博士論文

論文題目： Besov and Triebel-Lizorkin spaces associated to non-negative self-adjoint operators
（非負自己共役作用素に関する Besov 及び Triebel-Lizorkin 空間）

氏　　名： Guorong HU (胡 国荣)

ABSTRACT

Besov and Triebel-Lizorkin Spaces Associated to Non-negative Self-adjoint Operators

Guorong Hu

Let (X, ρ, μ) be a metric measure space satisfying the doubling, reverse doubling, and non-collapsing conditions. Let $\mathscr{L}$ be a non-negative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel satisfies the pointwise Gaussian upper bound. In this thesis, we develop the Besov spaces $B^{s,\mathscr{L}}_{p,q}(X)$ and the Triebel-Lizorkin spaces $F^{s,\mathscr{L}}_{p,q}(X)$ associated to $\mathscr{L}$ with complete range of the exponents s, p and q. Characterizations and properties of these spaces such as Peetre type maximal function characterization, continuous Littlewood-Paley characterization, atomic decomposition, complex interpolation, lifting property and embedding theorem are given. The homogeneous spaces $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ and $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$ are also discussed. In particular, the identification of $\dot{F}^{0,\mathscr{L}}_{p,2}(X)$ with various definitions of Hardy spaces associated to $\mathscr{L}$ is verified. In the special case where X is a stratified Lie group, these function spaces are applied to study the boundedness of singular integral operators.

Acknowledgement

I would like to thank my advisor Professor Hitoshi Arai for his guidance, constant encouragement, enlightening discussion and great support through the years.

Contents

Chapter 1

Introduction

Function spaces are useful tools for studying problems in harmonic analysis, partial differential equations, probability theory and many other areas of mathematics. In harmonic analysis, various classical function spaces such as Lebesgue, Sobolev, Bessel-potential, Hardy, BMO and Hölder-Zygmund spaces can be studied from a unifying perspective via the Littlewood-Paley theory. In particular, when one studies interpolations, embeddings, wavelet characterizations, Fourier multipliers or boundedness of singular integrals of/on these function spaces, it is convenient to regard them as special cases of Besov and Triebel-Lizorkin spaces which are defined via Littlewood-Paley decomposition.

On the other hand, the developments of many function spaces arising in harmonic analysis were originally tied to the properties of harmonic functions and the Laplacian $\Delta := -\sum_{j=1}^{n} \partial^2/\partial x_j^2$. For instance, one well-known characterization of the real Hardy spaces $H^p(\mathbb{R}^n)$ $(0 < p < \infty)$ states that a (bounded) distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^p(\mathbb{R}^n)$ if and only if $S_\Delta f \in L^p(\mathbb{R}^n)$, where $S_\Delta f$ is the square function (associated to the Laplacian Δ) defined by the area integral

$$S_\Delta f(x) := \left(\iint_{\Gamma(x)} \left| \frac{\partial}{\partial t} e^{-t\sqrt{\Delta}} f(y) \right|^2 t^{1-n} dydt \right)^{1/2}, \tag{1.1}$$

with $\Gamma(x) := \{(y,t) : |y-x| < t\}$. The harmonicity of the Possion integral $e^{-t\sqrt{\Delta}} f$ in the upper half-space $\mathbb{R}^{n+1}_+$ plays a role in deriving such a characterization. Besides Hardy spaces, the study of the Besov(-Lipschitz) spaces $B^s_{p,q}(\mathbb{R}^n)$ and $\dot{B}^s_{p,q}(\mathbb{R}^n)$, especially in the 1960s and 1970s, was also connected with the properties of harmonic functions and the Laplacian. Indeed, these spaces were usually done by considering the Possion integral $e^{-t\sqrt{\Delta}} f$ or the Gauss-Weierstrass integral $e^{-t\Delta} f$ of distributions f (see Bui [9], Flett [31], Johnson [57], Stein [74], Taibleson [81]; see also Saka [69] for a generalization to nolpotent Lie groups). Analogous results in the Triebel-Lizorkin case can be found in [10, 11, 12].

In the seminal paper of Fefferman and Stein [30], a real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, \infty)$ was systematically developed. The real-variable method made it possible to extend Hardy spaces to a much more general setting, which is called "spaces of homogeneous type" (see Coifman and Weiss [18]). We refer also to Han *et al.* [42, 43, 44, 45] for extensions of Besov and Triebel-Lizorkin spaces to spaces of homogeneous type. However, there are some important situations in which the classical real-variable function spaces are not the most suitable choices. For instance, the classical real-variable Hardy spaces $H^p(\mathbb{R}^n)$ seem not applicable when one studies problems related to the divergence form elliptic operator $\mathscr{L} f = -\operatorname{div}(A\nabla f)$ with bounded complex coefficients. In fact, the Riesz transform $\nabla \mathscr{L}^{-1/2}$ associated to $\mathscr{L}$ may not be bounded from the classical Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Therefore, it seems reasonable to introduce function spaces adapted to a linear operator $\mathscr{L}$ which generalizes the Laplacian Δ, in much the same way that the classical Hardy spaces, Besov spaces and Triebel-Lizorkin spaces are adapted to the Laplacian.

Auscher, Duong and McIntoshi [4] first introduced a class of Hardy space $H^1_{\mathscr{L}}$ associated to an operator $\mathscr{L}$ by means of the square function in (1.1) with the Poisson semigroup $e^{-t\sqrt{\Delta}}$ replaced by the semigroup $e^{-t\mathscr{L}}$, under the assumption that the heat kernel of $\mathscr{L}$ satisfies a pointwise Poisson upper bound. Then Duong and Yan [24] introduced BMO spaces associated to such an $\mathscr{L}$ and they proved in [25] that BMO spaces associated to the adjoint operator $\mathscr{L}^*$ is the dual space of $H^1_{\mathscr{L}}$. Recently, Auscher, McIntoshi and Russ [5] studied the Hardy space associated to the Hodge Laplacian on a Riemannian manifold, while Hofmann and Mayboroda [47] investigated Hardy spaces associated to a second order divergence form elliptic operator $\mathscr{L}$ on $\mathbb{R}^n$ with complex coefficients. The theory of the Hardy spaces $H^p_{\mathscr{L}}(X)$, $1 \le p < \infty$, on a metric space X associated to a non-negative self-adjoint operator $\mathscr{L}$ satisfying Davies-Gaffney estimates was developed in [46]. Function spaces associated to operators turn out to be useful for studying the boundedness of non-classical singular integrals (e.g., Riesz transform $\nabla\mathscr{L}^{-1/2}$ associated to a divergence form elliptic operator $\mathscr{L}$) which may not fall within the scope of the Calderón-Zygmund theory.

In the case that $\mathscr{L} = -\Delta + V$ is a Schrödinger operator with a locally integrable non-negative potential V, the H^p spaces associated to $\mathscr{L}$ was earlier investigated by Dziubański *et al.*; see [27, 28] and the references therein. In these works the spaces $H^p_{\mathscr{L}}(\mathbb{R}^n)$ were introduced by means of the radial maximal function associated to the semigroup $e^{-t\mathscr{L}}$, instead of using square functions. Note that the operator $\mathscr{L} = -\Delta + V$ satisfies the Davies-Gaffney estimates, and it was proved in [46] and [55] that for such a special operator $\mathscr{L}$ the Hardy spaces defined via square functions are equivalent to those defined via maximal functions. Hence, the general theory developed in [46] applies to this Schrödinger setting. However, the spaces $H^p_{\mathscr{L}}(\mathbb{R}^n)$ associated to $\mathscr{L} = -\Delta + V$ enjoy some interesting properties which may not be satisfied by Hardy spaces associated to general operators satisfying Davies-Gaffney estimates. For instance, if the potential V satisfies certain additional assumptions (e.g., reverse Hölder inequality), the space $H^1_{\mathscr{L}}(\mathbb{R}^n)$ associated to $\mathscr{L} = -\Delta+V$ is characterized by the (generalized) Riesz transform $\nabla(-\Delta+V)^{-1/2}$; see [26] for more details.

It is natural to ask whether one can establish a theory of Besov and Treibel-Lizorkin spaces associated to operators. To do this one first needs to generalize the classical Littlewood-Paley decomposition to operator settings. Recall that if ψ and φ are two functions in $\mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\hat{\varphi}$ is compact and bounded away from the origin, and

$$\hat{\psi}(\xi) + \sum_{j=1}^{\infty} \hat{\varphi}(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $\hat{}$ denotes the Fourier transform on $\mathbb{R}^n$, then for any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f = \psi * f + \sum_{j=1}^{\infty} \varphi_j * f \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{1.2}$$

where $\varphi_j(x) := 2^{jn}\varphi(2^j x)$ for $j \ge 1$. This reproducing identity is the starting point of establishing Besov and Triebel-Lizorkin spaces. However, when one considers function spaces associated to abstract operators, the Fourier transform is unavailable in general. Nevertheless, for (unbounded) self-adjoint operators, the function calculus can be regarded as a good substitute of the Fourier transform. To be precise, let (X, μ) be a measure space and consider a non-negative self-adjoint operator $\mathscr{L}$ on $L^2(X, d\mu)$. Let $\{E(\lambda) : \lambda \ge 0\}$ be the spectral resolution of $\mathscr{L}$. Given any bounded Borel measurable function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$, the operator $\Phi(\mathscr{L})$ defined by

$$\Phi(\mathscr{L}) = \int_0^\infty \Phi(\lambda) dE(\lambda)$$

is bounded on $L^2(X, d\mu)$. If Φ_0 and Φ are two function in $C^\infty(\mathbb{R}_{\geq 0})$ such that $\operatorname{supp}\Phi_0$ and $\operatorname{supp}\Phi$ are compact, $0 \notin \operatorname{supp}\Phi$, and

$$\Phi_0(\lambda) + \Phi(2^{-2j}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}_{\geq 0},$$

then by the spectral theory we know that for any $f \in L^2(X, d\mu)$,

$$f = \Phi_0(\mathscr{L})f + \sum_{j=1}^\infty \Phi(2^{-2j}\mathscr{L})f, \tag{1.3}$$

where the convergence of the sum is in $L^2(X, d\mu)$. This can be viewed as an analogy of the reproducing identity (1.2), though it is far from being sufficient to establish Besov and Triebel-Lizorkin spaces associated to $\mathscr{L}$. It is now understood that, to get well-defined Besov and Tribel-Lizorkin spaces via (1.3), one needs to have some size and smoothness estimates for the integral kernels of the operators $\Phi(2^{-2j}\mathscr{L})$ which ensure the "almost orthogonal estimates" for the integral kernels of the operators $\Phi(2^{-2j}\mathscr{L})\Phi(2^{-2\ell}\mathscr{L})$.

Very recently, Kerkyacharian and Petrushev in [59] proved that if the heat kernel of the non-negative self-adjoint operator $\mathscr{L}$ satisfies the Gaussian upper bound and the Höler continuity, and if the function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is sufficiently good, then the operator $\Phi(\mathscr{L})$ is an integral operator and its kernel satisfies appropriate size and smoothness estimates. These estimates enabled them to develop a theory of Besov and Triebel-Lizokin spaces associated to the operator $\mathscr{L}$. Let us describe a bit more precisely their work. Suppose (X, ρ, μ) is a locally compact metric measure space satisfying the doubling, reverse-doubling, and non-collapsing conditions. Suppose further that $\mathscr{L}$ is a non-negative self-adjoint operator on $L^2(X, d\mu)$ whose heat kernel satisfies the pointwise Gaussian upper bound and the Hölder continuity. Let Φ_0, Φ be two functions in $C^\infty(\mathbb{R}_{\geq 0})$ such that

$$\Phi_0^{(2\nu+1)}(0) = 0 \text{ for all } \nu \in \mathbb{N}_0, \tag{1.4}$$

$$\operatorname{supp}\Phi_0 \subset [0, 2], \quad |\Phi_0(\lambda)| \geq c > 0 \text{ for } \lambda \in [0, 2^{3/4}], \tag{1.5}$$

$$\operatorname{supp}\Phi \subset [2^{-1}, 2], \quad |\Phi(\lambda)| \geq c > 0 \text{ for } \lambda \in [2^{-3/4}, 2^{3/4}]. \tag{1.6}$$

We point out that Φ_0, Φ lie in $C^\infty(\mathbb{R}_{\geq 0})$ and satisfy (1.4)–(1.6) if and only if the functions Ψ_0, Ψ defined by

$$\Psi_0(\lambda) := \Phi_0(\sqrt{\lambda}), \quad \Psi(\lambda) := \Phi(\sqrt{\lambda})$$

lie in $C^\infty(\mathbb{R}_{\geq 0})$ and satisfy

$$\operatorname{supp}\Psi_0 \subset [0, 2^2], \quad |\Psi_0(\lambda)| \geq c > 0 \text{ for } \lambda \in [0, 2^{3/2}], \tag{1.7}$$

$$\operatorname{supp}\Psi \subset [2^{-2}, 2^2], \quad |\Psi(\lambda)| \geq c > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}]. \tag{1.8}$$

Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$. In [59], the Besov space $B_{p,q}^{s,\mathscr{L}}(X)$, with $s \in \mathbb{R}$ and $p \in (0, \infty]$ and $q \in (0, \infty]$, is defined as the collection of all distributions $f \in \mathcal{S}'_{\mathscr{L}}(X)$ (see Section 2.5 for the definition of $\mathcal{S}'_{\mathscr{L}}(X)$) such that

$$\|f\|_{B_{p,q}^{s,\mathscr{L}}(X)} := \left(\sum_{j=0}^\infty \left\| 2^{js}\Phi_j(\sqrt{\mathscr{L}})f \right\|_{L^p(X)}^q \right)^{1/q} < \infty,$$

and the Triebel-Lizorkin space $F^{s,\mathscr{L}}_{p,q}(X)$, with $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$, is defined as the collection of all distributions $f \in \mathcal{S}'_{\mathscr{L}}(X)$ such that

$$\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} := \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js}\Phi_j(\sqrt{\mathscr{L}})f \right|^q \right)^{1/q} \right\|_{L^p(X)} < \infty.$$

Kerkyacharian and Petrushev in [59] showed that these function spaces are independent of the choice of Φ_0, Φ as long as Φ_0, Φ satisfy (1.4)–(1.6), by using the smooth functional calculus related to the heat kernel and a generalized Peetre's maximal inequality established by themselves. Moreover, using elegant techniques on functional calculus and on the construction of frames, they also established embedding theorems, heat-kernel characterization and frame decomposition for these function spaces. Their theory applies in quite general situations such as uniformly elliptic divergence form operators on $\mathbb{R}^n$ with real symmetric coefficients, Riemannian manifolds with non-negative Ricci curvature and Lie groups of polynomial growth. However, the restriction that the heat kernel satisfies Hölder continuity makes some interesting operators fall outside the scope of their setting. For example, the heat kernel of the Schrödinger operator $-\Delta + V$, with $0 \le V \in L^1_{loc}(\mathbb{R}^n)$, enjoys Gaussian upper bound; however, in general its heat kernel does not satisfies the Hölder continuity. Also, the Dirichlet or Neumann heat kernels of some non-smooth domains enjoy the Gaussian upper bound but may not satisfy the Hölder continuity. See Section 2.6 for a more detailed discussion.

The primary goal of the current thesis is to generalize the work of Kerkyacharian and Petrushev [59]. To be precise, in this thesis we develop the Besov and Triebel-Lizorkin spaces on a doubling metric measure space (X,ρ,μ) associated to a non-negative self-adjoint operator $\mathscr{L}$ on $L^2(X,d\mu)$ whose heat kernel $p_t(x,y)$ satisfies the Gaussian upper bound but need not satisfy any condition on the regularity in the variables x and y. As we mentioned above, there are some interesting operators whose heat kernels satisfy Gaussian upper bound but may not satisfy the Hölder continuity. Thus, our setting is more general than that considered in [59]. Let us describe our definition of Besov and Triebel-Lizorkin spaces associated to operators. Let $s \in \mathbb{R}$ and $q \in (0,\infty]$. Let Φ_0, Φ be two functions in $\mathcal{S}(\mathbb{R}_{\ge 0})$ such that

$$|\Phi_0(\lambda)| \ge c > 0 \quad \text{on } \{0 \le \lambda \le 2^{3/2}\varepsilon\}, \tag{1.9}$$

$$|\Phi(\lambda)| \ge c > 0 \quad \text{on } \{2^{-3/2}\varepsilon \le \lambda \le 2^{3/2}\varepsilon\} \tag{1.10}$$

for some $\varepsilon > 0$, and

$$\text{the function } \lambda \mapsto \lambda^{-M}\Phi(\lambda) \text{ belongs to } \mathcal{S}(\mathbb{R}_{\ge 0}) \tag{1.11}$$

for some nonnegative integer $M > s/2$. Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$, $j \ge 1$. For $p \in (0,\infty]$, we define the Besov space $B^{s,\mathscr{L}}_{p,q}(X)$ as the collection of all distributions $f \in \mathcal{S}'_{\mathscr{L}}(X)$ such that

$$\|f\|_{B^{s,\mathscr{L}}_{p,q}(X)} := \left(\sum_{j=0}^{\infty} \left\| 2^{js}\Phi_j(\mathscr{L})f \right\|^q_{L^p(X)} \right)^{1/q} < \infty. \tag{1.12}$$

For $p \in (0,\infty)$, we define the Triebel-Lizorkin space $F^{s,\mathscr{L}}_{p,q}(X)$ as the collection of all distributions $f \in \mathcal{S}'_{\mathscr{L}}(X)$ such that

$$\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} := \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js}\Phi_j(\mathscr{L})f \right|^q \right)^{1/q} \right\|_{L^p(X)} < \infty. \tag{1.13}$$

To see that these function spaces are well-defined, we first need to show that different choices of Φ_0, Φ yield equivalent Besov and Triebel-Lizorkin quasi-norms, as long as Φ_0, Φ satisfy (1.9)–(1.11). However, since there is no assumption on the regularity of the heat kernel of $\mathscr{L}$, the generalized Petree's inequality established in [59, Lemma 6.4] fails in our setting. To overcome this obstacle, we follow the ideas of Bui *et al.* [11, 12], Rychkov [68] and Ullrich [84]. Note that the most important contribution of these papers is the characterization of classical Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$ using Littlewood-Paley decomposition (1.2) involving functions $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ which are not required to be band limited but only satisfy a Tauberian condition and a moment condition. In the present thesis we extend such type characterization to the operator setting. This approach enables us to remove the restriction that Φ_0, Φ have compact supports. (Compare (1.9)–(1.11) to (1.7)–(1.8)).

This thesis is organized as follows: In Chapter 2 we present notations and preliminaries. After introducing some notations in Section 2.1, we recall the notions of doubling, reverse doubling, and non-collapsing conditions for the metric measure space (X, ρ, μ) in Section 2.2 and recall the notions of Gaussian upper bound and Hölder continuity for heat kernels in Section 2.3. In Section 2.4 we recall an important result of Kerkyacharian and Petrushev concerning smooth functional calculus induced by the heat kernels. In Section 2.5 we recall the notions of test functions and distributions associated to operators which were first introduced by Kerkyacharian and Petrushev. In Section 2.6 we describe several examples.

In Chapter 3 we introduce Besov and Triebel-Lizorkin spaces associated to the operator $\mathscr{L}$ using the quasi-norms (1.12) and (1.13) in which Φ_0 and Φ are chosen to satisfy (1.9)–(1.11). The main result in this chapter is the well-definedness of our function spaces. More precisely, we show that a different choice of (Φ_0, Φ) yields equivalent Besov and Triebel-Lizorkin quasi-norms. We divide the proof into two steps. The first step (see Theorem 3.4) is to show that if (Φ_0, Φ) is a couple of functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying (1.9)–(1.11) then

$$\left(\sum_{j=0}^{\infty} \left\| 2^{js} [\Phi_j(\mathscr{L})]^*_a f \right\|^q_{L^p(X)} \right)^{1/q} \sim \left(\sum_{j=0}^{\infty} \left\| 2^{js} \Phi_j(\mathscr{L}) f \right\|^q_{L^p(X)} \right)^{1/q},$$

$$\left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} [\Phi_j(\mathscr{L})]^*_a f \right|^q \right)^{1/q} \right\|_{L^p(X)} \sim \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} \Phi_j(\mathscr{L}) f \right|^q \right)^{1/q} \right\|_{L^p(X)},$$

where $[\Phi_j(\mathscr{L})]^*_a f$ is the Peetre maximal functions defined by (3.3), and the notation $\sim$ means that the quantities on both sides are comparable. The second step (see Theorem 3.5) is to show that if (Φ_0, Φ) and $(\widetilde{\Phi}_0, \widetilde{\Phi})$ are two couples of functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying (1.9)–(1.11) then

$$\left(\sum_{j=0}^{\infty} \left\| 2^{js} [\widetilde{\Phi}_j(\mathscr{L})]^*_a f \right\|^q_{L^p(X)} \right)^{1/q} \sim \left(\sum_{j=0}^{\infty} \left\| 2^{js} [\Phi_j(\mathscr{L})]^*_a f \right\|^q_{L^p(X)} \right)^{1/q},$$

$$\left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} [\widetilde{\Phi}_j(\mathscr{L})]^*_a f \right|^q \right)^{1/q} \right\|_{L^p(X)} \sim \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} [\Phi_j(\mathscr{L})]^*_a f \right|^q \right)^{1/q} \right\|_{L^p(X)}.$$

Combining these two steps we see that our definition of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$ is independent of the choice of Φ_0, Φ, as long as Φ_0, Φ satisfy (1.9)–(1.11). In Section 3.3 we give some basic properties of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$, including the completeness of these spaces and the continuous embeddings $B^{s,\mathscr{L}}_{p,q}(X) \hookrightarrow \mathcal{S}'_{\mathscr{L}}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X) \hookrightarrow \mathcal{S}'_{\mathscr{L}}(X)$. In Section 3.4 we show the continuous Littlewood-Paley characterization which states that if Φ_0, Φ satisfy (1.9)–(1.11) then

$$\|f\|_{B^{s,\mathscr{L}}_{p,q}(X)} \sim \|\Phi_0(\mathscr{L})f\|_{L^p(X)} + \left(\int_0^1 t^{-sq}\big\|\Phi(t^2\mathscr{L})f\big\|^q_{L^p(X)}\frac{dt}{t}\right)^{1/q}, \tag{1.14}$$

$$\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} \sim \|\Phi_0(\mathscr{L})f\|_{L^p(X)} + \left\|\left(\int_0^1 t^{-sq}\big|\Phi(t^2\mathscr{L})f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}. \tag{1.15}$$

This characterization is very useful because it leads immediately to the heat kernel characterization of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$. Indeed, if we take

$$\Phi_0(\lambda) = e^{-\lambda} \quad \text{and} \quad \Phi(\lambda) = \lambda^M e^{-\lambda}$$

in (1.14) and (1.15), we get

$$\|f\|_{B^{s,\mathscr{L}}_{p,q}(X)} \sim \|e^{-\mathscr{L}}f\|_{L^p(X)} + \left(\int_0^1 t^{-sq}\big\|(t^2\mathscr{L})^M e^{-t^2\mathscr{L}}f\big\|^q_{L^p(X)}\frac{dt}{t}\right)^{1/q},$$

$$\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} \sim \|e^{-\mathscr{L}}f\|_{L^p(X)} + \left\|\left(\int_0^1 t^{-sq}\big|(t^2\mathscr{L})^M e^{-t^2\mathscr{L}}f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}.$$

In Chapter 4 we systematically discuss properties and characterizations of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$. First we establish the atomic decomposition of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$. Then, using the atomic decomposition and following the idea of [64], we show the complex interpolation property for $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$. We also obtain the lifting property and embedding theorem for these spaces. Finally, in Section 4.5 we point out that $F^{0,\mathscr{L}}_{p,2}(X)$ is identified with $L^p(X)$ for $p \in (1,\infty)$.

In Chapter 5 we introduce homogeneous Besov spaces $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ and homogeneous Triebel-Lizorkin spaces $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$. To define these homogeneous spaces, we need to introduce the new test function space $\mathcal{S}_{\infty,\mathscr{L}}(X)$. The key ingredient in this chapter is a homogeneous Calderón reproducing formula in the distribution space $\mathcal{S}'_{\infty,\mathscr{L}}(X)$ (see Proposition 5.5). In Section 5.3 we list some properties and characterizations of $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ and $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$. Most of the proofs of these properties and characterizations are skipped since they are analogous to their inhomogeneous versions given in Chapter 3. In Section 5.4 we show that $\dot{F}^{0,\mathscr{L}}_{p,2}(X)$ $(0<p<\infty)$ are characterized by the Lusin area integral. In Section 5.5 we show that $\dot{F}^{0,\mathscr{L}}_{p,2}(X)$ $(0<p\le 1)$ can also be identified with the atomic Hardy spaces $H^{p,q,M}_{\mathscr{L}}(X)$ associated to $\mathscr{L}$.

In Chapter 6 we apply our theory to the special setting of stratified Lie groups. It is well known that any stratified Lie group G satisfies the doubling, reverse doubling and non-collapsing conditions, and the heat kernel of any sub-Laplacian Δ on G satisfies the Gaussian upper bound. Hence, applying the general theory established in Chapter 3 and Chapter 5, we can define the Besov spaces $B^{s,\Delta}_{p,q}(G)$ and $\dot{B}^{s,\Delta}_{p,q}(G)$ and the Tribel-Lizorkin spaces $F^{s,\Delta}_{p,q}(G)$ and $\dot{F}^{s,\Delta}_{p,q}(G)$. In section 6.2 we prove that for any two sub-Laplacians Δ and $\widetilde{\Delta}$ on G, we have $B^{s,\Delta}_{p,q}(G) = B^{s,\widetilde{\Delta}}_{p,q}(G)$, $\dot{B}^{s,\Delta}_{p,q}(G) = \dot{B}^{s,\widetilde{\Delta}}_{p,q}(G)$, $F^{s,\Delta}_{p,q}(G) = F^{s,\widetilde{\Delta}}_{p,q}(G)$ and $\dot{F}^{s,\Delta}_{p,q}(G) = \dot{F}^{s,\widetilde{\Delta}}_{p,q}(G)$. This result tells us that the Besov and Triebel-Lizorkin spaces on G reflect properties of the group, not of the sub-Laplacian used for the construction of the Littlewood-Paley decomposition. In Section 6.3 we obtain the $\dot{B}^s_{p,q}(G)$- and $\dot{F}^s_{p,q}(G)$-boundedness of singular integral operators of convolution type on G.

In Chapter 7 we consider the maximal function characterization the space $\dot{F}^{0,\Delta}_{p,2}(X)$ in the special case that X is a Riemannian manifold and Δ is the Laplace-Beltrami operator on X. We show that in this case $\dot{F}^{0,\Delta}_{p,2}(X)$ can be identified with $H^p_{\max,\Delta}(X)$, where $H^p_{\max,\Delta}(X)$ is the Hardy space on X defined via the non-tangential or radial maximal functions associated to Δ.

Chapter 2

Notations and preliminaries

2.1 Notations

Throughout this thesis we assume that X is a locally compact metric space with a distance ρ, and μ is a positive regular Borel measure on X. To avoid repetition, we skip this assumption in all the subsequent statements.

We denote by $B(x,r)$ the open ball with center $x \in X$ and radius $r > 0$, and by $V(x,r)$ its measure $\mu(B(x,r))$.

The symbol $\mathbb{N}$ will denote the set of all positive integers while $\mathbb{N}_0$ will denote the set of all non-negative integers.

If σ is a positive number, we denote by $\lfloor \sigma \rfloor$ the largest integer less than or equal to σ. For $p \in (1,\infty)$, the conjugate exponent p' is defined by $1/p + 1/p' = 1$.

For $p \in (0,\infty)$, the Lebesgue space $L^p(X, d\mu)$ will be written in short $L^p(X)$.

Let $\mathbb{R}_{\geq 0} := [0,\infty)$ and $\mathbb{R}_{>0} := (0,\infty)$. If Φ is a smooth function on $\mathbb{R}_{\geq 0}$ and $\nu \in \mathbb{N}_0$, then we use $\Phi^{(\nu)}$ to denote the ν-th order derivative of Φ. In addition, we define the space $\mathcal{S}(\mathbb{R}_{\geq 0})$ by

$$\mathcal{S}(\mathbb{R}_{\geq 0}) := \big\{\Phi \in C^\infty(\mathbb{R}_{>0}) : \forall \nu \in \mathbb{N}_0, \Phi^{(\nu)} \text{ decays rapidly at infinity and } \lim_{\lambda \to 0^+} \Phi^{(\nu)}(\lambda) \text{ exists}\big\}.$$

Throughout this thesis, the letters C, c will denote positive constants, which are independent of the main parameters and not necessarily the same at each occurrence. By writing $A \lesssim B$, we mean $A \leq CB$. We also use $A \sim B$ to denote $A \lesssim B \lesssim A$. Some important constants will be denoted by $C_*, C_\dagger, C_\flat, C_\sharp, \cdots$, and they will remain unchanged throughout.

2.2 Doubling, reverse doubling, and non-collapsing conditions

One says that the metric measure space (X, ρ, μ) satisfies the *doubling condition*, if there exists a constant $C_* > 1$ such that

$$0 < V(x,2r) \leq C_* V(x,r) < \infty \tag{2.1}$$

for all $x \in X$ and $r \in (0, \infty)$. Notice that (2.1) implies

$$V(x, \lambda r) \leq C_* \lambda^d V(x, r) \tag{2.2}$$

for all $x \in X$, $r \in (0, \infty)$ and $\lambda \in [1, \infty)$, where $d = \log_2 C_* > 0$ is a constant playing the role of a dimension. Since $B(x, r) \subset B(y, \rho(x, y) + r)$, (2.2) yields that

$$V(x, r) \leq C_* \left(1 + \frac{\rho(x, y)}{r}\right)^d V(y, r) \tag{2.3}$$

for all $x, y \in X$ and $r \in (0, \infty)$.

One says that (X, d, μ) satisfies the *reverse doubling condition*, if there exists a constant $C_\dagger > 1$ such that

$$V(x, 2r) \geq C_\dagger V(x, r) \tag{2.4}$$

for all $x \in X$ and $r \in (0, \frac{\operatorname{diam} X}{3}]$. Note that (2.4) implies

$$V(x, \lambda r) \geq C_\dagger^{-1} \lambda^\varsigma V(x, r) \tag{2.5}$$

for all $x \in X$, $r \in (0, \infty)$, $\lambda \in [1, \infty)$ and $r \in (0, \frac{2 \operatorname{diam} X}{3\lambda}]$, where $\varsigma = \log_2 C_\dagger > 0$. It was shown in [19, Proposition 2.2] that the reverse doubling condition (2.4) is a consequence of the doubling condition (2.1) whenever X is connected.

One says that (X, d, μ) satisfies the *non-collapsing condition*, if there exists a constant $C_\flat > 0$ such that

$$\inf_{x \in X} V(x, 1) \geq C_\flat. \tag{2.6}$$

Note that (2.6) coupled with (2.1) imply that for all $r \in (0, 1]$

$$\inf_{x \in X} V(x, r) \geq C_*^{-1} C_\flat r^d. \tag{2.7}$$

2.3 Gaussian upper bound and Hölder continuity of the heat kernel

Let $\mathscr{L}$ be a non-negative self-adjoint operator with domain $D(\mathscr{L})$ dense in $L^2(X)$. Let $E(\lambda)$ be the spectral resolution of $\mathscr{L}$. For any bounded Borel measurable function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$, the operator

$$\Phi(\mathscr{L}) = \int_0^\infty \Phi(\lambda) dE(\lambda).$$

is bounded on $L^2(X)$. We assume that the associated semigroup $P_t = e^{-t\mathscr{L}}$ consists of integral operators with (heat) kernel $p_t(x, y)$. We say that the heat kernel of $\mathscr{L}$ satisfies the *Gaussian upper bound*, if there exist two constants $C_\sharp, c_\sharp > 0$ such that

$$|p_t(x, y)| \leq C_\sharp \frac{\exp\left\{-\frac{\rho^2(x,y)}{c_\sharp t}\right\}}{\sqrt{V(x, \sqrt{t}) V(y, \sqrt{t})}}. \tag{2.8}$$

for all $t \in (0,1]$ and $x, y \in X$. We say that the heat kernel of $\mathscr{L}$ satisfies the *Hölder continuity*, if there exists a constant $\alpha > 0$ such that

$$|p_t(x,y) - p_t(x,y')| \le C_\sharp \left(\frac{\rho(y,y')}{\sqrt{t}}\right)^\alpha \frac{\exp\left\{-\frac{\rho^2(x,y)}{c_\sharp t}\right\}}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \tag{2.9}$$

for all $t \in (0,1]$ and $x, y, y' \in X$ satisfying that $\rho(y,y') \le \sqrt{t}$.

2.4 Smooth functional calculus induced by the heat kernel

For $t, \sigma > 0$ and $x, y \in X$, we set

$$D_{t,\sigma}(x,y) := [V(x,t)V(y,t)]^{-1/2}\left(1 + \frac{\rho(x,y)}{t}\right)^{-\sigma}. \tag{2.10}$$

In addition, for $\Phi \in \mathcal{S}(\mathbb{R}_{\ge 0})$ and $m \in \mathbb{N}_0$, we put

$$\|\Phi\|_{(m)} := \sup_{\lambda \in \mathbb{R}_{\ge 0},\ 0 \le \nu \le m} (1+\lambda)^{m+d+1} |\Phi^{(\nu)}(\lambda)|. \tag{2.11}$$

Next we recall an important estimate obtained by Kerkyacharian and Petrushev [59].

Lemma 2.1. ([59, Theorem 3.4]) *Suppose (X, ρ, μ) satisfies the doubling condition* (2.1)*, reverse doubling condition* (2.4) *and non-collapsing condition* (2.6)*. Suppose $\mathscr{L}$ is a non-negative self-adjoint operator on $L^2(X)$ whose heat kernel satisfies the Gaussian upper bound* (2.8) *and the Hölder continuity* (2.9)*. Suppose $m \in \mathbb{N}_0$, $m \ge d+1$, $r \ge m+d+1$, $\Phi \in C^m(\mathbb{R}_{\ge 0})$, and there exists a constant $\widetilde{C} > 0$ such that*

$$|\Phi^{(\nu)}(\lambda)| \le \widetilde{C}(1+\lambda)^{-r}$$

for all $\lambda \in \mathbb{R}_{\ge 0}$ and $\nu \in \{0, 1, \cdots, m\}$. Suppose further that

$$\Phi^{(2\nu+1)}(0) = 0$$

for all $\nu \in \mathbb{N}_0$ with $2\nu + 1 \le m$. Then for any $t \in (0,1]$, $\Phi(t\sqrt{\mathscr{L}})$ is an integral operator with a kernel $K_{\Phi(t\sqrt{\mathscr{L}})}(x,y)$; moreover, there exists a constant $C > 0$ (depending on m) such that

$$\left|K_{\Phi(t\sqrt{\mathscr{L}})}(x,y)\right| \le C\widetilde{C} D_{t,m}(x,y) \tag{2.12}$$

for all $t \in (0,1]$ and $x, y \in X$, and

$$\left|K_{\Phi(t\sqrt{\mathscr{L}})}(x,y) - K_{\Phi(t\sqrt{\mathscr{L}})}(x,y')\right| \le C\widetilde{C}\left(\frac{\rho(y,y')}{t}\right)^\alpha D_{t,m}(x,y) \tag{2.13}$$

for all $t \in (0,1]$ and $x, y, y' \in X$ satisfying $\rho(y,y') \le t$.

Remark 2.2. If we do not assume the Hölder continuity for the heat kernel of $\mathscr{L}$, the estimate (2.13) fails but the estimate (2.12) still holds.

Observe that if $\Phi \in \mathcal{S}(\mathbb{R}_{\ge 0})$, then the function $\Psi : \mathbb{R}_{\ge 0} \to \mathbb{C}$ defined by $\Psi(\lambda) := \Phi(\lambda^2)$ also lies in $\mathcal{S}(\mathbb{R}_{\ge 0})$, and moreover, $\Psi^{(2\nu+1)}(0) = 0$ for all $\nu \in \mathbb{N}_0$. Also note that for any $m \in \mathbb{N}_0$, there exists a constant $C > 0$, which depends on m but is independent of Φ, such that $\|\Psi\|_{(m)} \le C\|\Phi\|_{(m)}$. By these facts, we can reformulate Lemma 2.1 as follows:

Lemma 2.3. *Suppose* (X, ρ, μ) *satisfies the doubling condition* (2.1)*, reverse doubling condition* (2.4) *and non-collapsing condition* (2.6)*. Suppose* $\mathscr{L}$ *is a non-negative self-adjoint operator on* $L^2(X)$ *whose heat kernel satisfies the Gaussian upper bound* (2.8) *and the Hölder continuity* (2.9)*. Then for any* $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ *and* $t \in (0,1]$*,* $\Phi(t^2\mathscr{L})$ *is an integral operator with a kernel* $K_{\Phi(t^2\mathscr{L})}(x,y)$*; moreover, for any* $m \in \mathbb{N}_0$ *with* $m \geq d+1$*, there is a constant* $C > 0$*, which depends on* m *but is independent of* Φ*, such that*

$$\left|K_{\Phi(t^2\mathscr{L})}(x,y)\right| \leq C\|\Phi\|_{(m)} D_{t,m}(x,y) \tag{2.14}$$

for all $t \in (0,1]$ *and* $x, y \in X$*, and*

$$\left|K_{\Phi(t^2\mathscr{L})}(x,y) - K_{\Phi(t^2\mathscr{L})}(x,y')\right| \leq C\|\Phi\|_{(m)} \left(\frac{\rho(y,y')}{t}\right)^{\alpha} D_{t,m}(x,y) \tag{2.15}$$

for all $t \in (0,1]$ *and* $x, y, y' \in X$ *satisfying* $\rho(y,y') \leq t$*.*

Remark 2.4. If we do not assume the Hölder continuity for the heat kernel of $\mathscr{L}$, the estimate (2.15) fails but the estimate (2.14) still holds.

2.5 Test functions and distributions associated to operators

We recall from [59] the notions of test functions and distributions on X associated to $\mathscr{L}$. The test function space $\mathcal{S}_{\mathscr{L}}(X)$ is defined as the collection of all functions $\phi \in \cap_{k\in\mathbb{N}_0} D(\mathscr{L}^k)$ such that

$$\mathcal{P}_{k,m}(\phi) := \operatorname*{ess\,sup}_{x\in X} (1+\rho(x,x_0))^m |\mathscr{L}^k\phi(x)| < \infty$$

for all $k, m \in \mathbb{N}_0$, where $x_0 \in X$ is arbitrary fixed point on X. Obviously, the definition of $\mathcal{S}_{\mathscr{L}}(X)$ is independent of the choice of x_0. So we fix x_0 once and for all. For our purpose it is convenient to introduce the following directed family of norms: For $k, m \in \mathbb{N}_0$ and $\phi \in \mathcal{S}_{\mathscr{L}}(X)$, we define

$$\mathcal{P}^*_{k,m}(\phi) := \sum_{\substack{0\leq j\leq k\\ 0\leq \ell\leq m}} \mathcal{P}_{j,\ell}(\phi).$$

It was shown in [59] that $\mathcal{S}_{\mathscr{L}}(X)$ is a Fréchet space. The space $\mathcal{S}'_{\mathscr{L}}(X)$ of distributions on X is defined as the space of all continuous linear functionals on $\mathcal{S}_{\mathscr{L}}(X)$. The action of $f \in \mathcal{S}'_{\mathscr{L}}(X)$ on $\phi \in \mathcal{S}_{\mathscr{L}}(X)$ will be denoted by $(f, \phi) := f(\phi)$. However, sometimes we will work with the sesquilinear version $\langle f, \phi\rangle = (f, \bar{\phi})$.

An important consequence of Lemma 2.3 and Remark 2.4 is the following

Corollary 2.5. *Suppose* (X, ρ, μ) *satisfies the doubling condition* (2.1)*, reverse doubling condition* (2.4) *and non-collapsing condition* (2.6)*. Suppose* $\mathscr{L}$ *is a non-negative self-adjoint operator on* $L^2(X)$ *whose heat kernel satisfies the Gaussian upper bound* (2.8)*. Let* $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$*. Then:*

(i) *For any* $t \in (0,1]$ *and for almost every fixed* $y \in X$*,* $K_{\Phi(t^2\mathscr{L})}(\cdot, y)$ *belongs to* $\mathcal{S}_{\mathscr{L}}(X)$*.*

(ii) *For any* $t \in (0,1]$ *and for almost every fixed* $x \in X$*,* $K_{\Phi(t^2\mathscr{L})}(x, \cdot)$ *belongs to* $\mathcal{S}_{\mathscr{L}}(X)$*.*

Proof. Let $t \in (0,1]$. From (5.14) in [59] we see that for almost every fixed $y \in X$ and for any $k \in \mathbb{N}_0$,

$$\mathscr{L}^k\left[K_{\Phi(t^2\mathscr{L})}(\cdot,y)\right] = K_{\mathscr{L}^k\Phi(t^2\mathscr{L})}(\cdot,y) = t^{-2k}K_{(t^2\mathscr{L})^k\Phi(t^2\mathscr{L})}(\cdot,y).$$

Hence, if m is an integer with $m \geq d+1$, we have by Lemma 2.3,

$$\begin{aligned} \left|\mathscr{L}^k\left[K_{\Phi(t^2\mathscr{L})}(\cdot,y)\right](x)\right| &= t^{-2k}\left|K_{(t^2\mathscr{L})^k\Phi(t^2\mathscr{L})}(x,y)\right| \\ &\lesssim t^{-2k}\|\lambda\mapsto\lambda^k\Phi(\lambda)\|_{(m)}D_{t,m}(x,y) \\ &\lesssim t^{-2k}\|\Phi\|_{(k+m)}D_{t,m}(x,y), \quad \text{for a.e. } x\in X. \end{aligned} \tag{2.16}$$

This implies that $K_{\Phi(t^2\mathscr{L})}(\cdot,y) \in \mathcal{S}_{\mathscr{L}}(X)$. Since $K_{\Phi(t^2\mathscr{L})}(x,\cdot) = \overline{K_{\Phi(t^2\mathscr{L})}(\cdot,x)}$, we also have $K_{\Phi(t^2\mathscr{L})}(x,\cdot) \in \mathcal{S}_{\mathscr{L}}(X)$ for almost every fixed x. □

Thanks to Corollary 2.5, it is now natural to define, for any $f \in \mathcal{S}'_{\mathscr{L}}(X)$ and $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$,

$$\Phi(\mathscr{L})f(x) := \big(f, K_{\Phi(\mathscr{L})}(x,\cdot)\big), \quad \text{for a.e. } x\in X.$$

This extends the domain of $\Phi(\mathscr{L})$ from $L^2(X)$ to $\mathcal{S}'_{\mathscr{L}}(X)$.

Lemma 2.6. (i) *Suppose $f \in \mathcal{S}'_{\mathscr{L}}(X)$ and $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$. Then there is a non-negative integer N such that for a.e. $x \in X$,*

$$|\Phi(\mathscr{L})f(x)| \leq C(1+\rho(x,x_0))^N. \tag{2.17}$$

In particular, $\Phi(\mathscr{L})f$ can be regarded as a distribution in $\mathcal{S}'_{\mathscr{L}}(X)$.

(ii) *Let $\Phi,\Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ and $\Upsilon := \Phi\Psi$. Then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\Phi(\mathscr{L})\big(\Psi(\mathscr{L})f\big) = \Psi(\mathscr{L})\big(\Phi(\mathscr{L})f\big) = \Upsilon(\mathscr{L})f \quad \text{in } \mathcal{S}'_{\mathscr{L}}(X). \tag{2.18}$$

Proof. (i) Since $f \in \mathcal{S}'_{\mathscr{L}}(X)$ and $K_{\Phi(\mathscr{L})}(x,\cdot) \in \mathcal{S}_{\mathscr{L}}(X)$, there exist $k_0, m_0 \in \mathbb{N}_0$ and a constant c_f (depending on f) such that for a.e. $x \in X$,

$$\begin{aligned} |\Phi(\mathscr{L})f(x)| &= \left|f\big(K_{\Phi(\mathscr{L})}(x,\cdot)\big)\right| \\ &\leq c_f\mathcal{P}^*_{k_0,m_0}\big(K_{\Phi(\mathscr{L})}(x,\cdot)\big) = c_f\sum_{\substack{0\leq k\leq k_0\\0\leq m\leq m_0}}\mathcal{P}_{j,\ell}\big(K_{\Phi(\mathscr{L})}(x,\cdot)\big) \\ &= c_f\sum_{\substack{0\leq k\leq k_0\\0\leq m\leq m_0}}\operatorname*{ess\,sup}_{y\in X}(1+\rho(y,x_0))^m\left|\mathscr{L}^k\left[K_{\Phi(\mathscr{L})}(x,\cdot)\right](y)\right| \\ &= c_f\sum_{\substack{0\leq k\leq k_0\\0\leq m\leq m_0}}\operatorname*{ess\,sup}_{y\in X}(1+\rho(y,x_0))^m\left|K_{\mathscr{L}^k\Phi(\mathscr{L})}(x,y)\right|. \end{aligned} \tag{2.19}$$

Here, for the last equality we used (5.14) in [59]. Since the function $\lambda\mapsto\lambda^k\Phi(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$, by Lemma 2.3 and (2.3) we have that for $k \in \{0,1,\cdots,k_0\}$

$$\begin{aligned} |K_{\mathscr{L}^k\Phi(\mathscr{L})}(x,y)| &\lesssim \|\lambda\mapsto\lambda^k\Phi(\lambda)\|_{(m_0+\lfloor d/2\rfloor+1)}[V(x,1)V(y,1)]^{-1/2}(1+\rho(x,y))^{-(m+\frac{d}{2})} \\ &\lesssim \|\lambda\mapsto\lambda^k\Phi(\lambda)\|_{(m_0+\lfloor d/2\rfloor+1)}[V(x,1)]^{-1}(1+\rho(x,y))^{-m_0} \\ &\lesssim \|\Phi\|_{(m_0+k_0+\lfloor d/2\rfloor+1)}[V(x,1)]^{-1}(1+\rho(x,x_0))^{m_0}(1+\rho(y,x_0))^{-m_0} \\ &\lesssim \|\Phi\|_{(m+k_0+\lfloor d/2\rfloor+1)}[V(x_0,1)]^{-1}(1+\rho(x,x_0))^{m_0+d}(1+\rho(y,x_0))^{-m_0} \\ &\sim (1+\rho(y,x_0))^{-m_0}. \end{aligned}$$

This together with (2.19) yield (2.17) with $N = m+d$.

(ii) By [67, Theorem 13.24], we know that for all $\phi \in \mathcal{S}_{\mathscr{L}}(X)$,

$$\Phi(\mathscr{L})\big(\Psi(\mathscr{L})\phi\big) = \Psi(\mathscr{L})\big(\Phi(\mathscr{L})\phi\big) = \Upsilon(\mathscr{L})\phi \quad \text{in } L^2(X). \tag{2.20}$$

Also note that by [59, Propostion 5.3] all of the three functions in (2.18) belong to the class $\mathcal{S}_{\mathscr{L}}(X)$. Thus (2.18) holds also in $\mathcal{S}_{\mathscr{L}}(X)$. The validness of (2.18) in the sense of distributions then follows by duality. □

2.6 Examples

As we mentioned in the introduction, Kerkyacharian and Petrushev [59] studied Besov and Triebel-Lizorkin spaces on (X, ρ, μ) associated to a non-negative self-adjoint operator $\mathscr{L}$ under the assumption that the heat kernel of $\mathscr{L}$ satisfies the Gaussian upper bound and the Hölder continuity, while in the current thesis the heat kernel $p_t(x, y)$ of the operator $\mathscr{L}$ is not assumed to have any regularity in the variables x and y. Thus, our setting is more general than that considered in [59]. In particular, the theory developed in the current thesis applies to all the examples described in [59]. Next we recall some of these examples. In addition, we also give some examples of operators whose heat kernel satisfies Gaussian upper bound but may not satisfy the Hölder continuity.

• *Uniformly elliptic divergence form operators on* $\mathbb{R}^n$. Let $\{a_{i,j}(x)\}_{1\le i,j\le n}$ be a matrix-valued function depending on $x \in \mathbb{R}^n$ such that

$$\begin{aligned} a_{i,j}(x) = a_{j,i}(x) &\quad \text{for all } 1 \le i, j \le n \text{ and a.e. } x \in \mathbb{R}^n, \\ a_{i,j} \in L^\infty(\mathbb{R}^n) &\quad \text{for all } 1 \le i, j \le n, \end{aligned}$$

and the following uniform ellipticity condition holds:

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \ge \theta|\xi|^2 \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and all } \xi \in \mathbb{R}^n, \tag{2.21}$$

where θ is a positive constant. Define a sesquilinear form Q on the product space $W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ by

$$Q(u, v) = \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} dx$$

for $u, v \in W^{1,2}(\mathbb{R}^n)$. Let $\mathscr{L}$ be the self-adjoint operator associated with Q. Then the domain of $\mathscr{L}$ is given by

$$D(\mathscr{L}) = \left\{ u \in W^{1,2}(\mathbb{R}^n) : \exists v \in L^2(\mathbb{R}^n) \text{ such that } Q(u, \varphi) = \int_{\mathbb{R}^n} v\bar{\varphi}, \quad \forall \varphi \in W^{1,2}(\mathbb{R}^n) \right\}.$$

Formally we can write

$$\mathscr{L} = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{i,j}(x) \frac{\partial}{\partial x_j} \right).$$

In this setting, the Gaussian upper and lower bounds of the heat kernel were obtained by Aronson and the Hölder regularity of the solutions is due to Nash [63].

• *Domains in* $\mathbb{R}^n$. Let Ω be a domain of $\mathbb{R}^n$. Let $\{a_{i,j}(x)\}_{1\le i,j\le n}$ be a matrix-valued function depending on $x \in \Omega$ such that

$$\begin{aligned} a_{i,j}(x) = a_{j,i}(x) &\quad \text{for all } 1 \le i, j \le n \text{ and a.e. } x \in \Omega, \\ a_{i,j} \in L^\infty(\Omega) &\quad \text{for all } 1 \le i, j \le n, \end{aligned}$$

and the following uniform ellipticity condition holds:

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n,$$

where θ is a positive constant. Let $\mathcal{V}$ be a linear space such that $C_0^\infty(\Omega) \subset \mathcal{V} \subset W^{1,2}(\Omega)$. Define a sesquilinear form on the product space $\mathcal{V} \times \mathcal{V}$ by

$$Q(u,v) = \int_\Omega \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} dx$$

for $u, v \in \mathcal{V}$. Let $\mathscr{L}$ the self-adjoint operator associated with Q. Then the domain of $\mathscr{L}$ is given by

$$D(\mathscr{L}) = \left\{ u \in \mathcal{V} : \exists v \in L^2(\Omega) \text{ such that } Q(u,\varphi) = \int_\Omega v\bar{\varphi},\ \forall \varphi \in \mathcal{V} \right\}.$$

Different choices of $\mathcal{V}$ correspond to different boundary conditions for the operator $\mathscr{L}$. For example, when $\mathcal{V}$ is chosen to be $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$, it corresponds to the Dirichlet boundary condition and the Neumann boundary condition, respectively. We denote by $\mathscr{L}_D$ and $\mathscr{L}_N$ the divergence operator subjecting to the Dirichlet boundary condition and the Neumann boundary condition, respectively.

Let $p_{t,\mathscr{L}_D}(x,y)$ and $p_{t,\mathscr{L}_N}(x,y)$ be the heat kernels of $\mathscr{L}_D$ and $\mathscr{L}_N$, respectively. It is well-known that (see, e.g., [20, Example 2.1.8]) $p_{t,\mathscr{L}_D}(x,y)$ always satisfies the Gaussian upper bound (2.8), without any conditions on smoothness of the boundary of Ω. However, to ensure the Gaussian upper bound of $p_{t,\mathscr{L}_N}(x,y)$ one need to impose suitable regularity condition to Ω. For instance, if Ω satisfies the extension property (i.e., there exists a bounded linear map $E : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^n)$ such that Eu is an extension of u from to R^n for all $u \in W^{1,2}(\Omega)$), then $p_{t,\mathscr{L}_N}(x,y)$ satisfies the Gaussian upper bound (see [20, Theorem 3.2.9] and [3, Theorem 4.4]). It is worth noting that every (locally) uniform domain satisfies the extension property, however, a domain satisfying the extension property need not be (locally) uniform (see [87] and the references therein). We also point out that the extension property implies that Ω satisfies the doubling property

$$|B^\Omega(x,2r)| \leq C|B^\Omega(x,r)|, \quad \forall x \in \Omega,\ \forall r \in (0, \operatorname{diam}(\Omega)),$$

where $B^\Omega(x,r) := \{x \in \Omega : |x-y| < r\}$. See

The Hölder continuity of $p_{t,\mathscr{L}_D}(x,y)$ and $p_{t,\mathscr{L}_N}(x,y)$ are more difficult to establish. For the Dirichlet boundary condition, it is shown in [22] that if Ω is bounded and satisfies the uniform outer ball condition, and $\mathscr{L} = -\Delta$, then $p_{t,\mathscr{L}_D}(x,y)$ satisfies the Hölder continuity. For the Neumann boundary condition, it is proved in [41] that if Ω is a uniform domain or a convex domain (not necessarily bounded) then $p_{t,\mathscr{L}_N}(x,y)$ satisfies the Hölder continuity.

• *Schrödinger type operators.* Let V be a locally integrable non-negative function on $\mathbb{R}^n$, which is not identically zero. Let $\mathcal{V} = \{u \in W^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V|u|^2 dv < \infty\}$. Let Q be the sesquilinear form on the product space $\mathcal{V} \times \mathcal{V}$, given by

$$Q(u,v) = \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} dx + \int_{\mathbb{R}^n} V u \bar{v} dx$$

for $u, v \in \mathcal{V}$. Simon proved in [73] that this sesquilinear form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$. Let $\mathscr{L}$ be the self-adjoint operator associated with Q. Then the domain of $\mathscr{L}$ is given by

$$D(\mathscr{L}) = \left\{ u \in \mathcal{V} : \exists v \in L^2(\mathbb{R}^n) \text{ such that } Q(u, \varphi) = \int_{\mathbb{R}^n} v\bar{\varphi},\ \forall \varphi \in \mathcal{V} \right\}.$$

Formally we can write $\mathscr{L} = -\Delta + V$ and call $\mathscr{L}$ the Schrödinger operator with potential V. Since V is non-negative and locally integrable, the Feynman-Kac formula yields that heat kernel $p_t(x, y)$ of $\mathscr{L}$ satisfies

$$0 \leq p_t(x, y) \leq (4\pi t)^{-n/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. That is, $p_t(x, y)$ satisfies the Gaussian upper bound.

In general, $p_t(x, y)$ does not satisfy the Hölder continuity condition. However, if one imposes appropriate conditions on the potential V, then $p_t(x, y)$ is Hölder continuous. For example, if one assume that V belongs to the reverse Hölder class RH^q for some $q > n/2$, that is, there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)dy \quad \text{for ever ball } B,$$

then for any $\alpha \in (0, \min\{1, 2 - n/q\})$, there exists constants $C, c > 0$ such that for all $t > 0$ and $x, y, y' \in \mathbb{R}^n$ satisfying $|y - y'| < \sqrt{t}$

$$|p_t(x, y) - p_t(x, y')| \leq Ct^{-n/2}\left(\frac{|y-y'|}{\sqrt{t}}\right)^\alpha \exp\left(\frac{-|x-y|^2}{ct}\right).$$

See [29, Theorem 4.11].

• *Riemannian manifolds with non-negative Ricci curvature.* Let M be a complete, connected, n-dimensional Riemannian manifolds with non-negative Ricci curvature. Let ρ be the geodesic distance, μ the Riemannian measure, and ∇ the Riemannian gradient on M. Denote by $|\cdot|$ the length in the tangent space. Let Δ be the Laplace-Beltrami operator, that is the positive self-adjoint operator on $L^2(M, d\mu)$ defined by the formal integration by parts $\langle \Delta f, f\rangle = \||\nabla f|\|_{L^2(X,d\mu)}$. Denote by $p_t(x, y)$ the heat kernel of M. By the Bishop-Gromov volume comparison theorem, we know that on such an manifold M there is a constant $C > 0$ such that for all $x \in M$ and $r' \geq r > 0$,

$$\frac{\mu(B(x, r'))}{\mu(B(x, r))} \leq C\left(\frac{r'}{r}\right)^n.$$

This implies that M satisfies the doubling condition. The reverse doubling condition then follows from the doubling condition and the connectedness of M (cf. [19, Proposition 2.2]).

Li and Yau [60] proved that the heat kernel $p_t(x, y)$ of M satisfies the following Gaussian upper and lower bounds:

$$C'\frac{\exp\left\{-\frac{\rho^2(x,y)}{c't}\right\}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}} \leq p_t(x, y) \leq C\frac{\exp\left\{-\frac{\rho^2(x,y)}{ct}\right\}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}} \tag{2.22}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Note that it was shown in [71] that the estimates (2.22) are equivalent to the so-called uniform parabolic Harnack principle, and they imply the Hölder continuity of $p_t(x, y)$.

• *Compact Riemannian manifolds.* Let M be a compact Riemannian manifold without boundary. In this case, the Ricci curvature of M is obviously bounded from below. Hence, by the Bishop-Gromov volume comparison theorem, M satisfies the doubling condition (2.1), and by the result of Li and Yau [60] the heat kernel of M satisfies the estimate (2.22) for $t \in (0, 1]$.

• *Lie groups of polynomial growth.* Let G be a connected unimodular Lie group and let μ be a fixed Haar measure on G. Denote by $\mathfrak{g}$ the Lie algebra of G. Let $\mathbb{X} = \{X_1, \cdots, X_k\}$ be left-invariant vector fields on G satisfying the Hörmander condition, that is, the X_i's and their commutators of all orders generate $\mathfrak{g}$. Let ρ be the Carnot-Carathéodory (control) distance on G associated to $\mathbb{X}$. For $x \in G$ and $r > 0$, let $B(x, r) := \{y \in G : \rho(x, y) < r\}$. Then for all $x \in G$ and $r > 0$, we have $\mu(B(x, r)) = \mu(B(e, r))$, where e is the identity element of G. We denote $V(r) := \mu(B(e, r))$. It was proved by Y. Guivarc'h [40] that either there exists an integer N such that

$$\forall r \in (1, \infty), \quad cr^N \leq V(r) < Cr^N$$

or

$$\forall r \in (1, \infty), \quad ce^{cr} \leq V(r) \leq Ce^{Cr}.$$

In the first case, we say that G is a Lie group of polynomial growth. For small r, by results of [62] we know that there exists an integer n, which is not necessarily the topological dimension of G, such that

$$\forall r \in (0, 1], \quad cr^n \leq V(r) \leq Cr^n.$$

From all of these, it follows that if G is Lie group of polynomial growth then it satisfies the doubling, reverse doubling, and non-collapsing conditions.

We denote by $\Delta_{\mathbb{X}} = -\sum_{i=1}^k X_i^2$ the sub-Laplacian on G associated with $\mathbb{X}$, and by $\nabla_{\mathbb{X}} = (X_1, \cdots, X_k)$ the gradient on G associated with $\mathbb{X}$. It was proved by Varopoulos [85] that G satisfies the (scaled) Poincaré inequality, namely, there exists $C > 0$ such that, for every ball $B = B(x, r)$ and every f with f, $\nabla_{\mathbb{X}} f$ locally square integrable,

$$\int_B |f - f_B|^2 d\mu \leq Cr^2 \int_B |\nabla_{\mathbb{X}} f|^2 d\mu, \tag{2.23}$$

where $f_B := \frac{1}{\mu(B)} \int_B f(x) d\mu(x)$. On the other hand, from [70] we see that the Li-Yau type estimate (2.22) is equivalent to the conjunction of the volume doubling condition and the Poincaré inequality (2.23). Therefore, the heat kernel of G satisfies the Gaussian upper bound and the Hölder continuity condition.

Recall that all simply connected nilpotent Lie groups are of polynomial volume growth. In particular, all stratified Lie groups and all H-type groups are Lie groups of polynomial volume growth.

• *Heat kernel on* $[-1, 1]$ *generated by the Jacobi operator.* Consider the interval $[-1, 1]$ endowed with the measure $d\mu(x) = w(x)dx$, where

$$w(x) := (1 - x)^\alpha (1 + x)^\beta, \quad \alpha, \beta > -1.$$

is the classical Jacobi weight. The Jacobi operator $\mathscr{L}$ is defied by

$$\mathscr{L} f(x) := -\frac{[w(x)a(x)f'(x)]'}{w(x)} \quad \text{with } a(x) := 1 - x^2$$

and $D(\mathscr{L}) = C^2[-1, 1]$. It is well known that (cf. [80]) $\mathscr{L} P_k = \lambda_k P_k$, where P_k $(k \in \mathbb{N}_0)$ is the (normalized) Jacobi polynomial of degree k, and $\lambda_k = k(k + \alpha + \beta + 1)$. Let ρ be an intrinsic metric on $[-1, 1]$ defined by $\rho(x, y) = |\arccos x - \arccos y|$. It is shown in [19] that the metric measure space $([-1, 1], \rho, \mu)$ satisfies the doubling condition (2.1), and the heat kernel of $\mathscr{L}$ satisfies the Gaussian upper bound (2.8) and the Hölder continuity condition (2.9).

Chapter 3

Besov and Triebel-Lizorkin spaces associated to operators

Throughout this chapter, we assume that the metric measure space (X, ρ, μ) satisfies the doubling condition (2.1), the reverse doubling condition (2.4), and the non-collapsing condition (2.6), and assume that $\mathscr{L}$ is a non-negative self-adjoint operator on $L^2(X)$ whose heat kernel $p_t(x, y)$ satisfies the pointwise Gaussian upper bound (2.8). We do not assume the Hölder continuity for $p_t(x, y)$ in the variables x and y. Our purpose in this chapter is to introduce and investigate Besov and Triebel-Lizorkin spaces associated to such an $\mathscr{L}$.

3.1 Definition of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$

Before we introduce Besov and Triebel-Lizorkin spaces associated to $\mathscr{L}$, we first define the classes $\mathcal{A}_M(\mathbb{R}_{\geq 0})$.

Definition 3.1. Let (Φ_0, Φ) be a couple of functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ and let $M \in \mathbb{N}_0$. We say that (Φ_0, Φ) belongs to the class $\mathcal{A}_M(\mathbb{R}_{\geq 0})$, if

$$|\Phi_0(\lambda)| \geq c > 0 \text{ on } \{0 \leq \lambda \leq 2^{3/2}\varepsilon\} \text{ and } |\Phi(\lambda)| \geq c > 0 \text{ on } \{2^{-3/2}\varepsilon \leq \lambda \leq 2^{3/2}\varepsilon\} \tag{3.1}$$

for some $\varepsilon > 0$, and

$$\text{the function } \lambda \mapsto \lambda^{-M}\Phi(\lambda) \text{ belongs to } \mathcal{S}(\mathbb{R}_{\geq 0}). \tag{3.2}$$

Remark 3.2. If $M \geq 1$, the condition (3.2) is equivalent to the following one:

$$\Phi^{(k)}(0) = 0 \quad \text{for } k = 0, 1, \cdots, M-1.$$

Example. Let $M \in \mathbb{N}_0$. Define $\Phi_0(\lambda) = e^{-\lambda}$ and $\Phi(\lambda) = \lambda^M e^{-\lambda}$ for $\lambda \in \mathbb{R}_{\geq 0}$. Then clearly $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$.

We introduce Besov and Triebel-Lizorkin spaces associated with $\mathscr{L}$ as follows:

Definition 3.3. (i) Let $s \in \mathbb{R}$, $p \in (0, \infty]$ and $q \in (0, \infty]$. Let $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ for some nonnegative integer $M > s/2$. Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$. We define the Besov space

$B^{s,\mathscr{L}}_{p,q}(X)$ as the collection of all distributions $f \in \mathcal{S}'_{\mathscr{L}}(X)$ such that

$$\|f\|_{B^{s,\mathscr{L}}_{p,q}(X)} := \left(\sum_{j=0}^{\infty} \left\| 2^{js} \Phi_j(\mathscr{L}) f \right\|^q_{L^p(X)} \right)^{1/q} < \infty.$$

(ii) Let $s \in \mathbb{R}$, $p \in (0, \infty)$ and $q \in (0, \infty]$. Let $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ for some nonnegative integer $M > s/2$. Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$. We define the Triebel-Lizorkin space $F^{s,\mathscr{L}}_{p,q}(X)$ as the collection of all distributions $f \in \mathcal{S}'_{\mathscr{L}}(X)$ such that

$$\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} := \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} \Phi_j(\mathscr{L}) f \right|^q \right)^{1/q} \right\|_{L^p(X)} < \infty.$$

3.2 Well-definedness and Peetre maximal function characterization

Given a couple (Φ_0, Φ) of functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$, a distribution $f \in \mathcal{S}'_{\mathscr{L}}(X)$, and a number $a > 0$, we define a system of Peetre type maximal functions by

$$[\Phi_j(\mathscr{L})]^*_a f(x) := \operatorname*{ess\,sup}_{y \in X} \frac{|\Phi_j(\mathscr{L}) f(y)|}{(1 + 2^j \rho(x,y))^a}, \quad x \in X, j \in \mathbb{N}_0, \tag{3.3}$$

where $\Phi_j(\cdot) := \Phi(2^{-2j} \cdot)$ for $j \geq 1$.

The following two theorems, which provide the Peetre type maximal function characterization of Besov and Triebel-Lizorkin spaces associated with $\mathscr{L}$, are the main results of this section.

Theorem 3.4. *Let $s \in \mathbb{R}$, $q \in (0, \infty]$ and let (Φ_0, Φ) be a couple of functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying (3.1). Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$.*

(i) *If $p \in (0, \infty]$ and $a > \frac{2d}{p}$, then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\left(\sum_{j=0}^{\infty} \left\| 2^{js} [\Phi_j(\mathscr{L})]^*_a f \right\|^q_{L^p(X)} \right)^{1/q} \sim \left(\sum_{j=0}^{\infty} \left\| 2^{js} \Phi_j(\mathscr{L}) f \right\|^q_{L^p(X)} \right)^{1/q}. \tag{3.4}$$

(ii) *If $p \in (0, \infty)$ and $a > \frac{2d}{\min\{p,q\}}$, then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} [\Phi_j(\mathscr{L})]^*_a f \right|^q \right)^{1/q} \right\|_{L^p(X)} \sim \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} \Phi_j(\mathscr{L}) f \right|^q \right)^{1/q} \right\|_{L^p(X)}. \tag{3.5}$$

Theorem 3.5. *Let $s \in \mathbb{R}$, $q \in (0, \infty]$ and $a > 0$. Let $(\Phi_0, \Phi), (\widetilde{\Phi}_0, \widetilde{\Phi}) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ for some nonnegative integer $M > s/2$. Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ and $\widetilde{\Phi}_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$.*

(i) *If $p \in (0, \infty]$, then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\left(\sum_{j=0}^{\infty} \left\| 2^{js} [\widetilde{\Phi}_j(\mathscr{L})]^*_a f \right\|^q_{L^p(X)} \right)^{1/q} \sim \left(\sum_{j=0}^{\infty} \left\| 2^{js} [\Phi_j(\mathscr{L})]^*_a f \right\|^q_{L^p(X)} \right)^{1/q}. \tag{3.6}$$

(ii) *If $p \in (0,\infty)$, then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} [\widetilde{\Phi}_j(\mathscr{L})]_a^* f \right|^q \right)^{1/q} \right\|_{L^p(X)} \sim \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js} [\Phi_j(\mathscr{L})]_a^* f \right|^q \right)^{1/q} \right\|_{L^p(X)}. \tag{3.7}$$

Combining Theorem 3.4 and 3.5, we get the following corollary:

Corollary 3.6. *Our definition of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$ is independent of the choice of $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$, as long as the nonnegative integer M is strictly larger than $s/2$.*

To prove Theorem 3.4 and Theorem 3.5 we need considerable preparation. First note that for any $\sigma > d$ there is a positive constant C (depending on σ) such that

$$\int_X \left(1 + \frac{\rho(x,y)}{t} \right)^{-\sigma} d\mu(y) \leq CV(x,t) \tag{3.8}$$

for all $t \in (0,\infty)$ and $x \in X$; see [19, Lemma 2.3]. This together with (2.3) yields that for any $\sigma > 3d/2$,

$$\|D_{t,\sigma}(x,\cdot)\|_{L^1(X,d\mu)} \leq C \tag{3.9}$$

uniformly for all $t \in (0,\infty)$ and $x \in X$, where $D_{t,\sigma}(x,y)$ is defined by (2.10).

Combining (2.8) and (2.11) from [19], we see that for any $\sigma > d$ there exists a constant C such that for all $s,t \in (0,\infty)$ and $x,y \in X$,

$$\int_X D_{t,\sigma}(x,z) D_{s,\sigma}(z,y) d\mu(z) \leq C \max\{(t^{-1}s)^d, (s^{-1}t)^d\} D_{t\vee s,\sigma}(x,y),$$

where $t \vee s := \max\{t,s\}$. However, for our purpose we need the following refinement:

Lemma 3.7. *For any $\sigma > 2d$, there exists a constant $C > 0$ such that for all $t,s > 0$ and all $x,y \in X$,*

$$\int_X D_{t,\sigma}(x,z) D_{s,\sigma}(z,y) d\mu(z) \leq C D_{t\vee s,\sigma-2d}(x,y). \tag{3.10}$$

Proof. By symmetry, we only need to show (3.10) for $t > s$. To do this, we write

$$\int_X D_{t,\sigma}(x,z) D_{s,\sigma}(z,y) d\mu(z) = \int_{D_1} + \int_{D_2} =: I_1 + I_2,$$

where $D_1 := \{z \in X : \rho(z,y) < \rho(x,y)/2\}$ and $D_2 := \{z \in X : \rho(z,y) \geq \rho(x,y)/2\}$. Observe that $\rho(x,y) \leq 2\rho(x,z)$ for all $z \in D_1$. This together with (2.3) yields that

$$\begin{aligned} \forall z \in D_1 : \quad D_{t,\sigma}(x,z) &\lesssim [V(x,t)V(z,t)]^{-1/2}(1 + t^{-1}\rho(x,z))^{-\sigma} \\ &\lesssim [V(x,t)V(y,t)]^{-1/2}(1 + t^{-1}\rho(z,y))^{d/2}(1 + t^{-1}\rho(x,z))^{-\sigma} \\ &\lesssim [V(x,t)V(y,t)]^{-1/2}(1 + t^{-1}\rho(x,y))^{-\sigma+d/2} \\ &= D_{t,\sigma-d/2}(x,y). \end{aligned} \tag{3.11}$$

Also note that, by (2.3) and the elementary inequality $1 + t^{-1}\rho(z,y) \leq C(1 + t^{-1}\rho(x,y))(1 + t^{-1}\rho(x,z))$, and taking into account that $\sigma > 2d$, we have

$$\begin{aligned} \forall z \in D_2 : \quad D_{t,\sigma}(x,z) &\lesssim [V(x,t)V(z,t)]^{-1/2}(1 + t^{-1}\rho(x,z))^{-d/2} \\ &\lesssim [V(x,t)V(y,t)]^{-1/2}(1 + t^{-1}\rho(z,y))^{d/2}(1 + t^{-1}\rho(x,z))^{-d/2} \\ &\lesssim [V(x,t)V(y,t)]^{-1/2}(1 + t^{-1}\rho(x,y))^{d/2}. \end{aligned} \tag{3.12}$$

From (3.11) and (3.9), it follows that

$$I_1 \lesssim D_{t,\sigma-d/2}(x,y) \int_{D_1} D_{s,\sigma}(z,y)d\mu(y) \lesssim D_{t,\sigma-d/2}(x,y). \tag{3.13}$$

To estimate I_2, note that by (3.12) we have

$$I_2 \lesssim [V(x,t)V(y,t)]^{-1/2}(1+t^{-1}\rho(x,y))^{d/2} \int_{\Omega_2} D_{s,\sigma}(z,y)d\mu(y). \tag{3.14}$$

Suppose first that $\rho(x,y) \le t$. In this case we have $1+t^{-1}\rho(x,y) \sim 1$, and hence by (3.9)

$$I_2 \lesssim [V(x,t)V(y,t)]^{-1/2} \int_X D_{s,\sigma}(z,y)d\mu(y) \lesssim [V(x,t)V(y,t)]^{-1/2} \sim D_{t,\sigma}(x,y).$$

If, instead, $\rho(x,y) > t$, then we decompose the set D_2 into $D_2 = \bigcup_{k=0}^{\infty} E_k$, where $E_k := \{z \in X : 2^{k-1}\rho(x,y) \le \rho(z,y) < 2^k\rho(x,y)\}$. By using (2.2) and (2.3) we can estimate as follows:

$$\begin{aligned}
\int_{D_2} D_{s,\sigma}(z,y)d\mu(z) &\lesssim V(y,s)^{-1} \int_{D_2} (1+s^{-1}\rho(z,y))^{-\sigma+d/2} d\mu(z) \\
&\lesssim V(y,s)^{-1} s^{\sigma-d/2} \sum_{k=0}^{\infty} \int_{E_k} \rho(z,y)^{-\sigma+d/2} d\mu(z) \\
&\le s^{\sigma-d/2} \sum_{k=0}^{\infty} [2^{k-1}\rho(x,y)]^{-\sigma+d/2} V(y,s)^{-1} V(y,2^k\rho(x,y)) \\
&\le s^{\sigma-d/2} \sum_{k=0}^{\infty} [2^{k-1}\rho(x,y)]^{-\sigma+d/2} \left(\frac{2^k\rho(x,y)}{s}\right)^d \\
&\lesssim t^{\sigma-3d/2}\rho(x,y)^{-\sigma+3d/2} \\
&\sim (1+t^{-1}\rho(x,y))^{-\sigma+3d/2}.
\end{aligned}$$

Here, for the last inequality we used that $t^{-1}\rho(x,y) \sim 1+t^{-1}\rho(x,y)$, which follows from $\rho(x,y) > t$. Inserting the above estimate into (3.14) we obtain

$$I_2 \lesssim D_{t,\sigma-2d}(x,y). \tag{3.15}$$

Combining (3.13) and (3.15), we arrive at (3.10) and the proof is thus completed. □

Lemma 3.8. *Suppose that $\Phi, \Psi \in \mathcal{S}(\mathbb{R}_{\ge 0})$ and suppose further that*

$$\textit{the function } \lambda \mapsto \lambda^{-M}\Psi(\lambda) \textit{ belongs to } \mathcal{S}(\mathbb{R}_{\ge 0}),$$

where $M \in \mathbb{N}_0$. Then for any $m \in \mathbb{N}_0$ with $m > \max\{2d, d+1\}$, there exists a constant $C > 0$ such that for all $j, \ell \in \mathbb{N}_0$ with $j \le \ell$,

$$\begin{aligned}
&\left|K_{\Phi(2^{-2j}\mathscr{L})\Psi(2^{-2\ell}\mathscr{L})}(x,y)\right| \\
&\le C\|\Phi\|_{(m+M)} \|\lambda \mapsto \lambda^{-M}\Psi(\lambda)\|_{(m)} 2^{2(j-\ell)M} D_{2^{-j},m-2d}(x,y).
\end{aligned} \tag{3.16}$$

Proof. Note that

$$\begin{aligned}
&\left|K_{\Phi(2^{-2j}\mathscr{L})\Psi(2^{-2\ell}\mathscr{L})}(x,y)\right| \\
&= 2^{2(j-\ell)M} \left|K_{(2^{-2j}\mathscr{L})^M\Phi(2^{-2j}\mathscr{L})(2^{-2\ell}\mathscr{L})^{-M}\Psi(2^{-2\ell}\mathscr{L})}(x,y)\right|
\end{aligned}$$

$$\leq 2^{2(j-\ell)M} \int_X \left|K_{(2^{-2j}\mathscr{L})^M \Phi(2^{-2j}\mathscr{L})}(x,z)\right| \left|K_{(2^{-2\ell}\mathscr{L})^{-M}\Psi(2^{-2\ell}\mathscr{L})}(z,y)\right| d\mu(z).$$

By (2.14) we have

$$\left|K_{(2^{-2j}\mathscr{L})^M \Phi(2^{-2j}\mathscr{L})}(x,z)\right| \leq C\|\lambda \mapsto \lambda^M \Phi(\lambda)\|_{(m)} D_{2^{-j},m}(x,z) \leq C\|\Phi\|_{(m+M)} D_{2^{-j},m}(x,z)$$

and

$$\left|K_{(2^{-2\ell}\mathscr{L})^{-M}\Psi(2^{-2\ell}\mathscr{L})}(z,y)\right| \leq C\|\lambda \mapsto \lambda^{-M}\Psi(\lambda)\|_{(m)} D_{2^{-\ell},m}(z,y).$$

These two estimates along with Lemma 3.7 yield (3.16). □

Lemma 3.9. *Suppose* (Φ_0, Φ) *is a couple of functions in* $\mathcal{S}(\mathbb{R}_{\geq 0})$ *satisfying* (3.1). *Then there exists another couple* (Ψ_0, Ψ) *of functions in* $\mathcal{S}(\mathbb{R}_{\geq 0})$ *such that*

$$\operatorname{supp}\Psi_0 \in [0, 2^2\varepsilon], \quad |\Psi_0(\lambda)| \geq c > 0 \text{ for } \lambda \in [0, 2^{3/2}\varepsilon],$$
$$\operatorname{supp}\Psi \subset [2^{-2}\varepsilon, 2^2\varepsilon], \quad |\Psi(\lambda)| \geq c > 0 \text{ for } \lambda \in [2^{-3/2}\varepsilon, 2^{3/2}\varepsilon],$$

and

$$\Phi_0(\lambda)\Psi_0(\lambda) + \sum_{j=1}^{\infty} \Phi(2^{-2j}\lambda)\Psi(2^{-2j}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}_{\geq 0}.$$

Proof. Choose nonnegative functions $\Theta, \Upsilon \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\operatorname{supp}\Theta \subset [0, 2^2\varepsilon], \quad \Theta(\lambda) \geq c > 0 \text{ for } \lambda \in [0, 2^{3/2}\varepsilon],$$
$$\operatorname{supp}\Upsilon \subset [2^{-2}\varepsilon, 2^2\varepsilon], \quad \Upsilon(\lambda) \geq c > 0 \text{ for } \lambda \in [2^{-3/2}\varepsilon, 2^{3/2}\varepsilon].$$

Then we put

$$\Xi(\lambda) := \Theta(\lambda)|\Phi_0(\lambda)|^2 + \sum_{j=1}^{\infty} \Upsilon(2^{-2j}\lambda)|\Phi(2^{-2j}\lambda)|^2, \quad \lambda \in \mathbb{R}_{\geq 0}. \tag{3.17}$$

By (3.1) and by our choice of Θ, Υ, there exists a constant $c' > 0$ such that

$$|\Theta(\lambda)||\Phi_0(\lambda)|^2 \geq c' \quad \text{for } \lambda \in [0, 2^{3/2}\varepsilon],$$
$$|\Upsilon(\lambda)||\Phi(\lambda)|^2 \geq c' \quad \text{for } \lambda \in [2^{-3/2}\varepsilon, 2^{3/2}\varepsilon].$$

Also note that for any $\lambda \in [2^{3/2}\varepsilon, \infty)$ there exists a positive integer j such that $2^{-2j}\lambda \in [2^{-3/2}\varepsilon, 2^{3/2}\varepsilon]$. Hence the function Ξ is strictly positive on $\mathbb{R}_{\geq 0}$ with a strictly positive lower bound. Moreover, since for any $\lambda \in \mathbb{R}_{\geq 0}$ the number of those j for which $\Upsilon(2^{-2j}\lambda) \neq 0$ is no more than 2, i.e., the sum on the right-hand side of (3.17) is in fact a finite sum for any fixed λ, we see that $\Xi \in C^\infty(\mathbb{R}_{>0})$, and for any $k \in \mathbb{N}_0$, $\lim_{\lambda \to 0^+} \Xi^{(k)}(\lambda)$ exists and $\Xi^{(k)}$ is a bounded function on $\mathbb{R}_{\geq 0}$.

Now define the functions $\Psi_0, \Psi : \mathbb{R}_{\geq 0} \to \mathbb{C}$ respectively by

$$\Psi_0(\lambda) := \frac{\Theta(\lambda)\overline{\Phi_0(\lambda)}}{\Xi(\lambda)} \quad \text{and} \quad \Psi(\lambda) := \frac{\Upsilon(\lambda)\overline{\Phi(\lambda)}}{\Xi(\lambda)}.$$

Then it is straightforward to verify that Ψ_0, Ψ satisfy the desired properties. □

Lemma 3.10. *Suppose* Φ_0, Φ *are functions in* $\mathcal{S}(\mathbb{R}_{\geq 0})$ *such that* $\operatorname{supp}\Phi_0$ *and* $\operatorname{supp}\Phi$ *are compact,* $0 \notin \operatorname{supp}\Phi$*, and*

$$\Phi_0(\lambda) + \sum_{j=1}^{\infty} \Phi(2^{-2j}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}_{\geq 0}. \tag{3.18}$$

Then for any $f \in \mathcal{S}'_{\mathscr{L}}(X)$,

$$f = \Phi_0(\mathscr{L})f + \sum_{j=1}^{\infty} \Phi(2^{-2j}\mathscr{L})f,$$

where the convergence of the sum is in the sense of $\mathcal{S}'_{\mathscr{L}}(X)$.

Proof. By duality, it suffices to show that for all $\phi \in \mathcal{S}_{\mathscr{L}}(X)$,

$$\phi = \Phi_0(\mathscr{L})\phi + \sum_{j=1}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi \quad \text{in} \quad \mathcal{S}_{\mathscr{L}}(X).$$

To do this, we first show that the series $\sum_{j=1}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi$ converges in the topology of $\mathcal{S}_{\mathscr{L}}(X)$. Write, for each $k, M \in \mathbb{N}_0$ and $j \in \mathbb{N}$,

$$\mathscr{L}^k\big[\Phi(2^{-2j}\mathscr{L})\phi\big] = 2^{-2jM}\big[(2^{-2j}\mathscr{L})^{-M}\Phi(2^{-2j}\mathscr{L})\big]\mathscr{L}^{k+M}\phi.$$

Since Φ has compact support and $0 \notin \operatorname{supp}\Phi$, the function $\lambda \mapsto \lambda^{-M}\Psi(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$ for all $M \in \mathbb{N}_0$. Hence it follows from (2.14) that for any $m \in \mathbb{N}_0$ with $m \geq d+1$, there exits a constant $C > 0$ (depending on m) such that for all $M \in \mathbb{N}_0$, $j \in \mathbb{N}$ and a.e. $x, y \in X$,

$$\big|K_{(2^{-2j}\mathscr{L})^{-M}\Phi(2^{-2j}\mathscr{L})}(x,y)\big| \leq C\|\lambda \mapsto \lambda^{-M}\Phi(\lambda)\|_{(m)} D_{2^{-j},m}(x,y),$$

Also, by the definition of $\mathcal{S}_{\mathscr{L}}(X)$, we have, for a.e. $y \in X$,

$$|\mathscr{L}^{k+M}\phi(y)| \leq \mathcal{P}_{k+M,m}(\phi)(1+\rho(y,x_0))^{-m} \leq C\mathcal{P}_{k+M,m}(\phi)V(x_0,1)D_{1,m-n/2}(y,x_0).$$

From these estimates and (3.10), it follows that for any $m \in \mathbb{N}_0$ with $m \geq \lfloor 3d \rfloor + 2$ ($\geq 2d+1$), there exits a constant $C > 0$ (depending on m) such that for all $k, M \in \mathbb{N}_0$, $j \in \mathbb{N}$ and a.e. $x \in X$,

$$\begin{aligned}
&\big|\mathscr{L}^k\big[\Phi(2^{-2j}\mathscr{L})\phi\big](x)\big| \\
&\leq C2^{-2jM}\int_X \big|K_{(2^{-2j}\mathscr{L})^{-M}\Phi(2^{-2j}\mathscr{L})}(x,y)\big||\mathscr{L}^{k+M}\phi(y)|d\mu(y) \\
&\leq C2^{-2jM}\|\lambda \mapsto \lambda^{-M}\Phi(\lambda)\|_{(m)}\mathcal{P}_{k+M,m}(\phi)\int_X D_{2^{-j},m}(x,y)D_{1,m-d/2}(y,x_0)d\mu(y) \\
&\leq C2^{-2jM}\|\lambda \mapsto \lambda^{-M}\Phi(\lambda)\|_{(m)}\mathcal{P}_{k+M,m}(\phi)D_{1,m-5d/2}(x,x_0) \\
&\leq C2^{-2jM}\|\lambda \mapsto \lambda^{-M}\Phi(\lambda)\|_{(m)}\mathcal{P}_{k+M,m}(\phi)(1+\rho(x,x_0))^{-m+3d}.
\end{aligned} \tag{3.19}$$

Replacing m by the integer $m + \lfloor 3d \rfloor + 2$, and multiplying both sides by $(1+\rho(x,x_0))^m$, we obtain that for all $k, m \in \mathbb{N}_0$,

$$\mathcal{P}_{k,m}\left(\Phi(2^{-2j}\mathscr{L})\phi\right) \leq C2^{-2jM}\|\lambda \mapsto \lambda^{-M}\Phi(\lambda)\|_{(m+\lfloor 3d \rfloor+2)}\mathcal{P}_{k+M,m+\lfloor 3d \rfloor+2}(\phi). \tag{3.20}$$

Hence, by choosing $M \geq 1$, we have

$$\sum_{j=1}^{\infty}\mathcal{P}_{k,m}\left(\Phi(2^{-2j}\mathscr{L})\phi\right) \leq C\|\lambda \mapsto \lambda^{-M}\Phi(\lambda)\|_{(m+\lfloor 3d \rfloor+2)}\mathcal{P}_{k+M,m+\lfloor 3d \rfloor+2}(\phi).$$

This implies that the series $\sum_{j=1}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi$ converges in the topology of $\mathcal{S}_{\mathscr{L}}(X)$. Hence (since $\mathcal{S}_{\mathscr{L}}(X)$ is Fréchet space), there exists $\psi \in \mathcal{S}_{\mathscr{L}}(X)$ such that $\Psi(\mathscr{L})\phi + \sum_{j=1}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi$ converges in the topology of $\mathcal{S}_{\mathscr{L}}(X)$ to ψ. On the other hand, by (3.18) and the spectral theorem (cf. [66, Theorem VII.2]), we have

$$\Phi_0(\mathscr{L})\phi + \sum_{j=1}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi = \phi,$$

which holds in the sense of $L^2(X)$-norm. Therefore, $\psi = \phi$. This completes the proof. □

Lemma 3.11. (see [68, Lemma 2]) *Let $0 < p, q \leq \infty$ and $\delta > 0$. Let $\{g_j\}_{j=0}^{\infty}$ be a sequence of nonnegative measurable function on X and put*

$$G_\ell(x) = \sum_{j=0}^{\infty} 2^{-|j-\ell|\delta} g_j(x), \quad x \in X, \ \ell \in \mathbb{N}_0.$$

Then, there is a constant C depending only on p, q, δ such that

$$\big\|\{G_\ell\}_{\ell=0}^{\infty}\big\|_{\ell^q(\ell^p)} \leq C\big\|\{g_j\}_{j=0}^{\infty}\big\|_{\ell^q(L^p)},$$
$$\big\|\{G_\ell\}_{\ell=0}^{\infty}\big\|_{L^p(\ell^q)} \leq C\big\|\{g_j\}_{j=0}^{\infty}\big\|_{L^p(\ell^q)}.$$

Here, $\ell^q(L^p)$ and $L^p(\ell^q)$ are the spaces of all sequences $\{h_j\}_{j=0}^{\infty}$ of measurable functions on X with the finite quasi-norms

$$\big\|\{h_j\}_{j=0}^{\infty}\big\|_{\ell^q(L^p)} := \big\|\{\|h_j\|_{L^p(X)}\}_{j=0}^{\infty}\big\|_{\ell^q},$$
$$\big\|\{h_j\}_{j=0}^{\infty}\big\|_{L^p(\ell^q)} := \big\|\big\|\{h_j(\cdot)\}_{j=0}^{\infty}\big\|_{\ell^q}\big\|_{L^p(X)}.$$

We now give the proof of Theorem 3.4.

Proof of Theorem 3.4. We follow the idea developed by Rychkov [68] and Ullrich [84]. Since Φ_0, Φ satisfy (3.1), by Lemma 3.9 there exist $\Psi_0, \Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\operatorname{supp} \Psi_0 \subset [0, 2^2\varepsilon], \quad \operatorname{supp} \Psi \subset [2^{-2}\varepsilon, 2^2\varepsilon],$$

and

$$\Phi_0(\lambda)\Psi_0(\lambda) + \sum_{j=1}^{\infty} \Phi(2^{-2j}\lambda)\Psi(2^{-2j}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}_{\geq 0}.$$

Setting $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ and $\Psi_j(\lambda) := \Psi(2^{-2j}\lambda)$ for $j \geq 1$, we can rewrite the above equality as

$$\sum_{j=0}^{\infty} \Phi_j(\lambda)\Psi_j(\lambda) = 1, \quad \forall \lambda \in \mathbb{R}_{\geq 0}.$$

Replacing λ by $2^{-2\ell}\lambda$, we get that for all $\ell \in \mathbb{N}_0$,

$$\sum_{j=0}^{\infty} \Phi_j(2^{-2\ell}\lambda)\Psi_j(2^{-2\ell}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}_{\geq 0}.$$

The last inequality along with Lemma 3.10 yields that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,

$$f = \sum_{j=0}^{\infty} \Phi_j(2^{-2\ell}\mathscr{L})\Psi_j(2^{-2\ell}\mathscr{L})f \quad \text{in } \mathcal{S}'_{\mathscr{L}}(X).$$

Hence, for $\ell \in \mathbb{N}_0$, we have the pointwise representation

$$\Phi_\ell(\mathscr{L})f(y) = \sum_{j=0}^{\infty} \Phi_\ell(\mathscr{L})\Phi_j(2^{-2\ell}\mathscr{L})\Psi_j(2^{-2\ell}\mathscr{L})f(y), \quad \text{a.e. } y \in X. \tag{3.21}$$

For $j, \ell \in \mathbb{N}_0$, let us define

$$\Lambda_{j,\ell}(\lambda) := \begin{cases} \Phi_0(2^{-2\ell}\lambda) & \text{if } j = 0 \text{ and } \ell \in \mathbb{N}_0, \\ \Phi_\ell(\lambda) & \text{if } j \in \mathbb{N} \text{ and } \ell \in \mathbb{N}_0. \end{cases}$$

Observe that

$$\Phi_\ell(\lambda)\Phi_j(2^{-2\ell}\lambda) = \Lambda_{j,\ell}(\lambda)\Phi_{j+\ell}(\lambda), \quad j, \ell \in \mathbb{N}_0.$$

Substituting this into (3.21), and using Lemma 2.6 (ii), we obtain the pointwise representation (in what follows we omit the range of the integration if it is X)

$$\begin{aligned} \Phi_\ell(\mathscr{L})f(y) &= \sum_{j=0}^{\infty} \Lambda_{j,\ell}(\mathscr{L})\Phi_{j+\ell}(\mathscr{L})\Psi_j(2^{-2\ell}\mathscr{L})f(y) \\ &= \sum_{j=0}^{\infty} \Psi_j(2^{-2\ell}\mathscr{L})\Lambda_{j,\ell}(\mathscr{L})\Phi_{j+\ell}(\mathscr{L})f(y) \\ &= \sum_{j=0}^{\infty} \int K_{\Psi_j(2^{-2\ell}\mathscr{L})\Lambda_{j,\ell}(\mathscr{L})}(y,z)\Phi_{j+\ell}(\mathscr{L})f(z)d\mu(z), \quad \text{a.e. } y \in X. \end{aligned} \tag{3.22}$$

Let N be any positive integer such that $N \geq 3d+1$. Since the function Ψ vanishes near the origin, the function $\lambda \mapsto \lambda^{-N}\Psi(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$. Hence by Lemma 3.8 there exists a constant $C > 0$ (depending on N) such that

$$\begin{aligned} &\left|K_{\Psi_j(2^{-2\ell}\mathscr{L})\Lambda_{j,\ell}(\mathscr{L})}(y,z)\right| \\ &= \begin{cases} \left|K_{\Psi_0(2^{-2\ell}\mathscr{L})\Phi_0(2^{-2\ell}\mathscr{L})}(y,z)\right| & \text{if } j = 0 \text{ and } \ell \in \mathbb{N}_0 \\ \left|K_{\Psi(2^{-2j}\mathscr{L})\Phi_0(\mathscr{L})}(y,z)\right| & \text{if } j \in \mathbb{N} \text{ and } \ell = 0 \\ \left|K_{\Psi(2^{-2(j+\ell)}\mathscr{L})\Phi(2^{-2\ell}\mathscr{L})}(y,z)\right| & \text{if } j \in \mathbb{N} \text{ and } \ell \in \mathbb{N} \end{cases} \\ &\leq \begin{cases} C\|\Phi_0\Psi_0\|_{(N)} D_{2^{-\ell},N}(y,z) & \text{if } j = 0 \text{ and } \ell \in \mathbb{N}_0 \\ C\|\Phi_0\|_{(2N)}\|\lambda \mapsto \lambda^{-N}\Psi(\lambda)\|_{(N)} 2^{-2jN} D_{1,N-2d}(y,z) & \text{if } j \in \mathbb{N} \text{ and } \ell = 0 \\ C\|\Phi\|_{(2N)}\|\lambda \mapsto \lambda^{-N}\Psi(\lambda)\|_{(N)} 2^{-2jN} D_{2^{-\ell},N-2d}(y,z) & \text{if } j \in \mathbb{N} \text{ and } \ell \in \mathbb{N} \end{cases}. \end{aligned}$$

This shows that for all $j, \ell \in \mathbb{N}_0$,

$$\left|K_{\Psi_j(2^{-2\ell}\mathscr{L})\Lambda_{j,\ell}(\mathscr{L})}(y,z)\right| \leq C2^{-2jN} D_{2^{-\ell},N-2d}(y,z),$$

where the last constant C depends on N, Φ_0, Ψ_0, Φ and Ψ, but is independent of $\ell \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$. Inserting this into (3.22) and using (2.3), we obtain that for a.e. $y \in X$,

$$\begin{aligned} |\Phi_\ell(\mathscr{L})f(y)| &\leq C\sum_{j=0}^{\infty} 2^{-2jN} \int D_{2^{-\ell},N-2d}(y,z)|\Phi_{j+\ell}(\mathscr{L})f(z)|d\mu(z) \\ &\leq C\sum_{j=0}^{\infty} 2^{-2jN} \int \frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|}{V(z,2^{-\ell})(1+2^\ell d(y,z))^{N-5d/2}} d\mu(z). \end{aligned} \tag{3.23}$$

Replacing ℓ by $k+\ell$, and then multiplying on both sides with 2^{-2kN}, we get that for all $k, \ell \in \mathbb{N}_0$ and a.e. $y \in X$,

$$\begin{aligned} &2^{-2kN}|\Phi_{k+\ell}(\mathscr{L})f(y)| \\ &\leq C\sum_{j=0}^{\infty} 2^{-2(j+k)N} \int \frac{|\Phi_{j+k+\ell}(\mathscr{L})f(z)|}{V(z,2^{-(k+\ell)})(1+2^{k+\ell}\rho(y,z))^{N-5d/2}} d\mu(z) \end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{j=0}^{\infty}2^{-2(j+k)N}\int\frac{|\Phi_{j+k+\ell}(\mathscr{L})f(z)|}{V(z,2^{-(j+k+\ell)})(1+2^{\ell}\rho(y,z))^{N-5d/2}}d\mu(z) && (3.24)\\
&= C\sum_{j=k}^{\infty}2^{-2jN}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|}{V(z,2^{-(j+\ell)})(1+2^{\ell}\rho(y,z))^{N-5d/2}}d\mu(z)\\
&\leq C\sum_{j=0}^{\infty}2^{-2jN}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|}{V(z,2^{-(j+\ell)})(1+2^{\ell}\rho(y,z))^{N-5d/2}}d\mu(z), && (3.25)
\end{aligned}$$

where N can be taken arbitrarily large. Now let us introduce the maximal type functions

$$M_{\ell,N}f(x):=\sup_{k\in\mathbb{N}_0}\operatorname*{ess\,sup}_{y\in X}2^{-2kN}\frac{|\Phi_{k+\ell}(\mathscr{L})f(y)|}{(1+2^{\ell}\rho(x,y))^{N-5d/2}},\quad \ell\in\mathbb{N}_0,N>0,x\in X. \tag{3.26}$$

Then it follows that for all $r\in(0,1]$, $\ell\in\mathbb{N}_0$, $N\geq 3d+1$ and all $x\in X$,

$$\begin{aligned}
&M_{\ell,N}f(x)\\
&\leq C_N\sum_{j=0}^{\infty}2^{-2jN}\int\operatorname*{ess\,sup}_{y\in X}\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|}{V(z,2^{-(j+\ell)})(1+2^{\ell}\rho(y,z))^{N-5d/2}(1+2^{\ell}\rho(x,y))^{N-5d/2}}d\mu(z)\\
&\leq C_N\sum_{j=0}^{\infty}2^{-2jN}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|}{V(z,2^{-(j+\ell)})(1+2^{\ell}\rho(x,z))^{N-5d/2}}d\mu(z)\\
&\leq C_N\sum_{j=0}^{\infty}2^{-2jNr}\left(2^{-2jN}\operatorname*{ess\,sup}_{y\in X}\frac{|\Phi_{j+\ell}(\mathscr{L})f(y)|}{(1+2^{\ell}\rho(x,y))^{N-5d/2}}\right)^{1-r} && (3.27)\\
&\qquad\times\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^{r}}{V(z,2^{-(j+\ell)})(1+2^{\ell}\rho(x,z))^{(N-5d/2)r}}d\mu(z)\\
&\leq C_N\sum_{j=0}^{\infty}2^{-2jNr}\big[M_{\ell,N}f(x)\big]^{1-r}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^{r}}{V(z,2^{-(j+\ell)})(1+2^{\ell}\rho(x,z))^{(N-5d/2)r}}d\mu(z)
\end{aligned}$$

where, for the second inequality we used the elementary inequalities

$$\begin{aligned}
&(1+2^{\ell}\rho(x,z))\leq C(1+2^{\ell}\rho(x,y))(1+2^{\ell}\rho(y,z))\quad\text{for all } x,y,z\in X,\\
&|\Phi_{j+\ell}(\mathscr{L})f(z)|\leq|\Phi_{j+\ell}(\mathscr{L})f(z)|^{r}(1+2^{\ell}\rho(x,z))^{(N-5d/2)(1-r)} && (3.28)\\
&\qquad\times\left(\operatorname*{ess\,sup}_{y\in X}\frac{|\Phi_{j+\ell}(\mathscr{L})f(y)|}{(1+2^{\ell}\rho(x,y))^{N-5d/2}}\right)^{1-r}\quad\text{for a.e. } z\in X.
\end{aligned}$$

Hence, if $M_{\ell,N}f(x)<\infty$ we obtain from (3.27)

$$\big[M_{\ell,N}f(x)\big]^{r}\leq C_N\sum_{j=0}^{\infty}2^{-2jNr}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^{r}}{V(z,2^{-(j+\ell)})(1+2^{\ell}\rho(x,z))^{(N-5d/2)r}}d\mu(z), \tag{3.29}$$

where C_N is a constant independent of x,f,ℓ. We claim that there exists $N^f\in\mathbb{N}_0$ such that $M_{\ell,N}f(x)<\infty$ for all $\ell\in\mathbb{N}_0$ and $N\geq N^f$. Indeed, by the definition of $\mathcal{S}'_{\mathscr{L}}(X)$, there exist $m_0,k_0\in\mathbb{N}_0$ and $c_f>0$ such that for a.e. $y\in X$,

$$\begin{aligned}
&|\Phi_{k+\ell}(\mathscr{L})f(y)|=\big|f\big(K_{\Phi_{k+\ell}(\mathscr{L})}(y,\cdot)\big)\big|\\
&\leq c_f\mathcal{P}^{*}_{k_0,m_0}\big(K_{\Phi_{k+\ell}(\mathscr{L})}(y,\cdot)\big)\\
&= c_f\sum_{\substack{0\leq j\leq k_0\\0\leq\eta\leq m_0}}\mathcal{P}_{j,\eta}\big(K_{\Phi_{k+\ell}(\mathscr{L})}(y,\cdot)\big)\\
&= c_f\sum_{\substack{0\leq j\leq k_0\\0\leq\eta\leq m_0}}\operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{\eta}\big|\mathscr{L}^{j}\big(K_{\Phi_{k+\ell}(\mathscr{L})}(y,\cdot)\big)(z)\big|
\end{aligned}$$

$$= c_f \sum_{\substack{0\le j\le k_0\\ 0\le \eta\le m_0}} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{\eta}\big|K_{\mathscr{L}^j\Phi_{k+\ell}(\mathscr{L})}(y,z)\big|$$

$$\le c_f \sum_{0\le j\le k_0} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}\big|K_{\mathscr{L}^j\Phi_{k+\ell}(\mathscr{L})}(y,z)\big|$$

$$= \begin{cases} c_f \displaystyle\sum_{0\le j\le k_0} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}\big|K_{\mathscr{L}^j\Phi_0(\mathscr{L})}(y,z)\big| & \text{if } k+\ell=0\\ c_f \displaystyle\sum_{0\le j\le k_0} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}2^{2(k+\ell)j}\big|K_{(2^{-2(k+\ell)}\mathscr{L})^j\Phi(2^{-2(k+\ell)}\mathscr{L})}(y,z)\big| & \text{if } k+\ell>0 \end{cases}$$

$$\le \begin{cases} c_{f,N} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}D_{1,N}(y,z) & \text{if } k+\ell=0\\ c_{f,N} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}2^{2(k+\ell)k_0}D_{2^{-(k+\ell)},N}(y,z) & \text{if } k+\ell>0 \end{cases}$$

$$\le \begin{cases} c_{f,N} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}(1+\rho(y,z))^{-N+\frac{d}{2}} & \text{if } k+\ell=0\\ c_{f,N} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}2^{2(k+\ell)k_0}[V(x_0,2^{-(k+\ell)})]^{-1}(1+2^{k+\ell}\rho(y,z))^{-N+\frac{d}{2}} & \text{if } k+\ell>0 \end{cases}$$

$$\le \begin{cases} c_{f,N} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}(1+\rho(y,z))^{-N+\frac{d}{2}} & \text{if } k+\ell=0\\ c_{f,N} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}2^{2(k+\ell)k_0}2^{(k+\ell)d}(1+2^{k+\ell}\rho(y,z))^{-N+\frac{d}{2}} & \text{if } k+\ell>0 \end{cases}.$$

Here, for the last inequality we used the non-collapsing condition. Hence, assuming $N > \max\{m_0+\frac{5d}{2}, k_0+\frac{d}{2}\}$, we estimate as follows:

$$\begin{aligned}
&M_{\ell,N}f(x)\\
&\le \sup_{k\in\mathbb{N}_0}\operatorname*{ess\,sup}_{y\in X} 2^{-2kN}\frac{|\Phi_{k+\ell}(\mathscr{L})f(y)|}{(1+2^{\ell}\rho(x,y))^{N-5d/2}}\\
&\le c_{f,N}\sup_{k\in\mathbb{N}_0}\operatorname*{ess\,sup}_{y\in X} 2^{-2kN}\\
&\quad\times\begin{cases} \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}(1+\rho(y,z))^{-N+\frac{d}{2}}(1+\rho(x,y))^{-N+\frac{5d}{2}} & \text{if } k+\ell=0\\ \operatorname*{ess\,sup}_{z\in X}(1+\rho(z,x_0))^{m_0}2^{2(k+\ell)k_0}2^{(k+\ell)d}(1+2^{k+\ell}\rho(y,z))^{-N+\frac{d}{2}}(1+2^{\ell}\rho(x,y))^{-N+\frac{5d}{2}} & \text{if } k+\ell>0 \end{cases}\\
&\le c_{f,N}\sup_{k\in\mathbb{N}_0}2^{-2kN}\begin{cases} (1+\rho(x,x_0))^{m_0} & \text{if } k+\ell=0\\ 2^{2(k+\ell)k_0}2^{(k+\ell)d}(1+\rho(x,x_0))^{m_0} & \text{if } k+\ell>0\end{cases}\\
&<\infty.
\end{aligned}$$

This implies that if $N>\max\{m_0+5d/2, k_0+d/2, 3d+1\} := N^f$ then $M_{\ell,N}f(x)<\infty$ for all $\ell\in\mathbb{N}_0$ and all $x\in X$. Therefore (3.29) together with the obvious inequality $|\Phi_\ell(\mathscr{L})f(x)| \le M_{\ell,N}f(x)$ (a.e. $x\in X$) yields that for all $N>N^f$ and for a.e. $x\in X$,

$$|\Phi_\ell(\mathscr{L})f(x)|^r \le c\sum_{j=0}^{\infty}2^{-2jNr}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^{\ell}\rho(x,z))^{(N-5d/2)r}}d\mu(z) \tag{3.30}$$

with $c=C_N$ independent of x, f and ℓ. Observe that the sum in the right-hand side of (3.30) decreases as N increases. Therefore, (3.30) is valid for all $N\in\mathbb{N}_0$ with

$$c = C_{N,f} = \begin{cases} C_{N^f} & \text{if } 0\le N\le N^f\\ C_N & \text{if } N>N^f\end{cases}$$

depending on N and f. We want to obtain (3.30) with c independent of f. For this purpose, we start from (3.30) which is valid for all $N\in\mathbb{N}$ but in which $c=C_{N,f}$ depends on N and f, apply the same argument as used from (3.24) to (3.25), and switch to the maximal function (3.26) with the aid of (3.28). Thus we get (3.29) with a constant depends on f. Untill now we have seen that if RHS(3.29) (= RHS(3.30))$<\infty$ then $M_{\ell,N}f(x)<\infty$.

Fix arbitrary $N\in\mathbb{N}_0$ with $N\geq 3d+1$. To prove (3.30) we may assume RHS(3.30)$<\infty$, since otherwise (3.30) is trivial. Hence, by the preceding remarks we have $M_{\ell,N}f(x)<\infty$. Therefore from (3.27) we deduce (3.29) with the constant C_N independent of f. Finally, form (3.29) and the obvious inequality $|\Phi_\ell(\mathscr{L})f(x)|\leq M_{\ell,N}f(x)$ (a.e. $x\in X$), we obtain (3.30).

Note that (3.30) also holds for $r\in(1,\infty)$. Indeed, it follows from (5.10) (with N replaced by $N+\lfloor 5d/2\rfloor+\lfloor 2d\rfloor+3$) that for a.e. $x\in X$,

$$
\begin{aligned}
&|\Phi_\ell(\mathscr{L})f(x)|\\
&\leq C_N\sum_{j=0}^{\infty}2^{-2j(N+\lfloor 5d/2\rfloor+\lfloor 2d\rfloor+3)}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{N+\lfloor 2d\rfloor+2}}d\mu(z)\\
&= C_N\sum_{j=0}^{\infty}2^{-2j(N+\lfloor 5d/2\rfloor+\lfloor 2d\rfloor+3)}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|}{[V(z,2^{-\ell})]^{1/r}(1+2^\ell\rho(x,z))^{N}}\\
&\qquad\qquad\times\frac{1}{[V(z,2^{-\ell})]^{1/r'}(1+2^\ell\rho(x,z))^{\lfloor 2d\rfloor+2}}d\mu(z)\\
&\leq C_N\sum_{j=0}^{\infty}2^{-2j(N+\lfloor 5d/2\rfloor+\lfloor 2d\rfloor+3)}\left(\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{Nr}}d\mu(z)\right)^{1/r}\\
&\qquad\qquad\times\left(\int\frac{1}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{(\lfloor 2d\rfloor+2)r'}}d\mu(z)\right)^{1/r'}\\
&\leq C_N\sum_{j=0}^{\infty}2^{-2jN}\left(\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{Nr}}d\mu(z)\right)^{1/r}2^{-2j(\lfloor 5d/2\rfloor+\lfloor 2d\rfloor+3)}\\
&\leq C_N\left(\sum_{j=0}^{\infty}2^{-2jNr}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{Nr}}d\mu(z)\right)^{1/r}\left(\sum_{j=1}^{\infty}2^{-2j(\lfloor\lfloor 5d/2\rfloor+\lfloor 2d\rfloor+3)r'}\right)^{1/r'}\\
&\leq C_N\left(\sum_{j=0}^{\infty}2^{-2jNr}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(x,z))^{Nr}}d\mu(z)\right)^{1/r}\\
&\leq C_N\left(\sum_{j=0}^{\infty}2^{-2jNr}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(x,z))^{(N-5d/2)r}}d\mu(z)\right)^{1/r},
\end{aligned}
$$

where we applied Hölder's inequality twice.

Now, choosing $N\geq a+5d/2$ in (3.30), it follows that for all $r\in(0,\infty)$, all $\ell\in\mathbb{N}_0$ and all $x\in X$,

$$
\begin{aligned}
&\left\{[\Phi_\ell(\mathscr{L})]_a^*f(x)\right\}^r\\
&=\operatorname*{ess\,sup}_{y\in X}\frac{|\Phi_\ell(\mathscr{L})f(y)|^r}{(1+2^\ell\rho(x,y))^{ar}}\\
&\lesssim\sum_{j=0}^{\infty}2^{-2jNr}\int\operatorname*{ess\,sup}_{y\in X}\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(y,z))^{(N-5d/2)r}(1+2^\ell\rho(x,y))^{ar}}d\mu(z)\\
&\leq\sum_{j=0}^{\infty}2^{-2jNr}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(x,z))^{ar}}d\mu(z)\\
&\lesssim\sum_{j=0}^{\infty}2^{-2jNr}2^{jd}\int\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{ar}}d\mu(z).
\end{aligned}
\tag{3.31}
$$

Here, for the second inequality we used (3.28) and for last inequality we applied (2.2). Since $a > \frac{2d}{p}$ (resp. $a > \frac{2d}{\min(p,q)}$), we may choose and fix $r \in (0,p)$ (resp. $r \in (0, \min(p,q))$) such that $ar > 2d$. Then, by using (2.2) and (2.3) the last integral can be estimated as follows.

$$\begin{aligned}
&\int \frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{ar}}d\mu(z) \lesssim \frac{1}{V(x,2^{-\ell})}\int \frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{(1+2^\ell\rho(x,z))^{ar-d}}d\mu(z)\\
&= \frac{1}{V(x,2^{-\ell})}\left(\int_{B(x,2^{-\ell})} + \sum_{k=1}^{\infty}\int_{B(x,2^{k-\ell})\setminus B(x,2^{k-\ell-1})}\right)\frac{|\Phi_{j+\ell}(\mathscr{L})f(z)|^r}{(1+2^\ell\rho(x,z))^{ar-d}}d\mu(z)\\
&\lesssim \frac{1}{V(x,2^{-\ell})}\left(\int_{B(x,2^{-\ell})}|\Phi_{j+\ell}(\mathscr{L})f(z)|^r d\mu(z) + \sum_{k=1}^{\infty}2^{-k(ar-d)}\int_{B(x,2^{k-\ell})}|\Phi_{j+\ell}(\mathscr{L})f(z)|^r d\mu(z)\right)\\
&\lesssim \frac{1}{V(x,2^{-\ell})}\int_{B(x,2^{-\ell})}|\Phi_{j+\ell}(\mathscr{L})f(z)|^r d\mu(z)\\
&\quad + \sum_{k=1}^{\infty}2^{-k(ar-2d)}\frac{1}{V(x,2^{k-\ell})}\int_{B(x,2^{k-\ell})}|\Phi_{j+\ell}(\mathscr{L})f(z)|^r d\mu(z)\\
&\lesssim M_{HL}\big(|\Phi_{j+\ell}(\mathscr{L})f(z)|^r\big)(x),
\end{aligned}$$

where M_{HL} is the Hardy-Littlewood maximal operator defined by

$$M_{HL}f(x) := \sup_{r>0}\sup_{y\in B(x,r)}\frac{1}{V(y,r)}\int_{B(y,r)}|f(z)|d\mu(z), \quad x \in X.$$

Substituting this into (3.31) gives that, for all $\ell \in \mathbb{N}_0$ and all $x \in X$,

$$\begin{aligned}
\big\{2^{-\ell s}[\Phi_\ell(\mathscr{L})]_a^* f(x)\big\}^r &\lesssim \sum_{j=0}^{\infty}2^{-j(2Nr-sr-d)}M_{HL}\big[\big(2^{-(j+\ell)s}|\Phi_{j+\ell}(\mathscr{L})f|\big)^r\big](x)\\
&= \sum_{j=\ell}^{\infty}2^{-(j-\ell)(2Nr-sr-d)}M_{HL}\big[\big(2^{-js}|\Phi_j(\mathscr{L})f|\big)^r\big](x)\\
&\le \sum_{j=0}^{\infty}2^{-|j-\ell|(2Nr-sr-d)}M_{HL}\big[\big(2^{-js}|\Phi_j(\mathscr{L})f|\big)^r\big](x).
\end{aligned}$$

If we apply Lemma 3.11 in spaces $\ell^{q/r}(L^{p/r})$ and $L^{p/r}(\ell^{q/r})$, we get

$$\big\|\big\{2^{\ell s}[\Phi_\ell(\mathscr{L})]_a^* f\big\}_{\ell=0}^{\infty}\big\|_{\ell^q(L^p)} \le C\big\|\big\{M_r\left[2^{js}\Phi_j(\mathscr{L})f\right]\big\}_{j=0}^{\infty}\big\|_{\ell^q(L^p)} \tag{3.32}$$

and

$$\big\|\big\{2^{\ell s}[\Phi_\ell(\mathscr{L})]_a^* f\big\}_{\ell=0}^{\infty}\big\|_{\ell^q(L^p)} \le C\big\|\big\{M_r\left[2^{js}\Phi_j(\mathscr{L})f\right]\big\}_{j=0}^{\infty}\big\|_{\ell^q(L^p)}. \tag{3.33}$$

where we denoted $M_r(g) := \big(M_{HL}(|g|^r)\big)^{1/r}$.

The Fefferman-Stein vector-valued maximal inequalities on spaces of homogeneous type (cf. [39]) yields that

$$M_r : \ell^q(L^p) \to \ell^q(L^p), \qquad r < p \le \infty,\ 0 < q \le \infty, \tag{3.34}$$

$$M_r : L^p(\ell^q) \to L^p(\ell^q), \qquad r < p < \infty,\ r < q \le \infty. \tag{3.35}$$

Since $r \in (0,p)$ (resp. $r \in (0, \min(p,q))$), by applying (3.34) (resp. (3.35)) to the right-hand side of (3.32) (resp. (3.33)), we get the desired (3.4) (resp. (3.5)). The proof of Theorem 3.4 is therefore completed. □

We next turn to the proof of Theorem 3.5.

Proof of Theorem 3.5. Since Φ_0, Φ satisfy (3.1), by Lemma 3.9 there exists $\Psi_0, \Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\operatorname{supp} \Psi_0 \subset [0, 2^2\varepsilon], \quad \operatorname{supp} \Psi \subset [2^{-2}\varepsilon, 2^2\varepsilon],$$

and

$$\Psi_0(\lambda)\Phi_0(\lambda) + \sum_{j=1}^{\infty} \Psi(2^{-2j}\lambda)\Phi(2^{-2j}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}_{\geq 0}.$$

Setting $\Psi_j(\lambda) := \Psi(2^{-2j}\lambda)$ and $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$, we can rewrite the above equality as

$$\sum_{j=0}^{\infty} \Psi_j(\lambda)\Phi_j(\lambda) = 1, \quad \forall \lambda \in \mathbb{R}_{\geq 0}.$$

Then it follows from Lemma 3.10 that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,

$$f = \sum_{j=0}^{\infty} \Psi_j(\mathscr{L})\Phi_j(\mathscr{L})f \quad \text{in } \mathcal{S}'_{\mathscr{L}}(X).$$

Consequently, for $\ell \in \mathbb{N}_0$ and a.e. $y \in X$,

$$\widetilde{\Phi}_\ell(\mathscr{L})f(y) = \sum_{j=0}^{\infty} \widetilde{\Phi}_\ell(\mathscr{L})\Psi_j(\mathscr{L})\Phi_j(\mathscr{L})f(y).$$

It follows that

$$\begin{aligned}
|\widetilde{\Phi}_\ell(\mathscr{L})f(y)| &\leq \sum_{j=0}^{\infty} |\widetilde{\Phi}_\ell(\mathscr{L})\Psi_j(\mathscr{L})\Phi_j(\mathscr{L})f(y)| \\
&\leq \sum_{j=0}^{\infty} \int \left|K_{\widetilde{\Phi}_\ell(\mathscr{L})\Psi_j(\mathscr{L})}(y,z)\right| |\Phi_j(\mathscr{L})f(z)| \, d\mu(z) \\
&\leq \sum_{j=0}^{\infty} [\Phi_j(\mathscr{L})]_a^* f(y) \int (1 + 2^j\rho(y,z))^a \left|K_{\widetilde{\Phi}_\ell(\mathscr{L})\Psi_j(\mathscr{L})}(y,z)\right| d\mu(z) \\
&= \sum_{j=0}^{\infty} [\Phi_j(\mathscr{L})]_a^* f(y) I_{j,\ell}(y),
\end{aligned} \tag{3.36}$$

where we have set

$$I_{j,\ell}(y) := \int (1 + 2^j\rho(y,z))^a \left|K_{\widetilde{\Phi}_\ell(\mathscr{L})\Psi_j(\mathscr{L})}(y,z)\right| d\mu(z).$$

To estimate $I_{j,\ell}(y)$ we consider two cases:

Case 1: $j < \ell$. In this case, by the fact that the function $\lambda \mapsto \lambda^{-M}\widetilde{\Phi}(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$ and by Lemma 3.8, we have

$$\left|K_{\widetilde{\Phi}_\ell(\mathscr{L})\Psi_j(\mathscr{L})}(y,z)\right| \lesssim 2^{2(j-\ell)M} D_{2^{-j},N-2d}(y,z),$$

where N can be taken arbitrarily large. Hence, choosing $N > a + \frac{7d}{2}$, it follows that

$$\begin{aligned}
I_{j,\ell}(y) &\lesssim 2^{2(j-\ell)M} \int (1 + 2^j\rho(y,z))^a D_{2^{-j},N-2d}(y,z) d\mu(z) \\
&= 2^{2(j-\ell)M} \int D_{2^{-j},N-a-2d}(y,z) d\mu(z) \\
&\lesssim 2^{2(j-\ell)M},
\end{aligned} \tag{3.37}$$

where for the last inequality we applied (3.9).

Case 2: $j \geq \ell$. In this case, we use the fact that $\lambda \mapsto \lambda^{-\widetilde{M}}\Psi(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$ for all nonnegative integers $\widetilde{M}$ (since Ψ vanishes near the origin) and by Lemma 3.8, we conclude that

$$\big|K_{\widetilde{\Phi}_\ell(\mathscr{L})\Psi_j(\mathscr{L})}(y,z)\big| \lesssim 2^{2(\ell-j)\widetilde{M}} D_{2^{-\ell},N-2d}(y,z),$$

where both $\widetilde{M}$ and N can be taken arbitrarily large. Hence in this case we have

$$\begin{aligned} I_{j,\ell}(y) &\lesssim 2^{2(\ell-j)\widetilde{M}} \int (1+2^j\rho(y,z))^a D_{2^{-\ell},N-2d}(y,z)d\mu(z) \\ &\lesssim 2^{2(\ell-j)(\widetilde{M}-\frac{a}{2})} \int (1+2^\ell\rho(y,z))^a D_{2^{-\ell},N-2d}(y,z)d\mu(z) \\ &= 2^{2(\ell-j)(\widetilde{M}-\frac{a}{2})} \int D_{2^{-\ell},N-a-2d}(y,z)d\mu(z) \\ &\lesssim 2^{2(\ell-j)(\widetilde{M}-\frac{a}{2})} \end{aligned} \tag{3.38}$$

where, for the last inequality we applied (3.9). Let us further observe that

$$\begin{aligned} [\Phi_j(\mathscr{L})]_a^* f(y) &\leq [\Phi_j(\mathscr{L})]_a^* f(x)(1+2^j\rho(x,y))^a \\ &\leq [\Phi_j(\mathscr{L})]_a^* f(x)(1+2^\ell\rho(x,y))^a \max\{1, 2^{(j-\ell)a}\}. \end{aligned} \tag{3.39}$$

Inserting (3.37)–(3.39) into (3.36) we get

$$\begin{aligned} 2^{\ell s}[\widetilde{\Phi}_\ell(\mathscr{L})]_a^* f(x) &\lesssim \sum_{j=0}^{\infty} 2^{js}[\Phi_j(\mathscr{L})]_a^* f(x) 2^{(\ell-j)s} \max(1, 2^{(j-\ell)a}) \begin{cases} 2^{2(j-\ell)M} & \text{if } j \leq \ell, \\ 2^{2(\ell-j)(\widetilde{M}-\frac{a}{2})} & \text{if} j > \ell. \end{cases} \\ &= \sum_{j=0}^{\infty} 2^{js}[\Phi_j(\mathscr{L})]_a^* f(x) \begin{cases} 2^{2(j-\ell)(M-\frac{s}{2})} & \text{if } j < \ell \\ 2^{2(\ell-j)(\widetilde{M}-a+\frac{s}{2})} & \text{if } j \geq \ell \end{cases}. \end{aligned}$$

Choosing $\widetilde{M} > a + |s|/2$ and setting $\delta := \min\{\widetilde{M} - a + \frac{s}{2}, M - \frac{s}{2}\} > 0$, we obtain

$$2^{\ell s}[\widetilde{\Phi}_\ell(\mathscr{L})]_a^* f(x) \lesssim \sum_{j=0}^{\infty} 2^{-2|j-\ell|\delta}\big\{2^{js}[\Phi_j(\mathscr{L})]_a^* f(x)\big\}.$$

Applying Lemma 3.11 gives the desired estimates (3.6) and (3.7). This completes the proof of Theorem 3.5. □

3.3 Basic properties

We have the following elementary properties for Besov and Triebel-Lizorkin spaces associated to operators:

Proposition 3.12. *Let $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty]$. Then:*

(i) *$\mathcal{S}_{\mathscr{L}}(X) \subset B^{s,\mathscr{L}}_{p,q}(X) \subset \mathcal{S}'_{\mathscr{L}}(X)$ and the inclusion maps are continuous.*

(ii) *The space $B^{s,\mathscr{L}}_{p,q}(X)$ is a quasi-Banach space.*

Proposition 3.13. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Then:*

(i) *$\mathcal{S}_{\mathscr{L}}(X) \subset F^{s,\mathscr{L}}_{p,q}(X) \subset \mathcal{S}'_{\mathscr{L}}(X)$ and the inclusion maps are continuous.*

(ii) *The space* $F^{s,\mathscr{L}}_{p,q}(X)$ *is a quasi-Banach space.*

We only prove Proposition 3.13. The proof of Proposition 3.12 is similar and we skip the details. Before we start the proof of Proposition 3.13, we recall a result from [59]. For $\lambda > 0$, we set

$$\Sigma_\lambda = \big\{f \in \mathcal{S}'_{\mathscr{L}}(X) : \Theta(\mathscr{L})f = f \text{ in } \mathcal{S}'_{\mathscr{L}}(X) \text{ for all } \Theta \in \mathcal{S}(\mathbb{R}_{\geq 0}) \text{ with } \Theta \equiv 1 \text{ on } [0,\lambda^2]\big\}.$$

Lemma 3.14. (see [59, Proposition 3.11]) *Suppose* $0 < p \leq q \leq \infty$. *Then there exists a constant* $C > 0$ *such that for all* $g \in \Sigma_\lambda$ *with* $\lambda \geq 1$

$$\|g\|_{L^q(X)} \leq C\lambda^{d(1/p-1/q)}\|g\|_{L^p(X)}.$$

Proof. (Due to [59]) Let $g \in \Sigma_\lambda$, where $\lambda \geq 1$. Set $\delta := \lambda^{-1} \leq 1$. Let $\Theta \in C^\infty_0(\mathbb{R}_{\geq 0})$ such that $\Theta = 1$ on $[0, 1]$. Then it follows from Lemma 2.1 that for any $\delta \in (0, 1]$ and $\sigma > 0$,

$$|K_{\Theta(\delta\sqrt{\mathscr{L}})}(x,y)| \lesssim D_{\delta,\sigma+d/2}(x,y) \leq [V(x,\delta)]^{-1}\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma}. \tag{3.40}$$

Suppose $1 < p < \infty$. Since $g \in \Sigma_\lambda$, we have, for a.e. $x \in X$,

$$g(x) = \Theta(\delta\sqrt{\mathscr{L}})g(x) = \int_X K_{\Theta(\delta\sqrt{\mathscr{L}})}(x,y)g(y)d\mu(y).$$

Hence, by using (3.40) with $\sigma > (d+1)/p'$, Hölder's inequality, (3.8) and the non-collapsing condition, we obtain

$$\begin{aligned} |g(x)| &\leq \|g\|_{L^p(X)}\left(\int_X [V(x,\delta)]^{-p'}\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma p'} d\mu(y)\right)^{1/p'} \\ &\lesssim \|g\|_{L^p(X)}[V(x,\delta)]^{-1/p}. \end{aligned} \tag{3.41}$$

Suppose now $0 < p \leq 1$. Then for a.e. $x \in X$,

$$\begin{aligned} |g(x)| &\leq \int_X [V(x,\delta)]^{-1}\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma}|g(y)|^p|g(y)|^{1-p}d\mu(y) \\ &\leq \|g\|^{1-p}_{L^\infty(X)}\|g\|^p_{L^p(X)}[V(x,\delta)]^{-1}. \end{aligned} \tag{3.42}$$

(3.41) together with (3.42) and the non-collapsing condition yield that for all $0 < p < \infty$

$$\|g\|_{L^\infty(X)} \lesssim \|g\|_{L^p(X)} \sup_{x\in X}[V(x,\delta)]^{-1/p} \lesssim \|g\|_{L^p(X)}\delta^{-d/p}. \tag{3.43}$$

So we have

$$\|g\|^q_{L^q(X)} = \int_X |g(x)|^{q-p}|g(x)|^p d\mu(x) \leq \|g\|^{q-p}_{L^\infty(X)}\|g\|^p_{L^p(X)} \lesssim \|g\|^{q-p}_{L^p}\delta^{-d(q-p)/p}\|g\|^p_{L^p(X)},$$

and hence

$$\|g\|_{L^q(X)} \lesssim \delta^{-d(1/p-1/q)}\|g\|_{L^p(X)} = \lambda^{d(1/p-1/q)}\|g\|_{L^p(X)}.$$

This completes the proof. □

Now we are ready to give the

Proof of Proposition 3.13. (i) We first show that $\mathcal{S}_{\mathscr{L}}(X) \subset F^{s,\mathscr{L}}_{p,q}(X)$ and the inclusion map is continuous. Indeed, let $\phi \in \mathcal{S}_{\mathscr{L}}(X)$. Let $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ where M, N are positive integers such that $M > \frac{s}{2}$ and $N \geq \frac{d+1}{p} + \lfloor 3d \rfloor + 2$. Then from (3.19) we see that for all $j \in \mathbb{N}_0$ and a.e. $x \in X$,

$$|\Phi_j(\mathscr{L})\phi(x)| \lesssim 2^{-2jM} \mathcal{P}_{M,N}(\phi)(1 + \rho(x, x_0))^{-(N-3d)}.$$

It follows that

$$\begin{aligned}
\|\phi\|_{F^{s,\mathscr{L}}_{p,q}(X)} &= \left\| \left(\sum_{j=0}^{\infty} \left(2^{js}|\Phi_j(\mathscr{L})\phi|\right)^q \right)^{1/q} \right\|_{L^p(X)} \\
&\lesssim \mathcal{P}_{M,N}(\phi) \left(\sum_{j=0}^{\infty} 2^{-2jq(M-\frac{s}{2})} \right)^{1/q} \left(\int (1+\rho(x,x_0))^{-(N-3d)p} d\mu(x) \right)^{1/p} \\
&\lesssim \mathcal{P}_{M,N}(\phi),
\end{aligned}$$

where for the last inequality we used (3.8) and $(N - 3d)p > d + 1$. This implies that $\mathcal{S}_{\mathscr{L}}(X) \subset F^{s,\mathscr{L}}_{p,q}(X)$ and the inclusion map is continuous.

Next we show that $F^{s,\mathscr{L}}_{p,q}(X) \subset \mathcal{S}'_{\mathscr{L}}(X)$. Let Φ_0, Φ be functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$|\Phi_0(\lambda)| \geq c > 0 \text{ for } \lambda \in [0, 2^{3/2}], \quad |\Phi(\lambda)| \geq c > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}],$$

and

$$\text{the function } \lambda \mapsto \lambda^{-M}\Phi(\lambda) \text{ belongs to } \mathcal{S}(\mathbb{R}_{\geq 0}),$$

where M is a sufficiently large positive integer which will be determined later. Then by Lemma 3.9 there exists $\Psi_0, \Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\begin{aligned}
\operatorname{supp} \Psi_0 \subset [0, 2^2], &\quad |\Psi_0(\lambda)| \geq c' > 0 \text{ for } \lambda \in [0, 2^{3/2}], \\
\operatorname{supp} \Psi \subset [2^{-2}, 2^2], &\quad |\Psi(\lambda)| \geq c' > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}],
\end{aligned}$$

and

$$\Phi_0(\lambda)\Psi_0(\lambda) + \sum_{j=1}^{\infty} \Phi(2^{-2j}\lambda)\Psi(2^{-2j}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}_{\geq 0}.$$

Hence, using Lemma 3.10, for any $f \in \mathcal{S}'_{\mathscr{L}}(X)$

$$f = \sum_{j=0}^{\infty} \Phi_j(\mathscr{L})\Psi_j(\mathscr{L}) f \quad \text{in } \mathcal{S}'_{\mathscr{L}}(X),$$

where we have set $\Phi_j(\cdot) := \Phi(2^{-2j}\cdot)$ and $\Psi_j(\cdot) := \Psi(2^{-2j}\cdot)$ for $j \geq 1$. It follows that

$$\langle f, \phi \rangle = \left\langle \sum_{j=0}^{\infty} \Phi_j(\mathscr{L})\Psi_j(\mathscr{L})f, \phi \right\rangle = \sum_{j=0}^{\infty} \langle \Psi_j(\mathscr{L})f, \Phi_j(\mathscr{L})\phi \rangle, \quad \phi \in \mathcal{S}_{\mathscr{L}}(X). \tag{3.44}$$

From (3.19) we see that for any sufficiently large positive integers M and N, there exits a constant $C > 0$ (depending on N) such that for all $j \geq 1$ and a.e. $x \in X$,

$$\left|\Phi_j(\mathscr{L})\phi(x)\right| \leq C 2^{-2jM} \|\lambda \mapsto \lambda^{-M}\Phi(\lambda)\|_{(N)} \mathcal{P}_{M,N}(\phi)(1 + \rho(x, x_0))^{-N+3d}. \tag{3.45}$$

Now we are ready to estimate the inner product in (3.44). We consider two cases:

Case 1: $1 < p < \infty$. Then applying Hölder's inequality we get for $j \in \mathbb{N}_0$

$$\begin{aligned} |\langle \Psi_j(\mathscr{L})f, \Phi_j(\mathscr{L})\phi\rangle| &\le \int 2^{js}|\Psi_j(\mathscr{L})f(x)|2^{-js}|\Phi_j(\mathscr{L})\phi(X)|d\mu(x) \\ &\le \left\|2^{js}\Psi_j(\mathscr{L})f(x)\right\|_{L^p(X)}\left\|2^{-js}\Phi_j(\mathscr{L})\phi\right\|_{L^{p'}(X)} \\ &\le \|f\|_{F^{s,\mathscr{L}}_{p,q}(X)}\left\|2^{-js}\Phi_j(\mathscr{L})\phi\right\|_{L^{p'}(X)}. \end{aligned} \tag{3.46}$$

Let us choose $M > \max\{0, -\frac{s}{2}\}$ and $N > \frac{d}{p'} + 4d$. Then, using (3.45) and (3.8), we obtain for $j \ge 1$

$$\begin{aligned} \left\|2^{-js}\Phi_j(\mathscr{L})\phi\right\|^{p'}_{L^{p'}(X)} &\lesssim 2^{-2jMp'}2^{-jsp'}[\mathcal{P}_{M,N}(\phi)]^{p'}\int(1+\rho(x,x_0))^{-(N-3d)p'}d\mu(x) \\ &\lesssim 2^{-j(2M+s)p'}[\mathcal{P}_{M,N}(\phi)]^{p'}. \end{aligned} \tag{3.47}$$

Analogously we have

$$\left\|\Phi_0(\mathscr{L})\phi\right\|^{p'}_{L^{p'}(X)} \lesssim [\mathcal{P}_{0,N}(\phi)]^{p'}. \tag{3.48}$$

Summing up (3.46)–(3.48), and taking into account (3.44), we obtain

$$|\langle f, \phi\rangle| \lesssim [\mathcal{P}_{M,N}(\phi) + \mathcal{P}_{0,N}(\phi)]\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} \le \mathcal{P}^*_{M,N}(\phi)\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)}.$$

Case 2: $0 < p \le 1$. We have for $\phi \in \mathcal{S}_{\mathscr{L}}(X)$ and $j \ge 1$

$$|\langle \Psi_j(\mathscr{L})f, \Phi_j(\mathscr{L})\phi\rangle| \le \|\Psi_j(\mathscr{L})f\|_{L^1(X)}\|\Phi_j(\mathscr{L})\phi\|_{L^\infty(X)}.$$

Since $\Psi_j(\mathscr{L})f \in \Sigma_{2^{j+1}}$ for all $j \in \mathbb{N}_0$, Lemma 3.14 yields that for $j \in \mathbb{N}_0$

$$\|\Psi_j(\mathscr{L})f\|_{L^1(X)} \lesssim (2^{j+1})^{d(1/p-1)}\|\Psi_j(\mathscr{L})f\|_{L^p(X)} \le 2^{jd(1/p-1)}\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)}.$$

On the other hand by (3.45) we have for $j \ge 1$ and a.e. $x \in X$

$$\begin{aligned} \left|\Phi_j(\mathscr{L})\phi(x)\right| &\lesssim 2^{-2jM}\|\lambda \mapsto \lambda^{-M}\Phi(\lambda)\|_{(N)}\mathcal{P}_{M,N}(\phi)(1+\rho(x,x_0))^{-N+3d} \\ &\lesssim 2^{-2jM}\mathcal{P}_{M,N}(\phi). \end{aligned}$$

For $j = 0$, we have the analogous estimate

$$\left|\Phi_0(\mathscr{L})\phi(x)\right| \lesssim \mathcal{P}_{0,N}(\phi), \quad a.e. x \in X.$$

Let us choose $M > d(1/p-1)$. Then summing up all these estimates and taking into account (3.44), we obtain

$$|\langle f, \phi\rangle| \lesssim [\mathcal{P}_{M,N}(\phi) + \mathcal{P}_{0,N}(\phi)]\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} \le \mathcal{P}^*_{M,N}(\phi)\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)},$$

as desired.

(ii) It is easy to see that $F^{s,\mathscr{L}}_{p,q}(X)$ is a quasi-normed space. We prove the completeness. Let $\{f_\ell\}_{\ell=1}^\infty$ be a fundamental sequence in $F^{s,\mathscr{L}}_{p,q}(X)$. Then the assertion (i) shows that $\{f_\ell\}_{\ell=1}^\infty$ is also a fundamental sequence in $\mathcal{S}'_{\mathscr{L}}(X)$. Since $\mathcal{S}'_{\mathscr{L}}(X)$ is Fréchet space (in particular, complete), we can find a limit element $f \in \mathcal{S}'_{\mathscr{L}}(X)$. Let Φ_0, Φ are functions in $\mathcal{S}(\mathbb{R}_{\ge 0})$ such that

$$|\Phi_0(\lambda)| \ge c > 0 \text{ for } \lambda \in [0, 2^{3/2}], \quad |\Phi(\lambda)| \ge c > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}],$$

and

$$\text{the function } \lambda \mapsto \lambda^{-M}\Phi(\lambda) \text{ belongs to } \mathcal{S}(\mathbb{R}_{\ge 0}),$$

where M is a positive integer such that $M > s/2$. Then $\Phi(2^{-2j}\mathscr{L})f_\ell$ (resp. $\Phi_0(\mathscr{L})f_\ell$) converges to $\Phi(2^{-2j}\mathscr{L})f$ (resp. $\Phi_0(\mathscr{L})f$) in $\mathcal{S}'_{\mathscr{L}}(X)$ and pointwise as $\ell \to \infty$. On the other hand, since $\{\Phi(2^{-2j}\mathscr{L})f_\ell\}_{\ell=1}^\infty$ (resp. $\{\Phi_0(\mathscr{L})f_\ell\}_{\ell=1}^\infty$) is a fundamental sequence in $L^p(X)$, by Lemma 3.14 it is also a fundamental sequence in $L^\infty(X)$. This shows that for $j \in \mathbb{N}_0$ the limiting element of $\{\Phi_j(\mathscr{L})f_\ell\}_{\ell=1}^\infty$ in $L^p(X)$ (which is the same as in $L^\infty(X)$) coincide with $\Phi_j(\mathscr{L})f$. Now it follows by standard arguments that f belongs to $F^{s,\mathscr{L}}_{p,q}(X)$ and that f_ℓ converges in $F^{s,\mathscr{L}}_{p,q}(X)$ to f. Hence, $F^{s,\mathscr{L}}_{p,q}(X)$ is a complete space. □

3.4 Continuous Littlewood-Paley characterization

The purpose of this section is to show the following continuous Littlewood-Paley characterization of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$:

Theorem 3.15. (i) *Let $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty]$. Let further $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ for some nonnegative integer $M > s/2$. Then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\|f\|_{B^{s,\mathscr{L}}_{p,q}(X)} \sim \|\Phi_0(\mathscr{L})f\|_{L^p(X)} + \left(\int_0^1 t^{-sq}\big\|\Phi(t^2\mathscr{L})f\big\|^q_{L^p(X)}\frac{dt}{t}\right)^{1/q}. \tag{3.49}$$

(ii) *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Let further $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ for some nonnegative integer $M > s/2$. Then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} \sim \|\Phi_0(\mathscr{L})f\|_{L^p(X)} + \left\|\left(\int_0^1 t^{-sq}\big|\Phi(t^2\mathscr{L})f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}. \tag{3.50}$$

Proof. We only give the proof of (ii) since the proof of (i) is similar. We first show that

$$\begin{aligned}&\|\Phi_0(\mathscr{L})f\|_{L^p(X,d\mu)} + \left\|\left(\int_0^1 t^{-sq}\big|[\Phi(t^2\mathscr{L})]^*_a f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}\\ &\sim \|\Phi_0(\mathscr{L})f\|_{L^p(X,d\mu)} + \left\|\left(\int_0^1 t^{-sq}\big|\Phi(t^2\mathscr{L})f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)},\end{aligned} \tag{3.51}$$

where

$$[\Phi(t^2\mathscr{L})]^*_a f(x) := \operatorname*{ess\,sup}_{y\in X}\frac{|\Phi(t^2\mathscr{L})f(y)|}{(1+t^{-1}\rho(x,y))^a}, \quad x \in X.$$

Note that for any $r > 0$ and $N \in \mathbb{N}_0$, there exists a constant $C_N > 0$ such that for all $f \in \mathcal{S}_{\mathscr{L}}(X)$, $\ell \in \mathbb{N}_0$, $t \in [1,2]$ and a.e. $x \in X$,

$$\big|\Phi_\ell(t^2\mathscr{L})f(x)\big|^r \leq C_N\sum_{j=0}^\infty 2^{-2jNr}\int_X\frac{|\Phi_{j+\ell}\big(t^2\mathscr{L}\big)f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(x,z))^{(N-5d/2)r}}d\mu(z). \tag{3.52}$$

Indeed, this follows from the argument used in the proof of (3.30) with slight modification. The estimate (3.52) implies immediately the following stronger estimate: for any $a > 0$ and $N \in \mathbb{N}$ with $N > a + 5d/2$,

$$\big|[\Phi_\ell(t^2\mathscr{L})]^*_a f(x)\big|^r \leq C_N\sum_{j=0}^\infty 2^{-2jNr}\int_X\frac{|\Phi_{j+\ell}\big(t^2\mathscr{L}\big)f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(x,z))^{ar}}d\mu(z),$$

where

$$[\Phi_\ell(t^2\mathscr{L})]_a^* f(x) := \operatorname*{ess\,sup}_{y\in X} \frac{\big|\Phi_\ell(t^2\mathscr{L})f(y)\big|}{(1+2^\ell t^{-1}\rho(x,y))^a}, \quad x\in X.$$

Hence we have

$$(2^{-\ell}t)^{-sr}\big|[\Phi_\ell(t^2\mathscr{L})]_a^* f(x)\big|^r \le C_N\sum_{j=0}^\infty 2^{-2jNr}2^{jsr}\int_X \frac{(2^{-(j+\ell)}t)^{sr}|\Phi_{j+\ell}\big(t^2\mathscr{L}\big)f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(x,z))^{ar}}d\mu(z),$$

If we choose $r<\min\{p,q\}$, we can apply the norm $\big(\int_1^2|\cdot|^{q/r}\frac{dt}{t}\big)^{r/q}$ on both sides use Minkowski's inequality for integrals, which yields that for all $\ell\in\mathbb{N}$,

$$\begin{aligned}
&\Big(\int_1^2 (2^{-\ell}t)^{-sq}\big|[\Phi_\ell(t^2\mathscr{L})]_a^* f(x)\big|^q\frac{dt}{t}\Big)^{r/q}\\
&\le C\sum_{j=0}^\infty 2^{-2jNr}2^{jsr}\int_X \frac{\big(\int_1^2(2^{-(j+\ell)}t)^{-sq}|\Phi_{j+\ell}(t^2\mathscr{L})f(z)|^q\frac{dt}{t}\big)^{r/q}}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(x,z))^{ar}}d\mu(z)\\
&= C\sum_{j=0}^\infty 2^{-2jNr}2^{jsr}\int_X \frac{\big(\int_1^2(2^{-(j+\ell)}t)^{-sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^q\frac{dt}{t}\big)^{r/q}}{V(z,2^{-(j+\ell)})(1+2^\ell\rho(x,z))^{ar}}d\mu(z)\\
&\le C\sum_{j=0}^\infty 2^{-2jNr}2^{jsr}2^{jd}\int_X \frac{\big(\int_1^2(2^{-(j+\ell)}t)^{-sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^q\frac{dt}{t}\big)^{r/q}}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{ar}}d\mu(z),
\end{aligned} \tag{3.53}$$

By using (2.2) and (2.3), we can estimate the last integral as follows:

$$\begin{aligned}
&\int_X \frac{\big(\int_1^2(2^{-(j+\ell)}t)^{-sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^q\frac{dt}{t}\big)^{r/q}}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{ar}}d\mu(z)\\
&\le \frac{C}{V(x,2^{-\ell})}\int_X \frac{\big(\int_1^2(2^{-(j+\ell)}t)^{-sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^q\frac{dt}{t}\big)^{r/q}}{(1+2^\ell\rho(x,z))^{ar-d}}d\mu(z)\\
&= \frac{C}{V(x,2^{-\ell})}\int_{B(x,2^{-\ell})} \frac{\big(\int_1^2(2^{-(j+\ell)}t)^{sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^q\frac{dt}{t}\big)^{r/q}}{(1+2^\ell\rho(x,z))^{ar-d}}d\mu(z)\\
&\qquad+\sum_{k=1}^\infty\int_{B(x,2^{k-\ell})\backslash B(x,2^{k-\ell-1})} \frac{\big(\int_1^2(2^{-(j+\ell)}t)^{sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^q\frac{dt}{t}\big)^{r/q}}{(1+2^\ell\rho(x,z))^{ar-d}}d\mu(z)\\
&\le \frac{C}{V(x,2^{-\ell})}\int_{B(x,2^{-\ell})}\Big(\int_1^2(2^{-(j+\ell)}t)^{sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^q\frac{dt}{t}\Big)^{r/q}d\mu(z)\\
&\qquad+C\sum_{k=1}^\infty\frac{2^{-k(ar-2d)}}{V(x,2^{k-\ell})}\int_{B(x,2^{k-\ell})}\Big(\int_1^2(2^{-(j+\ell)}t)^{sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^q\frac{dt}{t}\Big)^{r/q}d\mu(z)\\
&\le CM_{HL}\left[\Big(\int_1^2(2^{-(j+\ell)}t)^{sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(\cdot)|^q\frac{dt}{t}\Big)^{r/q}\right](x).
\end{aligned}$$

Substituting this into (3.53) gives that for all $\ell\in\mathbb{N}$,

$$\begin{aligned}
&\Big(\int_1^2(2^{-\ell}t)^{-sq}\big|[\Phi(2^{-2\ell}t^2\mathscr{L})]_a^* f(x)\big|^q\frac{dt}{t}\Big)^{r/q}\\
&\qquad\le C\sum_{j=0}^\infty 2^{-j(2Nr-sr-d)}M_{HL}\left[\Big(\int_1^2(2^{-(j+\ell)}t)^{sq}|\Phi(2^{-2(j+\ell)}t^2\mathscr{L})f(\cdot)|^q\frac{dt}{t}\Big)^{r/q}\right](x)\\
&\qquad\le C\sum_{j=\ell}^\infty 2^{-(j-\ell)(2Nr-sr-d)}M_{HL}\left[\Big(\int_1^2(2^{-j}t)^{sq}|\Phi(2^{-2j}t^2\mathscr{L})f(\cdot)|^q\frac{dt}{t}\Big)^{r/q}\right](x)
\end{aligned}$$

$$\leq C\sum_{j=1}^{\infty}2^{-|j-\ell|(2Nr-sr-d)}M_{HL}\left[\left(\int_1^2(2^{-j}t)^{sq}\big|\Phi(2^{-2j}t^2\mathscr{L})f(\cdot)\big|^q\frac{dt}{t}\right)^{r/q}\right](x).$$

Applying Lemma 3.11 in the space $L^{p/r}(\ell^{q/r})$ and the Fefferman-Stein vector-valued maximal inequalities on spaces of homogeneous type (cf. [39]), we obtain

$$\begin{aligned}
&\left\|\left(\int_0^1 t^{-sq}\big|[\Phi(t^2\mathscr{L})]_a^*f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}\\
&\quad=\left\|\left\{\left(\int_1^2(2^{-\ell}t)^{-sq}\big|[\Phi(2^{-2\ell}t^2\mathscr{L})]_a^*f\big|^q\frac{dt}{t}\right)^{r/q}\right\}_{\ell=1}^{\infty}\right\|_{L^{p/r}(\ell^{q/r})}^{1/r}\\
&\quad\leq C\left\|\left\{M_{HL}\left[\left(\int_1^2(2^{-j}t)^{sq}|\Phi(2^{-2j}t^2\mathscr{L})f(\cdot)|^q\frac{dt}{t}\right)^{r/q}\right]\right\}_{j=1}^{\infty}\right\|_{L^{p/r}(\ell^{q/r})}^{1/r}\\
&\quad\leq C\left\|\left\{\left(\int_1^2(2^{-j}t)^{sq}\big|\Phi(2^{-2j}t^2\mathscr{L})f\big|^q\frac{dt}{t}\right)^{r/q}\right\}_{j=1}^{\infty}\right\|_{L^{p/r}(\ell^{q/r})}^{1/r}\\
&\quad=C\left\|\left(\int_0^1 t^{-sq}\big|\Phi(t^2\mathscr{L})f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}.
\end{aligned}$$

This yields the direction "$\lesssim$" in (3.51). The inverse direction "$\gtrsim$" in (3.51) is obvious. Hence (3.51) is established.

Combining (3.51) and Theorem 3.4 we see that, to prove (3.50) it suffices to show

$$\begin{aligned}
&\left\|\left(\sum_{j=0}^{\infty}\big|2^{js}\Phi_j(\mathscr{L})f\big|^q\right)^{1/q}\right\|_{L^p(X)}\\
&\qquad\lesssim\|\Phi_0(\mathscr{L})f\|_{L^p(X)}+\left\|\left(\int_0^1 t^{-sq}\big|[\Phi(t^2\mathscr{L})]_a^*f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}
\end{aligned}\tag{3.54}$$

and

$$\begin{aligned}
&\|\Phi_0(\mathscr{L})f\|_{L^p(X)}+\left\|\left(\int_0^1 t^{-sq}\big|\Phi(t^2\mathscr{L})f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}\\
&\qquad\lesssim\left\|\left(\sum_{j=0}^{\infty}\big|2^{js}[\Phi_j(\mathscr{L})]_a^*f\big|^q\right)^{1/q}\right\|_{L^p(X)}.
\end{aligned}\tag{3.55}$$

We only give the proof (3.54) since the proof of (3.55) is similar.

Since Φ_0,Φ satisfy (3.1), by Lemma 3.9 there exists $\Psi_0,\Psi\in\mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\operatorname{supp}\Psi_0\subset[0,2^2\varepsilon],\quad \operatorname{supp}\Psi\subset[2^{-2}\varepsilon,2^2\varepsilon],$$

and

$$\Psi_0(\lambda)\Phi_0(\lambda)+\sum_{j=1}^{\infty}\Psi(2^{-2j}\lambda)\Phi(2^{-2j}\lambda)=1,\quad \lambda\in\mathbb{R}_{\geq 0}.$$

Setting $\Psi_j(\lambda) := \Psi(2^{-2j}\lambda)$ and $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$, we can rewrite the above equality as

$$\sum_{j=0}^{\infty} \Psi_j(\lambda)\Phi_j(\lambda) = 1, \quad \lambda \in \mathbb{R}_{\geq 0}.$$

Hence for all $t \in [1, 2]$,

$$\sum_{j=0}^{\infty} \Psi_j(t^2\lambda)\Phi_j(t^2\lambda) = 1, \quad \lambda \in \mathbb{R}_{\geq 0}.$$

Then it follows from Lemma 3.10 that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,

$$f = \sum_{j=0}^{\infty} \Psi_j(t^2\mathscr{L})\Phi_j(t^2\mathscr{L})f \quad \text{in } \mathcal{S}'_{\mathscr{L}}(X).$$

Consequently, for $\ell \in \mathbb{N}_0$ and $t \in [1, 2]$, we have the pointwise representation

$$\Phi_\ell(\mathscr{L})f(y) = \sum_{j=0}^{\infty} \Phi_\ell(\mathscr{L})\Psi_j(t^2\mathscr{L})\Phi_j(t^2\mathscr{L})f(y), \quad \text{a.e. } y \in X.$$

It follows that for a.e. $y \in X$,

$$\begin{aligned}
|\Phi_\ell(\mathscr{L})f(y)| &\leq \sum_{j=0}^{\infty} |\Phi_\ell(\mathscr{L})\Psi_j(t^2\mathscr{L})\Phi_j(t^2\mathscr{L})f(y)| \\
&\leq \sum_{j=0}^{\infty} \int \left|K_{\Phi_\ell(\mathscr{L})\Psi_j(t^2\mathscr{L})}(y,z)\right| \left|\Phi_j(t^2\mathscr{L})f(z)\right| d\mu(z) \\
&\leq \sum_{j=0}^{\infty} [\Phi_j(t^2\mathscr{L})]_a^* f(y) \int (1 + 2^j\rho(y,z))^a \left|K_{\Phi_\ell(\mathscr{L})\Psi_j(t^2\mathscr{L})}(y,z)\right| d\mu(z) \\
&= \sum_{j=0}^{\infty} [\Phi_j(t^2\mathscr{L})]_a^* f(y) I_{j,\ell,t}(y),
\end{aligned} \tag{3.56}$$

where we have set

$$I_{j,\ell,t}(y) := \int (1 + 2^j\rho(y,z))^a \left|K_{\Phi_\ell(\mathscr{L})\Psi_j(t^2\mathscr{L})}(y,z)\right| d\mu(z).$$

Similarly to the proof of Theorem 3.5, we have

$$I_{j,\ell,t} \leq \begin{cases} 2^{2(j-\ell)M} & \text{if } j < \ell \\ 2^{2(\ell-j)(\widetilde{M}-\frac{a}{2})} & \text{if } j \geq \ell \end{cases},$$

where the constant C is independent of $j, \ell \in \mathbb{N}_0$ and $t \in [1, 2]$. The constant $\widetilde{M}$ above can be taken arbitrarily large. Inserting this into (3.56) we get that for all $\ell \in \mathbb{N}$,

$$\begin{aligned}
\left|2^{\ell s}\Phi_\ell(\mathscr{L})f(y)\right| &\lesssim \sum_{j=0}^{\infty} 2^{js}[\Phi_j(t^2\mathscr{L})]_a^* f(y) 2^{(\ell-j)s} \begin{cases} 2^{2(j-\ell)M} & \text{if } j < \ell \\ 2^{2(\ell-j)(\widetilde{M}-\frac{a}{2})} & \text{if } j \geq \ell. \end{cases} \\
&= \sum_{j=0}^{\infty} 2^{js}[\Phi_j(t^2\mathscr{L})]_a^* f(y) \begin{cases} 2^{2(j-\ell)(M-\frac{s}{2})} & \text{if } j < \ell \\ 2^{2(\ell-j)(\widetilde{M}-\frac{a}{2}+\frac{s}{2})} & \text{if } j \geq \ell \end{cases}.
\end{aligned}$$

Choosing $S > a/2 + |s|/2$ and setting $\delta := \min\left\{\widetilde{M} - \frac{a}{2} + \frac{s}{2}, M - \frac{s}{2}\right\}$, we obtain

$$\left|2^{\ell s}\Phi_\ell(\mathscr{L})f(y)\right| \lesssim \sum_{j=0}^{\infty} 2^{-2|j-\ell|\delta}\left|2^{js}[\Phi_j(t^2\mathscr{L})]_a^* f(y)\right|.$$

This estimate holds uniformly for $t \in [1,2]$. Applying Lemma 3.11 yields

$$\begin{aligned}\left\|\left(\sum_{\ell=1}^{\infty}\left|2^{\ell s}\Phi_\ell(\mathscr{L})f\right|^q\right)^{1/q}\right\|_{L^p(X)} &\lesssim \left\|\left(\sum_{j=1}^{\infty}\int_1^2 \left|2^{js}[\Phi_j(t^2\mathscr{L})]_a^* f\right|^q \frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)} \\ &= \left\|\left(\int_0^1 t^{-sq}\left|[\Phi(t^2\mathscr{L})]_a^* f\right|^q \frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)},\end{aligned}$$

which gives the desired estimate (3.54). □

As a corollary of Theorem 3.15, we have the following heat kernel characterization of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$:

Corollary 3.16. (i) *Let $s \in \mathbb{R}$, $p \in (0,\infty]$, $q \in (0,\infty]$ and M be a nonnegative integer strictly larger that $s/2$. Then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\|f\|_{B^{s,\mathscr{L}}_{p,q}(X)} \sim \|e^{-\mathscr{L}}f\|_{L^p(X)} + \left(\int_0^1 t^{-sq}\left\|(t^2\mathscr{L})^M e^{-t^2\mathscr{L}}f\right\|^q_{L^p(X)}\frac{dt}{t}\right)^{1/q}.$$

(ii) *Let $s \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$ and M be a nonnegative integer strictly larger than $s/2$. Then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)} \sim \|e^{-\mathscr{L}}f\|_{L^p(X)} + \left\|\left(\int_0^1 t^{-sq}\left|(t^2\mathscr{L})^M e^{-t^2\mathscr{L}}f\right|^q \frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}.$$

Proof. Let Φ_0, Φ be functions on $\mathbb{R}_{\geq 0}$ given respectively by

$$\Phi_0(\lambda) := e^{-\lambda}, \quad \Phi(\lambda) := \lambda^M e^{-\lambda}, \quad \lambda \in \mathbb{R}_{\geq 0}.$$

Clearly $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$. Hence the conclusions follow from Theorem 3.15. □

Chapter 4

Further properties and characterizations of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$

Throughout this chapter, we assume that the metric measure space (X, ρ, μ) satisfies the doubling condition (2.1), the reverse doubling condition (2.4), and the non-collapsing condition (2.6), and assume that $\mathscr{L}$ is a non-negative self-adjoint operator on $L^2(X)$ whose heat kernel $p_t(x, y)$ satisfies the pointwise Gaussian upper bound (2.8). We do not assume the Hölder continuity for $p_t(x, y)$ in the variables x and y.

4.1 Atomic decomposition

In this section, we generalize the atomic decomposition of classical Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$ to our operator setting. To do this, we need the following analogue of the grid of Euclidean dyadic cubes on a metric measure space with doubling measure.

Lemma 4.1. ([15, Theorem 11]) *There exists a collection $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets of X, where I_k is some index set (possibly finite), and constants $\delta \in (0, 1)$ and $A_1, A_2 > 0$, such that*

(i) $\mu\big(X \setminus \cup_{\alpha \in I_k} Q^k_\alpha\big) = 0$ *for each fixed k and $Q^k_\alpha \cap Q^k_\beta = \emptyset$ if $\alpha \neq \beta$;*

(ii) *for any α, β, k, ℓ with $\ell \geq k$, either $Q^\ell_\beta \subset Q^k_\alpha$ or $Q^\ell_\beta \cap Q^k_\alpha = \emptyset$;*

(iii) *for each (k, α) and $\ell < k$, there exists a unique β such that $Q^k_\alpha \subset Q^\ell_\beta$;*

(iv) $\mathrm{diam}(Q^k_\alpha) \leq A_1 \delta^k$, *where* $\mathrm{diam}(Q^k_\alpha) := \sup\{\rho(x, y) : x, y \in Q^k_\alpha\}$;

(v) *each Q^k_α contains some ball $B(z^k_\alpha, A_2\delta^k)$, where $z^k_\alpha \in X$.*

The set Q^k_α can be thought of as a *dyadic cube* on X with diameter roughly δ^k and centered at z^k_α. We denote by $\mathscr{D}$ the family of all dyadic cubes on X. For $k \in \mathbb{Z}$, we set $\mathscr{D}_k = \{Q^k_\alpha : \alpha \in I_k\}$, so that $\mathscr{D} = \cup_{k \in \mathbb{Z}} \mathscr{D}_k$. For any dyadic cube $Q = Q^k_\alpha$, we denote by $z_Q := z^k_\alpha$ the "center" of Q. In the sequel, we assume without loss of generality that $\delta = 1/2$. If this is not the case, we need to replace 2^j in the definition of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$ by δ^{-j} and make some other necessary changes.

Definition 4.2. Let $K, S \in \mathbb{N}_0$, and let Q be a dyadic cube in $\mathscr{D}_k$, with $k \in \mathbb{N}_0$.

(A) In the case $k \in \mathbb{N}$, a function $a_Q \in L^2(X)$ is said to be a (K, S)-atom for Q if a_Q satisfies the following conditions for $m \in \{K, -S\}$.

(i) $a_Q \in D(\mathscr{L}^m)$;

(ii) $\operatorname{supp}(\mathscr{L}^m a_Q) \subset B(z_Q, (A_1 + 1)2^{-k})$;

(iii) $\operatorname*{ess\,sup}_{x\in X} |\mathscr{L}^m a_Q(x)| \le 2^{2km}[\mu(Q)]^{-1/2}$.

(B) In the case $k = 0$, a function a_Q is said to be a (K, S)-atom for Q if it satisfies (i)–(iii) for $m \in \{K, 0\}$.

Following [35] and [59], we define the sequences $b^s_{p,q}$ and $f^s_{p,q}$:

Definition 4.3. (i) Let $s \in \mathbb{R}$, $p \in (0, \infty]$ and $q \in (0, \infty]$. The sequence space $b^s_{p,q}$ consists of all sequences $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of complex scalars such that

$$\|w\|_{b^s_{p,q}} := \left(\sum_{k=0}^{\infty} 2^{ksq} \left[\sum_{Q\in\mathscr{D}_k} \left(|w_Q| [\mu(Q)]^{1/p-1/2} \right)^p \right]^{q/p} \right)^{1/q} < \infty.$$

(ii) Let $s \in \mathbb{R}$, $p \in (0, \infty)$ and $q \in (0, \infty]$. The sequence space $f^s_{p,q}$ consists of all sequences $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of complex scalars such that

$$\|w\|_{f^s_{p,q}} := \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} \sum_{Q\in\mathscr{D}_k} \left(|w_Q| [\mu(Q)]^{-1/2} \chi_Q \right)^q \right)^{1/q} \right\|_{L^p(X)} < \infty.$$

Here, χ_Q is the characteristic function of Q.

The atomic decomposition of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$ is stated in the following two theorems:

Theorem 4.4. *Let $s \in \mathbb{R}$, $p \in (0, \infty)$ and $q \in (0, \infty)$. Let $K, S \in \mathbb{N}_0$ such that $K > \frac{s}{2}$ and $S > \frac{d}{2p} - \frac{s}{2}$. Then there is a constant $C > 0$ such that for every sequence $\{a_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of (K, S)-atoms and every sequence $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of complex scalars,*

$$\left\| \sum_{k=0}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q \right\|_{B^{s,\mathscr{L}}_{p,q}(X)} \le C \|w\|_{b^s_{p,q}}. \tag{4.1}$$

Conversely, there is a constant $C' > 0$ such that given any distribution $f \in B^{s,\mathscr{L}}_{p,q}(X)$ and any $K, S \in \mathbb{N}_0$, there exist a sequence $\{a_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of (K, S)-atoms and a sequence $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of complex scalars such that

$$f = \sum_{k=0}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q,$$

where the sum converges in $\mathcal{S}'_{\mathscr{L}}(X)$, and moreover,

$$\|w\|_{b^s_{p,q}} \le C' \|f\|_{B^{s,\mathscr{L}}_{p,q}(X)}. \tag{4.2}$$

Theorem 4.5. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty)$. Let $K, S \in \mathbb{N}_0$ such that $K > \frac{s}{2}$ and $S > \frac{d}{2\min\{p,q\}} - \frac{s}{2}$. Then there is a constant $C > 0$ such that for every sequence $\{a_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of (K,S)-atoms and every sequence $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of complex scalars,*

$$\left\|\sum_{k=0}^{\infty}\sum_{Q\in\mathscr{D}_k} w_Q a_Q\right\|_{F^{s,\mathscr{L}}_{p,q}(X)} \le C\|w\|_{f^s_{p,q}}. \tag{4.3}$$

Conversely, there is a constant $C' > 0$ such that given any distribution $f \in F^{s,\mathscr{L}}_{p,q}(X)$ and any $K, S \in \mathbb{N}_0$, there exist a sequence $\{a_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of (K,S)-atoms and a sequence $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ of complex scalars such that

$$f = \sum_{k=0}^{\infty}\sum_{Q\in\mathscr{D}_k} w_Q a_Q,$$

where the sum converges in $\mathcal{S}'_{\mathscr{L}}(X)$, and moreover,

$$\|w\|_{f^s_{p,q}} \le C'\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)}. \tag{4.4}$$

We shall only give the proof of Theorem 4.5. The proof of Theorem 4.4 is similar and we omit the details. We need some lemmas.

Lemma 4.6. *Suppose $K, S \in \mathbb{N}_0$, Q is a dyadic cube in $\mathscr{D}_k$ with $k \in \mathbb{N}_0$, and a_Q is a (K,S)-atom for Q. Suppose further that $\Phi_0, \Phi \in \mathcal{S}(\mathbb{R}_{\ge 0})$ such that $\lambda^{-\max\{K,S\}}\Phi(\lambda) \in \mathcal{S}(\mathbb{R}_{\ge 0})$. Then for arbitrarily large positive integer N, the following estimate holds:*

$$|\Phi_j(\mathscr{L})a_Q(x)| \le \begin{cases} C_N 2^{2(j-k)S}[\mu(Q)]^{1/2} D_{2^{-j},N}(x,z_Q) & \text{if } 0 \le j \le k \\ C_N 2^{2(k-j)K}[\mu(Q)]^{1/2} D_{2^{-k},N}(x,z_Q) & \text{if } j \ge k \end{cases},$$

where C_N is a constant depending on N, and $\Phi_j(\cdot) := \Phi(2^{-2j}\cdot)$ for $j \ge 1$.

Proof. If $k \in \mathbb{N}$, then the conditions (i)–(iii) in Definition 5.13 and the fact that $\mu(Q) \sim V(z_Q, 2^{-k})$ together with (2.3) yield that for arbitrarily large positive integer N

$$\begin{aligned} &|\mathscr{L}^m a_Q(x)| \\ &\lesssim 2^{2km}[\mu(Q)]^{-1/2}(1 + 2^k\rho(x,z_Q))^{-N} \\ &\sim 2^{2km}[V(x,2^{-k})]^{1/2}[V(x,2^{-k})V(z_Q,2^{-k})]^{-1/2}(1+2^k\rho(x,z_Q))^{-N} \\ &\lesssim 2^{2km}[V(z_Q,2^{-k})]^{1/2}[V(x,2^{-k})V(z_Q,2^{-k})]^{-1/2}(1+2^k\rho(x,z_Q))^{-N+\frac{d}{2}} \\ &\sim 2^{2km}[\mu(Q)]^{1/2} D_{2^{-k},N-(d/2)}(x,z_Q), \end{aligned} \tag{4.5}$$

which holds for $m \in \{K, -S\}$. If $k = 0$ then (4.5) holds for $m \in \{K, 0\}$. To estimate $\Phi_j(\mathscr{L})a_Q$ we consider the following two cases:

Case 1: $k = 0$. In this case we consider the following two subcases:

Subcase 1.1: $k = 0$ and $j = 0$. By Lemma 2.3, (4.5) with $m = 0$, and Lemma 3.7, we have

$$\begin{aligned} \left|\Phi_j(\mathscr{L})a_Q(x)\right| &\le \int \left|K_{\Phi_0(\mathscr{L})}(x,y)\right||a_Q(y)|d\mu(y) \\ &\lesssim [\mu(Q)]^{1/2}\int D_{1,N}(x,y)D_{1,N-(d/2)}(y,z_Q)d\mu(y) \\ &\lesssim [\mu(Q)]^{1/2} D_{1,N-(5d/2)}(x,z_Q). \end{aligned}$$

Subcase 1.2: $k=0$ and $j\in\mathbb{N}$. We use the fact that $\lambda\mapsto\lambda^{-K}\Phi(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$ and apply Lemma 2.3, (4.5) with $m=K$, and Lemma 3.7, to get

$$\begin{aligned}\left|\Phi_j(\mathscr{L})a_Q(x)\right| &= 2^{-2jK}\left|(2^{-2j}\mathscr{L})^{-K}\Phi_j(\mathscr{L})(\mathscr{L}^K a_Q)(x)\right|\\ &\leq 2^{-2jK}\int\left|K_{(2^{-2j}\mathscr{L})^{-K}\Phi_j(\mathscr{L})}(x,y)\right|\left|\mathscr{L}^K a_Q(y)\right|d\mu(y)\\ &\lesssim 2^{-2jK}[\mu(Q)]^{1/2}\int D_{2^{-j},N}(x,y)D_{1,N-(d/2)}(y,z_Q)d\mu(y)\\ &\lesssim 2^{-2jK}[\mu(Q)]^{1/2}D_{1,N-(5d/2)}(x,z_Q).\end{aligned}$$

Case 2: $k\in\mathbb{N}$. In this case we consider the following three subcases:

Subcase 2.1: $k\in\mathbb{N}$ and $j=0$. By Lemma 2.3, (4.5) with $m=-S$, and Lemma 3.7, we have

$$\begin{aligned}\left|\Phi_0(\mathscr{L})a_Q(x)\right| &= \left|\mathscr{L}^S\Phi_0(\mathscr{L})(\mathscr{L}^{-S}a_Q)(x)\right|\\ &\leq \int\left|K_{\mathscr{L}^S\Phi_0(\mathscr{L})}(x,y)\right|\left|\mathscr{L}^{-S}a_Q(y)\right|d\mu(y)\\ &\lesssim 2^{-2kS}[\mu(Q)]^{1/2}\int D_{1,N}(x,y)D_{2^{-k},N-(d/2)}(y,z_Q)d\mu(y)\\ &\lesssim 2^{-2kS}[\mu(Q)]^{1/2}D_{1,N-(5d/2)}(x,z_Q).\end{aligned}$$

Subcase 2.2: $k\in\mathbb{N}$, $j\in\mathbb{N}$ and $j\leq k$. By Lemma 2.3, (4.5) with $m=-S$, and Lemma 3.7, we have

$$\begin{aligned}\left|\Phi_j(\mathscr{L})a_Q(x)\right| &= 2^{2jS}\left|(2^{-2j}\mathscr{L})^S\Phi_j(\mathscr{L})(\mathscr{L}^{-S}a_Q)(x)\right|\\ &\leq 2^{2jS}\int\left|K_{(2^{-2j}\mathscr{L})^S\Phi_j(\mathscr{L})}(x,y)\right|\left|\mathscr{L}^{-S}a_Q(y)\right|d\mu(y)\\ &\lesssim 2^{2(j-k)S}[\mu(Q)]^{1/2}\int D_{2^{-j},N}(x,y)D_{2^{-k},N-(d/2)}(y,z_Q)d\mu(y)\\ &\lesssim 2^{2(j-k)S}[\mu(Q)]^{1/2}D_{2^{-j},N-(5d/2)}(x,z_Q).\end{aligned}$$

Subcase 2.3: $k\in\mathbb{N}$, $j\in\mathbb{N}$ and $j>k$. We use the fact $\lambda\mapsto\lambda^{-K}\Phi(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$ and apply Lemma 2.3, (4.5) with $m=K$, and Lemma 3.7, to get

$$\begin{aligned}\left|\Phi_j(\mathscr{L})a_Q(x)\right| &= 2^{-2jK}\left|(2^{-2j}\mathscr{L})^{-K}\Phi_j(\mathscr{L})(\mathscr{L}^K a_Q)(x)\right|\\ &\leq 2^{-2jK}\int\left|K_{(2^{-2j}\mathscr{L})^{-K}\Phi_j(\mathscr{L})}(x,y)\right|\left|\mathscr{L}^K a_Q(y)\right|d\mu(y)\\ &\lesssim 2^{2(k-j)K}[\mu(Q)]^{1/2}\int D_{2^{-j},N}(x,y)D_{2^{-k},N-(d/2)}(y,z_Q)d\mu(y)\\ &\lesssim 2^{2(k-j)K}[\mu(Q)]^{1/2}D_{2^{-k},N-(5d/2)}(x,z_Q).\end{aligned}$$

Combining all the above cases yields the desired estimate. □

Lemma 4.7. *Let $M\in\mathbb{N}$* (resp. $M=0$)*. There exists $\Psi\in\mathcal{S}(\mathbb{R}_{\geq 0})$ (resp. $\Psi_0\in\mathcal{S}(\mathbb{R}_{\geq 0})$) such that the following conditions hold:*

(i) *The function $\lambda\mapsto\lambda^{-M}\Psi(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$*
(resp. the function $\lambda\mapsto\Psi_0(\lambda)$) belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$).

(ii) $|\Psi(\lambda)|\geq c>0$ *on* $\{2^{-3/2}\varepsilon\leq\lambda\leq 2^{3/2}\varepsilon\}$ *for some* $\varepsilon>0$
(resp. $|\Psi_0(\lambda)|\geq c>0$ *on* $\{0\leq\lambda\leq 2^{3/2}\varepsilon\}$ *for some* $\varepsilon>0$*).*

(iii) *For all integers $m \geq -M$ (resp. $m \geq 0$) and for all $j \in \mathbb{Z}$,*

$$\operatorname{supp} K_{(2^{-2j}\mathscr{L})^m\Psi(2^{-2j}\mathscr{L})} \subset \{(x,y) \in X \times X : \rho(x,y) < 2^{-j}\}$$
$$\big(\textit{resp.}\ \operatorname{supp} K_{(2^{-2j}\mathscr{L})^m\Psi_0(2^{-2j}\mathscr{L})} \subset \{(x,y) \in X \times X : \rho(x,y) < 2^{-j}\}\big),$$

where $K_{(2^{-2j}\mathscr{L})^m\Psi(2^{-2j}\mathscr{L})}$ (resp. $K_{(2^{-2j}\mathscr{L})^m\Psi_0(2^{-2j}\mathscr{L})}$) is the integral kernel of the operator $(2^{-2j}\mathscr{L})^m\Psi(2^{-2j}\mathscr{L})$ (resp. $(2^{-2j}\mathscr{L})^m\Psi_0(2^{-2j}\mathscr{L})$).

(iv) *For every integer $m \geq -M$ (resp. $m \geq 0$), there exists a constant $C = C(m)$ (depending on m) such that for all $j \in \mathbb{Z}$ and for a.e. $(x,y) \in X \times X$,*

$$|K_{(2^{-2j}\mathscr{L})^m\Psi(2^{-2j}\mathscr{L})}(x,y)| \leq C[V_{2^{-j}}(x)]^{-1}$$
$$\big(\textit{resp.}\ |K_{(2^{-2j}\mathscr{L})^m\Psi_0(2^{-2j}\mathscr{L})}(x,y)| \leq C[V_{2^{-j}}(x)]^{-1}\big),$$

Proof. Let $M \in \mathbb{N}$ (resp. $M = 0$). Let $\Theta \in \mathcal{S}(\mathbb{R}_{\geq 0})$ be a even function such that $\int_{-\infty}^{\infty} \Theta(\lambda)d\lambda \neq 0$ and $\operatorname{supp}\Theta \subset (-\tau,\tau)$ where $\tau > 0$ is sufficiently small. Set $\Gamma(\xi) := \hat{\Theta}(\xi)$ for $\xi \in \mathbb{R}$, and then put $\Upsilon(\xi) := \Gamma(\sqrt{\xi})$ for $\xi \in \mathbb{R}_{\geq 0}$. Finally, let us set $\Psi(\lambda) := \lambda^M\Upsilon(\lambda)$ (resp. $\Psi_0(\lambda) := \Upsilon(\lambda)$), $\lambda \in \mathbb{R}_{\geq 0}$. Since Γ is an even Schwartz function on $\mathbb{R}$, we have $\Upsilon \in \mathcal{S}(\mathbb{R}_{\geq 0})$, i.e., $\lambda^{-M}\Psi(\lambda) \in \mathcal{S}(\mathbb{R}_{\geq 0})$ (resp. $\Psi_0 \in \mathcal{S}(\mathbb{R}_{\geq 0})$). Also, since $\Upsilon(0) = \Gamma(0) = \hat{\Theta}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Theta(\lambda)d\lambda \neq 0$, we see that if $\varepsilon > 0$ is sufficiently small then $|\Psi(\lambda)| \geq c > 0$ on $\{2^{-3/2}\varepsilon \leq \lambda \leq 2^{3/2}\varepsilon\}$ (resp. $|\Psi_0(\lambda)| \geq c > 0$ on $\{0 \leq \lambda \leq 2^{3/2}\varepsilon\}$). Thus we have verified (i) and (ii). The conditions (iii) and (iv) follow from [17, Lemma 3.1] (see also [36, Lemma 2.3] and [79, Lemma 2]). □

Now we are ready to prove Theorem 4.5.

Proof of Theorem 4.5. Let $K, S \in \mathbb{N}_0$ such that $K > \frac{s}{2}$ and $S > \frac{d}{2\min\{p,q\}} - \frac{s}{2}$. Let $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ with $M \geq \max\{K,S\}$. Then by Lemma 4.6 and (2.3), we have, for a.e. $x \in X$,

$$\begin{aligned}
&2^{js}\left|\Phi_j(\mathscr{L})\left(\sum_{k=0}^{\infty}\sum_{Q\in\mathscr{D}_k} w_Q a_Q\right)(x)\right| \\
&\leq 2^{js}\sum_{k=0}^{\infty}\sum_{Q\in\mathscr{D}_k}|w_Q|\big|\Phi(2^{-2j}\mathscr{L})a_Q(x)\big| \\
&\leq 2^{js}\left(\sum_{k=0}^{j}\sum_{Q\in\mathscr{D}_k}2^{2(k-j)K}|w_Q|[\mu(Q)]^{1/2}D_{2^{-k},N}(x,z_Q)\right) \\
&\quad + 2^{js}\left(\sum_{k=j+1}^{\infty}\sum_{Q\in\mathscr{D}_k}2^{2(j-k)S}|w_Q|[\mu(Q)]^{1/2}D_{2^{-j},N}(x,z_Q)\right) \\
&= \sum_{k=0}^{j}2^{2(k-j)K}2^{(j-k)s}\sum_{Q\in\mathscr{D}_k}2^{ks}|w_Q|[\mu(Q)]^{1/2}D_{2^{-k},N}(x,z_Q) \\
&\quad + \sum_{k=j+1}^{\infty}2^{2(j-k)S}2^{(j-k)s}\sum_{Q\in\mathscr{D}_k}2^{ks}|w_Q|[\mu(Q)]^{1/2}D_{2^{-j},N}(x,z_Q) \\
&\lesssim \sum_{k=0}^{j}2^{2(k-j)K}2^{(j-k)s}\sum_{Q\in\mathscr{D}_k}2^{ks}|w_Q|[\mu(Q)]^{-1/2}(1+2^k\rho(x,z_Q))^{-N+\frac{d}{2}} \\
&\quad + \sum_{k=j+1}^{\infty}2^{2(j-k)S}2^{(j-k)s}\sum_{Q\in\mathscr{D}_k}2^{ks}|w_Q|[\mu(Q)]^{-1/2}(1+2^j\rho(x,z_Q))^{-N+\frac{d}{2}},
\end{aligned} \tag{4.6}$$

where N can be taken arbitrarily large.

Now let us set

$$\begin{aligned} S_0 :=& \{Q \subset \mathscr{D}_k : \rho(z_Q, x) < A_1 2^{-(j\wedge k)}\}, \\ S_m :=& \{Q \in \mathscr{D}_k : A_1 2^{m-1} 2^{-(j\wedge k)} \le \rho(z_Q, x) < A_1 2^m 2^{-(j\wedge k)}\}, \quad m \in \mathbb{N}, \\ B_m :=& \{z \in X : \rho(z, x) < A_1 2^{m+1} 2^{-(j\wedge k)}\}, \quad m \in \mathbb{N}_0, \end{aligned}$$

where the notation $j \wedge k$ denotes $\min\{j, k\}$, and A_1 is a constant as in Lemma 4.1. Observe that $Q \in S_m \Rightarrow Q \subset B_m$. Choose and fix $0 < r < \min\{p, q\}$ such that $2S + s - \frac{d}{r} > 0$. This is possible since $S > \frac{d}{2\min\{p,q\}} - \frac{s}{2}$. Then take $N > \frac{d}{2} + \frac{d}{r}$. We note that

$$\begin{aligned} &\sum_{Q\in\mathscr{D}_k} 2^{ks}|w_Q|[\mu(Q)]^{-1/2}(1 + 2^{j\wedge k}\rho(x, z_Q))^{-N+\frac{d}{2}} \\ &\lesssim \sum_{m=0}^{\infty} 2^{-m(N-\frac{d}{2})} \sum_{Q\in S_m} 2^{ks}|w_Q|[\mu(Q)]^{-1/2} \\ &\quad \le \sum_{m=0}^{\infty} 2^{-m(N-\frac{d}{2})} \left(\sum_{Q\in S_m} 2^{ksr}|w_Q|^r[\mu(Q)]^{-r/2} \right)^{1/r} \\ &= \sum_{m=0}^{\infty} 2^{-m(N-\frac{d}{2})} \left(\int_X \sum_{Q\in S_m} 2^{ksr}|w_Q|^r[\mu(Q)]^{-r/2}[\mu(Q)]^{-1}\chi_Q(z)d\mu(z) \right)^{1/r} \\ &= \sum_{m=0}^{\infty} 2^{-m(N-\frac{d}{2})} \left(\int_X \left[\sum_{Q\in S_m} 2^{ks}|w_Q|[\mu(Q)]^{-1/2}[\mu(Q)]^{-1/r}\chi_Q(z) \right]^r d\mu(z) \right)^{1/r} \\ &\lesssim \sum_{m=0}^{\infty} 2^{-m(N-\frac{d}{2})} \left(\frac{1}{\mu(B_m)} \int_{B_m} \left[\sum_{Q\in S_m} 2^{ks}|w_Q|[\mu(Q)]^{-1/2} \left(\frac{\mu(B_m)}{\mu(Q)}\right)^{1/r} \chi_Q(z) \right]^r d\mu(z) \right)^{1/r} \\ &\lesssim \sum_{m=0}^{\infty} 2^{-m(N-\frac{d}{2})+(m+k-j\wedge k)\frac{d}{r}} \left(\frac{1}{\mu(B_m)} \int_{B_m} \left[\sum_{Q\in S_m} 2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q(z) \right]^r d\mu(z) \right)^{1/r} \\ &\lesssim 2^{(k-j\wedge k)\frac{d}{r}} \left\{ M_{HL}\left[\left(\sum_{Q\in\mathcal{D}_k} 2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q \right)^r \right](x) \right\}^{1/r} \sum_{m=0}^{\infty} 2^{-m(N-\frac{d}{2}-\frac{d}{r})} \\ &\lesssim 2^{(k-j\wedge k)\frac{d}{r}} \left\{ M_{HL}\left[\left(\sum_{Q\in\mathcal{D}_k} 2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q \right)^r \right](x) \right\}^{1/r}, \end{aligned}$$

where we used that

$$\forall\ Q \in S_m : \quad \frac{\mu(B_m)}{\mu(Q)} \le \frac{\mu\big(B(z_Q, A_1 2^{m+2} 2^{-(j\wedge k)})\big)}{\mu(B(z_Q, A_2 2^{-k}))} \lesssim 2^{(m+k-j\wedge k)d},$$

and A_1, A_2 are constants as in Lemma 4.1. Inserting this into (4.6) gives that for a.e. $x \in X$

$$\begin{aligned} &2^{js} \left| \Phi_j(\mathscr{L}) \left(\sum_{k=0}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q \right)(x) \right| \\ &\lesssim \sum_{k=0}^{j} 2^{(k-j)(2K-s)} \left\{ M_{HL}\left[\left(\sum_{Q\in\mathscr{D}_k} 2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q \right)^r \right](x) \right\}^{1/r} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=j+1}^{\infty} 2^{(j-k)(2S+s-\frac{d}{r})}\left\{M_{HL}\left[\left(\sum_{Q\in\mathscr{D}_k} 2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right](x)\right\}^{1/r}\\
&=\sum_{k=0}^{\infty}\gamma(k-j)\left\{M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)(x)\right\}^{1/r},
\end{aligned}
$$

where the map $\gamma:\mathbb{Z}\to\mathbb{R}$ is defined by

$$
\gamma(\ell):=\begin{cases} 2^{\ell(2K-s)} & \text{if } \ell\le 0,\\ 2^{-\ell(2S+s-\frac{d}{r})} & \text{if } \ell\ge 1.\end{cases}
$$

From this estimate it follows that

$$
\begin{aligned}
&\left\|\sum_{k=0}^{\infty}\sum_{Q\in\mathscr{D}_k} w_Q a_Q\right\|_{F_{p,q}^{s,\mathscr{L}}(X)}\\
&\lesssim\left\|\left\{\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\gamma(j-k)\left\{M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)\right\}^{1/r}\right]^q\right\}^{1/q}\right\|_{L^p(X)}.
\end{aligned}
\tag{4.7}
$$

If $q\le 1$, the expression inside the L^p norm in the right-hand side of (4.7) can be estimated as follows:

$$
\begin{aligned}
&\left\{\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\gamma(j-k)\left\{M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)\right\}^{1/r}\right]^q\right\}^{1/q}\\
&\quad\le\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\gamma(j-k)^q\left\{M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)\right\}^{q/r}\right\}^{1/q}\\
&\quad=\left\{\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\gamma(j-k)^q\right)\left\{M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)\right\}^{q/r}\right\}^{1/q}\\
&\quad\lesssim\left\{\sum_{k=0}^{\infty}\left\{M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)\right\}^{q/r}\right\}^{1/q}.
\end{aligned}
$$

If $q>1$, the expression inside the L^p norm in the right-hand side of (4.7) can be estimated as follows:

$$
\begin{aligned}
&\left\{\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\gamma(j-k)\left\{M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)\right\}^{1/r}\right]^q\right\}^{1/q}\\
&\quad=\left\{\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\gamma(j-k)^{1/q'}\gamma(j-k)^{1/q}\left\{M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)\right\}^{1/r}\right]^q\right\}^{1/q}\\
&\quad\le\left\{\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\gamma(j-k)\right]^{q/q'}\left[\sum_{k=0}^{\infty}\left\{\gamma(j-k)M_{HL}\left(\sum_{Q\in\mathscr{D}_k}\left(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^r\right)\right\}^{q/r}\right]\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{\sum_{j=0}^{\infty}\left[\sum_{k=0}^{\infty}\gamma(j-k)\left\{M_{HL}\Big(\sum_{Q\in\mathscr{D}_k}\big(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\big)^r\Big)\right\}^{q/r}\right]\right\}^{1/q}\\
&= \left\{\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{\infty}\gamma(j-k)\Big)\left\{M_{HL}\Big(\sum_{Q\in\mathscr{D}_k}\big(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\big)^r\Big)\right\}^{q/r}\right\}^{1/q}\\
&\lesssim \left\{\sum_{k=0}^{\infty}\left\{M_{HL}\Big(\sum_{Q\in\mathscr{D}_k}\big(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\big)^r\Big)\right\}^{q/r}\right\}^{1/q}.
\end{aligned}
$$

Combining these, we conclude that

$$
\begin{aligned}
\left\|\sum_{k=0}^{\infty}\sum_{Q\in\mathscr{D}_k}w_Qa_Q\right\|_{F^{s,\mathscr{L}}_{p,q}(X)} &\lesssim \left\|\left\{\sum_{k=0}^{\infty}\left\{M_{HL}\Big(\sum_{Q\in\mathscr{D}_k}\big(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\big)^r\Big)\right\}^{q/r}\right\}^{1/q}\right\|_{L^p(X)}\\
&= \left\|\left\{\sum_{k=0}^{\infty}\left\{M_{HL}\Big(\sum_{Q\in\mathscr{D}_k}\big(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\big)^r\Big)\right\}^{q/r}\right\}^{r/q}\right\|_{L^{p/r}(X)}^{1/r}\\
&\lesssim \left\|\left\{\sum_{k=0}^{\infty}\left\{\sum_{Q\in\mathscr{D}_k}\big(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\big)^r\right\}^{q/r}\right\}^{r/q}\right\|_{L^{p/r}(X)}^{1/r}\\
&= \left\|\left\{\sum_{k=0}^{\infty}\sum_{Q\in\mathscr{D}_k}\big(2^{ks}|w_Q|[\mu(Q)]^{-1/2}\chi_Q\big)^q\right\}^{1/q}\right\|_{L^p(X)} = \|w\|_{f^s_{p,q}},
\end{aligned}
$$

where we used the Fefferman-Stein vector valued maximal inequality on spaces of homogeneous type (see [39]). This verifies (4.3).

We now turn to the converse of the statement. Let $K, S \in \mathbb{N}_0$. We choose $\Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ (resp. $\Psi_0 \in \mathcal{S}(\mathbb{R}_{\geq 0})$) such that Ψ (resp. Ψ_0) satisfies (i)–(iv) in Lemma 4.7 with $M \geq S$ (resp. $M = 0$). In particular, the couple (Ψ_0, Ψ) satisfies

$$|\Psi_0(\lambda)| \geq c > 0 \text{ on } \{0 \leq \lambda \leq 2^{3/2}\varepsilon\} \quad \text{and} \quad |\Psi(\lambda)| \geq c > 0 \text{ on } \{2^{-3/2}\varepsilon \leq \lambda \leq 2^{3/2}\varepsilon\}$$

for some $\varepsilon > 0$. Hence, by Lemma 3.9 it is possible to find $\Phi_0, \Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$
\begin{aligned}
\operatorname{supp}\Phi_0 \subset [0, 2^{3/2}\varepsilon], \quad |\Phi_0(\lambda)| \geq c' > 0 \text{ on } \{0 \leq \lambda \leq 2^{3/2}\varepsilon\},\\
\operatorname{supp}\Phi \subset [2^{-3/2}\varepsilon, 2^{3/2}\varepsilon], \quad |\Phi(\lambda)| \geq c' > 0 \text{ on } \{2^{-3/2}\varepsilon \leq \lambda \leq 2^{3/2}\varepsilon\},
\end{aligned}
$$

and

$$\sum_{j=0}^{\infty}\Psi_j(\lambda)\Phi_j(\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}_{\geq 0}, \tag{4.8}$$

where we have set $\Psi_j(\cdot) := \Psi(2^{-2j}\cdot)$ and $\Phi_j(\cdot) := \Phi(2^{-2j}\cdot)$ for $j \geq 1$. Clearly $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ for all $M \in \mathbb{N}_0$. Hence (Φ_0, Φ) can be used to define $F^{s,\mathscr{L}}_{p,q}(X)$. From (4.8) and Lemma 3.10, it follows that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,

$$f = \sum_{j=0}^{\infty}\Psi_j(\mathscr{L})\Phi_j(\mathscr{L})f \quad \text{in } \mathcal{S}'_{\mathscr{L}}(X). \tag{4.9}$$

If $Q \in \mathscr{D}_0$, we set

$$\widetilde{w}_Q := [\mu(Q)]^{1/2} \left(\operatorname*{ess\,sup}_{y\in Q} \left|\Phi_0(\mathscr{L})f(y)\right| \right) \left(\max_{m\in\{K,0\}} \operatorname*{ess\,sup}_{x\in X} \int_Q \left| K_{\mathscr{L}^m \Psi_0(\mathscr{L})}(x,y)\right| d\mu(y) \right)$$

$$\widetilde{a}_Q := \frac{1}{\widetilde{w}_Q} \int_Q K_{\Psi_0(\mathscr{L})}(x,y)\Phi_0(\mathscr{L})f(y)d\mu(y),$$

while if $Q \in \mathscr{D}_j$ with $j \geq 1$, we set

$$\widetilde{w}_Q := [\mu(Q)]^{1/2} \left(\operatorname*{ess\,sup}_{y\in Q} \left|\Phi_j(\mathscr{L})f(y)\right| \right) \left(\max_{m\in\{K,-S\}} \operatorname*{ess\,sup}_{x\in X} \int_Q \left| K_{(2^{-2j}\mathscr{L})^m \Psi_j(\mathscr{L})}(x,y)\right| dy \right)$$

$$\widetilde{a}_Q := \frac{1}{\widetilde{w}_Q} \int_Q K_{\Psi_j(\mathscr{L})}(x,y)\Phi_j(\mathscr{L})f(y)d\mu(y).$$

Then it follows from (4.9) that

$$\begin{aligned} f &= \sum_{j=0}^{\infty} \int_X K_{\Psi_j(\mathscr{L})}(x,y)\Phi_j(\mathscr{L})f(y)d\mu(y) \\ &= \sum_{j=0}^{\infty} \sum_{Q\in\mathscr{D}_j} \int_Q K_{\Psi_j(\mathscr{L})}(x,y)\Phi_j(\mathscr{L})f(y)d\mu(y) \\ &= \sum_{j=0}^{\infty} \sum_{Q\in\mathscr{D}_j} \widetilde{w}_Q \widetilde{a}_Q, \end{aligned}$$

where the sum converges in $\mathcal{S}'_{\mathscr{L}}(X)$.

Since $\widetilde{a}_Q$ can be expressed as $\widetilde{a}_Q = \frac{1}{\widetilde{w}_Q}\Psi_j(\mathscr{L})\left[\left(\Phi_j(\mathscr{L})f\right)\chi_Q\right]$, and since Ψ (resp. Ψ_0) satisfies the condition (i) in Lemma 4.7 with $M \geq S$ (resp. $M = 0$), we have $\widetilde{a}_Q \in D(\mathscr{L}^K) \cap D(\mathscr{L}^{-S})$ (resp. $\widetilde{a}_Q \in D(\mathscr{L}^K)$) whenever $Q \in \mathscr{D}_j$ with $j \geq 1$ (resp. $j = 0$). Moreover, if $Q \in \mathscr{D}_j$ with $j \geq 1$ (resp. $j = 0$), then

$$\begin{aligned} \mathscr{L}^m \widetilde{a}_Q &= \frac{1}{\widetilde{w}_Q} \mathscr{L}^m \Psi_j(\mathscr{L})\left[\left(\Phi_j(\mathscr{L})f\right)\chi_Q\right] \\ &= \frac{2^{2jm}}{\widetilde{w}_Q} \int_Q K_{(2^{-2j}\mathscr{L})^m \Psi_j(\mathscr{L})}(x,y)\Phi_j(\mathscr{L})f(y)d\mu(y) \end{aligned}$$

holds for $m \in \{K, -S\}$ (resp. $m \in \{K, 0\}$). Hence, by using the conditions (i)–(iv) in Lemma 4.7 it is straightforward to verify that for any $Q \in \cup_{j\in\mathbb{N}_0}\mathscr{D}_j$, $\widetilde{a}_Q$ is a (K,S)-atom multiplied by a constant independent of Q.

Now, for any $Q \in \cup_{j\in\mathbb{N}_0}\mathscr{D}_j$, we set $w_Q := C\widetilde{w}_Q$ and $a_Q := C^{-1}\widetilde{a}_Q$, where $C > 0$ is a sufficiently large constant independent of Q. Then a_Q is a (K,S)-atom, and moreover,

$$f = \sum_{j=0}^{\infty} \sum_{Q\in\mathscr{D}_j} w_Q a_Q,$$

where the sum converges in $\mathcal{S}'_{\mathscr{L}}(X)$.

It remains to verify (4.4). Indeed, by our choice of Ψ_0 and Ψ and by the conditions (iii) and (iv) in Lemma 4.7, we have

$$\operatorname{supp} K_{(2^{-2j}\mathscr{L})^m \Psi_j(\mathscr{L})} \subset \left\{(x,y) \in X \times X : \rho(x,y) < 2^{-j}\right\}$$

and

$$\left|K_{(2^{-2j}\mathscr{L})^m\Psi_j(\mathscr{L})}(x,y)\right| \le C[V(x,2^{-j})]^{-1}, \quad \text{a.e. } (x,y)\in X\times X,$$

both of which are valid for $m\in\{K,-S\}$ (resp. $m\in\{K,0\}$) if $j\ge 1$ (resp. $j=0$). In the last inequality C is a positive constant independent of $j\in\mathbb{N}_0$. Hence, for every $j\in\mathbb{N}_0$ and $Q\in\mathscr{D}_j$, we have

$$\begin{aligned}|w_Q| &\lesssim [\mu(Q)]^{1/2}\left(\operatorname*{ess\,sup}_{y\in Q}\left|\Phi_j(\mathscr{L})f(y)\right|\right)\operatorname*{ess\,sup}_{\rho(x,z_Q)\le(A_1+1)2^{-j}}\int_Q[V(x,2^{-j})]^{-1}d\mu(y)\\ &\lesssim [\mu(Q)]^{1/2}\left(\operatorname*{ess\,sup}_{y\in Q}\left|\Phi_j(\mathscr{L})f(y)\right|\right),\end{aligned}$$

where we applied (2.2) and (2.3), and A_1 is the constant as in Lemma 4.1.

We now choose $a>\frac{2d}{\min(p,q)}$, and note that

$$\begin{aligned}&\sum_{Q\in\mathscr{D}_j}\left(2^{js}|w_Q|[\mu(Q)]^{-1/2}\chi_Q(x)\right)^q \lesssim \sum_{Q\in\mathscr{D}_j}\left(\operatorname*{ess\,sup}_{y\in Q}2^{js}|\Phi_j(\mathscr{L})f(y)|\chi_Q(x)\right)^q\\ &\lesssim \operatorname*{ess\,sup}_{y\in B(x,2A_12^{-j})}\left[2^{js}|\Phi_j(\mathscr{L})f(y)|\right]^q \lesssim \left[\operatorname*{ess\,sup}_{y\in X}\frac{2^{js}|\Phi_j(\mathscr{L})f(y)|}{(1+2^j\rho(x,y))^a}\right]^q = 2^{js}[\Phi_j(\mathscr{L})]^*_a f(x),\end{aligned}$$

which along with Theorem 3.4 and the fact $(\Phi_0,\Phi)\in\mathcal{A}_M(\mathbb{R}_{\ge0})$ for all $M\in\mathbb{N}_0$ yields that

$$\|w\|_{f^s_{p,q}}\lesssim\|f\|_{F^{s,\mathscr{L}}_{p,q}(X)}.$$

The proof of Theorem 4.5 is thus completed. □

Corollary 4.8. *Suppose $s\in\mathbb{R}$, $p\in(0,\infty)$ and $q\in(0,\infty)$. Then $\mathcal{S}_{\mathscr{L}}(X)$ is dense in $F^{s,\mathscr{L}}_{p,q}(X)$ and is dense in $F^{s,\mathscr{L}}_{p,q}(X)$.*

Proof. We only deal with the case of Triebel-Lizorkin spaces since the case of Besov spaces is the same. From Proposition 3.12 and Proposition 3.13, we see that $\mathcal{S}_{\mathscr{L}}(X)$ is a subset of $B^{s,\mathscr{L}}_{p,q}(X)$ and of $F^{s,\mathscr{L}}_{p,q}(X)$ for $s\in\mathbb{R}$, $p\in(0,\infty)$ and $q\in(0,\infty)$. Let $f\in F^{s,\mathscr{L}}_{p,q}(X)$. Let $\varepsilon>0$. Since sequences $w=\{w_Q\}_{Q\in\cup_{k\in\mathbb{N}_0}\mathscr{D}_k}$ with finite support (i.e., only finitely many scalars w_Q is not zero) is dense in $f^s_{p,q}$ (see for example [8]), by the atomic decomposition there exists $g\in F^{s,\mathscr{L}}_{p,q}(X)$ such that $\|f-g\|_{F^{s,\mathscr{L}}_{p,q}(X)}<\varepsilon$, and for any $k\in\mathbb{N}_0$,

$$\operatorname{supp}\mathscr{L}^k g \text{ is a bounded set,}\quad\text{and } \operatorname*{ess\,sup}_{x\in X}|\mathscr{L}^k g(x)|\le C(k), \tag{4.10}$$

where $C(k)$ is a constant depending on k.

Next, let Φ_0,Φ be functions in $\mathcal{S}(\mathbb{R}_{\ge0})$ such that

$$\begin{aligned}\operatorname{supp}\Phi_0\in[0,2^2],\quad |\Phi_0(\lambda)|\ge c>0 \text{ for } \lambda\in[0,2^{3/2}],\\ \operatorname{supp}\Phi\subset[2^{-2},2^2],\quad |\Phi(\lambda)|\ge c>0 \text{ for } \lambda\in[2^{-3/2},2^{3/2}],\end{aligned}$$

and

$$\sum_{j=0}^\infty\Phi_j(\lambda)=1\quad\text{for all } \lambda\in\mathbb{R}_{\ge0},$$

where we have set $\Phi_j(\cdot):=\Phi(2^{-2j}\cdot)$ for $j\ge1$. Set $g_N:=\sum_{j=0}^N\Phi_j(\mathscr{L})g$, $N\in\mathbb{N}$. We claim that $g_N\in\mathcal{S}_{\mathscr{L}}(X)$ for all $N\in\mathbb{N}$. To see this, we fix j, and let k,m be arbitrary nonnegative integers.

By (4.10) we have

$$|g(y)| \lesssim (1+\rho(y,x_0))^{-(m+3d)} \sim [V(x_0,1)]^{-1}(1+\rho(y,x_0))^{-(m+3d)} \lesssim D_{1,m+5d/2}(y,x_0).$$

Also, using Lemma 2.3 we have

$$|K_{\mathscr{L}^k\Phi_j(\mathscr{L})}(x,y)| \lesssim D_{1,m+5d/2}(x,y)$$

It then follows from Lemma 3.7 that for a.e. $x \in X$,

$$\begin{aligned}|\mathscr{L}^k\Phi_j(\mathscr{L})g(x)| &\le \int_X |K_{\mathscr{L}^k\Phi_j(\mathscr{L})}(x,y)||g(y)|d\mu(y)\\ &\lesssim \int_X D_{1,m+5d/2}(x,y)D_{1,m+5d/2}(y,x_0)d\mu(y)\\ &\lesssim D_{1,m+d/2}(x,x_0) \lesssim (1+\rho(x,x_0))^{-m}.\end{aligned}$$

This shows that $\mathcal{P}_{k,m}\big(\Phi_j(\mathscr{L})g\big) < \infty$, and hence $g_N \in \mathcal{S}_{\mathscr{L}}(X)$ for all $N \in \mathbb{N}$.

On the other hand, the argument in Step 5 of the proof of [83, Theorem 2.3.3] shows that g_N approximates f in $F^{s,\mathscr{L}}_{p,q}(X)$, as $N \to \infty$. Hence there is a sufficiently large integer N_0 such that $\|g - g_{N_0}\|_{F^{s,\mathscr{L}}_{p,q}(X)} < \varepsilon$.

Summing up all of these we see that $g_{N_0} \in \mathcal{S}_{\mathscr{L}}(X)$ and

$$\|f - g_{N_0}\|_{F^{s,\mathscr{L}}_{p,q}(X)} \le \|f-g\|_{F^{s,\mathscr{L}}_{p,q}(X)} + \|g - g_{N_0}\|_{F^{s,\mathscr{L}}_{p,q}(X)} \le 2\varepsilon.$$

This shows that $\mathcal{S}_{\mathscr{L}}(X)$ is dense in $F^{s,\mathscr{L}}_{p,q}(X)$. □

4.2 Complex interpolation

Let $A = \{z \in \mathbb{C} | 0 < \operatorname{Re} z < 1\}$ be a strip in the complex plane. Its closure $\{z \in \mathbb{C} | 0 \le \operatorname{Re} z \le 1\}$ is denoted by $\overline{A}$. We say that f is an $\mathcal{S}'_{\mathscr{L}}(X)$-analytic function in A if the following properties are satisfied:

(i) For every fixed $z \in A$ we have $f(z) \in \mathcal{S}'_{\mathscr{L}}(X)$.

(ii) For every $\Phi \in \mathcal{S}(\mathbb{R}_{\ge 0})$ with compact support in $\mathbb{R}_{\ge 0}$ and for almost every fixed $x \in X$, the function $z \mapsto \Phi(\mathscr{L})(f(z))(x)$ is a uniformly continuous and bounded in $\overline{A}$,

(iii) For every $\Phi \in \mathcal{S}(\mathbb{R}_{\ge 0})$ with compact support in $\mathbb{R}_{\ge 0}$ and for almost every fixed $x \in X$, the function $z \mapsto \Phi(\mathscr{L})(f(z))(x)$ is analytic in A.

Following [83], we introduce the following two definitions:

Definition 4.9. Let $-\infty < s_0 < \infty$, $-\infty < s_1 < \infty$, $0 < q_0 \le \infty$ and $0 < q_1 \le \infty$.

(i) If $0 < p_0 \le \infty$ and $0 < p_1 \le \infty$, we define $F\big(B^{s_0,\mathscr{L}}_{p_0,q_0}(X), B^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)$ to be the space of all $\mathcal{S}'_{\mathscr{L}}(X)$-analytic functions f in A such that

$$\|f\|_{F(B^{s_0,\mathscr{L}}_{p_0,q_0}(X), B^{s_1,\mathscr{L}}_{p_1,q_1}(X))} := \max_{\ell \in \{0,1\}} \sup_{t \in \mathbb{R}} \|f(\ell + it)\|_{B^{s_\ell,\mathscr{L}}_{p_\ell,q_\ell}(X)} < \infty.$$

(ii) If $0 < p_0 < \infty$ and $0 < p_1 < \infty$, we define $F\big(F^{s_0,\mathscr{L}}_{p_0,q_0}(X), F^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)$ to be the space of all $\mathcal{S}'_{\mathscr{L}}(X)$-analytic functions f in A such that

$$\|f\|_{F(F^{s_0,\mathscr{L}}_{p_0,q_0}(X),F^{s_1,\mathscr{L}}_{p_1,q_1}(X))} := \max_{\ell\in\{0,1\}} \sup_{t\in\mathbb{R}} \|f(\ell+it)\|_{F^{s_\ell,\mathscr{L}}_{p_\ell,q_\ell}(X)} < \infty.$$

Definition 4.10. Let $-\infty < s_0 < \infty$, $-\infty < s_1 < \infty$, $0 < q_0 \le \infty$, $0 < q_1 \le \infty$ and $0 < \theta < 1$.

(i) If $0 < p_0 \le \infty$ and $0 < p_1 \le \infty$, we define

$$\begin{aligned}&\big(B^{s_0,\mathscr{L}}_{p_0,q_0}(X), B^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)_\theta\\ &\qquad := \big\{g\in\mathcal{S}'_{\mathscr{L}}(X)\big|\exists f\in F\big(B^{s_0,\mathscr{L}}_{p_0,q_0}(X), B^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big) \text{ with } g=f(\theta)\big\}.\end{aligned} \tag{4.11}$$

and

$$\|g\|_{(B^{s_0,\mathscr{L}}_{p_0,q_0}(X),B^{s_1,\mathscr{L}}_{p_1,q_1}(X))_\theta} := \inf \|f\|_{F(B^{s_0,\mathscr{L}}_{p_0,q_0}(X),B^{s_1,\mathscr{L}}_{p_1,q_1}(X))},$$

where the infimum is taken over all admissible functions f in the sense of (4.11).

(ii) If $0 < p_0 < \infty$ and $0 < p_1 < \infty$, we define

$$\begin{aligned}&\big(F^{s_0,\mathscr{L}}_{p_0,q_0}(X), F^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)_\theta\\ &\qquad := \big\{g\in\mathcal{S}'_{\mathscr{L}}(X)\big|\exists f\in F\big(F^{s_0,\mathscr{L}}_{p_0,q_0}(X), F^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big) \text{ with } g=f(\theta)\big\}.\end{aligned} \tag{4.12}$$

and

$$\|g\|_{(F^{s_0,\mathscr{L}}_{p_0,q_0}(X),F^{s_1,\mathscr{L}}_{p_1,q_1}(X))_\theta} := \inf \|f\|_{F(F^{s_0,\mathscr{L}}_{p_0,q_0}(X),F^{s_1,\mathscr{L}}_{p_1,q_1}(X))},$$

where the infimum is taken over all admissible functions f in the sense of (4.12).

Lemma 4.11. ([83, Lemma 2.4.6/2]) *Let* $A = \{z|0 < \operatorname{Re} z < 1\}$, $\overline{A} = \{z|0 \le \operatorname{Re} z \le 1\}$ *and* $0 < r < \infty$. *Then there exists two functions* $\mu_0(\theta,t)$ *and* $\mu_1(\theta,t)$ *in* $(0,1)\times\mathbb{R}$ *such that*

$$|g(z)|^r \le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} |g(it)|^r \mu_0(\theta,t)dt\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} |g(1+it)|^r \mu_1(\theta,t)dt\right)^{\theta} \tag{4.13}$$

where $\theta = \operatorname{Re} z$ *for any analytic function* $g(z)$ *in* A *which is uniformly continuous and bounded in* $\overline{A}$. *Furthermore, if* $0 < \theta < 1$ *then*

$$\frac{1}{1-\theta}\int_{\mathbb{R}} \mu_0(\theta,t)dt = \frac{1}{\theta}\int_{\mathbb{R}} \mu_1(\theta,t)dt = 1. \tag{4.14}$$

The main result of this section is the following theorem:

Theorem 4.12. *Let* $-\infty < s_0 < \infty$, $-\infty < s_1 < \infty$, $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < q_0 < \infty$, $0 < q_1 < \infty$ *and* $0 < \theta < 1$. *If* s, p, q *are given by* $s = (1-\theta)s_0 + \theta s_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$, *then*

$$\big(B^{s_0,\mathscr{L}}_{p_0,q_0}(X), B^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)_\theta = B^{s,\mathscr{L}}_{p,q}(X), \tag{4.15}$$

$$\big(F^{s_0,\mathscr{L}}_{p_0,q_0}(X), F^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)_\theta = F^{s,\mathscr{L}}_{p,q}(X), \tag{4.16}$$

and the corresponding quasi-norms are equivalent.

Proof. We only prove (4.16) since the proof of (4.15) is similar.

Step 1. We follow the method of [83] to prove that

$$\big(F^{s_0,\mathscr{L}}_{p_0,q_0}(X), F^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)_\theta \subset F^{s,\mathscr{L}}_{p,q}(X). \tag{4.17}$$

Let $f \in F\big(F^{s_0,\mathscr{L}}_{p_0,q_0}(X), F^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)$. Let Φ_0, Φ be functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying that

$$\operatorname{supp}\Phi_0 \subset [0,2^2], \quad |\Psi_0(\lambda)| \geq c > 0 \text{ for } \lambda \in [0, 2^{3/2}],$$
$$\operatorname{supp}\Phi \subset [2^{-2},2^2], \quad |\Phi(\lambda)| \geq c > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}].$$

For $k \geq 1$, we set $\Phi_k(\cdot) := \Phi(2^{-2k}\cdot)$. Also, for $k \in \mathbb{N}_0$, we put $g_k(x,z) = \Phi_k(\mathscr{L})(f(z))(x)$. Let $0 < r < \min\{p_0, q_0, p_1, q_1\}$. Then

$$\begin{aligned} \|f(\theta)\|_{F^{s,\mathscr{L}}_{p,q}(X)} &= \left[\int_X \left(\sum_{k=0}^\infty \big|2^{ks} g_k(x,\theta)\big|^q\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} \\ &= \left[\int_X \left(\sum_{k=0}^\infty \big(\big|2^{ks} g_k(x,\theta)\big|^r\big)^{\frac{q}{r}}\right)^{\frac{r}{q}\cdot\frac{p}{r}} dx\right]^{\frac{r}{p}\cdot\frac{1}{r}}. \end{aligned} \tag{4.18}$$

Let $\mu_0(\theta,t)$ and $\mu_1(\theta,t)$ be as in Lemma 4.11. Then we set

$$a_k(x) = \frac{1}{1-\theta}\int_{\mathbb{R}} |g_k(x,it)|^r \mu_0(\theta,t)dt \tag{4.19}$$

and

$$b_k(x) = \frac{1}{\theta}\int_{\mathbb{R}} |g_k(x,1+it)|^r \mu_1(\theta,t)dt. \tag{4.20}$$

Applying (4.13) in Lemma 4.11 with $2^{ks}g_k(x,z)$ instead of $g(z)$ (where $x \in X$ is fixed), Hölder's inequality and [83, Lemma 1, p. 68], we have

$$\begin{aligned} \left[\sum_{k=0}^\infty \big(\big|2^{ks} g_k(x,\theta)\big|^r\big)^{\frac{q}{r}}\right]^{\frac{r}{q}} &\leq \left[\sum_{k=0}^\infty \big(2^{ksr} a_k^{1-\theta}(x) b_k^\theta(x)\big)^{\frac{q}{r}}\right]^{\frac{r}{q}} \\ &\leq \left(\sum_{k=0}^\infty 2^{ks_0q_0} a_k^{\frac{q_0}{r}}(x)\right)^{\frac{r}{q_0}(1-\theta)} \left(\sum_{k=0}^\infty 2^{ks_1q_1} b_k^{\frac{q_1}{r}}(x)\right)^{\frac{r}{q_1}\theta} \\ &= \big\|\{2^{ks_0r}a_k(x)\}_{k=0}^\infty\big\|_{\ell^{q_0/r}}^{1-\theta} \big\|\{2^{ks_1r}b_k(x)\}_{k=0}^\infty\big\|_{\ell^{q_1/r}}^{\theta}. \end{aligned} \tag{4.21}$$

From (4.19) and Minkowski's inequality it follows that

$$\begin{aligned} \big\|\{2^{ks_0r}a_k(x)\}_{k=0}^\infty\big\|_{\ell^{q_0/r}} &\leq \frac{1}{1-\theta}\int_{\mathbb{R}} \big\|\{\big|2^{ks_0}g_k(x,it)\big|^r\}_{k=0}^\infty\big\|_{\ell^{q_0/r}} \mu_0(\theta,t)dt \\ &= \frac{1}{1-\theta}\int_{\mathbb{R}} \left(\sum_{k=0}^\infty 2^{ks_0q_0}|g_k(x,it)|^{q_0}\right)^{\frac{1}{q_0}r} \mu_0(\theta,t)dt. \end{aligned}$$

Also,

$$\big\|\{2^{ks_1r}b_k(x)\}_{k=0}^\infty\big\|_{\ell^{q_1/r}} \leq \frac{1}{\theta}\int_{\mathbb{R}} \left(\sum_{k=0}^\infty 2^{ks_1q_1}|g_k(x,1+it)|^{q_1}\right)^{\frac{1}{q_1}r} \mu_1(\theta,t)dt.$$

Inserting these two estimates and (4.21) into (4.18) gives

$$\|f(\theta)\|_{F^{s,\mathscr{L}}_{p,q}(X)} \leq \left[\int_X \left(\frac{1}{1-\theta}\int_R \left(\sum_{k=0}^\infty 2^{ks_0q_0}|g_k(x,it)|^{q_0}\right)^{\frac{1}{q_0}r} \mu_0(\theta,t)dt\right)^{(1-\theta)\frac{p}{r}}\right.$$

$$\times\left(\frac{1}{\theta}\int_{\mathbb{R}}\left(\sum_{k=0}^{\infty}2^{ks_1q_1}|g_k(x,1+it)|^{q_1}\right)^{\frac{1}{q_1}r}\mu_1(\theta,t)dt\right)^{\theta\frac{p}{r}}dx\Bigg]^{\frac{r}{p}\cdot\frac{1}{r}}.$$

Hence, applying Holder's inequality we have

$$\begin{aligned}&\|f(\theta)\|_{F^{s,\mathscr{L}}_{p,q}(X)}\\ &\le\left[\int_X\left(\frac{1}{1-\theta}\int_R\left(\sum_{k=0}^{\infty}2^{ks_0q_0}|g_k(x,it)|^{q_0}\right)^{\frac{1}{q_0}r}\mu_0(\theta,t)dt\right)^{\frac{p_0}{r}}dx\right]^{\frac{r}{p_0}\cdot\frac{1-\theta}{r}}\\ &\quad\times\left[\int_X\left(\frac{1}{\theta}\int_{\mathbb{R}}\left(\sum_{k=0}^{\infty}2^{ks_1q_1}|g_k(x,1+it)|^{q_1}\right)^{\frac{1}{q_1}r}\mu_1(\theta,t)dt\right)^{\frac{p_1}{r}}dx\right]^{\frac{r}{p_1}\cdot\frac{\theta}{r}}.\end{aligned}\tag{4.22}$$

Since $\|\cdot\|_{L^{p_0/r}(X)}$ and $\|\cdot\|_{L^{p_1/r}(X)}$ are norms, by the Minkowski's inequality we can estimate (4.22) from above by

$$\begin{aligned}\|f(\theta)\|_{F^{s,\mathscr{L}}_{p,q}(X)}\le&\left(\frac{1}{1-\theta}\int_{\mathbb{R}}\left[\int_X\left(\sum_{k=0}^{\infty}2^{ks_0q_0}|g_k(x,it)|^{q_0}\right)^{\frac{r}{q_0}\cdot\frac{p_0}{r}}dx\right]^{\frac{r}{p_0}}\mu_0(\theta,t)dt\right)^{\frac{1-\theta}{r}}\\ &\times\left(\frac{1}{\theta}\int_{\mathbb{R}}\left[\int_X\left(\sum_{k=0}^{\infty}2^{ks_1q_1}|g_k(x,1+it)|^{q_1}\right)^{\frac{r}{q_1}\cdot\frac{p_1}{r}}dx\right]^{\frac{r}{p_1}}\mu_1(\theta,t)dt\right)^{\frac{\theta}{r}}.\end{aligned}$$

In view of Lemma 4.11, this yields

$$\begin{aligned}\|f(\theta)\|_{F^{s,\mathscr{L}}_{p,q}(X)}&\le\left[\sup_{t\in\mathbb{R}}\left\|\{2^{ks_0}g_k(\cdot,it)\}_{k=0}^{\infty}\right\|^{1-\theta}_{L^{P_0}(\ell^{q_0})}\right]\left[\sup_{t\in\mathbb{R}}\left\|\{2^{ks_1}g_k(\cdot,1-it)\}_{k=0}^{\infty}\right\|^{\theta}_{L^{p_1}(\ell^{q_1})}\right]\\ &\le\|f(z)\|_{F(F^{s_0,\mathscr{L}}_{p_0,q_0}(X),F^{s_1,\mathscr{L}}_{p_1,q_1}(X))}.\end{aligned}$$

Hence (4.17) is established.

Step 2. We prove that

$$F^{s,\mathscr{L}}_{p,q}(X)\subset\left(F^{s_0,\mathscr{L}}_{p_0,q_0}(X),F^{s_1,\mathscr{L}}_{p_1,q_1}(X)\right)_{\theta}.\tag{4.23}$$

In this step we follow the idea of Noi and Sawano [64]. Let $g\in F^{s,\mathscr{L}}_{p,q}(X)$. Then by Theorem 4.5 g admits a decomposition

$$g=\sum_{k=0}^{\infty}\sum_{\alpha\in I_k}\lambda_{k,\alpha}a_{k,\alpha},\tag{4.24}$$

where each $a_{k,\alpha}$ is an atom for the dyadic cube Q^k_α, each $\lambda_{k,\alpha}$ is a scalar, the sum converges in $\mathcal{S}'_{\mathscr{L}}(X)$, and

$$\left\|\left(\sum_{k=0}^{\infty}\sum_{\alpha\in I_k}\left(2^{ks}|\lambda_{k,\alpha}|[\mu(Q^k_\alpha)]^{-1/2}\chi_{Q^k_\alpha}\right)^q\right)^{1/q}\right\|_{L^p(X)}\lesssim\|g\|_{F^{s,\mathscr{L}}_{p,q}(X)}.$$

For $x\in X$ and $z\in\overline{A}$, we define

$$\rho_1(z):=\frac{qs}{q_0}(1-z)+\frac{qs}{q_1}z-s_0(1-z)-s_1z,$$

$$\rho_2(z) := \frac{q}{q_0}(1-z) + \frac{q}{q_1}z,$$
$$\rho_3(z) := \left(\frac{p}{qp_0} - \frac{1}{q_0}\right)(1-z) + \left(\frac{p}{qp_1} - \frac{1}{q_1}\right)z.$$

In addition, for $k \in \mathbb{N}_0$ and $\alpha \in I_k$, we define the holomorphic function $\Lambda_{k,\alpha}$ by

$$\Lambda_{k,\alpha}(z) = \left\{ \frac{1}{\mu(Q^k_\alpha)} \int_{Q^k_\alpha} 2^{k\frac{\rho_1(z)}{K}} |\lambda_{k,\alpha}|^{\frac{\rho_2(z)}{K}} [\mu(Q^k_\alpha)]^{-\frac{\rho_2(z)}{2K}} \left(\sum_{j=0}^k \sum_{\beta\in I_j} R_{j,\beta}{}^q \chi_{Q^j_\alpha}(y) \right)^{\frac{\rho_3(z)}{K}} d\mu(y) \right\}^K,$$

where K is a large integer, and $R_{k,\alpha} := 2^{ks}|\lambda_{k,\alpha}|[\mu(Q^k_\alpha)]^{-1/2}$. Abbreviate $\sum_{j=0}^k \sum_{\beta\in I_j} R_{j,\beta}{}^q \chi_{Q^j_\beta}(y)$ to $S_k(y)$. Then we have, for $\ell = 0, 1$,

$$\begin{aligned}
&|2^{ks_\ell}\Lambda_{k,\alpha}(\ell+it)| \\
&= \left| \frac{1}{\mu(Q^k_\alpha)} \int_{Q^k_\alpha} 2^{\frac{k}{K}[s_\ell - s_0 + \ell(s_0-s_1) + i(s_0-s_1)t]} R_{k,\alpha}{}^{\frac{\rho_2(\ell+it)}{K}} S_k(y)^{\frac{\rho_3(\ell+it)}{K}} d\mu(y) \right|^K \\
&\lesssim \left\{ \frac{1}{\mu(Q^k_\alpha)} \int_{Q^k_\alpha} \left| 2^{\frac{k}{K}[s_\ell - s_0 + \ell(s_0-s_1) + i(s_0-s_1)t]} R_{k,\alpha}{}^{\frac{\rho_2(\ell+it)}{K}} S_k(y)^{\frac{\rho_3(\ell+it)}{K}} \right| d\mu(y) \right\}^K \\
&= \left\{ \frac{1}{\mu(Q^k_\alpha)} \int_{Q^k_\alpha} R_{k,\alpha}{}^{\frac{q}{Kq_\ell}} S_k(y)^{\frac{\rho_3(\ell)}{K}} d\mu(y) \right\}^K \\
&= \left\{ \frac{1}{\mu(Q^k_\alpha)} \int_{Q^k_\alpha} R_{k,\alpha}{}^{\frac{q}{Kq_\ell}} S_k(y)^{\frac{\rho_3(\ell)}{K}} \chi_{Q^k_\alpha}(y) d\mu(y) \right\}^K .
\end{aligned}$$

Hence, if $x \in Q^k_\alpha$, we have

$$\begin{aligned}
|2^{ks_\ell}\Lambda_{k,\alpha}(\ell+it)| &\lesssim \left[M_{HL}\left(R_{k,\alpha}{}^{\frac{q}{Kq_\ell}} S_k{}^{\frac{\rho_3(\ell)}{K}} \chi_{Q^k_\alpha} \right)(x) \right]^K \\
&\le \left[M_{HL}\left(\sum_{\beta\in I_k} R_{k,\beta}{}^{\frac{q}{Kq_\ell}} S_k{}^{\frac{\rho_3(\ell)}{K}} \chi_{Q^k_\beta} \right)(x) \right]^K .
\end{aligned}$$

Consequently, we obtain for all $x \in X$

$$\sum_{\alpha\in I_k} |2^{ks_\ell}\Lambda_{k,\alpha}(\ell+it)|\chi_{Q^k_\alpha}(x) \lesssim \left[M_{HL}\left(\sum_{\beta\in I_k} R_{k,\beta}{}^{\frac{q}{Kq_\ell}} S_k{}^{\frac{\rho_3(\ell)}{K}} \chi_{Q^k_\beta} \right)(x) \right]^K .$$

This estimate along with the Fefferman-Stein vector-valued inequality on spaces of homogeneous type (see [39]) yields that for $\ell = 0, 1$,

$$\begin{aligned}
&\left\| \left\{ \sum_{k=0}^\infty \left(2^{ks_\ell} \sum_{\alpha\in I_k} |\Lambda_{k,\alpha}(\ell+it)|\chi_{Q^k_\alpha} \right)^{q_\ell} \right\}^{1/q_\ell} \right\|_{L^{p_\ell}} \\
&\lesssim \left\| \left\{ \sum_{k=0}^\infty \left[M_{HL}\left(\sum_{\beta\in I_k} R_{k,\beta}{}^{\frac{q}{Kq_\ell}} S_k{}^{\frac{\rho_3(\ell)}{K}} \chi_{Q^k_\beta} \right) \right]^{Kq_\ell} \right\}^{1/q_\ell} \right\|_{L^{p_\ell}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\| \left\{ \sum_{k=0}^{\infty} \left[\sum_{\beta\in I_k} R_{k,\beta}{}^{\frac{q}{Kq_\ell}} S_k{}^{\frac{\rho_3(\ell)}{K}} \chi_{Q^k_\beta} \right]^{Kq_\ell} \right\}^{1/q_\ell} \right\|_{L^{p_\ell}} \\
&= \left\| \left\{ \sum_{k=0}^{\infty} \left[\sum_{\beta\in I_k} R_{k,\beta}{}^{\frac{q}{Kq_\ell}} \left(\sum_{j=0}^{k} \sum_{\gamma\in I_j} R_{j,\gamma}{}^{q} \chi_{Q^j_\gamma} \right)^{\frac{\rho_3(\ell)}{K}} \chi_{Q^k_\beta} \right]^{Kq_\ell} \right\}^{1/q_\ell} \right\|_{L^{p_\ell}} \\
&= \left\| \left\{ \sum_{k=0}^{\infty} \sum_{\beta\in I_k} R_{k,\beta}{}^{q} \left(\sum_{j=0}^{k} \sum_{\gamma\in I_j} R^q_{j,\gamma} \chi_{Q^j_\gamma} \right)^{\rho_3(\ell)q_\ell} \chi_{Q^k_\beta} \right\}^{1/q_\ell} \right\|_{L^{p_\ell}} \\
&= \left\| \left\{ \sum_{k=0}^{\infty} \sum_{\beta\in I_k} R_{j,\beta}{}^{q} \left(\sum_{j=0}^{k} \sum_{\gamma\in I_k} R^q_{j,\gamma} \chi_{Q^j_\gamma} \right)^{\left(\frac{p}{qp_\ell}-\frac{1}{q_\ell}\right)q_\ell} \chi_{Q^k_\beta} \right\}^{1/q_\ell} \right\|_{L^{p_\ell}}.
\end{aligned}
$$

By [83, p. 68, Lemma 1] (see also [64, Lemma 2.17]),

$$
\sum_{k=0}^{\infty} \sum_{\beta\in I_k} R_{j,\beta}{}^{q} \left(\sum_{j=0}^{k} \sum_{\gamma\in I_j} R^q_{j,\gamma} \chi_{Q^j_\gamma} \right)^{\left(\frac{p}{qp_\ell}-\frac{1}{q_\ell}\right)q_\ell} \chi_{Q^k_\beta} \lesssim \left(\sum_{k=0}^{\infty} \sum_{\beta\in I_k} R^q_{k,\beta} \chi_{Q^k_\beta} \right)^{\frac{pq_\ell}{qp_\ell}}.
$$

Hence

$$
\begin{aligned}
&\left\| \left\{ \sum_{k=0}^{\infty} \left(2^{ks_\ell} \sum_{\alpha\in I_k} |\Lambda_{k,\alpha}(\ell+it)| \chi_{Q^k_\alpha} \right)^{q_\ell} \right\}^{1/q_\ell} \right\|_{L^{p_\ell}} \\
&\qquad \lesssim \left\| \left(\sum_{k=0}^{\infty} \sum_{\beta\in I_k} R^q_{k,\beta} \chi_{Q^k_\beta} \right)^{\frac{p}{qp_\ell}} \right\|_{L^{p_\ell}} \\
&\qquad = \left\| \left(\sum_{k=0}^{\infty} \sum_{\beta\in I_k} \left(2^{ks} |\lambda_{k,\beta}| [\mu(Q^k_\beta)]^{-1/2} \chi_{Q^k_\beta}\right)^q \right)^{\frac{1}{q}} \right\|_{L^p}^{\frac{p}{p_\ell}} \lesssim \|f\|^{p/p_\ell}_{F^{s,\mathscr{L}}_{p,q}(X)}.
\end{aligned}
\tag{4.25}
$$

Now we define

$$
f(z,x) = \sum_{k=0}^{\infty} \sum_{\alpha\in I_k} \Lambda_{k,\alpha}(z) a_{k,\alpha}(x).
$$

Then by (4.24), (4.25) and a homogeneity argument we have

$$
f(\theta) = g
$$

and

$$
\max_{\ell\in\{0,1\}} \sup_{t\in\mathbb{R}} \|f(\ell+it)\|_{F^{s_\ell,\mathscr{L}}_{p_\ell,q_\ell}(X)} \lesssim \|g\|_{F^{s,\mathscr{L}}_{p,q}(X)}.
$$

This exactly says that

$$
\|g\|_{(F^{s_0,\mathscr{L}}_{p_0,q_0}(X),F^{s_1,\mathscr{L}}_{p_1,q_1}(X))_\theta} \lesssim \|g\|_{F^{s,\mathscr{L}}_{p,q}(X)}.
$$

Hence (4.23) is verified and the proof of the theorem is completed. □

4.3 Lifting property

The purpose of this section is to prove the following result:

Theorem 4.13. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty)$. Then:*

(i) *For any $\sigma \in \mathbb{R}$, $(I+\mathscr{L})^\sigma$ is an isomorphism of $B^{s,\mathscr{L}}_{p,q}(X)$ to $B^{s-2\sigma,\mathscr{L}}_{p,q}(X)$.*

(ii) *For any $\sigma \in \mathbb{R}$, $(I+\mathscr{L})^\sigma$ is an isomorphism of $F^{s,\mathscr{L}}_{p,q}(X)$ to $F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$.*

To prove this theorem we need some lemmas.

Lemma 4.14. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty)$. Then:*

(i) *For any $\sigma \in \mathbb{R}$, $\mathrm{Dom}\left((I+\mathscr{L})^\sigma\right) \cap B^{s,\mathscr{L}}_{p,q}(X)$ is dense in $B^{s,\mathscr{L}}_{p,q}(X)$.*

(ii) *For any $\sigma \in \mathbb{R}$, $\mathrm{Dom}\left((I+\mathscr{L})^\sigma\right) \cap F^{s,\mathscr{L}}_{p,q}(X)$ is dense in $F^{s,\mathscr{L}}_{p,q}(X)$.*

Proof. From Corollary 4.8 we see that $\mathcal{S}_{\mathscr{L}}(X)$ is dense in $F^{s,\mathscr{L}}_{p,q}(X)$. Hence it suffices to show that $\mathcal{S}_{\mathscr{L}}(X) \subset \mathrm{Dom}\left((I+\mathscr{L})^\sigma\right)$ for any $\sigma \in \mathbb{R}$. Note that the latter is trivial for the case $\sigma \le 0$, since in this case we have $\mathrm{Dom}\left((I+\mathscr{L})^\sigma\right) = L^2(X)$. Assume now $\sigma > 0$. Let $m = \lfloor \sigma \rfloor + 1$, and set $\Phi(\lambda) = (1+\lambda)^\sigma (1+\lambda^m)^{-1}$, $\lambda \in \mathbb{R}_{\ge 0}$. Then $(1+\lambda)^\sigma = \Phi(\lambda)(1+\lambda^m)$. Hence by [67, Theorem 13.24 (b)] $\Phi(\mathscr{L})(1+\mathscr{L}^m) \subset (1+\mathscr{L})^\sigma$. In particular,

$$\mathrm{Dom}\left(\Phi(\mathscr{L})(1+\mathscr{L}^m)\right) \subset \mathrm{Dom}\left((1+\mathscr{L})^\sigma\right). \tag{4.26}$$

On the other hand, since $\Phi \in L^\infty(\mathbb{R}_{\ge 0})$, we have $\mathrm{Dom}\left(\Phi(\mathscr{L})\right) = L^2(X)$ and hence

$$\begin{aligned}\mathrm{Dom}\left(\Phi(\mathscr{L})(I+\mathscr{L}^m)\right) &= \left\{ f \in L^2(X) \middle| f \in \mathrm{Dom}\left(I+\mathscr{L}^m\right), (I+\mathscr{L}^m)f \in \mathrm{Dom}\left(\Phi(\mathscr{L})\right)\right\} \\ &= \mathrm{Dom}(I+\mathscr{L}^m).\end{aligned}$$

Combining this with (4.26) we get $\mathrm{Dom}\left(I+\mathscr{L}^m\right) \subset \mathrm{Dom}\left((I+\mathscr{L})^\sigma\right)$. It follows that $\mathcal{S}_{\mathscr{L}}(X) \subset \mathrm{Dom}\left((I+\mathscr{L})^\sigma\right)$. This completes the proof. ☐

Lemma 4.15. *Let Φ_0, Φ be functions in $\mathcal{S}(\mathbb{R}_{\ge 0})$ such that*

$$\begin{aligned}\operatorname{supp}\Phi_0 &\in [0,2^2], \quad |\Phi_0(\lambda)| \ge c > 0 \text{ for } \lambda \in [0,2^{3/2}], \\ \operatorname{supp}\Phi &\subset [2^{-2},2^2], \quad |\Phi(\lambda)| \ge c > 0 \text{ for } \lambda \in [2^{-3/2},2^{3/2}],\end{aligned}$$

and

$$\sum_{k=0}^{\infty} \Phi_k(\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}_{\ge 0}, \tag{4.27}$$

where we have set $\Phi_k(\cdot) := \Phi(2^{-2k}\cdot)$ for $k \ge 1$. Let $\vec{\Psi} = \{\Psi_k\}_{k=0}^\infty$ be a system of functions in $\mathcal{S}(\mathbb{R}_{\ge 0})$ having compact supports such that

$$0 \notin \operatorname{supp}\Psi_k \quad \text{for all } k \ge 1.$$

Let $s \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty)$, and let L be a positive number such that $L > \frac{3d}{2} + \frac{3d}{\min\{p,q\}} + \frac{|s|}{2} + 2$. Then there exists a positive number C such that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,

$$\left(\sum_{k=0}^{\infty} 2^{ksq} \left\|\Psi_k(\mathscr{L})f\right\|^q_{L^p(X)}\right)^{1/q} \le C(\vec{\Psi}) \left(\sum_{k=0}^{\infty} 2^{ksq} \left\|\Phi_k(\mathscr{L})f\right\|^q_{L^p(X)}\right)^{1/q}$$

and

$$\left\|\left(\sum_{k=0}^{\infty}2^{ksq}\left|\Psi_k(\mathscr{L})f\right|^q\right)^{1/q}\right\|_{L^p(X)}\le C(\vec{\Psi})\left\|\left(\sum_{k=0}^{\infty}2^{ksq}\left|\Phi_k(\mathscr{L})f\right|^q\right)^{1/q}\right\|_{L^p(X)},$$

where

$$C(\vec{\Psi}):=\sup_{\substack{\lambda\in[0,\infty)\\0\le\nu\le L}}(1+\lambda)^L\left|\frac{d^\nu\Psi_0}{d\lambda^\nu}(\lambda)\right|+\sup_{\substack{\lambda\in(0,\infty)\\0\le\nu\le L\\k=1,2,\cdots}}(\lambda^L+\lambda^{-L})\left|\frac{d^\nu[\Psi_k(2^{2k}\cdot)]}{d\lambda^\nu}(\lambda)\right|.$$

Proof. Let Θ_0,Θ be functions in $\mathcal{S}(\mathbb{R}_{\ge 0})$ such that

$$\operatorname{supp}\Theta_0\subset[0,2^4],\quad \Theta_0(\lambda)=1 \text{ for } \lambda\in\operatorname{supp}\Phi_0$$
$$\operatorname{supp}\Theta\subset[2^{-4},2^4],\quad \Theta(\lambda)=1 \text{ for } \lambda\in\operatorname{supp}\Phi.$$

Set $\Theta_k(\lambda):=\Theta(2^{-2k}\lambda)$ for $k\ge1$. By (4.27) and Lemma 3.10, for any $\mathcal{S}'_{\mathscr{L}}(X)$

$$f=\sum_{\ell=0}^{\infty}\Phi_\ell(\mathscr{L})f\quad\text{in } \mathcal{S}'_{\mathscr{L}}(X).$$

Hence for all $k\in\mathbb{N}_0$ and a.e. $y\in X$,

$$\Psi_k(\mathscr{L})f(y)=\sum_{\ell=0}^{\infty}\Psi_k(\mathscr{L})\Theta_\ell(\mathscr{L})\Phi_\ell(\mathscr{L})f(y).$$

It follows that

$$\begin{aligned}|\Psi_k(\mathscr{L})f(y)|&\le\sum_{\ell=0}^{\infty}\int\left|K_{\Psi_k(\mathscr{L})\Theta_\ell(\mathscr{L})}(z,y)\right||\Phi_\ell(\mathscr{L})f(z)|d\mu(z)\\&\le[\Phi_\ell(\mathscr{L})]^*_af(y)\int(1+2^\ell\rho(y,z))^a\left|K_{\Psi_k(\mathscr{L})\Theta_\ell(\mathscr{L})}(z,y)\right|d\mu(z)\\&=[\Phi_\ell(\mathscr{L})]^*_af(y)I_{k,\ell}(y)\\&\le[\Phi_\ell(\mathscr{L})]^*_af(x)(1+2^\ell\rho(x,y))^aI_{k,\ell}(y),\end{aligned}$$

where we have set

$$I_{k,\ell}(y):=\int(1+2^\ell\rho(y,z))^a\left|K_{\Psi_k(\mathscr{L})\Theta_\ell(\mathscr{L})}(z,y)\right|d\mu(z).$$

Hence for all $k\in\mathbb{N}_0$ and all $x\in X$,

$$\begin{aligned}2^{ks}[\Psi_k(\mathscr{L})]^*_af(x)&\lesssim\sum_{\ell=0}^{\infty}2^{(k-\ell)s}2^{\ell s}[\Phi_\ell(\mathscr{L})]^*_af(x)\sup_{y\in X}\frac{(1+2^\ell\rho(x,y))^a}{(1+2^k\rho(x,y))^a}I_{k,\ell}(y)\\&\le\sum_{\ell=0}^{\infty}2^{(k-\ell)s}2^{\ell s}[\Phi_\ell(\mathscr{L})]^*_af(x)\max\{1,2^{(\ell-k)a}\}\sup_{y\in X}I_{k,\ell}(y),\end{aligned}\tag{4.28}$$

where we used the inequality

$$\frac{1+2^\ell\rho(x,y)}{1+2^k\rho(x,y)}\le\max\{1,2^{\ell-k}\}.$$

We now estimate $\sup_{y\in X} I_{k,\ell}(y)$. Let a be a positive number such that $a > \frac{2d}{\min\{p,q\}}$, and let N be a positive integer such that $N > \frac{3d}{2} + a$ and $2L - 2N - |s| - a > 0$. This is possible since $L > \frac{3d}{2} + \frac{3d}{\min\{p,q\}} + \frac{|s|}{2} + 2$. To estimate $I_{k,\ell}(y)$ we consider the following cases:

Case 1: $k \in \{1, 2, \cdots\}$ and $\ell \in \{1, 2, \cdots\}$. Let $\Upsilon(\lambda) := \Theta(\lambda)\Psi_k(2^\ell\lambda)$, $\lambda \in \mathbb{R}_{\geq 0}$. Then by Lemma 2.3, for all sufficiently large positive integer N we have

$$\left|K_{\Theta(2^{-\ell}\mathscr{L})\Psi_k(\mathscr{L})}(y,z)\right| = \left|K_{\Upsilon(2^{-\ell}\mathscr{L})}(y,z)\right| \lesssim \|\Upsilon\|_{(N)} D_{2^{-\ell},N}(y,z).$$

It follows that

$$\begin{aligned}
I_{k,\ell}(y) &= \int (1 + 2^\ell \rho(y,z))^a \left|K_{\Theta_\ell(\mathscr{L})\Psi_k(\mathscr{L})}(z,y)\right| d\mu(z) \\
&\lesssim \|\Upsilon\|_{(N)} \int D_{2^{-\ell},N-a}(y,z) d\mu(z) \\
&\lesssim \|\Upsilon\|_{(N)} \\
&= \sup_{0\leq\nu\leq N} \sup_{\lambda\in\mathbb{R}_{\geq 0}} (1+\lambda)^{N+d+1} \left|\Upsilon^{(\nu)}(\lambda)\right| \\
&\leq \sup_{0\leq\nu\leq N} \sup_{\lambda\in\mathbb{R}_{\geq 0}} (1+\lambda)^{N+d+1} \sum_{\nu_1+\nu_2=\nu} C_\nu^{\nu_1} \left|\frac{d^{\nu_1}\Theta}{d\lambda^{\nu_1}}(\lambda)\right| \left|\frac{d^{\nu_2}[\Psi_k(2^{2\ell}\cdot)]}{d\lambda^{\nu_2}}(\lambda)\right| \\
&= \sup_{0\leq\nu\leq N} \sup_{\lambda\in\mathbb{R}_{\geq 0}} (1+\lambda)^{N+d+1} \sum_{\nu_1+\nu_2=\nu} 2^{2(\ell-k)\nu_2} \left|\frac{d^{\nu_1}\Theta}{d\lambda^{\nu_1}}(\lambda)\right| \left|\frac{d^{\nu_2}[\Psi_k(2^{2k}\cdot)]}{d\lambda^{\nu_2}}(2^{2(\ell-k)}\lambda)\right| \\
&\lesssim C(\vec{\Psi}) \sup_{0\leq\nu\leq N} \sup_{\lambda\in\mathbb{R}_{\geq 0}} (1+\lambda)^{N+d+1} \sum_{\nu_1+\nu_2=\nu} 2^{2(\ell-k)\nu_2} \left|\frac{d^{\nu_1}\Theta}{d\lambda^{\nu_1}}(\lambda)\right| \left(|2^{2(\ell-k)}\lambda|^L + |2^{2(\ell-k)}\lambda|^{-L}\right)^{-1} \\
&\lesssim C(\vec{\Psi}) 2^{2|\ell-k|N} 2^{-2|\ell-k|L} \\
&= C(\vec{\Psi}) 2^{-2|\ell-k|(L-N)}.
\end{aligned}$$

Case 2: $k = 0$ and $\ell \in \{1, 2, \cdots\}$. Similarly to Case 1 we have

$$\begin{aligned}
I_{0,\ell}(y) &\leq \sup_{0\leq\nu\leq N} \sup_{\lambda\in\mathbb{R}_{\geq 0}} (1+\lambda)^{N+d+1} \sum_{\nu_1+\nu_2=\nu} C_\nu^{\nu_1} \left|\frac{d^{\nu_1}\Theta}{d\lambda^{\nu_1}}(\lambda)\right| \left|\frac{d^{\nu_2}[\Psi_0(2^{2\ell}\cdot)]}{d\lambda^{\nu_2}}(\lambda)\right| \\
&= \sup_{0\leq\nu\leq N} \sup_{\lambda\in\mathbb{R}_{\geq 0}} (1+\lambda)^{N+d+1} \sum_{\nu_1+\nu_2=\nu} 2^{2\ell\nu_2} \left|\frac{d^{\nu_1}\Theta}{d\lambda^{\nu_1}}(\lambda)\right| \left|\frac{d^{\nu_2}\Psi_0}{d\lambda^{\nu_2}}(2^{2\ell}\lambda)\right| \\
&\lesssim C(\vec{\Psi}) \sup_{0\leq\nu\leq N} \sup_{\lambda\in\mathbb{R}_{\geq 0}} (1+\lambda)^{N+d+1} \sum_{\nu_1+\nu_2=\nu} 2^{2\ell\nu_2} \left|\frac{d^{\nu_1}\Theta}{d\lambda^{\nu_1}}(\lambda)\right| \left(1 + 2^{2\ell}\lambda\right)^{-L} \\
&\lesssim C(\vec{\Psi}) 2^{2\ell N} 2^{-2\ell L} \\
&= C(\vec{\Psi}) 2^{-2\ell(L-N)}.
\end{aligned}$$

Case 3: $k \in \{1, 2, \cdots\}$ and $\ell = 0$. By Lemma 2.3 we can estimate as follows:

$$\begin{aligned}
I_{k,0}(y) &= \int (1 + \rho(y,z))^a \left|K_{\Theta_0(\mathscr{L})\Psi_k(\mathscr{L})}(z,y)\right| d\mu(z) \\
&\lesssim \|\Theta_0\Psi_k\|_{(N)} \int D_{1,N-a}(y,z)| d\mu(z) \\
&\lesssim \|\Theta_0\Psi_k\|_{(N)} \\
&= \sup_{0\leq\nu\leq N} \sup_{\lambda\in\mathbb{R}_{\geq 0}} (1+\lambda)^{N+d+1} \left|\frac{d^\nu(\Theta_0\Psi_k)}{d\lambda^\mu}(\lambda)\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{0\le\nu\le N}\sup_{\lambda\in\mathbb{R}_{\ge0}}(1+\lambda)^{N+d+1}\sum_{\nu_1+\nu_2=\nu}C^{\nu_1}_{\nu}\left|\frac{d^{\nu_1}\Theta_0}{d\lambda^{\nu_1}}(\lambda)\right|\left|\frac{d^{\nu_2}\Psi_k}{d\lambda^{\nu_2}}(\lambda)\right|\\
&= \sup_{0\le\nu\le N}\sup_{\lambda\in\mathbb{R}_{\ge0}}(1+\lambda)^{N+d+1}\sum_{\nu_1+\nu_2=\nu}C^{\nu_1}_{\nu}\left|\frac{d^{\nu_1}\Theta_0}{d\lambda^{\nu_1}}(\lambda)\right|\left|\frac{d^{\nu_2}\Psi_k(2^{2k}\cdot)}{d\lambda^{\nu_2}}(2^{-2k}\lambda)\right|2^{-2k\nu_2}\\
&\lesssim C(\vec{\Psi})\sup_{0\le\nu\le N}\sup_{\lambda\in\mathbb{R}_{>0}}(1+\lambda)^{N+d+1}\sum_{\nu_1+\nu_2=\nu}C^{\nu_1}_{\nu}\left|\frac{d^{\nu_1}\Theta_0}{d\lambda^{\nu_1}}(\lambda)\right|\left(|2^{-2k}\lambda|^{L}+|2^{-2k}\lambda|^{-L}\right)^{-1}2^{-2k\nu_2}\\
&\lesssim C(\vec{\Psi})\sup_{0\le\nu\le N}\sup_{\lambda\in\mathbb{R}_{>0}}(1+\lambda)^{N+d+1}\sum_{\nu_1+\nu_2=\nu}C^{\nu_1}_{\nu}\left|\frac{d^{\nu_1}\Theta_0}{d\lambda^{\nu_1}}(\lambda)\right|2^{-2kL}\lambda^{L}2^{-2k\nu_2}\\
&\lesssim C(\vec{\Psi})2^{-2kL}.
\end{aligned}
$$

Case 4: $k=\ell=0$. Again by Lemma 2.3 we have

$$
\begin{aligned}
I_{0,0}(y) &= \int(1+\rho(y,z))^{a}\left|K_{\Theta_0(\mathscr{L})\Psi_0(\mathscr{L})}(y,z)\right|d\mu(z)\\
&\lesssim \|\Theta_0\Psi_0\|_{(N)}\int D_{1,N-a}(y,z)d\mu(z)\\
&\lesssim \|\Theta_0\Psi_0\|_{(N)}\\
&= \sup_{0\le\nu\le N}\sup_{\lambda\in\mathbb{R}_{\ge0}}(1+\lambda)^{N+d+1}|\partial^{\nu}_{\lambda}(\Theta_0\Psi_0)(\lambda)|\\
&\lesssim C(\vec{\Psi}).
\end{aligned}
$$

Summing up all these cases, we obtain that

$$
\operatorname*{ess\,sup}_{y\in X} I_{k,\ell}(y)\lesssim C(\vec{\Psi})2^{-2|\ell-k|(L-N)}.
$$

This along with (4.28) yields that for all $k\in\mathbb{N}_0$ and all $x\in X$,

$$
\begin{aligned}
2^{ks}[\Psi_k(\mathscr{L})]^*_a f(x) &\lesssim \sum_{\ell=0}^{\infty}2^{(k-\ell)s}2^{\ell s}[\Phi_\ell(\mathscr{L})]^*_a f(x)\max\{1,2^{(\ell-k)a}\}2^{-2|\ell-k|(L-N)}\\
&\lesssim 2^{-|\ell-k|(2L-2N-|s|-a)}2^{\ell s}[\Phi_\ell(\mathscr{L})]^*_a f(x).
\end{aligned}
$$

From this, Lemma 3.11, and Theorem 3.5, it follows that

$$
\begin{aligned}
\left(\sum_{k=0}^{\infty}2^{ksq}\left\|\Psi_k(\mathscr{L})f\right\|^q_{L^p(X)}\right)^{1/q} &\le \left(\sum_{\ell=0}^{\infty}2^{\ell sq}\left\|[\Psi_\ell(\mathscr{L})]^*_a f\right\|^q_{L^p(X)}\right)^{1/q}\\
&\lesssim \left(\sum_{\ell=0}^{\infty}2^{\ell sq}\left\|[\Phi_\ell(\mathscr{L})]^*_a f\right\|^q_{L^p(X)}\right)^{1/q} \lesssim \left(\sum_{\ell=0}^{\infty}2^{\ell sq}\left\|\Phi_\ell(\mathscr{L})f\right\|^q_{L^p(X)}\right)^{1/q}
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\left(\sum_{k=0}^{\infty}2^{ksq}\left|\Psi_k(\mathscr{L})f\right|^q\right)^{1/q}\right\|_{L^p(X)} &\le \left\|\left(\sum_{k=0}^{\infty}2^{ksq}\left|[\Psi_k(\mathscr{L})]^*_a f\right|^q\right)^{1/q}\right\|_{L^p(X)}\\
&\lesssim \left\|\left(\sum_{k=0}^{\infty}2^{ksq}\left|[\Phi_k(\mathscr{L})]^*_a f\right|^q\right)^{1/q}\right\|_{L^p(X)} \lesssim \left\|\left(\sum_{k=0}^{\infty}2^{ksq}\left|\Phi_k(\mathscr{L})f\right|^q\right)^{1/q}\right\|_{L^p(X)}.
\end{aligned}
$$

This completes the proof of Lemma 4.15. □

Proof of Theorem 4.13. First we prove that $(I+\mathscr{L})^\sigma$ is a bounded mapping from $F^{s,\mathscr{L}}_{p,q}(X)$ to $F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$. Let Φ_0, Φ be functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\operatorname{supp}\Phi_0 \in [0,2^2], \quad |\Phi_0(\lambda)| \geq c > 0 \text{ for } \lambda \in [0, 2^{3/2}],$$
$$\operatorname{supp}\Phi \subset [2^{-2},2^2], \quad |\Phi(\lambda)| \geq c > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}].$$

Set $\Phi_k(\lambda) := \Phi(2^{-2k}\lambda)$ for $k \geq 1$. Define a system $\vec{\Psi} = \{\Psi\}_{=0}^{\infty}$ of functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ by

$$\Psi_k(\lambda) := 2^{-2k\sigma}\Phi_k(\lambda)(1+\lambda)^\sigma, \quad k \in \mathbb{N}_0.$$

It is easy to see that $C(\vec{\Psi}) < \infty$. Then by Lemma 4.15 we have that for all $f \in \operatorname{Dom}((I+\mathscr{L})^\sigma) \cap F^{s,\mathscr{L}}_{p,q}(X)$

$$\begin{aligned}
\|(I+\mathscr{L})^\sigma f\|_{F^{s-2\sigma,\mathscr{L}}_{p,q}(X)} &= \left\| \left(\sum_{k=0}^{\infty} 2^{k(s-2\sigma)q} \left| \Phi_k(\mathscr{L})(I+\mathscr{L})^\sigma f \right|^q \right)^{1/q} \right\|_{L^p(X)} \\
&= \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} \left| \Psi_k(\mathscr{L}) f \right|^q \right)^{1/q} \right\|_{L^p(X)} \\
&\lesssim \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} \left| \Phi_k(\mathscr{L}) f \right|^q \right)^{1/q} \right\|_{L^p(X)} = \|f\|_{F^{s,\mathscr{L}}_{p,q}(X)}.
\end{aligned} \tag{4.29}$$

Since $\operatorname{Dom}\big(I+\mathscr{L}\big)^\sigma\big) \cap F^{s,\mathscr{L}}_{p,q}(X)$ is dense in $F^{s,\mathscr{L}}_{p,q}(X)$, $(I+\mathscr{L})^\sigma)$ extends to a bounded linear operator from $F^{s,\mathscr{L}}_{p,q}(X)$ to $F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$. We denote this extension by T_σ. By the same argument, $(I+\mathscr{L})^{-\sigma}$ extends to a bounded linear operator $T_{-\sigma}$ from $F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ to $F^{s,\mathscr{L}}_{p,q}(X)$.

Next we show that for any $\sigma \in \mathbb{R}$, the mapping $T_\sigma : F^{s,\mathscr{L}}_{p,q}(X) \to F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ is injective. Indeed, assume $f \in F^{s,\mathscr{L}}_{p,q}(X)$ such that $T_\sigma f$ is the zero element of $F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$. By Lemma 4.14 we can find a sequence f_ℓ in $\operatorname{Dom}\big((I+\mathscr{L})^\sigma\big) \cap F^{s,\mathscr{L}}_{p,q}(X)$ which converges in $F^{s,\mathscr{L}}_{p,q}(X)$ to f. Then by the boundedness of $(I+\mathscr{L})^\sigma$, $(I+\mathscr{L})^\sigma f_\ell$ converges in $F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ to the zero element. Since $(I+\mathscr{L})^\sigma f_\ell \in \operatorname{Dom}\big((I+\mathscr{L})^{-\sigma}\big) \cap F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ and $f_\ell = (I+\mathscr{L})^{-\sigma}((I+\mathscr{L})^\sigma f_\ell)$, the boundedness of $(I+\mathscr{L})^{-\sigma}$ yields that f_ℓ converges in $F^{s,\mathscr{L}}_{p,q}(X)$ to the zero element. Therefore, f is the zero element in $F^{s,\mathscr{L}}_{p,q}(X)$. This proves that $T_\sigma : F^{s,\mathscr{L}}_{p,q}(X) \to F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ is injective.

Now we show that $T_\sigma : F^{s,\mathscr{L}}_{p,q}(X) \to F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ is surjective. Indeed, given $f \in F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$, we let f_ℓ be a sequence in $\operatorname{Dom}\big((I+\mathscr{L})^{-\sigma}\big) \cap F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ which converges in $F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ to f. Then the boundedness of $(I+\mathscr{L})^{-\sigma}$ yields that $(I+\mathscr{L})^{-\sigma} f_\ell$ converges in $F^{s,\mathscr{L}}_{p,q}(X)$. Denote this limit by g. We claim that $T_\sigma g = f$. Indeed, since $f_\ell = (I+\mathscr{L})^\sigma((I+\mathscr{L})^{-\sigma} f_\ell)$, it follows from the boundedness of $(I+\mathscr{L})^\sigma$ that f_ℓ converges to $T_\sigma g$ in $F^{s,\mathscr{L}}_{p,q}(X)$. Hence $T_\sigma g = f$ in $F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$. This proves that T_σ is surjective.

The above arguments also show that both $T_{-\sigma} \circ T_\sigma$ and $T_\sigma \circ T_{-\sigma}$ are identity operators on $F^{s,\mathscr{L}}_{p,q}(X)$. Furthermore, by an easy density argument we see that (4.29) holds for all $f \in F^{s,\mathscr{L}}_{p,q}(X)$, provided that $(I+\mathscr{L})^\sigma$ in (4.29) is replaced by T_σ. Thus, $T_\sigma : F^{s,\mathscr{L}}_{p,q}(X) \to F^{s-2\sigma,\mathscr{L}}_{p,q}(X)$ is an isomorphism, and $\|T_\sigma f\|_{F^{s-2\sigma,\mathscr{L}}_{p,q}(X)}$ is an equivalent quasi-norm of $F^{s,\mathscr{L}}_{p,q}(X)$. □

4.4 Embedding theorem

The purpose of this section is to prove the following result:

Theorem 4.16. (i) *Let $0 < p_0 \le p_1 < \infty$, $0 < q \le \infty$ and $-\infty < s_1 \le s_0 < \infty$. Then we have the continuous embedding*

$$B^{s_0,\mathscr{L}}_{p_0,q}(X) \subset B^{s_1,\mathscr{L}}_{p_1,q}(X) \quad \text{if } s_0 - d/p_0 = s_1 - d/p_1$$

(ii) *Let $0 < p_0 < p_1 < \infty$, $0 < q \le \infty$, $0 < r \le \infty$ and $-\infty < s_1 < s_0 < \infty$. Then we have the continuous embedding*

$$F^{s_0,\mathscr{L}}_{p_0,q}(X) \subset F^{s_1,\mathscr{L}}_{p_1,r}(X) \quad \text{if } s_0 - d/p_0 = s_1 - d/p_1$$

Proof. We only prove the assertion (ii) since the proof of (i) is similar. We follow Jawerth [54]. By the lifting property (Theorem 4.13), we may assume $s_0 = 0$. Moreover, we may assume $q = \infty$ and $0 < r < 1$. Let $f \in F^{0,\mathscr{L}}_{p_0,\infty}(X)$ with $\|f\|_{F^{0,\mathscr{L}}_{p_0,\infty}(X)} = 1$. Let Φ_0, Φ be functions in $\mathcal{S}(\mathbb{R}_{\ge 0})$ such that

$$\begin{aligned} &\operatorname{supp}\Phi_0 \in [0,2^2], \quad |\Phi_0(\lambda)| \ge c > 0 \text{ for } \lambda \in [0, 2^{3/2}],\\ &\operatorname{supp}\Phi \subset [2^{-2},2^2], \quad |\Phi(\lambda)| \ge c > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}], \end{aligned}$$

Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \ge 1$. Since $\Phi_j(\mathscr{L})f \in \Sigma_{2^{j+1}}$ for all $j \in \mathbb{N}_0$, by Lemma 3.14 we have that for $p \in (0,\infty)$ and $j \in \mathbb{N}_0$,

$$\|\Phi_j(\mathscr{L})f\|_{L^\infty(X)} \lesssim 2^{jd/p}\|\Phi_j(\mathscr{L})\|_{L^p(X)} \le 2^{jd/p}\|f\|_{F^{0,\mathscr{L}}_{p,\infty}(X)} = 2^{jd/p}.$$

It follows that, for any fixed integer $N \in \mathbb{N}_0$ and for a.e. $x \in X$,

$$\left(\sum_{j=0}^{N} 2^{jsr}\big|\Phi_j(\mathscr{L})f(x)\big|^r\right)^{1/r} \le C\left(\sum_{j=0}^{N} 2^{jsr}2^{jdr/p}\right)^{1/r} \le C2^{dN/p_1}, \tag{4.30}$$

where C is a constant independent of N. On the other hand, since $s_1 < 0$, we have that for a.e. $x \in X$,

$$\left(\sum_{j=N}^{\infty} 2^{js_1r}|\Phi_j(\mathscr{L})f(x)|^r\right)^{1/r} \le C2^{s_1N}\sup_{j\in\mathbb{N}_0}|\Phi_j(\mathscr{L})f(x)|. \tag{4.31}$$

We write

$$\|f\|^{p_1}_{\dot F^{s_1}_{p_1,r}} = p_1\int_0^\infty t^{p_1-1}\left|\left\{x : \left(\sum_{j=0}^{\infty} 2^{js_1r}|\Phi_j(\mathscr{L})f(x)|^r\right)^{1/r} > t\right\}\right| dt. \tag{4.32}$$

Let us split the range of the integration in (4.32) into $(0,(2C)^{1/r})$ and $((2C)^{1/r},\infty)$, where C is the same constant as in (4.30). By (4.31) with $N = 0$, we have

$$\begin{aligned} &\int_0^{(2C)^{1/r}} t^{p_1-1}\left|\left\{x : \left(\sum_{j=0}^{\infty} 2^{js_1r}|\Phi_j(\mathscr{L})f(x)|^r\right)^{1/r} > t\right\}\right| dt\\ &\le c\int_0^{c'(2C)^{1/r}} t^{p_0-1}\left|\left\{x : \sup_{j\in\mathbb{N}_0}|\Phi_j(\mathscr{L})f(x)| > t\right\}\right| dt\\ &\le c''\|f\|_{F^{0,\mathscr{L}}_{p_0,\infty}(X)} = c''. \end{aligned} \tag{4.33}$$

If $t > (2C)^{1/r}$, we choose N in (4.30) to be the largest non-negative integer such that $C2^{Nd/p_1} \leq t/2$. Now (4.30) coupled with (4.31) yield that

$$\left|\left\{x : \left(\sum_{j=0}^{\infty} 2^{js_1 r}|\Phi_j(\mathscr{L})f(x)|^r\right)^{1/r} > t\right\}\right| \leq \left|\left\{x : \left(\sum_{j=N}^{\infty} 2^{js_1 r}|\Phi_j(\mathscr{L})f(x)|^r\right)^{1/r} > \frac{t}{2}\right\}\right|$$
$$\leq \left|\left\{x : \sup_{j\in\mathbb{N}_0} |\Phi_j(\mathscr{L})f(x)| > ct2^{-Ns_1}\right\}\right|.$$

Since $t2^{-Ns_1} \sim t^{1-s_1p_1/d} \sim t^{p_1/p_0}$, it follows from the above estimate that

$$\begin{aligned}
&\int_{(2C)^{1/r}}^{\infty} t^{p_1-1}\left|\left\{x : \left(\sum_{j=0}^{\infty} 2^{js_1 r}|\Phi_j(\mathscr{L})f(x)|^r\right)^{1/r} > t\right\}\right| dt \\
&\leq \int_{c(2C)^{1/r}}^{\infty} t^{p_1-1}\left|\left\{x : \sup_{j\in\mathbb{N}_0} |\Phi_j(\mathscr{L})f(x)| > t^{p_1/p_0}\right\}\right| dt \\
&\leq c' \int_{0}^{\infty} t^{p_0-1}\left|\left\{x : \sup_{j\in\mathbb{N}_0} |\Phi_j(\mathscr{L})f(x)| > t\right\}\right| dt \\
&\leq c''\|f\|_{F^{0,\mathscr{L}}_{p_0,\infty}(X)} = c''.
\end{aligned} \tag{4.34}$$

Combining (4.32), (4.33) and (4.34) we get

$$\|f\|^{p_1}_{F^{s_1,\mathscr{L}}_{p_1,r}(X)} \leq c\|f\|^{p_1}_{F^{0,\mathscr{L}}_{p_0,\infty}(X)}, \tag{4.35}$$

which holds for all $f \in F^{0,\mathscr{L}}_{p_0,\infty}(X)$ with $\|f\|_{F^{0,\mathscr{L}}_{p_0,\infty}(X)} = 1$. By a homogeneity argument, we further see that (4.35) holds for all $f \in F^{0,\mathscr{L}}_{p_0,\infty}(X)$. The proof of the theorem is completed. □

4.5 The identification $F^{0,\mathscr{L}}_{p,q}(X) = L^p(X)$ for $1 < p < \infty$

Our aim in this section is to show the following theorem:

Theorem 4.17. *Let $p \in (1, \infty)$. Then $F^{0,\mathscr{L}}_{p,2}(X) = L^p(X)$ with equivalent norms.*

The identification $F^{0,\mathscr{L}}_{p,2}(X) = L^p(X)$ is proved in [59, Thorem 7.8] under the additional assumption that the heat kernel of $\mathscr{L}$ satisfies the Hölder continuity estimate. To see that $F^{0,\mathscr{L}}_{p,2}(X) = L^p(X)$ remains valid for those operators $\mathscr{L}$ whose heat kernel only satisfy pointwise Gaussian upper bound, we need the following lemma:

Lemma 4.18. (see [23, Theorem 3.1]) *Suppose $F \in C^k(\mathbb{R}_{\geq 0})$ for some $k \geq \lfloor d/2 \rfloor + 1$, and*

$$\sup_{\lambda\in\mathbb{R}_{\geq 0}} |\lambda^\nu F^{(\nu)}(\lambda)| < \infty \quad \textit{for any } \nu \in \{0, 1, \cdots, k\}.$$

Then the operator $F(\mathscr{L})$ is bounded on $L^p(X)$ for $1 < p < \infty$.

Proof of Theorem 4.17. The proof if the same as the proof of [59, Theorem 7.8] except that one needs to replace [59, Theorem 7.9] used in the proof of [59, Theorem 7.8] by Lemma 4.17 stated above. □

Chapter 5

Homogeneous function spaces associated to operators

Throughout this chapter, we assume that the metric measure space (X, ρ, μ) satisfies the doubling condition (2.1), the reverse doubling condition (2.4), and the non-collapsing condition (2.6), and assume that $\mathscr{L}$ is a non-negative self-adjoint operator on $L^2(X)$ whose heat kernel $p_t(x, y)$ satisfies the pointwise Gaussian upper bound (2.8) for $t \in (0, \infty)$. We assume in addition that $\mu(X) = \infty$. We do not assume the Hölder continuity for $p_t(x, y)$ in the variables x and y.

5.1 Spaces of test functions and distributions

To treat homogeneous Besov and Triebel-Lizorkin spaces associated to operators, we need to use appropriate spaces of test functions and distributions which are different from those used to treat inhomogeneous function spaces.

Definition 5.1. The test function space $\mathcal{S}_{\infty,\mathscr{L}}(X)$ is defined as the collection of all functions $\phi \in \cap_{k\in\mathbb{Z}} D(\mathscr{L}^k)$ such that

$$\mathcal{P}_{k,m}(\phi) := \operatorname*{ess\,sup}_{x\in X} (1 + \rho(x, x_0))^m |\mathscr{L}^k \phi(x)| < \infty$$

for all $k \in \mathbb{Z}$ and all $m \in \mathbb{N}_0$, where $x_0 \in X$ is arbitrary fixed point on X.

Obviously, the definition of $\mathcal{S}_{\infty,\mathscr{L}}(X)$ is independent of the choice of x_0. So we fix x_0 once and for all. For our purpose it is convenient to introduce the following directed family of norms: For $k, m \in \mathbb{N}_0$ and $\phi \in \mathcal{S}_{\infty,\mathscr{L}}(X)$, we define

$$\mathcal{P}^*_{k,m}(\phi) := \sum_{\substack{-k\le j\le k\\ 0\le \ell\le m}} \mathcal{P}_{j,\ell}(\phi).$$

The space $\mathcal{S}'_{\infty,\mathscr{L}}(X)$ is defined as the collection of all continuous linear functionals on $\mathcal{S}_{\infty,\mathscr{L}}(X)$. The action of $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$ on $\phi \in \mathcal{S}_{\infty,\mathscr{L}}(X)$ will be denoted by $(f, \phi) := f(\phi)$. However, sometimes we will work with the sesquilinear version $\langle f, \phi\rangle = (f, \bar{\phi})$.

Proposition 5.2. *$\mathcal{S}_{\infty,\mathscr{L}}(X)$ is a Fréchet space.*

Proof. To prove that $\mathcal{S}_{\infty,\mathscr{L}}(X)$ is a Fréchet space we only have to establish the completeness of $\mathcal{S}_{\infty,\mathscr{L}}(X)$. Let $\{\phi_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $\mathcal{S}_{\infty,\mathscr{L}}(X)$, i.e. $\mathcal{P}_{k,m}(\phi_j - \phi_\ell) \to 0$ as $j,\ell \to \infty$ all $k \in \mathbb{Z}$ and $m \in \mathbb{N}_0$. Choose $m \in \mathbb{N}_0$ so that $m \geq (d+1)/2$. Then clearly for any $k \in \mathbb{Z}$,

$$\begin{aligned}\|\mathscr{L}^k\phi_j - \mathscr{L}^k\phi_\ell\|_{L^2(X)} &\leq \mathcal{P}_{k,m}(\phi_j - \phi_\ell)\int_X (1+\rho(x,x_0))^{-d-1}d\mu(x)\\ &\lesssim V(x_0,1)\mathcal{P}_{k,m}(\phi_j - \phi_\ell),\end{aligned}$$

where we used (3.8). Therefore, $\|\mathscr{L}^k\phi_j - \mathscr{L}^k\phi_\ell\|_{L^2(X)} \to 0$ as $j,\ell \to \infty$ and by the completeness of $L^2(X)$ there exists $\psi_k \in L^2(X)$ such that $\|\mathscr{L}^k\phi_j - \psi_k\|_{L^2(X)} \to 0$ as $j \to \infty$. Write $\phi := \psi_0$. From $\|\phi_j - \phi\|_{L^2(X)} \to 0$, $\|\mathscr{L}^k\phi_j - \psi_k\|_{L^2(X)} \to 0$, and the fact that $\mathscr{L}^k$ being a self-adjoint operator is closed [66] it follows that $\phi \in D(\mathscr{L}^k)$ and

$$\|\mathscr{L}^k\phi_j - \mathscr{L}^k\phi\|_{L^2(X)} \to 0 \quad \text{as } j \to \infty \text{ for all } k \in \mathbb{Z}. \tag{5.1}$$

On the other hand, $\|\mathscr{L}^k\phi_j - \psi_k\|_{L^\infty(X)} \to 0$ as $j,\ell \to \infty$, and from the completeness of $L^\infty(X)$ the sequence $\{\mathscr{L}^k\phi_j\}_{j=1}^{\infty}$ converges in $L^\infty(X)$. This and (5.1) yield

$$\|\mathscr{L}^k\phi_j - \mathscr{L}^k\phi\|_{L^\infty(X)} \to 0 \quad \text{as } j \to \infty \text{ for all } k \in \mathbb{Z}.$$

In turn, this along with $\mathcal{P}_{k,m}(\phi_j - \phi_\ell) \to 0$ as $j,\ell \to \infty$ implies $\mathcal{P}_{k,m}(\phi_j - \phi) \to 0$ as $j \to \infty$ which confirms the completeness of $\mathcal{S}_{\infty,\mathscr{L}}(X)$. □

Proposition 5.3. *Suppose Φ is a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ such that for all $M \in \mathbb{N}$, the functions $\lambda \mapsto \lambda^{-M}\Phi(\lambda)$ belong to $\mathcal{S}(\mathbb{R}_{\geq 0})$. Then:*

(i) *For almost every fixed $y \in X$, $K_{\Phi(\mathscr{L})}(\cdot,y)$ belongs to $\mathcal{S}_{\infty,\mathscr{L}}(X)$.*

(ii) *For almost every fixed $x \in X$, $K_{\Phi(\mathscr{L})}(x,\cdot)$ belongs to $\mathcal{S}_{\infty,\mathscr{L}}(X)$.*

Proof. From (5.14) in [59] we see that for almost every fixed $y \in X$ and for any $k \in \mathbb{Z}$,

$$\mathscr{L}^k\big[K_{\Phi(\mathscr{L})}(\cdot,y)\big] = K_{\mathscr{L}^k\Phi(\mathscr{L})}(\cdot,y).$$

Hence, if m is an integer with $m \geq d+1$, we have by Lemma 2.3

$$\begin{aligned}\big|\mathscr{L}^k\big[K_{\Phi(\mathscr{L})}(\cdot,y)\big](x)\big| = \big|K_{\mathscr{L}^k\Phi(\mathscr{L})}(x,y)\big| &\lesssim \|\lambda \mapsto \lambda^k\Phi(\lambda)\|_{(m)}D_{1,m}(x,y)\\ &\lesssim \|\lambda \mapsto \lambda^k\Phi(\lambda)\|_{(m)}[V(y,1)]^{-1}(1+d(x,y))^{-m+\frac{d}{2}}, \quad \text{for a.e. } x \in X.\end{aligned}$$

This shows that $K_{\Phi(\mathscr{L})}(\cdot,y) \in \mathcal{S}_{\infty,\mathscr{L}}(X)$. Since $K_{\Phi(\mathscr{L})}(x,\cdot) = \overline{K_{\Phi(\mathscr{L})}(\cdot,x)}$, we also have $K_{\Phi(\mathscr{L})}(x,\cdot) \in \mathcal{S}_{\infty,\mathscr{L}}(X)$ for almost every fixed $x \in X$. □

If $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$ and if Φ is a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ such that for all $M \in \mathbb{N}$ the functions $\lambda \mapsto \lambda^{-M}\Phi(\lambda)$ belong to $\mathcal{S}(\mathbb{R}_{\geq 0})$, then (thanks to Proposition 5.3) it is natural to define

$$\Phi(\mathscr{L})f(x) := \big(f, K_{\Phi(\mathscr{L})}(x,\cdot)\big), \quad \text{for a.e. } x \in X.$$

This extends the domain of $\Phi(\mathscr{L})$ from $L^2(X)$ to $\mathcal{S}'_{\infty,\mathscr{L}}(X)$.

Lemma 5.4. *Let $\{E(\lambda) : \lambda \geq 0\}$ be spectral resolution of $\mathscr{L}$. Then the spectral measure of $\{0\}$ is zero, i.e., the point $\lambda = 0$ may be neglected in the spectral resolution.*

Proof. Assume by contradiction that $E(\{0\}) \neq 0$, then there exists $g \in L^2(X)$ such that $f := E(\{0\})g$ is not the zero element in $L^2(X)$. Since $E(\{0\})$ is a an orthogonal projection,

$$E(\{0\})f = E(\{0\})E(\{0\})g = E(\{0\})g = f.$$

It follows that

$$e^{-t\mathscr{L}}f = \int_0^\infty e^{-t\lambda}dE(\lambda)f = \int_0^\infty e^{-t\lambda}dE(\lambda)E(\{0\})f = \int_{\{0\}} e^{-t\lambda}dE(\lambda)f = E(\{0\})f = f,$$

which implies

$$\begin{aligned}
\|f\|_{L^\infty(X)} = \|e^{-t\mathscr{L}}f\|_{L^\infty(X)} &\leq \sup_{x\in X}\int_X |p_t(x,y)||f(y)|d\mu(y) \\
&\leq \sup_{x\in X}\|f\|_{L^2(X)}\left(\int_X |p_t(x,y)|^2 d\mu(y)\right)^{1/2} \\
&\lesssim \sup_{x\in X}\|f\|_{L^2(X,d\mu)}\left(\int_X \frac{1}{[V(x,\sqrt{t})]^2}\left(1+\frac{d(x,y)}{\sqrt{t}}\right)^{-N} d\mu(y)\right)^{1/2} \\
&\lesssim \sup_{x\in X}\frac{\|f\|_{L^2(X,d\mu)}}{[V(x,\sqrt{t})]^{1/2}} = \frac{\|f\|_{L^2(X)}}{\inf_{x\in X}[V(x,\sqrt{t})]^{1/2}} \\
&\lesssim t^{-\kappa/4}\|f\|_{L^2(X)} \to 0 \quad \text{as } t\to\infty,
\end{aligned}$$

where we used the reverse-doubling and the non-collapsing conditions. Hence $f = 0$ in $L^2(X)$, which leads to a contradiction. Therefore we must have $E(\{0\}) = 0$. □

The following Calderón reproducing formula is a homogeneous counterpart of Lemma 3.10. It plays an important role in establishing homogeneous Besov and Tribel-Lizorkin spaces associated to $\mathscr{L}$.

Proposition 5.5. *Suppose $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$, Φ vanishes near the origin, and*

$$\sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\lambda) = 1 \quad \textit{for all } \lambda \in \mathbb{R}_{>0}. \tag{5.2}$$

Then for any $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$,

$$f = \sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\mathscr{L})f \quad \textit{in } \ \mathcal{S}'_{\infty,\mathscr{L}}(X).$$

Proof. By duality, it suffices to show that for all $\phi \in \mathcal{S}_{\infty,\mathscr{L}}(X)$,

$$\phi = \sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi \quad \text{in } \mathcal{S}_{\infty,\mathscr{L}}(X).$$

To do this, we first show that the sum $\sum\limits_{j=-\infty}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi$ converges in the topology of $\mathcal{S}_{\infty,\mathscr{L}}(X)$. For this purpose we write

$$\sum_{j=-\infty}^{\infty} \mathcal{P}_{k,m}\left(\Phi(2^{-2j}\mathscr{L})\phi\right)$$

$$
\begin{aligned}
&= \sum_{j=-\infty}^{0} \mathcal{P}_{k,m}\left(\Phi(2^{-2j}\mathscr{L})\phi\right) + \sum_{j=1}^{\infty} \mathcal{P}_{k,m}\left(\Phi(2^{-2j}\mathscr{L})\phi\right) \\
&= \sum_{j=-\infty}^{0} \operatorname*{ess\,sup}_{x\in X}(1+\rho(x,x_0))^m \left|\mathscr{L}^k \Phi(2^{-2j}\mathscr{L})\phi(x)\right| \\
&\quad + \sum_{j=1}^{\infty} \operatorname*{ess\,sup}_{x\in X}(1+\rho(x,x_0))^m \left|\mathscr{L}^k \Phi(2^{-2j}\mathscr{L})\phi(x)\right| \\
&= \sum_{j=-\infty}^{0} 2^{2j(m+1)} \operatorname*{ess\,sup}_{x\in X}(1+\rho(x,x_0))^m \left|(2^{-2j}\mathscr{L})^{m+1}\Phi(2^{-2j}\mathscr{L})(\mathscr{L}^{k-m-1}\phi)(x)\right| \\
&\quad + \sum_{j=1}^{\infty} 2^{-2j} \operatorname*{ess\,sup}_{x\in X}(1+\rho(x,x_0))^m \left|(2^{-2j}\mathscr{L})^{-1}\Phi(2^{-2j}\mathscr{L})(\mathscr{L}^{k+1}\phi)(x)\right|.
\end{aligned}
$$

Note that if $j \le 0$ then by Lemma 2.3 and Lemma 3.7

$$
\begin{aligned}
&\left|(2^{-2j}\mathscr{L})^{m+1}\Phi(2^{-2j}\mathscr{L})(\mathscr{L}^{k-m-1}\phi)(x)\right| \\
&\le \int_X \left|K_{(2^{-2j}\mathscr{L})^{m+1}\Phi(2^{-2j}\mathscr{L})}(x,y)\right|\left|(\mathscr{L}^{k-m-1}\phi)(y)\right| d\mu(y) \\
&\lesssim \int_X D_{2^{-j},m+5d/2}(x,y) D_{1,m+5d/2}(y,x_0)\, d\mu(y) \\
&\lesssim D_{2^{-j},m+d/2}(x,x_0) \lesssim [V(x_0,2^{-j})]^{-1}(1+2^j\rho(x,x_0))^{-m} \\
&\lesssim 2^{-jm}(1+\rho(x,x_0))^{-m}, \quad \text{for a.e. } x\in X,
\end{aligned}
$$

while if $j \ge 1$ then

$$
\begin{aligned}
&\left|(2^{-2j}\mathscr{L})^{-1}\Phi(2^{-2j}\mathscr{L})(\mathscr{L}^{k+1}\phi)(x)\right| \\
&\le \int_X \left|K_{(2^{-2j}\mathscr{L})^{-1}\Phi(2^{-2j}\mathscr{L})}(x,y)\right| |\mathscr{L}^{k+1}\phi(y)|\, d\mu(y) \\
&\lesssim \int_X D_{2^{j},m+5d/2}(x,y) D_{1,m+5d/2}(y,x_0)\, d\mu(y) \\
&\lesssim D_{1,m+d/2}(x,x_0) \lesssim (1+\rho(x,x_0))^{-m}, \quad \text{for a.e. } x\in X.
\end{aligned}
$$

Hence

$$
\sum_{j=-\infty}^{\infty} \mathcal{P}_{k,m}\left(\Phi(2^{-2j}\mathscr{L})\phi\right) \lesssim \sum_{j=-\infty}^{0} 2^{j(m+2)} + \sum_{j=1}^{\infty} 2^{-2j} < \infty,
$$

which yields that sum $\sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi$ converges in the topology of $\mathcal{S}_{\infty,\mathscr{L}}(X)$. By the completeness of $\mathcal{S}_{\infty,\mathscr{L}}(X)$, there exists $\psi \in \mathcal{S}_{\infty,\mathscr{L}}(X)$ such that

$$
\sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi = \psi \quad \text{in } \mathcal{S}_{\infty,\mathscr{L}}(X).
$$

On the other hand, by (5.2) and the spectral theorem (cf. [66, Theorem VII.2]) and Lemma 5.4, we have

$$
\sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\mathscr{L})\phi = \phi \quad \text{in } L^2(X).
$$

Therefore, $\psi = \phi$ in $\mathcal{S}_{\infty,\mathscr{L}}(X)$. This completes the proof. □

5.2 Definition of $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ and $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$

We now introduce homogeneous Besov and Triebel-Lizorkin spaces associated with $\mathscr{L}$:

Definition 5.6. Let $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\operatorname{supp}\Phi \subset [2^{-2}, 2^2] \quad \text{and} \quad |\Phi(\lambda)| \geq c > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}]. \tag{5.3}$$

Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \in \mathbb{Z}$.

(i) If $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty]$, we define the homogeneous Besov space $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ as the collection of all distributions $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$ such that

$$\|f\|_{\dot{B}^{s,\mathscr{L}}_{p,q}(X)} := \left(\sum_{j=-\infty}^{\infty} \left\| 2^{js}\Phi_j(\mathscr{L})f \right\|^q_{L^p(X)} \right)^{1/q} < \infty.$$

(ii) If $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$, we define the homogeneous Triebel-Lizorkin space $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$ as the collection of all distributions $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$ such that

$$\|f\|_{\dot{F}^{s,\mathscr{L}}_{p,q}(X)} := \left\| \left(\sum_{j=-\infty}^{\infty} \left| 2^{js}\Phi_j(\mathscr{L})f \right|^q \right)^{1/q} \right\|_{L^p(X)} < \infty.$$

Given a function $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying (5.3), a distribution $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$, and a positive number a, we define a system of Peetre type maximal functions by

$$[\Phi_j(\mathscr{L})]^*_a f(x) := \operatorname*{ess\,sup}_{y\in X} \frac{|\Phi_j(\mathscr{L})f(y)|}{(1+2^j\rho(x,y))^a}, \quad x \in X, j \in \mathbb{Z},$$

where $\Phi_j(\cdot) := \Phi(2^{-2j}\cdot)$ for $j \in \mathbb{Z}$.

The following two theorems are homogeneous counterparts of Theorem 3.4 and Theorem 3.5 respectively. Their proofs are analogous to those of Theorem 3.4 and Theorem 3.5 respectively and are thus skipped.

Theorem 5.7. *Let Φ be a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying* (5.3)*. Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \in \mathbb{Z}$.*

(i) *If $s \in \mathbb{R}$, $p \in (0,\infty]$, $q \in (0,\infty]$ and $a > \frac{2d}{p}$, then for all $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$,*

$$\left(\sum_{j=-\infty}^{\infty} \left\| 2^{js}[\Phi_j(\mathscr{L})]^*_a f \right\|^q_{L^p(X)} \right)^{1/q} \sim \left(\sum_{j=-\infty}^{\infty} \left\| 2^{js}\Phi_j(\mathscr{L})f \right\|^q_{L^p(X)} \right)^{1/q}.$$

(ii) *If $s \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$ and $a > \frac{2d}{\min\{p,q\}}$, then for all $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$,*

$$\left\| \left(\sum_{j=-\infty}^{\infty} \left| 2^{js}[\Phi_j(\mathscr{L})]^*_a f \right|^q \right)^{1/q} \right\|_{L^p(X)} \sim \left\| \left(\sum_{j=-\infty}^{\infty} \left| 2^{js}\Phi_j(\mathscr{L})f \right|^q \right)^{1/q} \right\|_{L^p(X)}.$$

Theorem 5.8. *Let $\Phi, \widetilde{\Phi}$ be functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ both of which satisfy* (5.3)*. Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ and $\widetilde{\Phi}_j(\lambda) := \widetilde{\Phi}(2^{-2j}\lambda)$ for $j \in \mathbb{Z}$.*

(i) *If $s \in \mathbb{R}$, $p \in (0,\infty]$, $q \in (0,\infty]$ and $a > 0$, then for all $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$,*

$$\left(\sum_{j=-\infty}^{\infty} \left\|2^{js}[\widetilde{\Phi}_j(\mathscr{L})]_a^* f\right\|_{L^p(X)}^q\right)^{1/q} \sim \left(\sum_{j=-\infty}^{\infty} \left\|2^{js}[\Phi_j(\mathscr{L})]_a^* f\right\|_{L^p(X)}^q\right)^{1/q}.$$

(ii)*If $s \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$ and $a > 0$, then for all $f \in \mathcal{S}'_{\infty,\mathscr{L}}(X)$,*

$$\left\|\left(\sum_{j=-\infty}^{\infty} \left|2^{js}[\widetilde{\Phi}_j(\mathscr{L})]_a^* f\right|^q\right)^{1/q}\right\|_{L^p(X)} \sim \left\|\left(\sum_{j=-\infty}^{\infty} \left|2^{js}[\Phi_j(\mathscr{L})]_a^* f\right|^q\right)^{1/q}\right\|_{L^p(X)}.$$

Combining Theorem 5.7 and 5.8, we get the following corollary:

Corollary 5.9. *The definition of $B^{s,\mathscr{L}}_{p,q}(X)$ and $F^{s,\mathscr{L}}_{p,q}(X)$ are independent of the choice of Φ, as long as $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ and Φ satisfies* (5.3).

5.3 Properties and characterizations

In this section we list some properties and characterizations of $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ and $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$. All the statements can be proved similarly as their inhomogeneous versions given in Chapter 3. Thus we will skip all the proofs.

Proposition 5.10. *Let $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty]$.*

(i) *$\mathcal{S}_{\infty,\mathscr{L}}(X) \subset \dot{B}^{s,\mathscr{L}}_{p,q}(X) \subset \mathcal{S}'_{\infty,\mathscr{L}}(X)$ and the inclusion maps are continuous.*

(ii) *The space $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ is a quasi-Banach space.*

Proposition 5.11. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$.*

(i) *$\mathcal{S}_{\infty,\mathscr{L}}(X) \subset \dot{F}^{s,\mathscr{L}}_{p,q}(X) \subset \mathcal{S}'_{\infty,\mathscr{L}}(X)$ and the inclusion maps are continuous.*

(ii) *The space $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$ is a quasi-Banach space.*

Theorem 5.12. *Let Φ be a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying* (5.3).

(i) *If $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty]$, then*

$$\|f\|_{\dot{B}^{s,\mathscr{L}}_{p,q}(X)} \sim \left(\int_0^\infty t^{-sq}\left\|\Phi(t^2\mathscr{L})f\right\|_{L^p(X)}^q \frac{dt}{t}\right)^{1/q}.$$

(ii) *If $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$, then*

$$\|f\|_{\dot{F}^{s,\mathscr{L}}_{p,q}(X)} \sim \left\|\left(\int_0^\infty t^{-sq}\left|\Phi(t^2\mathscr{L})f\right|^q \frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}.$$

Definition 5.13. Let $K, S \in \mathbb{N}_0$, and let Q be a dyadic cube in $\mathscr{D}_k$, with $k \in \mathbb{Z}$. A function $a_Q \in L^2(X)$ is said to be a (homogeneous) (K,S)-atom for Q if a_Q satisfies the following conditions for $m \in \{K, -S\}$.

(i) $a_Q \in D(\mathscr{L}^m)$;

(ii) $\operatorname{supp}(\mathscr{L}^m a_Q) \subset B(z_Q, (A_1+1)2^{-k})$;

(iii) $\operatorname*{ess\,sup}_{x\in X} |\mathscr{L}^m a_Q(x)| \le 2^{2km}[\mu(Q)]^{-1/2}$.

Following [35], we define the sequences $\dot{b}^s_{p,q}$ and $\dot{f}^s_{p,q}$:

Definition 5.14. (i) Let $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty]$. The sequence space $\dot{b}^s_{p,q}$ consists of all sequences $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of complex scalars such that

$$\|w\|_{\dot{b}^s_{p,q}} := \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \left[\sum_{Q\in\mathscr{D}_k} \left(|w_Q|[\mu(Q)]^{1/p-1/2}\right)^p \right]^{q/p} \right)^{1/q} < \infty.$$

(ii) Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. The sequence space $\dot{f}^s_{p,q}$ consists of all sequences $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of complex scalars such that

$$\|w\|_{\dot{f}^s_{p,q}} := \left\| \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \sum_{Q\in\mathscr{D}_k} \left(|w_Q|[\mu(Q)]^{-1/2}\chi_Q\right)^q \right)^{1/q} \right\|_{L^p(X)} < \infty.$$

Here, χ_Q is the characteristic function of Q.

The atomic decomposition of $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ and $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$ is stated in the following two theorems:

Theorem 5.15. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty)$. Let $K, S \in \mathbb{N}_0$ such that $K > \frac{s}{2}$ and $S > \frac{d}{2p} - \frac{s}{2}$. Then there is a constant $C > 0$ such that for every sequence $\{a_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of (K,S)-atoms and every sequence $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of complex scalars,*

$$\left\| \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q \right\|_{\dot{B}^{s,\mathscr{L}}_{p,q}(X)} \le C\|w\|_{\dot{b}^s_{p,q}}.$$

Conversely, there is a constant C' such that given any distribution $f \in \dot{B}^{s,\mathscr{L}}_{p,q}(X)$ and any $K, S \in \mathbb{N}_0$, there exist a sequence $\{a_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of (K,S)-atoms and a sequence $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of complex scalars such that

$$f = \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q,$$

where the sum converges in $\mathcal{S}'_{\infty,\mathscr{L}}(X)$, and moreover,

$$\|w\|_{\dot{b}^s_{p,q}} \le C'\|f\|_{\dot{B}^{s,\mathscr{L}}_{p,q}(X)}.$$

Theorem 5.16. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty)$. Let $K, S \in \mathbb{N}_0$ such that $K > \frac{s}{2}$ and $S > \frac{d}{2\min\{p,q\}} - \frac{s}{2}$. Then there is a constant $C > 0$ such that for every sequence $\{a_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of (K,S)-atoms and every sequence $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of complex scalars,*

$$\left\| \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q \right\|_{\dot{F}^{s,\mathscr{L}}_{p,q}(X)} \le C\|w\|_{\dot{f}^s_{p,q}}.$$

Conversely, there is a constant C' such that given any distribution $f \in F^{s,\mathscr{L}}_{p,q}(X)$ and any $K, S \in \mathbb{N}_0$, there exist a sequence $\{a_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of (K,S)-atoms and a sequence $w = \{w_Q\}_{Q\in\cup_{k\in\mathbb{Z}}\mathscr{D}_k}$ of *complex scalars such that*

$$f = \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathscr{D}_k} w_Q a_Q,$$

where the sum converges in $\mathcal{S}'_{\infty,\mathscr{L}}(X)$, *and moreover,*

$$\|w\|_{\dot{f}^s_{p,q}} \le C' \|f\|_{\dot{F}^{s,\mathscr{L}}_{p,q}(X)}.$$

Using atomic decomposition, one can prove the following result:

Proposition 5.17. *Let* $s \in \mathbb{R}$, $p \in (0,\infty)$ *and* $q \in (0,\infty)$. *Then* $\mathcal{S}_{\infty,\mathscr{L}}(X)$ *is dense in* $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ *and is dense in* $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$.

The complex interpolation property of $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ and $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$ is stated as follows:

Theorem 5.18. *Let* $-\infty < s_0 < \infty$, $-\infty < s_1 < \infty$, $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < q_0 < \infty$, $0 < q_1 < \infty$ *and* $0 < \theta < 1$. *If* s, p, q *are given by* $s = (1-\theta)s_0 + \theta s_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1$, *then*

$$\big(\dot{B}^{s_0,\mathscr{L}}_{p_0,q_0}(X), \dot{B}^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)_\theta = \dot{B}^{s,\mathscr{L}}_{p,q}(X),$$
$$\big(\dot{F}^{s_0,\mathscr{L}}_{p_0,q_0}(X), \dot{F}^{s_1,\mathscr{L}}_{p_1,q_1}(X)\big)_\theta = \dot{F}^{s,\mathscr{L}}_{p,q}(X),$$

and the corresponding quasi-norms are equivalent.

We have the following lifting property:

Theorem 5.19. *Let* $s \in \mathbb{R}$, $p \in (0,\infty)$ *and* $q \in (0,\infty)$. *Then:*

(i) *For any* $\sigma \in \mathbb{R}$, $\mathscr{L}^\sigma$ *is an isomorphism of* $\dot{B}^{s,\mathscr{L}}_{p,q}(X)$ *to* $\dot{B}^{s-2\sigma,\mathscr{L}}_{p,q}(X)$.

(ii) *For any* $\sigma \in \mathbb{R}$, $\mathscr{L}^\sigma$ *is an isomorphism of* $\dot{F}^{s,\mathscr{L}}_{p,q}(X)$ *to* $\dot{F}^{s-2\sigma,\mathscr{L}}_{p,q}(X)$.

We also have the following embedding theorem:

Theorem 5.20. (i) *Let* $0 < p_0 \le p_1 < \infty$, $0 < q \le \infty$ *and* $-\infty < s_1 \le s_0 < \infty$. *Then we have the continuous embedding*

$$\dot{B}^{s_0,\mathscr{L}}_{p_0,q}(X) \subset \dot{B}^{s_1,\mathscr{L}}_{p_1,q}(X) \quad \textit{if } s_0 - d/p_0 = s_1 - d/p_1.$$

(ii) *Let* $0 < p_0 < p_1 < \infty$, $0 < q \le \infty$, $0 < r \le \infty$ *and* $-\infty < s_1 < s_0 < \infty$. *Then we have the continuous embedding*

$$\dot{F}^{s_0,\mathscr{L}}_{p_0,q}(X) \subset \dot{F}^{s_1,\mathscr{L}}_{p_1,r}(X) \quad \textit{if } s_0 - d/p_0 = s_1 - d/p_1.$$

Moreover, we have the following useful result:

Theorem 5.21. *Let* $s \in \mathbb{R}$ *and* $q \in (0,\infty]$, *and let* M *be a nonnegative integer such that* $M > s/2$. *Let* Ψ *be a function in* $\mathcal{S}(\mathbb{R}_{\ge 0})$ *such that*

$$|\Psi(\lambda)| \ge c > 0 \quad \textit{on } \{2^{-3/2}\varepsilon \le \lambda \le 2^{3/2}\varepsilon\} \tag{5.4}$$

for some $\varepsilon > 0$, *and*

$$\textit{the function } \lambda \mapsto \lambda^{-M}\Psi(\lambda) \textit{ belongs to } \mathcal{S}(\mathbb{R}_{\ge 0}). \tag{5.5}$$

Set $\Psi_j(\lambda) := \Psi(2^{-2j}\lambda)$ *for* $j \in \mathbb{Z}$. *Then the following statements are valid:*

(i) *If* $p \in (0,\infty]$ *and* $a > \frac{2d}{p}$, *then for all* $f \in L^2(X)$,

$$\|f\|_{\dot B^{s,\mathscr{L}}_{p,q}(X)} \sim \left(\sum_{j=-\infty}^{\infty} \big\|2^{js}[\Psi_j(\mathscr{L})]^*_a f\big\|^q_{L^p(X)}\right)^{1/q} \sim \left(\sum_{j=-\infty}^{\infty} \big\|2^{js}\Psi_j(\mathscr{L})f\big\|^q_{L^p(X)}\right)^{1/q}$$
$$\sim \left(\int_0^\infty t^{-sq}\big\|[\Psi(t^2\mathscr{L})]^*_a f\big\|^q_{L^p(X)}\frac{dt}{t}\right)^{1/q} \sim \left(\int_0^\infty t^{-sq}\big\|\Psi(t^2\mathscr{L})f\big\|^q_{L^p(X)}\frac{dt}{t}\right)^{1/q}.$$

(ii) *If* $p \in (0,\infty)$ *and* $a > \frac{2d}{\min\{p,q\}}$, *then for all* $f \in L^2(X)$,

$$\|f\|_{\dot F^{s,\mathscr{L}}_{p,q}(X)} \sim \left\|\left(\sum_{j=-\infty}^{\infty} \big|2^{js}[\Psi_j(\mathscr{L})]^*_a f\big|^q\right)^{1/q}\right\|_{L^p(X)} \sim \left\|\left(\sum_{j=-\infty}^{\infty} \big|2^{js}\Psi_j(\mathscr{L})f\big|^q\right)^{1/q}\right\|_{L^p(X)}$$
$$\sim \left\|\left(\int_0^\infty t^{-sq}\big|[\Psi(t^2\mathscr{L})]^*_a f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)} \sim \left\|\left(\int_0^\infty t^{-sq}\big|\Psi(t^2\mathscr{L})f\big|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p(X)}.$$

The above theorem along with the fact that $L^2(X)\cap\dot B^{s,\mathscr{L}}_{p,q}(X)$ (resp. $L^2(X)\cap\dot F^{s,\mathscr{L}}_{p,q}(X)$) is dense in $\dot B^{s,\mathscr{L}}_{p,q}(X)$ (resp. $\dot F^{s,\mathscr{L}}_{p,q}(X)$) (cf. Proposition 5.17) yields the following

Corollary 5.22. *Let* $s \in \mathbb{R}$ *and* $q \in (0,\infty]$, *and let* Ψ *be the same as in Theorem 5.21.*

(i) *If* $p \in (0,\infty]$, *then* $\dot B^{s,\mathscr{L}}_{p,q}(X)$ *is isometric to the completion of the space*

$$\left\{ f \in L^2(X) : \left(\sum_{j=-\infty}^{\infty} \big\|2^{js}\Psi_j(\mathscr{L})f\big\|^q_{L^p(X)}\right)^{1/q} < \infty \right\}$$

in the quasi-norm

$$\|f\|^*_{\dot B^{s,\mathscr{L}}_{p,q}(X)} := \left(\sum_{j=-\infty}^{\infty} \big\|2^{js}\Psi_j(\mathscr{L})f\big\|^q_{L^p(X)}\right)^{1/q}.$$

(ii) *If* $p \in (0,\infty)$, *then* $\dot F^{s,\mathscr{L}}_{p,q}(X)$ *is isometric to the completion of the space*

$$\left\{ f \in L^2(X) : \left\|\left(\sum_{j=-\infty}^{\infty} \big|2^{js}\Psi_j(\mathscr{L})f\big|^q\right)^{1/q}\right\|_{L^p(X)} < \infty \right\}$$

in the quasi-norm

$$\|f\|^*_{\dot F^{s,\mathscr{L}}_{p,q}(X)} := \left\|\left(\sum_{j=-\infty}^{\infty} \big|2^{js}\Psi_j(\mathscr{L})f\big|^q\right)^{1/q}\right\|_{L^p(X)}.$$

5.4 Area integral characterization of $\dot F^{0,\mathscr{L}}_{p,2}(X)$ for $0 < p < \infty$

Recently, Hardy spaces $H^p_{\mathscr{L}}(X)$ on a metric measure space X associated to a non-negative self-adjoint operator $\mathscr{L}$ satisfying Davies-Gaffney estimates were studied by Hofmann *et al.* [46] and by Jiang and Yang [56]. Since the pointwise Gaussian upper bound estimate (2.8) implies the Davies-Gaffney estimate (see e.g. [59, Proposition 2.7]), the theory developed in [46] and [56] can be applied to the setting of the present paper. Let us recall from [46] and [56] the definition of $H^p_{\mathscr{L}}(X)$. Set

$$H^2(X) := \overline{\mathcal{R}(\mathscr{L})} = \overline{\{\mathscr{L}u \in L^2(X) : u \in D(\mathscr{L})\}}.$$

Then $L^2(X) = H^2(X) \oplus \mathcal{N}(\mathscr{L})$, where $\mathcal{N}(\mathscr{L})$ stands for the null space of $\mathscr{L}$. For $0 < p < \infty$, the Hardy space $H^p_{\mathscr{L}}(X)$ is defined as the completion of

$$\{f \in H^2(X) : S_{\mathscr{L}}(f) \in L^p(X)\}$$

in the quasi-norm

$$\|f\|_{H^p_{\mathscr{L}}(X)} := \|S_{\mathscr{L}}(f)\|_{L^p(X)},$$

where $\mathcal{S}_{\mathscr{L}} f$ is the Lusin area integral defined by

$$S_{\mathscr{L}}(f)(x) := \left(\iint_{\Gamma(x)} |t^2 \mathscr{L} e^{-t^2\mathscr{L}} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2},$$

with $\Gamma(x) := \{(y,t) \in X \times (0,\infty) : \rho(y,x) < t\}$. Since the heat kernel of $\mathscr{L}$ obeys the Gaussian upper bound, by a result in [4] we know that $H^p_{\mathscr{L}}(X) = L^p(X)$ for all $p \in (1,\infty)$.

It is worth pointing out that, under the assumption that $\mu(X) = \infty$, one has $H^2(X) = L^2(X)$. Indeed, from Lemma 5.4 we see that 0 is not an eigenvalue of $\mathscr{L}$, i.e., $\mathcal{N}(\mathscr{L}) = \{0\}$. This along with $L^2(X) = H^2(X) \oplus \mathcal{N}(\mathscr{L})$ yields that $H^2(X) = L^2(X)$.

The aim of this section is to show the following theorem:

Theorem 5.23. *Let $p \in (0,\infty)$. Then $\dot{F}^{0,\mathscr{L}}_{p,2}(X) = H^p_{\mathscr{L}}(X)$ with equivalent quasi-norms.*

For the proof of this theorem we need some preparation. For $f \in L^2(X)$, $a > 0$, $t > 0$ and $x \in X$, we define

$$G^*_{a,\mathscr{L}}(f)(x) := \left(\int_0^\infty \int_X \frac{|t^2 \mathscr{L} e^{-t^2\mathscr{L}} f(y)|^2}{(1 + t^{-1}\rho(x,y))^{2a}} \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}.$$

and

$$M^*_{a,\mathscr{L}}(f)(x,t) := \operatorname*{ess\,sup}_{y \in X} \frac{|t^2 \mathscr{L} e^{-t^2\mathscr{L}} f(y)|}{(1 + t^{-1}\rho(x,y))^a}.$$

Lemma 5.24. *Let $0 < p < \infty$ and let $a > d/\min\{p,2\}$. Then there is a constant $C > 0$ such that for all $f \in L^2(X)$,*

$$\|G^*_{a,\mathscr{L}}(f)\|_{L^p(X)} \le C \|S_{\mathscr{L}}(f)\|_{L^p(X)}.$$

Proof. For the proof, we refer the reader to [13, Theorem 3.5]. See also [36, Lemma 3.1]. □

Lemma 5.25. *Let $0 < p < \infty$ and $a > 0$. Then there exists a constant $C > 0$ such that for all $f \in L^2(X)$,*

$$\|S_{\mathscr{L}}(f)\|_{L^p(X)} \le C \left\| \left(\int_0^\infty \left[M^*_{a,\mathscr{L}}(f)(\cdot,t) \right]^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(X)}.$$

Proof. Note that for all $a > 0$, $t > 0$, and $x \in X$, we have

$$\frac{1}{V(x,t)} \int_{B(x,t)} |t^2 \mathscr{L} e^{-t^2\mathscr{L}} f(y)|^2 d\mu(y)$$

$$\le \operatorname*{ess\,sup}_{y\in B(x,t)} |t^2\mathscr{L}e^{-t^2\mathscr{L}}f(y)|^2 \le 2^{2a} \operatorname*{ess\,sup}_{y\in B(x,t)} \frac{|t^2\mathscr{L}e^{-t^2\mathscr{L}}f(y)|^2}{(1+t^{-1}\rho(x,y))^{2a}}$$
$$\le 2^{2a} \operatorname*{ess\,sup}_{y\in X} \frac{|t^2\mathscr{L}e^{-t^2\mathscr{L}}f(y)|^2}{(1+t^{-1}\rho(x,y))^{2a}} = 2^{2a}\big[M^*_{a,\mathscr{L}}(f)(x,t)\big]^2.$$

Applying the norm $\int_0^\infty |\cdot|\frac{dt}{t}$ on both sides, we get

$$\big[S_{\mathscr{L}}(f)(x)\big]^2 \le 2^{2a}\int_0^\infty \big[M^*_{a,\mathscr{L}}(f)(x,t)\big]^2\frac{dt}{t}.$$

This yields the desired estimate. □

Lemma 5.26. *For any $r>0$, $a>0$ and $N\in\mathbb{N}_0$ with $N>a+5d/2$, there is a constant $C>0$ such that for all $f\in L^2(X)$, $\ell\in\mathbb{Z}$, $t\in[1,2]$ and a.e. $x\in X$,*

$$\big[M^*_{a,\mathscr{L}}(f)(x,2^{-\ell}t)\big]^r \le C\sum_{j=-\infty}^{\infty} 2^{-2jNr}\int_X \frac{|\Psi\big(2^{-2(j+\ell)}t^2\mathscr{L}\big)f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{ar}}d\mu(z), \tag{5.6}$$

where Ψ is a function in $\mathcal{S}(\mathbb{R}_{\ge 0})$ defined by $\Psi(\lambda):=\lambda e^{-\lambda}$, $\lambda\in\mathbb{R}_{\ge0}$.

Proof. We follow the ideas of [68] and [84]. Clearly, $|\Psi(\lambda)|>0$ on $\{1/4<\lambda<4\}$. Let us fix an arbitrary $\Gamma\in\mathcal{S}(\mathbb{R}_{\ge0})$ with the property that $|\Gamma(\lambda)|>0$ on $\{0\le\lambda<4\}$. Then there exist $\Phi,\Theta\in\mathcal{S}(\mathbb{R}_{\ge0})$ such that $\operatorname{supp}\Phi\in[0,4]$, $\operatorname{supp}\Theta\subset[1/4,4]$, and

$$\Phi(\lambda)\Gamma(\lambda)+\sum_{j=1}^{\infty}\Theta(2^{-2j}\lambda)\Psi(2^{-2j}\lambda)=1,\quad \forall\lambda\in\mathbb{R}_{\ge0}.$$

By replacing λ with $2^{-2\ell}t^2\lambda$, we get for all $\ell\in\mathbb{Z}$, $t\in[1,2]$, and $\lambda\in\mathbb{R}_{\ge0}$,

$$\Phi(2^{-2\ell}t^2\lambda)\Gamma(2^{-2\ell}t^2\lambda)+\sum_{j=1}^{\infty}\Theta(2^{-2j}2^{-2\ell}t^2\lambda)\Psi(2^{-2j}2^{-2\ell}t^2\lambda)=1.$$

It then follows from the spectral theorem that for any $f\in L^2(X,d\mu)$,

$$f=\Phi(2^{-2\ell}t^2\mathscr{L})\Gamma(2^{-2\ell}t^2\mathscr{L})f+\sum_{j=1}^{\infty}\Theta(2^{-2(j+\ell)}t^2\mathscr{L})\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f$$

holds in $L^2(X)$-norm. Hence, for a.e. $y\in X$, we have

$$\begin{aligned}\Psi(2^{-2\ell}t^2\mathscr{L})f(y) &= \Phi(2^{-2\ell}t^2\mathscr{L})\Gamma(2^{-2\ell}t^2\mathscr{L})\Psi(2^{-2\ell}t^2\mathscr{L})f(y)\\ &\quad+\sum_{j=1}^{\infty}\Psi(2^{-2\ell}t^2\mathscr{L})\Theta(2^{-2(j+\ell)}t^2\mathscr{L})\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(y)\\ &=\int_X K_{\Phi(2^{-2\ell}t^2\mathscr{L})\Gamma(2^{-2\ell}t^2\mathscr{L})}(y,z)\Psi(2^{-2\ell}t^2\mathscr{L})f(z)d\mu(z)\\ &\quad+\sum_{j=1}^{\infty}\int_X K_{\Psi(2^{-2\ell}t^2\mathscr{L})\Theta(2^{-2(j+\ell)}t^2\mathscr{L})}(y,z)\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)d\mu(z).\end{aligned} \tag{5.7}$$

Let m be an integer such that $m\ge\{d+1,5d/2+a\}$. Since Θ vanishes near the origin, the function $\lambda\mapsto\lambda^{-N}\Theta(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\ge0})$. Hence by Lemma 3.8 there exists a constant $C>0$

(depending on N and m) such that for all $\ell \in \mathbb{Z}$, $j \in \{1, 2, \cdots\}$, and $t \in [1, 2]$,

$$\begin{aligned}
&\left|K_{\Psi(2^{-2\ell}t^2\mathscr{L})\Theta(2^{-2(j+\ell)}t^2\mathscr{L})}(y,z)\right| \\
&\qquad \leq C\|\Psi(t^2\cdot)\|_{(m+N)}\|\lambda \mapsto (t^2\lambda)^{-N}\Theta(t^2\lambda)\|_{(m)}2^{-2jN}D_{2^{-\ell},m-2d}(y,z).
\end{aligned}$$

Obviously, for fixed N and m, there is a constant C depending on Ψ, Θ, m and N such that

$$\sup_{t\in[1,2]} \|\Psi(t^2\cdot)\|_{(m+N)}\|\lambda \mapsto (t^2\lambda)^{-N}\Theta(t^2\lambda)\|_{(m)} \leq C.$$

Hence

$$\left|K_{\Psi(2^{-2\ell}t^2\mathscr{L})\Theta(2^{-2(j+\ell)}t^2\mathscr{L})}(y,z)\right| \leq C2^{-2jN}D_{2^{-\ell},m-2d}(y,z), \tag{5.8}$$

where the constant C depends on Ψ, Θ, m and N, but is independent of $\ell \in \mathbb{Z}$, $j \in \{1, 2, \cdots\}$ and $t \in [1, 2]$. Analogously we have

$$\left|K_{\Phi(2^{-2\ell}t^2\mathscr{L})\Gamma(2^{-2\ell}t^2\mathscr{L})}(y,z)\right| \leq CD_{2^{-\ell},m-2d}(y,z). \tag{5.9}$$

Inserting (5.8) and (5.9) into (5.7) and using (2.3), we get

$$\begin{aligned}
|\Psi(2^{-2\ell}t^2\mathscr{L})f(y)| &\leq C\sum_{j=0}^{\infty} 2^{-2jN}\int_X D_{2^{-\ell},m-2d}(y,z)|\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|d\mu(z) \\
&\leq C\sum_{j=0}^{\infty} 2^{-2jN}\int_X \frac{|\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|}{V(z,2^{-\ell})(1+2^\ell\rho(y,z))^{m-5d/2}}d\mu(z) \\
&\leq C\sum_{j=\ell}^{\infty} 2^{-2(j-\ell)N}\int_X \frac{|\Psi(2^{-2j}t^2\mathscr{L})f(z)|}{V(z,2^{-\ell})(1+2^\ell\rho(y,z))^{a}}d\mu(z).
\end{aligned} \tag{5.10}$$

Let $r \in (0, 1]$. If we divide both sides of (5.10) by $(1 + 2^\ell t^{-1}\rho(x, y))^a$, in the left-hand side take the essential supremum over $y \in X$, in the right-hand use the following inequalities:

$$\begin{gathered}
(1+2^\ell t^{-1}\rho(x,y)) \sim (1+2^\ell\rho(x,y)), \\
(1+2^\ell\rho(x,y))(1+2^\ell\rho(y,z)) \geq (1+2^\ell\rho(x,z)), \\
|\Psi(2^{-2j}t^2\mathscr{L})f(z)| \leq |\Psi(2^{-2j}t^2\mathscr{L})f(z)|^r\left[M^*_{a,\mathscr{L}}(f)(x,2^{-j}t)\right]^{1-r}(1+2^jt^{-1}\rho(x,z))^{a(1-r)}, \\
\frac{(1+2^jt^{-1}\rho(x,z))^{a(1-r)}}{(1+2^\ell\rho(x,z))^a} \leq C\frac{2^{(j-\ell)a}}{(1+2^j\rho(x,z))^{ar}},
\end{gathered}$$

we obtain, for all $f \in L^2(X, d\mu)$, all $\ell \in \mathbb{Z}$, all $t \in [1, 2]$, and a.e. $x \in X$, the estimate

$$\begin{aligned}
&M^*_{a,\mathscr{L}}(f)(x,2^{-\ell}t) \\
&\leq C\sum_{j=\ell}^{\infty} 2^{-2(j-\ell)(N-a/2)}\int_X \frac{|\Psi(2^{-2j}t^2\mathscr{L})f(z)|^r}{V(z,2^{-j})(1+2^j\rho(x,z))^{ar}}d\mu(z)\left[M^*_{a,\mathscr{L}}(f)(x,2^{-j}t)\right]^{1-r}.
\end{aligned}$$

Hence, replacing N with $N + \lfloor a/2\rfloor + \lfloor d/2r\rfloor + 2$, and using [68, Lemma 3], we get

$$\begin{aligned}
\left[M^*_{a,\mathscr{L}}(f)(x,2^{-\ell}t)\right]^r &\leq C\sum_{j=\ell}^{\infty} 2^{-2(j-\ell)(N+\lfloor d/2r\rfloor+1)r}\int_X \frac{|\Psi(2^{-2j}t^2\mathscr{L})f(z)|^r}{V(z,2^{-j})(1+2^j\rho(x,z))^{ar}}d\mu(z) \\
&= C\sum_{j=0}^{\infty} 2^{-2j(N+\lfloor d/2r\rfloor+1)r}\int_X \frac{|\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^r}{V(z,2^{-(j+\ell)})(1+2^{j+\ell}\rho(x,z))^{ar}}d\mu(z) \\
&\leq C\sum_{j=0}^{\infty} 2^{-2j(N+\lfloor d/2r\rfloor+1)r}2^{jd}\int_X \frac{|\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{ar}}d\mu(z)
\end{aligned}$$

$$\leq C\sum_{j=0}^{\infty}2^{-2jN}\int_X \frac{|\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(x,z))^{ar}}d\mu(z).$$

This finishes the proof of (5.6) in the case $0<r\leq 1$.

The proof of (5.6) for $r>1$ is much easier. Indeed, from (5.10) (with $N+1$ instead of N, and with $a+(2d+1)/r'$ instead of a, where $1/r+1/r'=1$) it follows that

$$\begin{aligned}
|\Psi(2^{-2\ell}t^2\mathscr{L})f(y)| &\leq C\sum_{j=0}^{\infty}2^{-2j(N+1)}\int_X \frac{|\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|}{V(z,2^{-\ell})(1+2^\ell\rho(y,z))^{a+(2d+1)/r'}}d\mu(z)\\
&\leq C\sum_{j=0}^{\infty}2^{-2j(N+1)}\left(\int_X \frac{|\Psi(2^{-2(j+\ell)}t^2\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(y,z))^{ar}}d\mu(z)\right)^{1/r}\\
&\qquad\times\left(\int \frac{1}{V(z,2^{-\ell})(1+2^\ell\rho(y,z))^{(2d+1)}}d\mu(z)\right)^{1/r'}\\
&\leq C\left(\sum_{j=0}^{\infty}2^{-2jNr}\int \frac{|\Psi(2^{-(j+\ell)}t^2\mathscr{L})f(z)|^r}{V(z,2^{-\ell})(1+2^\ell\rho(y,z))^{ar}}d\mu(z)\right)^{1/r}\left(\sum_{j=0}^{\infty}2^{-2jr'}\right)^{1/r'},
\end{aligned}$$

where we used (3.8) and also applied Hölder's inequality for the integrals and the sums. Dividing both sides by $(1+2^\ell t^{-1}\rho(x,y))^a$, and using that

$$(1+2^\ell t^{-1}\rho(x,y))^{ar}(1+2^\ell\rho(y,z))^{ar}\gtrsim(1+2^\ell\rho(x,z))^{ar},$$

we get the desired estimate. □

Lemma 5.27. *Let $0<p<\infty$ and $a>2d/\min\{p,2\}$. Then for all $f\in L^2(X)$,*

$$\left\|\left(\int_0^\infty\left[M^*_{a,\mathscr{L}}(f)(\cdot,t)\right]^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p(X)}\sim\|f\|_{\dot{F}^{0,\mathscr{L}}_{p,2}(X)}.$$

Proof. Let $\Psi(\lambda):=\lambda e^{-\lambda}$, $\lambda\in\mathbb{R}_{\geq 0}$. Then Ψ satisfies (5.4) and (5.5) with $M=1$, and for all $f\in L^2(X)$, $t>0$ and $x\in X$, we have $[\Psi(t^2\mathscr{L})f]^*_a f(x)=M^*_{a,\mathscr{L}}(f)(x,t)$. Hence the desired conclusion follows from Theorem 5.21. □

Lemma 5.28. *For any $a>0$, there is a constant $C>0$ such that for all $f\in L^2(X)$ and $t\in[1,2]$,*

$$\left\|\left(\int_0^\infty\left[M^*_{a+d/2,\mathscr{L}}(f)(\cdot,t)\right]^2\frac{dt}{t}\right)^{1/2}\right\|_{L^p(X)}\leq C\|G^*_{a,\mathscr{L}}(f)\|_{L^p(X)}.$$

Proof. Let Ψ be the same as in Lemma 5.26. From Lemma 5.26 with $r=2$, we see that for any $a>0$ and any $N\in\mathbb{N}_0$ with $N>a+5d/2$, there is a constant $C>0$ such that for all $f\in L^2(X)$, $\ell\in\mathbb{Z}$, and $t\in[1,2]$,

$$\begin{aligned}
&\left[M^*_{\sigma+d/2,\mathscr{L}}(f)(x,2^{-\ell}t)\right]^2\\
&\qquad\leq C\sum_{j=0}^{\infty}2^{-4jN}\int_X \frac{|\Psi\big(2^{-2(j+\ell)}t^2\mathscr{L}\big)f(z)|^2}{V(z,2^{-\ell}t)(1+2^\ell t^{-1}\rho(x,z))^{2a+d}}d\mu(z)\\
&\qquad\leq C\sum_{j=\ell}^{\infty}2^{-4(j-\ell)N}\int_X \frac{|\Psi\big(2^{-2j}t^2\mathscr{L}\big)f(z)|^2}{V(z,2^{-j}t)(1+2^\ell t^{-1}\rho(x,z))^{2a+d}}d\mu(z)
\end{aligned}$$

$$\leq C \sum_{j=-\infty}^{\infty} 2^{-4|j-\ell|(N-a/2-d/4)} \int_X \frac{|\Psi(2^{-2j}t^2\mathscr{L})f(z)|^2}{(1+2^j t^{-1}\rho(x,z))^{2a}} \frac{d\mu(z)}{V(x,2^{-j}t)},$$

where for the last inequality we used (2.3) and the following inequality:

$$(1+2^j t^{-1}\rho(x,z))^{2a+d} \leq 2^{(j-\ell)(2a+d)}(1+2^\ell t^{-1}\rho(x,z))^{2a+d}, \quad \forall j \geq \ell.$$

Taking the norm $\int_1^2 |\cdot| \frac{dt}{t}$ on both sides, we get

$$\int_1^2 \left[M^*_{a+d/2,\mathscr{L}}(f)(x,2^{-\ell}t)\right]^2 \frac{dt}{t} \lesssim \sum_{j=-\infty}^{\infty} 2^{-4|j-\ell|(M-a/2-d/4)} \int_1^2 \int_X \frac{|\Psi(2^{-2j}t^2\mathscr{L})f(z)|^2}{(1+2^j t^{-1}\rho(x,z))^{2a}} \frac{d\mu(z)}{V(x,2^{-j}t)} \frac{dt}{t}.$$

Choose M such that $M > a/2 + d/4$. Then applying Lemma 3.11 in $L^{p/2}(\ell^1)$ we obtain

$$\left\| \left(\int_0^\infty \left[M^*_{a+d/2,\mathscr{L}}(f)(\cdot,t)\right]^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} = \left\| \left\{ \int_1^2 \left[M^*_{a+d/2,\mathscr{L}}(f)(\cdot,2^{-\ell}t)\right]^2 \frac{dt}{t} \right\}_{\ell=-\infty}^{\infty} \right\|_{L^{p/2}(\ell^1)}^{1/2}$$
$$\lesssim \left\| \left\{ \int_1^2 \int_X \frac{|\Psi(2^{-2j}t^2\mathscr{L})f(z)|^2}{(1+2^j t^{-1}\rho(\cdot,z))^{2a}} \frac{d\mu(z)}{V(\cdot,2^{-j}t)} \frac{dt}{t} \right\}_{j=-\infty}^{\infty} \right\|_{L^{p/2}(\ell^1)}^{1/2} = \left\| G^*_{a,\mathscr{L}}(f) \right\|_{L^p}.$$

This completes the proof. □

We are now ready to give the

Proof of Theorem 5.23. Let $0 < p < \infty$. Fix $a > d/\min\{p,2\}$ and $a' > 2d/\min\{p,2\}$. Then, by Theorem 5.21, Lemma 5.28, Lemmas 5.24, Lemma 5.25, and Lemma 5.27, we have that for all $f \in L^2(X)$,

$$\begin{aligned}
\|f\|_{\dot F^{0,\mathscr{L}}_{p,2}(X)} &\sim \left\| \left(\int_0^\infty |t^2\mathscr{L}e^{-t^2\mathscr{L}}f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(X)} \\
&\leq \left\| \left(\int_0^\infty \left[M^*_{a+d/2,\mathscr{L}}(f)(\cdot,t)\right]^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(X)} \\
&\lesssim \|G^*_{a,\mathscr{L}}(f)\|_{L^p(X)} \lesssim \|S_{\mathscr{L}}(f)\|_{L^p(X)} \\
&\lesssim \left\| \left(\int_0^\infty \left[M^*_{a',\mathscr{L}}(f)(\cdot,t)\right]^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(X)} \sim \|f\|_{\dot F^{0,\mathscr{L}}_{p,2}(X)}.
\end{aligned}$$

Hence $\|f\|_{\dot F^{0,\mathscr{L}}_{p,2}(X)} \sim \|S_{\mathscr{L}}(f)\|_{L^p(X)}$ for all $f \in L^2(X)$. Since $L^2(X) \cap \dot F^{0,\mathscr{L}}_{p,2}(X)$ is dense in $\dot F^{0,\mathscr{L}}_{p,2}(X)$, and $L^2(X) \cap H^p_{\mathscr{L}}(X)$ is dense in $H^p_{\mathscr{L}}(X)$, we have $\dot F^{0,\mathscr{L}}_{p,2}(X) = H^p_{\mathscr{L}}(X)$. □

5.5 Identification of $\dot F^{0,\mathscr{L}}_{p,2}(X)$ with atomic Hardy spaces $H^{p,q,M}_{\mathscr{L}}(X)$

Hofmann *et al.* [46] and Jiang and Yang [56] established the $(p,2,M)$-atomic decomposition for the Hardy spaces $\dot F^{0,\mathscr{L}}_{p,2}(X)$, $0 < p \leq 1$, by following the tent space approach of Coifman *et al.*

[16]. The purpose of this section is to present a (p,q,M)-atomic decomposition of $\dot{F}^{0,\mathscr{L}}_{p,2}(X)$, $0<p\le 1$. Our approach is different from that of [46] and [56]. To achieve our goal, we shall apply the Peetre maximal function characterization of $\dot{F}^{0,\mathscr{L}}_{p,2}(X)$. The main idea of this section comes from [14], [34] and [21].

Let us start by the following definition:

Definition 5.29. For any distribution $f\in\mathcal{S}'_{\mathscr{L}}(X)$ and any $k\in\mathbb{N}_0$, we define $\mathscr{L}^k f$ to be a distribution in $\mathcal{S}'_{\mathscr{L}}(X)$ given by

$$\langle \mathscr{L}^k f,\phi\rangle=\langle f,\mathscr{L}^k\phi\rangle,\quad \forall\phi\in\mathcal{S}_{\mathscr{L}}(X),$$

and we call $\mathscr{L}^k f$ a distribution derivative of f in the sense of $\mathcal{S}'_{\mathscr{L}}(X)$. Also, we say that a distribution $f\in\mathcal{S}'_{\mathscr{L}}(X)$ coincides with a measurable function $h:X\to\mathbb{C}$, if for every $\phi\in\mathcal{S}_{\mathscr{L}}(X)$ the function $h\phi$ lies in $L^1(X)$ and

$$(f,\phi)=\int_X h(x)\phi(x)d\mu(x).$$

If a distribution $f\in\mathcal{S}'_{\mathscr{L}}(X)$ coincides with some measurable function h, we shall consider the pointwise value of f, given naturally by

$$f(x):=h(x),\quad x\in X.$$

If a distribution $f\in\mathcal{S}'_{\mathscr{L}}(X)$ coincides with some function $h\in L^q(X)$, we will write $f\in L^q(X)$, and also set

$$\|f\|_{L^q(X)}:=\|h\|_{L^q(X)}.$$

Now we introduce the notion of (p,q,M)-atoms associated to $\mathscr{L}$.

Definition 5.30. Let $p\in(0,1]$, $q\in(1,\infty]$ and $M\in\mathbb{N}$. A distribution $a\in\mathcal{S}'_{\mathscr{L}}(X)$ is called a (p,q,M)-atom if there exist a distribution $b\in\mathcal{S}'_{\mathscr{L}}(X)$ and a ball $B=B(x_B,r_B)$ such that:

(i) $a=\mathscr{L}^M b$, where $\mathscr{L}^M b$ is the distribution derivative of b in the sense of $\mathcal{S}'_{\mathscr{L}}(X)$;

(ii) for every $m\in\{0,1,\cdots,M\}$, $\mathscr{L}^m b$ coincides with a measurable function on X;

(iii) for every $m\in\{0,1,\cdots,M\}$, $\operatorname{supp}\mathscr{L}^m b\subset B$;

(iv) for every $m\in\{0,1,\cdots,M\}$, $\|\mathscr{L}^m b\|_{L^q(X)}\le r_B^{2(M-m)}[V(x_B,r_B)]^{1/q-1/p}$.

We say that $f=\sum_{j=0}^\infty\gamma_j a_j$ is a (p,q,M)-atomic decomposition (of $f\in\mathcal{S}'_{\mathscr{L}}(X)$) if $\{\gamma_j\}_{j=0}^\infty$ is a sequence of complex scalars with $\sum_{j=0}^\infty|\gamma_j|^p<\infty$, each a_j is a (p,q,M)-atoms, and the sum converges in $\mathcal{S}'_{\mathscr{L}}(X)$. Set

$$H^{p,q,M}_{\mathscr{L}}(X)=\big\{f\in\mathcal{S}'_{\mathscr{L}}(X):\ f\text{ admits a }(p,q,M)\text{-atomic decomposition}\big\}$$

with the quasi-norm given by

$$\|f\|_{H^{p,q,M}_{\mathscr{L}}(X)}=\inf\left\{\left(\sum_{j=1}^\infty|\gamma_i|^p\right)^{1/p}:f=\sum_{j=1}^\infty\gamma_j a_j\text{ is a }(p,q,M)\text{-atomic decomposition}\right\}.$$

The goal of this section is to prove following theorem:

Theorem 5.31. *Suppose $p \in (0,1]$, $q \in (1,\infty)$, $M \in \mathbb{N}$ and $M \geq \lfloor \frac{d}{2p} - \frac{\varsigma}{2} \rfloor + 1$, where ς is the same constant as in* (2.5)*. Then*

$$\dot{F}^{0,\mathscr{L}}_{p,2}(X) = H^{p,q,M}_{\mathscr{L}}(X)$$

with equivalent quasi-norms.

Remark 5.32. If $q_1 < q_2$, then every (p, q_2, M)-atom is also a (p, q_1, M)-atom. Consequently, $H^{p,q_2,M}_{\mathscr{L}}(X) \subset H^{p,q_1,M}_{\mathscr{L}}(X)$, and the inclusion map is continuous.

Lemma 5.33. *Suppose Φ is a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying* (5.3)*, and $q \in (1,\infty)$. Then there exists a constant $C > 0$ such that*

$$\|f\|_{\dot{F}^{0,\mathscr{L}}_{q,2}(X)} \leq C\|f\|_{L^q(X,d\mu)}. \tag{5.11}$$

Proof. This follows immediately from the fact that $\dot{F}^{0,\mathscr{L}}_{q,2}(X) = L^q(X)$ for all $q \in (1,\infty)$, which is proved in [4]. Here we give a different proof. First note that (5.11) is valid for $q = 2$. To see this, we set

$$\Theta(\lambda) := \sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\lambda)\Phi(2^{-2j}\lambda), \quad \lambda \in \mathbb{R}_{\geq 0}.$$

Since Φ satisfies (5.3), we have $|\Phi(\lambda)| \lesssim \lambda^N$ for $\lambda \in (0,1)$, and $|\Phi(\lambda)| \lesssim \lambda^{-N}$ for $\lambda \in (1,\infty)$, where N can be taken to be arbitrarily large. Using these, it is easy to show that $\Theta \in L^\infty(\mathbb{R}_{\geq 0})$. Hence it follows from the spectral theory that for any $f \in L^2(X)$,

$$\begin{aligned}
\|f\|_{\dot{F}^{0,\mathscr{L}}_{q,2}(X)} \sim \int_X \sum_{j=-\infty}^{\infty} |\Phi(2^{-2j}\mathscr{L})f(x)|^2 d\mu(x) &= \left\langle \sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\mathscr{L})\Phi(2^{-2j}\mathscr{L})f, f \right\rangle \\
&\leq \left\| \sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\mathscr{L})\Phi(2^{-2j}\mathscr{L})f \right\|_{L^2(X)} \|f\|_{L^2(X)} \\
&= \|\Theta(\mathscr{L})f\|_{L^2(X)}\|f\|_{L^2(X)} \leq \|\Theta\|_{L^\infty(\mathbb{R}_{\geq 0})}\|f\|^2_{L^2(X)} \lesssim \|f\|^2_{L^2(X)}.
\end{aligned}$$

In order to show that (5.11) is valid for all $q \in (1,\infty)$, by vector-valued singular integral operator theory it suffices to verify that

$$\left\|\{K_{\Phi(2^{-2j}\mathscr{L})}(x,y)\}_{j=-\infty}^{\infty}\right\|_{\ell^2} \lesssim \frac{1}{V(x,\rho(x,y))} \quad \text{for all distinct } x,y \in X,$$

and that for some $\theta > 0$

$$\left\|\{K_{\Phi(2^{-2j}\mathscr{L})}(x,y) - K_{\Phi(2^{-2j}\mathscr{L})}(x,y')\}_{j=-\infty}^{\infty}\right\|_{\ell^2} \lesssim \left(\frac{\rho(y,y')}{\rho(x,y)}\right)^\theta \frac{1}{V(x,\rho(x,y))}$$

whenever $\rho(y,y') \leq \frac{1}{2}\rho(x,y)$. But these can be verified in a standard manner by using (2.14) and (2.15). We omit the details here. □

Now we are ready to give the

Proof of Theorem 5.31. We first show that $H^{p,q,M}_{\mathscr{L}}(X) \subset \dot{F}^{0,\mathscr{L}}_{p,2}(X)$ and the inclusion map is continuous. To do this, it suffices to show that there is a constant C such that for all (p,q,M)-atoms a,

$$\|a\|_{\dot{F}^{0,\mathscr{L}}_{p,2}(X)} \leq C.$$

Let a be a (p,q,M)-atom related to the ball $B=B(x_B,r_B)$, and Ψ be a function in $\mathcal{A}_M(\mathbb{R}_{\geq 0})$ satisfying (i)–(iv) in Lemma 4.7. We write

$$\|a\|_{\dot{F}^{0,\mathscr{L}}_{p,2}(X)} \sim \int_{B(x_B,2r_B)} \left(\sum_{j=-\infty}^{\infty} |\Psi(2^{-2j}\mathscr{L})a(y)|^2\right)^{p/2} d\mu(y)$$
$$+ \int_{X\backslash B(x_B,2r_B)} \left(\sum_{j=-\infty}^{\infty} |\Psi(2^{-2j}\mathscr{L})a(y)|^2\right)^{p/2} d\mu(y) =: I_1 + I_2.$$

Applying Hölder's inequality, Lemma 5.33 and (2.1), we get

$$I_1 \leq \|a\|^p_{\dot{F}^{0,\mathscr{L}}_{q,2}(X)} [V(x_B,2r_B)]^{1-p/q} \lesssim \|a\|^p_{L^q(X)} [V(x_B,r_B)]^{1-p/q} \lesssim 1.$$

To estimate I_2, note that if $y \in X\backslash B(x_B,2r_B)$, $z \in B(x_B,r_B)$, and $\rho(y,z) \leq 2^{-j}$, then (by the triangle inequality) $r_B \leq \rho(y,z) \leq 2^{-j}$. Thus,

$$\rho(y,x_B) \leq \rho(y,z) + \rho(z,x_B) \leq 2^{-j} + r_B \leq 2^{-j+1}.$$

Hence, for $y \in X\backslash B(x_B,2r_B)$, by the support property and size property of $K_{(2^{-2j}\mathscr{L})^M\Psi(2^{-2j}\mathscr{L})}(x,y)$, Hölder's inequality, and (2.5), we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} |\Psi(2^{-2j}\mathscr{L})a(y)|^2 \\
&= \sum_{j\leq -\log_2[\rho(y,x_B)]+1} \left| \int_{B(x_B,r_B)} a(z) K_{\Psi(2^{-2j}\mathscr{L})}(y,z) d\mu(z) \right|^2 \\
&= \sum_{j\leq -\log_2[\rho(y,x_B)]+1} 2^{4jM} \left| \int_{B(x_B,x_B)} b(z) K_{(2^{-2j}\mathscr{L})^M\Psi(2^{-2j}\mathscr{L})}(y,z) d\mu(z) \right|^2 \\
&\lesssim \sum_{j\leq -\log_2[\rho(y,x_B)]+1} 2^{4jM} [V(y,2^{-j})]^{-2} \left(\int_{B(x_B,r_B)} |b(z)| d\mu(z) \right)^2 \\
&\lesssim \sum_{j\leq -\log_2[\rho(y,x_B)]+1} 2^{4jM} [V(y,2^{-j})]^{-2} \|b\|^2_{L^q(X)} [V(x_B,r_B)]^{2/q'} \\
&\lesssim \sum_{j\leq -\log_2[\rho(y,x_B)]+1} 2^{4jM} [V(y,2^{-j})]^{-2} r_B^{4M} [V(x_B,r_B)]^{2/q-2/p} [V(x_B,r_B)]^{2/q'} \\
&\lesssim \sum_{j\leq -\log_2[\rho(y,x_B)]+1} 2^{4jM} [V(x_B,r_B)]^{-2} 2^{2j\varsigma} r_B^{2\varsigma} r_B^{4M} [V(x_B,r_B)]^{2-2/p} \\
&= r_B^{4M+2\varsigma} [V(x_B,r_B)]^{-2/p} \sum_{j\leq -\log_2[\rho(y,r_B)]+1} 2^{j(4M+2\varsigma)} \\
&\lesssim r_B^{4M+2\varsigma} [V(x_B,r_B)]^{-2/p} [\rho(y,x_B)]^{-(4M+2\varsigma)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
I_2 &\leq r_B^{p(4M+2\varsigma)/2} [V(x_B,r_B)]^{-1} \int_{X\backslash B(x,2r_B)} [\rho(y,x_B)]^{-p(4M+2\varsigma)/2} d\mu(y) \\
&= r_B^{p(4M+2\varsigma)/2} [V(x_B,r_B)]^{-1} \sum_{k=0}^{\infty} \int_{B(x_B,2^{k+2}r_B)\backslash B(x_B,2^{k+1}r_B)} [\rho(y,x_B)]^{-p(4M+2\varsigma)/2} d\mu(y)
\end{aligned}$$

$$\lesssim r_B^{p(4M+2\varsigma)/2}[V(x_B,r_B)]^{-1}\sum_{k=0}^{\infty}V(x_B,2^{k+2}r_B)(2^{k+1}r_B)^{-p(4M+2\varsigma)/2}$$
$$\lesssim [V(x_B,r_B)]^{-1}\sum_{k=0}^{\infty}V(x_B,r_B)2^{kd}2^{-kp(4M+2\varsigma)/2}$$
$$\lesssim 1.$$

Here, for the last inequality we used that $M > \frac{d}{2p} - \frac{\varsigma}{2}$.

Next we turn to the proof of $\dot{F}^{0,\mathscr{L}}_{p,2}(X) \subset H^{p,q,M}_{\mathscr{L}}(X)$. Since $L^2(X) \cap \dot{F}^{0,\mathscr{L}}_{p,2}(X)$ is dense in $\dot{F}^{0,\mathscr{L}}_{p,2}(X)$ (cf. Proposition 5.17), it suffices to show that any $f \in L^2(X) \cap \dot{F}^{0,\mathscr{L}}_{p,2}(X)$ admits a (p,q,M)-atomic decomposition.

Choose a function $\Psi \in \mathcal{A}_{M+1}(\mathbb{R}_{\geq 0})$ satisfying the conditions (i)–(iv) in Lemma 4.7 in which M is replaced by $M+1$. Then there exists a function $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that $\operatorname{supp}\Phi \subset [2^{-2}\varepsilon, 2^2\varepsilon]$, $|\Phi(\lambda)| > 0$ for $\lambda \in [2^{-3/2}\varepsilon, 2^{3/2}\varepsilon]$ for some $\varepsilon > 0$, and

$$\sum_{j=-\infty}^{\infty}\Psi(2^{-2j}\lambda)\Phi(2^{-2j}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}_{>0}.$$

Hence it follows by the spectral theorem (cf. [66, Theorem VII.2]) and Lemma 5.4 that for all $f \in L^2(X) \cap \dot{F}^{0,\mathscr{L}}_{p,2}(X)$,

$$f = \sum_{j=-\infty}^{\infty}\Psi(2^{-2j}\mathscr{L})\Phi(2^{-2j}\mathscr{L})f, \tag{5.12}$$

where the sum converges in $L^2(X)$ and hence, in the topology of $\mathcal{S}'_{\mathscr{L}}(X)$. Now define

$$\eta(x) = \left(\sum_{j=-\infty}^{\infty}\left|[\Phi(2^{-2j}\mathscr{L})]^*_a f(x)\right|^2\right)^{1/2},$$

where $a > 2d/p$. For every $k \in \mathbb{Z}$, set

$$\Omega_k = \{x \in X : \eta(x) > 2^k\}$$

and

$$\widetilde{\Omega}_k = \left\{x \in X : M(\chi_{\Omega_k})(x) > \frac{A_2^d}{2\cdot 3^d A_1^d}\right\},$$

where A_1, A_2 are positive numbers same as in Lemma 4.1. Note that by the Hardy-Littlewood maximal theorem

$$\mu(\widetilde{\Omega}_k) \leq C\mu(\Omega_k).$$

For every $k \in \mathbb{Z}$, we also set

$$\mathcal{R}_k = \{Q \in \mathscr{D} : \mu(Q \cap \Omega_k) > \mu(Q)/2, \quad \mu(Q \cap \Omega_{k+1}) \leq \mu(Q)/2\},$$

and denote

$$\mathcal{R}_k^{\max} = \{Q \in \mathcal{R}_k : \text{there is no } Q' \in \mathcal{R}_k \text{ such that } Q' \supset Q\}.$$

For each $Q \in \mathscr{D}$, we set

$$F_Q(x) = \Psi(2^{-2j_Q}\mathscr{L})\left[\chi_Q\Phi(2^{-2j}\mathscr{L})f\right](x) = \int_Q K_{\Psi(2^{-2j_Q}\mathscr{L})}(x,y)\Phi(2^{-2j_Q}\mathscr{L})f(y)d\mu(y).$$

Then by (5.12) we have

$$f=\sum_{j=-\infty}^{\infty}\sum_{Q\in\mathscr{D}_j}F_Q=\sum_{Q\in\mathscr{D}}F_Q, \tag{5.13}$$

the sum converging in $\mathcal{S}'_{\mathscr{L}}(X)$. Since

$$\mathscr{D}=\bigcup_{k=-\infty}^{\infty}\mathcal{R}_k=\bigcup_{k=-\infty}^{\infty}\bigcup_{Q_k^{\max}\in\mathcal{R}_k^{\max}}\bigcup_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}Q,$$

we can rewrite (5.13) as

$$f=\sum_{k=-\infty}^{\infty}\sum_{Q_k^{\max}\in\mathcal{R}_k^{\max}}\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}F_Q=\sum_{k=-\infty}^{\infty}\sum_{Q_k^{\max}\in\mathcal{R}_k^{\max}}\gamma_{Q_k^{\max}}\cdot a_{Q_k^{\max}},$$

where

$$\gamma_{Q_k^{\max}}:=\widetilde{C}\big[\mu\big(B_k^{\max}\big)\big]^{1/p-1/q}\left(\int_X\left(\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}\chi_Q(x)|\Phi(2^{-2j_Q}\mathscr{L})f(x)|^2\right)^{q/2}d\mu(x)\right)^{1/q},$$

$$a_{Q_k^{\max}}:=\frac{1}{\gamma_{Q_k^{\max}}}\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}F_Q,$$

and the ball $B_k^{\max}$ is defined by

$$B_k^{\max}=B\big(x_{B_k^{\max}},r_{B_k^{\max}}\big):=B\big(z_{Q_k^{\max}},(A_1+1)2^{-j_{Q_k^{\max}}}\big).$$

Here, $z_{Q_k^{\max}}$ denote the "center" of the dyadic cube $Q_k^{\max}$. We claim that, if the constant $\widetilde{C}$ is suitably chosen, then $a_{Q_k^{\max}}$ is a (p,q,M)-atom. To see this, set

$$\begin{aligned}b_{Q_k^{\max}}(x)&=\frac{1}{\gamma_{Q_k^{\max}}}\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}2^{-2Mj_Q}(2^{-2j_Q}\mathscr{L})^{-M}\Psi(2^{-2j_Q}\mathscr{L})\big[\chi_Q\Phi(2^{-2j_Q}\mathscr{L})f\big](x)\\&=\frac{1}{\gamma_{Q_k^{\max}}}\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}2^{-2Mj_Q}\int_Q K_{(2^{-2j_Q}\mathscr{L})^{-M}\Psi(2^{-2j_Q}\mathscr{L})}(x,y)\Phi(2^{-2j_Q}\mathscr{L})f(y)d\mu(y).\end{aligned}$$

Note that $b_{Q_k^{\max}}$ is well-defined since $\Psi\in\mathcal{A}_{M+1}(\mathbb{R}_{\geq 0})$. Observe that $a_{Q_k^{\max}}=\mathscr{L}^M b_{Q_k^{\max}}$ in $\mathcal{S}'_{\mathscr{L}}(X)$. For every integer $m\in\{0,1,\cdots,M\}$, the distribution derivative $\mathscr{L}^m b_{Q_k^{\max}}$ coincides with the function

$$\mathscr{L}^m b_{Q_k^{\max}}(x)=\frac{1}{\gamma_{Q_k^{\max}}}\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}2^{-2(M-m)j_Q}\int_Q K_{(2^{-2j_Q}\mathscr{L})^{m-M}\Psi(2^{-2j_Q}\mathscr{L})}(x,y)\Phi(2^{-2j_Q}\mathscr{L})f(y)d\mu(y).$$

From Lemma 4.1 and the support property of the kernel $K_{(2^{-2j_Q}\mathscr{L})^{m-M}\Psi(2^{-2j_Q}\mathscr{L})}(\cdot,\cdot)$ it follows that

$$\operatorname{supp}\mathscr{L}^m b_{Q_k^{\max}}\subset B_k^{\max},\quad\forall m\in\{0,1,\cdots,M\}.$$

To see the size condition of $\mathscr{L}^m b_{Q_k^{\max}}$, note that by Hölder's inequality, for any $h \in L^{q'}(X)$ satisfying $\|h\|_{L^{q'}(X)} \le 1$ we have

$$\begin{aligned}
&\left|\int_X \mathscr{L}^m b_{Q_k^{\max}}(x)\overline{h(x)}d\mu(x)\right| \\
&= \frac{1}{\gamma_{Q_k^{\max}}}\left|\int_X \left\{\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}} 2^{-2(M-m)j_Q}(2^{-2j_Q}\mathscr{L})^{m-M}\Psi(2^{-2j_Q}\mathscr{L})\big[\chi_Q\Phi(2^{-2j_Q}\mathscr{L})f\big](x)\right\}\overline{h(x)}d\mu(x)\right| \\
&= \frac{1}{\gamma_{Q_k^{\max}}}\left|\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}} 2^{-2(M-m)j_Q}\int_X \big\{(2^{-2j_Q}\mathscr{L})^{m-M}\Psi(2^{-2j_Q}\mathscr{L})\big[\chi_Q\Phi(2^{-2j_Q}\mathscr{L})f\big](x)\big\}\overline{h(x)}d\mu(x)\right| \\
&= \frac{1}{\gamma_{Q_k^{\max}}}\left|\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}} 2^{-2(M-m)j_Q}\int_X \big[\chi_Q\Phi(2^{-2j_Q}\mathscr{L})f\big](x)\overline{\big[\chi_Q(2^{-2j_Q}\mathscr{L})^{m-M}\Psi(2^{-2j_Q}\mathscr{L})h\big](x)}\mu(x)\right| \\
&\le \frac{1}{\gamma_{Q_k^{\max}}}2^{-2(M-m)j_{Q_k^{\max}}}\int_X\left(\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}\chi_Q(x)\big|\Phi(2^{-2j_Q}\mathscr{L})f(x)\big|^2\right)^{1/2} \\
&\qquad\times\left(\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}\chi_Q(x)\big|(2^{-2j_Q}\mathscr{L})^{m-M}\Psi(2^{-2j_Q}\mathscr{L})h(x)\big|^2\right)^{1/2} d\mu(x) \\
&\le \frac{1}{\gamma_{Q_k^{\max}}}2^{-2(M-m)j_{Q_k^{\max}}}\left\{\int_X\left(\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}\chi_Q(x)\big|\Phi(2^{-2j_Q}\mathscr{L})f(x)\big|^2\right)^{q/2}d\mu(x)\right\}^{1/q} \\
&\qquad\times\left\{\int_X\left(\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}\chi_Q(x)\big|(2^{-2j_Q}\mathscr{L})^{m-M}\Psi(2^{-2j_Q}\mathscr{L})h(x)\big|^2\right)^{q'/2}d\mu(x)\right\}^{1/q'} \\
&\le \frac{1}{\gamma_{Q_k^{\max}}}2^{-2(M-m)j_{Q_k^{\max}}}\left\{\int_X\left(\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}\chi_Q(x)\big|\Phi(2^{-2j_Q}\mathscr{L})f(x)\big|^2\right)^{q/2}d\mu(x)\right\}^{1/q} \\
&\qquad\times\left\{\int_X\left(\sum_{j=-\infty}^{\infty}\big|(2^{-2j}\mathscr{L})^{m-M}\Psi(2^{-2j}\mathscr{L})h(x)\big|^2\right)^{q'/2}d\mu(x)\right\}^{1/q'} \\
&\le \widetilde{C}(A_1+1)^{-2(M-m)}r_{B_k^{\max}}^{2(M-m)}\big[\mu\big(B_k^{\max}\big)\big]^{1/q-1/p}\|g_{\Theta,\mathscr{L}}(h)\|_{L^{q'}(X)} \\
&\le r_{B_k^{\max}}^{2(M-m)}\big[\mu\big(B_k^{\max}\big)\big]^{1/q-1/p},
\end{aligned}$$

where we have set $\Theta(\lambda) := \lambda^{m-M}\Psi(\lambda)$ and $C := (A_1+1)^{2(M-m)}\|g_{\Theta,\mathscr{L}}\|^{-1}_{L^{q'}(X)\to L^{q'}(X)}$, with $g_{\Theta,\mathscr{L}}$ the Littlewood-Paley function defined by

$$g_{\Theta,\mathscr{L}}f(x) := \left(\sum_{j=-\infty}^{\infty}|\Psi(2^{-2j}\mathscr{L})f(x)|^2\right)^{1/2}.$$

Here, the $L^{q'}(X)$-boundedness of the operator $g_{\Theta,\mathscr{L}}$ follows from Lemma 5.33 and the fact that $\Theta\in\mathcal{A}_1(\mathbb{R}_{\geq 0})$. Therefore, each $a_{Q_k^{\max}}$ is a (p,q,M)-atom related to the ball $B_k^{\max}$.

Now we claim that

$$\int_X\left(\sum_{Q\in\mathcal{R}_k}\chi_Q(x)|\Phi(2^{-2j_Q}\mathscr{L})f(x)|^2\right)^{q/2}d\mu(x)\lesssim 2^{qk}\mu(\Omega_k). \tag{5.14}$$

Assume this for a moment. Then Hölder's inequality applied to the sum yields

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}\sum_{Q_k^{\max}\in\mathcal{R}_k^{\max}}|\gamma_{Q_k^{\max}}|^p &\sim \sum_{k=-\infty}^{\infty}\sum_{Q_k^{\max}\in\mathcal{R}_k^{\max}}\left[\mu\big(B_k^{\max}\big)\right]^{1-p/q}\\
&\quad\times\left(\int_X\left(\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}\chi_Q(x)|\Phi(2^{-2j_Q}\mathscr{L})f(x)|^2\right)^{q/2}d\mu(x)\right)^{p/q}\\
&\lesssim \sum_{k=-\infty}^{\infty}\left(\sum_{Q_k^{\max}\in\mathcal{R}_k^{\max}}\mu\big(Q_k^{\max}\big)\right)^{1-p/q}\\
&\quad\times\left(\sum_{Q_k^{\max}\in\mathcal{R}_k^{\max}}\int_X\left(\sum_{\substack{Q\in\mathcal{R}_k\\ Q\subset Q_k^{\max}}}\chi_Q(x)|\Phi(2^{-2j_Q}\mathscr{L})f(x)|^2\right)^{q/2}d\mu(x)\right)^{p/q}\\
&= \sum_{k=-\infty}^{\infty}\left(\sum_{Q_k^{\max}\in\mathcal{R}_k^{\max}}\mu\big(Q_k^{\max}\big)\right)^{1-p/q}\\
&\quad\times\left(\int_X\left(\sum_{Q\in\mathcal{R}_k}\chi_Q(x)|\Phi(2^{-2j_Q}\mathscr{L})f(x)|^2\right)^{q/2}d\mu(x)\right)^{p/q}\\
&\lesssim \sum_{k=-\infty}^{\infty}\left(\sum_{Q_k^{\max}\in\mathcal{R}_k^{\max}}\mu\big(Q_k^{\max}\cap\Omega_k\big)\right)^{1-p/q}\left[2^{qk}\mu(\Omega_k)\right]^{p/q}\\
&\lesssim \sum_{k=-\infty}^{\infty}\left[\mu(\Omega_k)\right]^{1-p/q}\left[2^{qk}\mu(\Omega_k)\right]^{p/q}\\
&\leq \sum_{k=-\infty}^{\infty}2^{kp}\mu(\Omega_k)\\
&\leq \|\eta\|^p_{L^p(X)}\\
&\sim \|f\|^p_{\dot F^{0,\mathscr{L}}_{p,2}(X)}.
\end{aligned}$$

Here, for the last line we used the Peetre maximal function characterization of $\dot F^{0,\mathscr{L}}_{p,2}(X)$.

It thus remains to show (5.14). Note that

$$\bigcup_{Q\in\mathcal{R}_k} Q \subset \widetilde{\Omega}_k. \tag{5.15}$$

Indeed, for any $Q\in\mathcal{R}_k$ and for any $x\in Q$, by Lemma 4.1 and (2.2) we have

$$\begin{aligned} M_{HL}(\chi_{\Omega_k})(x) &\geq \frac{1}{\mu\big(B(x,2A_1 2^{-j_Q})\big)}\int_{B(x,2A_1 2^{-j_Q})}\chi_{\Omega_k}(y)d\mu(y)\\ &\geq \frac{1}{\mu\big(B(z_Q,3A_1 2^{-j_Q})\big)}\int_{B(z_Q,A_1 2^{-j_Q})}\chi_{\Omega_k}(y)d\mu(y)\\ &\geq \frac{\mu(Q\cap\Omega_k)}{(3A_1A_2^{-1})^d\mu(Q)}\geq \frac{A_2^d}{2\cdot 3^d A_1^d}. \end{aligned}$$

Hence $Q\subset\widetilde{\Omega}_k$. We also note that for all $Q\in\mathcal{R}_k$ and all $x\in Q$ the following inequality holds:

$$M_{HL}\big(\chi_{Q\cap\widetilde{\Omega}_k\backslash\Omega_{k+1}}\big)(x)\gtrsim 1\geq \chi_Q(x). \tag{5.16}$$

Indeed, by the fact that $Q\subset\widetilde{\Omega}_k$ we have

$$M_{HL}\big(\chi_{Q\cap\widetilde{\Omega}_k\backslash\Omega_{k+1}}\big)(x)\gtrsim \frac{1}{\mu(Q)}\int_Q \chi_{Q\cap\widetilde{\Omega}_k\backslash\Omega_{k+1}}(y)d\mu(y)\geq \frac{\mu(Q)-\frac{\mu(Q)}{2}}{\mu(Q)}\sim 1\geq\chi_Q(x).$$

From (5.15), (5.16), the Fefferman-Stein vector-valued inequality (cf. [39]), and the fact that $\mu(\widetilde{\Omega}_k)\leq C\mu(\Omega_k)$, it follows that

$$\begin{aligned} &\int_X\Bigg(\sum_{Q\in\mathcal{R}_k}\chi_Q(x)|\Phi(2^{-2j_Q}\mathscr{L})f(x)|^2\Bigg)^{q/2}d\mu(x)\\ &\quad\lesssim \int_X\Bigg(\sum_{Q\in\mathcal{R}_k}\sup_{w\in Q}|\Phi(2^{-2j_Q}\mathscr{L})f(w)|^2\big[M_{HL}(\chi_{Q\cap\widetilde{\Omega}_k\backslash\Omega_{k+1}})(x)\big]^2\Bigg)^{q/2}d\mu(x)\\ &\quad\lesssim \int_X\Bigg(\sum_{Q\in\mathcal{R}_k}\sup_{w\in Q}|\Phi(2^{-2j_Q}\mathscr{L})f(w)|^2\chi_{Q\cap\widetilde{\Omega}_k\backslash\Omega_{k+1}}(x)\Bigg)^{q/2}d\mu(x)\\ &\quad= \int_{\widetilde{\Omega}_k\backslash\Omega_{k+1}}\Bigg(\sum_{Q\in\mathcal{R}_k}\sup_{w\in Q}|\Phi(2^{-2j_Q}\mathscr{L})f(w)|^2\chi_Q(x)\Bigg)^{q/2}d\mu(x)\\ &\quad\leq \int_{\widetilde{\Omega}_k\backslash\Omega_{k+1}}\Bigg(\sum_{j=-\infty}^{\infty}\sum_{Q\in\mathscr{D}_j}\sup_{w\in Q}|\Phi(2^{-2j_Q}\mathscr{L})f(w)|^2\chi_Q(x)\Bigg)^{q/2}d\mu(x)\\ &\quad\lesssim \int_{\widetilde{\Omega}_k\backslash\Omega_{k+1}}\Bigg(\sum_{j=-\infty}^{\infty}\big|\big[\Phi(2^{-2j}\mathscr{L})\big]^*_\sigma f(x)\big|^2\Bigg)^{q/2}d\mu(x)\\ &\quad\lesssim 2^{qk}\mu(\Omega_k). \end{aligned}$$

This verifies (5.14) and completes the proof of Theorem 5.31. □

Chapter 6

Applications to stratified Lie groups

6.1 Preliminaries on stratified Lie groups

In this section we briefly review the basic notions concerning stratified Lie groups and their associated sub-Laplacians. For more details we refer the reader to the monograph by Folland and Stein [34]. A Lie group G is called a *stratified Lie group* if it is connected and simply connected, and its Lie algebra $\mathfrak{g}$ may be decomposed as a direct sum $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$, with $[V_1, V_k] = V_{k+1}$ for $1 \leq k \leq m-1$ and $[V_1, V_m] = 0$. Such a group G is clearly nilpotent, and thus it may be identified with $\mathfrak{g}$ (as a manifold) via the exponential map $\exp : \mathfrak{g} \to G$. Examples of stratified Lie groups include Euclidean spaces $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}^n$.

The algebra $\mathfrak{g}$ is equipped with a family of *dilations* $\{\delta_t : t > 0\}$ which are the algebra automorphisms defined by

$$\delta_t\left(\sum_{j=1}^{m} X_j\right) = \sum_{j=1}^{m} t^j X_j \qquad (X_j \in V_j).$$

Under our identification of G with $\mathfrak{g}$, δ_t may also be viewed as a map $G \to G$. We generally write tx instead of $\delta_t(x)$, for $x \in G$. We shall denote by

$$\kappa = \sum_{j=1}^{m} j[\dim(V_j)]$$

the *homogeneous dimension* of G.

A *homogeneous norm* on G is a continuous function $x \mapsto |x|$ from G to $\mathbb{R}_{\geq 0}$ smooth away from 0 (the group identity), vanishing only at 0, and satisfying $|x^{-1}| = |x|$ and $|tx| = t|x|$ for all $x \in G$ and $t > 0$. Homogeneous norms on G always exist and any two of them are equivalent. We assume G is provided with a fixed homogeneous norm. It satisfies a triangle inequality: there exists a constant $\gamma \geq 1$ such that $|xy| \leq \gamma(|x| + |y|)$ for all $x, y \in G$. If $x \in G$ and $r > 0$ we define the *ball of radius r about x* by $B(x, r) = \{y \in G : |y^{-1}x| < r\}$. The Lebesgue measure on $\mathfrak{g}$ induces a bi-invariant Haar measure dx on G. We fix the normalization of Haar measure by requiring that the measure of $B(0, 1)$ be 1. We shall denote the measure of any measurable $E \subset G$ by $|E|$. Clearly we have $|\delta_t(E)| = t^\kappa |E|$. Obviously, $(G, |\cdot|, dx)$ satisfies the doubling, reverse doubling, and non-collapsing conditions. All integrals on G are with respect to (the normalization of) Haar measure. Convolution is defined by

$$f * g(x) = \int f(y)g(y^{-1}x)dy = \int f(xy^{-1})g(y)dy.$$

We consider $\mathfrak{g}$ as the Lie algebra of all left-invariant vector fields on G, and let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$, obtained as a union of bases of the V_j's. In particular, $X_1, \cdots, X_\nu$, with $\nu = \dim(V_1)$, is a basis of V_1. We denote by $Y_1, \cdots, Y_n$ the corresponding basis for right-invariant vector fields, i.e.

$$Y_j f(x) = \frac{d}{dt} f(\exp(tX_j)x)|_{t=0}.$$

If $I = (i_1, \cdots, i_n) \in \mathbb{N}^n$ is a multi-index we set $X^I = X_1^{i_1} \cdots X_n^{i_n}$ and $Y^I = Y_1^{i_1} \cdots Y_n^{i_n}$. Moreover, we set

$$|I| = \sum_{k=1}^{n} i_k \quad \text{and} \quad d(I) = \sum_{k=1}^{n} d_k i_k,$$

where the integers $d_1 \leq \cdots \leq d_n$ are given according to that $X_k \in V_{d_k}$. Then X^I (resp. Y^I) is a left-invariant (resp. right-invariant) differential operator, homogeneous of degree $d(I)$, with respect to the dilations δ_t, $t > 0$.

A complex-valued function P on G is called a *polynomial* on G if $P \circ \exp$ is a polynomial on $\mathfrak{g}$. Let $\xi_1, \cdots, \xi_n$ be the basis for the linear forms on $\mathfrak{g}$ dual to the basis $X_1, \cdots, X_n$ for $\mathfrak{g}$, and set $\eta_j = \xi_j \circ \exp^{-1}$. From our definition of polynomials on G, $\eta_1, \cdots, \eta_n$ are generators of the algebra of polynomials on G. Thus, every polynomial on G can be written uniquely as

$$P = \sum_I a_I \eta^I, \quad a_I \in \mathbb{C}, \tag{6.1}$$

where all but finitely many of the coiefficients vanish, and $\eta^I = \eta^{i_1} \cdots \eta^{i_n}$. A polynomial of the type (6.1) is called *of homogeneous degree* L, where $L \in \mathbb{N}_0$, if $d(I) \leq L$ holds for all multi-indices I with $a_I \neq 0$. We let $\mathcal{P}$ denote the space of all polynomials on G, and let $\mathcal{P}_L$ denote the space of polynomials on G of homogeneous degree L. Obviously, the definition of $\mathcal{P}_L$ is independent of the choice of the basis $X_1, \cdots, X_n$, as long as this basis is obtained as a union of bases of the V_j's. Also note that $\mathcal{P}_L$ is invariant under left and right translations (see [34, Proposition 1.25]). A function $f : G \to \mathbb{C}$ is said to have vanishing moments of order L, if

$$\forall P \in \mathcal{P}_{L-1} : \quad \int_G f(x)P(x)dx = 0,$$

with the absolute convergence of the integral.

The Schwartz class on G is defined by

$$\mathcal{S}(G) := \left\{ \phi \in C^\infty(G) : P \frac{\partial^{|I|}\phi}{\partial \eta^{i_1} \cdots \partial \eta^{i_n}} \in L^\infty(G),\ \forall I \in \mathbb{N}_0^n,\ \forall P \in \mathcal{P} \right\};$$

that is, $\phi \in \mathcal{S}(G)$ if and only if $\phi \circ \exp$ is a Schwartz function on $\mathfrak{g} \equiv \mathbb{R}^n$. In view of [34, Proposition 1.25] and the remarks following it, we can replace $\frac{\partial^{|I|}}{\partial \eta^{i_1} \cdots \partial \eta^{i_n}}$ by X^I or Y^I in this definition without changing anything. $\mathcal{S}(G)$ is a Fréchet space whose topology is defined by any of a number of families of norms. In the present thesis, for our purpose it will be convenient to use the following family of norms: if $N \in \mathbb{N}_0$, we define

$$\|\phi\|_{(N)} := \sup_{|I| \leq N, x \in G} (1 + |x|)^{\kappa + N + d(I)} |X^I \phi(x)|.$$

The dual space $\mathcal{S}'(G)$ of $\mathcal{S}(G)$ is the space of tempered distributions on G. If $f \in \mathcal{S}'(G)$ and $\phi \in \mathcal{S}(G)$ we shall denote the evaluation of f on ϕ by (f, ϕ).

We use the notation $\mathcal{S}_\infty(G)$ to denote the space of all Schwartz functions on G with vanishing moments of all orders. $\mathcal{S}_\infty(G)$ is a subspace of $\mathcal{S}(G)$, with the relative topology. Since $\mathcal{S}_\infty(G)$ is the intersection of null spaces of a family of tempered distributions, it is a closed subspace. It is shown in [32] that the dual space $\mathcal{S}'_\infty(G)$ can be canonically identified with the factor space $\mathcal{S}'(G)/\mathcal{P}$.

For a basis $\mathbb{X} = \{X_1, \cdots, X_n\}$ of $\mathfrak{g}$ chosen as above, we define the sub-Laplacian $\Delta_{\mathbb{X}} := -\sum_{j=1}^{\nu} X_j^2$, where $\nu = \dim(V_1)$. When restricted to smooth functions with compact support, $\Delta_{\mathbb{X}}$ is non-negative and essentially self-adjoint. Its closure has domain $\{u \in L^2(G) : \Delta_{\mathbb{X}} u \in L^2(G)\}$, where $\Delta_{\mathbb{X}} u$ is taken in the sense of distributions. We denote this extension still by the symbol $\Delta_{\mathbb{X}}$. By the spectral theorem, $\Delta_{\mathbb{X}}$ admits a spectral resolution

$$\Delta_{\mathbb{X}} = \int_0^\infty \lambda dE(\lambda),$$

where $dE(\lambda)$ is the projection measure. If Φ is a bounded Borel measurable function on $\mathbb{R}_{\geq 0}$, the operator

$$\Phi(\Delta_{\mathbb{X}}) = \int_0^\infty \Phi(\lambda) dE(\lambda)$$

is bounded on $L^2(G)$, and commutes with left translations. Thus, by the Schwartz kernel theorem, there exists a tempered distribution $K_{\Phi(\Delta_{\mathbb{X}})}$ on G such that

$$\Phi(\Delta_{\mathbb{X}}) f = f * K_{\Phi(\mathbb{X})}, \quad \forall f \in \mathcal{S}(G).$$

An important fact proved by Hulanicki [53] is as in the following lemma.

Lemma 6.1. *If $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ then the distribution kernel $K_{\Phi(\Delta_{\mathbb{X}})}$ of $\Phi(\Delta_{\mathbb{X}})$ coincides with a function in $\mathcal{S}(G)$.*

For any function h on G and $t > 0$, we define the L^1-normalized dilation of h by

$$D_t h(x) = t^\kappa h(tx).$$

Note that 2-homogeneity of $\Delta_{\mathbb{X}}$ implies that the convolution kernel of the operator $\Phi(t^2 \Delta_{\mathbb{X}})$ coincides with $D_{t^{-1}} \phi$, for all $t > 0$.

For any function f on G, we define $\tilde{f}(x) = f(x^{-1})$. Then we have $f * g = \widetilde{\tilde{g} * \tilde{f}}$.

6.2 Besov and Triebel-Lizorkin spaces on stratified groups

Let $\mathbb{X} = \{X_1, \cdots, X_n\}$ be a basis of $\mathfrak{g}$, chosen as above, i.e., $X_1, \cdots, X_n$ is a union of bases of the V_j's. Let $\Delta_{\mathbb{X}}$ be the sub-Laplacian associated to $\mathbb{X}$. It is well-known that the semigroup semigroup $P_t = e^{-t\Delta_{\mathbb{X}}}$ consists of convolution operators with (heat) kernel $p_t(x)$ satisfying the following Gaussian upper bound: for all $x \in G$ and $t > 0$,

$$|p_t(x)| \leq C t^{-\kappa/2} e^{-|x|^2/(ct)},$$

where C and c are positive constants. See for instance [86, Theorem IV.4.2]. Let $B^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$, $\dot{B}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$, $F^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ and $\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ be (inhomogeneous and homogeneous) Besov and Triebel-Lizorkin spaces on G associated to $\Delta_{\mathbb{X}}$, defined according to the general theory established in Chapter 3 and Chapter 5. More precisely, these spaces are defined as follows:

Definition 6.2. (i) Let $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty]$. Let $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ for some nonnegative integer $M > s/2$. Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$. We define the Besov space $B^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ as the collection of all distributions $f \in \mathcal{S}'_{\Delta_{\mathbb{X}}}(G)$ such that

$$\|f\|_{B^{s,\Delta_{\mathbb{X}}}_{p,q}(G)} := \left(\sum_{j=0}^{\infty} \left\| 2^{js}\Phi_j(\Delta_{\mathbb{X}})f \right\|^q_{L^p(G)} \right)^{1/q} < \infty.$$

(ii) Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Let $(\Phi_0, \Phi) \in \mathcal{A}_M(\mathbb{R}_{\geq 0})$ for some nonnegative integer $M > s/2$. Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \geq 1$. We define the Triebel-Lizorkin space $F^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ as the collection of all distributions $f \in \mathcal{S}'_{\Delta_{\mathbb{X}}}(G)$ such that

$$\|f\|_{F^{s,\Delta_{\mathbb{X}}}_{p,q}(G)} := \left\| \left(\sum_{j=0}^{\infty} \left| 2^{js}\Phi_j(\Delta_{\mathbb{X}})f \right|^q \right)^{1/q} \right\|_{L^p(G)} < \infty.$$

Definition 6.3. Let $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\operatorname{supp}\Phi \subset [2^{-2}, 2^2] \quad \text{and} \quad |\Phi(\lambda)| \geq c > 0 \text{ for } \lambda \in [2^{-3/2}, 2^{3/2}]. \tag{6.2}$$

Set $\Phi_j(\lambda) := \Phi(2^{-2j}\lambda)$ for $j \in \mathbb{Z}$.

(i) If $s \in \mathbb{R}$, $p \in (0,\infty]$ and $q \in (0,\infty]$, we define the homogeneous Besov space $\dot{B}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ as the collection of all distributions $f \in \mathcal{S}'_{\infty,\Delta_{\mathbb{X}}}(X)$ such that

$$\|f\|_{\dot{B}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)} := \left(\sum_{j=-\infty}^{\infty} \left\| 2^{js}\Phi_j(\Delta_{\mathbb{X}})f \right\|^q_{L^p(G)} \right)^{1/q} < \infty.$$

(ii) If $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$, we define the homogeneous Triebel-Lizorkin space $\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ as the collection of all distributions $f \in \mathcal{S}'_{\infty,\Delta_{\mathbb{X}}}(G)$ such that

$$\|f\|_{\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)} := \left\| \left(\sum_{j=-\infty}^{\infty} \left| 2^{js}\Phi_j(\Delta_{\mathbb{X}})f \right|^q \right)^{1/q} \right\|_{L^p(G)} < \infty.$$

Lemma 6.4. *Suppose $M \in \mathbb{N}$ and Φ is a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ such that the function $\lambda \mapsto \lambda^{-M}\Phi(\lambda)$ belongs to $\mathcal{S}(\mathbb{R}_{\geq 0})$. Then the convolution kernel of $\Phi(\Delta_{\mathbb{X}})$ is a function in $\mathcal{S}(G)$ having vanishing moments of order $2M$. In particular, if $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ vanishes near the origin, then the convolution kernel of $\Phi(\Delta_{\mathbb{X}})$ has all vanishing moments.*

Proof. Let $P \in \mathcal{P}_{2M-1}$. Then the function $h(x) := \Delta^M_{\mathbb{X}}P(x)$ satisfies $h(tx) = (\Delta^M_{\mathbb{X}}P)(tx) = t^{-2M}\Delta^M_{\mathbb{X}}\big(P(t\cdot)\big)(x) = t^{-2M}\Delta^M_{\mathbb{X}}\big(t^{2M-1}P\big)(x) = t^{-1}(\Delta^M_{\mathbb{X}}P)(x) = t^{-1}h(x)$, which along with the fact that $P \in C^\infty(G)$ implies that $h \equiv 0$. Denoting by ϕ the convolution kernel of $\Phi(\Delta_{\mathbb{X}})$ and by ψ be the convolution kernel of $\Delta^{-M}_{\mathbb{X}}\Phi(\Delta_{\mathbb{X}})$, then $\phi = \Delta^M_{\mathbb{X}}\psi$, and hence

$$\int_G \phi(x)P(x)dx = \int_G (\Delta^M_{\mathbb{X}}\psi)(x)P(x)dx = \int_G \psi(x)(\Delta^M_{\mathbb{X}}P)(x)dx = 0.$$

This shows that ϕ has vanishing moments of order $2M$. □

Lemma 6.5. *Suppose L is a positive integer and ϕ, ψ are functions on G satisfying that*

$$\begin{aligned} |\phi(x)| &\leq C\frac{1}{(1+|x|)^{\kappa+L}} \quad \text{for all } x \in G, \\ \int_G \phi(x)P(x)dx &= 0, \quad \text{for all } P \in \mathcal{P}_{L-1}, \quad \text{and} \\ |Y^I\psi(x)| &\leq C\frac{1}{(1+|x|)^{\kappa+L+d(I)}} \quad \text{for all } x \in G \text{ and } 0 \leq d(I) \leq L. \end{aligned}$$

Then for any $\varepsilon \in (0,1)$, there is a constant $C > 0$ such that for all $j, j' \in \mathbb{Z}$ with $j \geq j'$,

$$|\phi_j * \psi_{j'}(x)| \lesssim 2^{-(j-j')(L-\varepsilon)}\frac{2^{j'\kappa}}{(1+2^{j'}|x|)^{\kappa+L}}.$$

where $\phi_j(x) := (D_{2^j}\phi)(x) = 2^{j\kappa}\phi(2^jx)$ and $\psi_{j'}(x) := (D_{2^{j'}}\psi)(x) = 2^{j'\kappa}\psi(2^{j'}x)$.

Proof. Let $y \mapsto P^{L-1}_{x,\psi_{j'}}(y)$ be the right Taylor polynomial of $\psi_{j'}$ at the point x. By the vanishing moments of ϕ we have

$$\begin{aligned} |\phi_j * \psi_{j'}(x)| =& \left| \int_G \phi_j(y)\left[\psi_{j'}(y^{-1}x) - P^{L-1}_{x,\psi_{j'}}(y^{-1})\right]dy \right| \\ \leq& \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} |\phi_j(y)|\left|\psi_{j'}(y^{-1}x) - P^{L-1}_{x,\psi_{j'}}(y^{-1})\right|dy \\ &+ \int_{|y| > \frac{2^{-j'}+|x|}{2\gamma b}} |\phi_j(y)||\psi_{j'}(y^{-1}x)|dy \\ &+ \int_{|y| > \frac{2^{-j'}+|x|}{2\gamma b}} |\phi_j(y)||P^{L-1}_{x,\psi_{j'}}(y^{-1})|dy \\ =:& I_1 + I_2 + I_3. \end{aligned}$$

By the stratified mean value theorem (cf. [34, p. 33]),

$$\begin{aligned} I_1 &\lesssim \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} |\phi_j(y)||y|^L \sup_{|y'| \leq b|y|, d(I)=L} |(Y^I\psi_{j'})(y'x)|dy \\ &= \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} |\phi_j(y)||y|^L \sup_{|y'| \leq b|y|, d(I)=L} 2^{j'(\kappa+L)}\left|(Y^I\psi)\left(2^{j'}(y'x)\right)\right|dy \\ &\lesssim \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} |\phi_j(y)||y|^L \sup_{|y'| \leq b|y|, d(I)=L} \frac{2^{j'(\kappa+L)}}{(1+2^{j'}|y'x|)^{\kappa+L+l}}dy \\ &\lesssim \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}|y|^L}{(2^{-j}+|y|)^{\kappa+L}} \sup_{|y'| \leq b|y|} \frac{2^{-j'}}{(2^{-j'}+|y'x|)^{\kappa+L+1}}dy \\ &\lesssim \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}|y|^L}{(2^{-j}+|y|)^{\kappa+L}} \frac{2^{-j'}}{(2^{-j'}+|x|)^{\kappa+L+1}}dy \\ &= \frac{2^{-j'}}{(2^{-j'}+|x|)^{\kappa+L}} \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j}+|y|)^{\kappa+L}} \frac{|y|^L}{2^{-j'}+|x|}dy \\ &\leq \frac{2^{-j'}}{(2^{-j'}+|x|)^{\kappa+L}} \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j}+|y|)^{\kappa+1}} \frac{|y|}{2^{-j'}+|x|}dy \\ &\lesssim \frac{2^{-j'}}{(2^{-j'}+|x|)^{\kappa+L}} \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j}+|y|)^{\kappa+1}} \left(\frac{|y|}{2^{-j'}+|x|}\right)^{1-\varepsilon} dy \\ &\leq \frac{2^{-j'}}{(2^{-j'}+|x|)^{\kappa+L}} \int_{|y| \leq \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j}+|y|)^{\kappa+1}} \left(\frac{2^{-j}+|y|}{2^{-j'}+|x|}\right)^{1-\varepsilon} dy \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{-j'}}{(2^{-j'}+|x|)^{\kappa+L}} \int_{|y|\le \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j}+|y|)^{\kappa+\varepsilon}} \frac{1}{(2^{-j'}+|x|)^{1-\varepsilon}} dy \\
&\le \frac{2^{-j'}2^{j'(1-\varepsilon)}}{(2^{-j'}+|x|)^{\kappa+L}} \int_{|y|\le \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j}+|y|)^{\kappa+\varepsilon}} dy \\
&= \frac{2^{-j'}2^{j'(1-\varepsilon)}2^{-j(L-\varepsilon)}}{(2^{-j'}+|x|)^{\kappa+L}} \int_{|y|\le \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-j\varepsilon}}{(2^{-j}+|y|)^{\kappa+\varepsilon}} dy \\
&\le \frac{2^{-j'}2^{j'(1-\varepsilon)}2^{-j(L-\varepsilon)}}{(2^{-j'}+|x|)^{\kappa+L}} \int_G \frac{1}{(1+|y|)^{\kappa+\varepsilon}} dy \\
&\lesssim \frac{2^{-j'\varepsilon}2^{-j(L-\varepsilon)}}{(2^{-j'}+|x|)^{\kappa+L}} = 2^{-(j-j')(L-\varepsilon)} \frac{2^{j'\kappa}}{(1+2^{j'}|x|)^{\kappa+L}},
\end{aligned}
$$

where we used the fact that if $|y| \le \frac{2^{-j'}+|x|}{2\gamma b}$ and $|y'| \le b|y|$, then $2^{-j'}+|y'x| \gtrsim 2^{-j'}+|x|$.

For the estimation of I_2, we have

$$
\begin{aligned}
I_2 &\lesssim \int_{|y|> \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j}+|y|)^{\kappa+L}} \frac{2^{-j'L}}{(2^{-j'}+|y^{-1}x|)^{\kappa+L}} dy \\
&\le \int_{|y|> \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j'}+|x|)^{\kappa+L}} \frac{2^{-j'L}}{(2^{-j'}+|y^{-1}x|)^{\kappa+L}} dy \\
&\lesssim \frac{2^{-jL}}{(2^{-j'}+|x|)^{\kappa+L}} \int_G \frac{2^{-j'L}}{(2^{-j'}+|y^{-1}x|)^{\kappa+L}} dy \\
&= \frac{2^{-jL}}{(2^{-j'}+|x|)^{\kappa+L}} \int_G \frac{1}{(1+|y|)^{\kappa+L}} dy \\
&\lesssim \frac{2^{-jL}}{(2^{-j'}+|x|)^{\kappa+L}} = 2^{-(j-j')L} \frac{2^{j'\kappa}}{(1+2^{j'}|x|)^{\kappa+L}}.
\end{aligned}
$$

To estimate I_3, we note that by [6, Proposition 20.3.14] $P^{L-1}_{x,\psi_{j'}}$ is of the form

$$
P_{x,\psi_j}(y) = \psi_j(x) + \sum_{\ell=1}^{L-1} \sum_{k=1}^{\ell} \sum_{\substack{1\le i_1,\cdots,i_k\le n \\ d_{i_1}+\cdots+d_{i_k}=\ell}} \frac{\eta_{i_1}(y)\cdots\eta_{i_k}(y)}{k!} Y_{i_1}\cdots Y_{i_k}\psi_j(x),
$$

where the integers d_{i_k} are given by $d_{i_k} := \{\beta : X_{i_k} \in V_\beta\}$. Hence

$$
\begin{aligned}
I_3 &\lesssim \int_{|y|> \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-jL}}{(2^{-j}+|y|)^{\kappa+L}} \left(\sum_{0\le \ell\le L-1} \frac{2^{j'(\kappa+\ell)}|y|^\ell}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \right) dy \\
&\le \sum_{0\le \ell\le L-1} \frac{2^{-jL}2^{j'(\kappa+\ell)}}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \int_{|y|> \frac{2^{-j'}+|x|}{2\gamma b}} \frac{|y|^\ell}{(2^{-j}+|y|)^{\kappa+L}} dy \\
&\le \sum_{0\le \ell\le L-1} \frac{2^{-jL}2^{j'(\kappa+\ell)}}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \int_{|y|> \frac{2^{-j'}+|x|}{2\gamma b}} \frac{1}{(2^{-j}+|y|)^{\kappa+L-\ell}} dy \\
&= \sum_{0\le \ell\le L-1} \frac{2^{-jL}2^{j'(\kappa+\ell)}2^{j'(L-\ell-\varepsilon)}}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \int_{|y|> \frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-j'(L-\ell-\varepsilon)}}{(2^{-j}+|y|)^{\kappa+L-\ell}} dy \\
&\le \sum_{0\le \ell\le L-1} \frac{2^{-jL}2^{j'(\kappa+\ell)}2^{j'(L-\ell-\varepsilon)}}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \int_{|y|> \frac{2^{-j'}+|x|}{2\gamma b}} \frac{|y|^{(L-\ell-\varepsilon)}}{(2^{-j}+|y|)^{\kappa+L-\ell}} dy
\end{aligned}
$$

$$\begin{aligned}
&\leq \sum_{0\leq \ell \leq L-1} \frac{2^{-jL}2^{j'(\kappa+\ell)}2^{j'(L-\ell-\varepsilon)}}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \int_{|y|>\frac{2^{-j'}+|x|}{2\gamma b}} \frac{1}{(2^{-j}+|y|)^{\kappa+\varepsilon}} dy \\
&= \sum_{0\leq \ell \leq L-1} \frac{2^{-jL}2^{j'(\kappa+\ell)}2^{j'(L-\ell-\varepsilon)}2^{j\varepsilon}}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \int_{|y|>\frac{2^{-j'}+|x|}{2\gamma b}} \frac{2^{-j\varepsilon}}{(2^{-j}+|y|)^{\kappa+\varepsilon}} dy \\
&\leq \sum_{0\leq \ell \leq L-1} \frac{2^{-jL}2^{j'(\kappa+\ell)}2^{j'(L-\ell-\varepsilon)}2^{j\varepsilon}}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \int_{G} \frac{2^{j\kappa}}{(1+|2^{j}y|)^{\kappa+\varepsilon}} dy \\
&\lesssim \sum_{0\leq \ell \leq L-1} \frac{2^{-jL}2^{j'(\kappa+\ell)}2^{j'(L-\ell-\varepsilon)}2^{j\varepsilon}}{(1+|2^{j'}x|)^{\kappa+L+\ell}} \leq \sum_{0\leq \ell \leq L-1} \frac{2^{-(j-j')(L-\varepsilon)}}{(1+2^{j'}|x|)^{\kappa+L+\ell}} \\
&\lesssim 2^{-(j-j')(L-\varepsilon)} \frac{2^{j'\kappa}}{(1+2^{j'}|x|)^{\kappa+L}}.
\end{aligned}$$

This finishes the proof. □

Proposition 6.6. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty)$. Then:*

(i) *$\mathcal{S}(G)$ is a dense subspace of $B^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ and is a dense subspace of $F^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$.*

(ii) *$\mathcal{S}_\infty(G)$ is a dense subspace of $\dot{B}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ and is a dense subspace of $\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$.*

Proof. We only prove that $\mathcal{S}_\infty(G)$ is a dense subspace of $\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$, since other statements can be proved similarly.

We first show that $\mathcal{S}_\infty(G) \subset \dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$. Let $g \in \mathcal{S}_\infty(G)$ and let Φ be a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying (6.2). Let $\phi \in \mathcal{S}(G)$ be the convolution kernel of $\Phi(\Delta_{\mathbb{X}})$. Choose an integer L such that

$$L > \max\big\{(\kappa+1)/p - \kappa, (\kappa+1)/p - \kappa - s, -s, 0\big\}.$$

Then there exists sufficiently small $\varepsilon > 0$ such that

$$L - \varepsilon > \max\big\{(\kappa+1)/p - \kappa - s, -s, 0\big\}.$$

Since both g and ϕ are Schwartz functions with all moments vanishing, it follows from Lemma 6.5 that

$$\begin{aligned}
\forall j \leq 0: \quad & 2^{js}|\Phi(2^{-2j}\Delta_{\mathbb{X}})g(x)| = |g * \phi_j(x)| \lesssim 2^{js}2^{j(L-\varepsilon)} \frac{2^{j\kappa}}{(1+2^j|x|)^{\kappa+L}} \\
&\leq 2^{js}2^{j(L-\varepsilon)} \frac{2^{j\kappa}}{(1+2^j|x|)^{(\kappa+1)/p}} \lesssim 2^{j[L-\varepsilon+\kappa-(\kappa+1)/p+s]} \frac{1}{(1+|x|)^{(\kappa+1)/p}}
\end{aligned}$$

and

$$\begin{aligned}
\forall j > 0: \quad & 2^{js}|\Phi(2^{-2j}\Delta_{\mathbb{X}})g(x)| = 2^{js}|g * \phi_j(x)| = 2^{js}|\tilde{\phi}_j * \tilde{g}(x^{-1})| \\
&\lesssim 2^{-j(L-\varepsilon+s)} \frac{1}{(1+|x|)^{\kappa+L}} \leq 2^{-j(L-\varepsilon+s)} \frac{1}{(1+|x|)^{(\kappa+1)/p}}.
\end{aligned}$$

Hence, if we set

$$\delta := \min\big\{L - \varepsilon + \kappa - (\kappa+1)/p + s, L - \varepsilon + s\big\} > 0,$$

then

$$\|g\|_{\dot F^{s,\Delta_{\mathbb X}}_{p,q}(G)} = \left\| \left(\sum_{j=-\infty}^{\infty} \left(2^{-js}|\Phi(2^{-2j}\Delta_{\mathbb X})f|\right)^q \right)^{1/q} \right\|_{L^p(G)}$$
$$\le C_g \left(\sum_{j=-\infty}^{\infty} 2^{-|j|\delta q} \right)^{1/q} \left(\int_G \frac{1}{(1+|x|)^{\kappa+1}} dx \right)^{1/p} \le C_g.$$

This shows that every function in $\mathcal{S}_\infty(G)$ belongs to $\dot F^{s,\Delta_{\mathbb X}}_{p,q}(G)$.

Next we show that if $s\in\mathbb{R}$, $p\in(0,\infty)$ and $q\in(0,\infty)$ then $\mathcal{S}_\infty(G)$ is dense in $\dot F^{s,\Delta_{\mathbb X}}_{p,q}(G)$. From the proof of Corollary 4.8 we see that it suffices to show that if Φ is a function in $\mathcal{S}(\mathbb{R}_{\ge 0})$ vanishing near the origin and if g is a bounded (not necessarily continuous) function on G with compact support, then $\Phi(\Delta_{\mathbb X})g\in\mathcal{S}_\infty(G)$. For any $I\in\mathbb{N}_0^n$,

$$X^I\big(\Phi(\Delta_{\mathbb X})g\big)(x) = X^I(g*\phi)(x) = g*(X^I\phi)(x),$$

where $\phi\in\mathcal{S}(G)$ is the convolution kernel of $\Phi(\Delta_{\mathbb X})$. Since g is bounded function with compact support, for arbitrarily large positive integer N we have $|g(x)|\le(1+|x|)^{-N}$. Hence

$$|X^I\big(\Phi(\Delta_{\mathbb X})g\big)(x)| \le \int_G |\phi(y)||g(y^{-1}x)|dy \le \int_G \frac{1}{(1+|y|)^N}\frac{1}{(1+|y^{-1}x|)^N}dy$$
$$= \int_{|y|\le|x|/(2\gamma)} + \int_{|y|>|x|/(2\gamma)} =: I_1+I_2.$$

Note that if $|y|\le|x|/(2\gamma)$ then $|y^{-1}x|\ge|x|/\gamma-|y|\ge|x|/(2\gamma)$. Hence

$$I_1 \lesssim \frac{1}{(1+|x|)^N}\int_{|y|\le|x|/(2\gamma)}\frac{1}{(1+|y|)^N}dy \lesssim \frac{1}{(1+|x|)^N}.$$

For I_2, we have

$$I_2 \lesssim \frac{1}{(1+|x|)^N}\int_{|y|>|x|/(2\gamma)}\frac{1}{(1+|y^{-1}x|)^N}dy \lesssim \frac{1}{(1+|x|)^N}.$$

Therefore, $|X^I\big(\Phi(\Delta_{\mathbb X})g\big)(x)|\lesssim(1+|x|)^{-N}$, which shows that $\Phi(\Delta_{\mathbb X})g\in\mathcal{S}_\infty(G)$. It remains to show that $\Phi(\Delta_{\mathbb X})g$ have vanishing moments of arbitrary order. Let L be any non-negative integer, and let $P\in\mathcal{P}_L$. We have

$$\int_G \Phi(\Delta_{\mathbb X})g(x)P(x)dx = \int_G \Delta_{\mathbb X}^{-(L+1)}\Phi(\Delta_{\mathbb X})g(x)\Delta_{\mathbb X}^{L+1}P(x)dx = 0.$$

This completes the proof. □

The following theorem shows that Besov and Triebel-Lizorkin spaces on G are independent of the choice of the sub-Laplacian.

Theorem 6.7. *Suppose $\mathbb{X}=\{X_1,\cdots,X_n\}$ and $\widetilde X=\{\widetilde X_1,\cdots,\widetilde X_n\}$ are two bases of $\mathfrak{g}$, both of which are obtained as unions of the bases of the V_j's.*

(i) *If $s\in\mathbb{R}$, $p\in(0,\infty]$ and $q\in(0,\infty]$, then*

$$B^{s,\Delta_{\mathbb X}}_{p,q}(G) = B^{s,\Delta_{\widetilde{\mathbb X}}}_{p,q}(G) \quad\text{and}\quad \dot B^{s,\Delta_{\mathbb X}}_{p,q}(G) = \dot B^{s,\Delta_{\widetilde{\mathbb X}}}_{p,q}(G).$$

(ii) *If $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty)$, then*

$$F^{s,\Delta_{\mathbb{X}}}_{p,q}(G) = F^{s,\Delta_{\widetilde{\mathbb{X}}}}_{p,q}(G) \quad \textit{and} \quad \dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G) = \dot{F}^{s,\Delta_{\widetilde{\mathbb{X}}}}_{p,q}(G).$$

Proof. We only show that $\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G) = \dot{F}^{s,\Delta_{\widetilde{\mathbb{X}}}}_{p,q}(G)$ since the proofs of other statements are similar. By Proposition 6.6, $L^2(G) \cap \dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ is dense in $\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ and $L^2(G) \cap \dot{F}^{s,\Delta_{\widetilde{\mathbb{X}}}}_{p,q}(G)$ is dense in $\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$. Hence it suffices to show that for all $f \in L^2(G)$,

$$\|f\|_{\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)} \sim \|f\|_{\dot{F}^{s,\Delta_{\widetilde{\mathbb{X}}}}_{p,q}(G)}. \tag{6.3}$$

Now let $f \in L^2(G)$. Let Φ be a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying (6.2). Then there exists $\Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that $\operatorname{supp}\Psi \in [2^{-2}, 2^2]$ and

$$\sum_{j=-\infty}^{\infty} \Psi(2^{-2j}\lambda)\Phi(2^{-2j}\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}_{>0}.$$

From this, the spectral theorem (cf. [66, Theorem VII.2]) and Lemma 5.4 it follows that

$$f = \sum_{j=-\infty}^{\infty} \Psi(2^{-2j}\Delta_{\mathbb{X}})\Phi(2^{-2j}\Delta_{\mathbb{X}})f,$$

where the sum converges in $L^2(G)$. Hence we have the pointwise representation

$$\Phi(2^{-2\ell}\Delta_{\widetilde{X}})f(y) = \sum_{j=-\infty}^{\infty} \Phi(2^{-2\ell}\Delta_{\widetilde{X}})\Psi(2^{-2j}\Delta_{\mathbb{X}})\Phi(2^{-2j}\Delta_{\mathbb{X}})f(y), \quad y \in G.$$

Let ϕ (resp. $\tilde{\phi}$) be the convolution kernel of $\Phi(\Delta_{\mathbb{X}})$ (resp. $\Phi(\Delta_{\widetilde{\mathbb{X}}})$). Let $\phi_j(x) := 2^{j\kappa}\phi(2^j x)$ (resp. $\tilde{\phi}_j(x) = 2^{j\kappa}\tilde{\phi}(2^j x)$) for $j \in \mathbb{Z}$. Then for all $\ell \in \mathbb{Z}$ and $y \in G$,

$$f * \tilde{\phi}_\ell(y) = \sum_{j=-\infty}^{\infty} f * \phi_j * \psi_j * \tilde{\phi}_\ell(y).$$

It follows that

$$\begin{aligned} |f * \tilde{\phi}_\ell(y)| &\leq \sum_{j=-\infty}^{\infty} \int |f * \phi_j(z)||\psi_j * \tilde{\phi}_\ell(z^{-1}y)|dz \\ &\leq \sum_{j=-\infty}^{\infty} [\Phi(2^{-2j}\Delta_{\mathbb{X}})]^*_a f(y) \int_G (1 + 2^j|z|)^a |\psi_j * \tilde{\phi}_\ell(z)|dz \\ &= \sum_{j=-\infty}^{\infty} [\Phi(2^{-2j}\Delta_{\mathbb{X}})]^*_a f(y) I_{j,\ell}, \end{aligned} \tag{6.4}$$

where we have set

$$I_{j,\ell} := \int_G (1 + 2^j|z|)^a |\psi_j * \tilde{\phi}_\ell(z)|dz.$$

Since both $\psi^{(1)}$ and $\phi^{(2)}$ are Schwartz functions with vanishing moments of all orders, it follows from Lemma 3.1 that

$$\begin{aligned} I_{j,\ell} &\lesssim \int_G (1 + 2^j|z|)^a 2^{-|j-\ell|(L-\varepsilon)} 2^{(j\wedge\ell)\kappa}(1 + 2^{j\wedge\ell}|z|)^{-(\kappa+L)}dz \\ &\lesssim \int_G 2^{-|j-\ell|(L-\varepsilon-a)} 2^{(j\wedge\ell)\kappa}(1 + 2^{j\wedge\ell}|z|)^{-(\kappa+L-a)}dz \end{aligned}$$

$$\lesssim 2^{-|j-\ell|(L-\varepsilon-a)},$$

where the positive integer L is taken sufficiently large. Let us further observe that

$$\begin{aligned}[\Phi(2^{-2j}\Delta_{\mathbb{X}})]_a^* f(y) &\le [\Phi(2^{-2j}\Delta_{\mathbb{X}})]_a^* f(x)(1+2^j|y^{-1}x|)^a \\ &\lesssim [\Phi(2^{-2j}\Delta_{\mathbb{X}})]_a^* f(x)(1+2^\ell|y^{-1}x|)^a \max\{1, 2^{(j-\ell)a}\}.\end{aligned}$$

Putting these estimates into (6.4), multiplying both sides by $2^{\ell s}$, dividing both sides by $(1+2^\ell|y^{-1}x|)^a$ and then taking the supremum over $y \in G$, we obtain

$$2^{\ell s}[\Phi(2^{-2\ell}\Delta_{\widetilde{\mathbb{X}}})]_a^* f(x) \lesssim \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|(L-\varepsilon-2a-|s|)} 2^{js}[\Phi(2^{-2j}\Delta_{\mathbb{X}})]_a^* f(x).$$

Take $a > \frac{2a}{\min\{p,q\}}$, $L > 2a+|s|$ and take ε sufficiently small such that $L-\varepsilon-2a-|s| > 0$. Then it follows from Lemma 3.11 that

$$\left\| \left(\sum_{\ell=-\infty}^{\infty} \left| 2^{js}[\Phi(2^{-2\ell}\Delta_{\widetilde{\mathbb{X}}})]_a^* f \right|^q \right)^{1/q} \right\|_{L^p(G)} \lesssim \left\| \left(\sum_{j=-\infty}^{\infty} \left| 2^{js}[\Phi(2^{-2j}\Delta_{\mathbb{X}})]_a^* f \right|^q \right)^{1/q} \right\|_{L^p(G)}. \tag{6.5}$$

By symmetry, the inverse inequality of (6.5) is also valid. This along with Theorem 5.7 yields (6.3). □

Remark 6.8. From Theorem 6.7 we see that the spaces $B^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$, $\dot{B}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$, $F^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ and $\dot{F}^{s,\Delta_{\mathbb{X}}}_{p,q}(G)$ are independent of the choice of the sub-Laplacian $\Delta_{\mathbb{X}}$. Hence in what follows we will not specify the choice of $\Delta_{\mathbb{X}}$ and write $B^s_{p,q}(G)$, $\dot{B}^s_{p,q}(G)$, $F^s_{p,q}(G)$ and $\dot{F}^s_{p,q}(G)$ in short.

6.3 $\dot{B}^s_{p,q}(G)$- and $\dot{F}^s_{p,q}(G)$-boundedness of convolution operators

In this section we study boundedness of convolution operators on homogeneous Besov and Triebel-Lizorkin spaces on stratified Lie groups. Following [75, §5.3 in Chapter XIII], we introduce a class of singular convolution kernels as follows.

Definition 6.9. Let r be a positive integer. A *kernel of order* r is a distribution $K \in \mathcal{S}'(G)$ with the following properties:

(i) K coincides with a C^r function $K(x)$ away from the group identity 0 and enjoys the regularity condition:

$$|X^I K(x)| \le C_I |x|^{-\kappa-d(I)} \quad \text{for } |I| \le r \text{ and } x \ne 0. \tag{6.6}$$

(ii) K satisfies the cancellation condition: For all normalized bump function ϕ and all $R > 0$, we have

$$|\langle K, \phi^R \rangle| \le C, \tag{6.7}$$

where $\phi^R(x) = \phi(Rx)$, and C is a constant independent of ϕ and R. Here, by a normalized bump function we mean a function ϕ supported in $\{|x| < 1\}$ and satisfying

$$\left| X^I \phi(x) \right| \le 1, \quad \forall\ |I| \le N,\ \forall\ x \in G,$$

for some fixed positive integer N.

A convolution operator T with kernel of order r is called a *singular integral operator of order* r.

Remark 6.10. Using [34, Proposition 1.29], it is easy to verify that (6.6) is equivalent to the following condition:

$$|Y^I K(x)| \leq C_I |x|^{-\kappa-d(I)}, \quad \text{for } |I| \leq r \text{ and } x \neq 0.$$

Examples of such kernels include the class of distributions which are *homogeneous of degree* $-\kappa$ (see Folland [34, p. 11] for definition) and agree with C^∞ functions away from 0. Indeed, assume $K \in \mathcal{S}'(G)$ is such a distribution, then it is easy to verify that K satisfies the regularity condition (i) in Definition 4.1; moreover, from [34, Proposition 6.13] we see that K is a principle value distribution such that $\int_{\varepsilon<|x|<L} K(x)dx = 0$ for all $0 < \varepsilon < L < \infty$. Hence, for every normalized bump function ϕ, by the homogeneity of K we have

$$\begin{aligned} |\langle K, \phi^R \rangle| = |\langle K, \phi \rangle| &= \left| \lim_{\varepsilon \to 0} \int_{\varepsilon<|x|<2} K(x)[\phi(x) - \phi(0)]dx \right| \\ &\leq \int_{|x|<2} |K(x)||\phi(x) - \phi(0)|dx. \end{aligned}$$

Using stratified mean value theorem (cf. [34, Theorem 1.41]) and (6.6)–(6.7), it is easy to verify that the last integral converges absolutely and is bounded by a constant independent of ϕ and R. Hence K satisfies the condition (ii) in Definition 6.9.

Now we state the main result of this section.

Theorem 6.11. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q < \infty$, and let r be a positive integer such that $r > \frac{\kappa}{\min\{p,q\}} + |s|$. Suppose T is a singular integral operator of order r. Then T extends to a bounded operator on $\dot{B}^s_{p,q}(G)$ and on $\dot{F}^s_{p,q}(G)$.*

If $K \in \mathcal{S}'(G)$ and $t > 0$, we define $D_t K$ as the tempered distribution given by

$$\langle D_t K, \phi \rangle = \langle K, \phi(t^{-1} \cdot) \rangle, \quad \phi \in \mathcal{S}(G).$$

For the proof of Theorem 6.11, we will need the following lemma, in which b is the same positive constant as in [34, Corollary 1.44].

Lemma 6.12. *Let r be a positive integer. Suppose K is a kernel of order r, and ϕ is a smooth function supported in $B(0, 1/(100\gamma b^r))$ with vanishing moments of order r. Then, there exists a constant $C > 0$ such that for all $j \in \mathbb{Z}$ and $x \in G$, we have*

$$|(D_{2^j} K) * \phi(x)| \leq C(1 + |x|)^{-\kappa-r} \tag{6.8}$$

and

$$|\phi * (D_{2^j} K)(x)| \leq C(1 + |x|)^{-\kappa-r}. \tag{6.9}$$

Moreover, both $\phi * (D_{2^j} K)$ and $(D_{2^j} K) * \phi$ have vanishing moments of the same order as ϕ.

Proof. Recall that the convolution of $\phi \in \mathcal{S}(G)$ with $K \in \mathcal{S}'(G)$ is defined by $\phi * K(x) := \langle K, ({}^x\phi)^\sim \rangle$, where ${}^x\phi$ is the function given by ${}^x\phi(z) = \phi(xz)$, and as before $\tilde{f}(x) := f(x^{-1})$ for any function $f : G \to \mathbb{C}$. From [34, p. 38] we see that $\phi * (D_{2^j} K)$ are C^∞ functions, $j \in \mathbb{Z}$. We claim that for every x with $|x| \leq \frac{1}{2\gamma}$, the function $z \mapsto ({}^x\phi)^\sim(z)$ is a normalized bump function multiplied with a constant independent of x. Indeed, using the quasi-triangle inequality satisfied by the homogeneous norm it is easy to verify that the function $z \mapsto ({}^x\phi)^\sim(z)$ is supported in

$B(0,1)$; moreover, since $|x| \le \frac{1}{2\gamma}$ and since (by [34, Proposition 1.29])

$$Y^I = \sum_{\substack{|J|\le|I|\\ d(J)\ge d(I)}} P_{I,J} X^J,$$

where $P_{I,J}$ are polynomials of homogeneous degree $d(J)-d(I)$, we have

$$\begin{aligned} |X^I[({}^x\phi)^{\sim}](z)| = |Y^I({}^x\phi)(z^{-1})| &\lesssim \sum_{\substack{|J|\le|I|\\ d(J)\ge d(I)}} |P_{I,J}(z^{-1})||X^J({}^x\phi)(z^{-1})| \\ &= \sum_{\substack{|J|\le|I|\\ d(J)\ge d(I)}} |P_{I,J}(z^{-1})||(X^J\phi)(xz^{-1})| \le C_I. \end{aligned}$$

Here C_I is a constant depending on I but not on x. Hence the claim is true. Thus, by the condition (ii) in Definition 6.9, there exits a constant $C>0$ such that for all $j\in\mathbb{Z}$ and all x with $|x|\le\frac{1}{2\gamma}$,

$$\big|\phi * (D_{2^j}K)(x)\big| = \big|\phi * (D_{2^j}K)(x)\big| = \big|\big(D_{2^j}K,\ ({}^x\phi)^{\sim}\big)\big| = \big|\big(K,\ ({}^x\phi)^{\sim}(2^{-j}\cdot)\big)\big| \le C. \tag{6.10}$$

Let now $|x|>\frac{1}{2\gamma}$. Let $y\in\operatorname{supp}\phi$. Denote by $P^{r-1}_{x,D_{2^j}K}$ the right Taylor polynomial of $D_{2^j}K$ at x of homogeneous degree $r-1$ (see [34, pp. 26-27]). Then by the right-invariant version of [34, Corollary 1.44], we have

$$\big|(D_{2^j}K)(y^{-1}x) - P^{r-1}_{x,D_{2^j}K}(y^{-1})\big| \le C|y|^r \sup_{\substack{|z|\le b^r|y|\\ d(I)=r}} |Y^I(D_{2^j}K)(zx)|. \tag{6.11}$$

Observe that for $y\in\operatorname{supp}\phi$ and $|z|\le b^r|y|$ we have $zx\in G\backslash\{0\}$. Thus, for all I with $d(I)=r$ and all z with $|z|\le b^r|y|$, by using (6.6) (with K replaced by $D_{2^j}K$) we have

$$\big|Y^I(D_{2^j}K)(zx)\big| = 2^{jr}\big|(Y^IK)(2^j(zx))\big| \lesssim 2^{j(\kappa+r)}|2^j(zx)|^{-\kappa-r} \lesssim |zx|^{-\kappa-r}$$

Inserting this into (6.11) we obtain

$$\big|(D_{2^j}K)(y^{-1}x) - P^{r-1}_{x,D_{2^j}K}(y^{-1})| \le C|y|^r \sup_{|z|\le b^r|y|} |zx|^{-\kappa-r}. \tag{6.12}$$

Notice that for $|x|\ge\frac{1}{2\gamma}$, $y\in\operatorname{supp}\phi$ and $|z|\le b^{r-|I|}|y|$, we have $|zx|\sim|x|$. Thus, by using the vanishing moments of ϕ and (6.12), we have

$$\begin{aligned} \big|\phi*(D_{2^j}K)(x)\big| &= \left|\int \phi(y)(D_{2^j}K)(y^{-1}x)dy\right| \\ &\le \int |\phi(y)|\big|(D_{2^j}K)(y^{-1}x) - P^{r-1}_{x,D_{2^j}K}(y^{-1})\big|dy \\ &\lesssim \sup_{|z|\le b^r|y|} \int |zx|^{-\kappa-r}|y|^r|\phi(y)|dy \\ &\lesssim |x|^{-\kappa-r}\int |y|^r|\phi(y)|dy \\ &\lesssim |x|^{-\kappa-r}. \end{aligned} \tag{6.13}$$

Combining (6.10) and (6.13), we get (6.8).

The estimate (6.9) follows from (6.8), the fact $(D_{2^j}K)*\phi(x) = \tilde{\phi}*(D_{2^j}\widetilde{K})(x^{-1})$, and $Y^I\widetilde{K}(x) = (-1)^{|I|}(X^IK)(x^{-1})$ (cf. [34, p. 44]).

It is straightforward to verify that both $\phi * (D_{2^j}K)$ and $(D_{2^j}K) * \phi$ have vanishing moments of the same order as ϕ. The proof of Lemma 6.12 is therefore completed. □

Lemma 6.13. *Suppose $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$, Φ vanishes near the origin, and*

$$\sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\lambda) = 1 \quad \textit{for all } \lambda \in \mathbb{R}_{>0}. \tag{6.14}$$

Then for any $\psi \in \mathcal{S}_\infty(G)$,

$$\psi = \sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\Delta_{\mathbb{X}})\psi,$$

where the sum converges in $\mathcal{S}_\infty(G)$.

Proof. We first show that $\mathcal{S}(G) * \mathcal{S}_\infty(G) \subset \mathcal{S}_\infty(G)$. For any $P \in \mathcal{P}$ and $f \in \mathcal{S}_\infty(G)$, by the unimodularity of G we have $\langle P, \tilde{f}\rangle = \langle \tilde{P}, f\rangle = 0$. Hence for any $g \in \mathcal{S}(G)$ and $f \in \mathcal{S}_\infty(G)$, we have for all polynomials P on G that

$$\langle g * f, P\rangle = \langle g, P * \tilde{f}\rangle = \langle g, 0\rangle = 0.$$

This shows that $g * f \in \mathcal{S}_\infty(G)$.

We next show that the sum $\sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\Delta_{\mathbb{X}})\phi$ converges in the topology of $\mathcal{S}_\infty(G)$. To do this, let ϕ be the convolution kernel of $\Phi(\Delta_{\mathbb{X}})$ and let $\phi_j = D_{2^j}\phi$, $j \in \mathbb{Z}$. For any given nonnegative integer N, we let N' be another integer such that $N' > N + 2mN$. Since both ψ and ϕ are Schwartz functions with all moments vanishing, it follows by Lemma 6.5 that

$$\begin{aligned}
&\|\Phi(2^{-2j}\Delta_{\mathbb{X}})\psi\|_{(N)} = \|\psi * \phi_j\|_{(N)} \\
&= \sup_{|I|\leq N} (1+|x|)^{\kappa+N+d(I)} \left|X^I\left[\psi * (D_{2^j}\phi)\right](x)\right| \\
&= \sup_{|I|\leq N} (1+|x|)^{\kappa+N+d(I)} 2^{jd(I)} \left|\psi * \left[D_{2^j}(X^I\phi)\right](x)\right| \\
&\lesssim \sup_{|I|\leq N} (1+|x|)^{\kappa+N+d(I)} 2^{jd(I)} \|\psi\|_{(N')} \|X^I\phi\|_{(N')} 2^{-|j|N'} \frac{2^{(j\wedge 0)\kappa}}{(1+2^{(j\wedge 0)}|x|)^{\kappa+N'}} \\
&\leq (1+|x|)^{\kappa+N+mN} 2^{|j|mN} \|\psi\|_{(N')} \|\phi\|_{(N'+N)} 2^{-|j|N'} \frac{2^{(j\wedge 0)\kappa}}{(1+2^{(j\wedge 0)}|x|)^{\kappa+N+mN}} \\
&\leq (1+|x|)^{\kappa+N+mN} 2^{|j|mN} \|\psi\|_{(N')} \|\phi\|_{(N'+N)} 2^{-|j|N'} 2^{|j|(N+mN)} \frac{1}{(1+|x|)^{\kappa+N+mN}} \\
&= \|\psi\|_{(N')} \|\phi\|_{(N'+N)} 2^{-|j|(N'-N-2mN)}.
\end{aligned}$$

This implies that $\sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\Delta_{\mathbb{X}})\phi$ converges the topology of $\mathcal{S}_\infty(G)$. Hence (since $\mathcal{S}_\infty(G)$ is complete) there exists $\eta \in \mathcal{S}_\infty(G)$ such that $\sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\Delta_{\mathbb{X}})\phi$ converges in the topology of $\mathcal{S}_\infty(G)$ to η. On the other hand, by (6.14), the spectral theorem (cf. [66, Theorem VII.2]) and Lemma 5.4, we have

$$\psi = \sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\Delta_{\mathbb{X}})\psi,$$

where the sum converges in L^2 norm. Therefore, $\eta = \psi$, which completes the proof. □

The proof of Theorem 6.11 also relies on the existence of smooth functions with compact support and having vanishing moments of arbitrarily high order.

Lemma 6.14. *Given any nonnegative integer M and any positive number δ, there exists a function $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ satisfying the following conditions:*

(i) $|\Psi(\lambda)| \geq c > 0$ *on* $\{2^{-3/2}\varepsilon \leq \lambda \leq 2^{3/2}\varepsilon\}$ *for some* $\varepsilon > 0$.

(ii) *The function* $\lambda \mapsto \lambda^{-M}\Psi(\lambda)$ *belongs to* $\mathcal{S}(\mathbb{R}_{\geq 0})$.

(iii) *The convolution kernel of* $\Phi(\Delta_{\mathbb{X}})$*, denoted by* ϕ*, is supported in the ball* $\{|x| < \delta\}$.

Proof. From the appendix of [38] we see that there exists $\Theta \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that $\Theta(0) = 1$ and Θ has compact support. Now let us define $\Phi(\lambda) = (t^{-2}\lambda)^M\Theta(t^{-2}\lambda)$, $\lambda \in \mathbb{R}_{\geq 0}$, where t is a positive number. Clearly Φ satisfies (ii). Since $\Theta(0) = 1$, (i) is also satisfied. Let ϕ and θ be the convolution kernels of $\Phi(\Delta_{\mathbb{X}})$ and $\Theta(\Delta_{\mathbb{X}})$, respectively. Then $\phi(x) = D_t(\Delta_{\mathbb{X}}^M\theta)(x) = t^{\kappa}(\Delta_{\mathbb{X}}^k\theta)(tx)$. Hence, if we take t sufficiently large, then (iii) is true. ☐

We are now ready to give the

Proof of Theorem 6.11. Choose a function $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ which satisfies conditions (i)–(iii) in Lemma 6.14 with $L = r$ and $\delta = 1/(100\gamma b^r)$. The condition (i) guarantees the existence of a function $\Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\operatorname{supp}\Psi \subset [2^{-2}\varepsilon, 2^2\varepsilon], \quad |\Psi(\lambda)| \geq c > 0 \text{ on } [2^{-3/2}\varepsilon, 2^{3/2}\varepsilon],$$

and

$$\sum_{j=-\infty}^{\infty} \Psi(2^{-2j}\lambda)\Phi(2^{-2j}\lambda) = 1, \quad \forall \lambda \in \mathbb{R}_{>0}. \tag{6.15}$$

Since (by Lemma 6.6) $\mathcal{S}_\infty(G)$ is a dense subspace of $\dot{B}^s_{p,q}(G)$ and of $\dot{F}^s_{p,q}(G)$, we only need to show that for all $f \in \mathcal{S}_\infty(G)$,

$$\|f * K\|_{\dot{B}^s_{p,q}(G)} \leq C\|f\|_{\dot{B}^s_{p,q}(G)} \quad \text{and} \quad \|f * K\|_{\dot{F}^s_{p,q}(G)} \leq C\|f\|_{\dot{F}^s_{p,q}(G)}. \tag{6.16}$$

Let $f \in \mathcal{S}_\infty(G) \subset L^2(G)$. By (6.15), the spectral theorem (cf. [66, Theorem VII.2]) and Lemma 6.13, we have

$$f = \sum_{j=-\infty}^{\infty} \Phi(2^{-2j}\Delta_{\mathbb{X}})\Psi(2^{-2j}\Delta_{\mathbb{X}})f,$$

where the sum converges in the topology of $\mathcal{S}_\infty(G)$. If we denote by ϕ and ψ the convolution kernels of $\Phi(\Delta_{\mathbb{X}})$ and $\Psi(\Delta_{\mathbb{X}})$ respectively, then

$$f = \sum_{j=-\infty}^{\infty} f * \psi_j * \phi_j,$$

where the sum converges in the topology of $\mathcal{S}(G)$. This yields that

$$f * K = \sum_{j=-\infty}^{\infty} f * \psi_j * \phi_j * K,$$

where the sum converges in the topology of $\mathcal{S}'(G)$. Hence for any $\ell \in \mathbb{Z}$ we have the pointwise representation

$$f * K * \phi_\ell(x) = \sum_{j=-\infty}^{\infty} f * \psi_j * \phi_j * K * \phi_\ell(x), \quad x \in G. \tag{6.17}$$

To estimate $|\phi_j * K * \phi_\ell|$, we write

$$\begin{aligned}\psi_j * K * \psi_\ell &= (D_{2^j}\phi) * K * (D_{2^\ell}\phi)\\ &= \begin{cases} D_{2^j}[\phi * (D_{2^{-j}}K)] * (D_{2^\ell}\phi) & \text{if } j \geq \ell,\\ (D_{2^j}\phi) * D_{2^\ell}[(D_{2^{-\ell}}K) * \phi] & \text{if } j < \ell.\end{cases}\end{aligned}$$

For $j \geq \ell$, we can use the size conditions and moment conditions on $\psi^{(1)} * (D_{2^{-j}}K)$ (obtained in Lemma 6.12) and the smooth conditions on ϕ, while for $j > \ell$ we can use the size conditions and moment conditions on $(D_{2^{-\ell}}K) * \phi$ and the smooth conditions on ϕ. Thus by Lemma 6.5 we conclude that

$$\begin{aligned}|\phi_j * K * \phi_\ell(y)| &\lesssim \begin{cases} 2^{-(j-\ell)(r-\varepsilon)}\frac{2^{\ell\kappa}}{(1+2^\ell|y|)^{\kappa+r}} & \text{if } j \geq \ell\\ 2^{-(\ell-j)(r-\varepsilon)}\frac{2^{j\kappa}}{(1+2^j|y|)^{\kappa+r}} & \text{if } j < \ell\end{cases}\\ &= 2^{-|j-\ell|(r-\varepsilon)}\frac{2^{(j\wedge\ell)\kappa}}{(1+2^{j\wedge\ell}|y|)^{\kappa+r}},\end{aligned}$$

where $\varepsilon > 0$ can be taken arbitrarily small. This and (6.17) yields that

$$\begin{aligned}&|f * K * \phi_\ell(x)|\\ &\leq \sum_{j=-\infty}^{\infty}\int |f * \psi_j(z)||\phi_j * K * \phi_\ell(z^{-1}x)|dz\\ &\lesssim \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|(r-\varepsilon)}\int \frac{2^{(j\wedge\ell)\kappa}|f * \psi_j(z)|}{(1+2^{j\wedge\ell}|z^{-1}x|)^{\kappa+r}}dz\\ &\lesssim \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|(r-\varepsilon)}\max\{2^{(j-\ell)a},1\}\left[\sup_{z\in G}\frac{|f * \psi_j(z)|}{(1+2^j|z^{-1}x|)^a}\right]\int \frac{2^{(j\wedge\ell)\kappa}}{(1+2^{j\wedge\ell}|z^{-1}x|)^{\kappa+r-a}}dz,\end{aligned}$$

where for the last inequality we used that

$$(1+2^j|z^{-1}x|)^a \lesssim \max\{2^{(j-\ell)a},1\}(1+2^{j\wedge\ell}|z^{-1}x|)^a.$$

It follows that

$$\begin{aligned}&2^{\ell s}|f * K * \phi_\ell(x)|\\ &\lesssim \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|(r-\varepsilon)}\max\{2^{(j-\ell)a},1\}2^{(j-\ell)s}2^{js}[\Psi(2^{-2j}\Delta_{\mathbb{X}})]^*_a f(x)\int \frac{2^{(j\wedge\ell)\kappa}}{(1+2^{j\wedge\ell}|z^{-1}x|)^{\kappa+r-a}}dz\\ &\leq \sum_{j=-\infty}^{\infty} 2^{-|j-\ell|(r-\varepsilon-a-|s|)}2^{js}[\Psi(2^{-2j}\Delta_{\mathbb{X}})]^*_a f(x)\int \frac{2^{(j\wedge\ell)\kappa}}{(1+2^{j\wedge\ell}|z^{-1}x|)^{\kappa+r-a}}dz.\end{aligned}$$

By the hypothesis we can choose a and ε such that $a > \kappa/\min\{p,q\}$ and $r - \varepsilon - a - |s| > 0$. Then it follows from Lemma 3.11 that

$$\left(\sum_{\ell=-\infty}^{\infty} 2^{\ell sq}\|f * K * \phi_\ell\|^q_{L^p(X)}\right)^{1/q} \lesssim \left(\sum_{\ell=-\infty}^{\infty} 2^{\ell sq}\|[\Psi(2^{-2j}\Delta_{\mathbb{X}})]^*_a f\|^q_{L^p(X)}\right)^{1/q},$$

$$\left\|\left(\sum_{\ell=-\infty}^{\infty} 2^{\ell sq}\left|f * K * \phi_\ell\right|^q\right)^{1/q}\right\|_{L^p(G)} \lesssim \left\|\left(\sum_{j=-\infty}^{\infty} 2^{jsq}\left|[\Psi(2^{-2j}\Delta_{\mathbb{X}})]_a^* f\right|^q\right)^{1/q}\right\|_{L^p(G)}.$$

This together with Theorem 5.21 yields (6.16) and completes the proof of Theorem 6.11. □

Corollary 6.15. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty)$, and let k be a nonnegative integer. Then for all $f \in \mathcal{S}'_\infty(G) \equiv \mathcal{S}'(G)/\mathcal{P}$, we have*

$$\|f\|_{\dot{B}^s_{p,q}(G)} \sim \sum_{d(I)=k} \|X^I f\|_{\dot{B}^{s-k}_{p,q}(G)}, \tag{6.18}$$

$$\|f\|_{\dot{F}^s_{p,q}(G)} \sim \sum_{d(I)=k} \|X^I f\|_{\dot{F}^{s-k}_{p,q}(G)}. \tag{6.19}$$

Proof. We only show (6.19) since the proof of (6.18) is analogous. Note that by the Poincaré-Birkhoff-Witt theorem (cf. [7, I.2.7]), the operators X^I form a basis of the algebra of the left-invariant differential operators on G. By this fact and the stratification of G, it suffices to show that

$$\|f\|_{\dot{F}^s_{p,q}(G)} \sim \sum_{j=1}^{\nu} \|X_j f\|_{\dot{F}^{s-1}_{p,q}(G)}. \tag{6.20}$$

To this end, we first note that when restricted to Schwartz functions, each $X_j \Delta_{\mathbb{X}}^{-1/2}$ is a convolution operator whose distribution kernel is homogeneous of degree $-\kappa$ and coincides with a smooth function in $G\backslash\{0\}$. This follows from the fact that the operator $\Delta_{\mathbb{X}}^{-1/2}$ is a convolution operator whose distribution kernel is homogeneous of degree $-\kappa+1$ and coincides with a smooth function in $G\backslash\{0\}$ (see [33, Proposition 3.17]). Hence, by Theorem 6.11, each $X_j \Delta_{\mathbb{X}}^{-1/2}$ extends to a bounded operator on $\dot{B}^s_{p,q}(G)$ and on $\dot{F}^s_{p,q}(G)$. From this and the lifting property (Theorem 5.19), we deduce that

$$\|X_j f\|_{\dot{F}^{s-1}_{p,q}(G)} = \|(X_j \Delta_{\mathbb{X}}^{-1/2})\Delta_{\mathbb{X}}^{1/2} f\|_{\dot{F}^{\alpha-1}_{p,q}(G)} \lesssim \|\Delta_{\mathbb{X}}^{1/2} f\|_{\dot{F}^{s-1}_{p,q}(G)} \sim \|f\|_{\dot{F}^s_{p,q}(G)}.$$

Hence $\sum_{j=1}^{\nu} \|X_j f\|_{\dot{F}^{s-1}_{p,q}(G)} \lesssim \|f\|_{\dot{F}^s_{p,q}(G)}$. To see the converse, we need to use [33, Lemma 4.12], which asserts that there exist tempered distributions $K_1, \cdots, K_\nu$ which are homogeneous of degree $-\kappa+1$ and coincide with smooth functions in $G\backslash\{0\}$ such that $f = \sum_{j=1}^{\nu}(X_j f) * K_j$ for all $f \in \mathcal{S}(G)$. By this result and Theorem 5.19, we have, at least for f of the form $f = \Phi(\Delta_{\mathbb{X}})g$ where $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ vanishes near the origin and g is bounded on G with compact support (the space of such functions f is dense in $\dot{F}^s_{p,q}(G)$),

$$\begin{aligned} \|f\|_{\dot{F}^s_{p,q}(G)} &= \left\|\Delta_{\mathbb{X}}(\Delta_{\mathbb{X}}^{-1/2} f)\right\|_{\dot{F}^{s-1}_{p,q}(G)} = \left\|\Delta_{\mathbb{X}}(f * R_1)\right\|_{\dot{F}^{s-1}_{p,q}(G)} \\ &= \left\|\Delta_{\mathbb{X}}\left(\sum_{j=1}^{\nu}(X_j f) * K_j * R_1\right)\right\|_{\dot{F}^{s-1}_{p,q}(G)} = \left\|\sum_{j=1}^{\nu}(X_j f) * \Delta_{\mathbb{X}}(K_j * R_1)\right\|_{\dot{F}^{s-1}_{p,q}(G)}, \end{aligned} \tag{6.21}$$

where R_1 is the convolution kernel of the operator $\Delta_{\mathbb{X}}^{-1/2}$. As is shown in [33, p. 190], each $\Delta_{\mathbb{X}}(K_j * R_1)$ is a distribution homogeneous of degree $-\kappa$ and coinciding with a smooth function away from 0. Hence it follows by Theorem 6.11 that

$$\left\|(X_j f) * \Delta_{\mathbb{X}}(K_j * R_1)\right\|_{\dot{F}^{s-1}_{p,q}(G)} \lesssim \|X_j f\|_{\dot{F}^{s-1}_{p,q}(G)}.$$

Inserting this into (6.21), we obtain $\|f\|_{\dot{F}^s_{p,q}(G)} \lesssim \sum_{j=1}^{\nu} \|X_j f\|_{\dot{F}^{s-1}_{p,q}(G)}$. Thus (6.20) is established and the proof is completed. □

Chapter 7

Maximal characterization of $F^{0,\Delta}_{p,2}(X)$ on Riemannian manifolds

7.1 The maximal Hardy spaces $H^p_{\max,\mathscr{L}}(X)$

Throughout this section, we assume that the metric measure space (X, ρ, μ) satisfies the doubling condition (2.1), the reverse doubling condition (2.4), and the non-collapsing condition (2.6), and assume that $\mathscr{L}$ is a non-negative self-adjoint operator on $L^2(X)$ whose heat kernel $p_t(x, y)$ satisfies the Gaussian upper bound (2.8) and the Hölder continuity (2.9) for $t \in (0, \infty)$.

The purpose of this section is to establish the maximal Hardy spaces associated to $\mathscr{L}$. We first introduce the radial, non-tangential and grand maximal functions associated to $\mathscr{L}$:

Definition 7.1. For $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$, $N \in \mathbb{N}_0$, and $x \in X$, define

$$M^0_{\Phi,\mathscr{L}} f(x) := \sup_{t>0} |\Phi(t^2\mathscr{L}) f(x)|, \quad M_{\Phi,\mathscr{L}} f(x) := \sup_{t>0} \sup_{\rho(y,x)<t} |\Phi(t^2\mathscr{L}) f(y)|,$$

and

$$M_{N,\mathscr{L}} f(x) := \sup_{\|\Phi\|_{(N)} \leq 1} M_{\Phi,\mathscr{L}} f(x),$$

where $\|\Phi\|_{(N)}$ is defined by (2.10).

We now introduce Hardy spaces associated to $\mathscr{L}$ by means of grand maximal functions:

Definition 7.2. For $p \in (0, 1]$, we define the Hardy space $H^p_{\mathscr{L}}(X)$ associated to $\mathscr{L}$ as

$$H^p_{\max,\mathscr{L}}(X) := \{f \in \mathcal{S}'_{\mathscr{L}}(X) : M_{N_p,\mathscr{L}} f \in L^p(X)\}$$

with the quasi-norm given by

$$\|f\|_{H^p_{\max,\mathscr{L}}(X)} := \|M_{N_p,\mathscr{L}} f\|_{L^p(X)},$$

where

$$N_p := \lfloor 2d/p \rfloor + \lfloor 3d/2 \rfloor + 4. \tag{7.1}$$

The following theorem, which says that $H^p_{\max,\mathscr{L}}(X)$ are equivalently characterized by radial and non-tangential maximal functions, is the main result of the present section.

Theorem 7.3. *Suppose $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$, $\Phi(0) \neq 0$, and $0 < p \leq 1$. Then for any $f \in \mathcal{S}'_{\mathscr{L}}(X)$ the following conditions are equivalent:*

(i) $f \in H^p_{\max,\mathscr{L}}(X)$.

(ii) $M_{\Phi,\mathscr{L}} f \in L^p(X)$.

(iii) $M^0_{\Phi,\mathscr{L}} f \in L^p(X)$.

Moreover, the following quasi-norm equivalence is valid:

$$\|M_{N_p,\mathscr{L}} f\|_{L^p(X)} \sim \|M_{\Phi,\mathscr{L}} f\|_{L^p(X)} \sim \|M^0_{\Phi,\mathscr{L}} f\|_{L^p(X)}.$$

For the proof of Theorem 7.3 we need a sequence of lemmas.

Lemma 7.4. *Suppose $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ and $\Phi(0) = 1$. Then for any $\Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ and $N \in \mathbb{N}_0$, there exist a family $\{\Theta_{(s)}\}_{0\leq s\leq 1}$ of functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ and a constant $C > 0$ such that:*

(i) $\Psi(\lambda) = \int_0^1 \Theta_{(s)}(\lambda)\Phi(s^2\lambda)ds$ *for all* $\lambda \in \mathbb{R}_{\geq 0}$.

(ii) $\int_X \left(1 + \frac{\rho(x,y)}{t}\right)^N \left|K_{\Theta_{(s)}(t^2\mathscr{L})}(x,y)\right| d\mu(y) \leq Cs^N \|\Psi\|_{(2N+\lfloor 3d/2\rfloor+3)}$ *for all* $t > 0$ *and* $x \in X$.

Proof. We follow [34, Theorem 4.9]. Fix $N \in \mathbb{N}_0$. Let $\{\Omega_{(s)}\}_{0\leq s\leq 1}$ be the unique family of functions in $\mathcal{S}(\mathbb{R}_{\geq 0})$ such that

$$\partial_s^{N+1}\left[\Phi(s^2\lambda)^{N+2}\right] = \Phi(s^2\lambda)\Omega_{(s)}(\lambda), \quad \forall s \in [0,1],\ \forall \lambda \in \mathbb{R}_{\geq 0}. \tag{7.2}$$

Notice that $\Omega_{(s)}$ has the expression

$$\Omega_{(s)}(\lambda) = \sum_{j_1+\cdots+j_{N+1}=N+1} C_{j_1,\cdots,j_k} \partial_s^{j_1}\left[\Phi(s^2\lambda)\right] \cdots \partial_s^{j_k}\left[\Phi(s^2\lambda)\right], \tag{7.3}$$

where each $C_{j_1,\cdots,j_k}$ is a non-negative integer. Choose $\Xi \in C^\infty([0,1])$ such that

$$\begin{aligned} \Xi(s) = s^N/N! &\quad \text{for all } s \in [0,1/2],\\ 0 \leq \Xi(s) \leq s^N/N! &\quad \text{for all } s \in [1/2,1],\\ \partial_s^j \Xi(1) = 0 &\quad \text{for all } j \in \{0,1,\cdots,N+1\}. \end{aligned}$$

Then we set

$$\Theta_{(s)}(\lambda) = (-1)^{N+1}\Xi(s)\Omega_{(s)}(\lambda)\Psi(\lambda) - \left[\partial_s^{N+1}\Xi(s)\right]\Phi(s^2\lambda)^{N+1}\Psi(\lambda), \quad \lambda \in \mathbb{R}_+. \tag{7.4}$$

Clearly, $\Theta_{(s)} \in \mathcal{S}(\mathbb{R}_{\geq 0})$ for every $s \in [0,1]$. We claim that (i) and (ii) hold for this choice of $\Theta_{(s)}$.

First we verify (i). Consider the integral

$$I(\lambda) = (-1)^{N+1}\int_0^1 \Xi(s)\left\{\partial_s^{N+1}\left[\Phi(s^2\lambda)^{N+2}\right]\right\}\Psi(\lambda)ds, \quad \lambda \in \mathbb{R}_+. \tag{7.5}$$

Integrating by parts $N+1$ times and noting that the boundary terms in the first N integration by parts vanish, we obtain

$$I(\lambda) = -\left[\partial_s^N \Xi(s)\right]\Phi(s^2\lambda)^{N+2}\Psi(\lambda)\Big|_{s=0}^1 + \int_0^1 \left[\partial_s^{N+1}\Xi(s)\right]\Phi(s^2\lambda)^{N+2}\Psi(\lambda)ds$$

$$= \Psi(\lambda) + \int_0^1 \left[\partial_s^{N+1}\Xi(s)\right]\Phi(s^2\lambda)^{N+2}\Psi(\lambda)ds,$$

where we used that $\Phi(0) = 1$. Hence by (7.5), (7.2) and (7.4) we have

$$\Psi(\lambda) = I(\lambda) - \int_0^1 \left[\partial_s^{N+1}\Xi(s)\right]\Phi(s^2\lambda)^{N+2}\Psi(\lambda)ds = \int_0^1 \Theta_{(s)}(\lambda)\Phi(s^2\lambda)ds.$$

Next we verify (ii). Since $\Xi(s)$ equals a constant for $s \in [0, 1/2]$, we have $|\partial_s^{N+1}\Xi(s)| \leq Cs^N$ for all $s \in [0, 1]$. From this fact, (7.3) and (7.4), it is not difficult to see that for every $m \in \mathbb{N}_0$,

$$\|\Theta_{(s)}\|_{(m)} \leq Cs^N \|\Psi\|_{(m+N+1)}, \tag{7.6}$$

where the constant C depends on Φ and m, but is independent of $s \in [0, 1]$ and Ψ. Take $m = N + \lfloor 3d/2 \rfloor + 2$ $(\geq d + 1)$. Then it follows from (2.14), (7.6) and (3.9) that

$$\begin{aligned}
&\int_X \left(1 + \frac{\rho(x,y)}{t}\right)^N \left|K_{\Theta_{(s)}(t^2\mathscr{L})}(x,y)\right| d\mu(y) \\
&\leq C\|\Theta_{(s)}\|_{(m)} \int_X \left(1 + \frac{\rho(x,y)}{t}\right)^N D_{t,m}(x,y)d\mu(y) \\
&= C\|\Theta_{(s)}\|_{(m)} \int_X D_{t,,m-N}(x,y)d\mu(y) \\
&\leq Cs^N \|\Psi\|_{(m+N+1)} = Cs^N \|\Psi\|_{(2N+\lfloor 3d/2 \rfloor+3)}.
\end{aligned}$$

This verifies (ii) and completes the proof. □

Lemma 7.5. *Suppose $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$ and $\Phi(0) = 1$. Then for any $N \in \mathbb{N}_0$ there exists a constant $C > 0$ such that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$ and $x \in X$,*

$$M_{2N+\lfloor 3d/2 \rfloor+3,\mathscr{L}} f(x) \leq C T^N_{\Phi,\mathscr{L}} f(x), \tag{7.7}$$

where

$$T^N_{\Phi,\mathscr{L}} f(x) = \sup_{y \in X, t>0} |\Phi(t^2\mathscr{L})f(y)| \left(1 + \frac{\rho(x,y)}{t}\right)^{-N}. \tag{7.8}$$

Proof. For any given $\Psi \in \mathcal{S}(\mathbb{R}_{\geq 0})$, write $\Psi(\cdot) = \int_0^1 \Theta_{(s)}(\cdot)\Phi(s^2\cdot)ds$ as in Lemma 7.4. Then for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $t \in (0, \infty)$, and $y \in X$, we have

$$\begin{aligned}
\Psi(t^2\mathscr{L})f(y) &= \int_0^1 \Theta_{(s)}(t^2\mathscr{L})\Phi(s^2t^2\mathscr{L})f(y)ds \\
&= \int_0^1 \int_X \Phi(s^2t^2\mathscr{L})f(z)K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)d\mu(z)ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\Psi(t^2\mathscr{L})f(y)| &\leq \int_0^1 \int_X |\Phi(s^2t^2\mathscr{L})f(z)||K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)|d\mu(z)ds \\
&\leq T^N_{\Phi,\mathscr{L}} f(x) \int_0^1 \int_X \left(1 + \frac{\rho(x,z)}{st}\right)^N |K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)|d\mu(z)ds \\
&\leq T^N_{\Phi,\mathscr{L}} f(x) \int_0^1 \int_X s^{-N} \left(1 + \frac{\rho(x,y) + \rho(y,z)}{t}\right)^N \left|K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)\right| d\mu(z)ds.
\end{aligned}$$

Note that if $y \in B(x,t)$ then $1 + \frac{\rho(x,y)+\rho(y,z)}{t} < 2\big(1 + \frac{\rho(y,z)}{t}\big)$. Hence by Lemma 7.4 (ii), we have

$$\begin{aligned} M_{\Psi,\mathscr{L}} f(x) &\leq 2^N T^N_{\Phi,\mathscr{L}} f(x) \int_0^1 \int_X s^{-N} \left(1 + \frac{\rho(y,z)}{t}\right)^N |K_{\Theta_{(s)}(t^2\mathscr{L})}(y,z)| d\mu(z) ds \\ &\leq C\|\Psi\|_{(2N+\lfloor 3d/2 \rfloor+3)} T^N_{\Phi,\mathscr{L}} f(x), \end{aligned}$$

which yields the desired inequality (7.7). □

Lemma 7.6. *For any $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$, $p \in (0,1]$, and $N \in \mathbb{N}_0$ with $N > d/p$, there exists a constant $C > 1$ such that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$,*

$$C^{-1}\|M_{\Phi,\mathscr{L}} f\|_{L^p(X)} \leq \|T^N_{\Phi,\mathscr{L}} f\|_{L^p(X)} \leq C\|M_{\Phi,\mathscr{L}} f\|_{L^p(X)},$$

where $T^N_{\Phi,\mathscr{L}} f$ is defined by (7.8).

Proof. Obviously, $M_{\Phi,\mathscr{L}} f(x) \leq 2^N T^N_{\Phi,\mathscr{L}} f(x)$ for every $x \in X$, so the first inequality holds as long as $C > 2^N$. To see the second inequality, set $q = d/N$, so that $q < p$. Observe that

$$|\Phi(t^2\mathscr{L}) f(y)| \leq M_{\Phi,\mathscr{L}} f(z) \quad \text{whenever } z \in B(y,t).$$

From this and (2.2) it follows that

$$\begin{aligned} |\Phi(t^2\mathscr{L}) f(y)|^q &\leq \frac{1}{V(y,t)} \int_{B(y,t)} \big[M_{\Phi,\mathscr{L}} f(z)\big]^q d\mu(z) \\ &\leq \frac{V(x, t+\rho(x,y))}{V(y,t)} \frac{1}{V(x, t+d(x,y))} \int_{B(x,t+\rho(x,y))} \big[M_{\Phi,\mathscr{L}} f(z)\big]^q d\mu(z) \\ &\lesssim \left(1 + \frac{\rho(x,y)}{t}\right)^n M_{HL}\big(\big[M_{\Phi,\mathscr{L}} f(\cdot)\big]^q\big)(x), \end{aligned}$$

where M_{HL} is the Hardy-Littlewood maximal operator. Since $N = d/q$ this says that for all $x \in X$

$$\big[T^N_{\Phi,\mathscr{L}} f(x)\big]^q \lesssim M_{HL}\big(\big[M_{\Phi,\mathscr{L}} f(\cdot)\big]^q\big)(x).$$

Then, since $p/q > 1$, the Hardy-Littlewood maximal theorem yields

$$\int_X \big[T^N_{\Phi,\mathscr{L}} f(x)\big]^p d\mu(x) \lesssim \int_X \big\{M_{HL}\big(\big[M_{\Phi,\mathscr{L}} f(\cdot)\big]^q\big)(x)\big\}^{p/q} d\mu(x) \lesssim \int_X \big[M_{\Phi,\mathscr{L}} f(x)\big]^p d\mu(x).$$

This completes the proof. □

For our purpose we introduce two auxiliary maximal type functions: for $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$, $K \in \mathbb{N}_0$, $N \in \mathbb{N}_0$ and $\varepsilon \in (0,1]$, we set

$$\begin{aligned} M^{\varepsilon K}_{\Phi,\mathscr{L}} f(x) &= \sup_{0<t<1/\varepsilon} \sup_{y \in B(x,t)} |\Phi(t^2\mathscr{L}) f(y)| \left(\frac{t}{t+\varepsilon}\right)^K (1 + \varepsilon\rho(y,x_0))^{-K}, \\ T^{\varepsilon NK}_{\Phi,\mathscr{L}} f(x) &= \sup_{0<t<1/\varepsilon} \sup_{y \in X} |\Phi(t^2\mathscr{L}) f(y)| \left(1 + \frac{\rho(x,y)}{t}\right)^{-N} \left(\frac{t}{t+\varepsilon}\right)^K (1 + \varepsilon\rho(y,x_0))^{-K}. \end{aligned}$$

Lemma 7.7. *For any $\Phi \in \mathcal{S}(\mathbb{R}_{\geq 0})$, $p \in (0,1]$, and $N \in \mathbb{N}_0$ with $N > d/p$, there exists $C > 0$ such that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $\varepsilon \in (0,1]$ and $K \in \mathbb{N}_0$,*

$$\|T^{\varepsilon NK}_{\Phi,\mathscr{L}} f\|_{L^p(X)} \leq C\|M^{\varepsilon K}_{\Phi,\mathscr{L}} f\|_{L^p(X)}.$$

Proof. The proof is the same as that of Lemma 7.6 and is thus skipped. □

Lemma 7.8. *For any $\Phi \in \mathcal{S}_{\mathscr{L}}(X)$, $p \in (0,1]$, and $f \in \mathcal{S}'_{\mathscr{L}}(X)$, there exists $K \in \mathbb{N}_0$ such that $M^{\varepsilon K}_{\Phi,\mathscr{L}} f \in L^p(X) \cap L^\infty(X)$ for $0 < \varepsilon \le 1$.*

Proof. By the definition of $\mathcal{S}'_{\mathscr{L}}(X)$ there exist $k_0, m_0 \in \mathbb{N}_0$ such that

$$|\Phi(t^2\mathscr{L})f(y)| = \big|\big(f, K_{\Phi(t^2\mathscr{L})}(y,\cdot)\big)\big| \le C\mathcal{P}^*_{k_0,m_0}\big(K_{\Phi(t^2\mathscr{L})}(y,\cdot)\big). \tag{7.9}$$

Let $M \in \mathbb{N}_0$ such that $M \ge \max\{m_0 + d/2, d+1\}$. Then by (2.16) and (2.3), we have

$$\begin{aligned}
&\mathcal{P}^*_{k_0,m_0}\big(K_{\Phi(t^2\mathscr{L})}(y,\cdot)\big)\\
&= \sum_{\substack{0\le k\le k_0\\ 0\le m\le m_0}} \sup_{z\in X}(1+\rho(z,x_0))^m \big|\mathscr{L}^k\big[K_{\Phi(t^2\mathscr{L})}(y,\cdot)\big](z)\big|\\
&\le C\sum_{0\le k\le k_0}\sup_{z\in X}(1+\rho(z,x_0))^{m_0} t^{-2k}\|\Phi\|_{(k+M)} D_{t,M}(y,z)\\
&\le C\sum_{0\le k\le k_0}\sup_{z\in X}\frac{t^{-2k}(1+\rho(z,x_0))^{m_0}}{V(z,t)}\left(1+\frac{\rho(y,z)}{t}\right)^{-M+d/2}\\
&\le C\sum_{0\le k\le k_0}\sup_{z\in X}\frac{t^{-2k}(1+\rho(z,x_0))^{m_0}}{V(z,t)}\left(1+\frac{\rho(y,z)}{t}\right)^{-m_0}.
\end{aligned} \tag{7.10}$$

Note that if $t \in (0,1]$, then by (2.7) and the triangle inequality for the distance ρ we have

$$\begin{aligned}
&\sum_{0\le k\le k_0}\sup_{z\in X}\frac{t^{-2k}(1+\rho(z,x_0))^{m_0}}{V(z,t)}\left(1+\frac{\rho(y,z)}{t}\right)^{-m_0}\\
&\le C\sup_{z\in X} t^{-(2k_0+d)}\left(1+\frac{\rho(z,x_0)}{t}\right)^{m_0}\left(1+\frac{\rho(y,z)}{t}\right)^{-m_0}\\
&\le Ct^{-(2k_0+d)}\left(1+\frac{\rho(y,x_0)}{t}\right)^{m_0}\\
&\le Ct^{-(2k_0+d+m_0)}(1+\rho(y,x_0))^{m_0}.
\end{aligned} \tag{7.11}$$

If $t \in (1, 1/\varepsilon]$, then from (2.5) and the triangle inequality for the distance d it follows that

$$\begin{aligned}
&\sum_{0\le k\le k_0}\sup_{z\in X}\frac{t^{-2k}(1+\rho(z,x_0))^{m_0}}{V(z,t)}\left(1+\frac{\rho(y,z)}{t}\right)^{-m_0}\\
&\le Ct^{-\varsigma}(1+\rho(z,x_0))^{m_0}\left(1+\frac{\rho(y,z)}{t}\right)^{-m_0}\\
&\le Ct^{m_0-\varsigma}\left(1+\frac{\rho(z,x_0)}{t}\right)^{m_0}\left(1+\frac{\rho(y,z)}{t}\right)^{-m_0}\\
&\le Ct^{m_0-\varsigma}\left(1+\frac{\rho(y,x_0)}{t}\right)^{m_0}\\
&\le Ct^{m_0-\varsigma}(1+\rho(y,x_0))^{m_0}.
\end{aligned} \tag{7.12}$$

Also note that if $t \in (0,1]$ and $K \ge 2k_0 + d + m_0$ then

$$\left(\frac{t}{t+\varepsilon}\right)^K t^{-(2k_0+d+m_0)} \le \left(\frac{1}{t+\varepsilon}\right)^{2k_0+d+m_0} \le \varepsilon^{-(2k_0+d+m_0)}, \tag{7.13}$$

while if $t \in (1, 1/\varepsilon)$ then for any $K \in \mathbb{N}_0$

$$\left(\frac{t}{t+\varepsilon}\right)^K t^{m_0-\varsigma} \le t^{|m_0-\varsigma|} \le \varepsilon^{-|m_0-\varsigma|}, \tag{7.14}$$

We now choose $K \in \mathbb{N}_0$ such that $K \ge \max\{2k_0 + d + m_0,\ m_0 + d/p\}$. Then from (7.9)–(7.14) it follows that for any fixed $\varepsilon \in (0, 1]$ and for all $t \in (0, 1/\varepsilon]$

$$\begin{aligned} |\Phi(t^2\mathscr{L})f(y)| \left(\frac{t}{t+\varepsilon}\right)^K (1+\varepsilon\rho(y,x_0))^{-K} &\le |\Phi(t^2\mathscr{L})f(y)| \left(\frac{t}{t+\varepsilon}\right)^K \varepsilon^{-K}(1+\rho(y,x_0))^{-K} \\ &\le C(1+\rho(y,x_0))^{-K+m_0}, \end{aligned}$$

where the constant C depends on ε. Hence

$$\begin{aligned} M^{\varepsilon K}_{\Phi,\mathscr{L}} f(x) &\le C \sup_{0<t<1/\varepsilon} \sup_{y\in B(x,t)} (1+\rho(y,x_0))^{-K+m_0} \\ &\le C \sup_{0<t<1/\varepsilon} \sup_{y\in B(x,t)} (1+\rho(x,x_0))^{-K+m_0}(1+\rho(x,y))^{K-m_0} \\ &\le C(1+\rho(x,x_0))^{-(K-m_0)}, \end{aligned}$$

where the constant C depends on ε. Since $p(K - m_0) > d$, it follows by (3.8) that $M^{\varepsilon K}_{\mathscr{L},\Phi} f \in L^p(X) \cap L^\infty(X)$. □

We also need the following auxiliary function: if $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $\Phi \in \mathcal{S}(\mathbb{R}_{\ge 0})$, $K \in \mathbb{N}_0$, $N \in \mathbb{N}_0$, and $0 < \varepsilon \le 1$, we set

$$\widetilde{M}^{\varepsilon K}_{\Phi,\mathscr{L}} f(x) = \sup_{0<t<1/\varepsilon} \sup_{y\in B(x,t)} \left(\sup_{z\in B(y,t)} \frac{t^\alpha |\Phi(t^2\mathscr{L})f(z) - \Phi(t^2\mathscr{L})f(y)|}{\rho(z,y)^\alpha} \right) \left(\frac{t}{t+\varepsilon}\right)^K (1+\varepsilon\rho(y,x_0))^{-K},$$

where $\alpha > 0$ is the same constant as in (2.9).

Lemma 7.9. *Suppose $\Phi \in \mathcal{S}(\mathbb{R}_{\ge 0})$ with $\Phi(0) = 1$. Then for any $N \in \mathbb{N}_0$ and $K \in \mathbb{N}_0$ there exists $C > 0$ such that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$, $\varepsilon \in (0, 1]$, and $x \in X$,*

$$\widetilde{M}^{\varepsilon K}_{\Phi,\mathscr{L}} f(x) \le C T^{\varepsilon NK}_{\Phi,\mathscr{L}} f(x).$$

Proof. Fix $K, N \in \mathbb{N}_0$. By Lemma 7.4 and its proof, we can write

$$\Phi(\cdot) = \int_0^1 \Theta_{(s)}(\cdot)\Phi(s^2\cdot) f ds, \tag{7.15}$$

where $\{\Theta_{(s)}\}_{0\le s\le 1}$ is a family of functions in $\mathcal{S}(\mathbb{R}_{\ge 0})$ with the following property: for any $m \in \mathbb{N}_0$ there exists a constant C (depending on Φ, m, N and K) such that

$$\|\Theta_{(s)}\|_{(m)} \le C s^{N+K} \quad \text{for all } s \in [0, 1]. \tag{7.16}$$

From (7.15) it follows that for all $f \in \mathcal{S}'_{\mathscr{L}}(X)$ and $t \in (0, \infty)$

$$\Phi(t^2\mathscr{L})f = \int_0^1 \Theta_{(s)}(t^2\mathscr{L})\Phi(s^2t^2\mathscr{L}) f ds, \tag{7.17}$$

which holds pointwisely and also in the sense of distributions in $\mathcal{S}'_{\mathscr{L}}(X)$. We fix $m \in \mathbb{N}_0$ such that $m \ge 3d/2 + N + K + 1$, and fix arbitrary $x \in X$. Let $t \in (0, 1/\varepsilon)$, $y \in B(x, t)$, and $z \in B(y, t)$.

By (7.16) and (2.15), we have

$$|K_{\Theta_{(s)}(t^2\mathscr{L})}(z,w) - K_{\Theta_{(s)}(t^2\mathscr{L})}(y,w)| \le Cs^{N+K}\left(\frac{\rho(z,y)}{t}\right)^{\alpha} D_{t,m}(y,w). \tag{7.18}$$

By this kernel estimate, (7.17), and (3.9), we can estimate as follows:

$$\begin{aligned}
&\frac{t^\alpha|\Phi(t^2\mathscr{L})f(z) - \Phi(t^2\mathscr{L})f(y)|}{\rho(z,y)^\alpha} \\
&= \frac{t^\alpha}{\rho(z,y)^\alpha}\left|\int_X \Phi(s^2t^2\mathscr{L})f(w)K_{\Theta_{(s)}(t^2\mathscr{L})}(z,w)d\mu(w) - \int_X \Phi(s^2t^2\mathscr{L})f(w)K_{\Theta_{(s)}(t^2\mathscr{L})}(y,w)d\mu(w)\right| \\
&\le \int_0^1\int_X \frac{t^\alpha\left|\Phi(s^2t^2\mathscr{L})f(w)\right|\left|K_{\Theta_{(s)}(t^2\mathscr{L})}(z,w) - K_{\Theta_{(s)}(t^2\mathscr{L})}(y,w)\right|}{\rho(z,y)^\alpha}d\mu(w)ds \\
&\lesssim \int_0^1\int_X \left|\Phi(s^2t^2\mathscr{L})f(w)\right| s^{N+K}D_{t,m}(y,w)d\mu(w)ds \\
&\lesssim T^{\varepsilon NK}_{\Phi,\mathscr{L}}f(x)\int_0^1\int_X s^{N+K}\left(1+\frac{\rho(x,w)}{st}\right)^N\left(\frac{st}{st+\varepsilon}\right)^{-K}(1+\varepsilon\rho(w,x_0))^K D_{t,m}(y,w)d\mu(w)ds \\
&\lesssim T^{\varepsilon NK}_{\Phi,\mathscr{L}}f(x)\left(\frac{t}{t+\varepsilon}\right)^{-K}\int_X\left(1+\frac{\rho(x,w)}{t}\right)^N(1+\varepsilon\rho(w,x_0))^K D_{t,m}(y,w)d\mu(w) \\
&\lesssim T^{\varepsilon NK}_{\Phi,\mathscr{L}}f(x)\left(\frac{t}{t+\varepsilon}\right)^{-K}\int_X\left(1+\frac{\rho(x,y)}{t}\right)^N\left(1+\frac{\rho(y,w)}{t}\right)^N(1+\varepsilon\rho(w,x_0))^K D_{t,m}(y,w)d\mu(w) \\
&\lesssim T^{\varepsilon NK}_{\Phi,\mathscr{L}}f(x)\left(\frac{t}{t+\varepsilon}\right)^{-K}\int_X(1+\varepsilon\rho(y,x_0))^K(1+\varepsilon\rho(y,w))^K D_{t,m-N}(y,w)d\mu(w) \\
&\lesssim T^{\varepsilon NK}_{\Phi,\mathscr{L}}f(x)\left(\frac{t}{t+\varepsilon}\right)^{-K}(1+\varepsilon\rho(y,x_0))^K\int_X D_{t,m-N-K}(y,w)d\mu(w) \\
&\lesssim T^{\varepsilon NK}_{\Phi,\mathscr{L}}f(x)\left(\frac{t}{t+\varepsilon}\right)^{-K}(1+\varepsilon\rho(y,x_0))^K,
\end{aligned}$$

where for the last inequality we used (3.9) and that $m-N-K>3d/2$. From this the desired inequality follows immediately. □

Now we are ready to give the proof the the main theorem.

Proof of Theorem 7.3. Clearly, (i) ⇒ (ii) ⇒ (iii) and

$$\|M^0_{\Phi,\mathscr{L}}f\|_{L^p(X)} \le \|M_{\Phi,\mathscr{L}}f\|_{L^p(X)} \le \|\Phi\|_{(N_p)}\|M_{N_p,\mathscr{L}}f\|_{L^p(X)}$$

for all $f\in\mathcal{S}'_{\mathscr{L}}(X)$. In addition, Lemma 7.5 and Lemma 7.6 yield that (ii) ⇒ (i) and $\|M_{N_p,\mathscr{L}}f\|_{L^p(X)} \lesssim \|M_{\Phi,\mathscr{L}}f\|_{L^p(X)}$. Hence, it remains to show that (iii) ⇒ (ii) and $\|M_{\Phi,\mathscr{L}}f\|_{L^p(X)} \lesssim \|M^0_{\Phi,\mathscr{L}}f\|_{L^p(X)}$.

Suppose now $f\in\mathcal{S}'_{\mathscr{L}}(X)$ such that $M^0_{\Phi,\mathscr{L}}f\in L^p(X)$. By Lemma 7.8, we can choose K so large that $M^{\varepsilon K}_{\Phi,\mathscr{L}}f\in L^p(X)\cap L^\infty(X)$ for $0<\varepsilon\le 1$. Then by Lemma 7.7 and Lemma 7.9, we have $\widetilde{M}^{\varepsilon K}_{\Phi,\mathscr{L}}f\in L^p(X)$ and $\|\widetilde{M}^{\varepsilon K}_{\Phi,\mathscr{L}}f\|_{L^p(X)} \le C_1\|M^{\varepsilon K}_{\Phi,\mathscr{L}}f\|_{L^p(X)}$, where C_1 is independent of $\varepsilon\in(0,1]$. For given $\varepsilon\in(0,1]$ we set

$$\Omega_\varepsilon = \{x\in X : \widetilde{M}^{\varepsilon K}_{\Phi,\mathscr{L}}f(x) \le C_2 M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x)\},$$

where $C_2 = 2^{1/p}C_1$. Note that

$$\int_X \left[M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x)\right]^p d\mu(x) \le 2\int_{\Omega_\varepsilon}\left[M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x)\right]^p d\mu(x). \tag{7.19}$$

Indeed, this follows from

$$\int_{\Omega_\varepsilon^c}\left[M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x)\right]^p d\mu(x) \le C_2^{-p}\int_{\Omega_\varepsilon^c}\left[\widetilde{M}^{\varepsilon K}_{\Phi,\mathscr{L}}f(x)\right]^p d\mu(x) \le (C_1/C_2)^p\int_X\left[M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x)\right]^p d\mu(x)$$

and $(C_1/C_2)^p = 1/2$.

We claim that for $0 < r < p$ there exists $C_3 > 0$, independent of ε, such that

$$M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x) \le C_3\left\{M_{HL}\left(\left[M^0_{\Phi,\mathscr{L}}f(\cdot)\right]^r\right)(x)\right\}^{1/r} \quad \text{for all } x\in\Omega_\varepsilon. \tag{7.20}$$

Once this claim is established, the required inequality $\|M_{\Phi,\mathscr{L}}f\|_{L^p(X)} \lesssim \|M^0_{\Phi,\mathscr{L}}f\|_{L^p(X)}$ will follow from the Hardy-Littlewood maximal theorem and the monotone convergence theorem (see, for instance, [75, Chapter 3] and [34, Chapter 4] for details).

Let us now prove the claim. Fix any $x\in\Omega_\varepsilon$. By the definition of $M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x)$, there exist $y\in X$ and $t>0$ such that $\rho(y,x) < t < 1/\varepsilon$ and

$$|\Phi(t^2\mathscr{L})f(y)|\left(\frac{t}{t+\varepsilon}\right)^K(1+\varepsilon\rho(y,x_0))^{-K} \ge \frac{1}{2}M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x). \tag{7.21}$$

We fix such y and t. Then by the definitions of $\widetilde{M}^{\varepsilon K}_{\Phi,\mathscr{L}}f$ and Ω_ε, we have

$$\begin{aligned}
&\sup_{z\in B(y,t)}\frac{t^\alpha|\Phi(t^2\mathscr{L})f(z)-\Phi(t^2\mathscr{L})f(y)|}{\rho(z,y)^\alpha}\\
&\le\left(\frac{t}{t+\varepsilon}\right)^{-K}(1+\varepsilon\rho(y,x_0))^K\widetilde{M}^{\varepsilon K}_{\mathscr{L},\Phi}f(x)\\
&\le C_2\left(\frac{t}{t+\varepsilon}\right)^{-K}(1+\varepsilon\rho(y,x_0))^K M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x)\\
&\le C_3|\Phi(t^2\mathscr{L})f(y)|,
\end{aligned} \tag{7.22}$$

where $C_3 = 2C_2$. Let $C_4 \ge \max(1,(2C_3)^{1/\alpha})$. Then we note that

$$|\Phi(t^2\mathscr{L})f(z)| \ge \frac{1}{2}|\Phi(t^2\mathscr{L})f(y)| \quad \text{for all } z\in B(y,t/C_4). \tag{7.23}$$

Indeed, since $d(z,y) < t/C_4 < t$, it follows from (7.22) that

$$\begin{aligned}
|\Phi(t^2\mathscr{L})f(z)-\Phi(t^2\mathscr{L})f(y)| &\le C_3\frac{\rho(z,y)^\alpha}{t^\alpha}|\Phi(t^2\mathscr{L})f(y)|\\
&\le C_3C_4^{-\alpha}|\Phi(t^2\mathscr{L})f(y)| \le \frac{1}{2}|\Phi(t^2\mathscr{L})f(y)|,
\end{aligned}$$

which yields (7.23). Now (7.23) together with (7.21) gives that

$$|\Phi(t^2\mathscr{L})f(z)| \ge \frac{1}{4}M^{\varepsilon K}_{\Phi,\mathscr{L}}f(x) \quad \text{for all } z\in B(y,t/C_4).$$

Also, since $C_4\ge 1$ and $\rho(y,x)<t$, we have $B(y,t/C_4)\subset B(x,2t)$. Therefore,

$$\begin{aligned}
M_{HL}\left(\left[M^0_{\mathscr{L},\Phi}f(\cdot)\right]^r\right)(x) &\ge \frac{1}{V(x,2t)}\int_{B(x,2t)}\left[M^0_{\Phi,\mathscr{L}}f(z)\right]^r d\mu(z)\\
&\ge \frac{1}{V(x,2t)}\int_{B(x,2t)}|\Phi(t^2\mathscr{L})f(z)|^r d\mu(z)\\
&\ge \frac{V(y,t/C_4)}{V(x,2t)}\frac{1}{V(y,t/C_4)}\int_{B(y,t/C_4)}|\Phi(t^2\mathscr{L})f(z)|^r d\mu(z)
\end{aligned}$$

$$\gtrsim \left[M^{\varepsilon K}_{\Phi,\mathscr{L}} f(x)\right]^r.$$

This establishes the claim and finishes the proof of Theorem 7.3. □

7.2 The identification $\dot{F}^{0,\Delta}_{p,2}(X) = H^p_{\max,\Delta}(X)$ on Riemannian manifolds

In this section, we consider the particular case that X is a Riemannian manifold. To be more precise, let X be a complete Riemannian manifold with C^∞-smooth Riemannian metric g_{jk}. Let d be the geodesic distance, μ the Riemannian measure, and ∇ the Riemannian gradient on X. Denote by $|\cdot|$ the length in the tangent space. Let Δ be the Laplace-Beltrami operator, that is the positive self-adjoint operator on $L^2(X, d\mu)$ defined by the formal integration by parts $\langle \Delta f, f\rangle = \||\nabla f|\|_{L^2(X,d\mu)}$. Denote by $p_t(x, y)$ the heat kernel of X.

We assume that the Riemannian manifold X satisfies the doubling condition (2.1), the reverse doubling condition (2.4), and the non-collapsing condition (2.6). Furthermore, we assume that $p_t(x, y)$ satisfies the Gaussian upper bound (2.8) and the Hölder continuity (2.9). In addition, assume that $\mu(X) = \infty$. It is well-known complete, non-compact, connected Riemannian manifolds with non-negative Ricci curvature satisfy all of these assumptions.

The purpose of this subsection is to prove the following result:

Theorem 7.10. *Let $p \in (0, 1]$. Then $\dot{F}^{0,\Delta}_{p,2}(X) = H^p_{\max,\Delta}(X)$ with equivalent quasi-norms.*

To prove Theorem 7.10 we need some preparation. Let $\mathcal{D}(X)$ and $\mathcal{D}'(X)$, respectively, denote the space of complex-valued smooth functions with compact support and the space of distributions, with the usual local convex topologies (cf. Schwartz [72]).

Lemma 7.11. (i) *$\mathcal{D}(X) \subset \mathcal{S}_\Delta(X)$ and the inclusion map is continuous.*

(ii) *$\mathcal{S}'_\Delta(X) \subset \mathcal{D}'(X)$ and the inclusion map is continuous.*

(iii) *The domain $D(\Delta)$ of Δ consists of all functions f in $L^2(X, d\mu)$ such that the distribution derivative Δf in the sense of $\mathcal{D}'(X)$ can be identified with a function in $L^2(X, d\mu)$.*

Proof. (i) is obvious, and (ii) follows from (i) by duality. For the proof of (iii), we refer the reader to [77, Lemma 2.1]. □

Lemma 7.12. *Let $f \in \mathcal{S}'_\Delta(X)$. Set*

$$u(x, t) := e^{-t^2\Delta} f(x) = \langle f, \overline{K_{e^{-t^2\Delta}}(x, \cdot)}\rangle, \quad x \in X,\ t \in (0, \infty).$$

Then $F(\cdot, t) \in C^\infty(X)$ with t fixed.

Proof. We fix $t > 0$. For any integer $\ell \geq 2$, let $\Delta^\ell u(\cdot, t)$ denote the distribution derivative of $u(\cdot, t)$ in the sense of $\mathcal{S}'_\Delta(X)$ (and hence also in the sense of $\mathcal{D}'(X)$, by Lemma 7.11 (ii)). Then $\Delta^\ell u(\cdot, t)$ coincides with the function

$$g_\ell(x) := \langle f, \overline{K_{\Delta^\ell e^{-t^2\Delta}}(x, \cdot)}\rangle = t^{-2\ell}\langle f, \overline{K_{\Phi(t^2\Delta)}(x, \cdot)}\rangle, \quad x \in X, \tag{7.24}$$

where Φ is a function in $\mathcal{S}(\mathbb{R}_{\geq 0})$ defined by $\Phi(\lambda) = \lambda^\ell e^{-\lambda}$. By using (2.15) it is not hard to show that g_ℓ is continuous on X. In particular, $g_\ell \in L^2_{\rm loc}(X, d\mu)$. Hence it follows from the interior regular theorem (cf. [61]) that $g_{\ell-1}$ belongs to the local Sobolev space $H^2_{\rm loc}(X)$. Applying the same theorem repeatedly, we obtain that $F(\cdot,t)\in H^{2\ell}_{\rm loc}(X)$. Since ℓ can be taken arbitrarily large, we have $F(\cdot,t)\in H^\infty_{\rm loc}(X)\subset C^\infty(X)$. □

Now we are ready to give the

Proof of Theorem 7.10. First we show that $\dot F^{0,\Delta}_{p,2}(X)\subset H^p_{\max,\Delta}(X)$. To do this, let $\Phi_0(\lambda)=e^{-\lambda}$, $\lambda\in\mathbb{R}_{\ge0}$. Clearly, $\Phi_0\in\mathcal{S}(\mathbb{R}_{\ge0})$ and $\Phi_0(0)\neq0$. Hence by Theorem 7.3 we know that $H^p_{\max,\Delta}(X)$ is the space of all $f\in\mathcal{S}'_\Delta(X)$ such that $\|M_{\Phi_0,\mathscr{L}}f\|_{L^p(X,d\mu)}<\infty$. Let M be a sufficiently large positive integer. By Theorem 5.31, any $f\in\dot F^{0,\mathscr{L}}_{p,2}(X)$ can be decomposed as $f=\sum_{j=0}^\infty\gamma_ja_j$, where each a_j is a $(p,2,M)$-atom, $\|\{\gamma_j\}_{j=0}^\infty\|_{\ell^p}\lesssim\|f\|_{\dot F^{0,\mathscr{L}}_{p,2}(X)}$, and the sum converges in $\mathcal{S}'_{\mathscr{L}}(X)$. Therefore, in order to show that $\dot F^{0,\Delta}_{p,2}(X)\subset H^p_{\max,\Delta}(X)$, it suffices to prove that there exists a constant $C>0$ such that for all $(p,2,M)$-atoms a, $\|M_{\Phi_0,\mathscr{L}}f\|_{L^p(X,d\mu)}\le C$. For the proof of the latter, we refer the reader to [46, Theorem 7.4 (i)] (for the case $p=1$) and to [55, p. 266] (for the case $p\in(0,1]$).

Next we show that $H^p_{\max,\Delta}(X)\subset\dot F^{0,\Delta}_{p,2}(X)$. To do this, let Φ and Ψ be two functions in $\mathcal{S}(\mathbb{R}_{\ge0})$ defined respectively by

$$\Phi(\lambda):=e^{-\lambda},\quad \Psi(\lambda):=\lambda e^{-\lambda},\quad \lambda\in\mathbb{R}_{\ge0}.$$

For $f\in\mathcal{S}'_\Delta(X)$ and $x\in X$, we define

$$f^*(x):=M_\Phi f(x)+M_\Psi f(x)=\sup_{(y,t)\in\Gamma(x)}\big(|e^{-t^2\Delta}f(y)|+|t^2\Delta e^{-t^2\Delta}f(y)|\big),$$

where $\Gamma(x):=\{(y,t)\in X\times(0,\infty):\rho(y,x)<t\}$. By Theorem 7.3, we have

$$\begin{aligned}\|M_{N_p,\mathscr{L}}f\|_{L^p(X,d\mu)}&\lesssim\|M_{\Phi,\mathscr{L}}f\|_{L^p(X,d\mu)}\le\|f^*\|_{L^p(X,d\mu)}\\&\le\big(\|\Phi\|_{(N_p)}+\|\Psi\|_{(N_p)}\big)\|M_{N_p,\mathscr{L}}f\|_{L^p(X,d\mu)}.\end{aligned}$$

Hence

$$\|f\|_{H^p_{\max,\Delta}(X)}\sim\|f^*\|_{L^p(X,d\mu)}. \tag{7.25}$$

For $\beta>0$, $f\in\mathcal{S}'_\Delta(X)$, and $x\in X$, we define

$$\begin{aligned}S_\beta(f)(x)&:=\left(\iint_{\Gamma_\beta(x)}|\Psi(t^2\Delta)|^2\frac{dy}{V(x,t)}\frac{dt}{t}\right)^{1/2}\\&=\left(\iint_{\Gamma_\beta(x)}|t^2\Delta e^{-t^2\Delta}f(y)|^2\frac{dy}{V(x,t)}\frac{dt}{t}\right)^{1/2},\end{aligned}$$

where $\Gamma_\beta(x):=\{(y,t)\in X\times(0,\infty):\rho(y,x)<\beta t\}$. As in the classical setting, for any fixed $\beta>0$ we have

$$\|S_\beta(f)\|_{L^p(X,d\mu)}\sim\|S_1(f)\|_{L^p(X,d\mu)}. \tag{7.26}$$

See [16, Proposition 4]. For any $\beta>0$ and $f\in\mathcal{S}'_{\mathscr{L}}(X)$, we set

$$\widetilde S_\beta(f)(x)=\left(\iint_{\Gamma_\beta(x)}|t\nabla u(y,t)|^2\frac{dy}{V(x,t)}\frac{dt}{t}\right)^{1/2},$$

where $u(y,t):=e^{-t^2\Delta}f(y)$. The argument of [47, Lemma 5.4] (see also (6.2) in [47]) yields that

$$\|S_\beta(f)\|_{L^p(X,d\mu)}\lesssim\|\widetilde S_\beta(f)\|_{L^p(X,d\mu)}. \tag{7.27}$$

Combing (7.25), (7.26), (7.27) and the area integral characterization of $\dot{F}^{0,\Delta}_{p,q}(X)$ (Theorem 5.23), we see that, in order to prove $H^p_{\max,\Delta}(X) \subset \dot{F}^{0,\Delta}_{p,q}(X)$, it suffices to show that for some $\beta > 0$

$$\|\widetilde{S}_\beta(f)\|_{L^p(X,d\mu)} \lesssim \|f^*\|_{L^p(X,d\mu)}. \tag{7.28}$$

To this end, for $\beta > 0$, $0 < \varepsilon < R < \infty$, $f \in \mathcal{S}'_\Delta(X)$, and $x \in X$, we set

$$\widetilde{S}^{\varepsilon,R}_\beta(f)(x) := \left(\iint_{\Gamma^{\varepsilon,R}_\beta(x)} |t\nabla u(y,t)|^2 \frac{dy}{V(x,t)}\frac{dt}{t}\right)^{1/2},$$

where

$$\Gamma^{\varepsilon,R}_\beta(x) := \{(y,t) \in M \times (\varepsilon, R) : y \in B(x,\beta t)\} = \{(y,t) \in \Gamma_\beta(x) : \varepsilon < t < R\}.$$

The argument of [5, Lemma 7.6] with slight modification yields the following "good λ" inequality: there exists a constant $C > 0$ such that for all $0 < \gamma < 1$, $\lambda > 0$, $0 < \varepsilon < R < \infty$, and $f \in \mathcal{S}'_\Delta(X)$,

$$\mu\left(\left\{x \in X : \widetilde{S}^{\varepsilon,R}_{1/20} f(x) > 2\lambda, f^*(x) \le \gamma\lambda\right\}\right) \le C\gamma^2 \mu\left(\left\{x \in X : \widetilde{S}^{\varepsilon,R}_{1/2} f(x) > \lambda\right\}\right).$$

This key inequality, along with the fact that $\|\widetilde{S}^{\varepsilon,R}_{1/2}(f)\|_{L^p(X,d\mu)} \lesssim \|\widetilde{S}^{\varepsilon,R}_{1/20}(f)\|_{L^p(X,d\mu)}$ (cf. [16, Proposition 4]), yields that $\|\widetilde{S}^{\varepsilon,R}_{1/20}(f)\|_{L^p(X,d\mu)} \lesssim \|f^*\|_{L^p(X,d\mu)}$. Letting $\varepsilon \to 0$ and $R \to \infty$, by the Fatou lemma we get (7.28) with $\beta = 1/20$. The proof is thus completed. □

Bibliography

[1] H. Arai, *Area integrals for Riesz measures on the Siegel upper half space of type II*, Tohoku Math. J. (2) **44** (1992), 613–622.

[2] H. Arai, *Generalized Dirichlet growth theorem and applications to hypoelliptic and $\overline{\partial_b}$ equations*, Comm. Partial Differential Equations **22** (1997), 2061–2088.

[3] W. Arendt and A.F.M. ter Elst, *Gaussian estimates for second order elliptic operators with boundary conditions*, J. Operator Theory **1** (1997), 87–130.

[4] P. Auscher, X.-T. Duong and A. McIntosh, *Boundedness of Banach space valued singular integral operators and Hardy spaces*, unpublished manuscript, 2004.

[5] P. Auscher, A. McIntosh and E. Russ, *Hardy spaces of differential forms on Riemannian manifolds*, J. Geom. Anal. **18** (2008), 192–248.

[6] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*, Springer-Verlag, Berlin, 2007.

[7] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. I-III (Eléments de Math., Fasc. 26 et 37), Hermann, Paris, 1960 and 1972.

[8] M. Bownik and K.-P. Hu, *Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces*, Trans. Amer. Math. Soc. **358** (2005), 1469–510.

[9] H.-Q. Bui, *Harmonic functions, Riesz potentials, and the Lipschitz spaces of Herz*, Hiroshima Math. J. **9** (1979), 245–295.

[10] H.-Q, Bui, *Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures*, J. Funct. Anal. **55** (1984), 39–62.

[11] H.-Q. Bui, M. Paluszyński and M.H. Taibleson, *A maximal function characterization of Besov-Lipschitz and Triebel-Lizorkin spaces*, Studia Math. **119** (1996), 219–246.

[12] H.-Q. Bui, M. Paluszyński and M.H. Taibleson, *Characterization of the Besov-Lipschitz and Triebel-Lizorkin spaces. The case $q < 1$*, J. Fourier Anal. Appl. **3** (1997), 837–846.

[13] A. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Adv. Math. **16** (1975), 1–64.

[14] S.Y.A. Chang and R. Fefferman, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. (2) **112** (1980), 179–201.

[15] M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), 601–628.

[16] R.R. Coifman, Y. Meyer and E.M. Stein, *Some new functions and their applications to harmonic analysis*, J. Funct. Anal. **62** (1985), 304–315.

[17] M. Cowling and A. Sikora, *A spectral multiplier theorem on* $SU(2)$, Math. Z. **238** (2001), 1–36.

[18] R.R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

[19] T. Coulhon, G. Kerkyacharian and P. Petrushev, *Heat kernel generated frames in the setting of Dirichlet spaces*, J. Fourier Anal. Appl. **18** (2012), 995–1066.

[20] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[21] Y. Ding, Y. Han, G. Lu and X. Wu, *Boundedness of Singular Integrals on Multiparameter Weighted Hardy Spaces* $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$, Potential Anal. **37** (2012), 31–56.

[22] X.-T. Duong, S. Hofmann, D. Mitrea, M. Mitrea and L. Yan, *Hardy spaces and regularity for the inhomogeneous Dirichlet and Neumann problems*, Rev. Mat. Iberoam. **29** (2013), 183–236.

[23] X.-T. Duong, E. M. Ouhabaz and A. Sikora, *Plancherel-type estimates and sharp spectral multipliers*, J. Funct. Anal. **196** (2002), 443–485.

[24] X.-T. Duong and L. Yan, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. **58** (2005), 1375–1420.

[25] X.-T. Duong and L. Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. **18** (2005), 943–973.

[26] J. Dziubański and M. Preisner, *Riesz transform characterization of Hardy spaces associated with Schrödinger operators with compactly supported potentials*, Ark. Math. **48** (2010), 301–310.

[27] J. Dziubański and J. Zienkiewicz, *Hardy space* H^1 *associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iberoam. **15** (1999), 279–296.

[28] J. Dziubański and J. Zienkiewicz, H^p *spaces for Schrödinger operators, Fourier analysis and related topics* (Bpolhk edlewo, 2000), 45–53, Banach Center Publ. **56**, Polish Acad. Sci., Warsaw, 2002.

[29] J. Dziubański and J. Zienkiewicz, H^p *spaces associated with Schrödinger operators with potentials from reverse Hölder class*, Colloq. Math. **98** (2003), 5–38.

[30] C. Fefferman and E.M. Stein, H^p *spaces of several variables*, Acta Math. **129** (1972), 137–193.

[31] T.M. Flett. *Temperatures, Bessel potentials and Lipschitz spaces*, Proc. Lond. Math. Sot. (3) **22** (1971), 385–451.

[32] H. Führ and A. Mayeli, *Homogeneous Besov spaces on stratified Lie groups and their wavelet characterization*, J. Funct. Spaces Appl. **2012** (2012).

[33] G.B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Math. **13** (1975), 161–207.

[34] G.B. Folland and E.M. Stein, *Hardy Spaces on Homogeneous Groups*, Princetion Univ. Press, Princeton, NJ, 1982.

[35] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), 34–170.

[36] R. Gong and L. Yan, *Littlewood-Paley and spectral multipliers on weighted L^p spaces*, J. Geom. Anal. **24** (2014), 873–900.

[37] L. Grafakos, *Mordern Fourier Analysis*, second ed., Grad. Texts in Math., vol 250, Springer, New York, 2008.

[38] L. Grafakos and X. Li, *Bilinear operators on homogeneous groups*, J. Operator Theory **44** (2000), 63–90.

[39] L. Grafakos, L. Liu, and D. Yang, *Vector-valued singular integrals and maximal functions on spaces of homogeneous type*, Math. Scand. **104** (2009). 296–310.

[40] Y. Guivarc'h, *Croissance polynômiale et période des functions harmoniques*, Bull. Soc. Math. France **101** (1973), 149–152.

[41] P. Gyrya and L. Saloff-Coste, *Neumann and Dirichlet heat kernels in inner uniform domains*, Astérisque **336** (2011), viii+144 pp.

[42] Y. Han, *Triebel-Lizorkin spaces on spaces of homogeneous type*, Studia Math. **108** (1994), 247–273.

[43] Y. Han, D. Müller and D. Yang, *A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces*, Abstr. Appl. Anal. **2008**, Art. ID 893409, 250 pp.

[44] Y. Han and E.T. Sawyer, *Littlewood-Paley theory on spaces of homogeneous type and the classical function spaces*, Mem. Amer. Math. Soc. **110** (1994), no. 530, vi+126 pp.

[45] Y. Han and D. Yang, *Some new spaces of Besov and Triebel-Lizorkin type on homogeneous spaces*, Studia Math. **156** (2003), 67–97.

[46] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L. Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*, Mem. Amer. Math. Soc. **214** (2011), no. 1007, vi+78 pp.

[47] S. Hofmann and S. Mayboroda, *Hardy and BMO spaces associated to divergence form elliptic operators*, Math. Ann. **344** (2009), 37–116.

[48] G. Hu, *Littlewood-Paley characterization of weighted anisotropic Hardy spaces*, Taiwanese J. Math. **17** (2013), 675–700.

[49] G. Hu, *Homogeneous Triebel-Lizorkin spaces on stratified Lie groups*, J. Funct. Spaces Appl. (2013). Article ID 475103, 16 pp.

[50] G. Hu, *Besov and Triebel-Lizorkin spaces associated with non-negative self-adjoint operators*, J. Math. Anal. Appl. **411** (2014), 753–772.

[51] G. Hu, *Maximal Hardy spaces associated to non-negative self-adjoint operators*, Bull. Aust. Math. Soc. (to appear).

[52] G. Hu, *Littlewood-Paley characterization of Hardy spaces associated to self-adjoint operators*, submitted.

[53] A. Hulanicki, *A functional calculus for Rockland operators on nilpotent Lie groups*, Studia Math. **78** (1984), 253–266.

[54] B. Jawerth, *Some observations on Besov and Lizorkin-Triebel spaces*, Math. Scand. **40** (1977), 94-104.

[55] R. Jiang, X. Jiang and D. Yang, *Maximal function characterizations of Hardy spaces associated with Schrödinger operators on nilpotent Lie groups*, Rev. Mat. Complut **24** (2011), 251–275.

[56] R. Jiang and D. Yang, *Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates*, Commun. Contemp. Math. **13** (2011), 331–373.

[57] R. Johnson, *Temperatures, Riesz potentials. and the Lipschitz spaces of Herz*, Proc. Lond. Math. Soc. (3) **27** (1973), 290–316.

[58] T. Kato, *Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups*, Topics in functional analysis (essays dedicated to M. G. Kreĭn on the occasion of his 70th birthday), pp. 185–195, Adv. in Math. Suppl. Stud., 3, Academic Press, New York, 1978.

[59] G. Kerkyacharian and P. Petrushev, *Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces*, Trans. Amer. Math. Soc. **367** (2015), 121–189.

[60] P. Li and S.T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986) 153–201.

[61] J.L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications. Vol. I*, Springer-Verlag, New York-Heidelberg, 1972.

[62] A. Nagel, E.M. Stein and S. Walnger, *Balls and metrics defined by vector fields*, Acta Math. **155** (1985), 103–147.

[63] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954.

[64] T. Noi and Y. Sawano, *Complex interpolation of Besov spaces and Triebel-Lizorkin spaces with variable exponents*, J. Math. Anal. Appl. **387** (2012), 676–690.

[65] E.M. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, Vol. 31, Princeton University Press, 2005.

[66] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I: Functional Analysis*, Academic Press, Orlando, 1980.

[67] W Rudin, *Functional Analysis*, second ed., McGraw-Hill Inc, New York, 1991.

[68] V.S. Rychkov, *On a theorem of Bui, Paluszyński and Taibleson*, Proc. Steklov Inst. **227** (1999), 280–292.

[69] K. Saka, *Besov and Sobolev spaces on nilpotent Lie groups*, Tohoku Math. J. (2) **31** (1979), 383–437.

[70] L. Saloff-Coste, *A note on Poincaré, Sobolev and Harnack inequalities*, Int. Math. Res. Not. IMRN 1992, no. 2, 27–38.

[71] L. Saloff-Coste, *Parabolic Harnack inequality for divergence form second order differential operators*, Potential Anal. **4** (1995) 429–467.

[72] L. Schwartz, *Théorie des distributions*, Hermannn, Paris, 1966.

[73] B. Simon, *Maximal and minimal Schrödinger forms*, J. Operator Theory **1** (1979), 37–47.

[74] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[75] E.M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Series **45**, Princeton Univ. Press, Princeton, NJ, 1993.

[76] E.M. Stein and G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of H^p-spaces*, Acta Math. **103** (1960), 25–62.

[77] R.S. Strichartz, *Analysis of the Laplacian on complete Riemannian manifolds*, J. Funct. Anal. **52** (1983), 48–79.

[78] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math., vol 1381, Springer-Verlag, Berlin, 1989.

[79] A. Sikora and J. Wright, *Imaginary powers of Laplace operators*, Proc. Amer. Math. Soc. **129** (2001), 1745–1754.

[80] G. Szegö, *Orthogonal Polynomials*, Fourth edition, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, 1975.

[81] M.H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean nspace, I. Principal properties; II. Translation invariant operators, duality, and interpolation.* J. Math. Mech. **13** (1964), 407–479; **14** (1965), 821–839.

[82] H. Triebel, *Complex Interpolation and Fourier Multipliers for the Spaces $B^s_{p,q}$ and $F^s_{p,q}$ of Besov-Hardy-Sobolev Type: The case* $0 < p \leq \infty$, $0 < q \leq \infty$, Math. Z. **176** (1981), 495–510.

[83] H. Triebel. *Theory of Function Spaces*, Monographs in Mathematics, vol 78, Birkhäuser, Basel, 1983.

[84] T. Ullrich, *Continuous characterizations of Besov-Lizorkin-Triebel spaces and new interpretations as coorbits*, J. Funct. Spaces Appl. (2012). Article ID 163213, 47 pp.

[85] N. Varopoulos, *Fonctions harmoniques sur les groupes de Lie*, C. R. Acad. Sci. Paris, Série I, Math. **309** (1987), 519–521.

[86] N. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and geometry on groups*, Cambridge Univ. Press, Cambridge, 1992.

[87] S. Yang, *A Sobolev extension domain that is not uniform*, Manuscripta Math. **120** (2006), 241–251.