Liouville type theorems for the Navier-Stokes equations and applications

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博士論文(要約)

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（ナヴィエ・ストークス方程式に対するリウヴィル型定理とその応用）

許　本源

Liouville type theorems for the Navier-Stokes equations and applications

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Contents

Abstract

In this thesis we study the Liouville type results for solutions to the Navier-Stokes equations, that is, the nonexistence of nontrivial bounded global (or entire) solutions to the Navier-Stokes equations.

We mainly consider the stationary Navier-Stokes equations in three-dimensional whole space and the non-stationary Navier-Stokes equations in half plane.

In Chapter 1, we give a brief introduction to this thesis.

In Chapter 2, we consider stationary solutions to the three-dimensional Navier-Stokes equations for viscous incompressible flows in the presence of a linear strain. For certain class of strains we prove a Liouville type theorem under suitable decay conditions on vorticity fields.

In Chapter 3, we establish a Liouville type result for a backward global solution to the Navier-Stokes equations in the half plane with the no-slip boundary condition. No assumptions on spatial decay for the vorticity nor the velocity field are imposed. We study the vorticity equations instead of the original Navier-Stokes equations. As an application, we extend the geometric regularity criterion for the Navier-Stokes equations in the three-dimensional half space under the no-slip boundary condition.

Chapter 2 is essentially based on [1]. And Chapter 3 is essentially based on [2].

All sections, formulas and theorems, etc., are cited only in the chapter where they appear.

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Chapter 1

Introduction

1 The subject of the thesis

The subject of this thesis is to study the Liouville type results for solutions to the Navier-Stokes equations, that is, the nonexistence of nontrivial bounded global (or entire) solutions to the Navier-Stokes equations. We proved Liouville type theorems for solutions to the Navier-Stokes equations in the following scenes.

(1) Entire solutions to the stationary Navier-Stokes equations with a liner strain in three-dimensional space.

(2) Backward global solutions to the Navier-Stokes equations in the half plane subject to the Dirichlet boundary condition.

We state the results in Chapter 2 and Chapter 3 respectively. We also extend the geometric regularity criterion for the Navier-Stokes equations in the three-dimensional half space under the no-slip boundary condition as an application of our Liouville type theorem in Chapter 3.

2 Introduction to Chapter 2

In Chapter 2 we consider stationary solutions to the three-dimensional Navier-Stokes equations for viscous incompressible flows with a linear strain:

$$\begin{cases} -\Delta U + Mx\cdot\nabla U + MU + U\cdot\nabla U + \nabla P = 0 & x\in\mathbb{R}^3, \\ \nabla\cdot U = 0 & x\in\mathbb{R}^3, \end{cases} \tag{$\mathrm{NS_M}$}$$

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \lambda_i \in \mathbb{R}. \tag{2.1}$$

Here $U(x) = (U_1(x), U_2(x), U_3(x))$ represents the velocity field, $P(x)$ is the pressure field, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, and each λ_i is a given real number.

The system ($\mathrm{NS_M}$) is closely related with the original Navier-Stokes equations. For example, the first equation of the system ($\mathrm{NS_M}$) is formally obtained by considering the stationary solution to the Navier-Stokes equations of the form $U(x) + Mx$. If the trace of M, denoted by $\mathrm{Tr}(M)$ in the sequel, is equal to zero then the second equation of the system ($\mathrm{NS_M}$) is also recovered. This represents stationary solutions to the Navier-Stokes equations with a background flow Mx (which we called it linear strain). Even in the case $\mathrm{Tr}(M) \neq 0$, the system ($\mathrm{NS_M}$) is derived from the Navier-Stokes equations through self-similar solutions. For precise transform formula please see the Section 1 of Chapter 2. This observation shows that the system ($\mathrm{NS_M}$) describes three important classes of solutions to Navier-Stokes equations depending on the eigenvalues λ_i of M. And the sign of eigenvalues is closely related to the existence (or nonexistence) for nontrivial entire solutions. Our goal is to clarify this relation.

One important problem in three-dimensional Navier-Stokes equations is that: does blow-up phenomenon occur in finite time? When $\lambda_1 = \lambda_2 = \lambda_3 > 0$, the system ($\mathrm{NS_M}$) is called "Leray's equation", for it was suggested in [6] to prove the existence of blow-up solutions to Navier-Stokes equations by constructing backward self-similar solutions. For this particular case it was proved in [8] that any weak solution to Leray's equation in $L^3(\mathbb{R}^3)$ must be trivial. This result declared that Leray's idea does not give the construction of blow-up solutions to the Navier-Stokes equations. Although the eigenvalues λ_i in literature, such as [7, 8, 10] are assumed to be positive and identical, one can apply the method especially in [10] for proving the nonexistence of nontrivial solutions to the system ($\mathrm{NS_M}$) even when the eigenvalues are all positive but does not coincide with each other. On the other hand, when $\lambda_1 = \lambda_2 = \lambda_3 < 0$, the system ($\mathrm{NS_M}$) describes the forward self-similar solutions to Navier-Stokes equations, and their existence is already well known. For example, see [2, 4, 5, 9]. Furthermore, when $\lambda_1 < 0, \lambda_2 < 0, \sum_{i=1}^{3} \lambda_i = 0$, the system ($\mathrm{NS_M}$) has an explicit two-dimensional solution, called the Burgers vortex in [1].

In Chapter 2 we study the case when one of λ_i is negative and the other two are positive, for this case is essentially open in the literature. If λ_i is positive then the transport term $Mx \cdot \nabla$ possesses an expanding effect in x_i direction, which tends to trivialize solutions. Conversely, if λ_i is negative then the term $Mx \cdot \nabla$ induces a localization in x_i direction, bringing an effect to keep solutions nontrivial. Because of the expanding effect in two directions, one naturally expects that nontrivial (stationary) solutions tend to be absent in our case. However, the precise relation between the nonexistence and the size of the eigenvalues was not clarified. We contribute to this question by finding sufficient conditions for the nonexistence of nontrivial solutions in terms of the eigenvalues.

3 Introduction to Chapter 3

In Chapter 3 we establish a Liouville type result for a backward global solution to the Navier-Stokes equations in the half plane with the no-slip boundary condition. When we study evolution equations the Liouville problem for bounded *backward* solutions plays an important role in obtaining an a priori bound of *forward* solutions through a suitable scaling argument called a blow-up argument. Indeed, if one imposes a uniform continuity on the alignment of the vorticity direction, the blow-up limit of the three-dimensional (Navier-Stokes) flow must be a nontrivial bounded two-dimensional flow, and the problem is essentially reduced to the analysis of two-dimensional Liouville problem. If we assume that the possible blow-up is type I, then the limit flow is not allowed to be a constant. Thus the resolution of the Liouville problem is a crucial step to reach a contradiction. From this systematic argument we can exclude the possibility of type I blow-up for the original three-dimensional flows under a regularity condition on the vorticity direction.

Recently the paper [3] successfully completes the above argument when the velocity field satisfies the *perfect slip* boundary condition, but the problem was remained open for the case of the *no-slip* boundary condition. In Chapter 3 we prove a Liouville type theorem under *no-slip* boundary condition.

When we consider two-dimensional Liouville problem, one effective approach is to investigate vorticity equation. Different from the case of the whole space (or whole plane) or of the perfect slip boundary condition, maximum principle is no longer a useful tool to obtain an a priori bound of the vorticity field. We overcome this difficulty by using the boundary condition on the vorticity field. We also apply our Liouville type theorem to settle the problem left open in [3].

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