研 究 速 報

A Note on Possible Natural Frequencies of In-plane Swing of a Sagging Chain Consisting of Rigid Links

Shigeru NAKAGIRI*

1. Introduction

The motion of chain of flexible links has been investigated by numerical and experimental analysis[1)], and formulated by use of the Hamilton's principle in problems of multi-body dynamics where attempt was made to analyse the motion as the sum of rigid-body motion and small vibration in addition to geometrically nonlinear approach[2)]. Hence the author has been interested in whether or not the concept of eigenmodes holds for the swing of multiply interconnected rigid bodies, and a chain of rigid bodies has natural frequencies. It was reported in the preceding note[3)] that an eigenvalue problem can be constituted on the basis of the Lagrange equations of motion expressed by the resilience matrix due to gravitational force and the inertia matrix due to rotatory moment of inertia. The mode shapes of a chain hanging down from a support were depicted together with the possible natural frequencies. This note describes the eigenvalue problem of a chain of rigid links that saggs down between two supports.

2. Statement of the problem

Suppose that a chain consisting of N identical rigid links interconnected by frictionless joints without play saggs down freely between two supports, whose geometrical layout is specified, and swings in a vertical plane without twist. The chain has tensile stiffness only, and has neither compressive stiffness nor bending rigidity because any rotatory moment cannot be transferred through the joints and supports. Damping of all types is assumed negligibly small. The problem of this note is to find the possible natural frequencies of the chain when the mass S and rotatory moment of inertia I_O of each link of l in length and b in breadth are given together with the horizontal distance B and vertical distance H of the two supports as shown schematically in Fig.1.

*1 st Department, Institute of Industrial Science, University of Tokyo

3. Shape finding of a sagging chain at rest

In the first place the configuration of the chain at rest, in the vicinity of which the chain swings, is to be identified. This shape finding can be carried out by means of minimizing the gravitational energy under equality constraint conditions imposed to realize the support layout as specified. The functional for the Lagrange multiplier method to describe the minimization is expressed by Eq.(1), where ϕ_n indicates the link angle of the n-th link shown in Fig.1, and Σ the sum from n=1 to N hereafter. The first term of the right hand side of Eq.(1) is derived on the assumption that the acting forces are self-weight of each link that passes through the center of gravity taken at the mid-point of the link. The functional is non-dimensionalized by means of dividing the energy by Sgl, while μ_x and μ_y are the non-dimensionalized Lagrange multipliers by means of dividing the constraint forces in horizontal direction and vertical one by Sg. The gravitational constant is denoted by g. B/l is denoted by B', and H/l by H'.

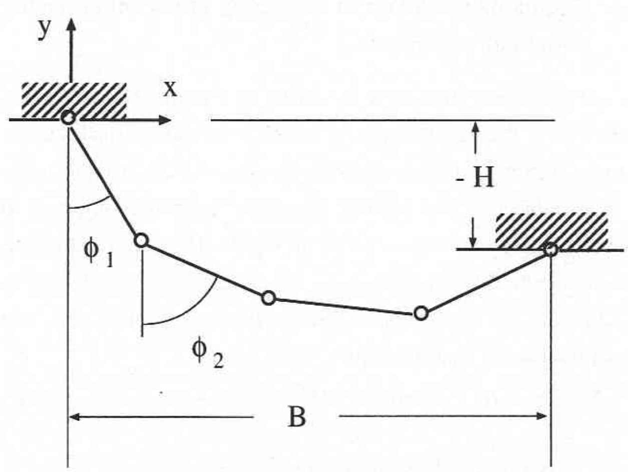

Fig. 1 Sagging chain of four links at rest

$$\Pi = -\Sigma\left(N - n + 0.5\right)\cos\phi_n + \mu_x\left(\Sigma\sin\phi_n - B'\right) + \mu_y\left(-\Sigma\cos\phi_n - H'\right) \quad \cdots\cdots\cdots (1)$$

The stationary conditions of the functional with respect to the link angles and multipliers result in their nonlinear simultaneous equations as follows.

$$\frac{\partial\Pi}{\partial\phi_n} = \left(N - n + 0.5 + \mu_y\right)\sin\phi_n + \mu_x\cos\phi_n = 0 \qquad n = 1 \sim N$$
$$\frac{\partial\Pi}{\partial\mu_x} = \Sigma\sin\phi_n - B' = 0$$
$$\frac{\partial\Pi}{\partial\mu_y} = -\Sigma\cos\phi_n - H' = 0 \quad \cdots\cdots (2)$$

These are solved by putting $\phi_n = \bar{\phi}_n + \Delta\phi_n$ and $\mu_j = \bar{\mu}_j + \Delta\mu_j$ (the mark of upper bar denotes the current guess and j is taken as either x or y) and through Eq.(3) for the determination of small unknowns $\Delta\phi_n$ and $\Delta\mu_j$ by the Newton method.

$$\begin{bmatrix} A & C \\ sym. & B \end{bmatrix}\begin{Bmatrix} \Delta\phi_n \\ \Delta\mu_j \end{Bmatrix} = -\{F\} \quad \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

The matrix $[A]$ is diagonal with the ingredients $A_{nn} = (N-n+0.5+\mu_y)\cos\phi_n - \mu_x\sin\phi_n$, the matrix $[B]$ is nill, and [C] is N x 2 matrix with the ingredients $C_{n1} = \cos\phi_n$ and $C_{n2} = \sin\phi_n$. The ingredients of the vector $\{F\}$ are the central terms of Eq.(2), whose ϕ_n and μ_j are replaced with $\bar{\phi}_n$ and $\bar{\mu}_j$. The shape of the chain at rest is found when the iteration of the Newton method is converged enough to fix the link angles of the standstill chain. The angles are not small necessarily and called rest link angle hereafter. This formulation hints that the chain shape is governed by N, B' and H' only.

4. Equations of motion of a sagging chain with equality constraint conditions

The total link angle ψ_n of the swinging n-th link is expressed as the sum of the rest link angle ϕ_n, which is constant as made known in the section 3, and dynamic (relative) one θ_n. The total link angle ψ_n is subject to two equality constraint conditions imposed to assure the support layout given by Eq.(4). The constraint conditions of the dynamic link angle θ_n are derived from Eq.(4) as Eq.(5) by use of the approximate formulae of $\sin\theta_n = \theta_n$ and $\cos\theta_n = 1$ for θ_n that is assumed small.

$$\Sigma\sin\psi_n = B', \quad -\Sigma\cos\psi_n = H' \quad \cdots\cdots\cdots\cdots\cdots (4)$$

$$\Sigma\,\theta_n\cos\phi_n = 0, \ \Sigma\,\theta_n\sin\phi_n = 0 \quad \cdots\cdots\cdots\cdots\cdots (5)$$

All the dynamic link angles θ_n cannot be taken as independent generalized coordinates q_n because of the constraint conditions of Eq.(5). In such cases of holonomic system on the conditions of Eq.(5) the Lagrange equations of motion are given in the following form by means of introducing Lagrange multipliers[4],

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_n}\right) - \frac{\partial T}{\partial q_n} + \frac{\partial}{\partial q_n}[C]\{\mu\} = Q_n \quad \cdots\cdots\cdots\cdots\cdots (6)$$

where T denotes the kinetic energy of the swinging chain, t the time, $[C]$ the constraint condition matrix derived from Eq.(5) to be found identical with the one in Eq.(3), $\{\mu\}$ the vector whose ingredients are dynamic Lagrange multipliers μ_x and μ_y, and Q_n the generalized forces associated with the generalized coordinates. For the sake of concrete and brief expression of the equations the chain consisting of three links is dealt with in this section. The kinetic energy is given by Eq.(7),

$$T = I_O\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right)/2 + \left(Sl^2/2\right)\left\{2\dot{\theta}_1^2 + 3\dot{\theta}_1\dot{\theta}_2\cos\left(\psi_2 - \psi_1\right) + \dot{\theta}_2^2 + \dot{\theta}_2\dot{\theta}_3\cos\left(\psi_3 - \psi_2\right) + \dot{\theta}_3\dot{\theta}_1\cos\left(\psi_1 - \psi_3\right)\right\} \quad \cdots\cdots (7)$$

where $I_O = I_C + Sl^2/4$ means the rotatory moment of inertia of a link around the upper joint while I_C is the one around the center of gravity of the link. The mark of upper dot indicates differentiation with respect to time. The virtual work δW done by the vertical gravitational forces acting at the center of gravity of each link is summarized in the following form for the virtual dynamic link angle $\delta\theta_n$, giving rise to the generalized forces of $Q_n = \delta W/\delta\theta_n$.

$$\delta W = \left(Sgl/2\right)\left(-5\sin\psi_1\delta\theta_1 - 3\sin\psi_2\delta\theta_2 - \sin\psi_3\,\delta\theta_3\right) \quad \cdots\cdots\cdots\cdots (8)$$

Substitution of Eqs.(7) and (8) into Eq.(6) gives rise to the equations of motion aimed at. The equations of motion obtained by differentiation with respect to θ_1, θ_2, and θ_3 are summarized as follows.

$$\begin{aligned}
&\left(I_O + 2Sl^2\right)\ddot{\theta}_1 + \left(Sl^2/2\right)\left\{3\ddot{\theta}_2\cos\left(\phi_2 - \phi_1\right) + \ddot{\theta}_3\cos\left(\phi_3 - \phi_1\right)\right\} + \left(Sgl\right)\left(2.5\sin\psi_1 + \mu_x\cos\psi_1 + \mu_y\sin\psi_1\right) = 0 \\
&\left(3Sl^2/2\right)\ddot{\theta}_1\cos\left(\phi_2 - \phi_1\right) + \left(I_O + Sl^2\right)\ddot{\theta}_2 + \left(Sl^2/2\right)\ddot{\theta}_3\cos\left(\phi_3 - \phi_2\right) + \left(Sgl\right)\left(1.5\sin\psi_2 + \mu_x\cos\psi_2 + \mu_y\sin\psi_2\right) = 0 \\
&\left(Sl^2/2\right)\left\{\ddot{\theta}_1\cos\left(\phi_3 - \phi_1\right) + \ddot{\theta}_2\cos\left(\phi_3 - \phi_2\right)\right\} + I_O\ddot{\theta}_3 + \left(Sgl\right)\left(0.5\sin\psi_3 + \mu_x\cos\psi_3 + \mu_y\sin\psi_3\right) = 0
\end{aligned} \quad (9)$$

In doing so, such approximation is used as $\ddot{\theta}_2\cos(\psi_2-\psi_1) = \ddot{\theta}_2\cos(\phi_2-\phi_1)$ derived on the assumption that $\ddot{\theta}_2(\theta_2-\theta_1)\sin(\phi_2-\phi_1)$ is negligibly small.

The unknown dynamic Lagrange mutipliers are assumed as the sum of the multipliers obtained at the rest state and made known together with the rest link angles in the section 3, $\bar{\mu}_x$ and $\bar{\mu}_y$, and small unknown increments $\Delta\mu_x$ and $\Delta\mu_y$. The last three terms of the left hand sides of Eq.(9) are rewritten in the form of Eq.(10), for instance in the case of the differentiation with respect to θ_1, by the formula of Eq.(11) that holds for the chain at rest and approximate equation of Eq.(12) based on the assumption that the products such as $\Delta\mu_j\theta_1$ are negligibly small.

$$\begin{aligned}(Sgl)\left(2.5\sin\psi_1+\mu_x\cos\psi_1+\mu_y\sin\psi_1\right)\\ =(Sgl)\left[\left\{\left(2.5+\bar{\mu}_y\right)\cos\phi_1-\bar{\mu}_x\sin\phi_1\right\}\theta_1\right.\\ \left.+\left(\Delta\mu_x\cos\phi_1+\Delta\mu_y\sin\phi_1\right)\right]\end{aligned} \qquad \cdots\cdots\cdots\cdots (10)$$

$$\left(2.5+\bar{\mu}_y\right)\sin\phi_1+\bar{\mu}_x\cos\phi_1=0 \qquad \cdots\cdots\cdots\cdots (11)$$

$$\Delta\mu_x\theta_1\sin\phi_1-\Delta\mu_y\theta_1\cos\phi_1\simeq 0 \qquad \cdots\cdots\cdots\cdots (12)$$

5. Derivation of eigenvalue problem with rectangular matrices

When Eqs.(9) and (10) are combined and arranged duly for N dynamic link angles and two incremental Lagrange multipliers, Eq.(13) is obtained as linearized equations of motion based on the assumption that those unknowns are small, and can be converted easily into an eigenvalue problem,

$$\left[[K],\ [C]\right]\begin{Bmatrix}\theta\\ \Delta\mu\end{Bmatrix}+\left[[M],\ [0]\right]\begin{Bmatrix}\ddot{\theta}\\ \Delta\ddot{\mu}\end{Bmatrix}=\{0\} \qquad \cdots\cdots\cdots\cdots (13)$$

where $[K]$ is the N x N real, diagonal matrix with the ingredients of $K_{nn}=(N\text{-}n+0.5+\alpha+\bar{\mu}_y)\cos\phi_n-\bar{\mu}_x\sin\phi_n$, $[M]$ the N x N real, symmetric matrix with the ingredients of $M_{nn}=(N\text{-}n+\alpha)\ \beta$ and $M_{nl}=(N\text{-}l+0.5)\ \beta\cos(\phi_l\text{-}\phi_n)$ for $l>n$, $\alpha=(4+b^2/l^2)/12$ and $\beta=l/g$. Equation (13) implies that the resultant eigenvalue problem is associated with N x $(N+2)$ matrix which is rectangular linewise. This type of eigenvalue problem with the rectangular matrices can be dealt with as follows. The essential part of the eigenvalue problem is expressed in the form of Eq.(14),

$$\left([K]-\lambda[M]\right)\{\theta\}+[C]\{\Delta\mu\}=\{0\} \qquad \cdots\cdots\cdots\cdots (14)$$

where the eigenvalue is denoted by $\lambda=(2\pi f)^2$, f being the natural frequency. $\{\theta\}$ and $\{\Delta\mu\}$ are component of the eigenvector. It is worthy to note that the equality constraint conditions of Eq.(5) takes the form of $[C]^T\{\theta\}=\{0\}$ with respect to the rectangular matrix $[C]$. The eigenvector component $\{\theta\}$ is determined in the following form by virtue of the Moore-Penrose generalized inverse, which is indicated by the superfix (-) hereafter, of any rectangular matrix[5].

$$\begin{aligned}\{\theta\}&=\left([C]^T\right)^{-}\{0\}+\left([I_N]-\left([C]^T\right)^{-}[C]^T\right)\{h\}\\ &=[D]\{h\}\end{aligned} \qquad \cdots\cdots\cdots\cdots (15)$$

$[I_N]$ denotes N-dimensional identity matrix, and $\{h\}$ is an arbitrary vector. This equation indicates that the mode shape of $\{\theta\}$ is governed by the matrix $[C]$, that is, the chain shape at rest, and that the i-th column vector $\{d_i\}$ of $[D]$ can be taken as the i-th eigenvector component $\{\theta\}^i$. By premultiplying $\{\theta\}^T$ to Eq.(14) and taking into account the formula of $([C]^T\{\theta\})^T\{\Delta\mu\}=\{\theta\}^T[C]\{\Delta\mu\}=\{0\}^T\{\Delta\mu\}=0$, the i-th eigenvalue λ^i related to the i-th eigenvector component $\{\theta\}^i$ can be calculated by the formula (16). Then the corresponding eigenvector component $\{\Delta\mu\}^i$ is determined as Eq.(17) as the least-square approximate solution of Eq.(14), since the exact solution may not exist in the case that $[C]$ is rectangular columnwise with N larger than 2[5].

$$\lambda^i=\{d_i\}^T[K]\{d_i\}\ /\ \{d_i\}^T[M]\{d_i\} \qquad \cdots\cdots\cdots\cdots (16)$$

$$\{\Delta\mu\}^i=-[C]^{-}\left([K]-\lambda^i[M]\right)\{d_i\} \qquad \cdots\cdots\cdots\cdots (17)$$

6. Numerical examples

The numerical examples are concerned with chains of 320 mm in full length, and the link breadth is set equal to 10 mm while the link length l is changed by N. g is taken equal to 9.8 m/sec^2. Figure 2 shows the comparison between the shape of a horizontal 5-link chain at rest by B = 300 mm and H = 0 mm (indicated by solid line hereafter) and the mode shape of the lowest natural frequency of 5.482 Hz (dotted line). The largest ingredient of $\{d_i\}$ is normalized equal to 0.1 to set the coordinates of the link joints in swing. Then the radius defined as the distance between the joints and the ones at rest is calculated. The link joints in swing are depicted with the radius magnified by factor of 5. Figure 3 illustrates the mode shapes of the minimum frequency and maximum frequency (broken line) of a horizontal 8-link chain of B = 150 mm. The minimum and maximum natural frequencies are shown in the figure. The magnification factor of the radius is set equal to 30 in the following figures. The mode shapes of the chain inclined by B = 170 mm and H = 60 mm are shown in Fig. 4. The

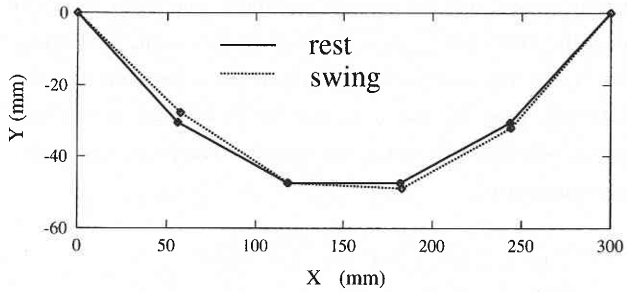

Fig. 2　Mode shapes of 5-link chain

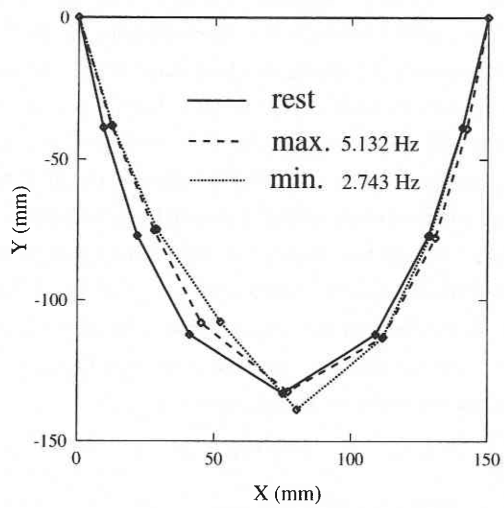

Fig. 3　Comparison of mode shapes of 8-link chain

effect of the link number is seen in Fig. 5 in the case of $N = 16$, $B = 150$ mm and $H = 150$ mm. From these figures it can be said for the swing of sagging chains that the sway mode is primary, the natural frequencies are neither dispersed widely nor changed largely even when inclined, and rather increased with the number of links.

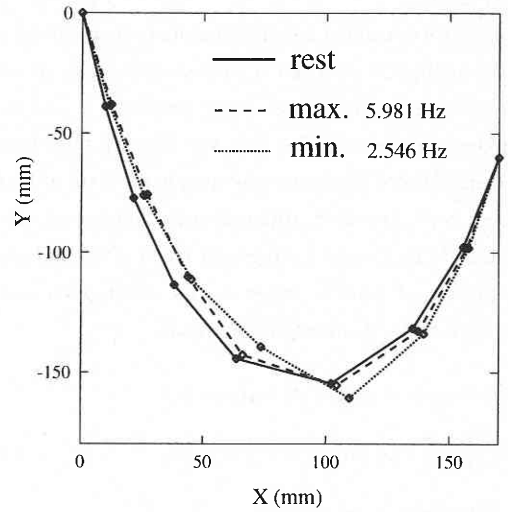

Fig. 4　Mode shapes of inclined 8-link chain

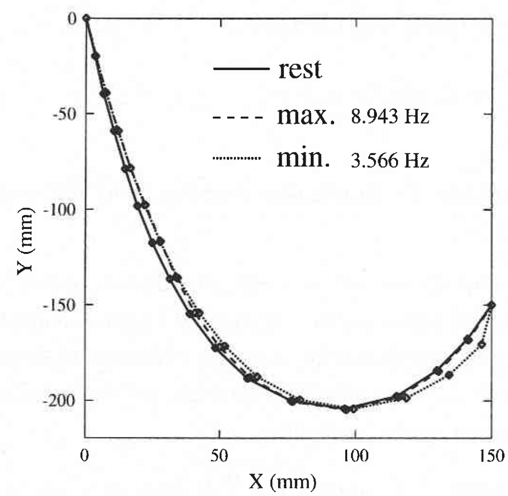

Fig. 5　Mode shapes of inclined 16-link chain

7. Conclusion

The eigenvalue problem of a chain that saggs down from two supports is constituted from the linearized Lagrange equations of motion formulated with the introduction of the Lagrange multiplier method and on the assumption that the dynamic link angles and multipliers are small. A formulation is devised to deal with the eigenvalue problem with rectangular matrices. The numerical examples hint that the natural frequencies are not dispersed widely compared with those of single-body beams.

(Manuscript received, February 17, 2000)

References

1) Giovagnoni, M.; A Numerical and Experimental Analysis of a Chain of Frexible Bodies, *ASME J. of Dynamic Systems, Measurement, and Control*, Vol. 116, No. 1 (1994), pp. 73-80.

2) Benati, M. and Morro, A.; Formulation of Equations of Motion for a Chain of Flexible Links Using Hamilton's Principle, *ASME J. of Dynamic Systems, Measurement, and Control*, Vol. 116, No. 1 (1994), pp. 81-88.

3) Nakagiri, S.; A Note on Possible Natural Frequencies of In-plane Swing of a Hanging Chain, *SEISAN-KENKYU*, Vol. 51, No. 9 (1999), pp. 674-677.

4) Shabana, A. A.; *Dynamics of Multibody Systems*, John Wiley & Sons, 1989.

5) Rao, C. R. and Mitra, S. K.; *Generalized Inverse of Matrices and its Applications*, John Wiley & Sons, 1971.