfying (B) to which Proposition 4.3.4 is not applicable

The pair $(\mathfrak{sl}(p+q,\mathbb{R}),\mathfrak{so}(p,q))$ (p,q: odd) is the c-dual of $(\mathfrak{su}(p,q),\mathfrak{so}(p,q))$. The pairs $(\mathfrak{sl}(2n,\mathbb{C}),\mathfrak{so}(2n,\mathbb{C}))$ and $(\mathfrak{so}(2n+1,\mathbb{C}),\mathfrak{so}(2n,\mathbb{C}))$ are the complexifications of $(\mathfrak{su}(2n-1,1),\mathfrak{so}(2n-1,1))$ and $(\mathfrak{so}(2n,1),\mathfrak{so}(2n-1,1))$, respectively. Hence, by Propositions 4.3.6 and 4.3.8, it is sufficient to verify that the restriction map $\mathrm{rest} : (P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is not surjective when $(\mathfrak{g},\mathfrak{h})$ is $(\mathfrak{su}(p,q),\mathfrak{so}(p,q))$ (p,q: odd) or $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{so}(p+r,q+s),\mathfrak{so}(p,q)\oplus\mathfrak{so}(r,s))$ (p,q: odd, $(r,s)\neq(0,0)$). If $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{su}(p,q),\mathfrak{so}(p,q))$ (p,q: odd), the nonsurjectivity follows from Proposition 4.3.1. If $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{so}(p+r,q+s),\mathfrak{so}(p,q)\oplus\mathfrak{so}(r,s))$ (p,q: odd, $(r,s)\neq(0,0)$), we have $\dim(P^{p+q-1}_{\mathfrak{g}^*})^{-\theta} < \dim(P^{p+q-1}_{\mathfrak{h}^*})^{-\theta}$, and $\mathrm{rest} : (P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ cannot be surjective.

We have completed the proof of Theorem 4.1.2. □

4.5 Examples of nonsymmetric homogeneous spaces that apply our method

So far, we have studied the case of semisimple symmetric spaces. In this section, we give some applications of Corollaries 4.3.1 and 4.3.4 to more general setting, namely, the case of homogeneous spaces of reductive type.

Example 4.5.1 (cf. [41, Cor. 1.6])**.** *There do not exist compact manifolds locally modelled on the following homogeneous spaces:*

(1) $\mathrm{SL}(n_1+\cdots+n_k,\mathbb{R})/(\mathrm{SL}(n_1,\mathbb{R})\times\cdots\times\mathrm{SL}(n_k,\mathbb{R}))$ $(n_1,n_2 \geqslant 3)$,

(2) $\mathrm{SL}(n_1+\cdots+n_k,\mathbb{C})/(\mathrm{SL}(n_1,\mathbb{C})\times\cdots\times\mathrm{SL}(n_k,\mathbb{C}))$ $(n_1,n_2 \geqslant 2)$,

(3) $\mathrm{SL}(n_1+\cdots+n_k,\mathbb{H})/(\mathrm{SL}(n_1,\mathbb{H})\times\cdots\times\mathrm{SL}(n_k,\mathbb{H}))$ $(n_1,n_2 \geqslant 2)$,

(4) $\mathrm{O}(p_1+\cdots+p_k,q_1+\cdots+q_k)/(\mathrm{O}(p_1,q_1)\times\cdots\times\mathrm{O}(p_k,q_k))$ (p_1,q_1: odd, $p_2 \geqslant 1$),

(5) $\mathrm{O}(n_1+\cdots+n_k,\mathbb{C})/(\mathrm{O}(n_1,\mathbb{C})\times\cdots\times\mathrm{O}(n_k,\mathbb{C}))$ $(n_1,n_2\geqslant 2$ or $n_1\geqslant 2$ is even, $n_2=1)$,

(6) $\mathrm{Sp}(n_1+\cdots+n_k+n,\mathbb{C})/(\mathrm{Sp}(n_1,\mathbb{C})\times\cdots\times\mathrm{Sp}(n_k,\mathbb{C}))$ $(n_1,n_2\geqslant 1)$.

Remark 4.5.2. In Example 4.5.1, (5) and (6) are already known not to admit compact Clifford–Klein forms; we can apply Kobayashi's method [23] to these cases.

Remark 4.5.3.

- The existence problem of compact Clifford–Klein forms of $\mathrm{SL}(n,\mathbb{K})/\mathrm{SL}(m,\mathbb{K})$ $(n>m\geqslant 2,\ \mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H})$ has been attracted considerable attention. The first result was obtained in [22] in the setting $n=3, m=2, \mathbb{K}=\mathbb{C}$. Some further results are in [2], [26], [35], [36], [53], [56], [61] and Chapter 5. For example, expanding the method of [61] and [35], Labourie–Zimmer [36] proved that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ does not admit a compact Clifford–Klein form if $n-m\geqslant 3$. Unfortunately, our method gives no information about this case.
- Benoist [2] proved that $\mathrm{SL}(p+q,\mathbb{R})/(\mathrm{SL}(p,\mathbb{R})\times\mathrm{SL}(q,\mathbb{R}))$ $(p,q\geqslant 1)$ does not admit a compact Clifford–Klein form if pq is even.
- By applying the method of [23], Kobayashi [26] gave many results that are similar to Example 4.5.1. See [26, Ex's 4.13.5–4.13.7].

Proof of Example 4.5.1. We have $d_3(\mathfrak{g})=1<2\leqslant d_3(\mathfrak{h})$ in (1) and (3), and $d_2(\mathfrak{g})=1<2\leqslant d_2(\mathfrak{h})$ in (2) and (6). So, we can apply Corollary 4.3.5 to these cases.

(4). We can prove this case by showing $\dim(P_{\mathfrak{g}^*}^{p_1+p_2-1})^{-\theta}=0<1\leqslant\dim(P_{\mathfrak{h}^*}^{p_1+p_2-1})^{-\theta}$. We here give another proof based on the following lemma:

Lemma 4.5.4 (cf. [41, Prop. 5.1])**.** *Let $(\mathfrak{g},\mathfrak{h})$ be a reductive pair with Cartan involution θ. Let l be a subalgebra of $\mathfrak{h}$ such that $\theta(\mathfrak{l})=\mathfrak{l}$.*

(1) *If the restriction map* $\mathrm{rest}:(P_{\mathfrak{h}^*})^{-\theta}\to(P_{\mathfrak{l}^*})^{-\theta}$ *is not surjective, neither is* $\mathrm{rest}:(P_{\mathfrak{g}^*})^{-\theta}\to(P_{\mathfrak{l}^*})^{-\theta}$.

(2) *Assume that* $\mathrm{rest}:(P_{\mathfrak{h}^*})^{-\theta}\to(P_{\mathfrak{l}^*})^{-\theta}$ *is surjective. If* $\mathrm{rest}:(P_{\mathfrak{g}^*})^{-\theta}\to(P_{\mathfrak{l}^*})^{-\theta}$ *is not surjective, neither is* $\mathrm{rest}:(P_{\mathfrak{g}^*})^{-\theta}\to(P_{\mathfrak{h}^*})^{-\theta}$.

Proof of Lemma 4.5.4. This lemma follows immediately from an observation that the linear map $\mathrm{rest}:(P_{\mathfrak{g}^*})^{-\theta}\to(P_{\mathfrak{l}^*})^{-\theta}$ is factorized as

$$(P_{\mathfrak{g}^*})^{-\theta}\xrightarrow{\mathrm{rest}}(P_{\mathfrak{h}^*})^{-\theta}\xrightarrow{\mathrm{rest}}(P_{\mathfrak{l}^*})^{-\theta}.$$

□

By Facts 4.1.1 and 4.2.2, it is enough to see that the restriction map $\mathrm{rest}:(P_{\mathfrak{g}^*})^{-\theta}\to(P_{\mathfrak{h}^*})^{-\theta}$ is not surjective for $(\mathfrak{g},\mathfrak{h})=(\mathfrak{so}(p_1+\cdots+p_k,q_1+\cdots+q_k),\mathfrak{so}(p_1,q_1)\oplus\cdots\oplus\mathfrak{so}(p_k,q_k))$ $(p_1,q_1,p_2\geqslant 1,\ p_1,q_1\text{: odd})$.

By Lemma 4.5.4 (2), it suffices to see that the restriction map is not surjective for $(\mathfrak{so}(p_1 + \cdots + p_k, q_1 + \cdots + q_k), \mathfrak{so}(p_1, q_1))$ $(p_1, q_1, p_2 \geqslant 1,\ p_1, q_1$: odd). Then, by Lemma 4.5.4 (1), we only need to see the nonsurjectivity for $(\mathfrak{so}(p_1 + 1, q_1), \mathfrak{so}(p_1, q_1))$ $(p_1, q_1 \geqslant 1,\ p_1, q_1$: odd). Now, we can apply Proposition 4.3.1.

(5). We see that

- If $n_1 = 2$ or $n_2 = 2$, then $d_1(\mathfrak{g}) = 0 < 1 \leqslant d_1(\mathfrak{h})$.
- If $n_1, n_2 \geqslant 3$, then $d_2(\mathfrak{g}) = 1 < 2 \leqslant d_2(\mathfrak{h})$.

We can apply Proposition 4.3.4 to these cases. Hence, it suffices to see that the restriction map rest : $(P_{\mathfrak{g}^*})^{-\theta} \to (P_{\mathfrak{h}^*})^{-\theta}$ is not surjective if $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(2n+m, \mathbb{C}), \mathfrak{so}(2n, \mathbb{C}))$ $(n, m \geqslant 1)$. If $m = 1$, the pair $(\mathfrak{g}, \mathfrak{h})$ is a semisimple symmetric pair and the nonsurjectivity is already proved in Section 4.4. Then the general case follows from Lemma 4.5.4 (1). Alternatively, we can see the nonsurjectivity from $\dim(P^{2n-1}_{\mathfrak{g}^*})^{-\theta} = 0 < 1 \leqslant \dim(P^{2n-1}_{\mathfrak{h}^*})^{-\theta}$. □

Example 4.5.5 (cf. [41, Rem. 5.2]). *More generally, if $n_1, n_2 \geqslant 3$, there do not exist compact manifolds locally modelled on*

- $\mathrm{SL}(n_1 + n_2 + n_3, \mathbb{R})/(\mathrm{SL}(n_1, \mathbb{R}) \times \mathrm{SL}(n_2, \mathbb{R}) \times H')$ *and*
- $\mathrm{SL}(n_1 + n_2 + n_3, \mathbb{R})/(\mathrm{S}(\mathrm{GL}(n_1, \mathbb{R}) \times \mathrm{GL}(n_2, \mathbb{R})) \times H')$

for any closed subgroup H' of $\mathrm{SL}(n_3, \mathbb{R})$ that is reductive in $\mathrm{SL}(n_3, \mathbb{R})$. The proof is the same as that of Example 4.5.1 (1). Similar results also hold for (2)–(6).

4.6 Nonlinear case

Let G/H be a homogeneous space of reductive type, i.e. G is a *linear* reductive Lie group with Cartan involution θ and H is a closed subgroup of G with finitely many connected components such that $\theta(H) = H$. As before, let $K = G^\theta$ and $K_H = H^\theta$ denote the corresponding maximal compact subgroups of G and H, respectively. Let $\pi : \tilde{G} \to G$ a covering map. Put $\tilde{H} = \pi^{-1}(H)$ and $\tilde{K_H} = \pi^{-1}(K_H)$. If π is an infinite covering, $\tilde{K_H}$ is noncompact. Assume that $\tilde{K_H}$ has finitely many connected components (this is always satisfied when $H = K$, for instance). We can then take a maximal compact subgroup C_H of $\tilde{H}$ so that $\mathfrak{c}_H \subsetneq \mathfrak{k}_H$ by the Cartan–Malcev–Iwasawa–Mostow theorem (see e.g. [6, Ch. VII, Th. 1.2] or [18, Ch. XV, Th. 3.1]), where $\mathfrak{c}_H$ is the Lie algebra of C_H . In this case, we have the following fact, which is stronger than Fact 4.1.1 in the linear setting:

Fact 4.6.1 (see Corollary 2.5.1). *If the homomorphism $i : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{c}_H; \mathbb{R})$ induced from the inclusion map $(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \hookrightarrow (\Lambda(\mathfrak{g}/\mathfrak{c}_H)^*)^{\mathfrak{c}_H}$ is* *not injective, then there does not exist a compact manifold locally modelled on the homogeneous space G/H (and, in particular, a compact Clifford–Klein form of G/H).*

By A. Borel's theorem [5], for any linear reductive Lie group G, the Riemannian symmetric space G/K of noncompact type admits a compact Clifford–Klein form. Let $\pi : \tilde{G} \to G$ be a covering map and put $\tilde{K} = \pi^{-1}(K)$. If $\tilde{G}$ is not linear, the proof of [5] does not work for $\tilde{G}/\tilde{K}$ because of the following two reasons:

- We cannot use Selberg's lemma [52, Lem. 8] to control the freeness of the action.
- If π is an infinite covering map, $\tilde{K}$ is noncompact. Hence a discrete subgroup of $\tilde{G}$ may not act properly on $\tilde{G}/\tilde{K}$.

The following example shows that the compactness of K is crucial.

Example 4.6.2. *Let G/K be a Hermitian symmetric space of noncompact type and $\pi : \tilde{G} \to G$ be a universal covering map. Put $\tilde{K} = \pi^{-1}(K)$. Then there does not exist a compact manifold locally modelled on $\tilde{G}/\tilde{K}$. In particular, $\tilde{G}/\tilde{K}$ does not admit a compact Clifford–Klein form.*

Remark 4.6.3. In the definition of a manifold locally modelled on a homogeneous space, we assumed that the transition functions satisfy the cocycle condition (see Section 2.2). Without this assumption, Example 4.6.2 is false. Note that a compact Clifford–Klein form always satisfies the cocycle condition for the transition functions.

Proof of Example 4.6.2. The Lie algebra of the maximal compact subgroup of $\tilde{K}$ is $\mathfrak{k}_{ss} = [\mathfrak{k}, \mathfrak{k}]$. Thus, by Fact 4.6.1, it suffices to see that the homomorphism

$$i : H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{k}_{ss}; \mathbb{R})$$

is not injective. Take a nonzero element X of the centre of $\mathfrak{k}$ so that $\mathrm{Stab}_{\mathfrak{g}}(X) = \mathfrak{k}$. We regard X as an element of $\mathfrak{g}^*$ via the Killing form B of $\mathfrak{g}$. Then, $\omega = dX$ is an element of $(\Lambda^2(\mathfrak{g}/\mathfrak{k})^*)^{\mathfrak{k}}$. Note that ω corresponds to the G-invariant Kähler form on G/K under the isomorphism $(\Lambda^2(\mathfrak{g}/\mathfrak{k})^*)^{\mathfrak{k}} \simeq \Omega^2(G/K)^G$. Since ω is nondegenerate on $\mathfrak{g}/\mathfrak{k}$, $[\omega]_{\mathfrak{g},\mathfrak{k}} \neq 0$ in $H^2(\mathfrak{g}, \mathfrak{k}; \mathbb{R})$. On the other hand, since $X \in ((\mathfrak{g}/\mathfrak{k}_{ss})^*)^{\mathfrak{k}_{ss}}$, we have $[\omega]_{\mathfrak{g},\mathfrak{k}_{ss}} = [dX]_{\mathfrak{g},\mathfrak{k}_{ss}} = 0$ in $H^2(\mathfrak{g}, \mathfrak{k}_{ss}; \mathbb{R})$. Therefore the above homomorphism i is not injective. $\square$

Remark 4.6.4. Atiyah–Schmid [1] applied Borel's theorem to construct the discrete series representations of a semisimple Lie group with finite centre. In its erratum the nonlinear case is discussed. Unfortunately, our method gives no information for the case of a nonlinear semisimple Lie group with finite centre.

Chapter 5

A cohomological obstruction to the existence of compact Clifford–Klein forms

5.1 Introduction

5.1.1 The existence problem of compact Clifford–Klein forms

A Clifford–Klein form is a double coset space $\Gamma\backslash G/H$, where G is a Lie group, H a closed subgroup of G, and Γ a discrete subgroup of G acting properly and freely on G/H. It admits a natural structure of a manifold locally modelled on G/H. If $\Gamma\backslash G/H$ is a Clifford–Klein form, a discrete subgroup Γ of G is called a discontinuous group for G/H.

It is one of the central open problems in the study of Clifford–Klein forms to determine all homogeneous spaces admitting *compact* Clifford–Klein forms. In the last three decades, this problem attracted considerable attention, and a number of obstructions to the existence of compact Clifford–Klein forms were found. Some of these obstructions are based on a homomorphism

$$\eta : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\Gamma\backslash G/H; \mathbb{R}) \qquad (\mathfrak{g} = \mathrm{Lie}(G),\ \mathfrak{h} = \mathrm{Lie}(H)),$$

which imposes a restriction on cohomology of compact Clifford–Klein forms ([30], [20], [3] and Theorem 2.1.2). In this chapter, we give a new obstruction arising from this homomorphism.

5.1.2 Main result

The main result of this chapter is as follows:

Theorem 5.1.1. *Let G be a connected linear Lie group and H its connected closed subgroup. Assume that $H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \neq 0$ $(N = \dim G - \dim H)$. Let K_H be a maximal compact subgroup of H and T_H a maximal torus of K_H. Let $I^\bullet = \bigoplus_{n \in \mathbb{N}} I^n$ be the graded ideal of $H^\bullet(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})$ generated by*

$$\bigoplus_{C, p} \operatorname{im}(i : H^p(\mathfrak{g}, \mathfrak{c}; \mathbb{R}) \to H^p(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})),$$

where the direct sum runs all connected compact subgroups C of G containing T_H and all $p > N + \dim K_H - \dim C$. If

$$\operatorname{im}(i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})) \subset I^N$$

holds, G/H does not admit a compact Clifford–Klein form.

Remark 5.1.2. We do not know if Theorem 5.1.1 applies to a more general case of manifolds locally modelled on G/H.

The key to the proof of Theorem 5.1.1 is an upper-bound estimate for cohomological dimensions of discontinuous groups (Lemma 5.2.2), which was established by Kobayashi [20] in the reductive case. It imposes another restriction on cohomology of Clifford–Klein forms. We prove Theorem 5.1.1 by linking these two restrictions.

Remark 5.1.3. In [23], Kobayashi gave an obstruction to the existence of compact Clifford–Klein forms by combining the estimate for cohomological dimensions with the criterion for proper actions. As far as the author understands, his and our obstructions do not include each other.

5.1.3 New examples of a homogeneous space without compact Clifford–Klein forms

Theorem 5.1.1 provides some new examples of homogeneous spaces that do not admit compact Clifford–Klein forms. For example, an irreducible symmetric space G/H does not have a compact Clifford–Klein form if the corresponding symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is as in Table 5.1 (see Corollaries 5.6.1, 5.6.2 and 5.6.5):

$\mathfrak{g}$	$\mathfrak{h}$	Conditions
$\mathfrak{sl}(p+q, \mathbb{C})$	$\mathfrak{su}(p, q)$	$p, q \geqslant 1$
$\mathfrak{sl}(p+q, \mathbb{R})$	$\mathfrak{so}(p, q)$	$p, q \geqslant 1$
$\mathfrak{sl}(p+q, \mathbb{H})$	$\mathfrak{sp}(p, q)$	$p, q \geqslant 1$
$\mathfrak{so}(p+q, \mathbb{C})$	$\mathfrak{so}(p, q)$	$p, q \geqslant 2,\ (p, q) \neq (2, 2)$
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n, 1)$	$n \geqslant 1$

$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}^*(2n)$	$n \geqslant 3$
$\mathfrak{so}(p+r, q)$	$\mathfrak{so}(p, q) \oplus \mathfrak{so}(r)$	$p, q, r \geqslant 1,\ q : \text{odd}$
$\mathfrak{sp}(p+q, \mathbb{C})$	$\mathfrak{sp}(p, q)$	$p, q \geqslant 1$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(-14)}$	—
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2, 2)$	—
$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-5)}$	—
$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$	—
$\mathfrak{e}_{8,\mathbb{C}}$	$\mathfrak{e}_{8(-24)}$	—
$\mathfrak{f}_{4,\mathbb{C}}$	$\mathfrak{f}_{4(-20)}$	—

Table 5.1: Irreducible symmetric spaces without compact Clifford–Klein forms

In particular, the nonexistence of a compact Clifford–Klein form of

$$\mathrm{SO}_0(p+1, q)/\,\mathrm{SO}_0(p, q) \qquad (p, q \geqslant 1,\ q : \text{odd})$$

is rephrased as:

Corollary 5.1.4. *If $p, q \geqslant 1$ and q is odd, then there does not exist a compact complete pseudo-Riemannian manifold of signature (p, q) with constant positive sectional curvature.*

We also obtain some nonsymmetric examples: for example,

$$\mathrm{SL}(n, \mathbb{R})/\,\mathrm{SL}(m, \mathbb{R}) \qquad (n > m \geqslant 2,\ m : \text{even})$$

does not admit a compact Clifford–Klein form (see Corollary 5.6.7).

Remark 5.1.5. We mention some related nonexistence results which can be obtained by previously known methods:

- **Calabi–Markus [10], Wolf [59], [60], Kobayashi [20]:** The following homogeneous spaces do not admit infinite discontinuous groups. In particular, they do not admit compact Clifford–Klein forms:
  - $\mathrm{SO}(p+q, \mathbb{C})/\,\mathrm{SO}_0(p, q) \quad (p, q \geqslant 1,\ |p-q| \leqslant 1)$,
  - $\mathrm{SO}_0(p+r, q+s)/(\mathrm{SO}_0(p, q) \times \mathrm{SO}_0(r, s))$ $(p \geqslant q \geqslant 1,\ r \geqslant s \geqslant 0,\ (r, s) \neq (0, 0))$.
- **Kulkarni [33]:** A homogeneous space
  - $\mathrm{SO}_0(p+1, q)/\,\mathrm{SO}_0(p, q) \quad (p, q \geqslant 1,\ p, q : \text{odd})$

does not admit a compact Clifford–Klein form.

- **Kobayashi [22], [23], [26]:** The following homogeneous spaces do not admit compact Clifford–Klein forms:
  - $\mathrm{SL}(2p,\mathbb{C})/\mathrm{SU}(p,p) \quad (p \geqslant 1)$,
  - $\mathrm{SL}(2p,\mathbb{R})/\mathrm{SO}_0(p,p) \quad (p \geqslant 1)$,
  - $\mathrm{SO}_0(p+r,q+s)/(\mathrm{SO}_0(p,q)\times\mathrm{SO}_0(r,s)) \quad (p,q,r,s \geqslant 1)$,
  - $\mathrm{SO}_0(p+r,q)/(\mathrm{SO}_0(p,q)\times\mathrm{SO}(r)) \quad (p+r > q \geqslant 1)$,
  - $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R}) \quad (n > 3[(m+1)/2],\ m \geqslant 2)$.
- **Zimmer [61], Labourie–Mozes–Zimmer [35], Labourie–Zimmer [36]:** A homogeneous space
  - $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R}) \quad (n-3 \geqslant m \geqslant 2)$

  does not admit a compact Clifford–Klein form. If, in addition, $n \geqslant 2m$, there does not exist a compact manifold locally modelled on this homogeneous space.
- **Shalom [53]:** A homogeneous space
  - $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(2,\mathbb{R}) \quad (n \geqslant 4)$

  does not admit a compact Clifford–Klein form.
- **Margulis [37], Oh [46]:** Let $\alpha_n : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ denote the real n-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{R})$. Then, a homogeneous space
  - $\mathrm{SL}(n,\mathbb{R})/\alpha_n(\mathrm{SL}(2,\mathbb{R})) \quad (n \geqslant 4)$

  does not admit a compact Clifford–Klein form.
- **Benoist [2], Okuda [47]:** The following homogeneous spaces do not admit non-virtually abelian discontinuous groups. In particular, they do not admit compact Clifford–Klein forms:
  - $\mathrm{SL}(p+q,\mathbb{C})/\mathrm{SU}(p,q) \quad (p,q \geqslant 1,\ |p-q| \leqslant 1)$,
  - $\mathrm{SL}(p+q,\mathbb{R})/\mathrm{SO}_0(p,q) \quad (p,q \geqslant 1,\ |p-q| \leqslant 1)$,
  - $\mathrm{SL}(p+q,\mathbb{H})/\mathrm{Sp}(p,q) \quad (p,q \geqslant 1,\ |p-q| \leqslant 1)$,
  - $\mathrm{SO}(4p+2,\mathbb{C})/\mathrm{SO}_0(2p+2,2p) \quad (p \geqslant 1)$,
  - $\mathrm{SO}_0(p+q+1,p+q+1)/(\mathrm{SO}_0(p+1,p)\times\mathrm{SO}_0(q,q+1))$ $(p \geqslant 1,\ q \geqslant 0,\ p+q : \text{even})$,
  - $\mathrm{SL}(m+1,\mathbb{R})/\mathrm{SL}(m,\mathbb{R}) \quad (m \geqslant 2,\ m : \text{even})$.

- **Kobayashi–Ono [30], Kobayashi [20], Benoist–Labourie [3], Morita (Chapters 2–4):** There do not exist compact manifolds locally modelled on the following homogeneous spaces. In particular, they do not admit compact Clifford–Klein forms:
  - $\mathrm{SL}(p+q,\mathbb{R})/\mathrm{SO}_0(p,q) \quad (p,q \geqslant 1,\ p,q : \mathrm{odd})$,
  - $\mathrm{SO}_0(p+r,q+s)/(\mathrm{SO}_0(p,q)\times\mathrm{SO}_0(r,s))$
    $(p,q,r \geqslant 1,\ s \geqslant 0,\ p,q : \mathrm{odd})$.

Remark 5.1.6. Now we mention some homogeneous spaces admitting compact Clifford–Klein forms:

- **Borel–Harish-Chandra [7], Mostow–Tamagawa [45], Borel [5]:** Every Riemannian symmetric space G/K admits a compact Clifford–Klein form.
- **Kulkarni [33], Kobayashi [22], [26], Kobayashi–Yoshino [31]:** The following homogeneous spaces admit compact Clifford–Klein forms.
  - $\mathrm{SO}(8,\mathbb{C})/\mathrm{SO}_0(7,1)$,
  - $\mathrm{SO}_0(p+r,q)/(\mathrm{SO}_0(p,q)\times\mathrm{SO}(r))$
    $((p,q,r)=(1,2n,1),(3,4n,1),(1,4,2),(1,4,3),(7,8,1),\ n\geqslant 1)$.

Remark 5.1.7. While the author was preparing the manuscript [44], Tholozan [56, ver. 1] proved the nonexistence of compact Clifford–Klein forms of some homogeneous spaces, such as

(1) $\mathrm{SO}_0(p+r,q)/\mathrm{SO}_0(p,q) \quad (p,q,r \geqslant 1,\ q : \mathrm{odd})$,

(2) $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R}) \quad (n > m \geqslant 2,\ m : \mathrm{even})$,

(3) $\mathrm{SL}(p+q,\mathbb{R})/\mathrm{SO}_0(p,q) \quad (p,q \geqslant 1,\ p+q : \mathrm{odd})$,

(4) $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(m,\mathbb{C}) \quad (n > m \geqslant 2,\ m : \mathrm{even})$,

(5) $\mathrm{SO}(p+q,\mathbb{C})/\mathrm{SO}_0(p,q) \quad (p,q \geqslant 1,\ p+q : \mathrm{odd})$.

Our results are sharper for (3), (5) and the same for (1)–(2), (4). Actually, (4) had been proved by earlier methods [23], [41] too. In [56, ver. 2], he gave a proof of the nonexistence of compact Clifford–Klein forms of the classical symmetric spaces listed in Table 5.1.

5.1.4 Outline of this chapter

In Section 5.2, we recall some basic facts on cohomology of Clifford–Klein forms, including the upper-bound estimate for cohomological dimensions of discontinuous groups and the definition of the homomorphism η from relative Lie algebra cohomology to de Rham cohomology. The proof of Theorem 5.1.1 is given in Section 5.3. The rest of this chapter is devoted to constructing examples of a homogeneous space to which Theorem 5.1.1 is applicable. In Section 5.4, we apply H. Cartan's theorem [11] on relative Lie algebra cohomology of reductive pairs to Theorem 5.1.1. We shall see that, if G/H is a homogeneous space of reductive type satisfying some invariant-theoretic condition, then Theorem 5.1.1 is applicable to G/H. In Section 5.5, we give a way to find semisimple symmetric spaces satisfying the condition obtained in Section 5.4 by using the notion of ε-families, introduced by Oshima–Sekiguchi [50]. Finally, in Section 5.6, we give examples of a homogeneous space without compact Clifford–Klein forms by applying the results in Sections 5.4 and 5.5.

5.2 Preliminaries for the proof of Theorem 5.1.1

We recall some basic facts on cohomology of Clifford–Klein forms. They are used in Section 5.3.

5.2.1 Orientability of Clifford–Klein forms

Let G be a Lie group, H a closed subgroup of G, and Γ a discrete subgroup of G acting properly and freely on G/H. The local system of orientation of $\Gamma\backslash G/H$ is isomorphic to $\Gamma\backslash G \times_H \mathbb{R}$, where H acts on $\mathbb{R}$ via $H \to \{\pm 1\}$, $h \mapsto \operatorname{sgn}\det \operatorname{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)$. Thus, $\Gamma\backslash G/H$ is orientable if H is connected.

5.2.2 Maximal compact subgroups of Lie groups

The following fact is fundamental for the computation of cohomology of homogeneous spaces and Clifford–Klein forms:

Fact 5.2.1 (Cartan–Malcev–Iwasawa–Mostow, [6, Ch. VII, Th. 1.2], [18, Ch. XV, Th. 3.1])**.** *Let G be a Lie group with finitely many connected components. Then,*

(1) *Every compact subgroup of G is contained in some maximal compact subgroup.*

(2) *Any two maximal compact subgroups of G are conjugate.*

(3) *Let K be a maximal compact subgroup of G. Then, there exist linear subspaces $V_1, \ldots, V_s$ of $\mathfrak{g}$ such that*

$$V_1 \times \cdots \times V_s \times K \to G, \qquad (v_1, \ldots, v_s, k) \mapsto \exp(v_1) \ldots \exp(v_s) k$$

is a diffeomorphism.

5.2.3 Cohomological dimensions of discontinuous groups

Recall that the real cohomological dimension $\mathrm{cd}_{\mathbb{R}}(\Gamma)$ of a discrete group Γ is defined as

$$\mathrm{cd}_{\mathbb{R}}(\Gamma) = \sup\{p \in \mathbb{N} : H^p(\Gamma; V) \neq 0 \text{ for some } \mathbb{R}\Gamma\text{-module } V\}.$$

Let G be a connected Lie group, H a connected closed subgroup of G, and Γ a torsion-free discrete subgroup of G acting properly (and therefore freely) on G/H. We put

$$\mathrm{cd}_{\mathbb{R}}(\Gamma; G/H) = \sup\{p \in \mathbb{N} : H^p(\Gamma\backslash G/H; \mathcal{V}) \neq 0 \text{ for some } \mathbb{R}\Gamma\text{-module } V\},$$

where $\mathcal{V}$ denotes the local system $V \times_\Gamma G/H$ on $\Gamma\backslash G/H$. Remind that $\mathrm{cd}_{\mathbb{R}}(\Gamma; G/K)$ is nothing but $\mathrm{cd}_{\mathbb{R}}(\Gamma)$, where K is a maximal compact subgroup of G, because G/K is a classifying space of Γ by Fact 5.2.1.

Lemma 5.2.2. *Let G, H and Γ be as above. Put $N = \dim G - \dim H$. Let K and K_H be maximal compact subgroups of G and H, respectively. Then,*

(1) $\mathrm{cd}_{\mathbb{R}}(\Gamma; G/H) \leqslant N$*; equality is attained if and only if the Clifford–Klein form $\Gamma\backslash G/H$ is compact.*

(2) $\mathrm{cd}_{\mathbb{R}}(\Gamma; G/H) = \mathrm{cd}_{\mathbb{R}}(\Gamma) + \dim K - \dim K_H$.

Proof. These are proved in [20, §5] when G/H is of reductive type. Our proof is along the same line.

(1) Since the Clifford–Klein form $\Gamma\backslash G/H$ is orientable, the Poincaré duality for $\Gamma\backslash G/H$ is stated as:

$$H^p(\Gamma\backslash G/H; \mathcal{V}) \simeq H^{\mathrm{BM}}_{N-p}(\Gamma\backslash G/H; \mathcal{V}),$$

where the right-hand side is the Borel–Moore homology. This immediately implies $\mathrm{cd}_{\mathbb{R}}(\Gamma; G/H) \leqslant N$, with equality if and only if $\Gamma\backslash G/H$ is compact.

(2) Take any $\mathbb{R}\Gamma$-module V. The Cartan–Leray spectral sequence [12, Ch. XVI, §9] for the Γ-action on G/H is:

$$E_2^{p,q} = H^p(\Gamma; H^q(G/H; V)) \Rightarrow H^{p+q}(\Gamma\backslash G/H; \mathcal{V}).$$

Since G is connected, its subgroup Γ acts trivially on $H^q(G/H; \mathbb{R})$. The E_2-term of the spectral sequence is thus rewritten as:

$$E_2^{p,q} = H^p(\Gamma; V) \otimes H^q(G/H; \mathbb{R}).$$

Therefore, we have

$$\mathrm{cd}_{\mathbb{R}}(\Gamma; G/H) = \mathrm{cd}_{\mathbb{R}}(\Gamma) + \sup\{q \in \mathbb{N} : H^q(G/H; \mathbb{R}) \neq 0\}.$$

Note that $\mathrm{cd}_{\mathbb{R}}(\Gamma) = \mathrm{cd}_{\mathbb{R}}(\Gamma; G/K) < \infty$ by (1). On the other hand,

Lemma 5.2.3. *The composition of an inclusion and a projection*

$$\pi \circ i : K/K_H \to G/K_H \to G/H$$

is a homotopy equivalence.

Proof. It directly follows from Fact 5.2.1 that the inclusion i is a homotopy equivalence. The projection π is a fibre bundle whose typical fibre H/K_H is contractible again by Fact 5.2.1, hence a homotopy equivalence ([14, Cor. 3.2]). □

Since K/K_H is an orientable compact manifold, we obtain from Lemma 5.2.3 that

$$\begin{aligned}\sup\{q \in \mathbb{N} : H^q(G/H; \mathbb{R}) \neq 0\} &= \sup\{q \in \mathbb{N} : H^q(K/K_H; \mathbb{R}) \neq 0\} \\ &= \dim K - \dim K_H.\end{aligned}$$ □

5.2.4 A homomorphism η from relative Lie algebra cohomology to de Rham cohomology

Let G be a Lie group, H a connected closed subgroup of G, and Γ a discrete subgroup of G acting properly and freely on G/H. We define $\eta : H^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^\bullet(\Gamma\backslash G/H; \mathbb{R})$ to be the homomorphism induced from the inclusion map

$$(\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}} \simeq \Omega(G/H)^G \hookrightarrow \Omega(G/H)^\Gamma \simeq \Omega(\Gamma\backslash G/H).$$

If a Clifford–Klein form $\Gamma\backslash G/H$ is compact,

$$\eta : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\Gamma\backslash G/H; \mathbb{R}) \qquad (N = \dim G - \dim H)$$

is injective (Section 2.3). Indeed, if $\Phi \in (\Lambda^N(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$ is nonzero, then $\eta([\Phi]) \in H^N(\Gamma\backslash G/H; \mathbb{R})$ is a cohomology class of a volume form, hence nonzero.

5.3 Proof of Theorem 5.1.1

Let us assume that $\mathrm{im}(i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})) \subset I^N$ and prove that G/H does not have a compact Clifford–Klein form. Here, as defined in Theorem 5.1.1, $I^\bullet$ is the graded ideal of $H^\bullet(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})$ generated by $\bigoplus_{C,p} \mathrm{im}(i :$

$H^p(\mathfrak{g}, \mathfrak{c}; \mathbb{R}) \to H^p(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R}))$, where the direct sum runs all connected compact subgroups C of G containing T_H and all $p > N + \dim K_H - \dim C$.

Suppose there were a discrete subgroup Γ of G such that $\Gamma\backslash G/H$ is a compact Clifford–Klein form. Such Γ is always finitely generated ([20, Lem. 2.1]). By Selberg's lemma [52, Lem. 8], we can assume Γ is torsion-free without loss of generality. We shall see that

$$\eta \circ i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R}) \to H^N(\Gamma\backslash G/T_H; \mathbb{R})$$

is a zero map and injective, which is impossible.

Let C be any compact connected subgroup of G containing T_H. Since Γ is torsion-free, $\Gamma\backslash G/C$ is a Clifford–Klein form. By Lemma 5.2.2, we have

$$\begin{aligned} \operatorname{cd}_{\mathbb{R}}(\Gamma; G/C) &= \operatorname{cd}_{\mathbb{R}}(\Gamma) + \dim K - \dim C \\ &= \operatorname{cd}_{\mathbb{R}}(\Gamma; G/H) + \dim K_H - \dim C \\ &= N + \dim K_H - \dim C. \end{aligned}$$

From the commutativity of the diagram

$$\begin{array}{ccc} H^p(\mathfrak{g}, \mathfrak{c}; \mathbb{R}) & \xrightarrow{\ i\ } & H^p(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R}) \\ \eta\downarrow & & \downarrow\eta \\ H^p(\Gamma\backslash G/C; \mathbb{R}) & \xrightarrow{\ \pi^*\ } & H^p(\Gamma\backslash G/T_H; \mathbb{R}), \end{array}$$

it follows that

$$\eta \circ i : H^p(\mathfrak{g}, \mathfrak{c}; \mathbb{R}) \to H^p(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R}) \to H^p(\Gamma\backslash G/T_H; \mathbb{R})$$

is a zero map for $p > N + \dim K_H - \dim C$. Therefore

$$I^\bullet \subset \ker(\eta : H^\bullet(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R}) \to H^\bullet(\Gamma\backslash G/T_H; \mathbb{R})).$$

In particular,

$$\eta \circ i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R}) \to H^N(\Gamma\backslash G/T_H; \mathbb{R})$$

is a zero map because $\operatorname{im}(i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})) \subset I^N$.

Consider another commutative diagram

$$\begin{array}{ccccc} H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) & & \xrightarrow{\ i\ } & & H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R}) \\ \eta\downarrow & & & & \downarrow\eta \\ H^N(\Gamma\backslash G/H; \mathbb{R}) & \xrightarrow{\ \pi^*\ } & H^N(\Gamma\backslash G/K_H; \mathbb{R}) & \xrightarrow{\ \pi^*\ } & H^N(\Gamma\backslash G/T_H; \mathbb{R}). \end{array}$$

As we recalled in Subsection 5.2.4,

$$\eta : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\Gamma\backslash G/H; \mathbb{R})$$

is injective. On the other hand, the projections $\pi : \Gamma\backslash G/K_H \to \Gamma\backslash G/H$ and $\pi : \Gamma\backslash G/T_H \to \Gamma\backslash G/K_H$ are fibre bundles with typical fibres H/K_H and K_H/T_H, respectively. The induced homomorphism on cohomology

$$\pi^* : H^N(\Gamma\backslash G/H;\mathbb{R}) \to H^N(\Gamma\backslash G/K_H;\mathbb{R})$$

is isomorphic since H/K_H is contractible (Fact 5.2.1), and

$$\pi^* : H^N(\Gamma\backslash G/K_H;\mathbb{R}) \to H^N(\Gamma\backslash G/T_H;\mathbb{R})$$

is injective by the splitting principle ([17, Th. 6.8.3]). Thus, the composition map

$$\eta \circ i : H^N(\mathfrak{g},\mathfrak{h};\mathbb{R}) \to H^N(\mathfrak{g},\mathfrak{t}_H;\mathbb{R}) \to H^N(\Gamma\backslash G/T_H;\mathbb{R})$$

is injective. This completes the proof of Theorem 5.1.1. □

5.4 A sufficient condition for Theorem 5.1.1 in the reductive case

5.4.1 Cartan's theorem

We say that $(\mathfrak{g},\mathfrak{h})$ is a real (resp. complex) reductive pair if $\mathfrak{g}$ is a real (resp. complex) reductive Lie algebra and $\mathfrak{h}$ is a real (resp. complex) subalgebra of $\mathfrak{g}$ that is reductive in $\mathfrak{g}$. In this chapter, we say that a homogeneous space G/H is of reductive type if G is a connected linear Lie group and H is a connected closed subgroup of G such that $(\mathfrak{g},\mathfrak{h})$ is a real reductive pair.

Relative Lie algebra cohomology of real or complex reductive pairs can be easily computed by H. Cartan's theorem [11]. Let us briefly recall the statement of the theorem (see [16], [48] for details). Let $(\mathfrak{g},\mathfrak{h})$ be a real or complex reductive pair. Let $P\mathfrak{g}^* = \bigoplus_{n\geqslant 1} P^{2n-1}\mathfrak{g}^*$ be the primitive subspace of $(\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ ([16, Ch. V, §5]). The inclusion $P\mathfrak{g}^* \subset (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ induces an isomorphism $\Lambda(P\mathfrak{g}^*) \simeq (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$. Fix a transgression $\tau : P^{2n-1}\mathfrak{g}^* \to (S^n\mathfrak{g}^*)^{\mathfrak{g}}$ in the Weil algebra of $\mathfrak{g}$ ([16, Ch. VI, §4]). We introduce a grading on an algebra $(S\mathfrak{h}^*)^{\mathfrak{h}} \otimes (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ by

$$\deg(Q\otimes\alpha) = 2\deg Q + \deg\alpha \qquad (Q \in (S\mathfrak{h}^*)^{\mathfrak{h}},\ \alpha \in (\Lambda\mathfrak{g}^*)^{\mathfrak{g}})$$

and define a differential δ on $(S\mathfrak{h}^*)^{\mathfrak{h}} \otimes (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}$ by

$$\delta(Q\otimes 1) = 0, \quad \delta(1\otimes\alpha) = -\tau(\alpha)|_{\mathfrak{h}} \otimes 1 \qquad (Q \in (S\mathfrak{h}^*)^{\mathfrak{h}},\ \alpha \in P\mathfrak{g}^*).$$

Cartan constructed a quasi-isomorphism of differential graded algebras (i.e. a homomorphism that induces isomorphism on cohomology)

$$\phi : ((S\mathfrak{h}^*)^{\mathfrak{h}} \otimes (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}, \delta) \to ((\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d)$$

([16, Ch. X, §2]). This ϕ is functorial in $\mathfrak{h}$, namely, a diagram

$$\begin{CD} ((S\mathfrak{h}^*)^{\mathfrak{h}} \otimes (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}, \delta) @>{\phi}>> ((\Lambda(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}, d) \\ @V{\mathrm{rest}\otimes 1}VV @VV{i}V \\ ((S\mathfrak{l}^*)^{\mathfrak{l}} \otimes (\Lambda\mathfrak{g}^*)^{\mathfrak{g}}, \delta) @>{\phi}>> ((\Lambda(\mathfrak{g}/\mathfrak{l})^*)^{\mathfrak{l}}, d) \end{CD}$$

commutes for any subalgebra $\mathfrak{l}$ of $\mathfrak{h}$ that is reductive in $\mathfrak{g}$, where rest : $(S\mathfrak{h}^*)^{\mathfrak{h}} \to (S\mathfrak{l}^*)^{\mathfrak{l}}$ denotes the restriction map.

5.4.2 A sufficient condition for Theorem 5.1.1 in terms of invariants

Proposition 5.4.1. *A homogeneous space G/H of reductive type satisfies the assumptions of Theorem 5.1.1 (and therefore does not admit a compact Clifford–Klein form) if there exist a connected compact subgroup C of G and a homomorphism of graded algebras $\phi : (S\mathfrak{h}_{\mathbb{C}}^*)^{\mathfrak{h}_{\mathbb{C}}} \to (S\mathfrak{c}_{\mathbb{C}}^*)^{\mathfrak{c}_{\mathbb{C}}}$ such that*

(i) $\dim C > \dim K_H$,

(ii) *C contains a maximal torus T_H of K_H, and*

(iii) *the diagram*

$$\begin{array}{ccccc} (S\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}} & \xrightarrow{\mathrm{rest}} & (S\mathfrak{h}_{\mathbb{C}}^*)^{\mathfrak{h}_{\mathbb{C}}} & \xrightarrow{\mathrm{rest}} & S(\mathfrak{t}_H)_{\mathbb{C}}^* \\ & \searrow^{\mathrm{rest}} & \downarrow{\scriptstyle \phi} & \nearrow_{\mathrm{rest}} & \\ & & (S\mathfrak{c}_{\mathbb{C}}^*)^{\mathfrak{c}_{\mathbb{C}}} & & \end{array}$$

commutes.

Proof. It suffices to see that $\mathrm{im}(i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})) \subset I^N$. By (iii),

$$\begin{array}{ccc} ((S\mathfrak{h}_{\mathbb{C}}^*)^{\mathfrak{h}_{\mathbb{C}}} \otimes (\Lambda\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}, \delta) & \xrightarrow{\mathrm{rest}\otimes 1} & (S(\mathfrak{t}_H)_{\mathbb{C}}^* \otimes (\Lambda\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}, \delta) \\ \downarrow{\scriptstyle \phi\otimes 1} & \nearrow_{\mathrm{rest}\otimes 1} & \\ ((S\mathfrak{c}_{\mathbb{C}}^*)^{\mathfrak{c}_{\mathbb{C}}} \otimes (\Lambda\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}, \delta) & & \end{array}$$

is a commutative diagram of differential graded algebras. The induced commutative diagram on cohomology

$$\begin{array}{ccc} H^\bullet(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}; \mathbb{C}) & \xrightarrow{i} & H^\bullet(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{t}_H)_{\mathbb{C}}; \mathbb{C}) \\ \downarrow{\scriptstyle \phi\otimes 1} & \nearrow_{i} & \\ H^\bullet(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}}; \mathbb{C}) & & \end{array}$$

implies

$$\begin{aligned}&\operatorname{im}(i : H^N(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}; \mathbb{C}) \to H^N(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{t}_H)_{\mathbb{C}}; \mathbb{C}))\\ &\quad\subset \operatorname{im}(i : H^N(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}}; \mathbb{C}) \to H^N(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{t}_H)_{\mathbb{C}}; \mathbb{C})),\end{aligned}$$

or equivalently,

$$\operatorname{im}(i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})) \subset \operatorname{im}(i : H^N(\mathfrak{g}, \mathfrak{c}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})),$$

while

$$\operatorname{im}(i : H^N(\mathfrak{g}, \mathfrak{c}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})) \subset I^N$$

by (i). This completes the proof. □

5.5 The case of semisimple symmetric spaces

5.5.1 Semisimple symmetric pairs

Let $\mathfrak{g}$ be a real semisimple Lie algebra and σ an involution of $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{g}^\sigma$ and $\mathfrak{q} = \mathfrak{g}^{-\sigma}$ be the fixed point sets of σ and $-\sigma$, respectively. We call $(\mathfrak{g}, \mathfrak{h})$ a semisimple symmetric pair. We say that $(\mathfrak{g}, \mathfrak{h})$ is irreducible if $\mathfrak{g}$ is simple or $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{l} \oplus \mathfrak{l}, \Delta\mathfrak{l})$ for some real simple Lie algebra $\mathfrak{l}$. Every semisimple symmetric pair can be uniquely written as a direct sum of irreducible ones.

Take a Cartan involution θ of $\mathfrak{g}$ such that $\theta\sigma = \sigma\theta$. Put $\mathfrak{k} = \mathfrak{g}^\theta$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$. We have a direct sum decomposition $\mathfrak{g} = (\mathfrak{k}\cap\mathfrak{h})\oplus(\mathfrak{k}\cap\mathfrak{q})\oplus(\mathfrak{p}\cap\mathfrak{h})\oplus(\mathfrak{p}\cap\mathfrak{q})$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. For $\alpha \in \mathfrak{a}^*$, we put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [Y, X] = \alpha(Y)X \text{ for any } Y \in \mathfrak{a}\}.$$

Then $\Sigma = \{\alpha \in \mathfrak{a}^* : \mathfrak{g}_\alpha \neq 0\} \smallsetminus \{0\}$ satisfies the axioms of root system ([51, Th. 5]). We call Σ the restricted root system of $(\mathfrak{g}, \mathfrak{h})$. If $\mathfrak{h} = \mathfrak{k}$, then Σ is nothing but the restricted root system of the real semisimple Lie algebra $\mathfrak{g}$. We fix a simple system Ψ of Σ and write Σ^+ for the set of positive roots with respect to Ψ.

5.5.2 ε-families of semisimple symmetric pairs

Let us review the notion of an ε-family of semisimple symmetric pairs, which was introduced by Oshima–Sekiguchi [50]. A map $\varepsilon : \Sigma \to \{\pm 1\}$ is called a signature of Σ if it satisfies

- $\varepsilon(-\alpha) = \varepsilon(\alpha)$ for any $\alpha \in \Sigma$, and
- $\varepsilon(\alpha)\varepsilon(\beta) = \varepsilon(\alpha + \beta)$ for any $\alpha, \beta \in \Sigma$ with $\alpha + \beta \in \Sigma$.

Given a signature ε of Σ, we define an involution σ_ε of $\mathfrak{g}$ by

$$\sigma_\varepsilon(X) = \begin{cases} \sigma(X) & (X \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})), \\ \varepsilon(\alpha)\sigma(X) & (X \in \mathfrak{g}_\alpha,\ \alpha \in \Sigma). \end{cases}$$

We write $\mathfrak{h}_\varepsilon = \mathfrak{g}^{\sigma_\varepsilon}$ and $\mathfrak{q}_\varepsilon = \mathfrak{g}^{-\sigma_\varepsilon}$. It is easily checked that σ_ε commutes with σ and θ, and $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}_\varepsilon$. Thus, Σ is also a restricted root system of the semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$. The set $F((\mathfrak{g}, \mathfrak{h})) = \{(\mathfrak{g}, \mathfrak{h}_\varepsilon) : \varepsilon \text{ is a signature of } \Sigma\}$ is called an ε-family of semisimple symmetric pairs ([50, §6]).

Let $\alpha \in \Sigma$. Since the involution $\theta\sigma$ leaves $\mathfrak{g}_\alpha$ invariant, we have a direct sum decomposition $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha^+ \oplus \mathfrak{g}_\alpha^-$, where $\mathfrak{g}_\alpha^\pm$ are the eigenspaces of $\theta\sigma$ with eigenvalues ± 1, respectively. Put $m^\pm(\alpha; \mathfrak{h}) = \dim \mathfrak{g}_\alpha^\pm$. Note that $m^\pm(\alpha; \mathfrak{h}) = m^\pm(-\alpha; \mathfrak{h})$. If ε is a signature of Σ, we have

$$m^\pm(\alpha; \mathfrak{h}_\varepsilon) = \begin{cases} m^\pm(\alpha; \mathfrak{h}) & \text{if } \varepsilon(\alpha) = 1, \\ m^\mp(\alpha; \mathfrak{h}) & \text{if } \varepsilon(\alpha) = -1. \end{cases}$$

A semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is said to be basic if $m^+(\alpha; \mathfrak{h}) \geqslant m^-(\alpha; \mathfrak{h})$ for any $\alpha \in \Sigma$ with $\alpha/2 \notin \Sigma$ ([50, Def. 6.4]). A typical example of a basic pair is a Riemannian symmetric pair $(\mathfrak{g}, \mathfrak{k})$. For any ε-family F of semisimple symmetric pairs, there exists a basic pair in F unique up to isomorphism ([50, Prop. 6.5]).

5.5.3 A characterization of the basic pairs

The following result should be known to experts, but we give a proof for the sake of completeness.

Lemma 5.5.1. *If a semisimple symmetric pair* $(\mathfrak{g}, \mathfrak{h})$ *is basic, an inequality* $\dim(\mathfrak{k} \cap \mathfrak{h}) \geqslant \dim(\mathfrak{k} \cap \mathfrak{h}_\varepsilon)$ *holds for any signature* ε *of* Σ*. Equality is attained if and only if* $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ *is also basic.*

Proof. There is a direct sum decomposition

$$\mathfrak{k} \cap \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{k} \cap \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma^+} \{X + \sigma(X) : X \in \mathfrak{g}_\alpha^+\}.$$

Hence,

$$\dim(\mathfrak{k} \cap \mathfrak{h}) = \dim(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{k} \cap \mathfrak{h}) + \sum_{\alpha \in \Sigma^+} m^+(\alpha; \mathfrak{h}).$$

Similarly,

$$\begin{aligned} \dim(\mathfrak{k} \cap \mathfrak{h}_\varepsilon) &= \dim(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{k} \cap \mathfrak{h}_\varepsilon) + \sum_{\alpha \in \Sigma^+} m^+(\alpha; \mathfrak{h}_\varepsilon) \\ &= \dim(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{k} \cap \mathfrak{h}) + \sum_{\substack{\alpha \in \Sigma^+, \\ \varepsilon(\alpha) = 1}} m^+(\alpha; \mathfrak{h}) + \sum_{\substack{\alpha \in \Sigma^+, \\ \varepsilon(\alpha) = -1}} m^-(\alpha; \mathfrak{h}). \end{aligned}$$

Notice that $\alpha/2 \notin \Sigma$ if $\varepsilon(\alpha) = -1$. Since $(\mathfrak{g}, \mathfrak{h})$ is basic, we have

$$\dim(\mathfrak{k} \cap \mathfrak{h}) - \dim(\mathfrak{k} \cap \mathfrak{h}_\varepsilon) = \sum_{\substack{\alpha \in \Sigma^+, \\ \varepsilon(\alpha) = -1}} (m^+(\alpha; \mathfrak{h}) - m^-(\alpha; \mathfrak{h})) \geqslant 0.$$

If equality is attained, then $m^+(\alpha; \mathfrak{h}) = m^-(\alpha; \mathfrak{h})$ for any $\alpha \in \Sigma^+$ with $\varepsilon(\alpha) = -1$. This implies that $m^\pm(\alpha; \mathfrak{h}_\varepsilon) = m^\pm(\alpha; \mathfrak{h})$ for any $\alpha \in \Sigma$, thus $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ is basic. Conversely, if $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ is basic, equality is clearly attained. □

5.5.4 Half-signatures

We say that $\delta : \Sigma \to \{\pm 1, \pm\sqrt{-1}\}$ is a half-signature of Σ if it satisfies

- $\delta(-\alpha) = \delta(\alpha)^{-1}$ for any $\alpha \in \Sigma$, and
- $\delta(\alpha)\delta(\beta) = \delta(\alpha + \beta)$ for any $\alpha, \beta \in \Sigma$ with $\alpha + \beta \in \Sigma$.

We remark that any map $\Psi \to \{\pm 1, \pm\sqrt{-1}\}$ (resp. $\Psi \to \{\pm 1\}$) is uniquely extended to a half-signature (resp. signature) of Σ. Hence, for each signature ε of Σ, there exist 2^r half-signatures δ such that $\delta^2 = \varepsilon$ ($r = \dim \mathfrak{a}$). Given a half-signature δ of Σ, we define an automorphism f_δ of $\mathfrak{g}_\mathbb{C}$ by

$$f_\delta(X) = \begin{cases} X & (X \in (\mathfrak{z}_\mathfrak{g}(\mathfrak{a}))_\mathbb{C}), \\ \delta(\alpha) X & (X \in (\mathfrak{g}_\alpha)_\mathbb{C},\ \alpha \in \Sigma) \end{cases}$$

and put $\mathfrak{g}_\delta = \{X \in \mathfrak{g} : f_\delta(X) = X\}$.

Lemma 5.5.2 (cf. [49, Lem. 1.3])**.** *Let ε be a signature of Σ and δ a half-signature of Σ such that $\delta^2 = \varepsilon$. Then,*

(1) $f_\delta(\mathfrak{h}_\mathbb{C}) = (\mathfrak{h}_\varepsilon)_\mathbb{C}$.

(2) $\mathfrak{h} \cap \mathfrak{g}_\delta = \mathfrak{h}_\varepsilon \cap \mathfrak{g}_\delta$.

Proof. These immediately follow from

$$\begin{aligned} \mathfrak{h} =& \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \cap \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma^+} \{X + \sigma(X) : X \in \mathfrak{g}_\alpha\}, \\ \mathfrak{h}_\varepsilon =& \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \cap \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Sigma^+, \\ \varepsilon(\alpha) = 1}} \{X + \sigma(X) : X \in \mathfrak{g}_\alpha\} \\ & \qquad \oplus \bigoplus_{\substack{\alpha \in \Sigma^+, \\ \varepsilon(\alpha) = -1}} \{X - \sigma(X) : X \in \mathfrak{g}_\alpha\}, \\ \mathfrak{g}_\delta =& \mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \oplus \bigoplus_{\substack{\alpha \in \Sigma^+, \\ \delta(\alpha) = 1}} \{X + \sigma(Y) : X, Y \in \mathfrak{g}_\alpha\}. \end{aligned}$$

□

5.5.5 Semisimple symmetric spaces

Let G be a connected linear semisimple Lie group whose Lie algebra is $\mathfrak{g}$. Suppose that the involution θ of $\mathfrak{g}$ lifts to G, and let H be an open subgroup of G^θ. The homogeneous space G/H is called a semisimple symmetric space associated with the semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$. In this chapter, we always assume that H is connected. Let θ be a Cartan involution of G such that $\theta\sigma = \sigma\theta$. Then $K = G^\theta$ and $K_H = K \cap H$ are maximal compact subgroups of G and H, respectively. Let ε be a signature of the restricted root system Σ of $(\mathfrak{g}, \mathfrak{h})$. The involution σ_ε of $\mathfrak{g}$ lifts to G ([54, Lem. 1.6]). Let H_ε be the identity component of G^{σ_ε}. Then $K_{H_\varepsilon} = K \cap H_\varepsilon$ is a maximal compact subgroup of H_ε.

5.5.6 A sufficient condition for Proposition 5.4.1 in terms of ε-families

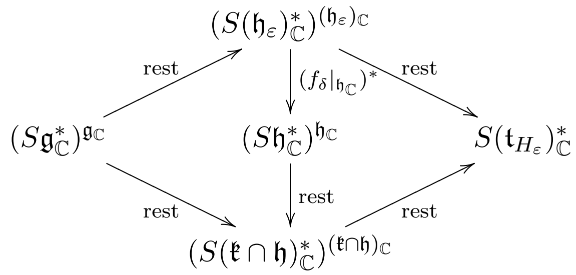
Now, we can prove:

Proposition 5.5.3. *Let G/H a semisimple symmetric space such that $(\mathfrak{g}, \mathfrak{h})$ is basic. Let ε be a signature of Σ such that $(\mathfrak{g}, \mathfrak{h}_\varepsilon)$ is not basic. Let δ be a half-signature of Σ such that $\delta^2 = \varepsilon$. If $\operatorname{rank}(\mathfrak{k} \cap \mathfrak{h}_\varepsilon) = \operatorname{rank}(\mathfrak{k} \cap \mathfrak{h}_\varepsilon \cap \mathfrak{g}_\delta)$, then G/H_ε satisfies the assumptions of Proposition 5.4.1 (and therefore does not admit a compact Clifford–Klein form).*

Proof. Let

$$(f_\delta|_{\mathfrak{h}_\mathbb{C}})^* : (S(\mathfrak{h}_\varepsilon)^*_\mathbb{C})^{(\mathfrak{h}_\varepsilon)_\mathbb{C}} \xrightarrow{\sim} (S\mathfrak{h}^*_\mathbb{C})^{\mathfrak{h}_\mathbb{C}}$$

be the isomorphism induced by $f_\delta|_{\mathfrak{h}_\mathbb{C}} : \mathfrak{h}_\mathbb{C} \xrightarrow{\sim} (\mathfrak{h}_\varepsilon)_\mathbb{C}$ (Lemma 5.5.2 (1)). Take a maximal torus T_{H_ε} of $K \cap H_\varepsilon$ such that $\mathfrak{t}_{H_\varepsilon} \subset \mathfrak{k} \cap \mathfrak{h}_\varepsilon \cap \mathfrak{g}_\delta$, which exists by the rank assumption. By Lemma 5.5.2 (2), we have $T_{H_\varepsilon} \subset K \cap H$. Consider the following diagram:



The right-hand side triangle clearly commutes since $f_\delta|_{(\mathfrak{t}_{H_\varepsilon})_\mathbb{C}}$ is the identity map of $(\mathfrak{t}_{H_\varepsilon})_\mathbb{C}$. Take a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. The restriction map rest : $(S\mathfrak{g}^*_\mathbb{C})^{\mathfrak{g}_\mathbb{C}} \to S\mathfrak{j}^*_\mathbb{C}$ is injective by Chevalley's restriction theorem ([9, Ch. VIII, §8, no. 3, Th. 1]), while $f_\delta|_{\mathfrak{j}_\mathbb{C}}$ is the identity map of $\mathfrak{j}_\mathbb{C}$. This shows that $f^*_\delta : (S\mathfrak{g}^*_\mathbb{C})^{\mathfrak{g}_\mathbb{C}} \xrightarrow{\sim} (S\mathfrak{g}^*_\mathbb{C})^{\mathfrak{g}_\mathbb{C}}$ is the identity map, and therefore the left-hand side triangle commutes. We conclude by Lemma 5.5.1 that G/H_ε satisfies the assumptions of Proposition 5.4.1. □

5.6 Examples

5.6.1 Nonbasic semisimple symmetric spaces of type (C, R)

Let us consider a semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ its real form (this case is called "type (C, R)" in [50]). Let ε be a signature of a restricted root system Σ of $(\mathfrak{g}, \mathfrak{h})$ and δ any half-signature such that $\delta^2 = \varepsilon$. It is easy to check that $\mathfrak{h}_\varepsilon$ is a real form of $\mathfrak{g}$. In this case, $\sqrt{-1}\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{k} \cap \mathfrak{h}_\varepsilon = \sqrt{-1}\mathfrak{p} \cap \mathfrak{q}_\varepsilon$ and is contained in $\mathfrak{k} \cap \mathfrak{h}_\varepsilon \cap \mathfrak{g}_\delta$. This implies $\operatorname{rank}(\mathfrak{k} \cap \mathfrak{h}_\varepsilon) = \operatorname{rank}(\mathfrak{k} \cap \mathfrak{h}_\varepsilon \cap \mathfrak{g}_\delta)$. From Proposition 5.5.3, we conclude:

Corollary 5.6.1. *Let G be a connected complex semisimple Lie group and H its connected real form. If the semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is not basic, G/H does not admit a compact Clifford–Klein form.*

For the reader's convenience, we list in Table 5.2 the nonbasic pair $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{g}$ is a simple complex Lie algebra and $\mathfrak{h}$ its real form (cf. [50, §§1, 6]). The sign $\star$ in Table 5.2 signifies that the nonexistence of a compact Clifford–Klein form seems to be a new result.

	$\mathfrak{g}$	$\mathfrak{h}$	Conditions
	$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sl}(2n, \mathbb{R})$	$n \geqslant 1$
$\star$	$\mathfrak{sl}(p+q, \mathbb{C})$	$\mathfrak{su}(p, q)$	$p, q \geqslant 1$
$\star$	$\mathfrak{so}(p+q, \mathbb{C})$	$\mathfrak{so}(p, q)$	$p, q \geqslant 2$, $(p, q) \neq (2, 2)$
$\star$	$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n, 1)$	$n \geqslant 1$
$\star$	$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}^*(2n)$	$n \geqslant 3$
	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{R})$	$n \geqslant 1$
$\star$	$\mathfrak{sp}(p+q, \mathbb{C})$	$\mathfrak{sp}(p, q)$	$p, q \geqslant 1$
	$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(6)}$	—
	$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(2)}$	—
$\star$	$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(-14)}$	—
	$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(7)}$	—
$\star$	$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-5)}$	—
$\star$	$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$	—
	$\mathfrak{e}_{8,\mathbb{C}}$	$\mathfrak{e}_{8(8)}$	—
$\star$	$\mathfrak{e}_{8,\mathbb{C}}$	$\mathfrak{e}_{8(-24)}$	—
	$\mathfrak{f}_{4,\mathbb{C}}$	$\mathfrak{f}_{4(4)}$	—
$\star$	$\mathfrak{f}_{4,\mathbb{C}}$	$\mathfrak{f}_{4(-20)}$	—
	$\mathfrak{g}_{2,\mathbb{C}}$	$\mathfrak{g}_{2(2)}$	—

Table 5.2: Irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ to which Corollary 5.6.1 is applicable

5.6.2 Half-signatures arising from hyperbolic elements

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple symmetric pair. As before, let σ be an involution of $\mathfrak{g}$ corresponding to $\mathfrak{h}$ and θ a Cartan involution of $\mathfrak{g}$ commuting with σ. We define $\mathfrak{h}^a = \mathfrak{g}^{\sigma\theta}$ $(= (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}))$ and $\mathfrak{q}^a = \mathfrak{g}^{-\sigma\theta}$ $(= (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}))$. The semisimple symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$ is called the associated pair of $(\mathfrak{g}, \mathfrak{h})$.

Corollary 5.6.2. *A semisimple symmetric space G/H does not admit a compact Clifford–Klein form if $\mathfrak{h}^a = \mathfrak{z}_{\mathfrak{g}}(X_0)$ for some $X_0 \in \mathfrak{p} \smallsetminus \{0\}$.*

Proof. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing X_0 and Σ the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We have a direct sum decomposition $\mathfrak{q}^a = \mathfrak{q}^a_+ \oplus \mathfrak{q}^a_-$, where

$$\mathfrak{q}^a_+ = \bigoplus_{\substack{\alpha \in \Sigma, \\ \alpha(X_0) > 0}} \mathfrak{g}_\alpha, \qquad \mathfrak{q}^a_- = \bigoplus_{\substack{\alpha \in \Sigma, \\ \alpha(X_0) < 0}} \mathfrak{g}_\alpha.$$

It is easily checked that $[\mathfrak{q}^a_+, \mathfrak{q}^a_+] = [\mathfrak{q}^a_-, \mathfrak{q}^a_-] = 0$, $[\mathfrak{q}^a_+, \mathfrak{q}^a_-] \subset \mathfrak{h}^a$. Hence, a map

$$\delta : \Sigma \to \{\pm 1, \pm\sqrt{-1}\}, \qquad \alpha \mapsto \begin{cases} 0 & \text{if } \alpha(X_0) = 0, \\ \sqrt{-1} & \text{if } \alpha(X_0) > 0, \\ -\sqrt{-1} & \text{if } \alpha(X_0) < 0. \end{cases}$$

is a half-signature of Σ. Put $\varepsilon = \delta^2$. By construction, we have $\mathfrak{h} = \mathfrak{k}_\varepsilon$ and $\mathfrak{h}^a = \mathfrak{g}_\delta$. Thus $\mathfrak{k} \cap \mathfrak{k}_\varepsilon = \mathfrak{k} \cap \mathfrak{k}_\varepsilon \cap \mathfrak{g}_\delta = \mathfrak{k} \cap \mathfrak{h}$. Now, Corollary 5.6.2 follows from Proposition 5.5.3 if we could prove that $(\mathfrak{g}, \mathfrak{h})$ is not basic. If $(\mathfrak{g}, \mathfrak{h})$ is basic, i.e. $\mathfrak{h}$ is isomorphic to $\mathfrak{k}$, then σ and θ are two commuting Cartan involutions of $\mathfrak{g}$, hence $\sigma = \theta$ ([19, Proof of Cor. 6.19]). Then $\mathfrak{g} = \mathfrak{h}^a = \mathfrak{z}_{\mathfrak{g}}(X_0)$, which is absurd. □

For the reader's convenience, we list in Table 5.3 the irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{h}^a = \mathfrak{z}_{\mathfrak{g}}(X_0)$ for some $X_0 \in \mathfrak{p} \smallsetminus \{0\}$ (cf. [50, §1]). The sign $\star$ in Table 5.3 signifies that the nonexistence of a compact Clifford–Klein form seems to be a new result. We remark that some examples such as $(\mathfrak{sl}(p+q, \mathbb{C}), \mathfrak{su}(p,q))$ $(p, q \geqslant 1)$ appear in both Table 5.2 and Table 5.3.

	$\mathfrak{g}$	$\mathfrak{h}$	Conditions
$\star$	$\mathfrak{sl}(p+q, \mathbb{C})$	$\mathfrak{su}(p,q)$	$p, q \geqslant 1$
$\star$	$\mathfrak{sl}(p+q, \mathbb{R})$	$\mathfrak{so}(p,q)$	$p, q \geqslant 1$

$\star$	$\mathfrak{sl}(p+q,\mathbb{H})$	$\mathfrak{sp}(p,q)$	$p,q \geqslant 1$
	$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}$	$n \geqslant 1$
$\star$	$\mathfrak{so}(n+2,\mathbb{C})$	$\mathfrak{so}(n,2)$	$n \geqslant 3$
$\star$	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}^*(2n)$	$n \geqslant 3$
	$\mathfrak{so}(p+1,q+1)$	$\mathfrak{so}(p,1) \oplus \mathfrak{so}(1,q)$	$p,q \geqslant 0,\ (p,q) \neq (0,0),(1,1)$
	$\mathfrak{so}(n,n)$	$\mathfrak{so}(n,\mathbb{C})$	$n \geqslant 3$
	$\mathfrak{so}^*(4n)$	$\mathfrak{sl}(n,\mathbb{H}) \oplus \mathbb{R}$	$n \geqslant 2$
	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$	$n \geqslant 1$
	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}$	$n \geqslant 1$
	$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbb{C})$	$n \geqslant 1$
$\star$	$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{e}_{6(-14)}$	—
$\star$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2,2)$	—
	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$	—
$\star$	$\mathfrak{e}_{7,\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$	—
	$\mathfrak{e}_{7(7)}$	$\mathfrak{sl}(4,\mathbb{H})$	—
	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$	—

Table 5.3: Irreducible symmetric pairs to which Corollary 5.6.2 is applicable

5.6.3 Examples obtained by direct computations

In Subsections 5.6.1 and 5.6.2, we systematically constructed examples of a homogeneous space that does not admit a compact Clifford–Klein form using Proposition 5.5.3. In this subsection, we shall give some examples via direct verification of Proposition 5.4.1.

To fix notations, we give explicit generators of $(S^k\mathfrak{g}_{\mathbb{C}}^*)^{\mathfrak{g}_{\mathbb{C}}}$ for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{so}(n,\mathbb{C})$. Define invariant polynomials $f_k \in (S^k(\mathfrak{gl}(n,\mathbb{C})^*))^{\mathfrak{gl}(n,\mathbb{C})}$ $(k = 1,2,\dots,n)$ on the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ by

$$\det(\lambda I_n - X) = \lambda^n + f_1(X)\lambda^{n-1} + f_2(X)\lambda^{n-2} + \cdots + f_n(X) \quad (X \in \mathfrak{gl}(n,\mathbb{C})).$$

For the convenience, we put $f_0 = 1$ and $f_k = 0$ for $k < 0$ and $k > n$. We use the same notation f_k for the restriction of f_k to $\mathfrak{sl}(n,\mathbb{C})$ or to $\mathfrak{so}(n,\mathbb{C})$. Then,

Fact 5.6.3 ([9, Ch. VIII, §13]). (1) *The $n-1$ elements $f_2, f_3, \dots, f_n$ generate the algebra $(S(\mathfrak{sl}(n,\mathbb{C})^*))^{\mathfrak{sl}(n,\mathbb{C})}$ and are algebraically independent. We have $f_1 = 0$.*

(2) *If* $n = 2m+1$, *the* m *elements* $f_2, f_4, \dots, f_{2m}$ *generate the algebra* $(S(\mathfrak{so}(n,\mathbb{C})^*))^{\mathfrak{so}(n,\mathbb{C})}$ *and are algebraically independent. We have* $f_1 = f_3 = \cdots = f_{2m+1} = 0$.

(3) *If* $n = 2m$, *the* m *elements* $f_2, f_4, \dots, f_{2m-2}, \tilde{f}$ *generate the algebra* $(S(\mathfrak{so}(n,\mathbb{C})^*))^{\mathfrak{so}(n,\mathbb{C})}$ *and are algebraically independent, where* $\tilde{f} \in (S^m(\mathfrak{so}(n,\mathbb{C})^*))^{\mathfrak{so}(n,\mathbb{C})}$ *is the Pfaffian of* $n \times n$ *skew-symmeric matrices. We have* $f_1 = f_3 = \cdots = f_{2m-1} = 0$ *and* $f_{2m} = \tilde{f}^2$.

Corollary 5.6.4. *A homogeneous space*

$$\mathrm{SO}_0(p+r,q)/\mathrm{SO}_0(p,q) \qquad (p,q,r \geqslant 1,\ q : \textit{odd})$$

does not admit a compact Clifford–Klein form.

Proof. Put $G = \mathrm{SO}_0(p+r,q)$ and $H = \mathrm{SO}_0(p,q)$. When p is odd, this has been already proved in Chapter 4 (see Table 4.1). Let p be even. Take a connected compact subgroup C of G to be

$$C = \mathrm{SO}(p+1) \times \mathrm{SO}(q) \subset \mathrm{SO}_0(p+r,q) = G$$

and define a homomorphism

$$\begin{aligned}\phi : (S\mathfrak{h}_{\mathbb{C}}^*)^{\mathfrak{h}_{\mathbb{C}}} &= (S(\mathfrak{so}(p+q,\mathbb{C})^*))^{\mathfrak{so}(p+q,\mathbb{C})} \\ &\to (S(\mathfrak{so}(p+1,\mathbb{C})^*))^{\mathfrak{so}(p+1,\mathbb{C})} \otimes (S(\mathfrak{so}(q,\mathbb{C})^*))^{\mathfrak{so}(q,\mathbb{C})} = (S\mathfrak{c}_{\mathbb{C}}^*)^{\mathfrak{c}_{\mathbb{C}}}\end{aligned}$$

by

$$\phi(f_{2k}) = \sum_{i+j=k} f_{2i} \otimes f_{2j} \qquad (1 \leqslant k \leqslant \frac{p+q-1}{2}).$$

Then C and ϕ satisfy the assumptions of Proposition 5.4.1. □

Corollary 5.6.5. *An irreducible symmetric space*

$$\mathrm{SO}_0(p+r,q)/(\mathrm{SO}_0(p,q) \times \mathrm{SO}(r)) \qquad (p,q,r \geqslant 1,\ q : \textit{odd})$$

does not admit a compact Clifford–Klein form.

Proof. This is immediate from Corollary 5.6.4 since $\mathrm{SO}(r)$ is compact. □

Remark 5.6.6. Corollary 5.6.5 seems new when p is even, q is odd, the inequality $p + r \leqslant q$ holds and $(p,q,r) \neq (2k, 2k+1, 1)$ $(k \geqslant 1)$. It was also previously known that $\mathrm{SO}_0(p+r,q)/(\mathrm{SO}_0(p,q) \times \mathrm{SO}(r))$ $(p,q,r \geqslant 1)$ does not admit a compact Clifford–Klein form if p, q are both odd, the inequality $p + r > q$ holds or $(p,q,r) = (2k, 2k+1, 1)$ $(k \geqslant 1)$. On the other hand, it admits a compact Clifford–Klein form if $(p,q,r) = (1,2n,1), (3,4n,1), (1,4,2), (1,4,3)$ or $(7,8,1)$ $(n \geqslant 1)$ (cf. Remarks 5.1.5 and 5.1.6). The remaining cases are, as far as the author knows, open.

Corollary 5.6.7. *A homogeneous space*

$$\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R}) \qquad (n > m \geqslant 2,\ m : \text{even})$$

does not admit a compact Clifford–Klein form.

Proof. Put $G = \mathrm{SL}(n,\mathbb{R})$ and $H = \mathrm{SL}(m,\mathbb{R})$. Take a connected compact subgroup C of G to be

$$C = \mathrm{SO}(m+1) \subset \mathrm{SL}(n,\mathbb{R}) = G$$

and define a homomorphism

$$\phi : (S\mathfrak{h}^*_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}} = (S(\mathfrak{sl}(m,\mathbb{C})^*))^{\mathfrak{sl}(m,\mathbb{C})} \to (S(\mathfrak{so}(m+1,\mathbb{C})^*))^{\mathfrak{so}(m+1,\mathbb{C})} = (S\mathfrak{c}^*_{\mathbb{C}})^{\mathfrak{c}_{\mathbb{C}}}$$

by $\phi(f_k) = f_k$ $(2 \leqslant f_k \leqslant m)$. Then C and ϕ satisfy the assumptions of Proposition 5.4.1. □

Remark 5.6.8. Corollary 5.6.7 seems new when $(n,m) = (2k+2, 2k)$ $(k \geqslant 2)$. It was also previously known that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ $(n > m \geqslant 2)$ does not admit a compact Clifford–Klein form if $n-3 \geqslant m$, $(n,m) = (2k+1, 2k)$ $(k \geqslant 1)$ or $(n,m) = (4,2)$ (cf. Remark 5.1.5). The remaining cases $(n,m) = (2k+3, 2k+1), (2k+2, 2k+1)$ $(k \geqslant 1)$ are, as far as the author knows, open.

5.6.4 Enlargement of Lie groups

The following lemma provides some other examples of a homogeneous space without compact Clifford–Klein forms.

Lemma 5.6.9. *Let G/H be a homogeneous space of reductive type satisfying the assumptions of Proposition 5.4.1. Let $\tilde{G}$ be a connected Lie group containing G as a closed subgroup. Let L be a connected closed subgroup of $\tilde{G}$ such that $G \cap L = \{1\}$, $L \subset Z_{\tilde{G}}(G)$ and $\tilde{G}/(H \times L)$ is a homogeneous space of reductive type. Then $\tilde{G}/(H \times L)$ satisfies the assumptions of Proposition 5.4.1 (and therefore does not admit a compact Clifford–Klein form).*

Proof. Let C be a connected compact subgroup of G and $\phi : (S\mathfrak{h}^*_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}} \to (S\mathfrak{c}^*_{\mathbb{C}})^{\mathfrak{c}_{\mathbb{C}}}$ a homomorphism of graded algebras satisfying the conditions (i)–(iii) of Proposition 5.4.1. Let K_L be a maximal compact subgroup of L and

$$\mathrm{rest} : (S\mathfrak{l}^*_{\mathbb{C}})^{\mathfrak{l}_{\mathbb{C}}} \to (S(\mathfrak{k}_L)^*_{\mathbb{C}})^{(\mathfrak{k}_L)_{\mathbb{C}}}$$

denote the restriction map. Let $\tilde{C} = C \times K_L$ and

$$\begin{aligned}\tilde{\phi} : (S(\mathfrak{h} \oplus \mathfrak{l})^*_{\mathbb{C}})^{(\mathfrak{h}\oplus\mathfrak{l})_{\mathbb{C}}} &= (S\mathfrak{h}^*_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}} \otimes (S\mathfrak{l}^*_{\mathbb{C}})^{\mathfrak{l}_{\mathbb{C}}} \\ &\xrightarrow{\phi\otimes\mathrm{rest}} (S\mathfrak{c}^*_{\mathbb{C}})^{\mathfrak{c}_{\mathbb{C}}} \otimes (S(\mathfrak{k}_L)^*_{\mathbb{C}})^{(\mathfrak{k}_L)_{\mathbb{C}}} = (S(\mathfrak{c} \oplus \mathfrak{k}_L)^*_{\mathbb{C}})^{(\mathfrak{c}\oplus\mathfrak{k}_L)_{\mathbb{C}}}.\end{aligned}$$

Then $\tilde{C}$ and $\tilde{\phi}$ satisfy the conditions (i)–(iii) with respect to $\tilde{G}/(H \times L)$. □

For instance,

$$\mathrm{SL}(p_1 + \cdots + p_n + q, \mathbb{R})/(\mathrm{SL}(p_1, \mathbb{R}) \times \cdots \times \mathrm{SL}(p_n, \mathbb{R}))$$
$$(n \geqslant 1,\ p_1, \ldots, p_n \geqslant 2,\ \prod_i p_i : \text{even},\ q \geqslant 1)$$

does not admit a compact Clifford–Klein form by Lemma 5.6.9 and the proof of Corollary 5.6.7.

5.6.5 Relation with Theorem 2.1.2 (2)

We proved in Chapter 2 the following result:

Fact 5.6.10 (Theorem 2.1.2 (2))**.** *Let G be a Lie group and H its closed subgroup with finitely many connected components. Let K_H be a maximal compact subgroup of H. If the homomorphism*

$$i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R}) \qquad (N = \dim G - \dim H)$$

is not injective, there is no compact manifold locally modelled on G/H.

If the homomorphism $i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$ is not injective,

$$\mathrm{im}(i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})) \subset I^N$$

trivially holds. Thus Theorem 5.1.1 yields the following corollary, which is slightly weaker than Theorem 2.1.2 (2):

Corollary 5.6.11. *Let G be a connected linear Lie group and H its connected closed subgroup. Let K_H be a maximal compact subgroup of H. If the homomorphism*

$$i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R}) \qquad (N = \dim G - \dim H)$$

is not injective, G/H does not admit a compact Clifford–Klein form.

Remark 5.6.12. Note that $i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})$ is injective if and only if so is $i : H^N(\mathfrak{g}, \mathfrak{h}; \mathbb{R}) \to H^N(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R})$, for $i : H^\bullet(\mathfrak{g}, \mathfrak{k}_H; \mathbb{R}) \to H^\bullet(\mathfrak{g}, \mathfrak{t}_H; \mathbb{R})$ is always injective by a variant of the splitting principle ([17, Th. 6.8.2]).

Bibliography

[1] M. Atiyah and W. Schmid, A geometric construction of the discrete series for semisimple Lie groups. *Invent. Math.* **42** (1977), 1–62; Erratum: *Invent. Math.* **54** (1979), 189–192.

[2] Y. Benoist, Actions propres sur les espaces homogènes réductifs. *Ann. of Math. (2)* **144** (1996), 315–347.

[3] Y. Benoist and F. Labourie, Sur les espaces homogènes modèles de variétés compactes. *Inst. Hautes Études Sci. Publ. Math.* **76** (1992), 99–109.

[4] M. Berger, Les espaces symétriques noncompacts. *Ann. Sci. École Norm. Sup. (3)* **74** (1957), 85–177.

[5] A. Borel, Compact Clifford–Klein forms of symmetric spaces. *Topology* **2** (1963), 111–122.

[6] ———, *Semisimple groups and Riemannian symmetric spaces.* Texts and Readings in Mathematics, 16. Hindustan Book Agency, New Delhi, 1998.

[7] A. Borel and Harish-Chandra, Arithmetic subgroups of algebraic groups. *Ann. of Math. (2)* **75** (1962), 485–535.

[8] R. Bott and A. Haefliger, On characteristic classes of Γ-foliations. *Bull. Amer. Math. Soc.* **78** (1972), 1039–1044.

[9] N. Bourbaki, *Lie groups and Lie algebras. Chapters 7–9. Translated from the 1975 and 1982 French originals by Andrew Pressley.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005.

[10] E. Calabi and L. Markus, Relativistic space forms. *Ann. of Math. (2)* **75** (1962), 63–76.

[11] H. Cartan, La transgression dans un groupe de Lie et dans un espace fibré principal. *Colloque de topologie (espaces fibrés), Bruxelles, 1950.* pp. 57–71. Georges Thone, Liège; Masson et Cie., Paris, 1951.

[12] H. Cartan and S. Eilenberg, *Homological algebra.* Princeton University Press, Princeton, N. J., 1956.

[13] K. Corlette and R. J. Zimmer, Superrigidity for cocycles and hyperbolic geometry. *Internat. J. Math.* **5** (1994), 273–290.

[14] A. Dold, Partitions of unity in the theory of fibrations. *Ann. of Math. (2)* **78** (1963), 223–255.

[15] D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras.* "Nauka", Moscow, 1984 (in Russian); English transl.: Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986, translated by A. B. Sosinskii.

[16] W. Greub, S. Halperin and R. Vanstone, *Connections, curvature, and cohomology. Volume III: Cohomology of principal bundles and homogeneous spaces.* Pure and Applied Mathematics, Vol. 47-III. Academic Press, New York–London, 1976.

[17] V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory. With an appendix containing two reprints by Henri Cartan.* Mathematics Past and Present. Springer-Verlag, Berlin, 1999.

[18] G. Hochschild, *The structure of Lie groups.* Holden-Day, Inc., San Francisco-London-Amsterdam, 1965.

[19] A. W. Knapp, *Lie groups beyond an introduction. Second edition.* Progress in Mathematics, 140. Birkhäuser Boston, Inc., Boston, MA, 2002.

[20] T. Kobayashi, Proper action on a homogeneous space of reductive type. *Math. Ann.* **285** (1989), 249–263.

[21] ———, Homogeneous spaces with indefinite-metric and discontinuous groups. *36th Geometry Symposium,* pp. 104–116, 1989 (in Japanese), available at:
`http://www.ms.u-tokyo.ac.jp/~toshi/texpdf/tk12-kika-sympo89.pdf`

[22] ———, Discontinuous groups acting on homogeneous spaces of reductive type. *Representation theory of Lie groups and Lie algebras (Fuji-Kawaguchiko, 1990),* pp. 59–75, World Sci. Publ., River Edge, NJ, 1992.

[23] ———, A necessary condition for the existence of compact Clifford–Klein forms of homogeneous spaces of reductive type. *Duke Math. J.* **67** (1992), 653–664.

[24] ———, On discontinuous groups acting on homogeneous spaces with noncompact isotropy subgroups. *J. Geom. Phys.* **12** (1993), 133–144.

[25] ———, Criterion for proper actions on homogeneous spaces of reductive groups. *J. Lie Theory* **6** (1996), 147–163,

[26] ———, Discontinuous groups and Clifford-Klein forms of pseudo-Riemannian homogeneous manifolds. *Algebraic and analytic methods in representation theory (Sønderborg, 1994),* Perspectives in Mathematics, vol. 17, Academic Press, San Diego, CA, 1997, 99–165.

[27] ———, Discontinuous groups for non-Riemannian homogeneous spaces. *Sugaku no Saisentan, challenge to the 21th century,* vol. 1, Springer-Verlag, Tokyo, 2002, pp. 18–73 (in Japanese); revised and enlarged from the English version in *Mathematics unlimited–2001 and beyond,* Springer, Berlin, 2001, pp. 723–747.

[28] ———, Introduction to actions of discrete groups on pseudo-Riemannian homogeneous manifolds. The 2000 Twente Conference on Lie Groups (Enschede). *Acta Appl. Math.* **73** (2002), 115–131.

[29] ———, On discontinuous group actions on non-Riemannian homogeneous spaces, *Sūgaku* **57** (2005), 267–281 (in Japanese); English transl.: *Sugaku Expositions* **22** (2009), 1–19, translated by M. Reid.

[30] T. Kobayashi and K. Ono, Note on Hirzebruch's proportionality principle. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **37** (1990), 71–87.

[31] T. Kobayashi and T. Yoshino, Compact Clifford–Klein forms of symmetric spaces — revisited. *Pure Appl. Math. Q.* **1** (2005), 591–663.

[32] J.-L. Koszul, Homologie et cohomologie des algèbres de Lie. *Bull. Soc. Math. France* **78** (1950), 65–127.

[33] R. S. Kulkarni, Proper actions and pseudo-Riemannian space forms. *Adv. in Math.* **40** (1981), 10–51.

[34] F. Labourie, Quelques résultats récents sur les espaces localement homogènes compacts. *Manifolds and geometry (Pisa, 1993), Sympos. Math., XXXVI,* Cambridge Univ. Press, Cambridge, 1996, 267–283.

[35] F. Labourie, S. Mozes and R. J. Zimmer, On manifolds locally modelled on non-Riemannian homogeneous spaces. *Geom. Funct. Anal.* **5** (1995), 955–965.

[36] F. Labourie and R. J. Zimmer, On the non-existence of cocompact lattices for $SL(n)/SL(m)$. *Math. Res. Lett.* **2** (1995), 75–77.

[37] G. Margulis, Existence of compact quotients of homogeneous spaces, measurably proper actions, and decay of matrix coefficients. *Bull. Soc. Math. France* **125** (1997), 447–456.

[38] G. Margulis, Problems and conjectures in rigidity theory. *Mathematics: frontiers and perspectives,* pp. 161–174, Amer. Math. Soc., Providence, RI, 2000.

[39] Y. Matsushima and S. Murakami, On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds. *Ann. of Math. (2)* **78** (1963), 365–416.

[40] S. Morita, *Geometry of characteristic classes.* Iwanami Shoten, Publishers, Tokyo, 1999 (in Japanese); English transl.: Translations of Mathematical Monographs 199, Iwanami Series in Modern Mathematics, American Mathematical Society, Providence, RI, 2001, translated by S. Morita.

[41] Y. Morita, A topological necessary condition for the existence of compact Clifford–Klein forms. *J. Differential Geom.* **100** (2015), 533–545.

[42] ———, Semisimple symmetric spaces without compact manifolds locally modelled thereon. *Proc. Japan. Acad. Ser. A Math. Sci.* **90** (2015), 29–33.

[43] ———, Homogeneous spaces of nonreductive type locally modelling no compact manifold. arXiv:1508.04862v1, to appear in *Publ. Res. Inst. Math. Sci.*

[44] ———, A cohomological obstruction to the existence of Clifford–Klein forms. *Selecta Math. (N.S.),* DOI: 10.1007/s00029-016-0295-1 (online first publication).

[45] G. D. Mostow and T. Tamagawa, On the compactness of arithmetically defined homogeneous spaces. *Ann. of Math. (2)* **76** (1962), 446–463.

[46] H. Oh, Tempered subgroups and representations with minimal decay of matrix coefficients. *Bull. Soc. Math. France* **126** (1998), 355–380.

[47] T. Okuda, Classification of semisimple symmetric spaces with proper $SL(2,\mathbb{R})$-actions. *J. Differential Geom.* **94** (2013), 301–342.

[48] A. L. Onishchik, *Topology of transitive transformation groups.* Johann Ambrosius Barth Verlag GmbH, Leipzig, 1994.

[49] T. Oshima and J. Sekiguchi, Eigenspaces of invariant differential operators on an affine symmetric space. *Invent. Math.* **57** (1980), 1–81.

[50] ———, The restricted root system of a semisimple symmetric pair. *Group representations and systems of differential equations (Tokyo, 1982),* pp. 433–497, Adv. Stud. Pure Math., 4, North-Holland, Amsterdam, 1984.

[51] W. Rossmann, The structure of semisimple symmetric spaces. *Canad. J. Math.* **31** (1979), 157–180.

[52] A. Selberg, On discontinuous groups in higher-dimensional symmetric spaces. *Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960),* pp. 147–164, Tata Institute of Fundamental Research, Bombay, 1960.

[53] Y. Shalom, Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group. *Ann. of Math. (2)* **152** (2000), 113–182.

[54] N. Shimeno, A compact imbedding of semisimple symmetric spaces. *J. Math. Sci. Univ. Tokyo* **3** (1996), 551–569.

[55] M. Takeuchi, On Pontrjagin classes of compact symmetric spaces. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **9** (1962), 313–328.

[56] N. Tholozan, Volume and non-existence of compact Clifford–Klein forms. arXiv:1511.09448v2, preprint.

[57] G. Warner, *Harmonic analysis on semi-simple Lie groups. I.* Die Grundlehren der mathematischen Wissenschaften, Band 188. Springer-Verlag, New York-Heidelberg, 1972.

[58] H. Whitney, Elementary structure of real algebraic varieties. *Ann. of Math. (2)* **66** (1957), 545–556.

[59] J. A. Wolf, The Clifford–Klein space forms of indefinite metric. *Ann. of Math. (2)* **75** (1962), 77–80.

[60] ———, Isotropic manifolds of indefinite metric. *Comment. Math. Helv.* **39** (1964), 21–64.

[61] R. J. Zimmer, Discrete groups and non-Riemannian homogeneous spaces. *J. Amer. Math. Soc.* **7** (1994), 159–168.