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A Tropical Characterization of Algebraic Subvarieties of Toric Varieties over Non-Archimedean Fields

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Abstract. We study the tropicalizations of analytic subvarieties of normal toric varieties over complete non-archimedean valuation fields. We show that a Zariski closed analytic subvariety of a normal toric variety is algebraic if its tropicalization is a finite union of polyhedra. Previously, the converse direction was known by the theorem of Bieri and Groves. Over the field of complex numbers, Madani, L. Nisse, and M. Nisse proved similar results for analytic subvarieties of tori.

1. Introduction

We study the tropicalizations of analytic subvarieties of normal toric varieties over complete non-archimedean valuation fields. We shall give a characterization of algebraic subvarieties of normal toric varieties in terms of their tropicalizations.

First, we recall the definition of the *tropicalization* of Zariski closed analytic subvarieties of normal toric varieties; see [Pay09-1, Section 2 and Section 3] for details. Let K be a complete non-archimedean valuation field with non-trivial absolute value $|\cdot|$. Let M be a free $\mathbb{Z}$-module of finite rank. Let Y_Σ be the normal toric variety over K associated to a fan Σ in $N_\mathbb{R} := \mathrm{Hom}_\mathbb{Z}(M, \mathbb{R})$. For each cone $\sigma \in \Sigma$, the torus orbit $O(\sigma)$ corresponding to σ is isomorphic to the torus $\mathrm{Spec}(K[\sigma^\perp \cap M])$. The *tropicalization map*

$$\mathrm{Trop}\colon O(\sigma)^{\mathrm{an}} \to (N_\sigma)_\mathbb{R} := \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{R})$$

is the proper surjective continuous map given by

$$\mathrm{Trop}(|\cdot|_x) := -\log|\cdot|_x \colon \sigma^\perp \cap M \to \mathbb{R}$$

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for $|\cdot|_x \in O(\sigma)^{\mathrm{an}}$. We define the tropicalization map

$$\mathrm{Trop}\colon Y_\Sigma^{\mathrm{an}} = \bigsqcup_{\sigma\in\Sigma} O(\sigma)^{\mathrm{an}} \to \bigsqcup_{\sigma\in\Sigma} (N_\sigma)_\mathbb{R}$$

by gluing the tropicalization maps $\mathrm{Trop}\colon O(\sigma)^{\mathrm{an}} \to (N_\sigma)_\mathbb{R}$ together; see Section 2 for details. Here, for an algebraic variety Z over K, we denote by Z^{an} the *Berkovich analytic space* associated to Z; see [Ber90, Theorem 3.4.1]. For an irreducible Zariski closed analytic subvariety $X \subset Y_\Sigma^{\mathrm{an}}$, the image $\mathrm{Trop}(X)$ of X is called the *tropicalization* of X.

Gubler showed that for any cone $\sigma \in \Sigma$, the intersection $\mathrm{Trop}(X) \cap (N_\sigma)_\mathbb{R}$ is a *locally finite* union of polyhedra [Gub07, Theorem 1.1]. Here, we identify $(N_\sigma)_\mathbb{R}$ with $\mathbb{R}^{\dim O(\sigma)}$ by taking a $\mathbb{Z}$-basis of $\mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{Z})$. On the other hand, if $X \subset Y_\Sigma^{\mathrm{an}}$ is the analytification of a Zariski closed algebraic subvariety of Y_Σ, Bieri and Groves showed that the intersection $\mathrm{Trop}(X) \cap (N_\sigma)_\mathbb{R}$ is a *finite* union of polyhedra for any cone $\sigma \in \Sigma$ [BG84, Theorem A]. (See also [EKL06, Theorem 2.2.3].)

For an irreducible Zariski closed analytic subvariety $X \subset Y_\Sigma^{\mathrm{an}}$, there is a unique cone $\sigma_X \in \Sigma$ such that $X \cap O(\sigma_X)^{\mathrm{an}}$ is a dense Zariski open analytic subvariety of X.

The main theorem of this paper is as follows:

THEOREM 1.1. *Let Y_Σ, $X \subset Y_\Sigma^{\mathrm{an}}$, and $\sigma_X \in \Sigma$ be as above. Assume that* $\mathrm{Trop}(X) \cap (N_{\sigma_X})_\mathbb{R}$ *is a finite union of polyhedra. Then X is the analytification of a Zariski closed algebraic subvariety of Y_Σ.*

Conversely, if X is algebraic then $\mathrm{Trop}(X) \cap (N_\sigma)_\mathbb{R}$ is a finite union of polyhedra in $(N_\sigma)_\mathbb{R}$ for any cone $\sigma \in \Sigma$ by the theorem of Bieri-Groves [BG84, Theorem A]. Hence, we get the following:

COROLLARY 1.2. *Let Y_Σ and $X \subset Y_\Sigma^{\mathrm{an}}$ be as above. Then X is the analytification of a Zariski closed algebraic subvariety of Y_Σ if and only if* $\mathrm{Trop}(X) \cap (N_\sigma)_\mathbb{R}$ *is a finite union of polyhedra in $(N_\sigma)_\mathbb{R}$ for any cone $\sigma \in \Sigma$.*

Chow's theorem over nonarchimedean fields follows from Theorem 1.1 as follows. When Y_Σ is the projective space, the analytic subvariety X is compact. By [Mar15, Theorem 1], one can see that the tropicalization

$\mathrm{Trop}(X)$ is a finite union of polyhedra. Hence, by Theorem 1.1, the analytic subvariety X is the analytification of a Zariski closed algebraic subvariety of Y_Σ.

In [MNN14], Madani, L. Nisse, and M. Nisse proved similar results for analytic subvarieties of tori over the field of complex numbers.

This paper is organized as follows. In Section 2, we recall basic notions of Berkovich analytic geometry. In Section 3, we recall basic notions of toric geometry and tropical geometry. We also recall that the tropicalizations of Zariski closed analytic subvarieties of tori can be calculated using initial forms. In Section 4, we prove Theorem 1.1 for analytic hypersurfaces in tori by calculating their tropicalizations explicitly in terms of initial forms of analytic functions. In Section 5, we prove Theorem 1.1 by using Payne's idea in [Pay09-2]. We use surjective homomorphisms of tori to reduce to the case of hypersurfaces. Finally, in Section 6, we give examples of the tropicalizations of analytic and algebraic hypersurfaces in the 2-dimensional torus.

2. Preliminaries on Berkovich Spaces

In this paper, we use the language of Berkovich analytic geometry. We refer to [Ber90] and [Ber93] for basic notations on Berkovich analytic geometry.

Let K be a complete non-archimedean valuation field with non-trivial absolute value $|\cdot|$, and K° (resp. k) its valuation ring (resp. residue field). Let $\overline{\cdot}\colon K^\circ \to k$ be the projection. We fix an algebraic closure K^{alg} of K. We also denote the extension of the absolute value $|\cdot|$ on K to K^{alg} by the same symbol. We put $\mathrm{val}(a) := -\log|a|$ for $a \in (K^{\mathrm{alg}})^\times$, $\mathrm{val}(0) := \infty$, and $\Gamma := \mathrm{val}((K^{\mathrm{alg}})^\times)$. There exists a group homomorphism $\varphi\colon \Gamma \to (K^{\mathrm{alg}})^\times$ such that $\mathrm{val}\circ\varphi = \mathrm{id}_\Gamma$, where id_Γ is the identity map on Γ [MS15, Lemma 2.1.15]. We fix such $\varphi\colon \Gamma \to (K^{\mathrm{alg}})^\times$ in this paper.

In this paper, *analytic spaces* mean Berkovich analytic spaces; see [Ber90] and [Ber93]. For a K-analytic space Z and a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$, the Zariski closed K-analytic subspace corresponding to $\mathcal{I}$ is denoted by $V(\mathcal{I}) \subset Z$; see [Ber90, Proposition 3.1.4].

For each scheme Z locally of finite type over K, the K-analytic space associated to Z is denoted by Z^{an}; see [Ber90, Theorem 3.4.1]. The scheme Z is reduced, pure d-dimensional, irreducible, or separated if and only if

Z^{an} has the same property; see [Duc17, Proposition 2.7.16] for irreducibility, [Ber90, Proposition 3.4.6] for separatedness, and [Ber90, Proposition 3.4.3] for the others.

In this paper, an algebraic variety over K means a reduced separated scheme of finite type over K. For an algebraic variety Z over K, a Zariski closed analytic subvariety of Z^{an} means a reduced Zariski closed K-analytic subspace of Z^{an}. We say that a Zariski closed analytic subvariety of $W \subset Z^{\mathrm{an}}$ is *algebraic* if W is the analytification of a Zariski closed algebraic subvariety of Z.

We recall basic properties of separatedness and relative boundaries of morphisms of Berkovich analytic spaces; see [Ber90, Definition 2.5.7 and Section 3.1] for the definition of relative boundaries.

Lemma 2.1. *Let W be a Zariski closed analytic subvariety of the analytification Z^{an} of an algebraic variety Z over K. Then for any K-analytic space U, a morphism $\phi\colon W \to U$ of K-analytic spaces is separated, and the relative boundary of ϕ is empty.*

Proof. First, we show that ϕ is separated. Since Z is separated, the analytification Z^{an} is separated by [Ber90, Proposition 3.4.6]. The Zariski closed analytic subvariety $W \subset Z^{\mathrm{an}}$ is separated by [Ber90, Proposition 3.1.5]. Hence, ϕ is separated by [Ber90, Proposition 3.1.5].

Second, we show that the relative boundary of ϕ is empty. The boundary of Z^{an} is empty by [Ber90, Theorem 3.4.1]. (We note that, in [Ber90], a K-analytic space is said to be closed if its boundary is empty; see [Ber90, Section 3.1, p.49].) Since the closed immersion $W \hookrightarrow Z^{\mathrm{an}}$ has no boundary by [Ber90, Corollary 2.5.13 (i) and Proposition 3.1.4 (i)], the boundary of W is empty by [Ber90, Proposition 3.1.3 (ii)]. Hence the relative boundary of ϕ is empty by [Ber90, Proposition 3.1.3 (ii)]. □

3. Tropicalizations of Analytic Varieties and Initial Forms

We recall some properties of normal toric varieties and their *tropicalizations*; see [CLS11, Chapter 3] for toric varieties and [Pay09-1, Section 2 and Section 3] for tropicalization. Let M be a free $\mathbb{Z}$-module of finite rank, and Y_Σ the normal toric variety over K associated to a fan Σ in $N_{\mathbb{R}} := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R})$. There is a natural bijection between the cones σ in Σ

and the torus orbits $O(\sigma)$ in Y_Σ. For each cone $\sigma \in \Sigma$, the torus orbit $O(\sigma)$ is isomorphic to the torus $\mathrm{Spec}(K[\sigma^\perp \cap M])$. Its Zariski closure $\overline{O(\sigma)}$ in Y_Σ is a normal toric variety over K containing it as the open dense torus orbit. The *tropicalization map*

$$\mathrm{Trop}\colon O(\sigma)^{\mathrm{an}} \to (N_\sigma)_\mathbb{R} := \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{R})$$

is the map given by

$$\mathrm{Trop}(|\cdot|_x) := -\log|\cdot|_x\colon \sigma^\perp \cap M \to \mathbb{R}$$

for $|\cdot|_x \in O(\sigma)^{\mathrm{an}}$. It is proper, surjective, and continuous; see [Pay09-1, Section 2]. We define the tropicalization map

$$\mathrm{Trop}\colon Y_\Sigma^{\mathrm{an}} = \bigsqcup_{\sigma\in\Sigma} O(\sigma)^{\mathrm{an}} \to \bigsqcup_{\sigma\in\Sigma}(N_\sigma)_\mathbb{R}$$

by gluing the tropicalization maps $\mathrm{Trop}\colon O(\sigma)^{\mathrm{an}} \to (N_\sigma)_\mathbb{R}$ together. We define a topology on the disjoint union $\bigsqcup_{\sigma\in\Sigma}(N_\sigma)_\mathbb{R}$ as follows. We extend the canonical topology on $\mathbb{R}$ to that on $\mathbb{R}\cup\{\infty\}$ so that $(a,\infty]$ for $a \in \mathbb{R}$ are a basis of neighborhoods of ∞. We also extend the addition on $\mathbb{R}$ to that on $\mathbb{R}\cup\{\infty\}$ by $a+\infty=\infty$ for $a \in \mathbb{R}\cup\{\infty\}$. For each cone $\sigma \in \Sigma$, we put $S_\sigma := \sigma^\vee \cap M$, where

$$\sigma^\vee := \{m \in M \otimes_\mathbb{Z} \mathbb{R} \mid n(m) \geq 0 \text{ for all } n \in \sigma\}.$$

Then $\mathbb{R}\cup\{\infty\}$ and S_σ are monoids. We consider the set of monoid homomorphisms $\mathrm{Hom}(S_\sigma, \mathbb{R}\cup\{\infty\})$ as a topological subspace of $(\mathbb{R}\cup\{\infty\})^{S_\sigma}$. We define a topology on $\bigsqcup_{\substack{\tau\in\Sigma\\ \tau\preceq\sigma}}(N_\tau)_\mathbb{R}$ by the canonical bijection

$$\mathrm{Hom}(S_\sigma, \mathbb{R}\cup\{\infty\}) \cong \bigsqcup_{\substack{\tau\in\Sigma\\ \tau\preceq\sigma}}(N_\tau)_\mathbb{R}.$$

Then we define a topology on $\bigsqcup_{\sigma\in\Sigma}(N_\sigma)_\mathbb{R}$ by gluing the topological spaces $\bigsqcup_{\substack{\tau\in\Sigma\\ \tau\preceq\sigma}}(N_\tau)_\mathbb{R}$ together. The definition of this topology on $\bigsqcup_{\sigma\in\Sigma}(N_\sigma)_\mathbb{R}$ makes sense since for any face $\rho \preceq \sigma$, the canonical embedding of the monoid homomorphism $\mathrm{Hom}(S_\rho, \mathbb{R}\cup\{\infty\})$ into $\mathrm{Hom}(S_\sigma, \mathbb{R}\cup\{\infty\})$ induces a homeomorphism from $\mathrm{Hom}(S_\rho, \mathbb{R}\cup\{\infty\})$ onto its image. We note that the tropicalization map

$$\mathrm{Trop}\colon Y_\Sigma^{\mathrm{an}} \to \bigsqcup_{\sigma\in\Sigma}(N_\sigma)_\mathbb{R}$$

is proper, surjective, and continuous; see [Pay09-1, Section 3]. For an irreducible Zariski closed analytic subvariety $X \subset Y_\Sigma^{\mathrm{an}}$, the image $\mathrm{Trop}(X)$ of X is called the *tropicalization* of X.

In [Gub07], Gubler showed that for any cone $\sigma \in \Sigma$, the subset $\mathrm{Trop}(X) \cap (N_\sigma)_\mathbb{R}$ of $(N_\sigma)_\mathbb{R}$ is a locally finite union of Γ-rational polyhedra. Here, $\Gamma = \mathrm{val}((K^{\mathrm{alg}})^\times)$. (See [MS15, Definition 2.3.2] for the definition of Γ-rational polyhedra.) Moreover, he showed that for a unique cone $\sigma_X \in \Sigma$ such that $X \cap O(\sigma_X)^{\mathrm{an}}$ is a dense Zariski open analytic subvariety of X, the subset $\mathrm{Trop}(X) \cap (N_{\sigma_X})_\mathbb{R}$ of $(N_{\sigma_X})_\mathbb{R}$ is a locally finite union of d-dimensional polyhedra, where d is the dimension of X [Gub07, Theorem 1.1]. Here, we identify $(N_\sigma)_\mathbb{R}$ with $\mathbb{R}^{\dim O(\sigma)}$ by taking a $\mathbb{Z}$-basis of $\mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{Z})$. When $X = Z^{\mathrm{an}} \subset Y_\Sigma^{\mathrm{an}}$ for a Zariski closed algebraic subvariety $Z \subset Y_\Sigma$, we have $\mathrm{Trop}(X) = \mathrm{Trop}(Z)$, and $\mathrm{Trop}(Z) \cap (N_\sigma)_\mathbb{R}$ is a finite union of polyhedra for any cone $\sigma \in \Sigma$; see [MS15, Theorem 3.3.5]. (See also [BG84, Theorem A] and [EKL06, Theorem 2.2.3]. See [MS15, Definition 3.2.1 and Section 6.2] for the definition of the tropicalizations of algebraic subvarieties of tori and toric varieties.)

Let M_1, M_2 be free $\mathbb{Z}$-modules of finite rank, and $\phi\colon \mathrm{Spec}(K[M_1]) \to \mathrm{Spec}(K[M_2])$ the homomorphism of algebraic tori over K induced by a homomorphism $M_2 \to M_1$. We denote by

$$\mathrm{Trop}(\phi^{\mathrm{an}})\colon \mathrm{Hom}_\mathbb{Z}(M_1, \mathbb{R}) \to \mathrm{Hom}_\mathbb{Z}(M_2, \mathbb{R})$$

the $\mathbb{R}$-linear map such that

$$\mathrm{Trop}(\phi^{\mathrm{an}}) \circ \mathrm{Trop} = \mathrm{Trop} \circ \phi^{\mathrm{an}};$$

see [Pay09-1, Section 1].

We shall introduce the initial forms of analytic functions on the analytification $(\mathbb{G}_m^r)^{\mathrm{an}}$ of the r-dimensional torus

$$\mathbb{G}_m^r := \mathrm{Spec}(K[T_1^{\pm 1}, \dots, T_r^{\pm 1}])$$

over K as follows. (See [MS15, Section 2.4] for the case of Laurent polynomials.) Let M' be the free abelian group generated by T_i ($1 \leq i \leq r$). We identify $\mathrm{Hom}_\mathbb{Z}(M', \mathbb{R})$ with $\mathbb{R}^r$ by sending $\phi \in \mathrm{Hom}_\mathbb{Z}(M', \mathbb{R})$ to $(\phi(T_1), \dots, \phi(T_r)) \in \mathbb{R}^r$. For $u = (u_1, \dots, u_r) \in \mathbb{Z}^r$, we put $T^u := T_1^{u_1} \cdots T_r^{u_r}$.

For a non-zero analytic function

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{O}) \setminus \{0\} \qquad (a_u \in K),$$

let

$$\mathrm{Trop}(f) \colon \mathbb{R}^r \to \mathbb{R}$$

be the piecewise linear function given by

$$\mathrm{Trop}(f)(w) := \min\{ \mathrm{val}(a_u) + \langle w, u \rangle \mid u \in \mathbb{Z}^r, a_u \neq 0 \} \qquad (w \in \mathbb{R}^r),$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^r$.

We note that a formal power series

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in K[[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]]$$

is an analytic function on $(\mathbb{G}_m^r)^{\mathrm{an}}$ if and only if for any $w \in \mathbb{R}^r$, we have $\lim_{|u| \to \infty} \mathrm{val}(a_u) + \langle w, u \rangle = \infty$, where we put $|u| := \sum_{i=1}^r |u_i|$. Hence the function $\mathrm{Trop}(f)$ is well-defined.

Definition 3.1. The *initial form* of f with respect to $w \in \mathbb{R}^r$ is the Laurent polynomial over the residue field k defined by

$$\mathrm{in}_w(f) := \sum_{\substack{u \in \mathbb{Z}^r,\ a_u \neq 0 \\ \mathrm{val}(a_u) + \langle w, u \rangle = \mathrm{Trop}(f)(w)}} \overline{a_u \varphi(-\mathrm{val}(a_u))} T^u \in k[T_1^{\pm 1}, \ldots, T_r^{\pm 1}].$$

Here, $\varphi \colon \Gamma \to (K^{\mathrm{alg}})^\times$ is the map fixed in Section 2.

Since $\lim_{|u| \to \infty} \mathrm{val}(a_u) + \langle w, u \rangle = \infty$, the Laurent polynomial $\mathrm{in}_w(f)$ is well-defined.

We also note that the initial form $\mathrm{in}_w(f)$ depends on the choice of $\varphi \colon \Gamma \to (K^{\mathrm{alg}})^\times$, but, in this paper, we focus only on whether the initial form $\mathrm{in}_w(f)$ is a monomial or not for each $w \in \mathbb{R}^r$. This does not depend on the choice of $\varphi \colon \Gamma \to (K^{\mathrm{alg}})^\times$.

Proposition 3.2. *For the Zariski closed analytic subvariety* $Z = V(\mathcal{I}) \subset (\mathbb{G}_m^r)^{\mathrm{an}}$ *corresponding to a coherent ideal sheaf* $\mathcal{I} \subset \mathcal{O}_{(\mathbb{G}_m^r)^{\mathrm{an}}}$*, we* *have*

$$\mathrm{Trop}(Z) = \{\, w \in \mathbb{R}^r \mid \mathrm{in}_w(f) \textit{ is not a monomial for any } f \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{I}) \setminus \{0\}\,\}.$$

Moreover, when $\mathcal{I}$ *is generated by a non-zero analytic function* $f \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{O}) \setminus \{0\}$*, we have*

$$\mathrm{Trop}(V(f)) = \{\, w \in \mathbb{R}^r \mid \mathrm{in}_w(f) \textit{ is not a monomial}\}.$$

Proof. For each affinoid domain U of $(\mathbb{G}_m^r)^{\mathrm{an}}$, $w \in \mathrm{Trop}(U)$, and $f \in \Gamma(U, \mathcal{I})$ with $\mathrm{Trop}^{-1}(w) \subset U$, when the initial form $\mathrm{in}_w(f)$ is not a monomial, the initial form $\mathrm{in}_w(g)$ is not a monomial for a function $g \in \Gamma(U, \mathcal{I})$ which is sufficiently close to f in $\Gamma(U, \mathcal{I})$. By [Van75, Theorem 3.1.1 and Example 3.1.3], the image of $\Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{I})$ in $\Gamma(U, \mathcal{I})$ is dense. Hence the first assertion follows from [Rab12, Theorem 7.8]. One can show that

$$\mathrm{in}_w(fg) = \mathrm{in}_w(f)\,\mathrm{in}_w(g)$$

for any $g \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{O}) \setminus \{0\}$ and any $w \in \mathbb{R}^r$ in the same way as [MS15, Lemma 2.6.2 (3)], where the equality is proved for Laurent polynomials. Hence the second assertion holds. □

4. Tropicalization of Analytic Hypersurfaces in Tori

In this section, we prove Theorem 1.1 for analytic hypersurfaces in the r-dimensional torus $(\mathbb{G}_m^r)^{\mathrm{an}}$.

First, we show that the tropicalization of an analytic hypersurface in $(\mathbb{G}_m^r)^{\mathrm{an}}$ is the $(r-1)$-skeleton (i.e., the union of cells of dimension less than or equal to $r-1$) of the polyhedral complex associated to the analytic function defining the analytic hypersurface. (See [MS15, Proposition 3.1.6 and Remark 3.1.7] for the case of algebraic hypersurfaces.) See [MS15, Section 2.3] for the terminology of polyhedral geometry used in this paper.

For a non-zero analytic function

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{O}) \setminus \{0\} \qquad (a_u \in K),$$

we write $\Sigma_{\mathrm{Trop}(f)}$ for the coarsest polyhedral complex in $\mathbb{R}^r$ containing

$$\sigma_u := \{\, w \in \mathbb{R}^r \mid \mathrm{Trop}(f)(w) = \mathrm{val}(a_u) + \langle w, u \rangle \,\}$$

for every $u \in \mathbb{Z}^r$ satisfying $a_u \neq 0$, where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^r$; see [MS15, Definition 2.5.5]. The polyhedral complex $\Sigma_{\mathrm{Trop}(f)}$ is pure r-dimensional and its support is $\mathbb{R}^r$.

LEMMA 4.1. *For a non-zero analytic function*

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{O}) \setminus \{0\} \qquad (a_u \in K),$$

let $V(f) \subset (\mathbb{G}_m^r)^{\mathrm{an}}$ *be the analytic hypersurface defined by* f. *Then the tropicalization* $\mathrm{Trop}(V(f))$ *is the* $(r-1)$*-skeleton of* $\Sigma_{\mathrm{Trop}(f)}$, *i.e., the union of cells of* $\Sigma_{\mathrm{Trop}(f)}$ *of dimension less than or equal to* $r-1$.

PROOF. One can prove this lemma in the same way as in the case of Laurent polynomials; see [MS15, Proposition 3.1.6 and Remark 3.1.7]. □

We shall now prove Theorem 1.1 for analytic hypersurfaces in $(\mathbb{G}_m^r)^{\mathrm{an}}$.

THEOREM 4.2. *Let* $f \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{O}) \setminus \{0\}$ *be a non-zero analytic function. Assume that* $\mathrm{Trop}(V(f))$ *is a finite union of polyhedra. Then* f *is a Laurent polynomial. In particular, the Zariski closed analytic subvariety* $V(f) \subset (\mathbb{G}_m^r)^{\mathrm{an}}$ *is the analytification of the algebraic hypersurface of* $\mathbb{G}_m^r$ *defined by* f.

PROOF. We put

$$f = \sum_{u \in \mathbb{Z}^r} a_u T^u \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{O}) \setminus \{0\}.$$

By Lemma 4.1, the $(r-1)$-skeleton of $\Sigma_{\mathrm{Trop}(f)}$ is a finite union of polyhedra. Since the polyhedral complex $\Sigma_{\mathrm{Trop}(f)}$ is pure r-dimensional and its support is $\mathbb{R}^r$, there are only finitely many maximal cells of $\Sigma_{\mathrm{Trop}(f)}$. We take a finite subset $\Lambda \subset \mathbb{Z}^r$ such that for each maximal cell $\sigma \in \Sigma_{\mathrm{Trop}(f)}$, there exists $u \in \Lambda$ satisfying $\sigma_u = \sigma$. Then we have $\bigcup_{u \in \Lambda} \sigma_u = \mathbb{R}^r$; in other words, for any $w \in \mathbb{R}^r$, there exists $u \in \Lambda$ such that

$$\mathrm{Trop}(f)(w) = \mathrm{val}(a_u) + \langle w, u \rangle.$$

We shall show that there are only finitely many $u \in \mathbb{Z}^r$ satisfying $a_u \neq 0$. Assume that there exist infinitely many $u \in \mathbb{Z}^r$ with $a_u \neq 0$. Then there exist $v = (v_1, \dots, v_r) \in \mathbb{Z}^r \setminus \Lambda$ and $1 \leq i \leq r$ such that $a_v \neq 0$ and $|v_i| > |u_i|$ for any $u = (u_1, \dots, u_r) \in \Lambda$. Take a real number $x_i \in \mathbb{R}$ such that

$$\mathrm{val}(a_v) + x_i v_i < \mathrm{val}(a_u) + x_i u_i$$

for any $u = (u_1, \dots, u_r) \in \Lambda$. Let $x := (0, \dots, 0, x_i, 0, \dots, 0) \in \mathbb{R}^r$ be the element such that the i-th entry is x_i and the j-th entry is 0 for $j \neq i$. Then we have

$$\mathrm{val}(a_v) + \langle x, v \rangle < \mathrm{val}(a_u) + \langle x, u \rangle$$

for any $u \in \Lambda$. Hence $x \in \mathbb{R}^r$ is not contained in $\bigcup_{u \in \Lambda} \sigma_u$, which contradicts $\bigcup_{u \in \Lambda} \sigma_u = \mathbb{R}^r$.

Consequently, there are only finitely many $u \in \mathbb{Z}^r$ satisfying $a_u \neq 0$. In other words, f is a Laurent polynomial. □

Remark 4.3. For an irreducible Zariski closed analytic subvariety $Z \subset (\mathbb{G}_m^r)^{\mathrm{an}}$ of codimension 1, there exists a non-zero analytic function $f \in \Gamma((\mathbb{G}_m^r)^{\mathrm{an}}, \mathcal{O}) \setminus \{0\}$ such that $Z = V(f)$. This easily follows from the fact that $Z \subset (\mathbb{G}_m^r)^{\mathrm{an}}$ is a Cartier divisor, and the line bundle on $(\mathbb{G}_m^r)^{\mathrm{an}}$ corresponding to Z is trivial; see [Lut16, Lemma 2.7.4].

5. Proof of the Main Theorem

In this section, we shall prove Theorem 1.1 by using surjective homomorphisms of tori to reduce to the case of hypersurfaces.

Let M be a free $\mathbb{Z}$-module of finite rank. Let Y_Σ be the normal toric variety over K associated to a fan Σ in $N_\mathbb{R} := \mathrm{Hom}_\mathbb{Z}(M, \mathbb{R})$, and X an irreducible Zariski closed analytic subvariety of Y_Σ^{an}. Take a unique cone $\sigma_X \in \Sigma$ such that $X \cap O(\sigma_X)^{\mathrm{an}}$ is a dense Zariski open analytic subvariety of X. We fix a $\mathbb{Z}$-basis $m_1, \dots, m_n$ of $(\sigma_X)^\perp \cap M$. By using this $\mathbb{Z}$-basis, we identify $(N_{\sigma_X})_\mathbb{R} := \mathrm{Hom}_\mathbb{Z}((\sigma_X)^\perp \cap M, \mathbb{R})$ with $\mathbb{R}^n$.

Assume that $\mathrm{Trop}(X) \cap (N_{\sigma_X})_\mathbb{R}$ is a finite union of polyhedra.

Lemma 5.1. *Assume that the Zariski closed analytic subvariety $X \cap O(\sigma_X)^{\mathrm{an}} \subset O(\sigma_X)^{\mathrm{an}}$ is algebraic. Then the analytic subvariety $X \subset Y_\Sigma^{\mathrm{an}}$ is algebraic.*

PROOF. Let $X' \subset O(\sigma_X)$ be the algebraic subvariety such that $X'^{\mathrm{an}} = X \cap O(\sigma_X)^{\mathrm{an}}$. By [Ber90, Proposition 3.4.4], we have $(\overline{X'})^{\mathrm{an}} = \overline{(X'^{\mathrm{an}})} = X$, where $\overline{X'}$ is the algebraic Zariski closure of X' in Y_Σ. Hence the analytic subvariety $X \subset Y_\Sigma^{\mathrm{an}}$ is algebraic. $\square$

By Lemma 5.1, it suffices to show that $X \cap O(\sigma_X)^{\mathrm{an}} \subset O(\sigma_X)^{\mathrm{an}}$ is algebraic.

We put $X' := X \cap O(\sigma_X)^{\mathrm{an}}$. Since $X' \subset X$ is Zariski open and X is irreducible, X' is irreducible. We put $d := \dim X' = \dim X$. By the $\mathbb{Z}$-basis $m_1, \dots, m_n$ of $(\sigma_X)^\perp \cap M$, we identify $O(\sigma_X)$ with $\mathbb{G}_m^n$. Let $\mathcal{I} \subset \mathcal{O}_{(\mathbb{G}_m^n)^{\mathrm{an}}}$ be the coherent ideal sheaf such that $V(\mathcal{I}) = X' \subset (\mathbb{G}_m^n)^{\mathrm{an}}$.

Let $\mathrm{Gr}(n-d-1, n)$ be the Grassmannian of $(n-d-1)$-dimensional subspaces of $\mathbb{Q}^n$, which is an integral variety over $\mathbb{Q}$; see [LB15, Section 5.3.2]. Let S be the set of surjective homomorphisms

$$\phi\colon \mathbb{G}_m^n \to \mathbb{G}_m^{d+1}$$

such that $\mathrm{Trop}(\phi^{\mathrm{an}})\colon \mathbb{R}^n \to \mathbb{R}^{d+1}$ is injective on every d-dimensional polyhedron contained in $\mathrm{Trop}(X')$.

LEMMA 5.2. *The subset*

$$\{\, \ker(\mathrm{Trop}(\phi^{\mathrm{an}})) \cap \mathbb{Q}^n \in \mathrm{Gr}(n-d-1, n)(\mathbb{Q}) \mid \phi \in S \,\}$$

is a dense Zariski open subset in $\mathrm{Gr}(n-d-1, n)(\mathbb{Q})$.

PROOF. For a d-dimensional Γ-rational polyhedron $P \subset \mathbb{R}^n$, we put

$$L(P) := \{\alpha(b-a) \in \mathbb{R}^n \mid a, b \in P,\ \alpha \in \mathbb{R}\}.$$

It is a linear subspace of $\mathbb{R}^n$ of dimension d. Since the polyhedron P is Γ-rational, the linear space $L(P)$ has a $\mathbb{R}$-basis $\{x_i \in \mathbb{Q}^n\}_{i=1}^d$. (See [MS15, Definition 2.3.2] for the definition of Γ-rational polyhedra.)

Hence we have

$$\begin{aligned}
&\{\, A \in \mathrm{Gr}(n-d-1, n)(\mathbb{Q}) \mid \text{the projection } \mathbb{R}^n \to \mathbb{R}^n/(A \otimes \mathbb{R}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{is injective on } P \,\} \\
=&\{\, A \in \mathrm{Gr}(n-d-1, n)(\mathbb{Q}) \mid A \cap L(P) = \{0\} \,\} \\
=&p^{-1}(\{\, B \in \mathbb{P}(\wedge^{n-d-1}\mathbb{Q}^n) \mid B \wedge \wedge^d L(P) \neq \{0\} \,\}),
\end{aligned}$$

where

$$p\colon\ \mathrm{Gr}(n-d-1,n)(\mathbb{Q}) \ni U \mapsto \wedge^{n-d-1}U \in \mathbb{P}(\wedge^{n-d-1}\mathbb{Q}^n)$$

is the Plücker embedding. Since

$$\{\, B \in \mathbb{P}(\wedge^{n-d-1}\mathbb{Q}^n) \mid B \wedge \wedge^d L(P) \neq \{0\}\,\}$$

is a nonempty Zariski open subset of $\mathbb{P}(\wedge^{n-d-1}\mathbb{Q}^n)$, the subset

$$\{\, A \in \mathrm{Gr}(n-d-1,n)(\mathbb{Q}) \mid \text{ the projection } \mathbb{R}^n \to \mathbb{R}^n/(A\otimes\mathbb{R}) \text{ is injective on } P\,\}$$

is a nonempty Zariski open subset of $\mathrm{Gr}(n-d-1,n)(\mathbb{Q})$. (Remind that the algebraic variety structure of the Grassmannian $\mathrm{Gr}(n-d-1,n)$ is defined by the Plücker embedding $p\colon\ \mathrm{Gr}(n-d-1,n) \to \mathbb{P}(\wedge^{n-d-1}\mathbb{Q}^n)$ [LB15, Theorem 5.2.1 and Theorem 5.2.3].) Since the Grassmannian $\mathrm{Gr}(n-d-1,n)$ is irreducible, the subset

$$\{\, A \in \mathrm{Gr}(n-d-1,n)(\mathbb{Q}) \mid \text{the projection } \mathbb{R}^n \to \mathbb{R}^n/(A\otimes\mathbb{R}) \text{ is injective on } P\,\}$$

is dense in $\mathrm{Gr}(n-d-1,n)(\mathbb{Q})$.

Since for any $A \in \mathrm{Gr}(n-d-1,n)(\mathbb{Q})$, there exists a surjective group homomorphism $\phi\colon \mathbb{G}_m^n \to \mathbb{G}_m^{d+1}$ such that $\ker(\mathrm{Trop}(\phi^{\mathrm{an}})) \cap \mathbb{Q}^n = A$, the subset

$$\begin{aligned}&\left\{ \ker(\mathrm{Trop}(\phi^{\mathrm{an}})) \cap \mathbb{Q}^n \in \mathrm{Gr}(n-d-1,n)(\mathbb{Q}) \;\middle|\; \begin{array}{l}\text{a surjective group homomorphism } \phi\colon \mathbb{G}_m^n \to \mathbb{G}_m^{d+1}\\ \mathrm{Trop}(\phi^{\mathrm{an}})\colon \mathbb{R}^n \to \mathbb{R}^{d+1} \text{ is injective on } P\end{array}\right\}\\ =&\{\, A \in \mathrm{Gr}(n-d-1,n)(\mathbb{Q}) \mid \text{ the projection } \mathbb{R}^n \to \mathbb{R}^n/(A\otimes\mathbb{R}) \text{ is injective on } P\,\}\end{aligned}$$

is a dense Zariski open subset in $\mathrm{Gr}(n-d-1,n)(\mathbb{Q})$. (We note that for a surjective group homomorphism $\phi\colon \mathbb{G}_m^n \to \mathbb{G}_m^{d+1}$, the linear map $\mathrm{Trop}(\phi^{\mathrm{an}})\colon \mathbb{R}^n \to \mathbb{R}^{d+1}$ conincides with the projection $\mathbb{R}^n \to \mathbb{R}^n/\ker(\mathrm{Trop}(\phi^{\mathrm{an}}))$.)

Since $\mathrm{Trop}(X')$ is a finite union of d-dimensional Γ-rational polyhedra, the assertion follows. □

LEMMA 5.3. *For any homomorphism $\phi \in S$, the image $\phi^{\mathrm{an}}(X') \subset (\mathbb{G}_m^{d+1})^{\mathrm{an}}$ is a Zariski closed analytic subvariety.*

PROOF. For each d-dimensional polyhedron P contained in $\mathrm{Trop}(X')$, the $\mathbb{R}$-linear map

$$\mathrm{Trop}(\phi^{\mathrm{an}})|_P \colon P \to \mathbb{R}^{d+1}$$

is injective. Hence $\mathrm{Trop}(\phi^{\mathrm{an}})|_P$ induces a homeomorphism from P onto $\mathrm{Trop}(\phi^{\mathrm{an}})(P)$.

For any affinoid domain U of $(\mathbb{G}_m^{d+1})^{\mathrm{an}}$, since $\mathrm{Trop}(U)$ is bounded, the intersection

$$\mathrm{Trop}(U) \cap \mathrm{Trop}(\phi^{\mathrm{an}})(P)$$

is bounded. Hence the intersection

$$\mathrm{Trop}(\phi^{\mathrm{an}})^{-1}(\mathrm{Trop}(U)) \cap P$$

is bounded. Since $\mathrm{Trop}(X')$ is a finite union of d-dimensional polyhedra, the intersection

$$\mathrm{Trop}(\phi^{\mathrm{an}})^{-1}(\mathrm{Trop}(U)) \cap \mathrm{Trop}(X')$$

is bounded. Hence

$$\begin{aligned}\mathrm{Trop}((\phi^{\mathrm{an}})^{-1}(U) \cap X') &\subset \mathrm{Trop}((\phi^{\mathrm{an}})^{-1}(U)) \cap \mathrm{Trop}(X') \\ &\subset \mathrm{Trop}(\phi^{\mathrm{an}})^{-1}(\mathrm{Trop}(U)) \cap \mathrm{Trop}(X')\end{aligned}$$

is bounded. Thus $\mathrm{Trop}((\phi^{\mathrm{an}})^{-1}(U) \cap X')$ is compact.

By [Pay09-1, Lemma 2.1], the tropicalization map

$$\mathrm{Trop}\colon (\mathbb{G}_m^n)^{\mathrm{an}} \to \mathbb{R}^n$$

is proper. Hence the closed subset

$$(\phi^{\mathrm{an}})^{-1}(U) \cap X' \subset \mathrm{Trop}^{-1}(\mathrm{Trop}((\phi^{\mathrm{an}})^{-1}(U) \cap X'))$$

is compact. It follows that for any compact subset $C \subset (\mathbb{G}_m^{d+1})^{\mathrm{an}}$, the subset $(\phi^{\mathrm{an}})^{-1}(C) \cap X'$ is compact. By Lemma 2.1, the morphism $\phi^{\mathrm{an}}|_{X'} \colon X' \to (\mathbb{G}_m^{d+1})^{\mathrm{an}}$ is separated. Hence the map of underlying topological spaces

$|X'| \to |(\mathbb{G}_m^{d+1})^{\mathrm{an}}|$ induced by $\phi^{\mathrm{an}}|_{X'}$ is proper; see [Ber90, Section 3.1, p.50]. Moreover, the relative boundary of $\phi^{\mathrm{an}}|_{X'}$ is empty by Lemma 2.1. Hence $\phi^{\mathrm{an}}|_{X'}$ is a proper morphism of Berkovich analytic spaces; see [Ber90, Section 3.1, p.50] for the definition of proper morphisms. By [Ber90, Proposition 3.3.6], the image $\phi^{\mathrm{an}}(X') \subset (\mathbb{G}_m^{d+1})^{\mathrm{an}}$ is a Zariski closed analytic subvariety. □

By Lemma 5.3, for each $\phi \in S$, we consider $\phi^{\mathrm{an}}(X')$ as a Zariski closed analytic subvariety of $(\mathbb{G}_m^{d+1})^{\mathrm{an}}$. Since X' is irreducible, its image $\phi^{\mathrm{an}}(X')$ is also irreducible. Since

$$\mathrm{Trop}(\phi^{\mathrm{an}}(X')) = \mathrm{Trop}(\phi^{\mathrm{an}})(\mathrm{Trop}(X'))$$

and $\phi \in S$, the tropicalization $\mathrm{Trop}(\phi^{\mathrm{an}}(X'))$ is a finite union of d-dimensional polyhedra. By [Gub07, Theorem 1.1], the irreducible Zariski closed analytic subvariety $\phi^{\mathrm{an}}(X') \subset (\mathbb{G}_m^{d+1})^{\mathrm{an}}$ is d-dimensional. By Remark 4.3, it is an analytic hypersurface in $(\mathbb{G}_m^{d+1})^{\mathrm{an}}$. Hence, by Theorem 4.2, it is the analytification of an algebraic subvariety of $\mathbb{G}_m^{d+1}$.

Therefore, for each $\phi \in S$, the Zariski closed analytic subvariety

$$W_\phi := (\phi^{\mathrm{an}})^{-1}(\phi^{\mathrm{an}}(X')) \subset (\mathbb{G}_m^n)^{\mathrm{an}}$$

is algebraic. We put

$$W := \bigcap_{\phi \in S} W_\phi.$$

Then W contains X'. Since $W_\phi \subset (\mathbb{G}_m^n)^{\mathrm{an}}$ is algebraic for every $\phi \in S$, the intersection W is algebraic. One can deduce that $X' \subset (\mathbb{G}_m^n)^{\mathrm{an}}$ is algebraic from the following:

Lemma 5.4. *The dimension of W is less than or equal to $d = \dim X' = \dim X$.*

Proof. Assume that there exists an irreducible component V of W of dimension greater than d. By [Gub07, Theorem 1.1], the tropicalization $\mathrm{Trop}(V)$ is a locally finite union of polyhedra of dimension greater than d. We take a polyhedron P of dimension $d+1$ contained in $\mathrm{Trop}(V)$.

Let S' be the set of surjective homomorphisms

$$\phi\colon \mathbb{G}_m^n \to \mathbb{G}_m^{d+1}$$

such that the $\mathbb{R}$-linear map $\mathrm{Trop}(\phi^{\mathrm{an}})\colon \mathbb{R}^n \to \mathbb{R}^{d+1}$ is injective on P. By [LB15, Theorem 5.2.3], the subset

$$\{\ker(\mathrm{Trop}(\phi^{\mathrm{an}})) \cap \mathbb{Q}^n \mid \phi \in S'\}$$

is dense Zariski open in $\mathrm{Gr}(n-d-1,n)(\mathbb{Q})$. Since S is also Zariski open dense in $\mathrm{Gr}(n-d-1,n)(\mathbb{Q})$, the intersection $S \cap S'$ is non-empty.

We take an element $\phi \in S \cap S'$. Since $\phi^{\mathrm{an}}(W_\phi) = \phi^{\mathrm{an}}(X')$, we have

$$\begin{aligned}\mathrm{Trop}(\phi^{\mathrm{an}})(\mathrm{Trop}(W_\phi)) &= \mathrm{Trop}(\phi^{\mathrm{an}}(W_\phi)) \\ &= \mathrm{Trop}(\phi^{\mathrm{an}}(X')) = \mathrm{Trop}(\phi^{\mathrm{an}})(\mathrm{Trop}(X')).\end{aligned}$$

Since $\phi \in S$, the image $\mathrm{Trop}(\phi^{\mathrm{an}})(\mathrm{Trop}(W_\phi))$ is a finite union of d-dimensional polyhedra. Since $\phi \in S'$, the polyhedron $\mathrm{Trop}(\phi^{\mathrm{an}})(P)$ is $(d+1)$-dimensional. Hence we have

$$\mathrm{Trop}(\phi^{\mathrm{an}})(P) \not\subset \mathrm{Trop}(\phi^{\mathrm{an}})(\mathrm{Trop}(W_\phi)).$$

Since $P \subset \mathrm{Trop}(V)$, we have

$$\mathrm{Trop}(\phi^{\mathrm{an}})(\mathrm{Trop}(V)) \not\subset \mathrm{Trop}(\phi^{\mathrm{an}})(\mathrm{Trop}(W_\phi)),$$

which contradicts $V \subset W \subset W_\phi$.

Consequently, the dimension of W is less than or equal to d. □

Proof of Theorem 1.1. Recall that $X' := X \cap O(\sigma_X)^{\mathrm{an}}$ and $d := \dim X' = \dim X$. By Lemma 5.4, the dimension of W is less than or equal to d. Since $X' \subset W$, the analytic subvariety X' is an irreducible component of W. Since W is algebraic, by [Duc17, Proposition 2.7.16], the Zariski closed analytic subvariety $X' \subset O(\sigma_X)^{\mathrm{an}}$ is algebraic. Therefore, by Lemma 5.1, the analytic subvariety $X \subset Y_\Sigma^{\mathrm{an}}$ is algebraic.

The proof of Theorem 1.1 is complete. □

6. Examples of the Tropicalizations of Algebraic and Analytic Hypersurfaces

In this section, we give two examples of the tropicalizations of hypersurfaces in the 2-dimensional torus $(\mathbb{G}_m^2)^{\mathrm{an}}$. The first example is analytic and not algebraic. It is a locally finite union of polyhedra, but the number of polyhedra is infinite. The second example is algebraic, and it is a finite union of polyhedra.

Let

$$f = \sum_{(i,j) \in \mathbb{Z}^2_{\geq 0}} a_{i,j} X^i Y^j \in \mathcal{O}((\mathbb{G}^2_m)^{\mathrm{an}}) \setminus \{0\} \qquad (a_{i,j} \in K)$$

be a non-zero analytic function on $(\mathbb{G}^2_m)^{\mathrm{an}} = (\operatorname{Spec} K[X^{\pm}, Y^{\pm}])^{\mathrm{an}}$ satisfying

$$\mathrm{val}(a_{i,j}) = i^2 + j^2 + ij - i - j$$

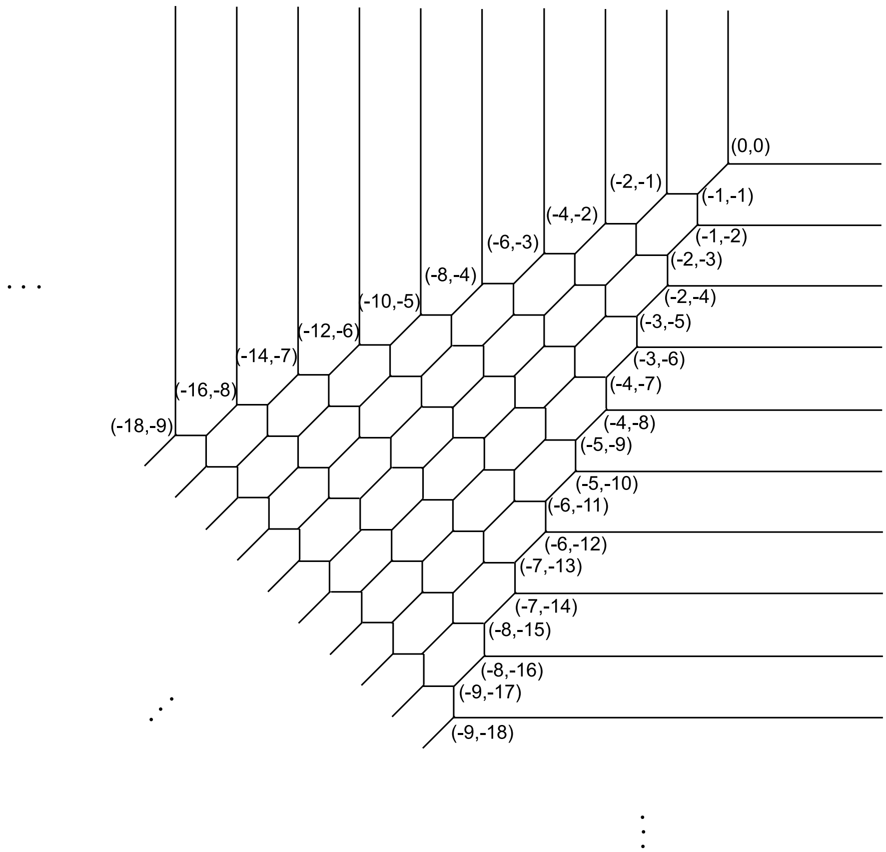

Fig. 1. The tropicalization of the analytic hypersurface $V(f) \subset (\mathbb{G}^2_m)^{\mathrm{an}}$.

for any $(i, j) \in \mathbb{Z}^2_{\geq 0}$. For each $(s, t) \in \mathbb{Z}^2_{\geq 0}$, we put

$$f_{s,t} := \sum_{\substack{0 \leq i \leq s \\ 0 \leq j \leq t}} a_{i,j} X^i Y^j \in K[X, Y] \setminus \{0\}.$$

First, we consider the tropicalization $\mathrm{Trop}(V(f)) \subset \mathbb{R}^2$. Since f has infinitely many non-zero terms, the analytic hypersurface $V(f) \subset (\mathbb{G}^2_m)^{\mathrm{an}}$ is not algebraic. The tropicalization $\mathrm{Trop}(V(f)) \subset \mathbb{R}^2$ is not a finite union of polyhedra; see Figure 1.

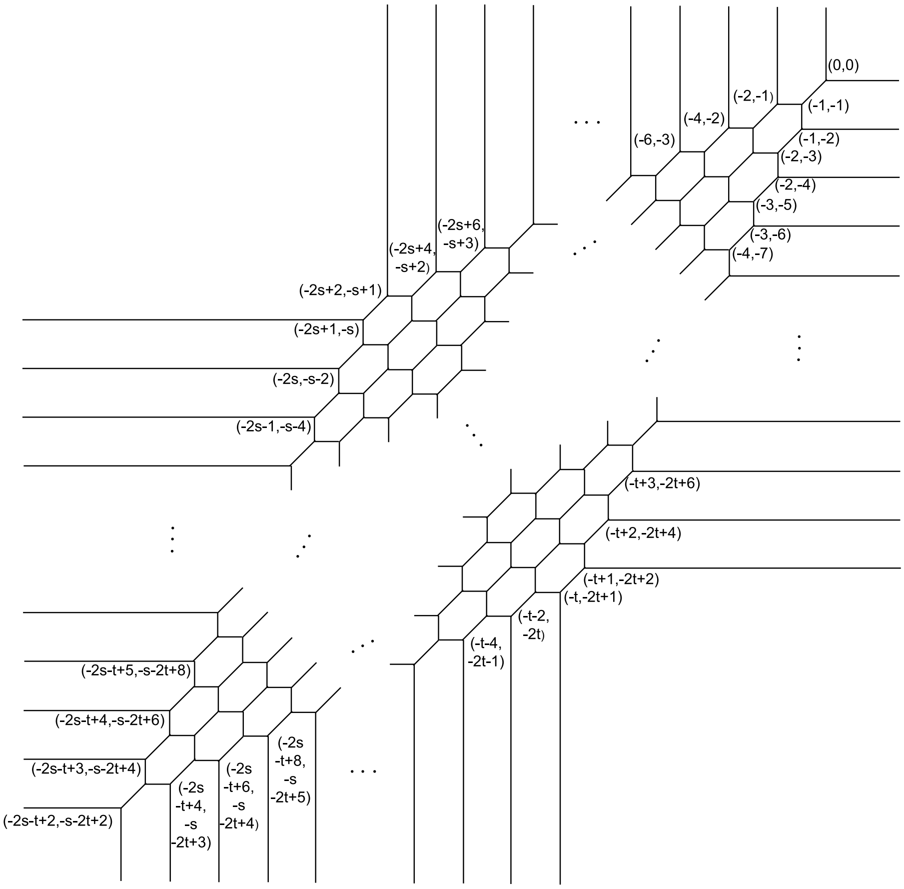

Fig. 2. The tropicalization of the algebraic hypersurface $V(f_{s,t}) \subset (\mathbb{G}^2_m)^{\mathrm{an}}$.

Next, we consider the tropicalization $\mathrm{Trop}(V(f_{s,t})) \subset \mathbb{R}^2$ for each $(s,t) \in \mathbb{Z}^2_{\geq 0}$. Since $f_{s,t}$ is a polynomial, the analytic hypersurface $V(f_{s,t}) \subset (\mathbb{G}_m^2)^{\mathrm{an}}$ is algebraic. The tropicalization $\mathrm{Trop}(V(f_{s,t})) \subset \mathbb{R}^2$ is a finite union of polyhedra; see Figure 2.

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