研 究 解 説

The Second Order Hydrodynamic Actions On A Flexible Body (Part II)

弾性浮体に働く二次の流体力学的作用（その２）

Wu YOUSHENG*, Hisaaki MAEDA** and Takeshi KINOSHITA**

呉 有 生・前 田 久 明・木 下 健

The contributions of the first order wave potentials and responses to the second order hydrodynamic actions for a flexible body are discussed. The velocity potetential, the hydrodynamic pressure and hydrodynamic forces are presented in Part I. The generalized forces in irregular waves and the equations of motion are presented in this paper.

1. Generalized forces in irregular waves

1.1 Irregular waves

According to Pierson[13] the irregular waves may be described by an aggregation of regular waves, namely

$$\zeta(t) = \sum_{j=1}^{N} \zeta_j \cos(\omega_{ej} t + \varepsilon_j) \tag{1.1}$$

Where $\omega_{ej} = \omega_j - k_j U\cos\beta$ is the encounter wave frequency, is a random phase angle, uniformly distributed in the regime of $(-\pi, \pi)$. ζ_j is the wave amplitude of the j-th wave component, which may be obtained from the wave energy spectrum $S_\zeta(\omega)$ in the form

$$\zeta_j = \sqrt{2S_\zeta(\omega_{ej})\Delta\omega_{ej}} \tag{1.2}$$

with $\Delta\omega_{ej}$ being the frequency interval corresponding to ω_{ej}. Following this expression, the first order potentials ϕ_0, ϕ_D and the principal coordinates P_k may be represented as

$$\phi_\alpha(x,y,z,t) = \sum_{j=1}^{N} \zeta_j \varphi_\alpha(x,y,z,\omega) e^{i(\omega_{ej} t + \varepsilon_j)} \tag{1.3a}$$

$$p_k(t) = \sum_{j=1}^{N} \zeta_j p_k(\omega_{ej}) e^{i(\omega_{ej} t + \varepsilon_j)} \tag{1.3b}$$

where ϕ_α denotes either or. ϕ_0 or ϕ_D.

1.2 The first order hydrodynamic forces

After using (1.3) for the formulas in Section 4.3 in the precious paper, the following expressions may be obtained.

(1) Wave exciting forces

$$\Xi_r^{(1)}(t) = \sum_{j=1}^{N} \zeta_j \xi_r \; e^{i(\omega_{ej} t + \varepsilon_j)} \tag{1.4a}$$

$$\xi_r(\omega_{ej}) = \rho \iint_S \bar{n} \cdot \bar{u}_r^o (i\omega_{ej} + \bar{W}\cdot\nabla)[\phi_o(t) + \phi_D(t)] dS \tag{1.4b}$$

(2) Radiation forces and hydrodynamic coefficients

$$H_r^{(1)}(t) = \sum_{k=1}^{m} \sum_{j=1}^{N} \zeta_j [\omega_{ej}^2 A_{rk}(\omega ej) - i\omega_{ej} B_{rk}(\omega_{ej})] \; p_k \; e^{i(\omega_{ej} t + \varepsilon_j)} \tag{1.5a}$$

where

$$\left.\begin{aligned} A_{rk}(\omega_e) &= \frac{1}{\omega_e^2} R_e \\ B_{rk}(\omega_e) &= -\frac{i}{\omega_e} I_m \end{aligned}\right\} \left[\rho \iint_S \bar{n} \cdot \bar{u}_r^o (i\omega_e + \bar{W}\cdot\nabla) \; \varphi_k(\omega_e) dS \right], \tag{1.5b}$$

(3) Restoring forces and coefficients

$$R_r^{(1)}(t) = \sum_{k=1}^{m} C_{rk} \sum_{j=1}^{N} \zeta_j p_{kj} \; e^{i(\omega_{ej} t + \varepsilon_j)}, \tag{1.6a}$$

$$C_{rk} = \rho \iint_S \bar{n} \bar{u}_r^o [g w_k + \tfrac{1}{2}(\bar{u}_k^o \cdot \nabla) W^2] dS, \tag{1.6b}$$

and

*China Ship Scientific Research Center, Wuxi, China

**2nd Department, Institute of Industriol Science, Unirersity of Tokyo

$$\Delta R_r(t) = \sum_{k=1}^{m} \Delta C_{rk} \sum_{j=1}^{N} \zeta_j p_{kj}\, e^{i(\omega_{ej}t+\varepsilon_j)}, \quad (1.7a)$$

$$\Delta C_{rk} = \rho \iint_S d_{rk}[gz' + \tfrac{1}{2}(W^2 - U^2)]dS. \quad (1.7b)$$

where

$$d_{r4} = w_r^o n_2 - v_r^o n_3, \quad d_{r5} = u_r^o n_3 - w_r^o n_1$$
$$d_{r6} = v_r^o n_1 - u_r^o n_2, \quad d_{rk} = 0 \quad (k \neq 4,5,6) \quad (1.8)$$

Among the hydrodynamic coefficients given above, A_{rk} and B_{rk} are respectively the hydrodynamic inertial and damping coefficients, C_{rk} and ΔC_{rk} are the frequency-independent restoring coefficients. A_{rk}, B_{rk} and C_{rk} are obviously the same as those in the linear hydroelasticity theory[4,5]. ΔC_{rk} is exceptional, which is due to the influence of the rigid body rotations to the steady-state fluid forces.

1.3 The second order hydrodynamic forces

Substitution of (3.3), (3.11), (4.5), (4.25 ~ 31) of the previous paper and (1.3) in (4.24) of the previous paper gives the second order hydrodynamic forces in the form

$$Z_r^{(2)} = J_{r0} + \sum_{i=0}^{N}\sum_{j=1}^{N} \zeta_i\zeta_j Q_{rij}\, e^{i[(\omega_{ei}-\omega_{ej})t+(\varepsilon_i-\varepsilon_j)]} + \sum_{i=0}^{N}\sum_{j=1}^{N} \zeta_i\zeta_j D_{rij}\, e^{i[(\omega_{ei}+\omega_{ej})t+(\varepsilon_i+\varepsilon_j)]} \quad (1.9)$$

where it is defined that

$$\varsigma_0 = 1, \quad \omega_0 = 0 = \varepsilon_0, \quad (1.10a)$$

$$J_{r0} = -\frac{\rho}{4g}\int_{C_W} (\vec{n}\cdot\vec{u}_r^o)(W^2 - U^2)^2 \frac{dl}{\sqrt{1-n_3^2}} \quad (1.10b)$$

$$Q_{r0j} = \bar{J}_r(\omega_{ej}) + \sum_{k=1}^{m} \bar{J}_{rk}(\omega_{ej})\bar{p}_k^{(1)}(\omega_{ej}), \quad (1.10c)$$

$$D_{r0j} = 3J_r(\omega_{ej}) + \sum_{k=1}^{m} J_{rk}(\omega_{ej})p_k^{(1)}(\omega_{ej}), \quad (1.10d)$$

$$\begin{aligned} Q_{rij} = {} & K_r^*(\omega_{ei},\omega_{ej}) + \tfrac{1}{2}f_r^*(\omega_{ei},\omega_{ej}) \\ & + \tfrac{1}{2}\sum_{k=1}^{m}\{\xi_{rk}(\omega_{ei})\bar{p}_k^{(1)}(\omega_{ej}) + h_{rk}^*(\omega_{ei},\omega_{ej})p_k^{(1)}(\omega_{ei}) \\ & + 2\bar{K}_{rk}(\omega_{ei},\omega_{ej})\bar{p}_k^{(1)}(\omega_{ej}) + 2K_{rk}(\omega_{ei},\omega_{ej})p_k^{(1)}(\omega_{ei}) \\ & + \tfrac{1}{2}p_k^{(1)}(\omega_{ei})\sum_{l=1}^{m}[\bar{p}_l^{(1)}(\omega_{ej})(q_{rkl}(\omega_{ei},\omega_{ej}) \\ & + t_{rkl}^*(\omega_{ei},\omega_{ej})) + g_{rkl}\bar{p}_l^{(1)}(\omega_{ej}) \\ & + 2G_{rkl}^*(\omega_{ei},\omega_{ej})\bar{p}_l^{(1)}(\omega_{ej})]\}, \end{aligned} \quad (1.10e)$$

$$\begin{aligned} D_{rij} = {} & K_r(\omega_{ei},\omega_{ej}) + \tfrac{1}{2}f_r(\omega_{ei},\omega_{ej}) \\ & + \tfrac{1}{2}\sum_{k=1}^{m}\{\xi_{rk}(\omega_{ei})p_k^{(1)}(\omega_{ej}) + h_{rk}(\omega_{ei},\omega_{ej})p_k^{(1)}(\omega_{ei}) \\ & + 2K_{rk}^*(\omega_{ei},\omega_{ej})p_k^{(1)}(\omega_{ej}) \\ & + \tfrac{1}{2}p_k^{(1)}(\omega_{ei})\sum_{l=1}^{m}[p_l^{(1)}(\omega_{ej})(q_{rkl}(\omega_{ei},\omega_{ej}) \\ & + t_{rkl}(\omega_{ei},\omega_{ej})) + g_{rkl}p_l^{(1)}(\omega_{ej}) \\ & + 2G_{rkl}(\omega_{ei},\omega_{ej})p_l^{(1)}(\omega_{ej})]\}. \end{aligned} \quad (1.10f)$$

The coefficients contained in these formulas, namely J_r, J_{rk}, ξ_{rk}, f_r, f_r^*, h_{rk}, h_{rk}^*, K_r, K_r^*, K_{rk}, $\bar{K}_{rk}$, K_{rk}^*, q_{rkl}, t_{rkl}, t_{rkl}^*, g_{rkl}, G_{rkl} and G_{rkl}^* are given in Appendix A.

Evidently, the second order forces provide the frequency independent components, the wave-frequency components, the sum and difference-frequency components. In deriving these formulas the following relation for multiplication of the real parts of two complex variables is employed.

$$\mathrm{Re}(X)\cdot\mathrm{Re}(Y) = \tfrac{1}{2}[\mathrm{Re}(X\cdot\bar{Y}) + \mathrm{Re}(X\cdot Y)].$$

Here in the formulas of (1.10) and those in Appendix A, an over bar $\bar{Y}$ is used to represent the conjugate of the complex variable Y.

2. Equations of Motion

2.1 Linear Equation

If only the first order excitations are included in the analysis, after extraction of the portion accounting for steady-state condition, the generalized equation of motion (3.13) in the previous paper may be written in the form

$$(\mathbf{a}+\mathbf{A})\ddot{\mathbf{p}} + (\mathbf{b}+\mathbf{B})\dot{\mathbf{p}} + (\mathbf{c}+\mathbf{C}+\mathbf{Cm})\mathbf{p} = \Xi^{(1)} \quad (2.1)$$

where **A, B, C** and **Cm** are the matrices with the components defined respectively by (1.5), (1.6), and (3.17) in the previous paper. The components of the wave excitation vector $\Xi^{(1)}$ is given by (1.4). The solution of this equation provides the first order principal coordinates and $p_k^{(1)}(\omega)$ and $p_k^{(1)}$ (k = 1, 2,..., m).

2.2 Non-linear equations of motion

Based on the solutions of the first order potentials and φ_0, φ_D and φ_k (k = 1, 2,..., m), and the first order principal coordinates $P_{k(1)}(\omega)$ (k = 1, 2,..., m), the second order forces may be calculated. The equations of motion for solving the principal coordinates P_k, where both the first order and the second order actions are included may be represented either in frequency domain, or in time domain.

2.2.1 Equations of motion in frequency domain

The hydroelastic equations of motion for a floating structure encountering the steady-state, first order and second order wave forces may be obtained in frequency domain by using (4.17~19) in Part I, (1.4~7), and (1.9) for (3.13) in Part I. The inclusion of non-linear mooring forces is not quite convenient in frequency domain. Therefore only the linear mooring forces described by (3.17) in Part I is introduced. This results in a set of equations in the form

$$\begin{aligned}\sum_{k=1}^{m}[a_{rk}+A_{rk})\ddot{p}_k(t)+(b_{rk}+B_{rk})\dot{p}_k(t)\\ +(c_{rk}+C_{rk}+\Delta C_{rk}+Cm_{rk})p_k(t)]\\ =\Xi_r^{(1)}(t)+Z_r^2(t)+(Z_r^{(0)}+G_r)\\ =(Z_r^{(0)}+G_r+J_{r0})\\ +\sum_{j=1}^{N}\zeta_j\xi_r(\omega_{ej})e^{i(\omega_{ej}t+\varepsilon_{ju})}\\ +\sum_{i=0}^{N}\sum_{j=1}^{N}\varsigma_i\varsigma_jQ_{rij}e^{i[(\omega_{ei}-\omega_{ej})t+(\varepsilon_i-\varepsilon_j)]}\\ +\sum_{i=0}^{N}\sum_{j=1}^{N}\varsigma_i\varsigma_jD_{rij}e^{i[(\omega_{ei}+\omega_{ej})t+(\varepsilon_i+\varepsilon_j)]}\end{aligned} \quad (2.2)$$

where r = 1, 2,..., m. $Z_{r(0)}$ G_r and J_{r0} are the steady-state forces given in (4.18), (3.18) in Part I and (1.10) respectively. The formulas for the first and second order force terms may be found in (1.4) and (1.10).

(2.2) shows that the solution of the total responses of the principal coordinates $P_k(k)$ consists of the steady-state components due to $Z_{r(0)}$, G_r and J_{r0}, the wave frequency components due to the terms of ξ_r, Q_{r0j} and D_{r0j}, the difference frequency components coming from the terms of Q_{rij}, and the sum frequency components coming from those of D_{rij}. It should be noted that when i = j in (2.2), the terms of Q_{rij} corresponds to the generalized mean draft force for r-th mode. The solutions of (2.2) may be decomposed in the form

$$\begin{aligned}p_k(t)=\bar{p}_k+\sum_{i=0}^{N}\sum_{j=1}^{N}\varsigma_i\varsigma_j\{p_k^-(\omega_{ij}^-)e^{i[\omega_{ij}^-t+\varepsilon_{ij}^-]}\\ +p_k^+(\omega_{ij}^+)e^{i[\omega_{ij}^+t+\varepsilon_{ij}^+]}\}\quad (k=1,2,\ldots,m)\end{aligned} \quad (2.3)$$

where

$$\omega_{ij}^-=\omega_i-\omega_j,\quad \varepsilon_{ij}^-=\varepsilon_i-\varepsilon_j,$$
$$\omega_{ij}^+=\omega_i-\omega_j,\quad \varepsilon_{ij}^+=\varepsilon_i+\varepsilon_j.$$

The results for $\bar{p}_k$, $p_k^-(\omega_{ij}^-)$ and $p_k^+(\omega_{ij}^+)$ may be numerically solved from the following equations.

$$\begin{aligned}\sum_{k=1}^{m}(c_{rk}+C_{rk}+\Delta C_{rk}+Cm_{rk})\bar{p}_k\\ =Z_r^{(0)}+G_r+J_{r0}\quad (r=1,2,\ldots,m)\end{aligned} \quad (2.4)$$

$$\begin{aligned}\sum_{k=1}^{m}[-(\omega_{ij}^-)^2(a_{rk}+A_{rk})+(i\omega_{ij}^-)(b_{rk}+B_{rk})\\ +(c_{rk}+C_{rk}+\Delta C_{rk}+Cm_{rk})]p_k^-(\omega_{ij}^-)=Q_{rij}\\ (r=1,2\ldots,m;\ i=0,1,\ldots,N;\ j=1,2,\ldots,N)\end{aligned} \quad (2.5)$$

$$\begin{aligned}\sum_{k=1}^{m}[-(\omega_{ij}^+)^2(a_{rk}+A_{rk})+(i\omega_{ij}^+)(b_{rk}+B_{rk})\\ +(c_{rk}+C_{rk}+\Delta C_{rk}+Cm_{rk})]p_k^+(\omega_{ij}^+)\\ =D_{rij}+\delta_{i0}\xi_r\\ (r=1,2\ldots,m;\ i=0,1,\ldots,N;\ j=1,2,\ldots,N)\end{aligned} \quad (2.6)$$

where δ_{r0} is the Kroneker delta function. The hydrodynamic coefficients A_{rk} and B_{rk} take their values at the corresponding frequencies ω_{ij}^- or ω_{ij}^+.

2.2.2 Equations of motion in time domain

After the inclusion of the second order wave forces, the three-dimensional hydroelastic equations of motion may be written in time domain in the form[6)]

$$\begin{aligned}\sum_{k=1}^{m}[(a_{rk}+\tilde{A}_{rk})\ddot{p}_k(t)+(b_{rk}+\tilde{B}_{rk})\dot{p}_k(t)\\ +(c_{rk}+\tilde{C}_{rk}+\tilde{C}'_{rk})p_k(t)\\ +\int_0^t K_{rk}(t-\tau)\dot{p}_k(\tau)d\tau]\\ =Z_r^{(0)}+G_r+\Xi_r^{(1)}(t)+Z_r^{(2)}(t)+\Delta_r(t)\end{aligned} \quad (2.7)$$

where $\tilde{A}_{rk}$, $\tilde{B}_{rk}$, $\tilde{C}_{rk}$ and $\tilde{C}'_{rk}$ are the time domain hydrodynamic coefficients. They are defined as follows

$$\begin{cases} \tilde{A}_{rk} = -\rho\iint_S a_r\psi_{1k}dS, \\ \tilde{B}_{rk} = -\rho\iint_S a_r\psi_{2k}dS + \rho\iint_S \bar{n}\cdot[(\bar{W}\cdot\nabla)\bar{u}_r^o \\ \qquad -(\bar{u}_r^o\cdot\nabla)\bar{W}]\psi_{1k}dS - \rho\oint_{C_w} a_r(\bar{I}\times\bar{n})\cdot\bar{W}\psi_{1k}dC, \end{cases} \tag{2.8}$$

$$\begin{cases} \tilde{C}_{rk} = C_{rk}, \\ \tilde{C}'_{rk} = \rho\iint_S \bar{n}\cdot[(\bar{W}\cdot\nabla)\bar{u}_r^o - (\bar{u}_r^o\cdot\nabla)\bar{W})]\psi_{2k}dS \\ \qquad -\rho\oint_{C_w} a_r(\bar{I}\times\bar{n})\cdot\bar{W}\psi_{2k}dC. \end{cases} \tag{2.9}$$

The retardation functions $K_{rk}(t)$ (r, k = 1, 2, ,..., m) have the form

$$K_{rk}(t) = -\rho\iint_S a_r\frac{\partial}{\partial x}\chi_k dS + \rho\iint_S \bar{n}\cdot[(\bar{W}\cdot\nabla)\bar{u}_r^o - (\bar{u}_r^o\cdot\nabla)\bar{W}]\chi_k dS - \rho\oint_{C_w} a_r(\bar{I}\times\bar{n})\cdot\bar{W}\chi_k dC \tag{2.10}$$

In the case $\bar{W} = -U\bar{i}$, it can be simplified that

$$\tilde{B}_{rk} = -\rho\iint_S a_r\psi_{2k}dS - \rho U\iint_S \psi_{1k}\bar{n}\cdot\frac{\partial}{\partial x}\bar{u}_r^o dS \tag{2.11}$$

$$\tilde{C}'_{rk} = -\rho\iint_S \psi_{2k}\bar{n}\cdot\frac{\partial}{\partial x}\bar{u}_r^o dS, \tag{2.12}$$

and

$$K_{rk}(t) = -\rho\iint_S a_r\frac{\partial\chi_k}{\partial x}dS - \rho U\iint_S \chi_k\bar{n}\cdot\frac{\partial\bar{u}_r^o}{\partial x}dS. \tag{2.13}$$

The time domain hydrodynamic coefficients $\tilde{A}_{rk}$, $\tilde{B}_{rk}$, $\tilde{C}_{rk}$ and $\tilde{C}'_{rk}$ are time and frequency independent. They only depend on the geometry of the wetted surface, the forward speed, and the dry modes of the structure. The following relations among the time domain hydrodynamic coefficients, the retardation functions, and the frequency domain hydrodynamic coefficients may also be used to calculate $\tilde{A}_{rk}$, $\tilde{B}_{rk}$ and $K_{rk}(t)$:

$K_{rk}(t)$:

$$\begin{cases} \tilde{A}_{rk} = A_{rk}(\omega) + \frac{1}{\omega}\int_0^\infty K_{rk}(\tau)\sin(\omega\tau)d\tau = A_{rk}(\infty), \\ \tilde{B}_{rk} = B_{rk}(\omega) - \int_0^\infty K_{rk}(\tau)\cos(\omega\tau)d\tau, \\ K_{rk}(t) = \frac{2}{\pi}\int_0^\infty \omega[A_{rk}(\infty) - A_{rk}(\omega)]\sin(\omega t)d\omega. \end{cases} \tag{2.14}$$

When the structure has no forward speed, it is found that

$$\tilde{B}_{rk} = \tilde{C}'_{rk} = 0.$$

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At the right side of the generalized equations of motion the steady-state forces, first order and second order forces may be obtained from (3.18), (4.18), (4.36) in Part I and (1.9). The response-dependent non-linear mooring forces now can be easily handled in the time domain analysis, and are all included in the term $\Delta_r(t)$.

3. Concluding remarks

The linear hydroelasticity theories have been developed and applied for years. The coupled analyses of the hydrodynamic and the structural problems allow the responses of marine structure to be examined in a unified manner. The mean drift forces, low frequency and sun frequency excitations induced by the second order wave actions, and the non-linear loads due to the instantaneous wetted surface effect, may give more influence to the structural distortions, than the rigid body motions. This may especially the case for a large bulkcarrier, a fast slender vehicle, and a very large floating structure. The existing methods of evaluating the second order wave forces have been widely employed for the predictions of motions and loads of rigid bodies. The formulas presented in this paper just show that the similar methods may be extended to the hydroelastic problems. More numerical effort is evidently needed. However there seems no significant difficulty to obtain the results with the accuracy similar to that achieved by the rigid body analyses.

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Appendix A. The coefficients of the second order forces

$$\xi_{rk}(\omega)=\rho\iint_S[d_{rk}+(\vec{n}\cdot\vec{u}_r^o)(\vec{u}_k^o\cdot\nabla)](i\omega+\vec{W}\cdot\nabla)[\varphi_o(\omega)+\varphi_D(\omega)]dS$$

$$f_r(\omega_i,\omega_j)=\frac{1}{2}\rho\iint_S(\vec{n}\cdot\vec{u}_r^o)\nabla[\varphi_o(\omega_i)+\varphi_D(\omega_i)]\cdot\nabla[\varphi_o(\omega_j)+\varphi_D(\omega_j)]dS$$

$$f_r^*(\omega_i,\omega_j)=\frac{1}{2}\rho\iint_S(\vec{n}\cdot\vec{u}_r^o)\nabla[\varphi_o(\omega_i)+\varphi_D(\omega_i)]\cdot\nabla[\bar{\varphi}_o(\omega_j)+\bar{\varphi}_D(\omega_j)]dS$$

$$h_{rk}(\omega_i,\omega_j)=\rho\iint_S(\vec{n}\cdot\vec{u}_r^o)\nabla[\varphi_o(\omega_j)+\varphi_D(\omega_j)]\cdot\nabla\varphi_k(\omega_i)dS$$

$$h_{rk}^*(\omega_i,\omega_j)=\rho\iint_S(\vec{n}\cdot\vec{u}_r^o)\nabla[\bar{\varphi}_o(\omega_j)+\bar{\varphi}_D(\omega_j)]\cdot\nabla\varphi_k(\omega_i)dS$$

$$q_{rkl}(\omega_i,\omega_j)=\rho\iint_S[d_{rl}+(\vec{n}\cdot\vec{u}_r^o)(\vec{u}_l^o\cdot\nabla)](i\omega_i+\vec{W}\cdot\nabla)\varphi_k(\omega_i)dS$$

$$t_{rkl}(\omega_i,\omega_j)=\frac{1}{2}\rho\iint_S(\vec{n}\cdot\vec{u}_r^o)\nabla\varphi_k(\omega_i)\cdot\nabla\varphi_l(\omega_j)dS$$

$$t_{rkl}^*(\omega_i,\omega_j)=\frac{1}{2}\rho\iint_S(\vec{n}\cdot\vec{u}_r^o)\nabla\varphi_k(\omega_i)\cdot\nabla\bar{\varphi}_l(\omega_j)dS$$

$$g_{rkl}=\rho\iint_S d_{rl}[gw_k^o+\frac{1}{2}(\vec{u}_k^o\cdot\nabla)W^2]dS+\rho\iint_S\alpha_{rkl}[gz'+\frac{1}{2}(W^2-U^2)]dS+\rho\iint_S(\vec{n}\cdot\vec{u}_r^o)e_{kl}dS$$

where d_{rk} $(r, k = 1,2,\dots,m)$ are defined in (4.36).

The coefficients e_{kl} and α_{rkl} are as follows.

$$e_{44}=-\frac{1}{2}(z'+\frac{1}{2}y'\frac{\partial}{\partial y'}W^2+\frac{1}{2}z'\frac{\partial}{\partial z'}W^2),$$
$$e_{55}=-\frac{1}{2}(z'+\frac{1}{2}x'\frac{\partial}{\partial x'}W^2+\frac{1}{2}z'\frac{\partial}{\partial z'}W^2),$$
$$e_{66}=-\frac{1}{4}(x'\frac{\partial}{\partial x'}W^2+y'\frac{\partial}{\partial y'}W^2),$$
$$e_{45}=e_{54}=\frac{1}{4}x'\frac{\partial}{\partial y'}W^2,$$
$$e_{46}=e_{64}=\frac{1}{2}(x'+\frac{1}{2}x'\frac{\partial}{\partial z'}W^2),$$
$$e_{56}=e_{65}=\frac{1}{2}(y'+\frac{1}{2}y'\frac{\partial}{\partial z'}W^2),$$
$$e_{4l}=v_l^o\quad(l\geq 7),$$
$$e_{5l}=-u_l^o\quad(l\geq 7),$$
$$e_{kl}=0\quad\text{(for all the other k and } l\text{)}.$$
$$\alpha_{r44}=-\frac{1}{2}(v_r^o n_2+w_r^o n_3),$$
$$\alpha_{r55}=-\frac{1}{2}(u_r^o n_1+w_r^o n_3),$$
$$\alpha_{r66}=-\frac{1}{2}(u_r^o n_1+v_r^o n_2),$$
$$\alpha_{r45}=\alpha_{r54}=\frac{1}{2}v_r^o,$$
$$\alpha_{r46}=\alpha_{r64}=\alpha_{r56}=\alpha_{r65}=\frac{1}{2}w_r^o,$$
$$\alpha_{rkl}=0\quad(k,\ l\neq 4,\ 5,\ 6).$$

$$J_{ro}=-\frac{\rho}{4g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)(W^2-U^2)^2\frac{dl}{\sqrt{1-n_3^2}}$$

$$J_r(\omega)=-\frac{\rho}{4g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)(W^2-U^2)(i\omega+\vec{W}\cdot\nabla)[\varphi_o(\omega)+\varphi_D(\omega)]\frac{dl}{\sqrt{1-n_3^2}}$$

$$\bar{J}_r(\omega)=-\frac{\rho}{4g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)(W^2-U^2)(-i\omega+\vec{W}\cdot\nabla)[\bar{\varphi}_o(\omega)+\bar{\varphi}_D(\omega)]\frac{dl}{\sqrt{1-n_3^2}}$$

$$J_{rk}(\omega)=-\frac{\rho}{4g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)[(i\omega+\vec{W}\cdot\nabla)\varphi_k(\omega)+\frac{1}{2}(\vec{u}_k^o\cdot\nabla)W^2-w_k^o]\frac{dl}{\sqrt{1-n_3^2}}$$

$$\bar{J}_{rk}(\omega)=-\frac{\rho}{4g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)[(-i\omega+\vec{W}\cdot\nabla)\bar{\varphi}_k(\omega)+\frac{1}{2}(\vec{u}_r^o\cdot\nabla)W^2-w_k^o]\frac{dl}{\sqrt{1-n_3^2}}$$

$$K_r(\omega_{ei},\omega_{ej})=-\frac{\rho}{2g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)(i\omega_{ei}+\vec{W}\cdot\nabla)[\varphi_o(\omega_{ei})+\varphi_D(\omega_{ei})](i\omega_{ej}+\vec{W}\cdot\nabla)[\varphi_o(\omega_{ej})+\varphi_D(\omega_{ej})]\frac{dl}{\sqrt{1-n_3^2}}$$

$$K_r^*(\omega_{ei},\omega_{ej})=-\frac{\rho}{2g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)(i\omega_{ej}+\vec{W}\cdot\nabla)[\varphi_0(\omega_{ei})+\varphi_D(\omega_{ei})](-i\omega_{ej}+\vec{W}\cdot\nabla)[\bar{\varphi}_o(\omega_{ej})+\bar{\varphi}_D(\omega_{ej})]\frac{dl}{\sqrt{1-n_3^2}}$$

$$K_{rk}(\omega_{ei},\omega_{ej})=-\frac{\rho}{2g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)(-i\omega_{ej}+\vec{W}\cdot\nabla)[\bar{\varphi}_o(\omega_{ej})+\bar{\varphi}_D(\omega_{ej})][(i\omega_{ei}+\vec{W}\cdot\nabla)\varphi_k(\omega_{ei})+\frac{1}{2}(\vec{u}_k^o\cdot\nabla)W^2-w_k^o]\frac{dl}{\sqrt{1-n_3^2}}$$

$$\bar{K}_{rk}(\omega_{ej},\omega_{ei})=-\frac{\rho}{2g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)(i\omega_{ei}+\vec{W}\cdot\nabla)[\varphi_o(\omega_{ei})+\varphi_D(\omega_{ei})][(-i\omega_{ej}+\vec{W}\cdot\nabla)\bar{\varphi}_k(\omega_{ej})+\frac{1}{2}(\vec{u}_k^o\cdot\nabla)W^2-w_k^o]\frac{dl}{\sqrt{1-n_3^2}}$$

$$K_{rk}^*(\omega_{ei},\omega_{ej})=-\frac{\rho}{g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)(i\omega_{ei}+\vec{W}\cdot\nabla)[\varphi_o(\omega_{ei})+\varphi_D(\omega_{ei})][(i\omega_{ej}+\vec{W}\cdot\nabla)\varphi_k(\omega_{ej})+\frac{1}{2}(\vec{u}_k^o\cdot\nabla)W^2-w_k^o]\frac{dl}{\sqrt{1-n_3^2}}$$

$$G_{rkl}(\omega_{ei},\omega_{ej})=-\frac{\rho}{2g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^o)[(i\omega_{ei}+\vec{W}\cdot\nabla)\varphi_k(\omega_{ei})+\frac{1}{2}(\vec{u}_k^o\cdot\nabla)W^2-w_k^o][(i\omega_{ej}+\vec{W}\cdot\nabla)\varphi_l(\omega_{ej})+\frac{1}{2}(\vec{u}_l^o\cdot\nabla)W^2-w_l^o]\frac{dl}{\sqrt{1-n_3^2}}$$

$$G_{rkl}^*(\omega_{ei},\omega_{ej})=-\frac{\rho}{2g}\int_{C_W}(\vec{n}\cdot\vec{u}_r^0)[(i\omega_{ei}+\vec{W}\cdot\nabla)\varphi_k(\omega_{ei})+\frac{1}{2}(\vec{u}_k^o\cdot\nabla)W^2-w_k^0][(-i\omega_{ej}+\vec{W}\cdot\nabla)\bar{\varphi}_l(\omega_{ej})+\frac{1}{2}(\vec{u}_l^o\cdot\nabla)W^2-w_l^o]\frac{dl}{\sqrt{1-n_3^2}}$$