研 究 速 報

A Note on Finite Element Interval Analysis (Part 2 Linear, Undamped Eigenvalue Problem)

有限要素区間解析に関するノート (第2報 線形無減衰固有値問題)

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1. Introduction

Interval analysis techniques were applied not only to estimation of charactersitics change of linear systems and electrical circuits whose components have tolerance[1)~3)], but also to evaluation of behavior fluctuation of structures whose parameters are uncertain[4)]. Interval analysis of eigenvalue problems has not been carried out yet, however, to the best of the author's knowledge. This paper presents a formulation of the finite element interval analysis in linear, undamped eigenvalue problem of structural vibration based on sensitivity analysis with respect to uncertain variables. The formulation is based on the finite element sensitivity analysis with respect to the system variables and is different from that of Koyluoglu et al[4)]. which is based on the technique of interval computation[5)].

2. Statement of problem

Suppose that we have a nominal design of an elastic structure, whose linear and undamped eigenvalue problem is given by Eq.(1) after treatment of the boundary conditions,

$$([K]-\lambda[M])\{\phi\}=\{0\} \tag{1}$$

where $[K]$ and $[M]$ denote the N-dimensional stiffness matrix and mass matrix subject to the uncertain variables ε_m defined by Eq.(2) for M structural parameters p_m that fluctuate uncertainly. λ and $\{\phi\}$ denote the eigenvalue and eigenvector of a single eigenmode, respectively.

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$$p_m=\bar{p}_m(1+\varepsilon_m) \tag{2}$$

The upper bar means the quantity defined at the nominal value of the structural parameters taken. It is assumed that the stiffness matrix and mass matrix are differentiable by the uncertain variables as well as the eigenpair.

The problem is to find the uncertain variables that earmark the interval of the eigenvalue and eigenvector components when a convex hull is specified as the domain in which the uncertain variables are conceived to exist[6)]. The interval arising from the rounding errors of computation is not dealt with in this note.

3. Finite element sensitivity analysis

Equations (3) and (4) are the Taylor series expansion of the stiffness matrix and mass matrix with respect to the uncertain variables and truncated at the first-order. The superfix I and suffix m denote the first-order sensitivity with respect to the m-th uncertain variable. Equations (5) and (6) stand for the variability of the eigenpair of an eigenmode under interest caused by the fluctuation of the stiffness and mass matrices.

$$[K]=[\bar{K}]+\sum_{m=1}^{M}[K_m{}^I]\,\varepsilon_m \tag{3}$$

$$[M]=[\bar{M}]+\sum_{m=1}^{M}[M_m{}^I]\,\varepsilon_m \tag{4}$$

$$\lambda=\bar{\lambda}+\sum_{m=1}^{M}\lambda_m{}^I\,\varepsilon_m \tag{5}$$

$$\{\phi\}=\{\bar{\phi}\}+\sum_{m=1}^{M}\{\phi_m{}^I\}\,\varepsilon_m \tag{6}$$

The nominal eigenpair is determined by Eq.(1), to which the nominal stiffness and mass matrices are input. Substituting Eqs.(3) through (6) into Eq.(1) and applying the perturbation technique, we obtain the eigenvalue sensitivity in the vicinity of the nominal structural parameters by Eq.(7),

$$\lambda_m{}^I = \{\bar{\phi}\}^T ([K_m{}^I] - \bar{\lambda}[M_m{}^I]) \{\bar{\phi}\} / \{\bar{\phi}\}^T [\bar{M}] \{\bar{\phi}\} \tag{7}$$

where the superfix T denotes transpose of matrix. Now that the eigenvector in this study is normalized so as to set $\{\phi\}^T [M] \{\phi\}$ always equal to unity, the eigenvector sensitivity is obtained on the basis of Eq.(8), in which the normalizing condition of eigenvector is added as an equality constraint condition for the eigenvector[7].

$$\begin{bmatrix} [\bar{K}] - \bar{\lambda}[\bar{M}] \\ 2\{\bar{\phi}\}^T [\bar{M}] \end{bmatrix} \{\phi_m{}^I\} = -\begin{bmatrix} [K_m{}^I] - \bar{\lambda}[M_m{}^I] - \lambda_m{}^I [M] \\ \{\bar{\phi}\}^T [M_m{}^I] \end{bmatrix} \{\bar{\phi}\} \tag{8}$$

4. Quadratic convex hull

The interval of the eigenvalue or eigenvector components is searched for a given interval of the uncertain variables, which can be constituted in various ways. The uncertain variables dealt with in stochastic structural mechanics are correlated more or less so that it is desirable to provide convex hull that is able to take correlation between the uncertain variables into account. Quadratic convex hull meets this purpose. Consequently we suppose that the conceivable domain of the uncertain variables is given in the form of a quadratic convex hull expressed by Eq.(9),

$$c^2 \{\varepsilon\}^T [W] \{\varepsilon\} - 1 \leqq 0 \tag{9}$$

Expanse and shape of the convex hull are specified by the parameter c and matrix $[W]$, respectively. These can be chosen judiciously and analogously somehow to variances and covariances of probabilistic variables.The matrix $[W]$ should be positive-definite for the purpose to set a convex hull as the conceivable domain of the uncertain variables[6].

5. Interval estimation by Lagrange multiplier method

The maximum and minimum of a convex function with respect to the variables confined in a convex hull take place on the boundary of the convex hull. This means that the maximum and minimum, whose difference is interval, can be searched by the Lagrange multiplier method for quadratic convex hull by constituting a functional π as stated in the following[6].

$$\pi = \{\varepsilon\}^T \{S\} + \nu (c^2 \{\varepsilon\}^T [W] \{\varepsilon\} - 1) \tag{10}$$

In case that the interval of eigenvalue is estimated, the first term of the right hand side of Eq.(10) stands for the eigenvalue change of Eq.(5) so that $\{S\}$ is the sensitivity vector consisting of the eigenvalue sensitivities $\lambda_m{}^I$.

The stationary condition of the functional π with respect to the unknowns ε_m and the Lagrange multiplier ν results in their nonlinear simultaneous equations (11) and (12). Introducing a vector $\{q\}$ defined by Eq.(13), we obtain $\{\varepsilon\}$ of Eq.(14) that satisfies Eq.(11).

$$\frac{\partial \pi}{\partial \{\varepsilon\}^T} = \{S\} + 2\nu c^2 [W] \{\varepsilon\} = \{0\} \tag{11}$$

$$\frac{\partial \pi}{\partial \nu} = c^2 \{\varepsilon\}^T [W] \{\varepsilon\} - 1 = 0 \tag{12}$$

$$\{q\} = [W]^{-1} \{S\} \tag{13}$$

$$\{\varepsilon\} = -\{q\} / 2\nu c^2 \tag{14}$$

The calculation of $\{q\}$ is available because the matrix $[W]$ is positive-definite. Substitution of Eq.(14) into Eq.(12) enables us to determine the Lagrange multiplier as given by Eq.(15) and to earmark the uncertain variables as Eq.(16) by substituting the Lagrange multiplier into Eq.(14).

$$\nu = \pm \sqrt{\{q\}^T [W] \{q\}} / 2c \tag{15}$$

$$\{\varepsilon^*\} = \mp \{q\} / c\sqrt{\{q\}^T [W] \{q\}} \tag{16}$$

Equation (16) earmarks the uncertain variables indicated by asterisk. Then the maximum and minimum of the eigenvalue are calculated by reanalysis of the eigenvalue problem based on the uncertain variables thus earmarked. The interval of each component of the eigenvector can be estimated by applying the process mentioned above to the component whose sensitivity is derived from Eq.(8). The interval of the eigenvalue and eigenvector components thus estimated satisfies the eigenvalue problem in the framework of the first-order approximation.

6. Numerical Example

The effect of uncertainty of elastic resilience and mass on the interval of the eigenpair of longitudinal vibration of an elastic bar is investigated in this study. Figure 1 illustrates the bar divided into two bar elements of unit length. In the modeling, lumped mass matrix is employed.The nominal spring constant and nominal half mass of each element are taken as $EA/l = 1$ and $\rho Al/2 = 1$ without loss of generality, respectively. The nominal eigenvalue problem of longitudinal vibration is expressed by Eq.(17) under the boundary condition that the left end of the bar is fixed. Two uncertain variables are dealt with, that is, the first one ε_1 is assigned to resilience of the element A and the second one ε_2 to mass of the element B. Equations (18) and (19) express the first-order approximation of the change of the stiffness matrix and mass matrix in the vicinity of the nominal matrices.

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\{\phi\} = \{0\} \tag{17}$$

$$[K] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\varepsilon_1 \tag{18}$$

$$[M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\varepsilon_2 \tag{19}$$

The sensitivity analysis of the eigenpair of the first eigenmode results in the following equations,

$$\lambda = 0.29289 + 0.25\varepsilon_1 - 0.21967\varepsilon_2 \tag{20}$$

$$\phi_1 = 0.5 - 0.0088388\varepsilon_1 - 0.21339\varepsilon_2 \tag{21}$$

$$\phi_2 = 0.70711 + 0.125\varepsilon_1 - 0.22855\varepsilon_2 \tag{22}$$

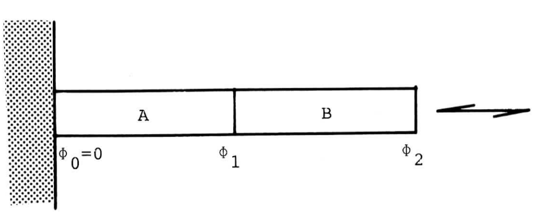

Fig. 1 Finite elements of bar under longitudinal vibration

where the first terms on the right hand side indicate the nominal eigenvalue and two components of the eigenvector. Suppose that the uncertain variables scatter in a circle of 0.1 in radius. The first-order approximation employed in Section 3 seems to be sufficient for such a small circle. Then, identity matrix is input as the matrix $[W]$ that represents the convex hull, and the parameter c is taken equal to 10.0.

Figure 2 depicts the conceivable domain of the uncertain variables by matted area within the circle and the contours of λ , ϕ_1 and ϕ_2 that are determined by Eqs.(20) to (22) and the positive terms of Eq.(16), and come in contact with the circular convex hull at (0.07512,-0.06600), (-0.03826, -0.09237) and (0.04796,-0.08770), respectively. These coordinates indicate the uncertain variables earmarked for the maximum of the eigenvalue and eigenvector components, while the minimum of them is determined by the negative terms of Eq.(16) in this case.

Figure 3 illustrates the ellipse in which the eigenvector components scatter corresponding to the uncertain resilience and mass in and on the circle of Fig. 2. The solid circles and squares in Fig. 3 indicate the intervals of ϕ_1 and ϕ_2 calculated by Eqs.(21) and (22) for the uncertain variables earmarked by themselves, respectively. These are in good agreement with the ellipse derived from the uncertain variables on the circle of the small convex hull. The solid triangles correspond to the interval of λ due to the uncertain variables earmarked by λ itself. The intervals of the eigenvalue and two eigenvector components due to the uncertain variables earmarked in different ways are listed in

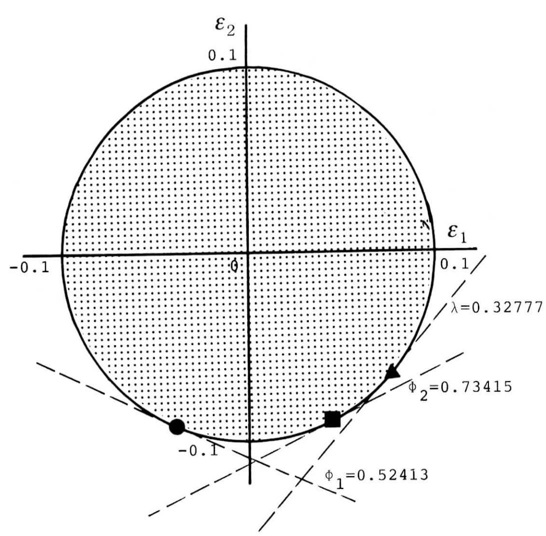

Fig. 2 Convex hull and contours of λ, ϕ_1 and ϕ_2

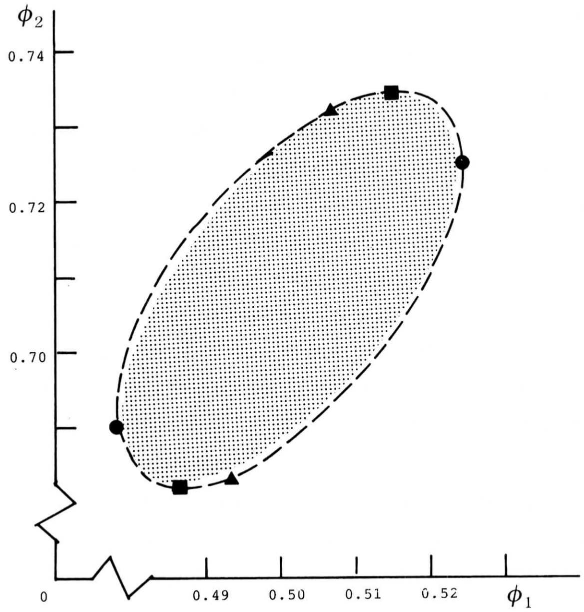

Fig. 3 Scatter of vector components ϕ_1 and ϕ_2

Table 1. The first line of Table 1 lists the intervals of λ, ϕ_1 and ϕ_2 calculated by the uncertain variables earmarked for the interval of the eigenvalue λ, while the second line and third line list the intervals of λ, ϕ_1 and ϕ_2 calculated due to ϕ_1 and ϕ_2, respectively. The first column of Table 1 lists the interval of λ, the second column that of ϕ_1, the third column that of ϕ_2, calculated by the uncertain variables earmarked for λ, ϕ_1 and ϕ_2, respectively. This table depicts that the interval of a quantity under interest takes the largest value at the uncertain variables earmarked for the change of the quantity.

7. Conclusion

A formulation is presented to estimate the interval of eigenvalue and eigenvector components on the basis of the finite element sensitivity analysis for the first-order approximation of the response changes under interest and the quadratic convex hull of uncertain variables. Uncertain variables are earmarked on the boundary of the input

	λ	Φ_1	Φ_2
λ	[0.26104, 0.32777]	[0.49311, 0.50870]	[0.68324, 0.73223]
Φ_1	[0.28350, 0.30324]	[0.47786, 0.52413]	[0.69192, 0.72349]
Φ_2	[0.26400, 0.32591]	[0.48585, 0.51630]	[0.68202, 0.73415]

Table 1 List of intervals of eigenpair.

convex hull for the interval of a quantity under interest. Interval of the other quantities are calculated consistently corresponding to the earmarked uncertain variables so as to satisfy the eigenvalue problem to the accuracy of the first-order approximation. The numerical example shows the validity of the proposed formulation and indicates that the estimated interval of a quantity takes the largest value at the uncertain variables earmarked for itself.

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