博士学位論文

New extensions in topological field theory with instanton effects

（インスタントン効果を含む位相的場の理論における新しい拡張）

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Abstract

In this thesis, we propose two extensions of a topological field theory. One is a construction of new observables. The other is a perturbation theory around a special point of the theory.

First, we construct the new observables in the supersymmetric quantum mechanics on a Riemaniann manifold. The observables of this theory correspond to the differential forms on the instanton moduli space. In our case, this space is the space of the gradient trajectories of the Morse function on the manifold, which is a subspace of the space of paths with both endpoints fixed. We consruct such differential forms by the mothod of iterated integrals. We find that the resulting observables are sensitive to the information of the non-commutativity of the fundamental group of the moduli space.

Second, we develop a proper method of a perturbation theory around a special limiting point of the topological field theory. It is known that in this special point, one can compute the correlation functions beyond the topological sector of the theory. This limiting point is characterized by the infinite value of a parameter λ of the theory. However, at this point $\lambda = \infty$, the theory becomes quite different from the original theory with a finte value of λ. To get desired information, we need to know the value of the correlation functions away from the point $\lambda = \infty$. We find that it can be achieved by a kind of perturbation theory around the point $\lambda = \infty$. This perturbation theory has properties different from the usual one for a quantum mechanics. We carry out the perturbation theory by the method of the resolvent. We find that the computation on the inifinite dimensional Hilbert space can be reduced to a finite dimensional matrix computation.

After reviewing some basic properties of topological field theories, we discuss the two extensions above.

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Chapter 1

Introduction and overview

In general, it is a difficult problem to analyze the dynamics of a quantum field theory (QFT), since a general QFT has infinite degrees of freedom and complicated interactions. In the early days of the development of the QFT, to compute physical correlation functions, people used perturbation theory that is typically represented by techniques of the Feynman diagrams or some other approximation methods. Since then, a number of approaches have been developed which enable one to compute some exact quantities in QFT. The theories of supersymmetric QFT's are representative of such approaches. In a supersymmetric theory, cancellations of the degrees of freedom between bosons and fermions are crucial. If the supersymmetry is not spontaneously broken, the theory has the lowest energy states that are called supersymmetric ground states. The supersymmetric ground states are also called BPS states in relation to the representation theory of supersymmetry algebras. These states are stable so that they are independent of the coupling constants or the potential of the theory.

In a series of papers [23], [24], [25], Witten considered a supersymmetric quantum mechanics that describes a particle moving on a manifold X. There he found that the BPS states of the supersymmetric quantum mechanics have close relationship with the topology of the manifold X. In [23], [24], He defined an invariant of the theory called the supersymmetric index or Witten index and showed that it coincides with the Euler characteristic of X. Besides, in [25], he showed that if one regards the potential of the particle as a Morse function on X and takes the effect of the tunneling into consideration, one can reproduce the Morse theory of the cohomology of X. Along these lines, supersymmetric quantum mechanics were used to give a physical explanation of the Atiyah-Singer index theorem [1]. Such an approach of using QFT's to study geometries and topologies of spaces made a tremendous impact on mathematicians. In turn, QFT's became more rigorous by treating them by mathematical or geometrical methods. In this way, the physics and the mathematics have been developed by influencing each other.

Under these circumstances, Witten pushed his ideas further and found a supersymmetric QFT where the physical states of the theory consist only of BPS states [27]. Consequently the physical correlations functions of this theory are independent of the metric and the coupling constants. QFT's with such properties are called topological field theories (TFT's). In [27], Witten constructed a theory that is obtained from four-dimensional Yang-Mills theory by applying an operation called "topological twist". The resulting theory is called the topological Yang-Mills theory. This theory has been used to study the topology of the moduli space of instantons. More precisely, it was shown that the correlation functions of this theory gives the topological invariants of the moduli space of instantons. The topology of the instanton moduli space had been independently studied by Donaldson by more mathematical means and the results of these two approaches coincided. Thus, the topological Yang-Mills theory is often called the Donaldson-Witten theory.

After these developments, applying the method of the topological twist to the two-dimensional supersymmetric sigma model [22], a two-dimensional TFT that describes the topology of the space of maps from a Riemann surface to a target space were obtained in [26]. This TFT is called the topological sigma model [2]. This theory subsequently developed into the so called topological string theory. It inspired the study of the mirror symmetry and many other interesting and important mathematical concepts.

In such developments of TFT's, there appeared a geometrical picture called the Mathai-Quillen formalism [21]. This formalism was originally found as a method for computing the Euler characteristic of manifolds. Later it was shown that one can construct TFT's by applying this formalism to the infinite dimensional space of fields. Mathai-Quillen formalism states that a TFT has a first order differential equation which is called the instanton equation or the BPS equation and the partition function of the theory coincides with the Euler characteristic of the space of the solution of the instanton equation, *i.e.*, the instanton moduli space. One can also show that the BPS states correspond to the closed forms on the moduli space and hence the correlation functions of the BPS state are represented by the integration of the closed forms over the moduli space. Therefore the BPS correlation functions are naturally topological quantities. They give the intersection numbers of the homology cycles in the moduli space.

As we mentioned above, given a TFT, there exists a corresponding physical supersymmetric QFT. The TFT can only compute a sector of the physical theory where only the BPS states appear. This sector is called the topological sector or BPS sector of the physical theory.

Frenkel, Losev, Nekrasov (FLN) [10] proposed a method of computing the correlation functions of the physical theory beyond the topological sector, still making use of the techniques of the topological theory. First they consider the case of the supersymmetric quantum mechanics and then applied the similar method to the 2D supersymmetric sigma model and the 4D supersymmetric Yang-Mills theory [12]. The reference [11] provides a short review of these works by the original authors.

It is clear, however, that such correlation functions beyond the topological sector depend on the metrics and the coupling constants of the theory. Therefore, their computational method is actually applicable only to a certain limit of the theory. In this limit, a real parameter λ of the theory goes to infinity, and it is shown that the limiting shape of the action becomes a delta functional supported on the instanton moduli space. Thus the contributions to the path integral are localized to the instanton moduli space and they can be given by the finite dimensional integrals which one can perform. .

However, in the limit $\lambda = \infty$, the theory becomes quite different from the original physical theory and it is difficult to obtain the information of the original theory with a finite value of λ. To get the desired information, we need to know the value of the correlation functions away from the point $\lambda = \infty$. One naturally expects that it can be achieved by a kind of perturbation theory around $\lambda = \infty$. But, as FLN showed, such a perturbation theory has properties quite different from the usual one for a quantum mechanics. Because of such nontrivial properties, the explicit computation using the perturbation theory has not been done.

In this thesis, we propose two new extensions based on the consideration of

[10]. One is a construction of new observables in the supersymmetric quantum mechanics. As we described above, the BPS observables of this theory are the closed forms on the instanton moduli space. In our case, this space is the space of the gradient trajectories of the Morse function. A nontrivial gradient trajectory must start from a critical point of the Morse function and end at another critical point. Therefore we will consider differential forms on the space of such trajectories, *i.e.*, the space of the paths with both endpoints fixed. There is a method for constructing such differential forms. It is known as the method of iterated integrals. This method has been developed by Chen [6], [7] and was used to study the de Rham theory of the path space. We apply this method to construct observables of the supersymmetric quantum mechanics. We will find that the resulting observables can detect the information of the non-commutativity of the fundamental group of the moduli space. Thus, using these observables, we can get the new information about the geometry of the moduli space.

The second extension is the development of the proper method to carry out the perturbation theory around the point $\lambda = \infty$. The basic reason why the usual perturbation method is not applicable is that the Hamiltonian of interest is not hermitean nor diagonalizable. Specifically, the Hamiltonian in the limit $\lambda = \infty$ has the Jordan block of the finite length. We will carry out the perturbation theory by the method of the resolvent [16]. We will see that due to the special form of the perturbed Hamiltonian, we can reduce the computation of the operator on the infinite dimensional Hilbert space to a finite dimensional matrix computation. This is the second of our new results. By this method we will compute the eigenvalues of the perturbed Hamiltonian to the second order.

Organization of the thesis This thesis is organized as follows. In chapter 2, we recall several basic facts about the supersymmetric quantum mechanics. We treat the supersymmetric mechanics describing a particle moving on a Riemannian manifold with a potential derived from a Morse function on the manifold. We show that the wave functions of the particle are localized around the critical points of the Morse function and they are approximated by the supersymmetric harmonic oscillator near the critical points.

In chapter 3, first we review general properties of TFT's of Witten type. Then we explain the Mathai-Quillen formalism of the TFT's. We see there the path integral of the TFT has a nice geometrical interpretation using the notion of the Poincaré duality. To construct the topological theory of the quantum mechanics, we need to apply the Mathai-Quillen formalism to the space of paths.

Therefore in chapter 4, we review the properties of the geometry of path space as a preparation for applying the Mathai-Quillen formalism. We also review the notion of iterated integrals as differential forms on the path space. These are identified with our new observables of the topological quantum mechanics.

In chapter 5, we apply the Mathai-Quillen formalism to the path space and obtain a theory of topological quantum mechanics. Then we compute, as an example, a correlation function that contains a new observable.

In chapter 6, we carry out the perturbation theory around the point $\lambda = \infty$. First we review the theory of the resolvent. Then using the perturbative expansion of the resolvent, we obtain an eigenprojections onto an eigenspace of the perturbed Hamiltonian. We see that the resulting eigenprojections can be represented by finite size matrices so that we can compute the perturbed eigenvalues. We carry out the perturbation theory to the second order.

In chapter 7, we conclude the thesis by summarizing and indicating the future directions.

Chapter 2

Supersymmetric quantum mechanics

We begin by reviewing some properties of the supersymmetric quantum mechanics which will be used in the later chapters.

2.1 Supersymmetry algebra

The supersymmetry is a symmetry between bosons and fermions. In the supersymmetric quantum mechanics, they are generated by two real, or one complex fermionic charges. The algebra generated by these charges are called supersymmetry algebra. In what follows, we recall the explicit form of the supersymmetry algebra.

Supersymmetry algebra is a $\mathbb{Z}_2$-graded algebra. The $\mathbb{Z}_2$-grading is defined by the operator $(-1)^F$, where F is called the fermion number and it takes the values 0 or 1 (mod2) on the operators. There are two real odd operators Q_1 and Q_2 called supercharges. There is a Hamiltonian operator H, which is an even operator.

$$(-1)^F Q_1 = -Q_1(-1)^F, \quad (-1)^F Q_2 = -Q_2(-1)^F. \tag{2.1}$$

$$(-1)^F H = H(-1)^F. \tag{2.2}$$

$$H = Q_1^2 = Q_2^2. \tag{2.3}$$

$$Q_1 Q_2 + Q_2 Q_1 = 0. \tag{2.4}$$

Define complex supercharges Q and Q^* by

$$Q := \frac{1}{\sqrt{2}}(Q_1 + iQ_2), \quad Q^* := \frac{1}{\sqrt{2}}(Q_1 - iQ_2). \tag{2.5}$$

They obey the commutation rules

$$\{Q, Q^*\} = 2H, \qquad \{Q, Q\} = \{Q^*, Q^*\} = 0. \tag{2.6}$$

Consequently, the complex supercharges are conserved. In other words, they commute with the Hamiltonian.

$$[H, Q] = [H, Q^*] = 0. \tag{2.7}$$

2.2 Supersymmetric quantum mechanics on a Riemannian manifold

The following is a recollection of the supersymmetric quantum mechanics on a Riemannian manifold.

Let X be a closed oriented n-dimensional Riemannian manifold equipped with a metric g. (Closedness of X is equivalent to the condition that X is compact and boundary-free.) A quantum mechanics that has X as a target space will be called a sigma model on X. It describes a quantum mechanics of a particle moving on the surface of X.

Let us choose a function on X, $f : X \to \mathbb{R}$, such that f has only isolated critical points. Such function is called a Morse function on X. (Critial points of f are defind as the points that have vanishing derivative of f, that is, $p \in X$ is a critical point if and only if $df(p) = 0$.)

Witten considered the following supersymmetric sigma model action on X.

$$\begin{aligned} S_{\text{phys}} = \int_I \Big(& \frac{1}{2}\lambda g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \frac{1}{2}\lambda g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} \\ & + i\pi_\mu D_t\psi^\mu - ig^{\mu\nu}\frac{D^2 f}{Dx^\nu Dx^\alpha}\pi_\mu\psi^\alpha + \frac{1}{4\lambda}R^{\mu\nu}{}_{\rho\sigma}\pi_\mu\pi_\nu\psi^\rho\psi^\sigma \Big), \end{aligned} \tag{2.8}$$

where,

$$D_t\psi^\mu(t) = \frac{d}{dt}\psi^\mu(t) + \Gamma^\mu_{\nu\rho}\dot{x}^\nu(t)\psi^\rho(t). \tag{2.9}$$

Here x^μ, $\mu = 1, \ldots, n$, are the coordinates of X. The real fermionic variables ψ^μ are the superpartners of x^μ, and other real fermionic variables π_μ are the conjugates of ψ^μ. And $\lambda > 0$ is a parameter. (This action has a slightly different normalization from Witten's original one.)

The space of states of this theory is identified with $\Omega^*(X)$, the space of complex-valued L_2-differential forms on X with the hermitean inner product

$$\langle \alpha|\beta\rangle = \int_X (\overline{\star\alpha}) \wedge \beta, \tag{2.10}$$

where, $\star$ is the Hodge dual with respect to g.

The supersymmetry algebra is generated by the operators:

$$\mathcal{Q} = d_\lambda = e^{-\lambda f} d e^{\lambda f} = d + \lambda df \wedge \tag{2.11}$$

$$\mathcal{Q}^* = (d_\lambda)^* = \frac{1}{\lambda} e^{\lambda f} d^* e^{-\lambda f} = \frac{1}{\lambda} d^* + \iota_{\nabla f}, \tag{2.12}$$

where $(\nabla f)^\mu = g^{\mu\nu}\partial_\nu f$ is the gradient vector field of the Morse function f. Here the operator d^* is defined as the adjoint of d with respect to a fixed metric g on X. But $\mathcal{Q}^*$ is the adjoint of $\mathcal{Q} = d_\lambda$ with respect to the metric λg, which explains the overall factor λ^{-1}.

The Hamiltonian H_λ is given by their anti-commutator:

$$\begin{aligned} H_\lambda &= \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^*\} \\ &= \frac{1}{2}(-\frac{1}{\lambda}\Delta + \lambda ||df||^2 + K_f), \end{aligned} \tag{2.13}$$

where, $K_f = (\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f})$. Recall that for a vector field v, we denote by $\mathcal{L}_v$ the Lie derivative acting on the differential forms. Using the Cartan's formula $\mathcal{L}_{\nabla f} = \{d, \iota_{\nabla f}\}$, this can be recasted into a more familiar form:

$$H_\lambda = \frac{1}{2}\Big(-\frac{1}{\lambda}\Delta + \lambda g^{\mu\nu}\partial_\mu f \partial_\nu f + \frac{D^2 f}{Dx^\mu Dx^\nu}[a^{*\mu}, a^\nu]\Big), \tag{2.14}$$

where, $a^\mu = g^{\mu\nu}\iota_{\partial/\partial x^\nu}$, and $a^{*\mu}$ is its adjoint, that is, $a^{*\mu} = dx^\mu \wedge$ (see Appendix A).

They satisfies the anti-commutation relations

$$\{a^{*\mu}, a^{\nu}\} = 2g^{\mu\nu}. \tag{2.15}$$

Thus, $a^{*\mu}$ and a^{μ} play the roles of the fermionic creation/annihilation operators.

2.3 An example: supersymmetric harmonic oscillator on $\mathbb{R}$

We choose $X = \mathbb{R}$ and $f = \frac{1}{2}\omega x^2$, $\omega \in \mathbb{R}$. Then, $df = \omega x dx$, and $\nabla f = \omega x \frac{d}{dx}$. There is one and only one critical point of f, that is the origin $x = 0$. The Hamiltonian takes the form

$$H_\lambda = -\frac{1}{2\lambda}\frac{d^2}{dx^2} + \frac{1}{2}\lambda\omega^2 x^2 + \frac{1}{2}\omega[a^*, a]. \tag{2.16}$$

The space of states $\mathcal{H}$ consists of two components, namely, the space of zero-forms and the space of one-forms on $\mathbb{R}$.

$$\mathcal{H} = \Omega(\mathbb{R}) = \Omega^0(\mathbb{R}) \oplus \Omega^1(\mathbb{R}). \tag{2.17}$$

The fermionic part of the Hamiltonian $\frac{1}{2}\omega[a^*, a]$ $(=: H_f)$ takes the value $-\frac{1}{2}\omega$ on zero-forms and $+\frac{1}{2}\omega$ on one-forms. While the bosonic part $-\frac{1}{2\lambda}\frac{d^2}{dx^2} + \frac{\lambda}{2}\omega^2 x^2$ $(=: H_b)$is a familiar harmonic oscillator. Therefore, the eigenvalues of H_b are given by

$$\frac{1}{2}|\omega|,\ \ \frac{3}{2}|\omega|, \ldots, \frac{2n+1}{2}|\omega|, \ldots \tag{2.18}$$

Corresponding eigenfunctions of H_b are given by

$$\Psi_n(x) = A_n e^{\frac{1}{2}\lambda|\omega|x^2} \frac{d^n}{dx^n} e^{-\lambda|\omega|x^2}, \tag{2.19}$$

where A_n are appropriate normalization constants that ensure

$$\langle \Psi_n | \Psi_n \rangle = \int_{\mathbb{R}} dx \overline{\Psi_n(x)} \Psi_n(x) = 1. \tag{2.20}$$

Now we consider the eigenvalues and eigenstates, in particular, the ground state of the total Hamiltonian $H_\lambda = H_b + H_f$. It is important that they depend on the sign of ω.

- For $\omega > 0$, the ground state is the zero-form $\Psi_0(x) = e^{-\frac{1}{2}\lambda\omega x^2}$.
- For $\omega < 0$, the ground state is the one-form $\Psi_0(x)dx = e^{-\frac{1}{2}\lambda|\omega|x^2}dx$.

In both cases, the corresponding eigenvalue is zero, that means, they are supersymmetric ground states. The excited states are given by $\Psi_n(x)$ and $\Psi_n(x)dx$, $n > 0$. Note that the eigenfunctions are localized around the origin. They exponentially decrease away from the origin.

2.4 Generalization to $\mathbb{R}^n$

The supersymmetric harmonic oscillator can be straightforwardly generalized to that on $X = \mathbb{R}^n$. Let us choose the Morse function on $\mathbb{R}^n$ as

$$f(x) = \sum_{\mu=1}^{n} \frac{1}{2}\omega_\mu (x^\mu)^2, \quad 0 \neq \omega_\mu \in \mathbb{R}. \tag{2.21}$$

The origin of $\mathbb{R}$ is one and only one critical point of f. Suppose, a given $1 \leq k \leq n$, we assume that $\omega_1, \ldots, \omega_k$ are negative and $\omega_{k+1}, \ldots, \omega_n$ are positive. The Hamiltonian (2.14) takes the form

$$H_\lambda = \sum_{\mu=1}^{n} \frac{-1}{2\lambda} \frac{\partial^2}{\partial (x^\mu)^2} + \frac{\lambda}{2}\omega_\mu^2 (x^\mu)^2 + \frac{1}{2}\omega_\mu [a^{*\mu}, a^\mu]. \tag{2.22}$$

The space of states $\mathcal{H}$ is decomposed to the direct sum of the space of m-forms, where $m = 1, \ldots, n$.

$$\mathcal{H} = \Omega^*(\mathbb{R}^n) = \Omega^1(\mathbb{R}) \oplus \cdots \oplus \Omega^n(\mathbb{R}). \tag{2.23}$$

The bosonic part of the Hamiltonian is the n-dimensional Harmonic oscillator and the fermionic part gives definite values depending on the signature of ω_μ's. It is easy to see that the ground state of the total Hamiltonian is

$$\Psi_0(x) = e^{-\frac{1}{2}\lambda(|\omega_1|(x^1)^2 + \cdots + |\omega_n|(x^n)^2)} dx^1 \wedge \cdots \wedge dx^k, \tag{2.24}$$

and corresponding eigenvalue is zero, i.e., this state is a supersymmetric ground state. There are no other supersymmetric ground state. The excited states have the obvious forms like

$$\Psi(x) = e^{\frac{1}{2}\lambda(|\omega_1|(x^1)^2+\ldots|\omega_n|(x^n)^2)}\partial_1^{n_1}\partial_2^{n_2}\ldots\partial_n^{n_2}e^{-\lambda(|\omega_1|(x^1)^2+\ldots|\omega_n|(x^n)^2)}dx^{l_1}\wedge\cdots\wedge dx^{l_s}. \tag{2.25}$$

2.5 General Riemannian manifold case

Now we consider the most general case. Since we are assuming X is compact, there are only finite a finite number of the critical points of the Morse function $f : X \to \mathbb{R}$. Let us focus on one of these critical points, say, $p \in X$. Then, it is known that we can find a local coordinate system around p such that the local expression of the Morse function f is written as

$$f(x) = f(p) + \sum_{\mu=0}^{n} \omega_\mu (x^\mu)^2 + \mathcal{O}((x^\mu)^3). \tag{2.26}$$

Suppose that the number of the negative eigenvalues of the Hessian of the Morse function at p is k. This non-negative integer k is called a *Morse index* of f at a critical point p. Then we may assume without loss of generality that $\omega_1, \ldots, \omega_k$ are negative and $\omega_{k+1}, \ldots, \omega_n$ are positive. Then the Hamiltonian is approximately given by the form of (2.22), that is, n-dimensional supersymmetric harmonic oscillator. Thus, for each critical point of f, we have one approximate supersymmetric ground state.

2.6 The main model: $X = \mathbb{CP}^1$

In this thesis, we will mainly study the supersymmetric quantum mechanics on $X = \mathbb{CP}^1$. It can be regard as a compactification of the complex plane in standard way: $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. We choose the Fubini-Study metric on $\mathbb{CP}^1$

$$g = \frac{dz d\bar{z}}{(1 + z\bar{z})^2}, \tag{2.27}$$

and the Morse function

$$f = \frac{1}{4}\frac{z\bar{z}-1}{z\bar{z}+1} \tag{2.28}$$

The Hamiltonian (2.14) takes the form

$$H_\lambda = -\frac{2}{\lambda}(1+z\bar{z})^2\partial_z\partial_{\bar{z}} + \frac{\lambda}{2}\frac{z\bar{z}}{(1+z\bar{z})^2} - \frac{z\bar{z}-1}{z\bar{z}+1}(F+\overline{F}-1), \tag{2.29}$$

where, F and $\overline{F}$ are the fermionic left and right charge operators that are defined by

$$F \cdot dz = 1, \ \ \overline{F} \cdot d\bar{z} = 1, \tag{2.30}$$

$$F \cdot d\bar{z} = \overline{F} \cdot dz = 0, \tag{2.31}$$

$$F \cdot 1 = \overline{F} \cdot 1 = 0. \tag{2.32}$$

Our Morse function has two critical points, $z = 0$, and $z = \infty$. Near $z = 0$ we have:

$$f = -\frac{1}{4} + \frac{1}{2}z\bar{z} + \dots, \tag{2.33}$$

while near $z = \infty$ we have

$$f = \frac{1}{4} - \frac{1}{2}w\bar{w} + \dots, \tag{2.34}$$

where $w = z^{-1}$ is a local coordinate near the point ∞. Therefore we shall find two approximate supersymmetric ground state corresponding to these two critical points. Furthermore, the origin $z = 0$ is a critical point with the Morse index zero, and the point at infinity $z = \infty$ is a critical point with the Morse index two. So the ground state localized around $z = 0$ must be a zero-form, and the one localized around $z = \infty$ must be a two form on $\mathbb{CP}^1$. It is easy to find these ground state. The one corresponding $z = 0$ is

$${}_0\Psi_{\text{vac}} = \sqrt{\frac{\lambda}{\pi(e^{\lambda/2} - e^{-\lambda/2})}}e^{-\lambda f}, \tag{2.35}$$

and the one corresponding to $z = \infty$ is

$$ {}_{\infty}\Psi_{\text{vac}} = \sqrt{\frac{\lambda}{\pi(e^{\lambda/2} - e^{-\lambda/2})}} e^{\lambda f} \omega_{FS}, \tag{2.36} $$

where,

$$ \omega_{FS} = \frac{dz d\overline{z}}{(1 + z\overline{z})^2} \tag{2.37} $$

is the Fubini-Study Kähler form. Here and below we use the notation

$$ dz d\overline{z} = d^2 z = i dz \wedge d\overline{z} = 2 dx \wedge dy, \quad z = x + iy. \tag{2.38} $$

It is known that these two states are not only approximate ground states, but true supersymmetric ground states of the Hamiltonian H_λ.

The excited states have the approximate forms near $z = 0$,

$$ {}_{0}\Psi_{n,\overline{n},p,\overline{p}} = {}_{0}A_{n,\overline{n}} \; e^{\lambda f} \; \partial_z^{\overline{n}} \partial_{\overline{z}}^{n} \; (e^{-2\lambda f})(dz)^p (dz)^{\overline{p}}, \tag{2.39} $$

and near $z = \infty$,

$$ {}_{\infty}\Psi_{n,\overline{n},p,\overline{p}} = {}_{\infty}A_{n,\overline{n}} \; e^{-\lambda f} \partial_w^{n} \partial_{\overline{w}}^{\overline{n}} \; (e^{2\lambda f})(dw)^p (d\overline{w})^{\overline{p}}, \quad n, \overline{n} \geq 0, \; p, \overline{p} = 0, 1, \tag{2.40} $$

where ${}_{0}A_{n,\overline{n}}$ and ${}_{\infty}A_{n,\overline{n}}$ are the appropriate normalization constants.

Chapter 3

Topological field theory and the Mathai-Quillen formalism

In this chapter, we introduce topological field theories (TFT's) and we give a brief general overview of their properties. We also review a beautiful construction method for a certain class of TFT's. This method is called Mathai-Quillen formalism [21] and gives us a geometric picture of TFT's.

3.1 Topological field theory of Witten type

A quantum field theory is called *topological* if there exists a set of observables (that we shall call topological observables) such that their correlation functions do not depend on the metric. If we denote these observables by $\mathcal{O}_i$, then

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{i_1} \dots \mathcal{O}_{i_n} \rangle = 0. \tag{3.1}$$

Here, the correlation function is defined by the path integral,

$$\langle \mathcal{O}_{i_1} \dots \mathcal{O}_{i_n} \rangle := \int D\phi \; \mathcal{O}_{i_1}(\phi) \dots \mathcal{O}_{i_n}(\phi) \; e^{-\frac{1}{\hbar} S[\phi]}, \tag{3.2}$$

where, $\hbar$ is an overall coupling constant.

In general, there are two types of quantum field theory that achieve the defining properties (3.1) of TFT's . One is called TFT's of Schwarz type. Both their actions and their observables are locally independent of the metric. Consequently, their correlation functions are independent of the metric. Another type is called TFT's of Witten type or Cohomological field theories (CohFT's), which are of interest for us. A CohFT do have a metric-dependent action. However, this is compensated by an extra symmetry that is called topological symmetry. In the following, we describe general properties of CohFT's.

Let us denote the generators of the topological symmetry by Q. Since Q is a symmetry of the theory, it satisfies $QS[\phi] = 0$ and $Q[D\phi] = 0$. Generally, we impose further conditions on Q. We take Q to be a nilpotent fermionic scalar charge,

$$Q^2 = 0. \tag{3.3}$$

One of the remarkable properties of the CohFT is that, the energy tensor $T_{\mu\nu}[\phi] = \frac{\delta S}{\delta g^{\mu\nu}}$ has the form

$$T_{\mu\nu} = Q\ G_{\mu\nu}[\phi], \tag{3.4}$$

where, $G_{\mu\nu}$ is some tensor. The topological observables $\mathcal{O}_i$ are observables that satisfy $Q\ \mathcal{O}_i = 0$. In addition, they are metric-independent. Then the correlation functions of the topological observables are found to be metric-independent:

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{i_1} \dots \mathcal{O}_{i_n} \rangle &= \langle \mathcal{O}_{i_1} \dots \mathcal{O}_{i_n} \left(Q G_{\mu\nu} \right) \rangle = \langle Q \left(\mathcal{O}_{i_1} \dots \mathcal{O}_{i_n} G_{\mu\nu} \right) \rangle \\ &= 0. \end{aligned} \tag{3.5}$$

Furthermore, if one of the $\mathcal{O}_i$'s has the form $\mathcal{O}_i = Q\Lambda_i$, topological correlations function that contain $Q\Lambda$ vanish.

$$\langle \mathcal{O}_{i_1} \dots (Q\Lambda) \dots \mathcal{O}_{i_n} \rangle = \langle Q \left(\mathcal{O}_{i_1} \dots \Lambda \dots \mathcal{O}_{i_n} \right) \rangle = 0. \tag{3.6}$$

It follows that if we replace some of the topological observable $\mathcal{O}$ with $\mathcal{O} + Q\Lambda$, the value of the correlation function does not change. Therefore, it is natural to identify $\mathcal{O}$ with $\mathcal{O} + Q\Lambda$ as topological observables. Taking into account this fact and the condition $Q^2 = 0$, it is natural to define the space of topological observables as the space of the cohomology classes of Q:

$$\{\ \mathcal{O}_i\ \} = \frac{\text{Ker } Q}{\text{Im } Q}\ . \tag{3.7}$$

An observable $\mathcal{O}$ that satisfies $Q\mathcal{O} = 0$ is called Q-closed, and it represents a cohomology class of Q. On the other hand, an observables $\mathcal{O}$ that has the form $\mathcal{O} = Q\Lambda$, for some Λ, is called Q-exact, and it is zero as the element of the cohomology class of Q.

A fermionic symmetry satisfying this property is called a BRST symmetry, and the fermionic charge Q that generates the BRST symmetry is called the BRST charge. The topological observables are often called the BPS observables.

In the most of CohFT's of our interest, not only the energy momentum tensor is Q-exact, but the action itself is Q-exact,

$$S[\phi] = Q\Psi[\phi]. \tag{3.8}$$

In this case, we can deduce further properties of the BPS correlation functions. They are independent from the overall coupling constant:

$$\begin{aligned} \frac{d}{d(\hbar^{-1})}\langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \rangle &= \frac{d}{d(\hbar^{-1})} \int D\phi\ \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n}\ e^{-\frac{1}{\hbar}S[\phi]} \\ &= \langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} Q\Psi \rangle \\ &= \langle Q\left(\mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \Psi\right) \rangle = 0. \end{aligned} \tag{3.9}$$

If we regard $\hbar$ as the Planck constant, this means that the exact computation of the correlation function is the same as the computation in the limit $\hbar \to 0$, that is, the semi-classical approximation. In this limit, the contributions to the path integral are localized around the classical solutions, the space of which are often finite dimensional. This is the notion of localization in CohFT. We will describe the further details of it in the next section.

Similarly, if the action consists of a sum of Q-exact terms,

$$S = t_1 Q\Psi_1 + t_2 Q\Psi_2 + \cdots + Q\Psi_k, \tag{3.10}$$

then, BPS correlation functions are independent from any of these coupling constants t_j,

$$\frac{\partial}{\partial t_j}\langle \mathcal{O}_{i_1} \ldots \mathcal{O}_{i_n} \rangle = 0, \quad j = 1, \ldots, k. \tag{3.11}$$

3.2 The Mathai-Quillen formalism of CohFT

It is known that a large class of topological field theories of Witten type, as known as Cohomological Field Theories (CohFT's), have a nice geometrical interpretation, which is called the Mathai-Quillen formalism [21], [4], [9]. This formalism enables us to interpret various properties of CohFT's in terms of geometry or topology. One of the most remarkable properties of CohFT is that the path integrals, or correlation functions of BPS observables can be computed by finite dimensional integrals over the moduli space of the BPS states. BPS observables are interpreted as closed differential forms on the Moduli space $\mathcal{M}$,

$$\begin{aligned} \langle \hat{\omega}_1 \dots \hat{\omega}_n \rangle &:= \int D\phi \;\; \hat{\omega}_1 \dots \hat{\omega}_n \;\; e^{-S[\phi]} \\ &= \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_n \;\; , \end{aligned} \tag{3.12}$$

where, $\hat{\omega}_i$, $i = 1, \dots, n$ are BPS observables and ω_i are corresponding closed differential forms on $\mathcal{M}$. The MQ formalism interpret this equation in terms of Poincaré duality. In this section, we will show how to construct CohFT's using MQ formalism and how the computation of BPS correlation functions are reduced to finite dimensional integrals over $\mathcal{M}$.

3.2.1 Zero-dimensional model

Here, we consider a following very simple model. Let us take $X = \mathbb{R}$ as the space of fields. For a vector bundle on X, let us choose the tangent bundle $E = TX = T\mathbb{R}$. Clearly, $T\mathbb{R}$ is a trivial bundle, $T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$, so that we can coordinatize $T\mathbb{R}$ globally by one local coordinate system (x, y). Suppose x parametrizes the base space and y parametrizes the fiber direction. Next, we have to choose a section of $T\mathbb{R}$. We choose it to be $y = s(x) = x$. The equation $s(x) = 0$ is called the BPS equation, or instanton equation. The moduli space is defined as a subspace of X such that $s(\mathcal{M}) = 0$, i.e., $\mathcal{M} := \{s^{-1}(0)\} \equiv \{x \in X | s(x) = 0\}$. In our case, $s(x) = x$ means $\mathcal{M} = \{0\} = \{\text{pt.}\}$. Next, as the BRST transformation, we take $Q = d = dx\frac{d}{dx}$, that is the exterior derivative on $\mathbb{R}$. Then we define the BPS observables of this model as the cohomology classes of Q, which are equal to the de Rham cohomology classes on $\mathbb{R}$. It has only one nontrivial cohomology class at degree zero, that is, a class of constant functions: $H^*(\mathbb{R}) = \delta_{*,0}\mathbb{R}$. We can take the constant function "1" as a representative of $H^0(\mathbb{R})$. In summary,

$$\text{Space of fields: } X = \mathbb{R}, \tag{3.13}$$
$$\text{Vector bundle: } E = TX = T\mathbb{R}, \tag{3.14}$$
$$\text{Instanton equation: } x = 0, \tag{3.15}$$
$$\text{Moduli space: } \mathcal{M} = \{0\} \subset X, \tag{3.16}$$
$$\text{BRST transformation: } Q = d = dx\frac{d}{dx} \ , \tag{3.17}$$
$$\text{Space of BPS observables: } H^*(X, Q) = H^*(\mathbb{R}, d) = \delta_{*,0}\mathbb{R}. \tag{3.18}$$

The moduli space $\mathcal{M}$ is a point, which is zero dimensional. We define the integral of 1 over $\mathcal{M}$ by,

$$\int_{\mathcal{M}=\{0\}} 1 := 1. \tag{3.19}$$

Then, we have an equality

$$\int_{\mathcal{M}} 1 = \int_{\mathbb{R}} 1 \wedge \delta(x)dx := \int_{-\infty}^{\infty} \delta(x)dx. \tag{3.20}$$

This equation says that the delta form $\delta(x)dx$ is a representative of the Poincaré dual of the point $\mathcal{M}$. We will describe the notion of the Poincaré duality in the next section.

Here, we have yet another representative of the Poincaré dual of $\mathcal{M}$,

$$\eta_{\mathcal{M},\lambda} := \sqrt{\frac{\lambda}{2\pi}}\, e^{-\frac{\lambda}{2}x^2}dx, \quad \lambda > 0. \tag{3.21}$$

For any $\lambda > 0$, the equality

$$\int_{\mathcal{M}} 1 = \int_{\mathbb{R}} 1 \wedge \eta_{\mathcal{M},\lambda} \tag{3.22}$$

holds. Hence, we have a one-parameter family $\eta_{\mathcal{M},\lambda}$ of representatives of the Poincaré dual of $\mathcal{M}$. Note that $\eta_{\mathcal{M},\lambda}$ tends to the delta form in the limit $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} \eta_{\mathcal{M},\lambda} = \delta(x)dx. \tag{3.23}$$

Here, let us make a short digression. In fact, this limit is very special, because for any *non-constant* function $f(x)$, $df(x) \neq 0$, the equality

$$\int_{\mathcal{M}} f(x) = \lim_{\lambda\to\infty} \int_{\mathbb{R}} f(x) \wedge \eta_\lambda \ \left(= \int_{-\infty}^{\infty} f(x)\delta(x)dx \right) \tag{3.24}$$

does hold. Where, we defined the integral of $f(x)$ over $\mathcal{M}$ by $\int_{\mathcal{M}} f(x) := f(0)$. While, for finite value of λ, this does not occur generally. This remark will be important for later chapters.

To return to the subject, let us manipulate the integral (3.22) and transform it to a form somewhat alike a path integral of a supersymmetric field theory. First, define a one-form Φ_λ on the total space of the vector bundle $E \cong \mathbb{R}^2$ as,

$$\Phi_\lambda := \sqrt{\frac{\lambda}{2}} e^{-\frac{\lambda}{2}y^2} dy. \tag{3.25}$$

Then, $\eta_{\mathcal{M},\lambda}$ is equal to the pullback of Φ_λ by the section $s : X \to E$,

$$\begin{aligned} \eta_{\mathcal{M},\lambda} = s^*\Phi_\lambda &\equiv \sqrt{\frac{\lambda}{2\pi}}\, e^{-\frac{\lambda}{2}s(x)^2}\, \frac{dy}{dx} dx \\ &= \sqrt{\frac{\lambda}{2\pi}}\, e^{-\frac{\lambda}{2}x^2}\, dx. \end{aligned} \tag{3.26}$$

Next, let us introduce two fermionic variables ψ, π and one bosonic variable p. We define the BRST transformation of these variables by fermionic transformations as,

$$Qx := \psi\ (= dx), \qquad Q\psi := 0, \tag{3.27}$$

$$Q\pi := p, \qquad Qp := 0. \tag{3.28}$$

Note that Q is clearly nilpotent, $Q^2 = 0$. Now we can find an equality[1]

$$\eta_{\mathcal{M},\lambda} = \frac{-1}{2\pi i} \int dp d\pi\ e^{-S_\lambda}, \tag{3.29}$$

where, $S_\lambda := Q\Psi_\lambda$, and

$$\Psi_\lambda := \pi(-ix + \frac{1}{2\lambda}p). \tag{3.30}$$

[1]The readers should not confuse the variable π with $\pi = 3.14$.

The explicit form of the Q-exact term S_λ is,

$$S_\lambda = -ipx + i\pi\psi + \frac{1}{2\lambda}p^2. \tag{3.31}$$

Thus, (3.22) can be written as,

$$\langle 1 \rangle := \frac{-1}{2\pi i}\int dxd\psi dpd\pi \; 1 \; e^{-(-ipx+i\pi\psi+\frac{1}{2\lambda}p^2)} = \int_{\mathcal{M}} 1. \tag{3.32}$$

If we compare this equation with (3.12), it looks like a path integral of a CohFT with Q-exact action $S_\lambda = Q\Psi_\lambda$.

3.2.2 Poincaré duality and Euler class

Poincaré duality

Poincaré duality is a duality between the homology and the cohomology of a manifold. It states that if X is an n-dimensional oriented closed manifold (compact and without boundary), then the kth homology group of X is isomorphic to the $(n-k)$th cohomology group of X, for all integers k.

$$H_k(X) \cong H^{n-k}(X). \tag{3.33}$$

In the language of de Rham cohomology theory, this statement becomes as follows. Let $\mathcal{M}$ be a k-dimensional closed submanifold in X. Then there exist a closed $(n-k)$-form $\eta_{\mathcal{M}}$, $d\eta_{\mathcal{M}} = 0$ that represents a $(n-k)$th de Rham cohomology class of X such that for any closed form ω, $d\omega = 0$, the equality

$$\int_{\mathcal{M}} i^*\omega = \int_X \omega \wedge \eta_{\mathcal{M}} \tag{3.34}$$

holds. Where, $i : \mathcal{M} \hookrightarrow X$ is the inclusion map, hence $i^*\omega$ is the restriction of ω on $\mathcal{M}$. The closed form $\eta_{\mathcal{M}}$ is called a Poincaré dual of $\mathcal{M}$. In the previous section, we have constructed a family of the Poincaré dual of $\mathcal{M} = \{0\} \subset \mathbb{R} = X$. In what follows, we will denote a Poincaré dual of a submanifold $\mathcal{M}$ by $\eta_{\mathcal{M}}$.

Euler class

Consider a real, oriented vector bundle $E \to X$ over an compact, oriented n-dimensional manifold X. Furthermore, assume that the rank of E is even and such that $\mathrm{rank}(E) = 2m \leq n$. The Euler class of E is an integral cohomology class $e(E) \in H^{2m}(X)$. There are a number of ways to think about $e(E)$. Followings are the two descriptions that are important for us.

- The first way is a topological description. Let $s : X \to E$ be any generic section of E, that is, transverse to the zero section. A theorem states that the Euler class of E is equal to the Poincaré dual of the zero locus of the generic section s,
$$e(E) = \eta_{\mathcal{M}} \in H^{2m}(X), \quad \mathcal{M} = \{s^{-1}(0)\}. \tag{3.35}$$

- The second way is a differential geometric description. Let ∇ be a connection on E. Using Chern-Weil theory, we can construct a representative $e_\nabla(E)$ of Euler class of E as,
$$e_\nabla(E) = \frac{1}{(2\pi)^m} \mathrm{Pf}\ (F_\nabla) \in H^{2m}(X), \tag{3.36}$$
where F_∇ is the curvature of ∇. We regard F_∇ as two-form valued $2m \times 2m$ anti-symmetric matrix and define its Pfaffian by
$$\mathrm{Pf}\ (F_\nabla) = \sum_{\sigma \in S_{2m}} \frac{(-1)^m}{2^m m!} \mathrm{sgn}\ (\sigma) F_\nabla^{\sigma(1)\sigma(2)} \wedge \cdots \wedge F_\nabla^{\sigma(2m-1)\sigma(2m)}. \tag{3.37}$$
It is known that the cohomology class of e_∇ is independent of the choice of ∇.

For more details, see [5].

3.2.3 The Mathai-Quillen formalism in finite dimensional spaces

The Mathai-Quillen formalism provides a representative of Euler class which interpolate above two descriptions. Here we retain the notation above. Mathai-Quillen [21] showed that a representative of the Euler class of the vector bundle

E over X is given by

$$e_{s,\nabla}(E) = \frac{1}{(2\pi)^m} \int dp_a d\pi_a \, e^{-(-is^a(x)p_a + \frac{1}{2\lambda}h^{ab}p_a p_b + i\pi_a \nabla s^a(\psi) + \frac{1}{4\lambda}F^{ab}_{\mu\nu}\pi_a\pi_b\psi^\mu\psi^\nu)} , \quad (3.38)$$

where we use the indices μ, ν for the coordinate on the base space X, and a, b for the coordinate on the fiber of the bundle E, and h^{ab} is a metric of E. We abbreviate F_∇ to F. We write $dx^\mu = \psi^\mu$. The variables p_a are bosonic and π_a are fermionic. $\lambda > 0$ is an arbitrary parameter. This expression $e_{s,\nabla}(E)$ is called the Mathai-Quillen representative of the Euler class $e(E)$.

Let us write this expression as

$$e_{s,\nabla}(E) = \frac{1}{(2\pi)^m} \int dp_a d\pi_a e^{-S}, \quad (3.39)$$

where

$$S = -is^a(x)p_a + \frac{1}{2\lambda}h^{ab}p_a p_b + i\pi_a \nabla s^a(\psi) + \frac{1}{4\lambda}F^{ab}_{\mu\nu}\pi_a\pi_b\psi^\mu\psi^\nu. \quad (3.40)$$

This can be recasted into the form

$$S = Q[\pi_a(-is^a(x) + \frac{1}{2\lambda}h^{ab}p_b)]. \quad (3.41)$$

Here Q is a fermionic transformation that is defined by

$$Qx^\mu := \psi^\mu, \quad (3.42)$$

$$Q\psi^\mu := 0, \quad (3.43)$$

$$Qp_a := p_a - A_{\mu}{}^{b}{}_{a}\pi_b\psi^\mu, \quad (3.44)$$

$$Q\pi_a := A_{\mu}{}^{b}{}_{a}\pi_b\psi^\mu - \frac{1}{2}F_{\mu\nu}{}^{b}{}_{a}\pi_b\psi^\mu\psi^\nu, \quad (3.45)$$

where $A_{\mu}{}^{b}{}_{a}$ is the connection one-form associated with the connection ∇. We can show that Q is nilpotent, $Q^2 = 0$. For the proof, see Appendix B. Therefore, Q can be regarded as the BRST transformation. The "action" S has a Q-exact form. This is the very property of the CohFT.

Recall that the Euler class is the Poincaré dual of the zero locus $\mathcal{M}$ of the generic section s. This means that, for any closed form ω on X, we have the equality

$$\int_{\mathcal{M}} i^*\omega = \int_X \omega \wedge e_{s,\nabla}(E). \tag{3.46}$$

If we write as $\omega = \omega(x)_{\mu_1\dots\mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$, (3.46) can be written as

$$\int_{\mathcal{M}} i^*\omega = \int dx^\mu d\psi^\mu dp_a d\pi_a \omega(x)_{\mu_1\dots\mu_k} \psi^{\mu_1} \dots \psi^{\mu_k} \, e^{-S}. \tag{3.47}$$

This looks like the localization property (3.12) of the path integral of the CohFT. The closed form ω is interpreted as a BPS observable of this CohFT.

Actually, a number of CohFT's can be obtain by formally applying the Mathai-Quillen formalism to the infinite dimensional field space. In the later chapter, we will do this for the space of paths and obtain the topological quantum mechanics.

Note that the "action" (3.41) consists of two Q-exact terms:

$$S = Q[i\pi_a s^a(x)] - \frac{1}{2\lambda} Q[h^{ab}\pi_a p_b]. \tag{3.48}$$

Therefore, by the argument of (3.10) and (3.11), the integral (3.47) is independent of λ. Moreover, by a similar argument as (3.24), if we take the limit of $\lambda \to \infty$, (3.47) does hold even if ω is not a closed form. FLN [10] made use of this property as a means of computation of the non-BPS correlation functions in the limit of the $\lambda \to \infty$ of the topological quantum mechanics.

Chapter 4

The path space geometry

We would like to use the Mathai-Quillen formalism to construct a topological version of the supersymmetric quantum mechanics. To do this, we need to know the geometry of the space of paths. Therefore, in this chapter, we review the details of the path space geometry.

4.1 The path space

Let X be a finite dimensional smooth Riemannian manifold equipped with a metric g. We denote the space of all the smooth paths on X by $\mathcal{P}X$. This space forms an infinite dimensional manifold and is called path space of X:

$$\mathcal{P}X := \{\gamma : I \longrightarrow X \mid \gamma \text{ is smooth}\}, \tag{4.1}$$

where, I is the "world line", which could be a finite interval $[t_i, t_f]$, or half-line $(-\infty, t_f]$, $[t_i, +\infty)$ or the entire line $(-\infty, +\infty)$. We will refer to the elements of I as "time". For simplicity, let us choose I as a unit interval: $I = [0, 1]$.

Let us also define a space of paths with both endpoints fixed as a subspace of $\mathcal{P}X$. We take $x_0 \in X$ as the initial point and $x_1 \in X$ as the final point. Then the fixed-endpoint path space $\mathcal{P}(X; x_0, x_1)$ is defined as follows:

$$\mathcal{P}(X; x_0, x_1) := \{\gamma \in \mathcal{P}X \mid \gamma(0) = x_0, \ \ \gamma(1) = x_1\}. \tag{4.2}$$

4.2 Vector fields on $\mathcal{P}X$

Here, we describe the tangent space of $\mathcal{P}X$ at a path γ, that is $T_\gamma \mathcal{P}X$. A tangent vector $v(\gamma) \in T_\gamma \mathcal{P}X$ represents an infinitesimal deformation of γ. Such a deformation can be represented by a family of tangent vectors along γ on X. Therefore $T_\gamma \mathcal{P}X$ is identified with $\Gamma^\infty(\gamma^* TX)$, i.e., the space of smooth sections of the pulled-back bundle $\gamma^* TX$. Generally, $v(\gamma)$ is written as the integral along the time, like

$$v(\gamma) = \int_0^1 dt\, v^\mu(\gamma(t)) \frac{\delta}{\delta\gamma^\mu(t)} \quad \in T_\gamma \mathcal{P}X, \tag{4.3}$$

where summation over $\mu = 1, \ldots, \dim X$ is implicit. This is analogous to the finite dimensional case

$$v(x) = \sum_{\mu=1}^{n} v^\mu(x) \frac{\partial}{\partial x^\mu} \quad \in T_x X. \tag{4.4}$$

So we can regard $\dfrac{\delta}{\delta\gamma^\mu(t)}$ as a basis vectors of $T_\gamma \mathcal{P}X$. A vector field on $\mathcal{P}X$ is of course, a smooth function of $\gamma \in \mathcal{P}X$ with values in tangent vectors $v(\gamma) \in T_\gamma \mathcal{P}X$. Note that, in the case of $\mathcal{P}(X; x_0, x_1)$, the coefficients of the tangent vector $v^\mu(\gamma(t))$ must satisfy $v^\mu(\gamma(0)) = v^\mu(\gamma(1)) = 0$, because both the endpoints are fixed.

The tangent bundle $T\mathcal{P}X$ admits a natural metric induced from g, the one on TX, as follows. For any two basis vectors $\frac{\delta}{\delta\gamma^\mu(t_1)}, \frac{\delta}{\delta\gamma^\nu(t_2)}$ of $T_\gamma \mathcal{P}X$, we define their inner product by

$$\langle \frac{\delta}{\delta\gamma^\mu(t_1)}, \frac{\delta}{\delta\gamma^\nu(t_2)} \rangle_\gamma := g_{\mu\nu}(\gamma(t_1)) \delta(t_1 - t_2). \tag{4.5}$$

Therefore, if $v_1(\gamma) = \int dt_1\, v_1^\mu(\gamma(t_1)) \frac{\delta}{\delta\gamma^\mu(t_1)}$ and $v_2(\gamma) = \int dt_2\, v_2^\nu(\gamma(t_2)) \frac{\delta}{\delta\gamma^\nu(t_2)}$ are two vector fields, their inner product can be computed as

$$\begin{aligned} \langle v_1, v_2 \rangle_\gamma &= \int dt_1 dt_2\, v_1^\mu(\gamma(t_1)) v_2^\nu(\gamma(t_2)) g_{\mu\nu} \delta(t_1 - t_2) \\ &= \int dt\, g_{\mu\nu} v_1^\mu(\gamma(t)) v_2^\nu(\gamma(t)). \end{aligned} \tag{4.6}$$

This defines the induced metric on $T\mathcal{P}X$.

4.3 Differential forms on $\mathcal{P}X$

Next, we define differential forms on $\mathcal{P}X$. Formally, the basis of the cotangent vector space of $\mathcal{P}X$ at γ are spanned by $\delta\gamma^{\mu}(t) \in T^*_{\gamma}\mathcal{P}X$, $t \in [0,1]$. Here we regard δ as the exterior derivative on $\mathcal{P}X$. By varying γ in $\mathcal{P}X$, the differential one-form $\delta\gamma^{\mu}(t)$ is defined. The pairing between $\delta\gamma^{\mu}(t_1)$ and a tangent vector $\frac{\delta}{\delta\gamma^{\nu}(t_2)}$ is defined as

$$\delta\gamma^{\mu}(t_1)\left[\frac{\delta}{\delta\gamma^{\nu}(t_2)}\right] := \delta^{\mu}_{\ \nu}\,\delta(t_1 - t_2). \tag{4.7}$$

Although the differential k-forms on $\mathcal{P}X$ can be generated by taking the k-fold wedge products of above one-forms, such objects are too abstract for our computation. It is difficult to think of the infinite dimensional manifold and the forms on it. Therefore we need some more concrete picture of the differential forms on the path space, that are explained in what follows.

Suppose, given an arbitrary non-negative integer n. Let us consider an open subset U of a Euclidean space $\mathbb{R}^n$ and a map

$$\phi : U \longrightarrow \mathcal{P}X \tag{4.8}$$

such that, upon defining $\varphi(t,x) := \phi(x)(t)$, $\quad \varphi : I \times U \longrightarrow X$ become a C^{∞}-map. Such pair (U,ϕ) is called a *local parameter system* of the path space $\mathcal{P}X$. This is analogous to a local coordinate system of a finite dimensional manifold. Using $\phi : U \to \mathcal{P}X$, we can pullback differential forms on $\mathcal{P}X$ to ones on the finite dimensional space U. Suppose ω is a differential form on $\mathcal{P}X$, we denote its pullback as $\omega_{\phi} := \phi^*\omega$. Of course, ω_{ϕ} becomes a form on U. If another open subset $V \subset \mathbb{R}^n$ and a C^{∞} map $f : V \longrightarrow U$ are given, $(V, \phi \circ f)$ is also a local parameter system of $\mathcal{P}X$. By the property of pullback, we have $f^*\omega_{\phi} = \omega_{\phi\circ f}$. Hence, by using this formula inversely, we can *define* differential forms on the path space as follows. For any given local parameter system (U,ϕ), a differential k-form ($k \leq n$) ω on $\mathcal{P}X$ is a k-form ω_{ϕ} on U such that for any $f : V \to U$ like above,

$$f^*\omega_{\phi} = \omega_{\phi\circ f} \tag{4.9}$$

is satisfied. The form defined as above is denoted by $\omega = \{\omega_{\phi}\}$. This definition of the forms on $\mathcal{P}X$ has the advantage that it can avoid the difficulty from the infinite-dimensionality of $\mathcal{P}X$.

4.4 The algebra of the differential forms on the path space

Let us choose a local parameter system (U, ϕ) and fix it. For a form $\omega = \{\omega_\phi\}$, we have

$$d\omega = \{d\omega_\phi\}. \tag{4.10}$$

This follows from the fact that the exterior derivative and the pullback commute. For two forms $\omega = \{\omega_\phi\}$ and $\tau = \{\tau_\phi\}$, their sum is defined as

$$\omega + \tau := \{\omega_\phi + \tau_\phi\}. \tag{4.11}$$

For a scalar $k \in \mathbb{R}$, the scalar multiplication of ω by k is

$$k\omega := \{k\omega_\phi\}. \tag{4.12}$$

The wedge product of ω and τ is

$$\omega \wedge \tau := \{\omega_\phi \wedge \tau_\phi\}. \tag{4.13}$$

We denote by $\Omega^k(\mathcal{P}X)$, the linear space which consists of all k-forms on $\mathcal{P}X$ and let $\Omega^*(\mathcal{P}X) := \bigoplus_{k\geq 0} \Omega^k(\mathcal{P}X)$. The construction above defines a structure of differential graded algebra on $\Omega^*(\mathcal{P}X)$. Since $d^2 = 0$ on $\Omega^*(\mathcal{P}X)$, it becomes a differential complex. By definition, this is the de Rham complex of $\mathcal{P}X$. Therefore, we can consider its cohomology, i.e., the de Rham cohomology of the path space $H^*_{DR}(\mathcal{P}X)$.

4.5 Various constructions of differential forms on $\mathcal{P}X$

Here we develop methods to construct the differential forms on the path space $\mathcal{P}X$. First we define the so-called evaluation maps. There are two kinds of such maps. One is the evaluation map with the time fixed, which is defined by

$$\mathrm{ev}_t : \mathcal{P}X \to X, \quad \gamma \mapsto \gamma(t), \tag{4.14}$$

where $t \in I$ is fixed. So there is a one-parameter family of the evaluation maps with the time fixed, $\{\mathrm{ev}_t\}$, $t \in I$. The other is the one with the time unfixed, which is defined by

$$\mathrm{ev} : I \times \mathcal{P}X \longrightarrow X, \quad (t, \gamma) \mapsto \gamma(t). \tag{4.15}$$

The simplest way to get forms on $\mathcal{P}X$ is to use $\mathrm{ev}_t : \mathcal{P}X \to X$. For each $t \in \mathbb{R}$, we can pullback forms on X to ones on $\mathcal{P}X$ by

$$\mathrm{ev}_t^* : \Omega^*(X) \longrightarrow \Omega^*(\mathcal{P}X), \quad \omega \mapsto \mathrm{ev}_t^* \omega. \tag{4.16}$$

We can also use $\mathrm{ev} : I \times \mathcal{P}X \to X$ to construct forms on $\mathcal{P}X$. First we pullback forms on X to ones on $I \times \mathcal{P}X$. Then by integrating them along the time direction, we get forms on $\mathcal{P}X$. To go into further detail, we need the notion of the local parameter systems of $\mathcal{P}X$ introduced in the previous section.

Let us choose a local parameter system $\phi : U \to \mathcal{P}X$, where U is an open subset of $\mathbb{R}^n$. Then a map

$$\phi_U : I \times U \longrightarrow X \tag{4.17}$$

is defined by the composition: $\phi_U(t, u) := (\mathrm{ev} \circ \phi)(t, u) = \phi(u)(t)$. When the fixed local parameter system (U, ϕ) is taken for granted, we often make an abuse of notation and simply write ev instead of ϕ_U. Suppose ω is a p-form on X, then $\mathrm{ev}^*\omega$ is a p-form on $I \times U$ and has the form

$$\mathrm{ev}^*\omega = dt \wedge \alpha + \beta, \tag{4.18}$$

where α and β are respectively a $(p-1)$-form and p-form on $I \times U$, and both of them do not contain dt. Note that we have $\alpha = \iota_{\frac{d}{dt}} \mathrm{ev}^*\omega$, where $\iota_{\frac{d}{dt}}$ is the interior product with respect to the vector field $\frac{d}{dt}$ (see appendix A). Using coordinate functions$(m^1, \ldots, m^n)$ of $U \subset \mathbb{R}^n$, we can write α as

$$\alpha = \alpha_{\mu^1 \ldots \mu^{p-1}}(t, m^1, \ldots, m^n)\, dm^{\mu_1} \wedge \cdots \wedge dm^{\mu_{p-1}}. \tag{4.19}$$

From the above representation, we define the integration of $\mathrm{ev}^*\omega$ along the time direction by

$$\begin{aligned}\int_I \mathrm{ev}^*\omega &:= \int_I dt\ \iota_{\frac{d}{dt}} \mathrm{ev}^*\omega \\ &= \Big(\int_0^1 \alpha_{\mu_1 \dots \mu_{p-1}}(t, m^1, \dots, m^n) dt \Big)\ dm^{\mu_1} \wedge \cdots \wedge dm^{\mu_{p-1}}. \end{aligned} \tag{4.20}$$

This is a $(p-1)$-form on $\mathcal{P}X$. The integral above is called a *fiber integration* with respect to the projection $p : I \times \mathcal{P}X \to \mathcal{P}X$. In this way, we can obtain a $(p-1)$-form on the path space $\mathcal{P}X$ from a p-form on X:

$$\omega \mapsto \int_I \mathrm{ev}^*\omega. \tag{4.21}$$

Here and subsequently, we will abbreviate $\int_I \mathrm{ev}^*\omega$ to $\int \omega$.

The above construction is generalized to the method of iterated integrals. By this method, from $\omega_1, \dots, \omega_k \in \Omega^*(X)$, we will get $\int \omega_1 \dots \omega_k$, which is $(p_1 + \cdots + p_k - k)$-form on $\mathcal{P}X$, where $p_1, \dots, p_k \geq 1$ are the form degrees of $\omega_1, \dots, \omega_k$, respectively. We describe the detail of the construction in the following.

We denote k-direct product of X by

$$X^k := \underbrace{X \times \cdots \times X}_{k}, \tag{4.22}$$

and let $\pi_j : X^k \longrightarrow X$, $1 \leq j \leq k$, be the projections onto jth component of X^k. The *cross product* of the forms $\omega_1, \dots, \omega_k$ is a form on M^k, which is defined by

$$\omega_1 \times \cdots \times \omega_k := \pi_1^*\omega_1 \wedge \cdots \wedge \pi_k^*\omega_k. \tag{4.23}$$

Let Δ_k be a k-simplex in $\mathbb{R}^k$,

$$\Delta_k = \{(t_1, \dots, t_k) \in \mathbb{R}^k | 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}. \tag{4.24}$$

Define k-fold evaluation map

$$\varphi : \Delta_k \times \mathcal{P}X \longrightarrow X^k \tag{4.25}$$

by $\varphi(t_1,\dots,t_k;\gamma) = (\gamma(t_1),\dots,\gamma(t_k))$. By definition, we have

$$\varphi^*(\omega_1 \times \cdots \times \omega_k) = \varphi^*\pi_1^*\omega_1 \wedge \cdots \wedge \varphi^*\pi_k^*\omega_k. \tag{4.26}$$

We are now attempting to define the fiber integration with respect to the projection $\Delta_k \times \mathcal{P}X \longrightarrow \mathcal{P}X$,

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k). \tag{4.27}$$

As the previous case, it is sufficient to define it for a local parameter system $\phi : U \longrightarrow \mathcal{P}X$, where U is an open subset of $\mathbb{R}^n$, $(k \leq n)$. Consider a following sequence of the maps

$$\Delta_k \times U \xrightarrow{1\times\phi} \Delta_k \times \mathcal{P}X \xrightarrow{\varphi} \mathcal{P}X. \tag{4.28}$$

As before, we abbreviate the composition $\varphi \circ (1 \times \phi)$ simply to φ (making an abuse of notation), and we write interior products ι_j instead of $\iota_{\frac{\partial}{\partial t_j}}$, $1 \leq j \leq k$. Let us denote the pulled back forms by

$$\omega_j(t_j) := \varphi^*\pi_j^*\omega_j, \quad 1 \leq j \leq k, \tag{4.29}$$

on $\Delta_k \times U$. These $\omega_j(t_j)$ have the forms

$$\omega_j(t_j) = dt_j \wedge \alpha_j + \beta_j, \tag{4.30}$$

where α_j and β_j are $(p_j - 1)$-forms and p_j-forms on $\Delta_k \times U$, respectively. All of them do not contain any of $dt_1,\dots,dt_k$. Note that, we have $\alpha_j = \iota_j\varphi^*\pi_j^*\omega_j$. The wedge product $\alpha_1 \wedge \cdots \wedge \alpha_k$ becomes a form on $\Delta_k \times U$, whose form degree is $p_1 + \cdots + p_k - k (=: p - k)$. We can represent it as

$$\alpha_1 \wedge \cdots \wedge \alpha_k = \alpha_{\mu_1\dots\mu_{p-k}}(t_1,\dots,t_k;m^1,\dots,m^k)\ dm^{\mu_1} \wedge \cdots \wedge dm^{\mu_{p-k}}. \tag{4.31}$$

Then we define

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k) := \Big(\int_{\Delta_k} \alpha_{\mu_1\dots\mu_{p-k}} dt_1 \dots dt_k\Big) dm^{\mu_1} \wedge \cdots \wedge dm^{\mu_{p-k}}, \tag{4.32}$$

where

$$\int_{\Delta_k} \alpha_{\mu_1 \ldots \mu_{p-k}} dt_1 \ldots dt_k$$
$$:= \int_0^1 dt_k \int_0^{t_k} dt_{k-1} \ldots \int_0^{t_2} dt_1 \, \alpha_{\mu_1 \ldots \mu_{p-k}}(t_1, \ldots, t_k; m^1, \ldots, m^n). \tag{4.33}$$

Thus, starting from forms $\omega_1, \ldots, \omega_k$ on X, whose degrees are $p_1, \ldots, p_k$, respectively, we obtain a form $\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$ on the path space $\mathcal{P}X$, whose degree is $p_1 + \cdots + p_k - k$. We denote it by

$$\int \omega_1 \ldots \omega_k, \tag{4.34}$$

and call it the iterated integral of the forms $\omega_1, \ldots, \omega_k$. The iterated integral defines a multilinear map

$$I : \Omega^{p_1} \times \cdots \times \Omega^{p_k} \longrightarrow \Omega^{p_1 + \cdots + p_k - k}(\mathcal{P}X). \tag{4.35}$$

4.6 The exterior derivative of iterated integrals

Let us focus on the fixed-endpoint path space $\mathcal{P}(X; x_0, x_1)$. We will describe the action of the exterior derivative of $\mathcal{P}(X; x_0, x_1)$ on the iterated integrals.

We have the following formula [17]: Let $\omega_1, \ldots, \omega_k$ be differential forms on X and regard the iterated integral $\int \omega_1 \ldots \omega_k$ as a differential form on $\mathcal{P}(X; x_0, x_1)$. Then the exterior derivative of it is given by

$$\begin{aligned} d \int \omega_1 \ldots \omega_k & \\ &= \sum_{j=1}^{k} (-1)^{p_{j-1}+1} \int \omega_1 \ldots \omega_{j-1} d\omega_j \omega_{j+1} \ldots \omega_k \\ &+ \sum_{j=1}^{k-1} (-1)^{p_j} \int \omega_1 \ldots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \ldots \omega_k. \end{aligned} \tag{4.36}$$

In particular, for $k = 1$

$$d \int \omega = - \int d\omega, \tag{4.37}$$

and for $k = 2$

$$\begin{aligned} d\int \omega_1\omega_2 = -\int d\omega_1\omega_2 + (-1)^{p_1+1}\int \omega_1 d\omega_2 \\ +(-1)^{p_1}\int(\omega_1 \wedge \omega_2). \end{aligned} \tag{4.38}$$

4.7 An invariant $F(\gamma)$ on $\mathcal{P}(X; x_0, x_1)$

Using the formulae above we obtain a following proposition: Let ω_i, $i = 1, \ldots, k$, be closed one-forms on X and let ϕ be a one-form on X. Suppose that there exists some constants c_{ij} , and they satisfy a relation

$$\sum_{i<j} c_{ij}\omega_i \wedge \omega_j + d\phi = 0. \tag{4.39}$$

Then, for a path $\gamma \in \mathcal{P}(X; x_0, x_1)$, the iterated integral

$$F(\gamma) := \sum_{i<j}\int_\gamma c_{ij}\omega_i\omega_j + \int_\gamma \phi \tag{4.40}$$

is depend only on the homotopy class of γ. The proof is straight forward by using (4.37) and (4.38).

$$\begin{aligned} dF(\gamma) &= \sum_{i<j} c_{ij}\, d\int_\gamma \omega_i\omega_j + d\int_\gamma \phi \\ &= -\sum_{i<j} c_{ij}\int_\gamma (\omega_i \wedge \omega_j) - \int_\gamma d\phi \\ &= -\int_\gamma \Big(\underbrace{\sum_{i<j} c_{ij}\omega_i \wedge \omega_j + d\phi}_{=0.}\Big) \\ &= 0. \end{aligned} \tag{4.41}$$

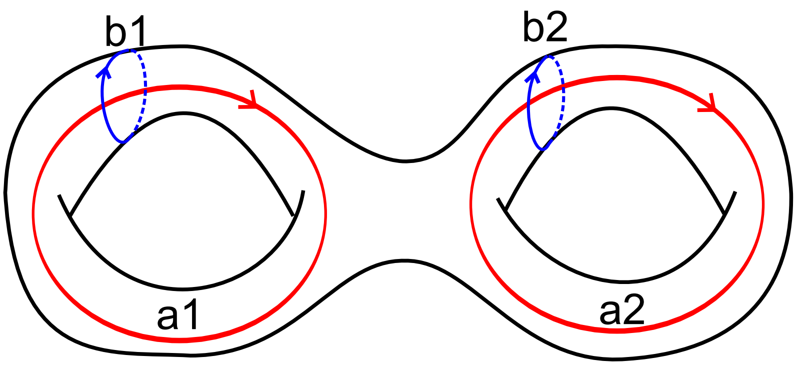
This means $F(\gamma)$ is locally constant on the path space $\mathcal{P}(X; x_0, x_1)$, i.e., it is invariant under continuous deformations of the path γ.

4.8 An example of $F(\gamma)$

Here we construct the easiest non-trivial example of $F(\gamma)$. We choose $X = \Sigma_2$, that is, the Riemann surface with the genus two. The cohomology of Σ_2 is given by

$$\begin{aligned} H^*(\Sigma_2) &= H^0(\Sigma_2) \oplus H^1(\Sigma_2) \oplus H^2(\Sigma_2) \\ &= \mathbb{Z} \oplus \mathbb{Z}^4 \oplus \mathbb{Z} . \end{aligned} \tag{4.42}$$

Hence, we have four closed two-forms. They are dual to the four cycles a_1, b_1, a_2, b_2 on Σ_2. We can choose the four closed one-forms $\theta_1, \eta_1, \theta_2, \eta_2$ representing the four



cohomology classes of $H^1(\Sigma_2)$ so that they satisfy

$$\begin{aligned} &\int_{a_i} \theta_j = \int_{b_i} \eta_j = \delta_{i,j} \\ &\int_{b_i} \theta_j = \int_{a_i} \eta_j = 0, \quad i, j = 1, 2. \end{aligned} \tag{4.43}$$

On the other hand, the second cohomology class $H^2(\Sigma_2)$ is generated by a volume two-form ω_{Σ_2}. It is clear that as the cohomology class, the equalities

$$[\theta_1 \wedge \eta_1] = [\theta_2 \wedge \eta_2] = [\omega_{\Sigma_2}] \tag{4.44}$$

hold. Hence the difference between $\theta_1 \wedge \eta_1$ and $\theta_2 \wedge \eta_2$ must be a exact form,

$$\theta_1 \wedge \eta_1 - \theta_2 \wedge \eta_2 = -d\phi, \tag{4.45}$$

where $-\phi$ is some one-form on Σ_2. Therefore we obtain an invariant on $\mathcal{P}(\Sigma_2; x_0, x_1)$,

$$F(\gamma) = \int_\gamma (\theta_1\eta_1 - \theta_2\eta_2) + \int_\gamma \phi. \tag{4.46}$$

In the later chapter, we will regard this invariant as an observable of quantum mechanics.

4.9 Moduli space as a subspace of the path space

In this section we describe a notion of a space of gradient trajectories of a Morse function. This space is defined as a subspace of the path space. This space will be important in the later chapter, where we consider a topological quantum mechanics sigma model.

Let us choose a Morse function f on X, that is, a smooth real function on X with a finite number of isolated critical points. Let us consider solutions of the following equation of the gradient flow of the Morse function f:

$$\frac{d\gamma^\mu(t)}{dt} = g^{\mu\nu}\partial_\nu f(\gamma(t)). \tag{4.47}$$

We call the solutions of this equation gradient trajectories of the Morse function f. If we define $s^\mu(\gamma) := \frac{d\gamma^\mu}{dt} - g^{\mu\nu}\partial_\nu f(\gamma)$, this can be regarded as a section of the tangent bundle $T\mathcal{P}X$ of the path space, i.e., $s : \mathcal{P}X \to T\mathcal{P}X$. We denote the space of the gradient trajectories as

$$\mathcal{M} := \{\gamma \in s^{-1}(0)\} = \left\{\gamma \in \mathcal{P}X \middle| \frac{d\gamma^\mu(t)}{dt} = g^{\mu\nu}\partial_\nu f(\gamma(t))\right\}. \tag{4.48}$$

This is a subspace of the path space $\mathcal{P}X$. We call $\mathcal{M}$ a moduli space. This is because in the later chapter, $\mathcal{M}$ will be regarded as the space of solutions of the instanton equation of the topological quantum mechanics.

Since we are considering a compact manifold X, a gradient trajectory must start from a critical point of f at time $t = -\infty$ and end at another critical point at time $t = +\infty$. Therefore the moduli space $\mathcal{M}$ consists of disjoint union of connected components $\mathcal{M}_{x_-,x_+}$:

$$\mathcal{M} = \bigcup_{x_-,x_+ \in \text{crit.pts.}} \mathcal{M}_{x_-,x_+}, \tag{4.49}$$

where, $\mathcal{M}_{x_-,x_+}$ is a space of gradient trajectories that starts from $\gamma(-\infty) = x_-$ and end at $\gamma(\infty) = x_+$, and $x_\pm$ are critical points of f. A theorem of Morse theory says that the dimension of each connected component of the moduli space $\mathcal{M}$ is given by

$$\dim \mathcal{M}_{x_-,x_+} = n_{x_+} - n_{x_-}, \tag{4.50}$$

where, $n_{x_\pm}$ are the Morse indices of the critical points $x_\pm$.

Now let us focus on one component of the moduli space, say, $\mathcal{M}_{x_-,x_+}$. This is a finite dimensional subspace of $\mathcal{P}(X; x_-, x_+)$, the space of paths with the both endpoints fixed. Let us denote the inclusion map by i,

$$i : \mathcal{M}_{x_-,x_+} \hookrightarrow \mathcal{P}(X; x_-, x_+). \tag{4.51}$$

Let us choose a local coordinate patch (U, ϕ) of $\mathcal{M}_{x_-,x_+}$,

$$\phi : U \xrightarrow{\simeq} \mathcal{M}_{x_-,x_+}, \tag{4.52}$$

where, U is an open subset of $\mathbb{R}^{\dim \mathcal{M}_{n_-,n_+}}$ and ϕ is a diffeomorphism. Then we can regard $(U, i \circ \phi)$ as a local parameter system of $\mathcal{P}(X; x_-, x_+)$. It covers the direction transverse to the moduli space $\mathcal{M}_{x_-,x_+}$. Composing these maps and the evaluation map, we can pullback differential forms on X to ones on $\mathcal{M}_{x_-,x_+}$ and represent them on the local coordinate of $\mathcal{M}_{x_-,x_+}$:

$$\begin{gathered} U \xrightarrow{\phi} \mathcal{M}_{x_-,x_+} \overset{i}{\hookrightarrow} \mathcal{P}(X; x_-, x_+) \xrightarrow{\mathrm{ev}_t} X \\ \Downarrow \\ \Omega^*(U) \xleftarrow{\phi^*} \Omega^*(\mathcal{M}_{x_-,x_+}) \xleftarrow{i^*} \Omega^*(\mathcal{P}(X; x_-, x_+)) \xleftarrow{\mathrm{ev}_t^*} \Omega^*(X). \end{gathered} \tag{4.53}$$

Example: $X = \mathbb{CP}^1$

As an example, let us construct some differential forms on the moduli space in case of a sphere $X = \mathbb{CP}^1$.

As in the section 2.6, we choose the metric g and the Morse function f on $\mathbb{CP}^1$ as,

$$g = \frac{dzd\bar{z}}{(1+z\bar{z})^2}, \tag{4.54}$$

and

$$f = \frac{1}{4}\frac{z\bar{z}-1}{z\bar{z}+1}. \tag{4.55}$$

Therefore, the corresponding gradient vector field is given by

$$\nabla f = z\partial_z + \bar{z}\partial_{\bar{z}}, \tag{4.56}$$

so that the instanton equations are given by

$$\frac{dz(t)}{dt} = z, \quad \frac{dz(t)}{dt} = \bar{z}. \tag{4.57}$$

These equations are easily solved and we get the solutions

$$z(t) = e^t m, \quad \bar{z}(t) = e^t \overline{m}. \tag{4.58}$$

Since these solutions are the complex conjugates to each other, it is sufficient to consider the one of them, say,

$$z(t) = e^t m. \tag{4.59}$$

Here, m is a point on $\mathbb{CP}^1$ where the flow $z(t)$ passes by at the time $t = 0$. It is given as the initial condition of the differential equations (4.57). If we choose $m \neq 0, \infty$, then the flow starts from $z(-\infty) = 0$ and ends at $z(\infty) = \infty$. Thus, $\mathcal{M}_{0,\infty} \cong \mathbb{CP}^1 - \{0, \infty\}$. If we choose $m = 0$, the flow is a constant map $z(t) \equiv 0$ and if $m = \infty$, then $z(t) \equiv \infty$. Thus, $\mathcal{M}_{0,0} = \{0\}$ and $\mathcal{M}_{\infty,\infty} = \{\infty\}$. In summary, we find the moduli space to be

$$\mathcal{M} = \mathcal{M}_{0,\infty} \cup \mathcal{M}_{0,0} \cup \mathcal{M}_{0,\infty} = \Big(\mathbb{CP}^1 - \{0,\infty\}\Big) \cup \{0\} \cup \{\infty\} \cong \mathbb{CP}^1. \tag{4.60}$$

and they have real dimensions $\dim \mathcal{M}_{0,\infty} = 2$, $\dim \mathcal{M}_{0,0} = 0$, and $\dim \mathcal{M}_{\infty,\infty} = 0$. Note that we can regard m as a local coordinate of the moduli space $\mathcal{M} \cong \mathbb{CP}^1$. This local coordinate corresponds the map ϕ in the description above,

$$\phi : \mathbb{CP}^1 \xrightarrow{\simeq} \mathcal{M}, \quad \phi(m) \mapsto z(t) = e^t m. \tag{4.61}$$

Next, we construct some differential forms on $\mathcal{M}$. Consider the following sequence of maps:

$$\mathbb{CP}^1 \xrightarrow{\phi} \mathcal{M} \overset{i}{\hookrightarrow} \mathcal{P}X \overset{\mathrm{ev}_t}{\to} X = \mathbb{CP}^1, \tag{4.62}$$

where, $i : \mathcal{M} \hookrightarrow \mathcal{P}X$ is the inclusion map. This sequence maps $m \in \mathbb{CP}^1 \cong \mathcal{M}$ to $z(t) = e^t m \in X$. Using this sequence of maps, we can pullback forms on X to forms on $\mathcal{M}$. For example, let us consider the Kähler two-form ω_2 on $X = \mathbb{CP}^1$. The explicit form of ω_2 is given by

$$\omega_2(z,\bar{z}) = \frac{idz \wedge d\bar{z}}{(1+z\bar{z})^2} \quad \in \Omega^2(X). \tag{4.63}$$

We can compute the pulled-back form as

$$\begin{aligned}(\hat{\omega}_2(t))(m,\overline{m}) &:= (\phi^* i^* \mathrm{ev}_t^* \omega_2)(m,\overline{m}) \equiv (\mathrm{ev}_t^* \omega_2)(m,\overline{m}) \\ &= \frac{i(e^t dm) \wedge (e^t d\overline{m})}{(1+(e^t m)(e^t \overline{m}))^2} \\ &= \frac{ie^{2t} dm \wedge d\overline{m}}{(1+e^{2t} m\overline{m})^2} \quad \in \Omega^2(\mathcal{M}).\end{aligned} \tag{4.64}$$

In the second equality, we abbreviate $(\mathrm{ev} \circ i \circ \phi)$ to ev. Similarly, by using the evaluation map with the time unfixed, $\mathrm{ev} : \mathbb{R} \times \mathcal{P}X \to X$, and by performing the fiber integration, we can construct a one-form $\int \omega_2$ on $\mathcal{M}$ as follows:

First, the pullback of ω_2 to $\mathbb{R} \times \mathcal{M}$ is computed as

$$\begin{aligned}(\phi^* i^* \mathrm{ev}^* \omega_2)(t;m,\overline{m}) &\equiv (\mathrm{ev}^* \omega_2)(t;m,\overline{m}) \\ &= \frac{id(e^t m) \wedge d(e^t m)}{(1+(e^t m)(e^t \overline{m}))^2} \\ &= \frac{ie^{2t}}{(1+e^{2t} m\overline{m})^2} \left\{ dm \wedge d\overline{m} - \overline{m} dt \wedge dm + m dt \wedge d\overline{m} \right\}.\end{aligned}$$

Then take the coefficients of dt by the interior product $\iota_{\frac{d}{dt}}$,

$$\alpha(t;m,\overline{m}) := (\iota_{\frac{d}{dt}} \mathrm{ev}^* \omega_2)(t;m,\overline{m}) = \frac{ie^{2t}}{(1+e^{2t} m\overline{m})^2}(-m d\overline{m} + m d\overline{m}). \tag{4.65}$$

Finally, by integrating it along the time direction we get a one-form on the moduli space,

$$\begin{aligned}\int_{z(t)=e^t m} \omega_2 &:= \int_{-\infty}^{\infty} dt\ \alpha(t, m, \overline{m}) \\ &= \int_{-\infty}^{\infty} dt \frac{ie^{2t}}{(1+e^{2t}m\overline{m})^2}(-\overline{m}dm + md\overline{m}) \\ &= \frac{i}{2}\left(-\frac{dm}{m} + \frac{d\overline{m}}{\overline{m}}\right) \quad \in \Omega^1(\mathcal{M}). \end{aligned} \tag{4.66}$$

This computation can be straightforwardly generalized to the iterated integrals.

Chapter 5

Topological quantum mechanics

In this chapter, first we construct the topological quantum mechanics on a Riemaniann manifold by the Mathai-Quillen formalism. Then in section 5.2 We construct new observables of the topological quantum mechanics by using the method of the iterated integrals and consider a correlation function that contains the new observable. The section 5.2 is one of our new result in this thesis.

In section 5.3 and 5.4 we review the Hamiltonian formalism of the topological quantum mechanics along the line of FLN [10].

5.1 Construction of topological quantum mechanics

Applying the Mathai-Quillen formalism, we construct a topological theory of quantum mechanics. Let X be a compact oriented n-dimensional Riemannian manifold equipped with a metric g. Let us choose a Morse function f on X and fix it:

$$f : X \longrightarrow \mathbb{R}. \tag{5.1}$$

- The space of fields is the path space $\mathcal{P}X$.
- Choose the vector bundle on the path space to be the tangent bundle $T\mathcal{P}X$.
- Choose the section of the tangent bundle to be gradient flow equation of the Morse function on X.

$$s^{\mu}(x(t)) := \frac{dx^{\mu}(t)}{dt} - g^{\mu\nu}\frac{\partial f(x(t))}{\partial x^{\nu}}. \tag{5.2}$$

The instanton moduli space $\mathcal{M} \subset \mathcal{P}X$ is therefore the space of gradient trajectories of the Morse function,

$$\begin{aligned}\mathcal{M} &= \{x \in s^{-1}(0)\} \\ &= \left\{x \in \mathcal{P}X \;\middle|\; \frac{dx^\mu(t)}{dt} = g^{\mu\nu}\partial_\nu f(x(t))\right\}.\end{aligned} \tag{5.3}$$

As we described in the previous chapter, since the gradient trajectories must start from a critical point and end at a critical point, the moduli space consists of disjoint components,

$$\mathcal{M} = \bigcup_{x_-,x_+ \in \text{crit.pts.}} \mathcal{M}_{x_-,x_+}. \tag{5.4}$$

The action has a Q-exact form.

$$S_\lambda = Q\,\Psi_\lambda, \tag{5.5}$$

where,

$$\Psi_\lambda = \int dt\; \pi_\mu(t)\left[-is^\mu(x(t)) + \frac{1}{2\lambda}g^{\mu\nu}p_\nu(t)\right], \quad \lambda \in \mathbb{R}_+. \tag{5.6}$$

Recall that x^μ and p_μ are bosonic variables, while ψ^μ and π_μ are fermionic variables. λ is a real positive parameter. The Q-action on these fields is defined by

$$Qx^\mu = \psi^\mu, \tag{5.7}$$

$$Q\pi_\mu = p_\mu - \Gamma^\nu_{\mu\rho}\pi_\nu\psi^\rho, \tag{5.8}$$

$$Q\psi^\mu = 0, \tag{5.9}$$

$$Qp_\mu = \Gamma^\nu_{\mu\rho}p_\nu\psi^\rho - \frac{1}{2}R^\nu{}_{\mu\rho\sigma}\pi_\nu\psi^\rho\psi^\sigma, \tag{5.10}$$

where, $\Gamma^\nu_{\mu\rho}$ is the Levi-Civita connection associated with the Riemannian metric g, and $R^\nu{}_{\mu\rho\sigma}$ is its curvature tensor. We can show straightforwardly that Q is nilpotent, $Q^2 = 0$. The action is

$$S_\lambda = \int dt\;\left[-ip_\mu s^\mu(x) + \frac{1}{2\lambda}g^{\mu\nu}p_\mu p_\nu + i\pi_\mu D_\nu s^\mu(x)\psi^\nu + \frac{1}{4\lambda}R^{\mu\nu}{}_{\rho\sigma}\pi_\mu\pi_\nu\psi^\rho\psi^\sigma\right]. \tag{5.11}$$

Substituting (5.2), we get

$$
\begin{aligned}
S_\lambda = \int dt \Bigg[& -ip_\mu \left(\frac{dx^\mu}{dt} - g^{\mu\nu}\partial_\nu f \right) + \frac{1}{2\lambda} g^{\mu\nu} p_\mu p_\nu \\
& + i\pi_\mu \left(D_t\psi^\mu - g^{\mu\nu} \frac{D^2 f}{Dx^\mu Dx^\alpha} \psi^\alpha \right) + \frac{1}{4\lambda} R^{\mu\nu}{}_{\rho\sigma} \pi_\mu \pi_\nu \psi^\rho \psi^\sigma \Bigg],
\end{aligned}
\tag{5.12}
$$

where,

$$
D_t\psi^\mu(t) = \frac{d}{dt}\psi^\mu(t) + \Gamma^\mu_{\nu\rho}\dot{x}^\nu(t)\psi^\rho(t). \tag{5.13}
$$

If we integrate out the auxiliary filed p_μ, we get

$$
\begin{aligned}
S_\lambda = \int dt \Bigg[& \frac{1}{2}\lambda g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{1}{2}\lambda g^{\mu\nu}\partial_\mu f \partial_\nu f \\
& + i\pi_\mu D_t\psi^\mu - i g^{\mu\nu}\pi_\mu D_\alpha(\partial_\nu f)\psi^\alpha + \frac{1}{4\lambda} R^{\mu\nu}{}_{\rho\sigma}\pi_\mu\pi_\nu\psi^\rho\psi^\sigma \Bigg] \\
& - \lambda \int dt\, \frac{df}{dt} \quad .
\end{aligned}
\tag{5.14}
$$

Note that this action differs from the physical action (2.8) by a topological term $-\lambda \int df$, that is,

$$
S_\lambda = S_{\text{phys}} - \lambda \int dt \frac{df}{dt}. \tag{5.15}
$$

The correlation function of the theory are given by path integrals:

$$
\begin{aligned}
& \langle x_f | e^{(t_n - t_f)H} \mathcal{O}_n e^{(t_{n-1}-t_n)H} \dots e^{(t_1 - t_2)H} \mathcal{O}_1 e^{(t_i - 1)H} | x_i \rangle = \\
& \int_{x(t_i)=x_i}^{x(t_f)=x_f} DxD\psi DpD\pi \mathcal{O}_1(t_1)\dots\mathcal{O}_n(t_n)\; e^{-S_\lambda}.
\end{aligned}
\tag{5.16}
$$

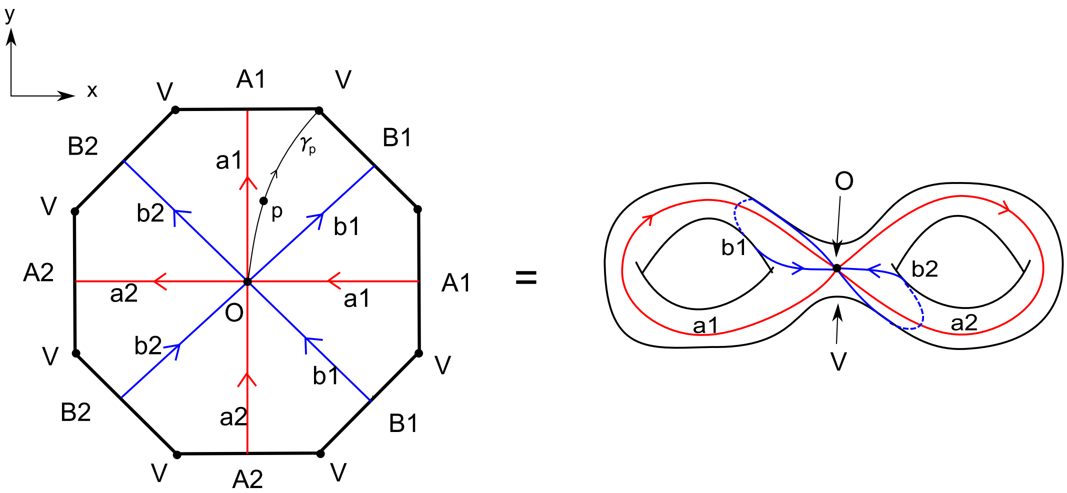
If $\langle\Psi'|$ and $|\Psi\rangle$ are states in the Hilbert space of the theory, the wave functions corresponding them are given by definition, $\Psi'(x) = \langle\Psi'|x\rangle$ and $\Psi(x) = \langle x|\Psi\rangle$. Therefore, the correlation function between these states is given by

$$
\begin{aligned}
& \langle \Psi' | e^{(t_n - t_f)H} \mathcal{O}_n e^{(t_{n-1}-t_n)H} \dots e^{(t_1 - t_2)H} \mathcal{O}_1 e^{(t_i - 1)H} | \Psi \rangle = \\
& \int_{X^2} \Psi'(x_f)\Psi(x_i) \int_{x(t_i)=x_i}^{x(t_f)=x_f} DxD\psi DpD\pi \mathcal{O}_1(t_1)\dots\mathcal{O}_n(t_n)\; e^{-S_\lambda}.
\end{aligned}
\tag{5.17}
$$

5.2 A correlation function with a new observable

We have described that in the Mathai-Quillen formalism of the CohFT, the BPS observables are given by closed differential form on the instanton moduli space. In our case, the moduli space is given by the space of gradient trajectories (5.3), (5.4). Let us focus on one component of $\mathcal{M}$, say, $\mathcal{M}_{x_-,x+}$. This is a subspace of the fixed-endpoint path space $\mathcal{P}(X;x_-,x_+)$. Therefore, we can regard the iterated integral as the observable of the topological quantum mechanics. This is one of our new proposals in this thesis. In particular, the invariant $F(\gamma)$ that we have described in the section 4.7 is closed zero-form on $\mathcal{P}(X;x_-,x_+)$, so that we can regard it as a new BPS observable of the topological quantum mechanics. Note that since $\mathcal{M}_{x_-,x_+}$ is a (finite dimensional) subspace of $\mathcal{P}(X;x_-,x_+)$, the restriction of $F(\gamma)$ onto $\mathcal{M}_{x_-,x_+}$ is of course a closed zero-form, too. Therefore, BPS correlation functions that contain $F(\gamma)$ are given by integrals of closed forms on $\mathcal{M}$ and they will be topological invariants of the theory.

In the following, we choose the target space as $X = \Sigma_2$ and we compute a correlation function that contains $F(\gamma)$ which we have constructed in the section 4.8. First we represent Σ_2 as an octagon with its edges are identified as the figure.



Let us choose a Morse function f on Σ_2 as follows: we define the value of f at given point $p \in \Sigma_2$ by

$$f(p) = |p - O|. \tag{5.18}$$

Then the origin O is a critical point of f with the Morse index zero, and the vertex V is a critical point with the Morse index two. The middle points of the edges A_1, A_2, B_1, B_2 are the critical points with the Morse index one.

We focus on the space of the gradient trajectories of f which start from O and end at V. We denote this space by $\mathcal{M}_{O,V}$. The space $\mathcal{M}_{O,V}$ has eight connected components so that $F(\gamma)$ can give different values for different components. But, it must be a constant on each component.

Let us consider a following BPS correlation function,

$$_V\langle\hat{\omega}(t)F(\gamma)\rangle_O = \int_{\mathcal{M}_{O,V}} \mathrm{ev}_t^* \; \omega \wedge F(\gamma), \tag{5.19}$$

where, $\omega := \delta^{(2)}(x - x_0, y - y_0) dx \wedge dy$ is a delta-two-form that has a support on a point $(x_0, y_0) =: p \in \Sigma_2$. Suppose that the point p is not on any of the lines a_1, a_2, b_1, b_2. Then, there exists unique gradient trajectory γ_p that passes p at the time t, that is, $\gamma_p(t) = p$ holds. The integral (5.19) is contributed only from this γ_p. So the value of the correlation function is proportional to $F(\gamma_p)$,

$$_V\langle\hat{\omega}(t)F(\gamma)\rangle_O \propto F(\gamma_p). \tag{5.20}$$

When the point p moves around on Σ_2, this value does not change as long as the corresponding γ_p stays in one component of $\mathcal{M}_{O,V}$, but it can change if γ_p go into a different component.

In order to compute the actual value of $F(\gamma)$ for a given γ, we need some more efforts. We leave it as a future problem.

5.3 Hamiltonian formalism

In this section, we will review a Hamiltonian formalism of our topological action. This means that we need to define the space of states of the model and realize our observables as linear operators acting on this space of states. Since our topological action differs from the physical action only by a topological term $-\lambda \int df$, we expect that the space of states of the topological theory could be realized as the same space of state as the physical theory, which is the space of differential forms on X.

The key formula for this construction is (5.17) that relates the Hamiltonian formalism matrix element with the Lagrangian formalism. Suppose, we would like to compute a topological theory matrix element of an observable $\widetilde{\mathcal{O}}(t) = e^{tH}\widetilde{\mathcal{O}}e^{-tH}$ between the states $\langle\widetilde{\Psi}'|$ and $|\widetilde{\Psi}\rangle$. We denote it as $\langle\widetilde{\Psi}'|e^{tH}\widetilde{\mathcal{O}}e^{-tH}|\widetilde{\Psi}\rangle_{\text{top}}$. Then the formula (5.17) says,

$$\begin{aligned}
&\langle\widetilde{\Psi}'|e^{tH}\widetilde{\mathcal{O}}e^{-tH}|\widetilde{\Psi}\rangle_{\text{top}} \\
&= \int_{X^2}\widetilde{\Psi}'(x_f)\widetilde{\Psi}(x_i)\int_{x(-\infty)=x_i}^{x(+\infty)=x_f}\widetilde{\mathcal{O}}(t)e^{-S_{\text{phys}}+\lambda\int_{-\infty}^{+\infty}dt\frac{df}{dt}} \\
&= \int_{X^2}\int_{x(-\infty)=x_i}^{x(+\infty)=x_f}\left[e^{\lambda f(x_f)}\widetilde{\Psi}'(x_f)\right]\ \left[e^{-\lambda f(x(t))}\widetilde{\mathcal{O}}(t)e^{+\lambda f(x(t))}\right]\left[e^{-\lambda f(x_i)}\widetilde{\Psi}(x_i)\right]\ e^{-S_{\text{phys}}} \\
&= \left\langle e^{\lambda f(x_f)}\widetilde{\Psi}'(x_f)\middle|e^{-\lambda f(x(t))}\widetilde{\mathcal{O}}(t)e^{+\lambda f(x(t))}\middle|e^{-\lambda f(x_i)}\widetilde{\Psi}(x_i)\right\rangle_{\text{phys}}. \qquad (5.21)
\end{aligned}$$

In the third line of (5.21), we have decomposed the integral $\lambda\int dt\frac{df}{dt}$ as

$$\begin{aligned}
\lambda\int_{-\infty}^{+\infty}\frac{df}{dt} &= \lambda\int_t^{\infty}dt\frac{df}{dt} + \lambda\int_{-\infty}^{t}dt\frac{df}{dt} \\
&= (\lambda f(x_f) - \lambda f(x(t)) + (\lambda f(x(t)) - \lambda f(x_i)). \qquad (5.22)
\end{aligned}$$

This is because, in the path integral, the observables should be arranged in the time order.

Therefore, if we define

$$\Psi' := e^{\lambda f}\widetilde{\Psi}',\ \ \Psi := e^{-\lambda f}\widetilde{\Psi},\ \ \mathcal{O}(t) := e^{-\lambda f}\widetilde{\mathcal{O}}(t)e^{\lambda f}, \qquad (5.23)$$

we obtain,

$$\langle\widetilde{\Psi}'|\widetilde{\mathcal{O}}(t)|\widetilde{\Psi}\rangle_{\text{top}} = \langle\Psi'|\mathcal{O}(t)|\Psi\rangle_{\text{phys}}. \qquad (5.24)$$

This means that the topological theory correlation function of $\widetilde{\mathcal{O}}(t) = e^{\lambda f(x(t))}\mathcal{O}(t)e^{-\lambda f(x(t))}$ between the "in" state $\widetilde{\Psi}(x_i) = e^{\lambda f(x_i)}\Psi(x_i)$ and the "out" state $\widetilde{\Psi}'(x_f) = e^{\lambda f(x_f)}\Psi'(x_f)$ is the same as the physical theory correlation function of $\mathcal{O}(t)$ between the states $\Psi(x_i)$ and $\Psi'(x_f)$. Thus, we can construct the matrix elements of the Hamiltonian formalism in the topological theory, by means of the physical theory matrix elements. The dictionary that interprets between the topological theory and physical theory is as follows:

- the "in" states get multiplied by $e^{\lambda f}$:

$$\Psi \mapsto \widetilde{\Psi}^{\text{in}} = e^{\lambda f}\Psi; \tag{5.25}$$

- the "out" states get multiplied by $e^{-\lambda f}$:

$$\Psi' \mapsto \widetilde{\Psi}'^{\text{out}} = e^{-\lambda f}\Psi'; \tag{5.26}$$

- the operators get conjugated:

$$\mathcal{O} \mapsto \widetilde{\mathcal{O}} = e^{\lambda f}\mathcal{O}e^{-\lambda f}. \tag{5.27}$$

Now we can find the topological theory operators corresponding $\mathcal{Q}$, $\mathcal{Q}^*$ and H_λ (see (2.11), (2.12), (2.13)). They are given by,

$$\widetilde{Q}_\lambda = e^{\lambda f}\mathcal{Q}e^{-\lambda f} = d, \tag{5.28}$$

$$\widetilde{Q}^*_\lambda = e^{\lambda f}\mathcal{Q}^*e^{-\lambda f} = 2\iota_{\nabla f} + \frac{1}{\lambda}d^*, \tag{5.29}$$

$$\widetilde{H}_\lambda = e^{\lambda f}H_\lambda e^{-\lambda f} = \frac{1}{2}\{\widetilde{Q}_\lambda, \widetilde{Q}^*_\lambda\} = \mathcal{L}_{\nabla f} - \frac{1}{2\lambda}\Delta. \tag{5.30}$$

5.4 The $\mathbb{CP}^1$ model

We apply the consideration above to the $\mathbb{CP}^1$ model that we described in the section 2.6. In this model recall that we have two towers of the states $\{{}_0\Psi_{n,\overline{n},p,\overline{p}}\}$ and $\{{}_\infty\Psi_{n,\overline{n},p,\overline{p}}\}$ that are localized around the point 0 and ∞, respectively. We get "in" states and "out" states from each of them,

$$\begin{aligned} {}_0\widetilde{\Psi}^{\text{in}}_{n,\overline{n},p,\overline{p}} &= e^{\lambda f}{}_0\Psi_{n,\overline{n},p,\overline{p}}, \quad {}_0\widetilde{\Psi}^{\text{out}}_{n,\overline{n},p,\overline{p}} = e^{-\lambda f}{}_0\Psi_{n,\overline{n},p,\overline{p}}, \\ {}_\infty\widetilde{\Psi}^{\text{in}}_{n,\overline{n},p,\overline{p}} &= e^{\lambda f}{}_\infty\Psi_{n,\overline{n},p,\overline{p}}, \quad {}_\infty\widetilde{\Psi}^{\text{out}}_{n,\overline{n},p,\overline{p}} = e^{-\lambda f}{}_\infty\Psi_{n,\overline{n},p,\overline{p}}. \end{aligned} \tag{5.31}$$

Substituting the expression (2.39) and (2.40) into the above and renormalizing the resulting states, we find them to be

$$
\begin{aligned}
{}_0\widetilde{\Psi}^{\text{in}}_{n,\overline{n},p,\overline{p}} &= (z^n \overline{z}^{\overline{n}})(dz)^p (d\overline{z})^{\overline{p}} + \mathcal{O}(\lambda^{-1}), && (5.32)\\
{}_0\widetilde{\Psi}^{\text{out}}_{n,\overline{n},p,\overline{p}} &= \frac{\lambda}{\pi(e^{\lambda/2} - e^{-\lambda/2})} \frac{1}{n!\overline{n}!} \partial_z^n \partial_{\overline{z}}^{\overline{n}}\, e^{-2\lambda f} (dz)^p (d\overline{z})^{\overline{p}} + \mathcal{O}(\lambda^{-1}). && (5.33)\\
{}_\infty\widetilde{\Psi}^{\text{in}}_{n,\overline{n},p,\overline{p}} &= \frac{\lambda}{\pi(e^{\lambda/2} - e^{-\lambda/2})} \frac{1}{n!\overline{n}!} \partial_w^n \partial_{\overline{w}}^{\overline{n}}\, e^{2\lambda f} (dw)^p (d\overline{w})^{\overline{p}} + \mathcal{O}(\lambda^{-1}), && (5.34)\\
{}_\infty\widetilde{\Psi}^{\text{out}}_{n,\overline{n},p,\overline{p}} &= w^n \overline{w}^{\overline{n}} (dw)^p (d\overline{w})^{\overline{p}} + \mathcal{O}(\lambda^{-1}). && (5.35)
\end{aligned}
$$

They take following forms in the limit $\lambda \to \infty$,

$$
\begin{aligned}
|n, \overline{n}, p, \overline{p}\rangle_{\mathbb{C}_0} &:= \lim_{\lambda\to\infty} {}_0\widetilde{\Psi}^{\text{in}}_{n,\overline{n},p,\overline{p}} = z^n \overline{z}^{\overline{n}}, && (5.36)\\
{}_0\langle n, \overline{n}, p, \overline{p}| &:= \lim_{\lambda\to\infty} {}_0\widetilde{\Psi}^{\text{out}}_{n,\overline{n},p,\overline{p}} = \frac{(-1)^{n+\overline{n}}}{n!\overline{n}!} \partial_z^n \partial_{\overline{z}}^{\overline{n}}\, \delta^{(2)}(z, \overline{z})(dz)^p (d\overline{z})^{\overline{p}}, && (5.37)\\
|n, \overline{n}, p, \overline{p}\rangle_\infty &:= \lim_{\lambda\to\infty} {}_\infty\widetilde{\Psi}^{\text{in}}_{n,\overline{n},p,\overline{p}} = \frac{1}{n!\overline{n}!} \partial_w^n \partial_{\overline{w}}^{\overline{n}}\, \delta^{(2)}(w, \overline{w})(dw)^p (d\overline{w})^{\overline{p}}, && (5.38)\\
{}_{\mathbb{C}_\infty}\langle n, \overline{n}, p, \overline{p}| &:= \lim_{\lambda\to\infty} {}_\infty\widetilde{\Psi}^{\text{out}}_{n,\overline{n},p,\overline{p}} = w^n \overline{w}^{\overline{n}}. && (5.39)
\end{aligned}
$$

Therefore, in this limit, the space of "in" states is isomorphic to the direct sum of two subspaces attached naturally to the critical points:

$$
\mathcal{H}^{\text{in}} \simeq \mathcal{H}^{\text{in}}_{\mathbb{C}_0} \oplus \mathcal{H}^{\text{in}}_\infty, \tag{5.40}
$$

where, $\mathcal{H}^{\text{in}}_{\mathbb{C}_0}$ is the space of the polynomial differential forms on $\mathbb{C}_0 := \mathbb{CP}^1 - \{\infty\}$:

$$
\mathcal{H}^{\text{in}}_{\mathbb{C}_0} := \mathbb{C}[z, \overline{z}] \otimes \Lambda[dz, d\overline{z}] = \text{Span}_{\mathbb{C}}\{\ |n, \overline{n}, p, \overline{p}\rangle_{\mathbb{C}_0}\ \}, \tag{5.41}
$$

and $\mathcal{H}^{\text{in}}_\infty$ is the space of the differentials of the delta-forms supported on $z = w^{-1} = \infty$:

$$
\mathcal{H}^{\text{in}}_\infty := \mathbb{C}[\partial_w, \partial_{\overline{w}}]\delta^{(2)}(w, \overline{w}) \otimes \Lambda[dw, d\overline{w}] = \text{Span}_{\mathbb{C}}\{\ |n, \overline{n}, p, \overline{p}\rangle_\infty\ \}. \tag{5.42}
$$

Similarly, the out space is isomorphic to the direct sum

$$
\mathcal{H}^{\text{out}} \simeq \mathcal{H}^{\text{out}}_{\mathbb{C}_\infty} \oplus \mathcal{H}^{\text{out}}_0, \tag{5.43}
$$

where, $\mathcal{H}^{\text{out}}_{\mathbb{C}_\infty}$ is the space of the polynomial differential forms on $\mathbb{C}_\infty := \mathbb{CP}^1 - \{0\}$:

$$\mathcal{H}^{\text{out}}_{\mathbb{C}_\infty} := \mathbb{C}[w,\overline{w}] \otimes \Lambda[dw, d\overline{w}] = \text{Span}_{\mathbb{C}}\{ \ {}_{\mathbb{C}_\infty}\langle n,\overline{n},p,\overline{p}| \ \}, \tag{5.44}$$

and $\mathcal{H}^{\text{out}}_0$ is the space of the differentials of the delta-forms supported on $z = 0$:

$$\mathcal{H}^{\text{out}}_0 := \mathbb{C}[\partial_z\partial_{\overline{z}}]\delta^{(2)}(z,\overline{z}) \otimes \Lambda[dz, d\overline{z}] = \text{Span}_{\mathbb{C}}\{ \ {}_0\langle n,\overline{n},p,\overline{p}| \ \}. \tag{5.45}$$

We can show that the basis

$$\{ \ |n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0}, \ \ |n,\overline{n},p,\overline{p}\rangle_\infty \ \} \quad \text{and} \quad \{ \ {}_0\langle n,\overline{n},p,\overline{p}|, \ \ {}_{\mathbb{C}_\infty}\langle n,\overline{n},p,\overline{p}| \} \tag{5.46}$$

are dual to each other up to a power of i (see [10]):

$$\begin{aligned} {}_0\langle m,\overline{m},r,\overline{r}|n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0} &= {}_{\mathbb{C}_\infty}\langle n,\overline{n},p,\overline{p}|n,\overline{n},p,\overline{p}\rangle_\infty \\ &= (-i)^p i^{\overline{p}} (-1)^{p\overline{r}} \delta_{n,m}\delta_{\overline{n},\overline{m}}\delta_{p,1-r}\delta_{\overline{p},1-\overline{r}} \ \ , \end{aligned} \tag{5.47}$$

$${}_{\mathbb{C}_\infty}\langle m,\overline{m},r,\overline{r}|n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0} = {}_0\langle m,\overline{m},r,\overline{r}|n,\overline{n},p,\overline{p}\rangle_\infty = 0. \tag{5.48}$$

A closer investigation in [10] shows that the action of the Hamiltonian (5.30) in the limit $\lambda \to \infty$ is given by,

$$\begin{aligned} \widetilde{H}_\infty|n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0} &= (n+\overline{n}+p+\overline{p})|n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0} \\ &\qquad - 2\pi|n+2p-1,\overline{n}+2\overline{p}-1,p,\overline{p}\rangle_\infty, \end{aligned} \tag{5.49}$$

$$\widetilde{H}_\infty|n,\overline{n},p,\overline{p}\rangle_\infty = (n+\overline{n}+2-p-\overline{p})|n,\overline{n},p,\overline{p}\rangle_\infty. \tag{5.50}$$

Thus, we see that the Hamiltonian $\widetilde{H}_\infty$ have Jordan blocks of the length two.

Chapter 6

Perturbation theory around the point $\lambda = \infty$

In the topological theory, we have shown that a complete basis of the space of states can be written down explicitly in the limit of $\lambda \to \infty$, so that non-BPS correlation functions can be computed exactly in this limit. We want to make use of these results in the physical theory.

Recall that the relations between the matrix elements of observables in the topological theory and the physical theory in the Hamiltonian formalism are

$$\langle \widetilde{\Psi}' | \widetilde{\mathcal{O}}(t) | \widetilde{\Psi} \rangle_{\text{top}} = \langle \Psi' | \mathcal{O}(t) | \Psi \rangle_{\text{phys}}. \tag{6.1}$$

where,

$$\widetilde{\Psi} = e^{\lambda f} \Psi, \qquad \widetilde{\Psi}' = e^{-\lambda f} \Psi', \qquad \widetilde{\mathcal{O}} = e^{\lambda f} \mathcal{O} e^{-\lambda f}. \tag{6.2}$$

However, these relations do not make any sense in the limit of $\lambda \to \infty$. Therefore, to connect the topological theory with the physical theory, we need to compute the matrix elements in the region of finite values of λ. We should do this by some kind of perturbation theory around the point $\lambda = \infty$.

FLN showed that the corresponding perturbation theory is unusual so that one can not use standard formulae of the perturbation theory of quantum mechanics. Because of this difficulty, this perturbation theory has not been done. We find that even in this case, we can use a more general method of perturbation theory that making use of resolvents of operators. The resulting formula of this general perturbation theory is known as Kato-Rellich formula. In the following sections, we use the Kato-Rellich formula and carry out the perturbation theory in the case $X = \mathbb{CP}^1$. This is one of our new result in this thesis.

6.1 Perturbation of the Hamiltonian in $\mathbb{CP}^1$ model

Here, we consider the $\mathbb{CP}^1$ model. For starters, it is natural to compute the energy eigenvalues of the Hamiltonian with finite values of λ by using the perturbation theory. We already know a complete set of the Hilbert space of the topological theory,

$$\mathcal{H}_{\text{in}} = \text{Span}_{\mathbb{C}}\{\ |n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0},\ \ |n,\overline{n},p,\overline{p}\rangle_{\infty}\ \}, \tag{6.3}$$

$$\mathcal{H}_{\text{out}} = \text{Span}_{\mathbb{C}}\{\ {}_0\langle n,\overline{n},p,\overline{p}|,\ \ {}_{\mathbb{C}_\infty}\langle n,\overline{n},p,\overline{p}|\ \}, \tag{6.4}$$

where, $n,\overline{n} \in \mathbb{Z}_{\geq 0}$, $p,\overline{p} = 0,1$. Their pairings are

$$\begin{aligned} {}_0\langle m,\overline{m},r,\overline{r}|n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0} &= {}_{\mathbb{C}_\infty}\langle n,\overline{n},p,\overline{p}|n,\overline{n},p,\overline{p}\rangle_{\infty} \\ &= (-i)^p i^{\overline{p}}(-1)^{p\overline{r}}\delta_{n,m}\delta_{\overline{n},\overline{m}}\delta_{p,1-r}\delta_{\overline{p},1-\overline{r}}\ , \end{aligned} \tag{6.5}$$

$${}_{\mathbb{C}_\infty}\langle m,\overline{m},r,\overline{r}|n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0} = {}_0\langle m,\overline{m},r,\overline{r}|n,\overline{n},p,\overline{p}\rangle_{\infty} = 0. \tag{6.6}$$

For simplicity, we will restrict ourselves to the subspace of zero-forms in $\mathcal{H}_{\text{in}}$, i.e., $p = \overline{p} = 0$. To simplify our notation, we will write $|n,\overline{n}\rangle_{\mathbb{C}_0}$ and $|n,\overline{n}\rangle_{\infty}$ for $|n,\overline{n},p,\overline{p}\rangle_{\mathbb{C}_0}$ and $|n,\overline{n},p,\overline{p}\rangle_{\infty}$. We will also write corresponding relevant "out" space basis as ${}_0\langle m,\overline{m}| := {}_0\langle m,\overline{m},2,2|$, and ${}_{\mathbb{C}_\infty}\langle m,\overline{m},| := {}_{\mathbb{C}_\infty}\langle m,\overline{m},2,2|$. Then, their parings have simple form,

$${}_0\langle m,\overline{m}|n,\overline{n}\rangle_{\mathbb{C}_0} = {}_{\mathbb{C}_\infty}\langle m,\overline{m}|n,\overline{n}\rangle_{\infty} = \delta_{n,m}\delta_{\overline{n},\overline{m}}, \tag{6.7}$$

$${}_{\mathbb{C}_\infty}\langle m,\overline{m}|n,\overline{n}\rangle_{\mathbb{C}_0} = {}_0\langle m,\overline{m}|n,\overline{n}\rangle_{\infty} = 0. \tag{6.8}$$

The total Hamiltonian has the form (see (5.30) and (2.29), we drop the symbol tilde to simplify the notation),

$$H = H_0 + \frac{1}{\lambda}H_1. \tag{6.9}$$

The action of H on our zero-form subspace is given by, (see [10]),

$$\begin{aligned}
H_0|n,\overline{n}\rangle_{\mathbb{C}_0} &= (n+\overline{n})|n,\overline{n}\rangle_{\mathbb{C}_0} - 2\pi|n-1,\overline{n}-1\rangle_\infty, && (6.10)\\
H_0|n,\overline{n}\rangle_\infty &= (n+\overline{n}+2)|n,\overline{n}\rangle_\infty, && (6.11)\\
H_1|n,\overline{n}\rangle_{\mathbb{C}_0} &= -2n\overline{n}(\ |n-1,\overline{n}-1\rangle_{\mathbb{C}_0} + 2|n,\overline{n}\rangle_{\mathbb{C}_0} + |n+1,\overline{n}+1\rangle_{\mathbb{C}_0}) \\
&\quad + 4\pi(n+\overline{n})(\ |n-1,\overline{n}-1\rangle_\infty + 2|n,\overline{n}\rangle_\infty + |n+1,\overline{n}+1\rangle_\infty), && (6.12)\\
H_1|n,\overline{n}\rangle_\infty &= -2(n+1)(\overline{n}+1)(|n-1,\overline{n}-1\rangle_\infty + 2|n,\overline{n}\rangle_\infty - |n+1,\overline{n}+1\rangle_\infty). && (6.13)
\end{aligned}$$

Note that the unperturbed Hamiltonian H_0 has Jordan blocks and hence, we can not diagonalize H_0. Consequently, H_0 is not hermitean.

We find a part of the explicit matrix form of H_0 and H_1 to be,

$$H_0 = \begin{array}{c} \\ |0,0\rangle_{\mathbb{C}_0} \\ |0,0\rangle_\infty \\ |1,1\rangle_{\mathbb{C}_0} \\ |1,1\rangle_\infty \\ |2,2\rangle_{\mathbb{C}_0} \\ |2,2\rangle_\infty \\ |3,3\rangle_{\mathbb{C}_0} \\ \dots \end{array}
\begin{array}{c}
\begin{array}{cccccccc} {}_0\langle 0,0| & {}_{\mathbb{C}_\infty}\langle 0,0| & {}_0\langle 1,1| & {}_{\mathbb{C}_\infty}\langle 1,1| & {}_0\langle 2,2| & {}_{\mathbb{C}_\infty}\langle 2,2| & {}_0\langle 3,3| & \dots \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & \\
0 & -2\pi & 2 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & \\
0 & 0 & 0 & -2\pi & 4 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & \\
0 & 0 & 0 & 0 & 0 & -2\pi & 6 & \\
 & & & & & & & \ddots
\end{pmatrix}
\end{array} \qquad (6.14)$$

$$H_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & -4 & 0 & -2 & 0 & 0 & 0 & \\
-2 & 8\pi & -4 & 16\pi & -2 & 8\pi & 0 & \\
0 & -8 & 0 & -16 & 0 & -8 & 0 & \\
0 & 0 & -8 & 16\pi & -16 & 32\pi & -8 & \\
0 & 0 & 0 & -18 & 0 & -36 & 0 & \\
0 & 0 & 0 & 0 & -18 & 24\pi & -36 & \\
 & & & & & & & \ddots
\end{pmatrix}. \qquad (6.15)$$

The corresponding perturbation theory is unusual, because normally one considers hermitean Hamiltonian which cannot have Jordan blocks. If H_0 were a hermitean operator, usually one would have assumed that an eigenvalue of the perturbed Hamiltonian $H = H_0 + \lambda^{-1}H_1$ could be expanded into *integer* power series of λ^{-1}, like

$$E = E^{(0)} + \lambda^{-1}E^{(1)} + \lambda^{-2}E^{(2)} + \dots . \tag{6.16}$$

However, in our case, we cannot justify this assumption. It entirely relies on the hermiticity of H_0 [16].

To gain some insight into our situation, let us observe a following simple model.

$$A = A_0 + \epsilon A_1, \tag{6.17}$$

where

$$A_0 = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.18}$$

We can immediately find the eigenvalues $\alpha_\pm$ of A, to be

$$\alpha_\pm = a \pm \sqrt{\epsilon + \epsilon^2}. \tag{6.19}$$

Clearly, $\alpha_\pm$ cannot be expanded into integer power series of ϵ. It can only be expanded into a *fractional* power series,

$$\alpha_\pm = a \pm (\epsilon^{1/2} + \frac{1}{2}\epsilon^{3/2} - \frac{1}{8}\epsilon^{5/2} + \dots). \tag{6.20}$$

Therefore we cannot assume a priori that energy eigenvalues of the Hamiltonian H are expanded into integer power series of λ^{-1}. To analyze our perturbation theory, we have to use a method different from usual quantum mechanical perturbation theory.

6.2 Perturbation method using the resolvent

We will analyze our unusual perturbation theory by a method using the resolvent of the Hamiltonian [16].

6.2.1 The theory of resolvents

First we briefly describe the general theory of resolvents of linear operators. Let H be a linear operator acting on a Hilbert space $\mathcal{H}$. We assume that H has a discrete spectrum. Note that such a operator H has unique canonical form

$$H = S + D, \tag{6.21}$$

$$S = \sum_n E_n P_n, \quad D = \sum_n D_n, \tag{6.22}$$

where, E_n's are the eigenvalues of H, P_n's are the eigenprojections for the eigenvalues E_n, and D_n's are eigennilpotent operators for E_n. This form is called the Jordan canonical form or the spectral representation of H.

Let ζ be a complex variable. The resolvent $R(\zeta)$ of H is the operator valued function defined by

$$R(\zeta) \equiv R(\zeta, H) := (H - \zeta)^{-1}. \tag{6.23}$$

The resolvent $R(\zeta)$ is well-defined for values of ζ which are not equal to any of the eigenvalues of H. In general, if ζ is equal to an eigenvalue E of H, $R(\zeta)$ has a pole at $\zeta = E$.

$R(\zeta)$ satisfies a following equation called the (first) resolvent equation,

$$R(\zeta_1) - R(\zeta_2) = (\zeta_1 - \zeta_2) R(\zeta_1) R(\zeta_2). \tag{6.24}$$

This can be derived easily as follows.

$$\begin{aligned}
\text{(L.H.S)} &= \frac{1}{H - \zeta_1} - \frac{1}{H - \zeta_2} \\
&= \frac{1}{H - \zeta_1}(H - \zeta_2)\frac{1}{H - \zeta_2} - \frac{1}{H - \zeta_1}(H - \zeta_1)\frac{1}{H - \zeta_2} \\
&= R(\zeta_1)(H - \zeta_2)R(\zeta_2) - R(\zeta_1)(H - \zeta_1)R(\zeta_2) \\
&= \cancel{R(\zeta_1)HR(\zeta_2)} - R(\zeta_1)\zeta_2 R(\zeta_2) - \cancel{R(\zeta_1)HR(z_2)} + R(\zeta_1)\zeta_1 R(\zeta_2) \\
&= R(\zeta_1)(\zeta_1 - \zeta_2)R(\zeta_2) \\
&= \text{(R.H.S)}.
\end{aligned} \tag{6.25}$$

Using this equation we can express a product of $R(\zeta)$'s as a sum of them.

The singular points, or poles of $R(\zeta)$, are equal to the eigenvalues of H. Consider a Laurent expansion around an eigenvalue $\zeta = E \in \mathbb{C}$. For simplicity, we assume $E = 0$ and write the expansion as

$$R(\zeta) = \sum_{n=-\infty}^{\infty} \zeta^n A_n. \tag{6.26}$$

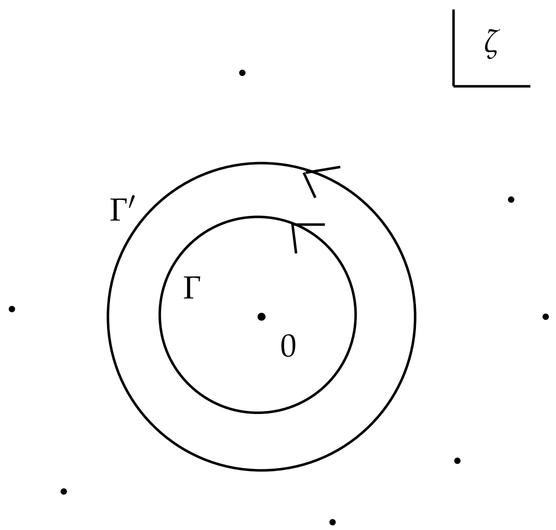
The coefficient operators A_n are given by

$$A_n = \frac{1}{2\pi i} \oint_\Gamma \zeta^{-n-1} R(\zeta) d\zeta \,, \tag{6.27}$$

where, Γ is a positively-oriented small circle enclosing $\zeta = 0$ but excluding other eigenvalues of H. A product of two of these coefficients can be computed as

$$\begin{aligned} A_n A_m &= \left(\frac{1}{2\pi i}\right)^2 \oint_{\Gamma'} \oint_\Gamma \zeta^{-n-1} \zeta^{-m-1} R(\zeta) R(\zeta') d\zeta d\zeta' \\ &= \left(\frac{1}{2\pi i}\right)^2 \oint_{\Gamma'} \oint_\Gamma \zeta^{-n-1} \zeta'^{-m-1} (\zeta' - \zeta)^{-1} [R(\zeta') - R(\zeta)] d\zeta d\zeta'. \end{aligned} \tag{6.28}$$

where, Γ' is a circle that are a little larger than Γ. We used the resolvent equation in the second line of (6.28).



After a short calculation, we find

$$A_n A_m = (\eta_n + \eta_m - 1) A_{n+m+1}, \tag{6.29}$$

where, the symbol η_n is defined by,

$$\eta_n = \begin{cases} 1, & \text{for } n \geq 0, \\ 0, & \text{for } n < 0. \end{cases} \tag{6.30}$$

In (6.29), setting $n = m = -1$, we get $A_{-1}^2 = -A_{-1}$. Thus, $-A_{-1}$ is a projection. We denote it as $P := -A_{-1}$. For $n, m < 0$, we get $-A_{-3} = A_{-2}^2$, $-A_{-4} = -A_{-3}A_{-2} = A_{-2}^3, \ldots$. Therefore, defining $D := -A_{-2}$, we have $A_{-k} = -D^{k-1}$, for $k \geq 2$. Similarly, for $n, m \geq 0$, we have $A_k = S^{k+1}$, $S := A_0$.

From the discussion above, we see that the Laurent expansion of $R(\zeta)$ around a general eigenvalues E_n of H takes the form

$$R(\zeta) = \sum_{k=1}^{\infty} \frac{-D_n^k}{(\zeta - E_n)^{k+1}} + \frac{-P_n}{\zeta - E_n} + \sum_{k=0}^{\infty} (\zeta - E_n)^k S_n^{k+1}. \tag{6.31}$$

It is known that the coefficient operators P_n and D_n coincide with the operators that appear in the spectral representation of H, c.f. (6.21), (6.22).

We have following equalities

$$HP_n = P_n H = P_n H P_n = E_n P_n + D_n, \quad n = 0, 1, 2, \ldots. \tag{6.32}$$

It is easy to see that HP_n has one and only one eigenvalue E_n. (E_n may have degenerate multiplicity.) For more details, see [16].

6.2.2 Perturbation theory of a resolvent

Suppose, we want to compute an eigenvalue of the perturbed Hamiltonian $H(\lambda^{-1}) = H_0 + \lambda^{-1}H_1$. To simplify the notation, we write $\epsilon := \lambda^{-1}$. We denote the resolvent of $H(\epsilon)$ by,

$$R(\zeta, \epsilon) := (H(\epsilon) - \zeta)^{-1}. \tag{6.33}$$

Note that the following equalities hold,

$$\begin{aligned} H(\epsilon) - \zeta &= H_0 - \zeta + \epsilon H_1 \\ &= [1 + \epsilon H_1 (H_0 - \zeta)^{-1}](H_0 - \zeta) \\ &= [1 + \epsilon H_1 R(\zeta)](H_0 - \zeta), \end{aligned} \tag{6.34}$$

where, $R(\zeta) = R(\zeta, 0) = (H_0 - \zeta)^{-1}$ is the resolvent of the unperturbed Hamiltonian. Taking the inverse of (6.34), we get a power series expansion of $R(\zeta, \epsilon)$,

$$\begin{aligned} R(\zeta, \epsilon) &= R(\zeta)[1 + \epsilon H_1 R(\zeta)]^{-1} \\ &= R(\zeta) \sum_{p=0}^{\infty} [\epsilon H_1 R(\zeta)]^p \\ &= R(\zeta) + \sum_{n=1}^{\infty} \epsilon^n R^{(n)}(\zeta), \end{aligned} \tag{6.35}$$

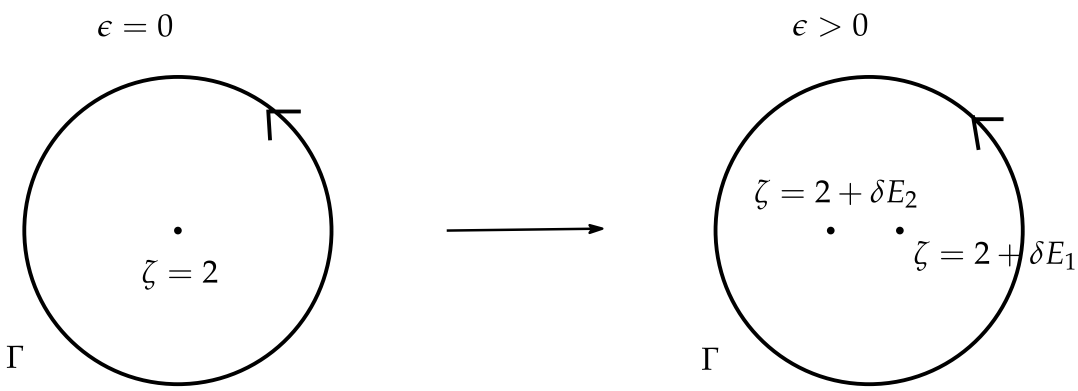
where,

$$R^{(n)}(\zeta) = (-1)^n \underbrace{R(\zeta) H_1 R(\zeta) H_1 \dots R(\zeta) H_1 R(\zeta)}_{n \ H_1 s.} \tag{6.36}$$

To compute eigenvalues of $H(\epsilon)$, we have to find the poles of $R(\zeta, \epsilon)$. If ϵ is sufficiently small, the positions of the poles of $R(\zeta, \epsilon)$ must be very close to that of $R(\zeta, 0)$. But, here we encounter a difficulty. The eigenvalues of H_0 are doubly degenerate. Therefore when we perturbed H_0 by H_1, generally, the degeneracy may be removed and the eigenvalues may split.

To make our picture more concrete, we will focus on an eigenvalue $E(0) = 2$ of H_0 and study how it varies as a function of ϵ. First, the Laurent expansion (6.31) of $R(\zeta)$ around the point $\zeta = 2$ takes the form,

$$R(\zeta) = \frac{-D_2}{(\zeta - 2)^2} + \frac{-P_2}{\zeta - 2} + \sum_{n=0}^{\infty} (\zeta - 2)^n S_2^{(n+1)}, \tag{6.37}$$



where,

$$D_2 = \begin{pmatrix} 0 & & & & & 0 & \\ & 0 & 0 & & & & \\ & -2\pi & 0 & & & & \\ & & & 0 & 0 & & \\ 0 & & & 0 & 0 & & \\ & & & & & \ddots \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & & & & & 0 & \\ & 1 & 0 & & & & \\ & 0 & 1 & & & & \\ & & & 0 & 0 & & \\ 0 & & & 0 & 0 & & \\ & & & & & \ddots \end{pmatrix}$$

$$S_2 = \begin{pmatrix} -\frac{1}{2} & & & & & 0 & \\ & 0 & 0 & & & & \\ & 0 & 0 & & & & \\ & & & \frac{1}{2} & 0 & & \\ 0 & & & \frac{\pi}{2} & \frac{1}{2} & & \\ & & & & & \ddots \end{pmatrix} \tag{6.38}$$

We can represent the eigenprojection of H_0 associated with the eigenvalue $E(0) = 2$ as

$$P_2 = -\frac{1}{2\pi i} \oint_\Gamma R(\zeta, 0) d\zeta. \tag{6.39}$$

If we replace $R(\zeta, 0)$ with $R(\zeta, \epsilon)$ for sufficiently small ϵ,

$$P_2(\epsilon) := -\frac{1}{2\pi i} \oint_\Gamma R(\zeta, \epsilon) d\zeta. \tag{6.40}$$

is a total eigenprojection associated with the two possibly split eigenvalues close to 2. The perturbation expansion of $P_2(\epsilon)$ is obtained by substituting (6.35) into

(6.40). This leads to,

$$P_2(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n P_2^{(n)}, \quad P_2^{(0)} = P_2. \tag{6.41}$$

$$P_2^{(n)} = \frac{(-1)^{n+1}}{2\pi i} \oint_{\Gamma} R(\zeta) H_1 R(\zeta) H_1 \dots R(\zeta) H_1 R(\zeta) d\zeta. \tag{6.42}$$

Furthermore, substituting the Laurent expansion (6.37) of $R(\zeta)$, we can perform the contour integral along Γ. The resulting formula is known as Kato-Rellich formula for the total eigenprojection. For example, we get

$$P_2^{(1)} = -D_2 H_1 S_2^2 - P_2 H_1 S_2 - S_2 H_1 P_2 - S_2^2 H_1 D_2. \tag{6.43}$$

$$P_2^{(2)} = \sum_{k_1+k_2+k_3=2} S_2^{(k_1)} H_1 S_2^{(k_2)} H_1 S_2^{(k_3)}, \tag{6.44}$$

where,

$$S^{(0)} = -P_2, \quad S_2^{(n)} = S_2^n, \quad S_2^{(-n)} = -D_2^n, \quad n \geq 1. \tag{6.45}$$

Now we can perturbatively compute the matrix $H(\epsilon)P_2(\epsilon)$. We will find that for each order of perturbation expansion of $P_2(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n P_2^{(n)}$, the resulting matrix $H(\epsilon)P_2(\epsilon)$ has only a finite number of nonzero entries. Therefore we can explicitly compute its characteristic polynomial and obtain the eigenvalues. We will perform this computation for the first and second orders.

The first order perturbation

We can straightforwardly compute (6.43) and obtain

$$H(\epsilon)P_2(\epsilon)_{\text{1st.}} =$$

$$\left(\begin{array}{ccccccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 2-4\epsilon-8\epsilon^2 & 0 & 2\epsilon-4\epsilon^2 & 0 & 0 & 0 & \\
-2\epsilon+4\epsilon^2 & -2\pi+8\pi\epsilon+64\pi\epsilon^2 & 2-4\epsilon-8\varepsilon^2 & -18\pi\epsilon+40\pi\epsilon^2 & 2\epsilon-4\epsilon^2 & -4\pi\epsilon+8\pi\epsilon^2 & 0 & \\
0 & 8\epsilon-64\epsilon^2 & 0 & -8\epsilon^2 & 0 & 0 & 0 & 0 \\
8\epsilon^2 & -8\pi\epsilon+64\pi\epsilon^2 & 8\epsilon-64\epsilon^2 & 64\pi\epsilon^2 & -8\epsilon^2 & 16\pi\epsilon^2 & 0 & \\
0 & -72\epsilon^2 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & -72\epsilon^2 & 0 & 0 & 0 & 0 & \\
\hline
 & & & 0 & & & & \ddots
\end{array}\right) \tag{6.46}$$

Thus, we have a finite-size matrix of 7×7. We find the relevant eigenvalues of this matrix to be

$$E(\epsilon) = 2 - 4\epsilon + \mathcal{O}(\epsilon^2), \quad \text{(doubly degenerate!)}. \tag{6.47}$$

The other five eigenvalues are of course zero up to the order $\mathcal{O}(\epsilon^2)$.

The second order perturbation

The expression in (6.44) has 21 terms, although we suppress showing all of them. Similar to the first order case, $H(\epsilon)P_2(\epsilon)_{\text{2nd.}}$ has a finite-size non-zero block of 8×8. The relevant eigenvalues are found to be

$$E(\epsilon) = 2 - 4\epsilon - 8\epsilon^2 + \mathcal{O}(\epsilon^3), \quad \text{(also doubly degenerate.)} \tag{6.48}$$

Chapter 7

Conclusions and Discussions

In this thesis, we proposed two extensions of the topological quantum mechanics on Riemannian manifolds based on the consideration of [10].

First we construct a class of new observable of topological quantum mechanics by using the method of iterated integrals. This method is one for a construction of differential forms on a space of paths. According to the Mathai-Quillen formalism of the cohomological field theory, observables of a cohomological field theory correspond to differential forms on the instanton moduli space of the theory. In the case of the topological quantum mechanics on a Riemannian manifold, the instanton moduli space is given by the space of the gradient trajectories of the Morse function on the manifold, which is a subspace of the space of paths on the manifold with the both endpoints fixed. Therefore the differential forms on the space of paths can be regarded as the new observables of the topological quantum mechanics. It is known that the method of the iterated integral can draw out the information about the non-commutativity of the fundamental group of the space. Thus, we expect that we can get new information about the geometry of the instanton moduli space by dealing with the new observables.

We gave a nontrivial example of the correlation function that contains this new observable. Though we focused only on the case of finite dimensional target space, FNL [12] shows the ways to regard the two-dimensional sigma model and the four-dimensional Yang-Mills theory as supersymmetric mechanics with the infinite dimensional target spaces. Therefore if we apply the method of iterated integrals to these infinite-dimensional model, we expect to have more interesting observables. They will draw out the information about the fundamental group of the instanton moduli of the sigma model and the Yang-Mills theory. It will be worth while to push forward this ideas.

Second, we carried out the unusual perturbation theory around the point $\lambda = \infty$ and obtained the energy eigenvalues of the first excited states of the Hamiltonian to the second order of the perturbation theory. The basic reason why this perturbation theory is unusual is that the unperturbed Hamiltonian is not hermitean nor diagonalizable. We saw that in this case, usual perturbation method of the quantum mechanics is not applicable. We avoided this difficulty by using the method of the resolvent. The computation of the Hamiltonian on the infinite Hilbert space was reduced to a finite dimensional matrix computation. We saw that the degeneracy of the eigenvalues were not removed and the Jordan block structure was still remaining. It is seemingly strange, but in fact it is reasonable because the difference of the almost degenerate energy eigenvalues are proportional to $e^{-\lambda|f(x_f)-f(x_i)|}$. That is, the degeneracy is removed by a non-perturbative effect. To compute this difference explicitly, we will need to take into account the effect of the instanton, or tunneling effect. It would be interesting to compute this non-perturbative effects explicitly.

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Appendix A

Manipulation of differential forms in a local coordinate

Let X be a n-dimensional Riemannian manifold and v, w be two vector fields on TX. Then, there are three basic operators acting on differential forms. Namely, the exterior derivative d, the interior product ι_v, and the Lie derivative $\mathcal{L}_v$. Here we recollect their properties in terms of a local coordinate system on X.

First, d, ι_v, and $\mathcal{L}_v$ satisfy the following algebra:

$$\{d, d\} = 2d^2 = 0, \tag{A.1}$$

$$\{d, \iota_v\} = \mathcal{L}_v, \tag{A.2}$$

$$\{\iota_v, \iota_w\} = 0, \tag{A.3}$$

$$[\mathcal{L}_v, \iota_w] = \iota_{[v,w]}, \tag{A.4}$$

$$[d, \mathcal{L}_v] = 0, \tag{A.5}$$

$$[\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]}. \tag{A.6}$$

The most important formula for us is (A.2), which is know as Cartan's formula.

Let us choose a local coordinate (U, x^μ) around a point on X. Then, U is diffeomorphic to $\mathbb{R}^n$. A k-form $\omega \in \Omega^k(X)$ is, on U, expressed as

$$\omega(x) = \frac{1}{k!}\omega_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}. \tag{A.7}$$

A vector field v is, on U, expressed as

$$v(x) = v^\mu(x)\frac{\partial}{\partial x^\mu}. \tag{A.8}$$

The exterior derivative and the interior products with respect to v are,

$$d = dx^\mu \frac{\partial}{\partial x^\mu} \equiv dx^\mu \partial_\mu, \tag{A.9}$$

$$\iota_v = v^\mu(x)\iota_{\partial/\partial x^\mu} \equiv v^\mu(x)\iota_\mu \tag{A.10}$$

The basic properties of ι_μ are,

$$\iota_\mu \, dx^\nu = \delta^\nu_\mu, \quad \iota_\mu f(x) = 0, \tag{A.11}$$

$$\iota_\mu(\omega \wedge \eta) = (\iota_\mu \omega) \wedge \eta + (-1)^k \omega \wedge (\iota_\mu \eta), \tag{A.12}$$

where, $f(x)$ is a zero-form, i.e., a function, and ω is a k-form. For example, let us choose

$$v(x) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{A.13}$$

and

$$\omega = \omega(x,y)dx \wedge dy. \tag{A.14}$$

Then, we can calculate as

$$\begin{aligned}
\iota_v \omega &= (x\iota_{\partial/\partial x} + y\iota_{\partial/\partial y})(\omega(x,y)dx \wedge dy) \\
&= x\omega(x,y)\iota_{\partial/\partial x}(dx \wedge dy) + y\omega(x,y)\iota_{\partial/\partial y}(dx \wedge dy) \\
&= x\omega(x,y)dy - y\omega(x,y)dx.
\end{aligned} \tag{A.15}$$

Cartan's formula is useful for calculation of Lie derivatives. For example,

$$
\begin{aligned}
\mathcal{L}_v\omega &= (d \circ \iota_v + \iota_v \circ \underbrace{d)\omega(x,y)dx \wedge dy}_{=0} \\
&= d[x\omega(x,y)dy - y\omega(x,y)dx] \\
&= \omega(x,y)dx \wedge dy + x\frac{\partial\omega}{\partial x}(x,y)dx \wedge dy - \omega(x,y)dy \wedge dx - y\frac{\partial\omega}{\partial y}(x,y)dy \wedge dx \\
&= [2\omega(x,y) + x\frac{\partial\omega}{\partial x}(x,y) - y\frac{\partial\omega}{\partial y}(x,y)]dx \wedge dy. \qquad \text{(A.16)}
\end{aligned}
$$

Appendix B

The nilpotency of the BRST transformation

Here we will prove that the BRST transformation that appears the Mathai-Quillen formalism is nilpotent, $Q^2 = 0$, on the space of fields $x^\mu, \psi^\mu, \pi_a, p_a$.

B.1 The definition of Q

Recall that the Q-transformation is defined by

$$Qx^\mu := \psi^\mu, \tag{B.1}$$

$$Q\psi^\mu := 0, \tag{B.2}$$

$$Qp_a := p_a - A_{\mu}{}^{b}{}_{a}\pi_b\psi^\mu, \tag{B.3}$$

$$Q\pi_a := A_{\mu}{}^{b}{}_{a}\pi_b\psi^\mu - \frac{1}{2}F_{\mu\nu}{}^{b}{}_{a}\pi_b\psi^\mu\psi^\nu. \tag{B.4}$$

The curvature F associated with the connection A is defined by

$$F := dA + A \wedge A. \tag{B.5}$$

That is in the component form,

$$F^b_{\ a} = \frac{1}{2}F_{\mu\nu}{}^{b}{}_{a}dx^\mu \wedge dx^\nu, \tag{B.6}$$

$$F_{\mu\nu}{}^{b}{}_{a} = \partial_\mu A_{\nu}{}^{b}{}_{a} - \partial_\nu A_{\mu}{}^{b}{}_{a} + A_{\mu}{}^{b}{}_{c}A_{\nu}{}^{c}{}_{a} - A_{\nu}{}^{b}{}_{c}A_{\mu}{}^{c}{}_{a}. \tag{B.7}$$

The curvature F satisfies the so-called Bianchi identity,

$$DF := dF + [A \wedge F] \equiv 0. \tag{B.8}$$

In the component form,

$$(\partial_\rho F_{\mu\nu}{}^b{}_a + A_\rho{}^b{}_c F_{\mu\nu}{}^c{}_a - A_\rho{}^c{}_a F_{\mu\nu}{}^b{}_c) dx^\mu \wedge dx^\nu \wedge dx^\rho. \tag{B.9}$$

B.2 The proof of $Q^2 = 0$

First note that, $Q^2 x^\mu = 0$ and $Q^2 \psi^\mu = 0$ is trivial.

Next, we will prove $Q^2 \pi_a = 0$.

$$\begin{aligned} Q^2 \pi_a &= Q(p_a - A_\mu{}^b{}_a \pi_b \psi^\mu) \\ &= \underbrace{Qp_a}_{(i)} \underbrace{-(QA_\mu{}^b{}_a)\pi_a\psi^\mu}_{(ii)} \underbrace{-A_\mu{}^b{}_a (Q\pi_b)\psi^\mu}_{(iii)} + A_\mu{}^b{}_a \pi_b (\underbrace{Q\psi^\mu}_{=0}). \end{aligned} \tag{B.10}$$

$$(i) = A_\mu{}^b{}_a p_b \psi^\mu - \frac{1}{2} F_{\mu\nu}{}^b{}_a \pi_b \psi^\mu \psi^\nu. \tag{B.11}$$

$$(ii) = -\partial_\nu A_\mu{}^b{}_a \pi_b \psi^\mu \psi^\nu. \tag{B.12}$$

$$\begin{aligned} (iii) &= -A_\mu{}^b{}_a (p_b - A_\nu^c{}_b \pi_c \psi^\nu)\psi^\mu \\ &= -A_\mu{}^b{}_a p_b \psi^\mu - \underbrace{A_\nu{}^c{}_b A_\mu{}^b{}_a \pi_c}_{\substack{\text{exchange dummy} \\ \text{subscripts } b \leftrightarrow c}} \psi^\mu \psi^\nu. \\ &= -A_\mu{}^b{}_a p_b \psi^\mu - A_\nu{}^b{}_c A_\mu{}^c{}_a \pi_b \psi^\mu \psi^\nu. \end{aligned} \tag{B.13}$$

$$
\begin{aligned}
Q^2\pi_a &= (i) + (ii) + (iii) \\
&= \left(\cancel{A_{\mu\ a}^{\ b} p_b \psi^\mu} - \frac{1}{2} F_{\mu\nu\ a}^{\ \ b} \pi_b \psi^\mu \psi^\nu\right) + \left(-\partial_\nu A_{\mu\ a}^{\ b} \pi_b \psi^\mu \psi^\nu\right) \\
&+ \left(\cancel{-A_{\mu\ a}^{\ b} p_b \psi^\mu} - A_{\nu\ c}^{\ b} A_{\mu\ a}^{\ c} \psi^\mu \psi^\nu\right) \\
&= -(\underbrace{\partial_\nu A_{\mu\ a}^{\ b} + A_{\nu\ b}^{\ c} A_{\mu\ a}^{\ b}}_{=-\frac{1}{2}F_{\mu\nu\ a}^{\ \ b}} + \frac{1}{2} F_{\mu\nu\ a}^{\ \ b}) \pi_b \psi^\mu \psi^\nu \\
&= 0.
\end{aligned}
\tag{B.14}
$$

Next we will show $Q^2 p_a = 0$.

$$
\begin{aligned}
Q^2 p_a &= Q(A_{\mu\ a}^{\ b} p_b \psi^\mu - \frac{1}{2} F_{\mu\nu\ a}^{\ \ b} \pi_b \psi^\mu \psi^\nu) \\
&= \underbrace{(QA_{\mu\ a}^{\ b}) p_b \psi^\mu}_{(i)} \underbrace{+ A_{\mu\ a}^{\ b} (Qp_b) \psi^\mu}_{(ii)} - A_{\mu\ a}^{\ b} p_b (\underbrace{Q\psi^\mu}_{=0}) \\
&\underbrace{-\frac{1}{2}(QF_{\mu\nu\ a}^{\ \ b}) \pi_b \psi^\mu \psi^\nu}_{(iii)} \underbrace{-\frac{1}{2} F_{\mu\nu\ a}^{\ \ b} (Q\pi_b) \psi^\mu \psi^\nu}_{(iv)} + \frac{1}{2} F_{\mu\nu\ a}^{\ \ b} \pi_b \underbrace{Q(\psi^\mu \psi^\nu)}_{=0}.
\end{aligned}
\tag{B.15}
$$

$$
(i) = -\partial_\nu A_{\mu\ a}^{\ b} p_b \psi^\mu \psi^\nu. \tag{B.16}
$$

$$
\begin{aligned}
(ii) &= \underbrace{A_{\mu\ a}^{\ b} (A_{\nu\ b}^{\ c} p_c \psi^\nu}_{\text{exchange dummy subscript } b\leftrightarrow c.} - \frac{1}{2} \underbrace{F_{\nu\rho\ b}^{\ \ c} \pi_c}_{} \psi^\nu \psi^\rho) \psi^\mu \\
&= -A_{\nu\ c}^{\ b} A_{\mu\ a}^{\ c} p_b \psi^\mu \psi^\nu - \frac{1}{2} A_{\mu\ a}^{\ c} F_{\nu\rho\ c}^{\ \ b} \pi_b \psi^\mu \psi^\nu \psi^\rho.
\end{aligned}
\tag{B.17}
$$

$$
\begin{aligned}
(iii) &= \frac{1}{2} \partial_\rho F_{\mu\nu\ a}^{\ \ b} \pi_b \psi^\mu \psi^\nu \psi^\rho. \\
(iv) &= -\frac{1}{2} F_{\mu\nu\ a}^{\ \ b} (p_b - A_{\rho\ b}^{\ c} \pi_c \psi^\rho) \psi^\mu \psi^\nu \\
&= -\frac{1}{2} F_{\mu\nu\ a}^{\ \ b} p_b \psi^\mu \psi^\nu + \frac{1}{2} \underbrace{A_{\rho\ b}^{\ c} F_{\mu\nu\ a}^{\ \ b} \pi_c}_{\text{exchange } b\leftrightarrow c} \psi^\mu \psi^\nu \psi^\rho \\
&= -\frac{1}{2} F_{\mu\nu\ a}^{\ \ b} p_b \psi^\mu \psi^\nu + \frac{1}{2} A_{\rho\ c}^{\ b} F_{\mu\nu\ a}^{\ \ c} \pi_b \psi^\mu \psi^\nu \psi^\rho.
\end{aligned}
\tag{B.18}
$$

$$
\begin{aligned}
Q^2 p_a &= (i) + (ii) + (iii) + (iv) \\
&= (-\partial_\nu A_{\mu}{}^{b}{}_{a} \underline{p_b \psi^\mu \psi^\nu}) \\
&+ (-A_{\nu}{}^{b}{}_{c} A_{\mu}{}^{c}{}_{a} \underline{p_b \psi^\mu \psi^\nu} - \frac{1}{2} \underbrace{A_{\mu}{}^{c}{}_{a} F_{\nu\rho}{}^{b}{}_{c} \underline{\underline{\pi_b \psi^\mu \psi^\nu \psi^\rho}}}_{\text{exchange } \mu \leftrightarrow \rho}) \\
&+ (\frac{1}{2} \partial_\rho F_{\mu\nu}{}^{b}{}_{a} \underline{\underline{\pi_b \psi^\mu \psi^\nu \psi^\rho}}) \\
&+ (-\frac{1}{2} F_{\mu\nu}{}^{b}{}_{a} \underline{p_b \psi^\mu \psi^\nu} + \frac{1}{2} A_{\rho}{}^{b}{}_{c} F_{\mu\nu}{}^{c}{}_{a} \underline{\underline{\pi_b \psi^\mu \psi^\nu \psi^\rho}}) \\
&= -\Big(\underbrace{\partial_\nu A_{\mu}{}^{b}{}_{a} + A_{\nu}{}^{b}{}_{c} A_{\mu}{}^{c}{}_{a}}_{= -\frac{1}{2} F_{\mu\nu}{}^{b}{}_{a}} + \frac{1}{2} F_{\mu\nu}{}^{b}{}_{a} \Big) p_b \psi^\mu \psi^\nu \\
&+ \frac{1}{2} \Big(\underbrace{\partial_\rho F_{\mu\nu}{}^{b}{}_{a} + A_{\rho}{}^{b}{}_{c} F_{\mu\nu}{}^{c}{}_{a} - A_{\rho}{}^{c}{}_{a} F_{\mu\nu}{}^{b}{}_{c}}_{\text{Bianchi identity}} \Big) \pi_b \psi^\mu \psi^\nu \psi^\rho \\
&= 0,
\end{aligned}
\tag{B.19}
$$

which completes the proof.

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