博士論文

論文題目　A new relationship between the dilatation of pseudo-Anosov braids and fixed point theory

（擬アノソフ組みひもの拡張率と固定点理論との新たな関係）

氏　　名　川島　夢人

A NEW RELATIONSHIP BETWEEN THE DILATATION OF PSEUDO-ANOSOV BRAIDS AND FIXED POINT THEORY

YUMEHITO KAWASHIMA

Abstract. A relation between the dilatation of pseudo-Anosov braids and fixed point theory was studied by Ivanov. In this paper we reveal a new relationship between the above two subjects by showing a formula for the dilatation of pseudo-Anosov braids by means of the representations of braid groups due to B. Jiang and H. Zheng.

1. Introduction

The purpose of this paper is to reveal a new relationship between the dilatation of pseudo-Anosov braids and fixed point theory. For this purpose we obtain a new formula to determine the dilatation of pseudo-Anosov braids from the representation $\zeta_{n,m}$ due to Jiang and Zheng [15].

Let us recall the notion of pseudo-Anosov braids. Let Σ_g be a closed surface of genus g and P_n be an n-point subset of Σ_g. We denote by $\Sigma_{g,n}$ the subset of Σ_g deleting P_n. We consider the case when $\Sigma_{g,n}$ has negative Euler characteristic. Let f be a homeomorphism of Σ_g fixing P_n setwise. We recall that f is *periodic* if f^k equals identity for some $k > 0$, and it is *reducible* if there exists an f-invariant closed 1-manifold $J \subset \Sigma_{g,n}$ whose complementary components in $\Sigma_{g,n}$ have negative Euler characteristic or else are Möbius bands. We refer to J as a *reduction* of f. Finally, f is *pseudo-Anosov* if there exists a number $\lambda > 1$ and a pair $\mathcal{F}^s$, $\mathcal{F}^u$ of transverse measured foliations with singularities modelled on k-prongs, $k = 1, 2, \dots$ in Figure 1 such that the equalities $f(\mathcal{F}^s) = (1/\lambda)\mathcal{F}^s$ and $f(\mathcal{F}^u) = \lambda\mathcal{F}^u$ hold. Furthermore, the one-prong singularities of these foliations are allowed to occur only at the punctures. For an isotopy class φ of homeomorphisms of Σ_g, φ is *periodic* if there exists a periodic element in φ. Similarly, φ is *reducible* if there exists a reducible element in φ and φ is *pseudo-Anosov* if there exists a pseudo-Anosov element in φ.

In [22], Thurston classified the isotopy classes of homeomorphisms on Σ_g fixing P_n into periodic, reducible and pseudo-Anosov types. Since we can regard the braid group B_n on n strands as the mapping class group of disk with n punctures, every element of B_n is also classified into periodic, reducible and pseudo-Anosov types. In [3], Bestvina and Handel obtained an algorithm which gave the classification for surface homeomorphisms. Using this algorithm, they established a method to calculate the dilatation of a pseudo-Anosov maping class φ.

Dilatations themselves are related to many fields and have been intensively studied by many authors. For example, it is known that the logarithm of the dilatation of pseudo-Anosov maps is the same as the topological entropy of pseudo-Anosov maps, which is an important subject in ergodic

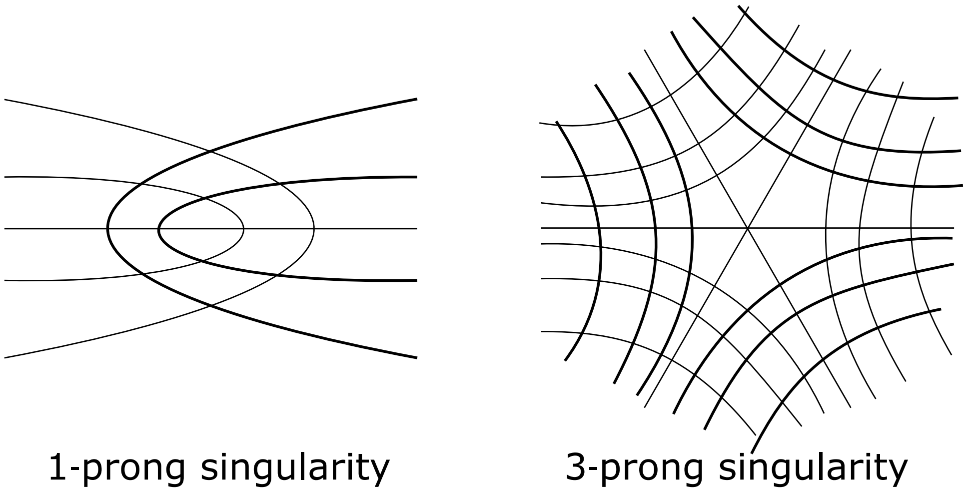

FIGURE 1. local chart around the singularities

theory. Also in [11], Ivanov showed that the logarithm of the asymptotic Nielsen number, which appeared in fixed point theory, coincides with the entropy. In this paper, we obtain a new formula to determine the dilatation of pseudo-Anosov braids from the representation $\zeta_{n,m}$ due to B. Jiang and H. Zheng [15].

The *growth rate* of a sequence $\{a_n\}$ of complex numbers is defined by

$$\operatorname*{Growth}_{n\to\infty} a_n = \max\left\{1, \limsup_{n\to\infty} |a_n|^{1/n}\right\}.$$

Let us notice that the above growth rate could be infinity. When the inequality $\operatorname*{Growth}_{n\to\infty} a_n > 1$ holds, we say that the sequence *grows exponentially.*

For any set S, $\mathbb{Z}S$ denotes the free abelian group with the specified basis S. If $x = \sum_{s\in S} k_s s$ is a finite sum, we define the *norm* of x in $\mathbb{Z}S$ by

$$\|x\| = \sum_{s\in S} |k_s|.$$

For any matrix $A = (a_{ij})$ with coefficients in $\mathbb{Z}S$, the norm of A is the matrix defined by $\|A\| = (\|a_{ij}\|)$ when a_{ij} is a finite sum for all i and j.

Let P_n be a finite subset of int D^2 of $n \geq 0$ points and we set $D_n = D^2\backslash P_n$. For integers $n, m \geq 0$, we consider three types of *configuration spaces* as follows: The space of m-tuples of distinct points in D_n denoted by

$$F_{n,m}(D^2) = \{(z_1, \ldots, z_m) \in (D_n)^m \mid z_i \neq z_j \text{ for all } i \neq j\},$$

the space of subsets of distinct m elements in D_n denoted by

$$\mathcal{C}_{n,m}(D^2) = F_{n,m}(D^2)/\mathcal{S}_m$$

and the space $IT_{n,m}(D^2)$ of pairs of disjoint subsets of n distinct elements and m distinct elements in D^2 denoted by

$$IT_{n,m}(D^2) = F_{0,n+m}(D^2)/\mathcal{S}_n \times \mathcal{S}_m,$$

where the symmetric group $\mathcal{S}_m$ acts on $F_{n,m}(D^2)$ by permuting components of an m-tuple and similarly, the subgroup $\mathcal{S}_n \times \mathcal{S}_m$ of $\mathcal{S}_{n+m}$ acts on

$F_{0,n+m}(D^2)$. We write $\{y_1,\dots,y_m\}$ and $(\{x_1,\dots,x_n\},\{y_1,\dots,y_m\})$ for the elements of $\mathcal{C}_{n,m}(D^2)$ and $IT_{n,m}(D^2)$ respectively.

We choose m distinct points $d_1,\dots,d_m$ in ∂D^2 and take a base point $c=\{d_1,\dots,d_m\}$ of $\mathcal{C}_{n,m}(D^2)$. Let $b=(P_n,c)$ be a base point of $IT_{n,m}(D^2)$. The *m-braid group on D_n* is defined by

$$\mathbf{B}_{n,m}(D^2)=\pi_1(\mathcal{C}_{n,m}(D^2),c)$$

and the *intertwining (n,m)-braid group on D^2* is defined by

$$\mathbf{E}_{n,m}(D^2)=\pi_1(IT_{n,m}(D^2),b).$$

We set

$$\mathcal{E}_{n,m}=\{\mu=(\mu_1,\dots,\mu_{n-1})\in\mathbb{N}^{n-1}\mid \mu_1+\cdots+\mu_{n-1}=m\}.$$

We construct a $\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]$-invariant free $\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]$-submodule $\mathcal{H}_F$ of a relative homology of the universal covering of some configuration space generated by certain m-dimensional subspaces corresponding to $\mu\in\mathcal{E}_{n,m}$. The precise definition is given in Section 4.1. The braid group B_n acts on the homology as the mapping class group and acts on $\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]$ by the right multiplication. Tensoring these two actions, B_n acts on

$$\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]\otimes_{\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]}\mathcal{H}_F$$

and we define a representation $\zeta_{n,m}$ by this action.

Let Γ be a group, $\mathbb{Z}\Gamma$ its group ring, Γ_c the set of conjugacy classes, $\mathbb{Z}\Gamma_c$ the free Abelian group generated by Γ_c, and $\pi_\Gamma:\mathbb{Z}\Gamma\to\mathbb{Z}\Gamma_c$ the natural projection. We suppose ζ is an endomorphism of a free $\mathbb{Z}\Gamma$-module satisfying $\zeta(v_i)=\sum_{j=1}^k a_{ij}\cdot v_j$ for a basis $\{v_1,\dots,v_k\}$. The *trace* of ζ is defined as

$$\operatorname{tr}_\Gamma\zeta=\pi_\Gamma\left(\sum_{i=1}^k a_{ii}\right)\in\mathbb{Z}\Gamma_c.$$

We note that, under the basis $\mathcal{E}_{n,m}$, all matrix elements of $\zeta_{n,m}(\beta)$ belong to $\mathbb{Z}\Gamma_{\beta,m}$, where $\Gamma_{\beta,m}$ is the subgroup of B_{n+m} generated by β and $\mathbf{B}_{n,m}(D^2)$. Therefore, $\zeta_{n,m}(\beta)$ can be naturally regarded as an endomorphism of the free $\mathbb{Z}\Gamma_{\beta,m}$-module generated by $\mathcal{E}_{n,m}$.

Our main result is stated as follows.

Theorem 1.1. *For any pseudo-Anosov braid $\beta\in B_n$, we denote by λ the dilatation of β. Then we obtain*

$$\operatorname*{Growth}_{k\to\infty}\left\|\operatorname{tr}_{\Gamma_{\beta^k,m}}\zeta_{n,m}(\beta^k)\right\|=\operatorname*{Growth}_{k\to\infty}\operatorname{tr}\left\|\zeta_{n,m}(\beta^k)\right\|=\lambda^m$$
$$\operatorname*{Growth}_{m\to\infty}\left\|\operatorname{tr}_{\Gamma_{\beta,m}}\zeta_{n,m}(\beta)\right\|=\lambda.$$

The representations $\zeta_{n,m}$ are related to homological representations of braid groups in the following way. For $m=1$, there exists a homomorphism $\rho_B:\mathbf{E}_{n,1}(D^2)\to\mathbb{Z}$ such that the representation induced by ρ_B is equivalent to the reduced Burau representation. Similarly for $m\ge 2$, there exists a homomorphism $\rho_{LKB}:\mathbf{E}_{n,m}(D^2)\to\mathbb{Z}\oplus\mathbb{Z}$ such that the representation induced by ρ_{LKB} is equivalent to Lawrence-Krammer-Bigelow representation. The Lawrence-Krammer-Bigelow representations of the braid groups were studied by Lawrence [21] in relation with Hecke algebra representations of the braid groups. In [4], [19] and [20], Bigelow and Krammer showed the faithfulness of the Lawrence-Krammer-Bigelow representation independently.

In [9], Fried proved that the entropy of pseudo-Anosov braids is bounded below by the logarithm of the spectral radius of the Burau matrix $B(t)$ of pseudo-Anosov braids after substituting a complex number of modulus 1 in place of t. In [18], Kolev proved the same estimation directly with different methods. The estimate will be called the *Burau estimate*. In [2], Band and Boyland showed that the spectral radius of the Burau matrix $B(t)$ of pseudo-Anosov braids after substituting the root of unity in place of t is the dilatation itself of pseudo-Anosov braids only if $t = -1$. Furthermore, Band and Boyland showed that the spectral radius of $B(-1)$ is the dilatation of pseudo-Anosov braids if and only if the invariant foliations for pseudo-Anosov maps in the classes of pseudo-Anosov braids have odd order singularities at all punctures and all interior singularities are even order.

In [17], Koberda proved that the square of the dilatation of pseudo-Anosov braids is bounded below by the spectral radius of Lawrence-Krammer-Bigelow representation $LKB(q, t)$ of pseudo-Anosov braids after substituting complex numbers of modulus 1 in place of q and t. In this paper we recover the following result of [9], [18] and [17].

Theorem 1.2. (Fried [9], Kolev [18] and Koberda [17]) *For a pseudo-Anosov braid β, the dilatation of β is equal to or greater than the spectral radius of the Burau matrix $B(t)$ of β after substituting a complex number of modulus 1 in place of t and the m-th power of the dilatation of β is equal to or greater than the spectral radius of the Lawrence-Krammer-Bigelow matrix $LKB_m(q, t)$ of β after substituting complex numbers of modulus 1 in place of q and t.*

This paper is organized as follows. In Section 2 we recall the definition of the topological entropy due to Adler, Konheim and McAndrew [1]. Then we recall how to define the topological entropy of self maps on metric spaces due to Bowen [7]. In Section 3, we review asymptotic fixed point theory. We recall asymptotic fixed point theory for compact spaces due to Jiang [14] and a version of relative Nielsen theory due to Jiang, Zhao and Zheng [16] and Jiang and Zheng [15]. In Section 4, we construct the representation $\zeta_{n,m}$ due to Jiang and Zheng [15] and state the relation between the trace of $\zeta_{n,m}$ and the number of essential fixed points of some good self map. In Section 5 we prove the main theorem using the relation among dilatation, entropy and fixed point theory. In Section 6 we recover from our main theorem the estimation of the dilatation of pseudo-Anosov braids in [9], [18] and [17] by means of the homological representation.

2. PRELIMINARIES

2.1. **Topological entropy.** The most widely used measure for the complexity of a dynamical system is the topological entropy. We refer the readers to [23] for an introductory treatment. We recall basic notions of the topological entropy due to Adler, Konheim and McAndrew [1]. Then we recall how to define the topological entropy of self maps on metric space due to Bowen [7]. Originally the topological entropy is defined in [1]. We recall [1] for the definition of the topological entropy. For any open cover α of X, let $N(\alpha)$ denote the number of sets in a subcover of minimal cardinality. Since X is compact and α is an open cover, there always exists a finite subcover of X in α. Let $H(\alpha)$ be the logarothm of $N(\alpha)$ and we call $H(\alpha)$ the *entropy* of α. For open covers α and β of X, their join is the open cover consisting of all sets of the form $A \cap B$ with $A \in \alpha$ and $B \in \beta$. Similarly, we can define the join $\bigvee_{i=1}^{n} \alpha_i$ of any finite collection $\{\alpha_i\}$ of open covers of X. For a continuous self map T of X, $T^{-1}\alpha$ denotes the open cover consisting of all sets $T^{-1}A$ with $A \in \alpha$. The *entropy* $h(T, \alpha)$ of a map T with respect to a cover α is defined as

$$\lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right).$$

The *topological entropy* $h(T)$ of a map T is defined as $\sup h(T, \alpha)$, where the supremum is taken over all open covers α.

For a compact surface X with negative Euler characteristic and a pseudo-Anosov homeomorphism f of X with the dilatation $\lambda > 1$,

$$h(f) = \log \lambda \tag{2.1}$$

is the minimal entropy in the homotopy class of f([8, p. 194]).

In [7], topological entropy is defined for self maps of a metric space X, which is not necessarily compact. Henceforth (X, d) is a metric space, not necessarily compact. $B(x; r)$ and $\overline{B}(x; r)$ denote the open and the closed ball centered at x and radius r respectively. We shall define the topological entropy for uniformly continuous maps $T : X \to X$. We denote by $UC(X, d)$ the space of all uniformly continuous maps of the metric space (X, d).

From now on T denotes a fixed element of $UC(X, d)$. If n is a natural number we can define a new metric d_n on X by

$$d_n(x, y) = \max_{0\le i\le n-1} d(T^i(x), T^i(y)).$$

The open ball centered at x and radius r in the metric d_n is

$$\bigcap_{i=0}^{n-1} T^{-i}B(T^i x; r).$$

For $\varepsilon > 0$ and a compact subset K of X, a subset F of X is said to (n, ε) *span* K with respect to T if for any element x of K, there exists an element y of F with $d_n(x, y) \le \varepsilon$. In other words, F is said to (n, ε) span K with respect to T if F satisfies the following condition

$$K \subset \bigcup_{y\in F} \bigcap_{i=0}^{n-1} T^{-i}\overline{B}(T^i y; \varepsilon).$$

Let $r_n(\varepsilon, K, T)$ denote the smallest cardinality of any (n, ε)-spanning set for K with respect to T. We set

$$r(\varepsilon, K, T) = \limsup_{n\to\infty} (1/n) \log r_n(\varepsilon, K, T)$$

and the *entropy* of T with respect to K is defined by

$$h_d(T, K) = \lim_{\varepsilon\to 0} r(\varepsilon, K, T).$$

Then the *entropy* of T is defined by

$$h_d(T) = \sup h_d(T, K),$$

where the supremum is taken over all compact subsets of X.

There exists another equivalent definition. A subset E of X is said to be (n, ε) *separated* with respect to T if for any distinct elements x, y of E, $d_n(x, y)$ is larger than ε. In other words, E is said to be (n, ε) *separated* with respect to T if for $x \in E$ the set

$$\bigcap_{i=0}^{n-1} T^i \overline{B}(T^i x; \varepsilon)$$

contains no other point of E. Let $s_n(\varepsilon, K, T)$ denote the largest cardinality of any (n, ε) separated subset of K with respect to T and we set

$$s(\varepsilon, K, T) = \limsup_{n \to \infty} (1/n) \log s_n(\varepsilon, K, T).$$

Then we have

$$h_d(T, K) = \lim_{\varepsilon \to 0} s(\varepsilon, K, T).$$

In [7], Bowen showed the equality $h(T) = h_d(T, X)$ when X is compact.

3. Asymptotic Nielsen theory for stratified maps

In [14], Jiang studied fixed point theory using mapping torus. In [16], Jiang, Zhao and Zheng studied fixed point theory for some good noncompact spaces. In [15], Jiang and Zheng studied fixed point theory for configuration spaces using the method in [16]. In this section we will review some of the relevant materials from [14], [15] and [16] about fixed point theory.

3.1. **Mapping torus.**

Subsections 3.1 and 3.2 are devoted to recall basic notions of fixed point theory due to [14]. In [14], Jiang studied fixed points by using mapping torus. Let X be a topological space and $f : X \to X$ be a continuous self map. We pick a base point $v \in X$ and a path w from v to $f(v)$. We denote by G the group $\pi_1(X, v)$ and let $f_G : G \to G$ be the composition

$$G = \pi_1(X, v) \xrightarrow{f_*} \pi_1(X, f(v)) \xrightarrow{w_*} \pi_1(X, v).$$

The *mapping torus* T_f of f is the space obtained from $X \times \mathbb{R}_+$ by identifying $(x, s+1)$ with $(f(x), s)$ for any element $x \in X$ and $s \in \mathbb{R}_+$, where $\mathbb{R}_+$ stands for the real interval $[0, \infty)$. On T_f there exists the natural semi-flow

$$\varphi : T_f \times \mathbb{R}_+ \to T_f,\ \varphi((x, s), t) = (x, s + t) \text{ for all } t \geq 0.$$

A point x of X and a positive number $\tau > 0$ determine the *time-τ orbit curve* $\varphi_{(x,\tau)} = \{\varphi_t(x, 0)\}_{0 \leq t \leq \tau}$ in T_f. We may identify X with the cross-section $X \times \{0\} \subset T_f$, then the map $f : X \to X$ is just the return map of the semi-flow φ.

We take the base point v of X as the base point of T_f. We define Γ to be the fundamental group $\pi_1(T_f, v)$ of T_f and let Γ_c be the set of conjugacy classes of Γ. Then Γ_c is independent of the base point of T_f and can be regarded as the set of free homotopy classes of closed curves in T_f. By the van Kampen Theorem, Γ is obtained from G by adding a new generator z

represented by the loop $\varphi_{(v,1)}w^{-1}$, and the relations $z^{-1}gz = f_G(g)$ for all $g \in G$:

$$\Gamma = \langle G, z \mid gz = zf_G(g) \text{ for all } g \in G\rangle.$$

In general, the map $\iota : G \to \Gamma$ induced by the inclusion $X \to T_f$ is not injective. However, if f is a homeomorphism, then ι is injective and is a section of the above exact sequence. Therefore there exists an exact sequence

$$1 \to \pi_1(X, v) \to \pi_1(T_f, v) \to \mathbb{Z} \to 1$$

if f is a homeomorphism.

We note that x is a fixed point of f if and only if its time-1 orbit curve is closed on the mapping torus T_f. For fixed points x and y of f, we define x and y to be in the same *fixed point class* if and only if their time-1 orbit curves are freely homotopic in T_f. Therefore every fixed point class $\mathbf{F}$ gives rise to a conjugacy class $\mathrm{cd}(\mathbf{F})$ in Γ_c, called the *coordinate* of $\mathbf{F}$. For a fixed point class $\mathbf{F}$ of f, the *fixed point index* $\mathrm{ind}(f, \mathbf{F})$ of f at $\mathbf{F}$ is the standard intersection number of the diagonal $\mathrm{diag}(X/\mathbf{F})$ of $(X/\mathbf{F}) \times (X/\mathbf{F})$ and the graph $\mathrm{graph}(f')$ of the map f' at $\mathbf{F}$, where f' is the induced map from f by the projection $X \to X/\mathbf{F}$. A fixed point class $\mathbf{F}$ is called *essential* if its index $\mathrm{ind}(f, \mathbf{F})$ is nonzero. The *generalized Lefschetz number* is defined as

$$L_\Gamma(f) = \sum_{\mathbf{F}} \mathrm{ind}(\mathbf{F}, f) \cdot \mathrm{cd}(\mathbf{F}),$$

where the summation is taken over all essential fixed point classes $\mathbf{F}$ of f. The *Nielsen number* $N_\Gamma(f)$ is the number of nonzero terms in $L_\Gamma(f)$ and the indices of the essential fixed point classes appear as the coefficients in $L_\Gamma(f)$. These invariants are homotopy invariants.

Remark 3.1. *We take an arbitrary path c from v to a fixed point x. In the light of the continuous map $H : I \times I \to T_f$ defined by $H(s,t) = (c(t), s)$, $\varphi_{(x,1)}$ is homotopic to the loop $c^{-1}\varphi_{(v,1)}f(c) = c^{-1}zwf(c)$ and we obtain*

$$\mathrm{cd}(x) = [[zwf(c)c^{-1}]],$$

where $[[\gamma]]$ is a free homotopy class obtained by γ.

3.2. **Periodic orbit classes.** In [14], Jiang studied the periodic orbit of f, i.e. the fixed points of the iterates of f.

The *periodic point set* of f is the set of points (x, n) in $X \times \mathbb{N}$ satisfying $x = f^n(x)$ and is denoted by $\mathrm{PP}f$. An *n-point* of f is a fixed point x of f^n. For an n-point x of f, an *n-orbit* of f at x is the f-orbit $\{x, f(x), \ldots, f^{n-1}(x)\}$ in X. A *primary* n-orbit is an n-orbit consisting of n distinct points. In other words, an n-orbit of f at x is a *primary* n-orbit if n is the least period of the periodic point x.

An *n-point class* of f is a fixed point class $\mathbf{F}^n$ of f^n. We recall from [12, Proposition III.3.3] that $f(\mathbf{F}^n)$ is also an n-point class, and the fixed point index $\mathrm{ind}(f(\mathbf{F}^n), f^n)$ of f^n at $f(\mathbf{F}^n)$ and the fixed point index $\mathrm{ind}(\mathbf{F}^n, f^n)$ of f^n at $\mathbf{F}^n$ are the same. Thus f acts as an index-preserving permutation among its n-point classes. An *n-orbit class* of f is the union of an orbit of this action. In other words, two points x and x' in $\mathrm{Fix}\, f^n$ are said to be in the same n-orbit class of f if and only if there exist natural numbers i and j such that $f^i(x)$ and $f^j(x')$ are in the same n-point class of f. The set

Fix f^n splits into a disjoint union of n-orbit classes. On the mapping torus T_f, we observe that (x, n) is in the periodic point set of f if and only if the time-n orbit curve $\varphi_{(x,n)}$ is closed. The free homotopy class $[[\varphi_{(x,n)}]] \in \Gamma_c$ of the closed curve $\varphi_{(x,n)}$ is called the Γ-*coordinate* of (x, n) and is denoted by $\mathrm{cd}_\Gamma(x, n)$. It follows from [13, §3] that periodic points (x, n) and (x', n') in the periodic point set of f have the same Γ-coordinate if and only if n and n' are the same and x and x' belong to the same n-orbit class of f. Therefore every n-orbit class $\mathbf{O}^n$ gives rise to a conjugacy class $\mathrm{cd}_\Gamma(\mathbf{O}^n)$ in Γ_c, called the Γ-*coordinate* of $\mathbf{O}^n$.

An important notion in the Nielsen theory for periodic orbits is the notion of reducibility. Suppose m is a divisor of n and m is less than n. If the n-orbit class $\mathbf{O}^n$ contains an m-orbit class $\mathbf{O}^m$, then for $x \in \mathbf{O}^m$, the closed curve $\varphi_{(x,n)}$ is the closed curve $\varphi_{(x,m)}$ traced n/m times and $\mathrm{cd}_\Gamma(\mathbf{O}^n)$ is the (n/m)-th power of $\mathrm{cd}_\Gamma(\mathbf{O}^m)$. An n-orbit class $\mathbf{O}^n$ is *reducible to period* m if $\mathrm{cd}_\Gamma(\mathbf{O}^n)$ has an (n/m)-th root and is *irreducible* if $\mathrm{cd}_\Gamma(\mathbf{O}^n)$ has no nontrivial root.

An n-orbit class $\mathbf{O}^n$ is called *essential* if its index $\mathrm{ind}(\mathbf{O}^n, f^n)$ is nonzero. For each natural number n, the generalized Lefschetz number with respect to Γ is defined as

$$L_\Gamma(f^n) = \sum_{\mathbf{O}^n} \mathrm{ind}(\mathbf{O}^n, f^n) \cdot \mathrm{cd}_\Gamma(\mathbf{O}^n) \in \mathbb{Z}\Gamma_c,$$

where the summation is taken over all essential n-orbit classes $\mathbf{O}^n$ of f. When we consider the case $n = 1$, 1-orbit classes of f are fixed point classes of f and the definition of generalized Lefschetz number with respect to Γ and the definition of generalized Lefschetz number in Section 3.1 coincide for $n = 1$. The *Nielsen number of n-orbits* $N_\Gamma(f^n)$ is the number of nonzero terms in $L_\Gamma(f^n)$ and the indices of the essential fixed point classes appear as the coefficients in $L_\Gamma(f^n)$. Clearly it is a lower bound for the number of n-orbits of f. The *Nielsen number of irreducible n-orbits* $NI_\Gamma(f^n)$ is the number of nonzero primary terms in $L_\Gamma(f^n)$. It is the number of irreducible essential n-orbit classes. It is a lower bound for the number of primary n-orbits of f. The generalized Lefschetz number with respect to Γ, the Nielsen number of n-orbits and the Nielsen number of irreducible n-orbits are homotopy invariants.

3.3. **Asymptotic Nielsen theory.** In [14] Jiang defines the *asymptotic Nielsen number* of f to be the growth rate of the Nielsen numbers

$$N^\infty(f) = \operatorname*{Growth}_{n\to\infty} N_\Gamma(f^n),$$

the *asymptotic irreducible Nielsen number* of f to be the growth rate of the Nielsen numbers of irreducible orbits

$$NI^\infty(f) = \operatorname*{Growth}_{n\to\infty} N_\Gamma(f^n)$$

and the *asymptotic absolute Lefschetz number* of f to be the growth rate of the norm of generalized Lefschetz numbers

$$L^\infty(f) = \operatorname*{Growth}_{n\to\infty} \|L_\Gamma(f^n)\|.$$

In [14] all these asymptotic numbers are shown to enjoy the homotopy invariance.

Remark 3.2. Since the inequality $NI_\Gamma(f) \leq N_\Gamma(f) \leq \|L_\Gamma(f)\|$ holds, we obtain $NI^\infty(f) \leq N^\infty(f) \leq L^\infty(f)$. In [14], Jiang showed that a sufficient condition for the equality $NI^\infty(f) = N^\infty(f)$ is that f satisfies the following Property of Essential Irreducibility: The number E_n of essentially irreducible n-point classes that are reducible is uniformly bounded in n. Also in [14], Jiang showed that a sufficient condition for the equality $N^\infty(f) = L^\infty(f)$ is that f satisfies the following Property of Bounded Index: The maximum absolute value B_n of the indices of n-point classes $\mathbf{F}^n$ is uniformly bounded in n. These conditions are not strong. For example, every homeomorphism of D_n satisfies the Property of Essential Irreducibility and the Property of Bounded Index.

In [11], Ivanov showed that the logarithm of the asymptotic Nielsen number $N^\infty(f)$ of a self map f coincides with the entropy of a self map f.

Theorem 3.3. (Ivanov [11]) *Let X be a compact surface with negative Euler characteristic and f be a self map of X. Then the entropy of f coincides with* $\log N^\infty(f)$.

For a compact surface X with negative Euler characteristic, we take a pseudo-Anosov homeomorphism f of X with the dilatation $\lambda > 1$. Then together with (2.1), we obtain that

$$h(f) = \log \lambda = \log N^\infty(f) \tag{3.1}$$

is the minimal entropy in the homotopy class of f.

3.4. **Nielsen theory for stratified maps.** In Section 3.1, the space X is always assumed to be compact. However, the configuration space $\mathcal{C}_{n,m}(D^2)$ is not compact. In [16], Jiang, Zhao and Zheng extended fixed point theory for some good noncompact space and using this, they developed Nielsen theory for $\mathcal{C}_{n,m}(D^2)$ in [15]. The Nielsen theory for stratified maps is a version of relative Nielsen theory. We recall basic notions of the Nielsen theory for stratified maps due to [15]. We refer the readers to [16] for a detailed treatment of this subject.

For a compact, connected polyhedron space W, let

$$\emptyset = W^0 \subset W^1 \subset \cdots \subset W^{m-1} \subset W^m = W$$

be a filtration of compact subpolyhedra. For $1 \leq k \leq m$, the subspace $W_k = W^k \setminus W^{k-1}$ is called the k-th stratum. A map $f : W \to W$ is called a *stratified map* if $f(W_k)$ is contained in W_k for all strata W_k. Two stratified maps $f, f' : W \to W$ are called *stratified homotopic* if there exists a homotopy $H : W \times I \to W$ such that H_0 equals f, H_1 equals f' and H_t is a stratified map for all t.

We define f_m to be a map restricting f on W_m. We will be concerned with fixed point classes of f_m in the top stratum. A free homotopy class of closed curves in T_{f_m}, represented by a closed curve γ, is said to be *related* to a lower stratum W_k if there exists a homotopy $H : S^1 \times I \to T_f$ such that H_0 equals γ, H_t is a closed curve in T_{f_m} for all $0 \leq t < 1$ and H_1 is a closed curve in $T_{f|_{W_k}}$. A fixed point class of f_m is called *degenerate* if its coordinate is related to some lower stratum W_k. Otherwise, it is called *non-degenerate*.

The *generalized Lefschetz number* of the stratified map f is defined as

$$L^s_\Gamma(f) = \sum_{\mathbf{F}_m} \mathrm{ind}(f_m, \mathbf{F}_m) \cdot \mathrm{cd}(\mathbf{F}_m) \in \mathbb{Z}\Gamma_c,$$

where the summation is taken over all non-degenerate fixed point class $\mathbf{F}_m$ of f_m. Let $N^s_\Gamma(f)$ be the number of nonzero terms in $L^s_\Gamma(f)$. It is the number of essential non-degenerate fixed point classes, and will be called the *Nielsen number* of the stratified map f.

The Nielsen fixed point theory has the natural version for stratified maps. The main result is that $L_\Gamma(f)$ and $N_\Gamma(f)$ are not changed by a stratified homotopy of the map f, which is proved in [16].

3.5. **Nielsen theory for finite invariant sets.** We recall basic notions of Nielsen theory for finite invariant sets due to [15]. In this subsection, we assume that X is a compact, connected, smooth manifold of dimension d and $f : X \to X$ is a self embedding. We fix a natural number m. We consider the symmetric product space

$$\mathrm{SP}^m X = X^m / \mathcal{S}_m.$$

Its points are written as $[x_1, \ldots, x_m]$, with repetition allowed. For an integer k satisfying $0 \leq k \leq m$, we define the subspace

$$\mathrm{SP}^{m,k} X = \{[x_1, \ldots, x_m] \in \mathrm{SP}^m X \mid \#\{x_1, \ldots, x_m\} \leq k\}.$$

Then we have a filtration

$$\emptyset = \mathrm{SP}^{m,0} X \subset \mathrm{SP}^{m,1} X \subset \cdots \subset \mathrm{SP}^{m,m-1} X \subset \mathrm{SP}^{m,m} X = \mathrm{SP}^m X.$$

For $1 \leq k \leq m$, the k-th stratum is $W_k = \mathrm{SP}^{m,k} X \setminus \mathrm{SP}^{m,k-1} X$. We notice that the top stratum is $\mathcal{C}_{0,m}(X)$.

The map f induces a map $\mathrm{SP}^m f : \mathrm{SP}^m X \to \mathrm{SP}^m X$ given by

$$\mathrm{SP}^m f([x_1, \ldots, x_m]) = [f(x_1), \ldots, f(x_m)].$$

Since f is an embedding, $\mathrm{SP}^m f$ is now a stratified map with respect to the above filtration. Hence the theory in the previous subsection is applicable.

We define $\widehat{f}$ to be the map restricting $\mathrm{SP}^m f$ on W_m. A fixed point $[x_1, \ldots, x_m]$ of $\widehat{f}$ corresponds to an f-invariant set consisting of precisely m distinct points. Thus, the number of non-degenerate, essential fixed point classes of $\widehat{f}$ is a lower bound for the number of such f-invariant sets for all embeddings isotopic to f.

Below is a useful criterion for the degeneracy of a fixed point class of $\widehat{f}$.

Proposition 3.4. (Jiang and Zheng [15]) *We suppose that X is a compact, connected smooth manifold of dimension d and $f : X \to X$ is a self embedding. Let $Q = \{x_1, \ldots, x_m\}$ be an f-invariant subset of X. We fix k satisfying $1 \leq k < m$. Let $\mathcal{D}$ denote the disjoint union of k copies of the d-dimensional disks. The coordinate of the fixed point $[x_1, \ldots, x_m]$ of $\widehat{f}$ is related to the k-th stratum W_k if and only if there exists an isotopy of embeddings $\{i_t : \mathcal{D} \to X\}_{0 \leq t \leq 1}$ such that $i_0 = f \circ i_1$, $Q \subset i_t(\mathcal{D})$ and each component of $i_t(\mathcal{D})$ contains at least one point of Q for all $0 \leq t \leq 1$.*

In Proposition 3.4, the components of $i_0(\mathcal{D})$ containing more than one point of Q are called *merging disks* of Q. The existence of merging disks of Q means that the f-invariant set Q can be merged into a smaller one by isotoping f in a neighborhood of these disks.

Given a nontrivial n-strand braid β, there exists a connecting isotopy $\{h_t : D^2 \to D^2\}_{0\leq t\leq 1}$ from id such that the curves $\{h_t(P_n)\}_{0\leq t\leq 1}$ represent the braid β. We set $f_\beta = h_1$. Jiang and Zheng figured out their key observation.

Proposition 3.5. (Jiang and Zheng [15]) *(1) The mapping torus of the induced map* $\widehat{f_\beta} : \mathcal{C}_{n,m}(D^2) \to \mathcal{C}_{n,m}(D^2)$ *can be identified with the space obtained from*

$$\left\{((h_t(P_n), \{y_1, \dots, y_m\}), t) \;\middle|\; y_i \in D^2 \setminus h_t(P_n), 0 \leq t \leq 1\right\} \subset IT_{n,m}(D^2) \times I$$

by identifying the top $\mathcal{C}_{n,m}(D^2) \times \{0\}$ *with the bottom* $\mathcal{C}_{n,m}(D^2) \times \{1\}$.
(2) Under the above identification, the fundamental group $\Gamma_{\beta,m}$ *of* $T_{\widehat{f_\beta}}$ *is isomorphic to the subgroup in* B_{n+m} *generated by* β *and* $\mathbf{B}_{n,m}(D^2)$.
(3) Moreover, when a fixed point of $\widehat{f_\beta}$ *corresponds to a finite* f_β*-invariant subset* Q *of* D_n*, the coordinate of the former is precisely* $[\beta_{P_n\cup Q}]$*, where* $\beta_{P_n\cup Q}$ *is the braid corresponding to the geometric braid* $\{h_t(P_n \cup Q)\}_{0\leq t\leq 1}$.

4. The representation $\zeta_{n,m}$ and fixed points

4.1. **The definition of $\zeta_{n,m}$.** In [6], Bigelow defined the triangle corresponding to the embedded edge for $m = 2$. Triangles are elements of the relative homology of some abelian covering of the configuration space $\mathcal{C}_{n,m}(D^2)$. In this subsection we define $\zeta_{n,m}$ due to Jiang and Zheng by using the lifts of triangles to the universal covering.

We introduce some relative homology of the universal covering of the configuration space $\mathcal{C}_{n,m}(D^2)$. Let $p : \widetilde{\mathcal{C}}_{n,m}(D^2) \to \mathcal{C}_{n,m}(D^2)$ be the universal covering of $\mathcal{C}_{n,m}(D^2)$ and fix $\widetilde{c} \in p^{-1}(c)$ as a base point of $\widetilde{\mathcal{C}}_{n,m}(D^2)$. For $\varepsilon > 0$, we define V_ε to be the set of points $\{x_1, \dots, x_m\}$ in $\mathcal{C}_{n,m}(D^2)$ such that at least one of the pair (x_i, x_j) is within distance ε of each other. We define $\widetilde{V_\varepsilon}$ to be the preimage of V_ε in $\widetilde{\mathcal{C}}_{n,m}(D^2)$. The relative homology $H_m(\widetilde{\mathcal{C}}_{n,m}(D^2), \partial\widetilde{\mathcal{C}}_{n,m}(D^2) \cup \widetilde{V_\varepsilon})$ is nested by inclusion.

For $\beta \in B_n$, $\widehat{f_\beta}$ has a unique lift $\widetilde{f_\beta} : (\widetilde{\mathcal{C}}_{n,m}(D^2), \widetilde{c}) \to (\widetilde{\mathcal{C}}_{n,m}(D^2), \widetilde{c})$ and induces an automorphism of the left $\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]$-module

$$\lim_{\varepsilon\to 0} H_m(\widetilde{\mathcal{C}}_{n,m}(D^2), \partial\widetilde{\mathcal{C}}_{n,m}(D^2) \cup \widetilde{V_\varepsilon}).$$

The induced automorphism is independent of the choice of the representative and denoted by $\widetilde{\beta}_*$.

The groups $\mathbf{B}_{n,m}(D^2)$ and $\mathbf{E}_{n,m}(D^2)$ can be regarded as subgroups of $\mathbf{B}_{0,n+m}(D^2)$. The intertwining (n, m)-braid group $\mathbf{E}_{n,m}(D^2)$ is the preimage of $\mathcal{S}_n \times \mathcal{S}_m$ under the canonical projection $\mathbf{B}_{0,n+m}(D^2) \to \mathcal{S}_{n+m}$. In addition, $\mathbf{B}_{n,m}(D^2)$ is the subgroup of $(n+m)$-braids in $\mathbf{E}_{n,m}(D^2)$ that become trivial by forgetting the last m strands. The intertwining (n, m)-braid group $\mathbf{E}_{n,m}(D^2)$ is isomorphic to the subgroup $E_{n,m}$ of B_{n+m} generated by

$$\sigma_1, \dots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1}, \dots, \sigma_{n+m-1}$$

and $\mathbf{B}_{n,m}(D^2)$ is isomorphic to the subgroup $B_{n,m}$ of B_{n+m} generated by

$$A_{1,n+1}, \ldots, A_{n,n+1}, \sigma_{n+1}, \ldots, \sigma_{n+m-1},$$

where A_{ij} is defined by

$$A_{ij} = \sigma_{j-1} \ldots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \ldots \sigma_{j-1}^{-1}.$$

Therefore B_n acts on $\mathbf{E}_{n,m}(D^2)$ by the right multiplication and so there exists an induced action of β on the $\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]$. Moreover, since $\mathbf{B}_{n,m}(D^2)$ is included in $\mathbf{E}_{n,m}(D^2)$, $\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]$ is a right $\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]$-module. Using the $\mathbb{Z}$-module automorphism $\widetilde{\beta}_*$ and the action on $\mathbf{E}_{n,m}(D^2)$ by B_n, we construct an automorphism $\beta \otimes \widetilde{\beta}_*$ on the left $\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]$-module

$$\mathbb{Z}[\mathbf{E}_{n,m}(D^2)] \otimes_{\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]} \lim_{\varepsilon \to 0} H_m(\widetilde{\mathcal{C}}_{n,m}(D^2), \partial\widetilde{\mathcal{C}}_{n,m}(D^2) \cup \widetilde{V}_\varepsilon)$$

by

$$(\beta \otimes \widetilde{\beta}_*)(h \otimes c) = h\beta \otimes \widetilde{\beta}_*(c).$$

Proposition 4.1. *For any $\beta \in B_n$, $\beta \otimes \widetilde{\beta}_*$ is a $\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]$-homomorphism.*

Proof. For every $\gamma \in \mathbf{E}_{n,m}(D^2)$, the equality

$$\begin{aligned}\gamma((\beta \otimes \widetilde{\beta}_*)(h \otimes c)) &= \gamma(h\beta \otimes \widetilde{\beta}_*(c)) = \gamma h\beta \otimes \widetilde{\beta}_*(c) \\ &= (\beta \otimes \widetilde{\beta}_*)(\gamma h \otimes c) = (\beta \otimes \widetilde{\beta}_*)(\gamma(h \otimes c)).\end{aligned}$$

holds. □

From now on, we define a representation $\zeta_{n,m}$ of B_n over the free left $\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]$-module generated by $\mathcal{E}_{n,m}$. The cardinality $d_{n,m}$ of the basis $\mathcal{E}_{n,m}$ is $\begin{pmatrix} n+m-2 \\ m \end{pmatrix}$.

We now introduce some other relative homology and an intersection pairing. Henceforth every path is a continuous map from $I = [0,1]$. For $\varepsilon > 0$, we define U_ε to be the set of points $\{x_1, \ldots, x_m\} \in \mathcal{C}_{n,m}(D^2)$ such that at least one of them is within distance ε of some puncture point. We define $\widetilde{U}_\varepsilon$ to be the preimage of p in $\widetilde{\mathcal{C}}_{n,m}(D^2)$. The relative homology $H_m(\widetilde{\mathcal{C}}_{n,m}(D^2), \widetilde{U_\varepsilon})$ is nested by inclusion.

We set

$$\begin{aligned}
p_i &= \left(\frac{i}{2n}, 0\right), P_n = \{p_1, \ldots, p_n\}, \\
d_j &= \left(\cos\frac{j}{3m}\pi, \sin\frac{j}{3m}\pi\right), c = \{d_1, \ldots, d_m\}, \\
N_i &= \left\{x = \frac{2i+1}{4n}\right\} \cap D^2, \alpha_i = \left\{(x,0) \;\middle|\; \frac{i}{2n} < x < \frac{i+1}{2n}\right\}, \\
z_i^j &= \left(\frac{2i+1}{4n}, \sin\frac{j}{3m}\pi\right)
\end{aligned}$$

and let α_i^j be a polygonal line connecting p_i, z_i^j and p_{i+1}. We call α_i^j *fork*. For $\mu \in \mathcal{E}_{n,m}$, we set

$$F_\mu = \left\{\{x_1, \ldots, x_m\} \in \mathcal{C}_{n,m}(D^2) \;\middle|\; \#(\{x_1, \ldots, x_m\} \cap N_i) = \mu_i\right\}$$

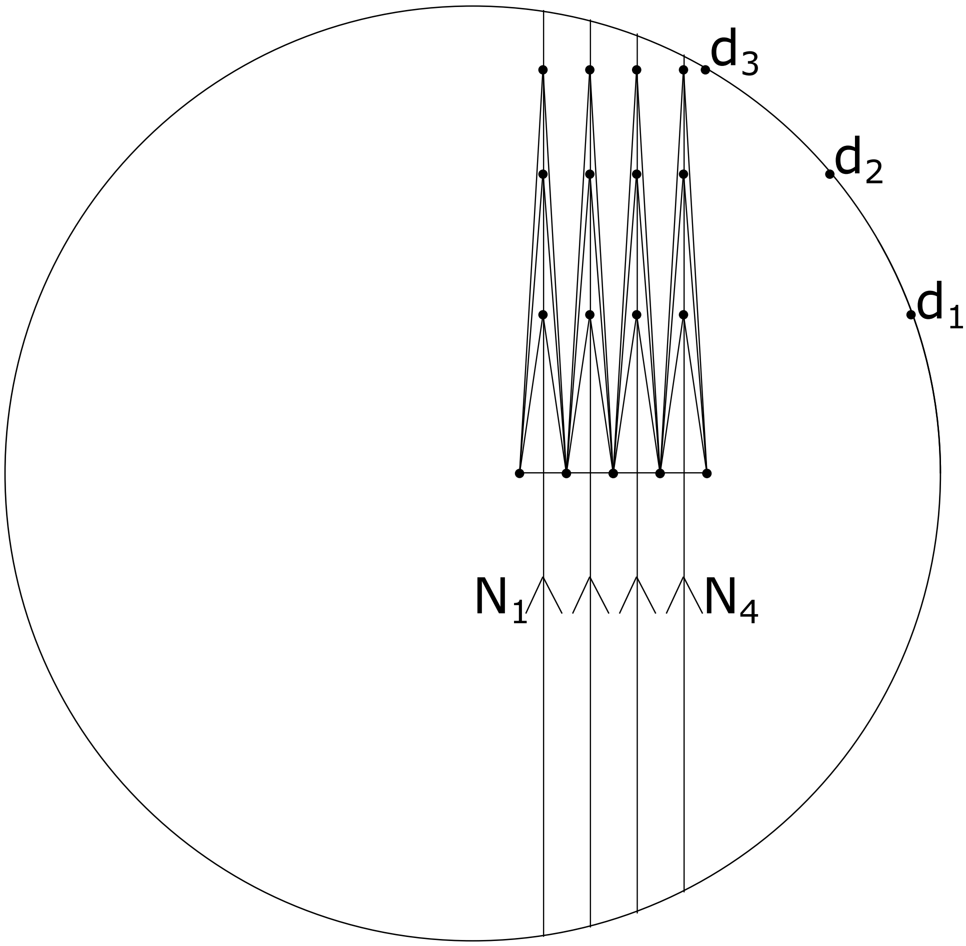

FIGURE 2. The picture for $n = 5$ and $m = 3$

and

$$S_\mu = \prod_{i=1}^{n-1} \prod_{j=u_i+1}^{u_{i+1}} \operatorname{int} \alpha_i^j,$$

where $u_i = \sum_{j=1}^{i-1} \mu_j$. We take line segments θ_j on D_n from c_j to z_i^j, where $u_i < j \leq u_{i+1}$. We notice that they are disjoint. Let z_μ be the endpoint of $\Theta_\mu = \{\theta_1, \ldots, \theta_m\}$. We take a lift $\widetilde{z}_\mu$ of z_μ so that the lift $\widetilde{\Theta}_\mu$ of Θ_μ is starting at $\widetilde{c}$ and ending at $\widetilde{z}_\mu$. We take lifts $\widetilde{F}_\mu$ and $\widetilde{S}_\mu$ of F_μ and S_μ containing $\widetilde{z}_\mu$ respectively. Let $[X]$ denote the element of certain relative homology corresponding to the m-dimensional subspace X of $\widetilde{\mathcal{C}}_{n,m}(D^2)$. We set

$$\mathcal{H}_F = \bigoplus_{\mu \in \mathcal{E}_{n,m}} \mathbb{Z}[\mathbf{B}_{n,m}(D^2)] \left[\widetilde{F}_\mu\right] \subset \lim_{\varepsilon \to 0} H_m(\widetilde{\mathcal{C}}_{n,m}(D^2), \partial\widetilde{\mathcal{C}}_{n,m}(D^2) \cup \widetilde{V}_\varepsilon)$$

and

$$\mathcal{H}_S = \bigoplus_{\mu \in \mathcal{E}_{n,m}} \mathbb{Z}[\mathbf{B}_{n,m}(D^2)] \left[\widetilde{S}_\mu\right] \subset \lim_{\varepsilon \to 0} H_m(\widetilde{\mathcal{C}}_{n,m}(D^2), \widetilde{U_\varepsilon}).$$

For $x \in \mathcal{H}_S$ and $y \in \mathcal{H}_F$, let $(x \cdot y) \in \mathbb{Z}$ denote the standard intersection number. In [6] for $m = 2$ and [5], Bigelow defined an intersection pairing. Similarly, we define an intersection pairing

$$\langle \cdot, \cdot \rangle : \mathcal{H}_S \times \mathcal{H}_F \to \mathbb{Z}[\mathbf{B}_{n,m}(D^2)]$$

by

$$\langle x, y \rangle = \sum_{\beta \in \mathbb{Z}[\mathbf{B}_{n,m}(D^2)]} (x \cdot \widetilde{\beta}_*(y))\beta.$$

We notice that $\left\langle \left[\widetilde{S}_\mu\right], \left[\widetilde{F}_\nu\right] \right\rangle$ equals 1 when $\mu = \nu$ and 0 otherwise. Therefore $\left\{\left[\widetilde{F}_\mu\right]\right\}_{\mu \in \mathcal{E}_{n,m}}$ is linearly independent. We define elements $d_{\mu\nu}^{(\beta)}$ of $\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]$ so that $\{d_{\mu\nu}^{(\beta)}\}_{\mu,\nu \in \mathcal{E}_{n,m}}$ satisfies the relations

$$\sum_\nu d_{\mu\nu}^{(\beta)} \left[\widetilde{F}_\nu\right] = \widetilde{\beta}_* \left(\left[\widetilde{F}_\mu\right] \right).$$

for all $\mu \in \mathcal{E}_{n,m}$. Using the intersection pairing, we obtain

$$d_{\mu\nu}^{(\beta)} = \tau \left(\left\langle \left[\widetilde{S}_\nu\right], \widetilde{\beta}_* \left(\left[\widetilde{F}_\mu\right] \right) \right\rangle \right), \tag{4.1}$$

where τ is an automorphism of $\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]$ with $\tau(\beta) = \beta^{-1}$. There exists a homomorphism

$$\zeta'_{n,m} : B_n \to \mathrm{Aut}_{\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]} \left(\mathbb{Z}[\mathbf{E}_{n,m}(D^2)] \otimes_{\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]} \mathcal{H}_F \right)$$

defined by $\zeta'_{n,m}(\beta) = (\beta \otimes \widetilde{\beta}_*)|_{\mathcal{H}_F}$. We notice that

$$\mathbb{Z}[\mathbf{E}_{n,m}(D^2)] \otimes_{\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]} \mathcal{H}_F \cong \bigoplus_{\mu \in \mathcal{E}_{n,m}} \mathbb{Z}[\mathbf{E}_{n,m}(D^2)] \left[\widetilde{F}_\mu\right]$$

and this gives the representation $\zeta_{n,m}$ to the matrix group

$$\mathrm{GL}(d_{n,m}, \mathbb{Z}[\mathbf{E}_{n,m}(D^2)]).$$

We set $\zeta_{n,m}(\beta) = (c_{\mu\nu}^{(\beta)})$ and notice that $c_{\mu\nu}^{(\beta)} = \beta d_{\mu\nu}^{(\beta)}$ in $\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]$.

Proposition 4.2. *The map $\zeta_{n,m}$ is a group homomorphism.*

Proof. For $\beta, \gamma \in B_n$, we obtain

$$\zeta_{n,m}(\beta)\zeta_{n,m}(\gamma) = \left(c_{\mu\nu}^{(\beta)}\right)\left(c_{\mu\nu}^{(\gamma)}\right) = \left(\beta d_{\mu\nu}^{(\beta)}\right)\left(\gamma d_{\mu\nu}^{(\gamma)}\right) = \left(\sum_\rho \beta d_{\mu\rho}^{(\beta)} \gamma d_{\rho\nu}^{(\gamma)}\right).$$

We notice that $f_{\beta\gamma} = f_\gamma \circ f_\beta$. Then we obtain

$$\begin{aligned}
\sum_\nu c^{(\beta\gamma)}_{\mu\nu} N_\nu &= \beta\gamma \cdot (\widetilde{\beta\gamma})_*(N_\mu) \\
&= \beta\gamma \cdot \widetilde{\gamma}_* \left(\sum_\rho d^{(\beta)}_{\mu\rho} N_\rho \right) \\
&= \beta\gamma \cdot \sum_\rho (\widehat{f_\beta})_*(d^{(\beta)}_{\mu\rho}) \widetilde{\gamma}_*(N_\rho) \\
&= \beta\gamma \sum_\rho \left(\gamma^{-1} d^{(\beta)}_{\mu\rho} \gamma \right) \left(\sum_\nu d^{(\gamma)}_{\rho\nu} N_\nu \right) \\
&= \sum_\nu \left(\sum_\rho \beta d^{(\beta)}_{\mu\rho} \gamma d^{(\gamma)}_{\rho\nu} \right) N_\nu \\
&= \sum_\nu \left(\sum_\rho c^{(\beta)}_{\mu\rho} c^{(\gamma)}_{\rho\nu} \right) N_\nu .
\end{aligned}$$

Therefore we obtain $\zeta_{n,m}(\beta)\zeta_{n,m}(\gamma) = \zeta_{n,m}(\beta\gamma)$. □

We recall the definition of trace. Let Γ be a group, $\mathbb{Z}\Gamma$ its group ring, Γ_c the set of conjugacy classes, $\mathbb{Z}\Gamma_c$ the free Abelian group generated by Γ_c, and $\pi_\Gamma : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma_c$ the natural projection. Let ζ be an endomorphism of a free $\mathbb{Z}\Gamma$-module satisfying $\zeta(v_i) = \sum_{j=1}^k a_{ij} \cdot v_j$ for a basis $\{v_1, \ldots, v_k\}$. The *trace* of ζ is defined as

$$\operatorname{tr}_\Gamma \zeta = \pi_\Gamma \left(\sum_{i=1}^k a_{ii} \right) \in \mathbb{Z}\Gamma_c.$$

We suppose $\zeta(u_i) = \sum_{j=1}^k b_{ij} \cdot u_j$ for another basis $\{u_1, \ldots, u_k\}$. Then there exist elements c_{ij} and d_{ij} such that $u_i = \sum_{j=1}^k c_{ij} \cdot v_j$ and $v_i = \sum_{j=1}^k d_{ij} \cdot u_j$. Then we obtain

$$\zeta(u_i) = \sum_{j=1}^k c_{ij}\zeta(v_j) = \sum_{l=1}^k \left(\sum_{j=1}^k c_{ij} a_{jl} \right) \cdot v_l = \sum_{m=1}^k \left(\sum_{j=1}^k \sum_{l=1}^k c_{ij} a_{jl} d_{lm} \right) \cdot u_m$$

and

$$\begin{aligned}
\pi_\Gamma \left(\sum_{i=1}^k b_{ii} \right) &= \pi_\Gamma \left(\sum_{i=1}^k \sum_{j=1}^k \sum_{l=1}^k c_{ij} a_{jl} d_{li} \right) = \pi_\Gamma \left(\sum_{j=1}^k \sum_{i=1}^k \sum_{l=1}^k d_{li} c_{ij} a_{jl} \right) \\
&= \pi_\Gamma \left(\sum_{j=1}^k a_{jj} \right).
\end{aligned}$$

Therefore the definition is independent of the choice of the basis. Let ζ and ξ be two endomorphisms of a free $\mathbb{Z}\Gamma$-module defined by $\zeta(v_i) = \sum_{j=1}^k a_{ij} \cdot v_j$ and $\xi(v_i) = \sum_{j=1}^k b_{ij} \cdot v_j$ for a basis $\{v_1, \ldots, v_k\}$. Then we obtain

$$\operatorname{tr}_\Gamma \zeta \circ \xi = \pi_\Gamma \left(\sum_{i=1}^k \sum_{j=1}^k a_{ij} b_{ji} \right) = \pi_\Gamma \left(\sum_{j=1}^k \sum_{i=1}^k b_{ji} a_{ij} \right) = \operatorname{tr}_\Gamma \xi \circ \zeta.$$

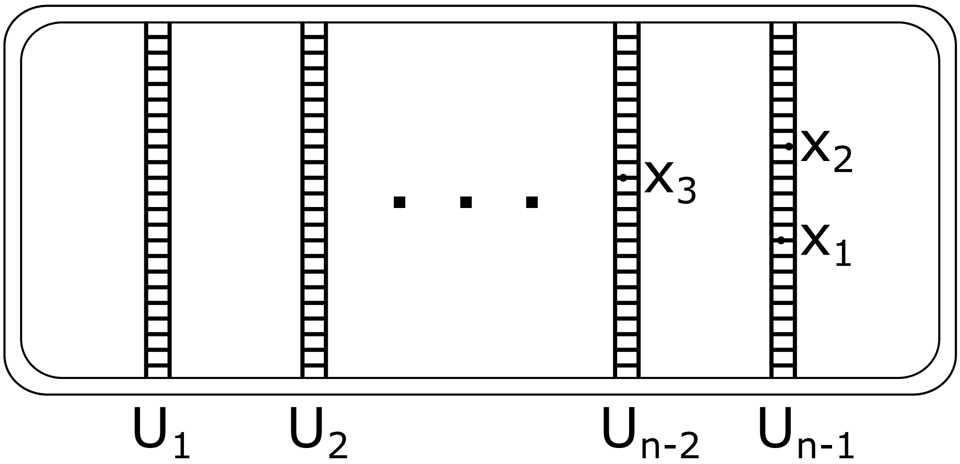

FIGURE 3. Decomposition of Y_n

We note that, under the basis $\mathcal{E}_{n,m}$, all matrix elements of $\zeta_{n,m}(\beta)$ belong to $\mathbb{Z}\Gamma_{\beta,m}$. Therefore $\zeta_{n,m}(\beta)$ can naturally be regarded as an endomorphism of the free $\mathbb{Z}\Gamma_{\beta,m}$-module generated by $\mathcal{E}_{n,m}$. In this way, the notations $\mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta)$ and $\mathrm{tr}_{\Gamma_{\beta^k,m}} \zeta_{n,m}(\beta^k)$ in the main theorem are well-defined.

Theorem 4.3. *For any pseudo-Anosov braid $\beta \in B_n$, we denote by λ the dilatation of β. Then we obtain*

$$\underset{k\to\infty}{\mathrm{Growth}} \left\| \mathrm{tr}_{\Gamma_{\beta^k,m}} \zeta_{n,m}(\beta^k) \right\| = \underset{k\to\infty}{\mathrm{Growth}}\, \mathrm{tr} \left\| \zeta_{n,m}(\beta^k) \right\| = \lambda^m,$$
$$\underset{m\to\infty}{\mathrm{Growth}} \left\| \mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta) \right\| = \lambda.$$

4.2. **The work of Jiang and Zheng.** The representation $\zeta_{n,m}$ is the same as the representation due to Jiang and Zheng [15]. We compactify D_n to a 2-disk with n holes and denote it by Y_n, and assume further that there exists a homeomorphism $\overline{f_\beta} : Y_n \to Y_n$ such that f_β is the map restricting $\overline{f_\beta}$ on $\mathrm{int}\, Y_n$. We identify $\mathrm{int}\, Y_n \cup \partial D^2$ with D_n. We decompose the surface Y_n into an anulus and $n-1$ foliated rectangles, as shown in Figure 3.

We define $U = U_1 \cup \cdots \cup U_{n-1}$ to be the union of the $n-1$ foliated open rectangles. We define a partial ordering on U such that $x_1 \prec x_2$ if either x_1 lies in a rectangle to the right of x_2 or x_1 lies in a strictly lower leaf of the same rectangle as x_2. For example, the order of the three points in Figure 3 is $x_1 \prec x_2 \prec x_3$.

We set

$$V = \left\{ \{x_1, \ldots, x_m\} \in \mathcal{C}_{m,0}(Y_n) \;\middle|\; x_i \in U, \begin{array}{l} \text{there exists } \eta \in \mathcal{S}_m \text{ s.t.} \\ x_{\eta(1)} \prec \cdots \prec x_{\eta(m)} \end{array} \right\}.$$

Then we have $V = \bigcup_{\mu \in \mathcal{E}_{n,m}} V_\mu$, where

$$V_\mu = \left\{ \{x_1, \ldots, x_m\} \in V \mid \#\{x_1, \ldots, x_m\} \cap U_i = \mu_i \right\}.$$

Each V_μ is connected; thus the elements of $\mathcal{E}_{n,m}$ are in one-to-one correspondence to the components of V.

Illustrated in Figure 4 and Figure 5 are two embeddings ϕ_i and $\overline{\phi}_i$, which can be understood as the action of the elementary mapping σ_i and σ_i^{-1}

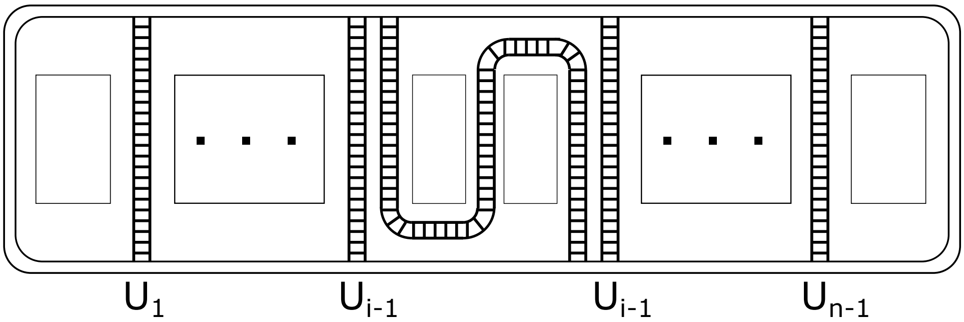

FIGURE 4. The image of the self map ϕ_i

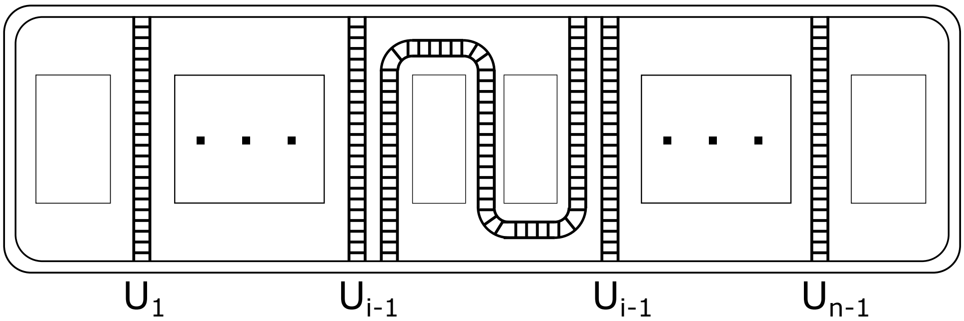

FIGURE 5. The image of the self map $\overline{\phi}_i$

on Y_n respectively. Both push the annulus outward, irrationally rotate the outmost boundary, keep the foliations of $(\phi_i)^{-1}(U)$ and $(\overline{\phi}_i)^{-1}(U)$, uniformly contract along the leaves of the foliations, and uniformly expand along the transversal direction.

For every $\phi \in \{\phi_1, \dots, \phi_{n-1}, \overline{\phi}_1, \dots, \overline{\phi}_{n-1}\}$, we have

$$V_\mu \cap \phi^{-1}(V_\nu) = \bigcup_{\eta \in \mathcal{S}_m} W^{(\phi)}_{\mu\nu\eta},$$

where

$$W^{(\phi)}_{\mu\nu\eta} = \left\{ x \in V_\mu \cap \phi^{-1}(V_\nu) \;\middle|\; \begin{array}{l} \text{there exist } x_1, \dots, x_m \text{ s.t.} \\ x = \{x_1, \dots, x_m\}, \\ x_{\eta(1)} \prec \cdots \prec x_{\eta(m)}, \\ \phi(x_1) \prec \cdots \prec \phi(x_m), \end{array} \right\}.$$

Each $W^{(\phi)}_{\mu\nu\eta}$ is connected; thus the elements of the set $\{\eta \in \mathcal{S}_m \mid W^{(\phi)}_{\mu\nu\eta} \neq \emptyset\}$ are in one-to-one correspondence to the components of $V_\mu \cap \phi^{-1}(V_\nu)$.

We choose a base point $b = \{b_1, \dots, b_m\}$ in $\operatorname{int} A$. For every element $x = \{x_1, \dots, x_m\}$ in V with $x_1 \prec \cdots \prec x_m$, the disjoint "descending" paths connecting b_k to x_k in Y_n give rise to a path γ_x in $\mathcal{C}_{n,m}(Y_n)$. Similarly, the disjoint "ascending" paths connecting b_k to $\phi(b_k)$ give rise to a path $\gamma_{\phi(b)}$ in $\mathcal{C}_{n,m}(Y_n)$. For every nonempty $W^{(\phi)}_{\mu\nu\eta}$, we choose a point $x \in W^{(\phi)}_{\mu\nu\eta}$ and $\alpha^{(\phi)}_{\mu\nu\eta}$ denotes the element of $\pi_1(\mathcal{C}_{n,m}(Y_n), b)$ represented by the loop $\gamma_{\phi(b)} \cdot \phi(\gamma_x) \cdot \gamma^{-1}_{\phi(x)}$. We note that $\alpha^{(\phi)}_{\mu\nu\eta}$ is independent of the choices of x, γ_x, $\gamma_{\phi(b)}$ and $\gamma_{\phi(x)}$.

In [15], Jiang and Zheng showed that the equations

$$\mu \cdot \zeta_{n,m}(\sigma_i) = \sum_{\nu \in \mathcal{E}_{n,m}} c_{\mu\nu}^{(i)} \cdot \nu,$$
$$\mu \cdot \zeta_{n,m}(\sigma_i^{-1}) = \sum_{\nu \in \mathcal{E}_{n,m}} d_{\mu\nu}^{(i)} \cdot \nu,$$

where

$$c_{\mu\nu}^{(i)} = (-1)^{\nu_i} \cdot \sigma_i \cdot \sum_{\eta : W_{\mu\nu\eta}^{(\phi)} \neq \emptyset} \operatorname{sgn} \eta \cdot \alpha_{\mu\nu\eta}^{(\phi_i)},$$
$$d_{\mu\nu}^{(i)} = (-1)^{\nu_i} \cdot \sigma_i^{-1} \cdot \sum_{\eta : W_{\mu\nu\eta}^{(\phi)} \neq \emptyset} \operatorname{sgn} \eta \cdot \alpha_{\mu\nu\eta}^{(\overline{\phi}_i)},$$

give rise to a group representation of B_n over the free $\mathbb{Z}B_{n+m}$ module generated by $\mathcal{E}_{n,m}$.

We take the base point b in $\Theta_\mu \cap A$. We can take the base point b independent of μ because of the definition of Θ_μ and A. Let Θ_b be a path from b to $\Theta_\mu(1)$ along Θ_μ and Θ'_b be a path from b to $\Theta_\mu(0)$ along Θ_μ. We identify $\pi_1(\mathcal{C}_{n,m}(D^2), c)$ with $\pi_1(\mathcal{C}_{0,m}(Y_n), b)$ by the map induced by Θ_b.

Proposition 4.4. *The representation defined above and the representation $\zeta_{n,m}$ give the same matrix for any braid under the above identification.*

Proof. We consider the case $\beta = \sigma_i$ and the case $\beta = \sigma_i^{-1}$ is similar. We notice that F_μ is given by shrinking V_μ along the leaves of foliations and then $\widehat{\phi}(W_{\mu\nu\eta}^{\phi})$ is homotopy equivalent to F_ν. Therefore the nonzero terms of $\widetilde{\sigma}_{i*}(\left[\widetilde{F_\mu}\right])$ are in one-to-one correspondence to the components of $V_\mu \cap \phi^{-1}(V_\nu)$, which are in one-to-one correspondence to the elements of the set $\{\eta \in \mathcal{S}_m \mid W_{\mu\nu\eta}^{(\phi_i)} \neq \emptyset\}$.

There exists a homotopy $\{H : D_n \times I \to D_n\}$ with $H(x,0) = \phi_i(x)$ and $H(x,1) = f_\beta(x)$ such that a map $H(\cdot,t)$ defined by $H(\cdot,t)(x) = H(x,t)$ is injective for any t. Let $\widehat{H} : \mathcal{C}_{n,m}(D^2) \times I \to \mathcal{C}_{n,m}(D^2)$ be the map defined by $\widehat{H}(\{x_1,\ldots,x_m\},t) = \{H(x_1,t),\ldots,H(x_m,t)\}$ and $\widehat{H}(x,\cdot)$ be the path defined by $\widehat{H}(x,\cdot)(t) = \widehat{H}(x,t)$.

For nonempty $W_{\mu\nu\eta}^{(\phi_i)}$, we take an element x in $W_{\mu\nu\eta}^{(\phi_i)} \cap F_\mu$. We take γ_x the composition of two paths Θ_b and the path from z_μ to x in F_μ. Since $\gamma_{\phi(b)}$ is homotopic to the composition of two paths Θ'_b and $\widehat{\phi}_i(\Theta'_b)^{-1}$ relative to the endpoints, the loop $\widehat{f_\beta}(\gamma_x)\gamma_{\widehat{f_\beta}(x)}^{-1}$ is identified with $\alpha_{\mu\nu\eta}^{(\phi_i)}$ by the above identification. Therefore $\alpha_{\mu\nu\eta}^{(\phi_i)}$ is the term of $\widetilde{\sigma}_{i*}(\widetilde{F_\mu})$ corresponding to $W_{\mu\nu\eta}^{(\phi_i)}$ and the signature is $(-1)^{\nu_i} \operatorname{sgn} \eta$. Finally, left multiplication of σ_i and tensoring σ_i from left induce the same action on $\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]$. Therefore $\zeta_{n,m}$ and the representation due to Jiang and Zheng [15] give the same matrix for all $\beta \in B_n$. □

In [15], Jiang and Zheng studied the relation between the forcing relation of braids and the trace of this representation. We review the result [15] of Jiang and Zheng. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-preserving homeomorphism and

$$\{h_t : \mathbb{R}^2 \to \mathbb{R}^2\}_{0 \le t \le 1}$$

be an isotopy with $h_0 = \mathrm{id}$ and $h_1 = f$. An f-invariant set $P = \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$ gives rise to a geometric braid

$$\{(h_t(x_i), t) \mid 0 \le t \le 1, 1 \le i \le n\}$$

in the cylinder $\mathbb{R}^2 \times [0,1]$. Indeed, the closed curve

$$\{(h_t(x_1), \ldots, h_t(x_n)) \mid 0 \le t \le 1\}$$

in the configuration space $\mathcal{C}_{n,m}(D^2)$ gives rise to a braid β_P in the n-strand braid group B_n. A braid β *forces* a braid γ if, for any orientation-preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ and any isotopy $\{h_t\} : \mathrm{id} \simeq f$, the existence of an f-invariant set P with $[\beta_P] = [\beta]$ guarantees the existence of an f-invariant set Q with $[\beta_Q] = [\gamma]$. A braid β' is an *extension* of β if β' is a disjoint union of β and another braid γ. We note that they are possibly intertwining. An extension β' is *forced by* β if, for any orientation-preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ and any isotopy $\{h_t\} : \mathrm{id} \simeq f$, the existence of an f-invariant set P with $[\beta_P] = [\beta]$ guarantees the existence of an additional f-invariant set $Q \subset \mathbb{R}^2 \setminus P$ with $[\beta_{P \cup Q}] = [\beta']$.

In $\mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta)$, there exist some unwanted terms. To describe them, we recall the Thurston classification theorem.

Theorem 4.5. (Thurston [22]) *Every homeomorphism $f : S \to S$ of a compact surface S is isotopic to a homeomorphism ϕ called Thurston representative such that either ϕ is periodic, pseudo-Anosov or there exists a system of disjoint simple closed curves $\gamma = \{\gamma_1, \ldots, \gamma_k\}$ in $\mathrm{int}\, S$ called reducing curves such that γ is invariant by ϕ and γ has a ϕ-invariant tubular neighborhood U such that each component of $S \setminus U$ has negative Euler characteristic and on each ϕ-component of $S \setminus U$, ϕ is either periodic or pseudo-Anosov.*

We suppose that $\beta' \in B_{n+m}$ is an extension of $\beta \in B_n$. Let ϕ be a Thurston representative determined by β'. We say β' is *collapsible* relative to β if there exists a system of reducing curves of ϕ such that one of them encloses none of the punctures corresponding to β. Similarly, we say β' is *peripheral* relative to β if there exists a system of reducing curves of ϕ such that one of them encloses precisely one of or all of the punctures corresponding to β. If an extension $\beta' \in \beta \cdot \pi_1(\mathbf{B}_{n,m}(D^2))$ of a braid $\beta \in B_n$ is collapsible relative to β, then we say the conjugacy class $[\beta']$ in $\Gamma_{\beta,m}$ is *collapsible* and if an extension $\beta' \in \beta \cdot \pi_1(\mathbf{B}_{n,m}(D^2))$ of a braid $\beta \in B_n$ is peripheral relative to β, then we say the conjugacy class $[\beta']$ in $\Gamma_{\beta,m}$ is *peripheral.* The relation between the forcing relation of braids and the trace of the representation defined above is written as follows.

Theorem 4.6. (Jiang and Zheng [15]) *We suppose that a braid $\beta' \in B_{n+m}$ is an extension of $\beta \in B_n$. Then β' is forced by β if and only if β' is neither collapsible nor peripheral relative to β and the conjugacy class $[\beta']$ has a nonzero coefficient in $\mathrm{tr}_{B_{n+m}} \zeta_{n,m}(\beta)$.*

4.3. **Trace of $\zeta_{n,m}$ and fixed points.** In this subsection, we prove the key lemma of the proof of main theorem. We define $\mathrm{eFix}\, f$ to be the set of essential fixed points of f. We choose a word $\beta = \tau_1 \ldots \tau_N$, where τ_i is an element of $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$. We put $\varphi_i = \phi_{j_i}$ if there exists a number j_i satisfying $\tau_i = \sigma_{j_i}$ and $\varphi_i = \overline{\phi}_{j_i}$ if there exists a number j_i satisfying

$\tau_i = \sigma_{j_i}^{-1}$. Then the embedding $g = \varphi_N \dots \varphi_1 : Y_n \to Y_n$ induces a map $\widehat{g} : B_{n,m}(Y_n) \to B_{n,m}(Y_n)$ stratified homotopic to $\widehat{\widehat{f_\beta}}$. It is immediate from the definition of ϕ_i and $\overline{\phi}_i$ that $\mathrm{Fix}\,\widehat{g}$ is a subset of V.

We prove the next lemma whose proof is similar to that of [15, Proposition 4.3.] by Jiang and Zheng.

Lemma 4.7. *There exists a positive number B such that we have the inequality*

$$\#\,\mathrm{eFix}(\widehat{g}^k) \leq \left\| \mathrm{tr}_{\Gamma_{\beta^k,m}}\, \zeta_{n,m}(\beta^k) \right\| \leq B \#\,\mathrm{eFix}(\widehat{g}^k).$$

Proof. Without loss of generality, we only have to prove the case $k = 1$. We note that each of the components W_μ^j of $\bigcup_{\mu \in \mathcal{E}_{n,m}} V_\mu \cap (\widehat{g})^{-1}(V_\mu)$ is homeomorphic to $\mathbb{R}^{2m}$. Since $\widehat{g}$ is a hyperbolic map on W_μ^j, there exists precisely one fixed point of $\widehat{g}$ on W_μ^j. Let $x_j \in W_\mu^j$ be the fixed point of $\widehat{g}$ on W_μ^j. We notice that the fixed point class containing x consists of one element x. We set

$$\alpha^g(x_j) = \gamma_{\widehat{g}(c)} \cdot (\widehat{g})(\gamma_{x_j}) \cdot \gamma_{x_j}^{-1}.$$

We obtain

$$\mathrm{cd}(x_j) = [[z\gamma_{\widehat{g}(c)} \cdot (\widehat{g})(\gamma_{x_j}) \cdot \gamma_{x_j}^{-1}]] = \beta[[\alpha^g(x_j)]] \in (\Gamma_{\beta,m})_c$$

by Remark 3.1 and recall that

$$\mathrm{ind}(\widehat{g}, x_j) = \langle \mathrm{diag}(\mathcal{C}_{n,m}(D^2)), \mathrm{graph}(\widehat{g}) \rangle|_{x_j}$$

is the definition of $\mathrm{ind}(\widehat{g}, x_j)$.

On the other hand, we take a lift $\widetilde{x}$ of x so that the lift $\widetilde{\gamma_x}$ of γ_x is starting at $\widetilde{c}$ and ending at $\widetilde{x}$. Then we obtain $\widetilde{g}(\widetilde{x_j}) = \alpha^g(x_j)\widetilde{x_j}$. Computing the fixed point index $\mathrm{ind}(\widehat{g}, x_j)$ of $\widehat{g}$ at x_j, we obtain

$$\mathrm{ind}(\widehat{g}, x_j) = (-1)^m \left(\alpha^g(x_j) \widetilde{S}_\mu \cdot (\widetilde{g})_*(\widetilde{F}_\mu) \right).$$

Therefore we obtain

$$(-1)^m [[c_{\mu\mu}^{(\beta)}]] = \sum_j \mathrm{ind}(\widehat{g}, x_j)\mathrm{cd}(x_j),$$

where $[[c]]$ is the element of the free abelian group $\mathbb{Z}(\Gamma_{\beta,m})_c$ projecting c, and

$$(-1)^m \,\mathrm{tr}_{\Gamma_{\beta,m}}\, \zeta_{n,m}(\beta) = \sum_{x \in \mathrm{Fix}\,\widehat{g}} \mathrm{ind}(\widehat{g}, x) \cdot \mathrm{cd}(x).$$

In the above equality, the number of nonzero terms in the right hand side is $\mathrm{eFix}(\widehat{g})$. By Remark 3.2, there exists a positive number B such that the inequality

$$\#\,\mathrm{eFix}(\widehat{g}) \leq \left\| \mathrm{tr}_{\Gamma_{\beta,m}}\, \zeta_{n,m}(\beta) \right\| \leq B \#\,\mathrm{eFix}(\widehat{g})$$

holds. □

We count the number of essential fixed points of $\widehat{g}^k$. Let $\{x_1, \dots, x_m\}$ be a fixed point of $\mathrm{Fix}\,(\widehat{g}^k)$. Then there exists an m-tuple $(n_1, \dots, n_m)$ of natural numbers with $\sum_{i=1}^m i n_i = m$ such that there exist n_i periodic orbits of g^k of period i in $\{x_1, \dots, x_m\}$ for all $1 \leq i \leq m$. Let A_m be the set of such

m-tuples and D_i^k be the number of essential periodic points of g^k of period i. Then there exist D_i^k/i periodic orbits of g^k of period i and we obtain

$$\#\,\mathrm{eFix}\,(\widehat{g}^k) = \sum_{(n_1,\ldots,n_m)\in A_m} \prod_{i=1}^{m} \begin{pmatrix} D_i^k/i \\ n_i \end{pmatrix}.$$

Remark 4.8. When we consider the period of periodic points of g^k of period i as periodic points of g, we notice that $D_i^k = D^1_{g(k,i)i}$, where $g(k,i)$ is the greatest common divisor of k and i. Moreover, if i a divisor of k then periodic orbits of g of period i is contained in some periodic orbits of g of period k and D_i^1/i is equal to or greater than D_k^1/k. Therefore we have

$$D_i^k/i = D^1_{g(k,i)i}/i \geq D^1_{ki}/l(ki),$$

where $l(k,i)$ is the least common multiplier, and we obtain

$$\#\,\mathrm{eFix}\,(\widehat{g}^k) \geq \sum_{(n_1,\ldots,n_m)\in A_m} \prod_{i=1}^{m} \begin{pmatrix} D^1_{ki}/l(k,i) \\ n_i \end{pmatrix}.$$

5. Proof of the main theorem

In this section we conclude the proof of main theorem. We denote by λ the dilatation of a pseudo-Anosov braid β.

Proposition 5.1. *For any pseudo-Anosov braid $\beta \in B_n$, the inequalities*

$$\begin{aligned} \underset{\mathrm{k}\to\infty}{\mathrm{Growth}} \left\| \mathrm{tr}_{\Gamma_{\beta^k,m}}\, \zeta_{n,m}(\beta^k) \right\| &\geq \lambda^m \\ \underset{\mathrm{m}\to\infty}{\mathrm{Growth}} \left\| \mathrm{tr}_{\Gamma_{\beta,m}}\, \zeta_{n,m}(\beta) \right\| &\geq \lambda \end{aligned}$$

hold.

Proof. We recall that $NI_{\Gamma_{\beta^k,1}}((g^k)^i)$ defined in Section 3.2 is a lower bound for the number of primary i-orbits of g^k. In other words, we have the inequality $D_i^k/i \geq NI_{\Gamma_{\beta^k,1}}(g^{ki})$. When we use this inequality and Remark 4.8, and consider the case $(n_1,\ldots,n_m) = (0,\ldots,0,1)$, we obtain the inequality

$$\begin{aligned} \left\| \mathrm{tr}_{\Gamma_{\beta,m}}\, \zeta_{n,m}(\beta^k) \right\| &\geq \#\,\mathrm{eFix}\,(\widehat{g}^k) \\ &= \sum_{(n_1,\ldots,n_m)\in A_m} \prod_{i=1}^{m} \begin{pmatrix} D_i^k/i \\ n_i \end{pmatrix} \\ &\geq \frac{D_m^k}{m} = \frac{D^1_{km}}{l(k,m)} \\ &\geq g(k,m) NI_{\Gamma_{\beta,1}}(g^{km}). \end{aligned}$$

Since g is homotopic to f_β, we obtain

$$\begin{aligned} \underset{\mathrm{k}\to\infty}{\mathrm{Growth}} \left\| \mathrm{tr}_{\Gamma_{\beta,m}}\, \zeta_{n,m}(\beta^k) \right\| &\geq \underset{\mathrm{k}\to\infty}{\mathrm{Growth}}\, g(k,m) NI_{\Gamma_{\beta,1}}(g^{km}) = \lambda^m, \\ \underset{\mathrm{m}\to\infty}{\mathrm{Growth}} \left\| \mathrm{tr}_{\Gamma_{\beta,m}}\, \zeta_{n,m}(\beta) \right\| &\geq \underset{\mathrm{m}\to\infty}{\mathrm{Growth}}\, NI_{\Gamma_{\beta,1}}(g^{m}) = \lambda. \end{aligned}$$

□

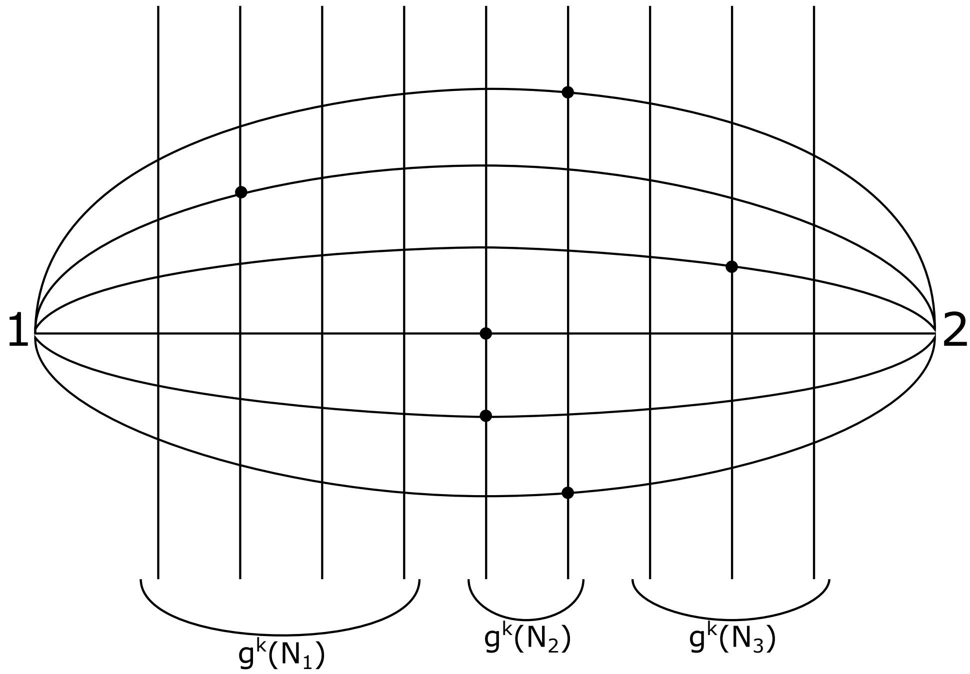

FIGURE 6. The case when $\nu_1 = 6, K^k_{11} = 4, K^k_{12} = 2, K^k_{13} = 3, \rho_{11} = 1, \rho_{12} = 4, \rho_{13} = 1$

Proposition 5.2. *For any pseudo-Anosov braid $\beta \in B_n$, the inequality*

$$\operatorname*{Growth}_{k\to\infty} \operatorname{tr} \left\| \zeta_{n,m}(\beta^k) \right\| \leq \lambda^m$$

holds.

Proof. By (4.1), the (μ,ν)-entry of $\|\zeta_{n,m}(\beta^k)\|$ is $\left\| \left\langle \left[\widetilde{S}_\nu \right], \widetilde{\beta}^k_* \left(\left[\widetilde{F}_\mu \right] \right) \right\rangle \right\|$. We notice that $\left\| \left\langle \left[\widetilde{S}_\nu \right], \widetilde{\beta}^k_* \left(\left[\widetilde{F}_\mu \right] \right) \right\rangle \right\|$ is equal to or less than the number of intersections of S_ν and $\widehat{g}^k(F_\mu)$. We define K^k_{ij} to be the number of intersections of α_i and $g^k(N_j)$ and set $A^k = \sum_{i,j} K^k_{ij}$. We set

$$M(n,\mu,\nu) = \left\{ \rho \in M(n-1,\mathbb{N}) \,\middle|\, \sum_{i=1}^{n-1} \rho_{ij} = \nu_j, \sum_{j=1}^{n-1} \rho_{ij} = \mu_i \right\}.$$

For every i,j and $\rho \in M(n,\mu,\nu)$, we can choose ρ_{ij} paths from ν_i forks and choose one intersection from K_{ij} intersections for each forks; see Figure 6. Therefore we obtain

$$\begin{aligned}\left\|\left\langle\left[\widetilde{S}_\nu\right],\widetilde{\beta}_*^k\left(\left[\widetilde{F}_\mu\right]\right)\right\rangle\right\| &\leq \sum_{\rho\in M(n,\mu,\nu)}\prod_{i=1}^{n-1}\nu_i!\prod_{j=1}^{n-1}\frac{1}{\rho_{ij}!}\left(K_{ij}^k\right)^{\rho_{ij}}\\ &\leq \left(\prod_{i=1}^{n-1}\nu_i!\right)\sum_{\rho\in M(n,\mu,\nu)}\prod_{i=1}^{n-1}\prod_{j=1}^{n-1}\frac{1}{\rho_{ij}!}\left(A^k\right)^{\rho_{ij}}\\ &= \left(\prod_{i=1}^{n-1}\nu_i!\right)\left(A^k\right)^m\sum_{\rho\in M(n,\mu,\nu)}\prod_{i=1}^{n-1}\prod_{j=1}^{n-1}\frac{1}{\rho_{ij}!}.\end{aligned}$$

Since

$$\left(\prod_{i=1}^{n-1}\nu_i\right)\sum_{\rho\in M(n,\mu,\nu)}\prod_{i=1}^{n-1}\prod_{j=1}^{n-1}\frac{1}{\rho_{ij}!}$$

is independent of k, we have

$$\operatorname*{Growth}_{k\to\infty}\left\|\left\langle\left[\widetilde{S}_\nu\right],\widetilde{\beta}_*^k\left(\left[\widetilde{F}_\mu\right]\right)\right\rangle\right\| \leq \left(\operatorname*{Growth}_{k\to\infty} A^k\right)^m.$$

It suffices to show $\operatorname*{Growth}_{k\to\infty} A^k \leq \lambda$. We set

$$U_i\cap g^{-1}(U_j)=\coprod_{l=1}^{K_{ij}^1} V_{ijl}$$

and take an open cover $\alpha=\{V_{ijk}\mid 1\leq i,j\leq n-1, 1\leq k\leq K_{ij}^1\}\cup A'$ of the compact set Y_n, where A' does not contain any intersections of $g^{-1}(\alpha_i)$ and N_j.

Lemma 5.3. *Each element of* $\bigvee_{p=0}^{k-1} g^{-p}(\alpha)$ *contains at most one intersection of* $g^{-k}(\alpha_j)$ *and* N_i.

Proof. Every nonempty element of $\bigvee_{p=0}^{k-1} g^{-p}(\alpha)$ can be written as

$$B=V_{i_0i_1l_1}\cap\cdots\cap g^{-k+1}(V_{i_{k-1}i_kl_k})$$

with $i_0=i$ and $i_k=j$. By the definition of ϕ and $\overline{\phi}$, $g^k|_B : B\to U_j$ is bijective. Therefore $(g^k|_B)^{-1}(\alpha_j)$ is one leaf of U_i and there exists only one intersection of $g^{-k}(\alpha_j)$ and N_i. □

It follows from Lemma 5.3 that

$$A^\ell=\sum_{i,j}K_{ij}^\ell\leq N\left(\bigvee_{i=0}^{\ell-1} g^{-i}(\alpha)\right)$$

and by (3.1), the growth rate of $N\left(\bigvee_{i=0}^{\ell-1} g^{-i}(\alpha)\right)$ is equal to or less than the dilatation of β. Therefore the proposition follows. □

Proposition 5.4. *For any pseudo-Anosov braid* $\beta\in B_n$*, the inequality*

$$\operatorname*{Growth}_{m\to\infty}\left\|\operatorname{tr}_{\Gamma_{\beta,m}}\zeta_{n,m}(\beta)\right\|\leq\lambda$$

holds.

Proof. By Lemma 4.7, $\left\|\mathrm{tr}_{\Gamma_{\beta^k,m}}\,\zeta_{n,m}(\beta^k)\right\|$ is equal to or greater than the number of essential fixed points of $\widehat{g}^k$. For $m = 1$, we notice that $\widehat{g}^k$ is g^k. Therefore $\left\|\mathrm{tr}_{\Gamma_{\beta^k,1}}\,\zeta_{n,1}(\beta^k)\right\|$ is equal to or greater than the number of essential periodic points of g whose period is a divisor of k. In particular, we obtain $\left\|\mathrm{tr}_{\Gamma_{\beta^k,1}}\,\zeta_{n,1}(\beta^k)\right\| \geq D^1_k/k$. Therefore we obtain

$$\begin{aligned}\left\|\mathrm{tr}_{\Gamma_{\beta,m}}\,\zeta_{n,m}(\beta)\right\| &\leq B\#\,\mathrm{eFix}\,\widehat{g} = B\sum_{(n_1,\ldots,n_m)\in A_m}\prod_{i=1}^{m}\binom{D^1_i/i}{n_i}\\ &\leq B\sum_{(n_1,\ldots,n_m)\in A_m}\prod_{i=1}^{m}\binom{\left\|\mathrm{tr}_{\Gamma_{\beta^i,1}}\,\zeta_{n,1}(\beta^i)\right\|}{n_i}.\end{aligned}$$

By Proposition 5.2, there exists a monotonically increasing sequence $\{a_i\}$ of real numbers such that

$$\left\|\mathrm{tr}_{\Gamma_{\beta^i,1}}\,\zeta_{n,1}(\beta^i)\right\| \leq (a_i\lambda)^i \text{ and } \limsup_{i\to\infty} a_i = 1$$

holds. Therefore we obtain

$$\begin{aligned}\left\|\mathrm{tr}_{\Gamma_{\beta,m}}\,\zeta_{n,m}(\beta)\right\| &\leq B\sum_{(n_1,\ldots,n_m)\in A_m}\prod_{i=1}^{m}(a_i\lambda)^{in_i}\\ &\leq B(a_m\lambda)^m S_m,\end{aligned}$$

where S_m is the number of elements of A_m.

Lemma 5.5. *The equality* $\lim_{m\to\infty} S_m^{1/m} = 1$ *holds.*

Proof. We suppose that $m - c_m > c_m d_m$, where

$$c_m = 4(\lfloor\sqrt[4]{m}\rfloor + 1)^2,\ d_m = 4(\lfloor\sqrt[4]{m}\rfloor + 2)$$

and $\lfloor x\rfloor$ is the floor function. Let C_m be the subset of A_m satisfying the following condition

$$\sum_{i=1}^{c_m} n_i = d_m \text{ and } n_{m-\sum_{i=1}^{c_m} in_i} = 1.$$

Then C_m is in one-to-one correspondence with the d_m-combinations with repetition from c_m elements. Therefore we obtain the inequality

$$\begin{aligned}S_m &\geq \binom{c_m + d_m - 1}{d_m} = \binom{4(\lfloor\sqrt[4]{m}\rfloor + 2)(\lfloor\sqrt[4]{m}\rfloor + 1)}{4(\lfloor\sqrt[4]{m}\rfloor + 2)}\\ &= \frac{4(\lfloor\sqrt[4]{m}\rfloor + 2)(\lfloor\sqrt[4]{m}\rfloor + 1)}{4(\lfloor\sqrt[4]{m}\rfloor + 2)} \times\cdots\times \frac{4(\lfloor\sqrt[4]{m}\rfloor + 1)^2}{1}\\ &\geq (\lfloor\sqrt[4]{m}\rfloor + 1)^{4(\lfloor\sqrt[4]{m}\rfloor+2)} \geq \sqrt[4]{m}^{4(\lfloor\sqrt[4]{m}\rfloor+2)} = m^{\lfloor\sqrt[4]{m}\rfloor+2}.\end{aligned}$$

We set

$$A_{m,k} = \{(n_1,\ldots,n_m)\in A_m \mid \max\{i \mid n_i \neq 0\} = k\}.$$

and let $S_{m,k}$ be the number of the elements of $A_{m,k}$. Then clearly

$$S_m = \sum_{k=1}^{m} S_{m,k}$$

holds and the recursion formula

$$S_{m+1,k+1} = S_{m,k} + S_{m-k,k+1} \tag{5.1}$$

follows from the equality $A_{m,k} = \coprod_{j=1}^{k} A_{m-k,j}$. Moreover, $S_{m,k}$ is less than the number of how to put m balls in distinct k boxes, which is m^k.

We assume that $\max_k S_{m,k} = S_{m,k_0}$. Since $S_m \leq m S_{m,k_0}$ holds, we obtain

$$m^{k_0} \geq S_{m,k_0} \geq \frac{1}{m} S_m \geq m^{\sqrt[4]{m}}$$

and $k_0 \geq \sqrt[4]{m}$. From (5.1), we obtain

$$S_{m,k_0} \leq S_{2(m-k_0),m-k_0} = S_{m-k_0}.$$

Since S_m is monotonically increasing for m, we obtain

$$S_m \leq m S_{m,k_0} \leq m S_{m-k_0} \leq m S_{m-\sqrt[4]{m}}.$$

There exists a natural number N such that the assumption holds for all $m \geq N$. We set $f(m) = m - \sqrt[4]{m}$ and $n_N(m) = \min\{i \mid f^i(m) \leq N\}$. Then we obtain $S_m \leq m^{n_N(m)} S_N$. We notice that if x is larger than $(\sqrt[4]{m}-1)^4$, then $x - f(x) = \sqrt[4]{x}$ is larger than $\sqrt[4]{m} - 1$. Therefore we obtain

$$f^{\lfloor \sqrt[4]{m}^2 - 2\sqrt[4]{m} + 2 \rfloor + 1}(m) \leq m - (\sqrt[4]{m} + (\sqrt[4]{m}-1)(\sqrt[4]{m}^2 - 2\sqrt[4]{m} + 2)) = (\sqrt[4]{m}-1)^4.$$

Therefore we obtain

$$n_N(m) \leq \sum_{k=1}^{\sqrt[4]{m}} \lfloor 4k^2 - 2k + 2 \rfloor + 1 \leq \sqrt[4]{m}(4\sqrt[4]{m}^2 - 2\sqrt[4]{m} + 3) \leq 4m^{3/4}$$

and

$$1 < \sqrt[m]{S_m} \leq (m^{n_N(m)} S_{f^{n_N(m)}(m)})^{1/m} \leq \sqrt[m]{S_N} m^{4m^{-\frac{1}{4}}}.$$

Since the limit $\lim_{m\to\infty} \sqrt[m]{S_N} m^{4/\sqrt[4]{m}}$ equals 1, squeeze theorem leads to the conclusion $\lim_{m\to\infty} S_m^{1/m} = 1$. □

By this lemma, we obtain

$$\limsup_{m\to\infty} \left\| \mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta) \right\|^{1/m} \leq \limsup_{m\to\infty} (B S_m)^{1/m} a_m \lambda = \lambda.$$

□

Proof of Theorem 1.1. Since we have the inequality $\mathrm{tr}(\|A\|) \geq \|\mathrm{tr}\, A\|$ for any matrix A with coefficients in Laurent polynomial ring, we obtain

$$\lambda^m \leq \operatorname*{Growth}_{k\to\infty} \left\| \mathrm{tr}_{\Gamma_{\beta^k,m}} \zeta_{n,m}(\beta^k) \right\| \leq \operatorname*{Growth}_{k\to\infty} \mathrm{tr} \left\| \zeta_{n,m}(\beta^k) \right\| \leq \lambda^m$$

by Proposition 5.1 and Proposition 5.2. Therefore we have

$$\operatorname*{Growth}_{k\to\infty} \left\| \mathrm{tr}_{\Gamma_{\beta^k,m}} \zeta_{n,m}(\beta^k) \right\| = \operatorname*{Growth}_{k\to\infty} \mathrm{tr} \left\| \zeta_{n,m}(\beta^k) \right\| = \lambda^m.$$

We have

$$\lambda \leq \operatorname*{Growth}_{m\to\infty} \left\| \mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta) \right\| \leq \lambda$$

by Proposition 5.1 and Proposition 5.4 and we have $\operatorname*{Growth}_{m\to\infty} \left\| \mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta) \right\| = \lambda$. □

6. Homological representation of braid groups

6.1. **Homological representation of braid groups.** In [21] Lawrence construct a monodromy representation of braid groups. We review the representation. We take a homomorphism

$$\rho_B : \mathbf{B}_{n,1}(D^2) \cong \langle \sigma_1, \dots, \sigma_{n-1}, \sigma_n^2 \rangle \to \mathbb{Z}$$

defined by $\rho_B(\sigma_i) = 0$ for all $1 \le i < n$ and $\rho_B(\sigma_n^2) = 1$. Let $p_B : \widetilde{D_n}^B \to D_n$ be the covering corresponding to $\operatorname{Ker} \rho_B$ and fix $\widetilde{d}^B \in p_B^{-1}(d_1)$. For an n-braid β, we take a representative f. Let

$$\widetilde{f}^B : (\widetilde{D_n}^B, \widetilde{d}^B) \to (\widetilde{D_n}^B, \widetilde{d}^B)$$

be the lift of f. Then $\widetilde{f}^B$ acts on $H_1(\widetilde{D_n}^B, \partial \widetilde{D_n}^B)$ as $\mathbb{Z}[\mathbb{Z}]$-homomorphism. The linear representation B defined by $B(\beta) = \widetilde{f}^B_*$ is called the *reduced Burau representation.* Let t denote the generator of covering transformation of $\widetilde{D_n}^B$ corresponding to $1 \in \mathbb{Z}$. Then the ring $\mathbb{Z}[\mathbb{Z}]$ is isomorphic to the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$ and $B(\beta)$ can be regarded as a matrix with coefficients in the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$. Similarly for $m \ge 2$, we take a homomorphism

$$\rho_{LKB} : \mathbf{B}_{n,m}(D^2) \cong \langle \sigma_1, \dots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1}, \dots, \sigma_{n+m-1} \rangle \to \mathbb{Z} \oplus \mathbb{Z}$$

defined by $\rho_{LKB}(\sigma_i) = 0 \oplus 0$ for all $1 \le i < n$, $\rho_{LKB}(\sigma_n^2) = 1 \oplus 0$ and $\rho_{LKB}(\sigma_{n+j}) = 0 \oplus 1$ for all $1 \le j < m$. Let $p_{LKB} : \widetilde{\mathcal{C}}^{LKB}_{n,m}(D^2) \to \mathcal{C}_{n,m}(D^2)$ be the covering corresponding to $\operatorname{Ker} \rho_{LKB}$ and fix $\widetilde{c}^{LKB} \in p_{LKB}^{-1}(c)$. For $\beta \in B_n$, we take a representative f. Let

$$\widetilde{f}^{LKB} : (\widetilde{\mathcal{C}}^{LKB}_{n,m}(D^2), \widetilde{c}^{LKB}) \to (\widetilde{\mathcal{C}}^{LKB}_{n,m}(D^2), \widetilde{c}^{LKB})$$

be the lift of $\widehat{f}$. Then $\widetilde{f}^{LKB}$ acts on $H_2(\widetilde{B}^{LKB}_{n,m}(D^2))$ as an $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$-homomorphism. The linear representation LKB_m defined by $LKB_m(\beta) = \widetilde{f}^{LKB}_*$ is called the *Lawrence-Krammer-Bigelow representations.* Let q and t denote the generator of covering transformation of $\widetilde{\mathcal{C}}^{LKB}_{n,m}(D^2)$ corresponding to $1 \oplus 0 \in \mathbb{Z} \oplus \mathbb{Z}$ and $0 \oplus 1 \in \mathbb{Z} \oplus \mathbb{Z}$ respectively. Then the ring $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$ is isomorphic to the Laurent polynomial ring $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ and $LKB_m(\beta)$ can be regarded as a matrix with coefficients in the 2-variable Laurent polynomial ring $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$.

The homological representation of braid groups has been also intensively studied. The Lawrence-Krammer-Bigelow representations of the braid groups were studied by Lawrence [21] in relation with Hecke algebra representations of the braid groups. In [4], [19] and [20], Bigelow and Krammer showed the faithfulness of the Lawrence-Krammer-Bigelow representation for $m = 2$ independently.

In [9], Fried showed how to estimate the entropy of a pseudo-Anosov braid by using the Burau matrix $B(t)$ of a pseudo-Anosov braid. In [18], Kolev proved the same estimation directly with different methods. The following theorem is the estimate and this estimate is called the *Burau estimate.*

Theorem 6.1. (Fried [9], Kolev [18]) *Let f be a homeomorphism of D^2 fixing P_n setwise and β be an n-braid represented by f. Then the topological entropy of f is equal to or greater than the logarithm of the spectral radius of* *the Burau matrix $B(t)$ of β after substituting a complex number of modulus 1 in place of t.*

If the inequality is an equality for $\eta = \eta_0$, then the Burau estimate is said to be *sharp at η_0*. In [2], Band and Boyland determined a necessary and sufficient condition when the Burau estimate is sharp at the root of unity.

Theorem 6.2. (Band and Boyland [2]) *For a pseudo-Anosov braid β, the Burau estimate is sharp at the root of unity η_0 only if $\eta_0 = -1$. Furthermore, the Burau estimate is sharp at -1 if and only if the invariant foliations for a pseudo-Anosov map in the class represented by β have odd order singularities at all punctures and all interior singularities are even order.*

In [17], Koberda shows the similar estimate by using Lawrence-Krammer-Bigelow representation.

Theorem 6.3. (Koberda [17]) *For a pseudo-Anosov braid β, the m-th power of the dilatation of β is equal to or greater than the spectral radius of the Lawrence-Krammer-Bigelow matrix $LKB_m(q,t)$ of β after substituting complex numbers of modulus 1 in place of q and t.*

6.2. **Homological estimation and Theorem 1.1.** In this section, we recover the estimation in [9], [18] and [17] using Theorem 1.1. If we have a homomorphism ρ from $\mathbf{E}_{n,m}(D^2)$ to some group G, we have an another representation $\rho_*(\zeta_{n,m})$ on the free $\mathbb{Z}[G]$-module defined by $\rho_*(\zeta_{n,m}) = (\rho_*(c_{\mu\nu}^{(\beta)}))$. Moreover, if G is a finitely generated free abelian group, $\mathbb{Z}[G]$ can be embedded in $\mathbb{C}$ and in this way, $\rho_*(\zeta_{n,m})$ gives rise to a linear representation $\rho'_*(\zeta_{n,m})$ over $\mathbb{C}$.

When $m = 1$, Let $\rho'_B : \mathbf{E}_{n,1}(D^2) \to \mathbb{Z}$ be a the homomorphism defined by $\rho'_B(\sigma_i) = 0$ for all $1 \le i < n$ and $\rho'_B(\sigma_i^2) = 1$. When $m \ge 2$, let $\rho'_{LKB} : \mathbf{E}_{n,m}(D^2) \to \mathbb{Z}\oplus\mathbb{Z}$ be a homomorphism defined by $\rho'_{LKB}(\sigma_i) = 0 \oplus 0$ for all $1 \le i < n$, $\rho'_{LKB}(\sigma_n^2) = 1 \oplus 0$ and $\rho'_{LKB}(\sigma_{n+j}) = 0 \oplus 1$. We consider the homomorphism from $\mathrm{Aut}_{\mathbb{Z}[\mathbf{E}_{n,m}(D^2)]}\left(\mathbb{Z}[\mathbf{E}_{n,m}(D^2)] \otimes_{\mathbb{Z}[\mathbf{B}_{n,m}(D^2)]} \mathcal{H}_F\right)$ induced by ρ'_{LKB}. Since $\rho'_{LKB}(\sigma_i)$ is $0 \oplus 0$ for all $1 \le i < n$, the action as the right multiplication becomes trivial and $(\rho'_{LKB})_*(\zeta_{n,m})$ is equivalent to the Lawrence-Krammer-Bigelow representations for all $m \ge 2$. Similarly, $(\rho'_B)_*(\zeta_{n,1})$ is equivalent to the reduced Burau representation.

For any matrix A with coefficients in n-variable Laurent polynomial ring and complex numbers $x_1, \ldots, x_n$, we denote by $A(x_1, \ldots, x_n)$ the matrix with coefficients in $\mathbb{C}$ substituting x_i for i-th variable. For any matrix A with coefficients in $\mathbb{C}$, we denote by $\mathrm{sr}\, A$ the spectral radius of A. We state the main result of this section.

Proposition 6.4. *For any matrix A with coefficients in the Laurent polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, we have*

$$\underset{k\to\infty}{\mathrm{Growth}} \left\| \mathrm{tr}\, A^k \right\| = \sup_{x_i \in S^1} \mathrm{sr}\, A(x_1, \ldots, x_n).$$

Let $I = (i_1, \ldots, i_n)$ be a multi index and $x^I = \prod_{k=0}^n x_k^{i_k}$.

Lemma 6.5. *We suppose* $f(x_1,\dots,x_n)=\sum_{i_1=0}^{M}\cdots\sum_{i_n=0}^{M}a_Ix^I$ *is an* n*-variable polynomial of degree* M*. Then we have the inequality*

$$\sum_I |a_I| \le (M+1)^n \sup_{x_k\in S^1}|f(x_1,\dots,x_n)|$$

Proof. First of all, we prove the case $n=1$. Then $f(x)$ is a polynomial $\sum_{i=0}^{M}a_ix^i$ of degree M. We consider the Vandermonde matrix

$$V=V_{M+1}(x_0,\dots,x_M)=\begin{pmatrix}1 & x_0 & \cdots & x_0^M\\ 1 & x_1 & \cdots & x_1^M\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_M & \cdots & x_M^M\end{pmatrix}.$$

Then we have $V\mathbf{a}=\mathbf{A}$, where

$$\mathbf{a}=\begin{pmatrix}a_0\\ \vdots\\ a_M\end{pmatrix}\text{ and }\mathbf{A}=\begin{pmatrix}f(x_0)\\ \vdots\\ f(x_M)\end{pmatrix}.$$

We denote by σ_m the m-th elementary symmetric function in the $(M+1)$ variables $x_0,\dots,x_M$. In other words, we have

$$\sigma_m=\sigma_m(x_0,\dots,x_M)=\sum_{\nu\in\mathcal{S}_m}x_{\nu(1)}\dots x_{\nu(m)}$$

for all $1\le m\le M+1$ and $\sigma_0=1$. We use the notation σ_m^i to denote the m-th elementary symmetric function in the M variables x_k with x_i missing. In other words, we have

$$\sigma_m^i=\sigma_m(x_0,\dots,x_{i-1},x_{i+1},\dots,x_M).$$

We set $V^{-1}=(v_{ij})_{0\le i,j\le M}$. It is well known (see [10]) that we have

$$v_{ij}=(-1)^i\frac{\sigma_{M-i}^j}{\prod_{k\ne j}(x_k-x_j)}$$

We put $\theta=\pi/M+1$ and $x_k=\exp(2\sqrt{-1}k\theta)$. Since x_i's are all the roots of $z^{M+1}-1=0$, we obtain $\sigma_m(x_0,\dots,x_M)=0$ for all $1\le m\le M$. Since the recursion formula $\sigma_{m+1}^i=\sigma_{m+1}-x_i\sigma_m^i$ holds, we obtain $\sigma_{m+1}^i=-x_i\sigma_m^i$ and $\sigma_m^i=(-x_i)^m$. We notice that $|x_k-x_j|=2\sin|k-j|\theta$. Then we obtain

$$|v_{ij}|=\left|(-1)^{i-1}\frac{\sigma_{M-i}^j}{\prod_{k\ne j}(x_k-x_j)}\right|=\frac{1}{\prod_{k=1}^{M}(2\sin k\theta)}.$$

Since we have $\mathbf{a} = V^{-1}\mathbf{A}$, we have the inequality

$$\begin{aligned}|a_i| &= \sum_{j=0}^{M} |v_{ij} f(x_j)| \\ &\leq \sum_{j=0}^{M} |v_{ij}||f(x_j)| \\ &\leq \frac{1}{\prod_{k=1}^{M}(2\sin k\theta)} \sum_{j=0}^{M} |f(x_j)| \\ &\leq \frac{M+1}{\prod_{k=1}^{M}(2\sin k\theta)} \max_k |f(x_k)| \\ &\leq \frac{M+1}{\prod_{k=1}^{M}(2\sin k\theta)} \sup_{x\in S^1} |f(x)|.\end{aligned}$$

Lemma 6.6. *The equality* $\prod_{k=1}^{M}(2\sin k\theta) = M+1$ *holds.*

Proof. We set

$$\cos(2n-1)\theta = \cos\theta f_n(\cos\theta),\ \sin 2n\theta = \sin 2\theta g_n(\cos\theta)$$

for $n \geq 1$. Since

$$\begin{cases} \cos(2n+3)\theta + \cos(2n-1)\theta = 2\cos 2\theta \cos(2n+1)\theta \\ \sin 2(n+2)\theta + \sin 2n\theta = 2\cos 2\theta \sin 2(n+1)\theta, \end{cases}$$

hold, we obtain recursion formulae $f_{n+2}(x) = 2(2x^2-1)f_{n+1}(x) - f_n(x)$ and $g_{n+2}(x) = 2(2x^2-1)g_{n+1}(x) - g_n(x)$. Moreover, because of the initial conditions $f_1(x) = 1$, $f_2(x) = 4x^2 - 3$, $g_1(x) = 1$ and $g_2(x) = 4x^2 - 2$, $f_n(x)$ and $g_n(x)$ are polynomials of degree $2(n-1)$. Solving the recursion formulae of leading coefficient and constant term, we find that the leading coefficients of $f_n(x)$ and $g_n(x)$ is 4^n, the constant term of $f_n(x)$ is $(2n-1)(-1)^{n-1}$ and the constant term of $g_n(x)$ is $n(-1)^{n-1}$.

There exist distinct $2(n-1)$ solutions

$$\pm\sin(k\pi/(2n-1)) = \cos(\pi/2 \pm k\pi/(2n-1))\ \ k = 1, \ldots, n-1$$

of $f_n(x) = 0$ and distinct $2(n-1)$ solutions

$$\pm\sin(k\pi/2n) = \cos(\pi/2 \pm k\pi/2n)\ \ k = 1, \ldots, n-1$$

of $g_n(x) = 0$. Vieta's formula implies $\prod_{k=1}^{M}(2\sin k\theta) = M+1$. □

Lemma 6.6 implies $\sum_{i=0}^{M} |a_i| \leq (M+1)\sup_{x\in S^1} |f(x)|$.

Now we consider the general case. For any n-variable polynomial

$$f(x_1, \ldots, x_n) = \sum_{i_1=0}^{M} \cdots \sum_{i_n=0}^{M} a_I x^I$$

of degree M, we set

$$f(x_1, \ldots, x_n) = \sum_{i_n=0}^{M} f_{i_n}(x_1, \ldots, x_{n-1}) x_n^{i_n}.$$

Then we obtain

$$\sup_{x_1,\dots,x_{n-1}\in S^1} \sum_i |f_i(x_1,\dots,x_{n-1})| \le (M+1) \sup_{x_1,\dots,x_n\in S^1} |f(x_1,\dots,x_n)|.$$

Repeating this n times shows the inequality

$$\sum_I |a_I| \le (M+1)^n \sup_{x_1,\dots,x_n\in S^1} |f(x_1,\dots,x_n)|.$$

□

Proof of Proposition 6.4. We notice that

$$\sup_{x_i\in S^1}\left|\sum_{i_1=m}^{M}\cdots\sum_{i_n=m}^{M} a_I x^I\right| = \sup_{x_i\in S^1}\left|\sum_{i_1=0}^{M-m}\cdots\sum_{i_n=0}^{M-m} a_I x^I\right|$$

holds. We denote by A a matrix with coefficients in n-variable Laurent polynomial ring. Let M and m be the maximum and minimum degree of all entries of A. Then the maximum degree of all entries of A^k is equal to or less than kM and the minimum degree of all entries of A^k is equal to or greater than km. Using Lemma 6.5, we obtain

$$\sup_{x_i\in S^1} |\operatorname{tr} A^k(x_1,\dots,x_n)| \le \left\|\operatorname{tr} A^k\right\| \le (k(M-m)+1)^n \sup_{x_i\in S^1} |\operatorname{tr} A^k(x_1,\dots,x_n)|.$$

Therefore we obtain

$$\operatorname*{Growth}_{k\to\infty} \left\|\operatorname{tr} A^k\right\| = \operatorname*{Growth}_{k\to\infty} \sup_{x_i\in S^1} |\operatorname{tr} A^k(x_1,\dots,x_n)|.$$

Cayley-Hamilton theorem shows

$$\operatorname{tr} A^k(x_1,\dots,x_n) = \lambda_1^k + \cdots + \lambda_N^k,$$

where $\lambda_1,\dots,\lambda_N$ are the eigenvalues of $A(x_1,\dots,x_n)$. Therefore we obtain

$$\operatorname*{Growth}_{k\to\infty} \sup_{x_i\in S^1} |\operatorname{tr} A^k(x_1,\dots,x_n)| = \sup_{x_i\in S^1} \operatorname{sr} A(x_1,\dots,x_n).$$

□

Using Proposition 6.4, we recover the estimation in [9], [18] and [17].

Corollary 6.7. *For a pseudo-Anosov braid β, the dilatation of β is equal to or greater than the spectral radius of the Burau matrix $B(t)$ of β after substituting a complex number of modulus 1 in place of t and the m-th power of the dilatation of β is equal to or greater than the spectral radius of the Lawrence-Krammer-Bigelow matrix $LKB_m(q,t)$ of β after substituting complex numbers of modulus 1 in place of q and t.*

Proof. Since $\left\|\operatorname{tr}(\rho)_*(\zeta_{n,m})(\beta^k)\right\|$ is equal to or less than $\left\|\operatorname{tr}_{\Gamma_{\beta^k,m}} \zeta_{n,m}(\beta^k)\right\|$, we obtain

$$\operatorname*{Growth}_{k\to\infty} \left\|\operatorname{tr}(\rho'_B)_*(\zeta_{n,1})(\beta^k)\right\| \le \lambda$$

and

$$\operatorname*{Growth}_{k\to\infty} \left\|\operatorname{tr}(\rho'_{LKB})_*(\zeta_{n,m})(\beta^k)\right\| \le \lambda^m.$$

From Proposition 6.4, we obtain

$$\operatorname*{Growth}_{k\to\infty} \left\|\operatorname{tr}(\rho'_B)_*(\zeta_{n,1})(\beta^k)\right\| = \sup_{t\in S^1} B(t)$$

and

$$\operatorname*{Growth}_{\mathrm{k}\to\infty} \left\| \operatorname{tr}(\rho'_{LKB})_*(\zeta_{n,m})(\beta^k) \right\| = \sup_{q,t\in S^1} LKB_m(q,t).$$

Therefore we obtain

$$\sup_{t\in S^1} B(t) \leq \lambda \quad \text{and} \quad \sup_{q,t\in S^1} LKB_m(q,t) \leq \lambda^m.$$

□

On the other hand, it is not known whether $\operatorname*{Growth}_{\mathrm{m}\to\infty} \|\operatorname{tr}(\rho_{LKB})_*(\zeta_{n,m})(\beta)\|$ is λ or not. If $\operatorname*{Growth}_{\mathrm{m}\to\infty} \|\operatorname{tr}(\rho_{LKB})_*(\zeta_{n,m})(\beta)\|$ is not necessarily λ, there exists some sufficient condition for $\operatorname*{Growth}_{\mathrm{m}\to\infty} \|\operatorname{tr}(\rho_{LKB})_*(\zeta_{n,m})(\beta)\| = \lambda$. Clearly the condition in Theorem 6.2 is a sufficient condition for the above equality. We want to reveal whether this sufficient condition is the best condition or not.

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