研 究 速 報

A Note on Possible Natural Frequencies of In-plane Swing of a Hanging Chain

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1. Introduction

Vibration of a solid, continuous body is governed by balance of resilience and inertia of the body. In the context of finite element analysis the natural frequencies of an elastic body in undamped, linear eigenvalue problem are calculated based on the stiffness matrix and mass matrix in the case that the resilience of the body is duly defined through stiffness. For example, flexural rigidity of beams, plates and shells plays an important role in the investigation of their natural frequencies of bending vibration.

On the other hand dynamics of multibody systems has been a focused subject in recent years since interconnected rigid and deformable bodies such as robotic manipulators and space structures have attracted industrial interest[1]. A chain is not a single body, but consists of multiple bodies, being able to be modeled as an assembly of links serially interconnected by revolute joints in the way that free rotation is allowed between the adjacent links because any mechanism of transferring moment from link to link is not provided. Consequently chains hardly have stiffness against compressive force and bending moment. This means that the concept of flexural rigidity cannot be applied to a whole chain no matter how stiff each link is.

Chains swing when exited. Then a question might be raised prior to such elucidation of dynamic behavior of chains as kinematic analysis of the swing motion, rebounding of chains[2] and dynamic analysis of spaghetti problem[3] ; Do chains have natural frequencies while countable resilience against bending is nonexistent? This note discusses the possibility to seek the natural frequencies of swing of a chain.

2. Statement of the problem

Suppose that a chain consists of N stiff links interconnected by revolute joints and hangs down vertically in a standstill due to gravitation. The top end of the chain is fixed in the manner that translational movement at the end is prohibited, but rotational movement of the chain is allowed in the x-y plane as shown in Fig.1. All the links are assumed without loss of generality to be identical, having the same moment of inertia I_o, mass S and length l. The moment of transverse inertia I_o is defined as the one around the upper joint of the link as mentioned later. The distance between the upper revolute joint and center of gravity of each link is set equal to $l/2$. The problem in this note is to seek the possible natural frequencies of the in-plane swing of the chain for given number of links, moment of inertia, mass and length. The effect of an anchor attached at the bottom end of the chain on the swing is also discussed. The mass of the anchor is denoted by αS. α is a nondimensional coefficient that stands for mass ratio of the anchor to a link. The followings are assumed.

a. The chain swings in a vertical plane without twist.

b. All damping is negligibly small.

c. The links are rigid and revolute joints are frictionless. Then

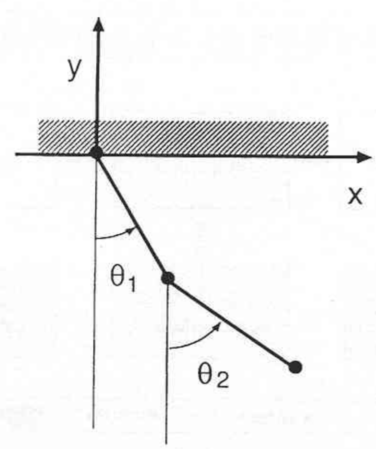

Fig.1 Swing of two-link chain

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the chain has infinite tensile stiffness only but no flexural rigidity as a whole because any moment cannot be transferred from link to link through the joint.

d. The links can be pictured as flat rectangulat plate with the joints when their moment of inertia and mass are calculated. The mass distribution in the links is uniform.

f. The mass of the anchor, when attached, is concentrated at the bottom end of the chain. Then the moment of inertia of the anchor is negligible.

3. Derivation of the equations of motion

When free swing of a hanging chain, only the top end of which is pinned with the freedom to rotate in the x-y plane, is discussed, the angle between the n-th link (numbered from the top end) to the vertical line θ_n, briefly called link angle hereafter, can be taken as independent generalized coordinate. For the sake of brevity the expressions in this section are written for the case of two links. The geometric constraints for the center of gravity of each link are expressed explicitly by the link angles owing to the assumption of rigid links. Equation (1) shows the constraints in the simple case of two links whose coordinates of the center of gravity are denoted by R_{x1} through R_{y2}. R_{xb} and R_{yb} denote the coordinates of the bottom end of the chain.

$$\begin{aligned} &R_{x1} = (l/2)\sin\theta_1, \quad R_{y1} = -(l/2)\cos\theta_1 \\ &R_{x2} = l(\sin\theta_1 + 0.5\sin\theta_2), \\ &R_{y2} = -l(\cos\theta_1 + 0.5\cos\theta_2) \\ &R_{xb} = l\sum_{n=1}^{N}\sin\theta_n, \quad R_{yb} = -l\sum_{n=1}^{N}\cos\theta_n \end{aligned} \qquad \cdots\cdots(1)$$

The velocity and virtual displacement of the coordinates, of the link 1 for example, are given as follows. The upper dot indicates differentiation with respect to time.

$$\begin{aligned} &\dot{R}_{x1} = (l/2)\dot{\theta}_1\cos\theta_1, \quad \dot{R}_{y1} = (l/2)\dot{\theta}_1\sin\theta_1 \\ &\delta R_{x1} = (l/2)\cos\theta_1\,\delta\theta_1, \quad \delta R_{y1} = (l/2)\sin\theta_1\delta\theta_1 \end{aligned} \qquad \cdots\cdots(2)$$

The Lagrange's equation employed to express the multibody dynamics of chain is written in the following form,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} - Q_n = 0, \quad n = 1\sim N \qquad \cdots\cdots(3)$$

where T denotes the kinetic energy of the chain as the sum of that due to translational motion and that due to rotational motion of two links, and Q_n is the component of the generalized force associated with the generalized coordinate q_n[1]. The kinetic energy is expressed by the sum of those of two links and an anchor as given below.

$$\begin{aligned} T &= \frac{1}{2}\left[S\left(\dot{R}_{x1}^2 + \dot{R}_{y1}^2\right) + I_c\dot{\theta}_1^2 + S\left(\dot{R}_{x2}^2 + \dot{R}_{y2}^2\right)\right. \\ &\quad \left. + I_c\dot{\theta}_2^2\right] + \frac{1}{2}\alpha S\left(\dot{R}_{xb}^2 + \dot{R}_{yb}^2\right) \\ &= \frac{1}{2}I_0\,\dot{\theta}_1^2 + \left(\frac{Sl^2}{2}\right)\left\{\dot{\theta}_1^2 + \dot{\theta}_1\,\dot{\theta}_2\cos(\theta_2 - \theta_1)\right\} \\ &\quad + \frac{1}{2}I_o\,\dot{\theta}_2^2 + \frac{1}{2}\alpha S\left(\dot{R}_{xb}^2 + \dot{R}_{yb}^2\right) \end{aligned} \qquad \cdots\cdots(4)$$

The mass of each link S is calculated as the product of mass density ρ of the material, length l, breadth b and thickness t. The moment of inertia of each link around the axis that is perpendicular to the x-y plane and passes through the center of gravity I_c is equal to $S(l^2+b^2)/12$. The moment of inertia around the axis that is parallel to the aforementioned axis and passes through the upper joint I_o is obtained as $I_c + Sl^2/4$.

The virtual work done by the virtual displacements is expressed by Eq. (5) for the forces acting at the center of gravity F_{x1} through F_{y2} and moments M_{c1} and M_{c2} around the center of gravity of each link. F_{xb} and F_{yb} are the force components acting on the anchor.

$$\begin{aligned} \delta W &= F_{x1}\delta R_{x1} + F_{y1}\delta R_{y1} + M_{c1}\delta\theta_1 \\ &\quad + F_{x2}\delta R_{x2} + F_{y2}\delta R_{y2} + M_{c2}\delta\theta_2 \\ &\quad + F_{xb}\delta R_{xb} + F_{yb}\delta R_{yb} \\ &= M_{o1}\delta\theta_1 + \left(F_{x2}\cos\theta_1 + F_{y2}\sin\theta_1\right)l\delta\theta_1 + M_{o2}\delta\theta_2 \\ &\quad + F_{xb}\delta R_{xb} + F_{yb}\delta R_{yb} \end{aligned} \qquad \cdots\cdots(5)$$

The moments of inertia in the above equation are modified from M_{cn} into M_{on} that concerns with the upper joint by the formula of Eq.(6).

$$M_{on} = \left(F_{xn}\cos\theta_n + F_{yn}\sin\theta_n\right)(l/2) + M_{cn}, \quad n = 1\sim 2 \qquad \cdots(6)$$

The component of generalized force associated with the link angle taken as generalized coordinate is obtained from the virtual work as follows.

$$Q_n = \frac{\partial W}{\partial \theta_n} \qquad \cdots\cdots(7)$$

The equations of motion with respect the link angles are obtained by means of carrying out the differentiation of the kinetic energy in Eq.(3) and substituting Eq.(7) into the result in the form of Eq.(8).

$$\begin{aligned} &\left\{I_O + (1+\alpha)Sl^2\right\}\ddot{\theta}_1 + (1+2\alpha)(Sl^2/2)\left\{\ddot{\theta}_2\cos(\theta_2-\theta_1)\right. \\ &\quad \left. - \dot{\theta}_1^2\sin(\theta_2-\theta_1)\right\} = -Sgl(3/2+\alpha)\sin\theta_1 \\ &(1+2\alpha)(Sl^2/2)\left\{\ddot{\theta}_1\cos(\theta_2-\theta_1) + \dot{\theta}_1^2\sin(\theta_2-\theta_1)\right\} + I_O\ddot{\theta}_2 \\ &\quad = -Sgl(1/2+\alpha)\sin\theta_2 \end{aligned} \qquad (8)$$

In the above derivation, only the influence of gravity is taken into account through $F_{y1}=F_{y2}=-Sg$ and $F_{yb}=-\alpha Sg$, while the other external loads are set equal to zero by $F_{x1}=F_{x2}=F_{xb}=0$ and $M_{c1}=M_{c2}=0$. The acceleration of gravity is denoted by g.

4. Constitution of Eigenvalue Problem

The equations of motion obtained in the section 3 can be used as the basis of a governing equation of dynamic analysis of chain motion in the future. In this section, however, attention is paid to the constitution of possible eigenvalue problem of the swing based on the equations of motion. As usual, $\cos\theta$ is approximated equal to unity and $\sin\theta$ equal to θ for small θ based on the hypothesis that the displacements and/or deformation of the chain can be taken small when eigenvalue problem is discussed. Furthemore, it is assumed that the terms with quadratic quantity such as $(\dot{\theta}_2^{\,2})\ \sin\ (\theta_2 - \theta_1)$ can be neglected. The reasoning of this assumption is that the angular velocity of the chain is nill at the full swing, and that the link angles are nill at the vertical line when the angular velocity is at the largest. Then the following approximate equations are obtained from Equation (8) and summarized in the matrix form of Eq.(9).

$$\begin{bmatrix} I_O + (2+2\alpha)(Sl^2/2) & (1+2\alpha)(Sl^2/2) \\ SYM. & I_0 + (2\alpha)(Sl^2/2) \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + Sgl \begin{bmatrix} 3/2+\alpha & 0 \\ SYM. & 1/2+\alpha \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \cdots\cdots\cdots (9)$$

$$\left(-\lambda [B] + [A]\right)\{\phi\} = \{0\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (10)$$

It is easily seen that Eq. (9) is modified into the eigenvalue problem of Eq.(10). The matrices [A] and [B] are N-dimensional, real and symmetric, giving rise to N real eigenvalues λ. The corresponding eigenvector is denoted by $\{\phi\}$. The matrix [A] is called resilience matrix, and [B] inertia matrix, respectively, hereafter.

Even in case that the number of links N is equal to or larger than 3, the matrices [A] and [B] can be constituted systematically by means of augmenting the matrices and merging the contribution of each link to the resilience matrix and inertia matrix in the similar manner with the augmentation used in the finite element method. The resilience matrix [A] is diagonal, and the n-th diagonal component is $\{(N - n) + 1/2 + \alpha\}Sgl$. The inertia matrix [B] is fully populated, and the n-th diagonal component is $I_o + (N - n + \alpha)Sl^2$, and the component of the m-th column ($m>n$) is $\{(N - m) + 1/2 + \alpha\}Sl^2$. These formulae show that the mass S does not affect the natural frequencies just as in the case of a single pendulum, and that the frequencies estimated by this formulation are governed by the acceleration of gravity, number and length of links and anchor mass.

5. Numerical examples

Suppose that we have a chain of 320 mm in total length L. The breadth and thickness of each link is constant as 10 mm and 4 mm, respectively, while the link length l is dependent on the number of links. The case of a physical pendulum is dealt with by the formulation by means of setting $N = 1$ and $\alpha=0$, and the natural frequency thus calculated gives rise to the identical result with the theoretical value of $f=(1/2\pi)\sqrt{MLg/2I}=1.0786$Hz for the moment of inertia around the top joint $I= 3.4087 \times 10^3$ kgmm^2, the total chain mass $M=9.984 \times 10^{-2}$ kg and $g= 9.8 \times 10^3$ mm/sec^2.

Figure 2 shows the four mode shapes of four-link chain without anchor, the four natrual frequencies being calculated as 1.06, 2.57, 4.47 and 7.36 Hz. The numerals in the figure indicate the mode order. The mode shape illustrated hereafter is normalized in the manner that the normalization of the eigenvector is to set the largest link angle equal to 0.1 and then the largest (in the absolute sense) transverse displacement of the joints is set equal to 1 mm. All the mode shapes in Fig.2 look alike that of bending vibration of a straight beam with flexural rigidity. Figure 3 illustrates the mode shapes of eight-link chain without anchor for the 1st (1.06 Hz), 3rd (3.99 Hz), 5th (7.80 Hz) and 8th (17.7 Hz) modes. The numerals in the parenthsis indicate the natural frequency. It is seen that the bottom link is almost in a standstill at the 8th mode, quite different from that of beam vibration. This tendency is confirmed clearly in Fig.4 for 32-link chain (1st/1.06 Hz and 32nd/63.3 Hz). Seven links from the bottom are in a standstill in the case of 16 links, and 22 links in a sandstill in the case of 32 links. On the other hand, regardless of the number of links, the natural frequency and mode shape of the first mode remain the same with those of a physical pendulum. Figure 5 shows the mode shapes of the first mode and 32nd mode obtained in the case of 32 links and $\alpha=100$, and Fig.6 the case of $\alpha=500$ in order to make clear the effect of the anchor on the mode shape. Figure 7 illustrates the effect of anchor mass on the natural frequency of the first mode calculated by 32 links. The dotted line in the figure indicates the natural frequency of a physical pendulum having the same moment of inertia I and mass M with the 32 links chain, and the chain line that of a single pendulum with the same cord length L, that is, $f=(1/2\pi)\sqrt{L/g}=0.8808$Hz. The swing of a hanging chain without flexural rigidity can be said characterized by the different mode shape at higher modes, namely some bottom links in a standstill, from that of the beam vibration with flexural rigidity.

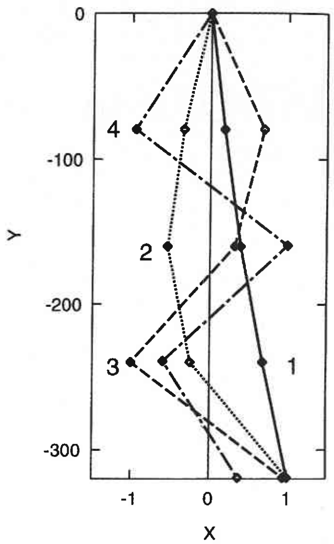

Fig.2　Eigenmodes of 4-link chain without anchor

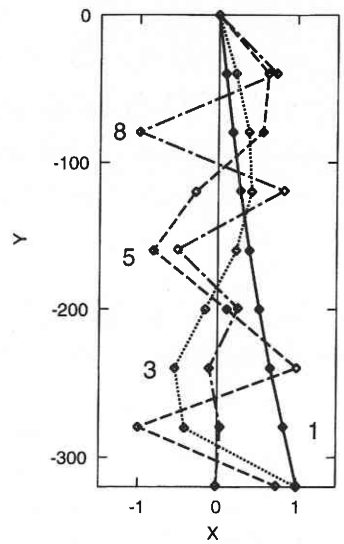

Fig.3　Eigenmodes of 8-link chain

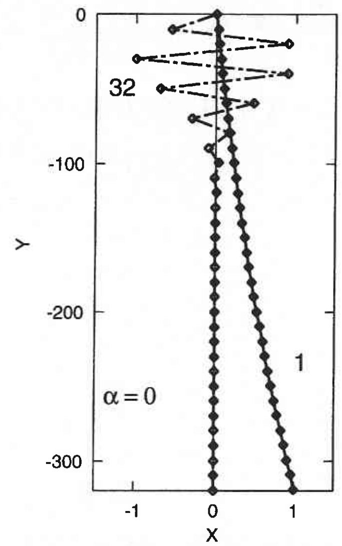

Fig.4　Eigenmodes of 32-link chain without anchor

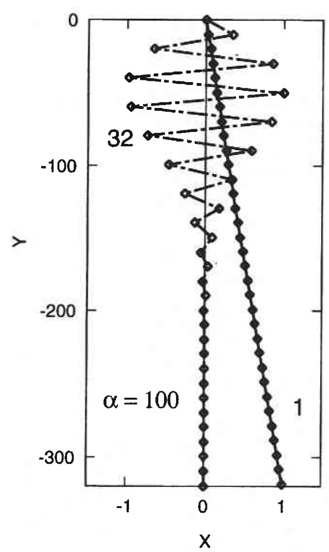

Fig.5　Eigenmodes of 32-link chain with anchor of α=100

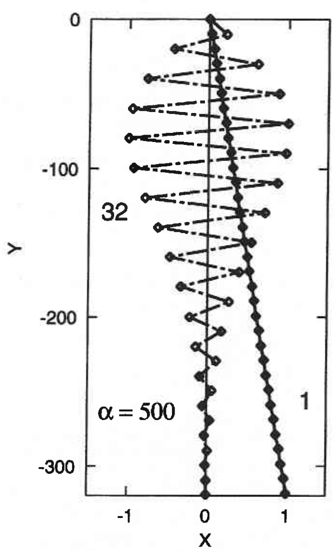

Fig.6　Eigenmodes of 32-link chain with anchor of α=500

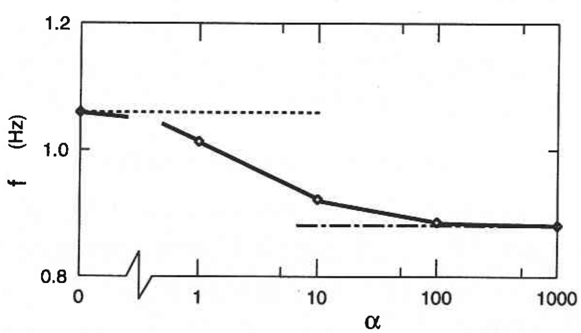

Fig.7　change of first-mode frequency with respect to anchor mass

6. Conclusion

A formulation is devised to guess the possible natural frequencies of a hanging chain that swings in a vertical plane under the influence of gravity only. The chain is modeled as an assembly of N identical links interconnected by frictionless revolute joints so that the chain has no flexural rigidity as a whole. The numerical examples show that the swing of the chain is characterized by the mode shape that the bottom side of the chain is in a standstill at the higher modes than eight.

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References

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