研　究　速　報

A Note on Finite Element Interval Analysis

(Part 1 Linear stiffness equation)

有限要素区間解析に関するノート（第一報　線形剛性方程式）

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1. Introduction

This paper proposes a method to estimate the interval of structural responses that arises from fluctuation involved in a structural system analyzed by the finite element method. Koyluoglu succeeded to demonstrate the validity of the technique of interval computation and linear programing for the interval mapping in structural mechanics in problem of linear stiffness equation of the finite element method[1]. His concerns are related to conservative estimation of the interval of each component of the nodal displacement vector affected by the uncertain components of the stiffness matrix and external loads of a static system.

Some components of the stiffness matrix are correlated with each other through the constitutive law of material and displacement function employed for the finite elements so that the components of the nodal displacement vector are also correlated with each other in order to make the stiffness equation hold, in other words, to satisfy the condition of stress equilibrium. This hints that interval analysis had better be carried out so as to satisfy the stiffness equation, especially in case that interval analysis of stresses in the structure is aimed at. This paper deals with the finite element interval analysis, in which consistent interval of the nodal displacements, in the sense that the stiffness equation is satisfied, is evaluated with respect to prescribed system variables.

2. Statement of problem

The followings are premised in this study concerned with a linear, elastic stiffness equation.

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1) Such system parameters can be identified that govern the variability of a structure. The system parameters stand for uncertain variables in case of stochastic structural analysis or design variables in case of structural shift synthesis.

2) We have a baseline structure prescribed by nominal system parameters, and the variability of the structural response with respect to the system parameters under interest can be estimated by the finite element sensitivity analysis.

3) The system parameters vary around their nominal value and are confined in a domain that their variability is conceived. The domain can be represented by a convex hull[2]. The range of the system parameter change is not so large as to change the nature of the structure.

Then, the problem is to estimate the upper bound and lower bound, in the other word, interval of the structural responses that is consistent with the stiffness equation and convex hull. In doing so, the interval is estimated by searching the earmark value of the system variables that specifies the interval on the convex hull.

3. Finite element sensitivity analysis

Suppose that a linear, elastic stiffness equation after treatment of boundary conditions is given by Eq. (1),

$$[K]\{U\} = \{F\} \tag{1}$$

where $[K]$ denotes the N-dimensional stiffness matrix, $\{U\}$ the nodal displacement vector, and $\{F\}$ the nodal load vector. It is assumed that their variability can be approximated by the Taylor series expansion with respect to M system variables ε_m, assigned to structural parameters p_m in the form of Eq. (2), in the vicinity of the nominal structural parameters and truncated at the first-order as given by Eqs. (3) and (4).

$$p_m = \bar{p}_m (1 + \varepsilon_m) \quad (2)$$

$$[K] = [\bar{K}] + \sum_{m=1}^{M} [K^I_m] \varepsilon_m \quad (3)$$

$$\{F\} = \{\bar{F}\} + \sum_{m=1}^{M} \{F^I_m\} \varepsilon_m \quad (4)$$

The upper bar indicates quantity evaluated at the nominal structural parameters, and the superfix I and suffix m stand for the first-order sensitivity with respect to m-th variable. In Eqs. (3) and (4), $[K^I_m]$ and $\{F^I_m\}$ are known.

The variability of the nodal displacement vector $\{U\}$ is assumed in the similar manner with the stiffness matrix as Eq. (5).

$$\{U\} = \{\bar{U}\} + \sum_{m=1}^{M} \{U^I_m\} \varepsilon_m \quad (5)$$

The unknowns $\{\bar{U}\}$ and $\{U^I_m\}$ are solved by substituting Eqs. (3), (4) and (5) into Eq. (1) as follows by the perturbation technique so as to satisfy the stiffness equation.

$$\{\bar{U}\} = [\bar{K}]^{-1} \{\bar{F}\} \quad (6)$$

$$\{U^I_m\} = [\bar{K}]^{-1} (\{F^I_m\} - [K^I_m] \{\bar{U}\}) \quad (7)$$

4. Choice of interval index

Now that the sensitivity analysis of the nodal displacement vector is carried out, the variability of quantity under interest can be estimated based on the sensitivity analysis. The characteristics of a displacement field is reflected in the strain energy stored in the field S so that it is possible to define an interval of global quantity such as the strain energy S subject to the system variable change. The first-order approximation of the variability of the strain energy S is given by Eq. (8) based on the results of Eqs. (6) and (7).

$$2S = \{U\}^T [K] \{U\} = \{\bar{U}\}^T [\bar{K}] \{\bar{U}\} + \sum_{m=1}^{M} (-\{\bar{U}\}^T [K^I_m] \{\bar{U}\}) \varepsilon_m = 2 (\bar{S} + \sum_{m=1}^{M} S^I_m \varepsilon_m) \quad (8)$$

On the other hand, if local interval of each displacement, that is, the n-th component U_n of the nodal displacement vector $\{U\}$ is needed, its first-order approximation is given by Eq. (8).

$$U_n = \bar{U}_n + \sum_{m=1}^{M} U^I_{nm} \varepsilon_m \quad (9)$$

Equation (8) means that a consistent index related to the strain energy can be devised, by which the interval of global structural response is specified in the form that the stiffness equation is satisfied approximately, whereas Eq. (9) is valid for straight estimation of the interval of local structural response.

5. Estimation of interval index by linear programing

The interval of the index representing structural responses can be estimated by the convex model[2)], when their variability evaluated in terms of the sensitivity analysis forms convex function of the structural parameters whose conceivable domain is input as a convex hull, since the maximum and minimum of a convex function confined in a convex hull lie on the hull boundary. The approximate variability of the strain energy of Eq. (8) is convex function, linear with respect to ε_m, as well as that of each displacement of Eq. (9). Suppose that the conceivable domain of the system variables is given in the form of siding conditions Eq. (10),

$$-c_{lm} \leq \varepsilon_m \leq c_{um} \quad (10)$$

where c_{lm} and c_{um} are positive constant standing for the lower bound and upper one of ε_m. This domain of rectangular prism forms a convex hull. Then, the search for the interval is stated in this simple case as

find

max. and. min. of objective function S or U_n

subject to $-c_{lm} \leq \varepsilon_m \leq c_{um}$

and can be carried out by the technique of linear programing. The earmark value of the system variables is obtained as the solution of the linear programing. Such case that the prescribed convex hull is quadratic will be discussed together with how to cope with the deficient first-order approximation later in the subsequent notes.

6. Numerical Example

Figure 1 illustrates an elastic bar under tension and discretized by two elements of unit length, the nominal spring constant EA / l of which is taken equal to unity without loss of generality. The left end is fixed, and the right end is subject to unit tension, giving rise to nominal

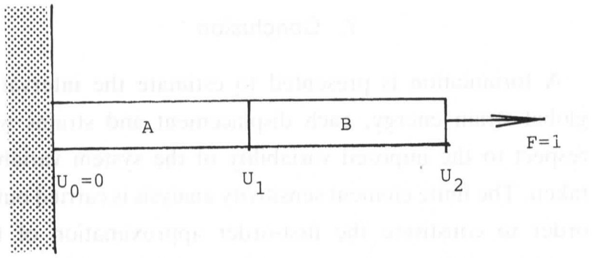

Figure 1 Finite elements of bar under tension

displacements of $\overline{U}_1 = 1$ and $\overline{U}_2 = 2$. Two system parameters are taken for the variability of spring constant of the two elements. The variability of the stiffness matrix is given by Eq. (11) exactly in this case, and that of the nodal displacement vector is computed as given by Eq. (12). The variability of the strain energy in the bar is summarized in the form of Eq. (13).

$$[K] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \varepsilon_1 + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \varepsilon_2 \tag{11}$$

$$\{U\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} + \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \varepsilon_1 + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} \varepsilon_2 \tag{12}$$

$$2S = 2 - \varepsilon_l - \varepsilon_2 \tag{13}$$

Figure 2 shows the convex hull by broken line and decisive contours of the index of S by solid line, which coincides with that of U_2, and the index of U_l by chain line. The amplitude of the system variables is taken equal to 10 % for both ε_1 and ε_2. The values of the contour of $2S$, U_2 and U_l are added in the figure. In this case, it is easily seen that the minimum and maximum of the objective function of the global strain energy S or the largest displacement U_2 take place at the same apex of ε_1 and ε_2, (0.1, 0.1) and (−0.1, −0.1) respectively, that is, the earmark value of the system variables. It is because both indices of S and U_2 are governed by the spring constants of the elements A and B under the given loading system, whereas the index of U_l is affected by ε_1 only, as shown by Eq. (12). U_l is governed by the spring constant of the element A. It means that the earmark value of ε_1 is determined for the interval of U_l, but that of ε_2 cannot be determined as shown by the parallel contour of U_l to the edge of the convex hull in Figure 2.

Figure 3 illustrates the range of the displacements of U_l and U_2 by matted area. The solid circle indicates the nominal value of U_l and U_2. The parenthesized figures stand for the value of ε_1 and ε_2 on the apex of the convex hull. Two apexes marked by blank circle are calculated exactly by reanalysis of the stiffness equation whose stiffness matrix is determined by the earmark value of the system variables. The other two apexes without any mark correspond to the relevant apexes of the convex hull. It can be said from this figure that, in case if the concept of interval is applied forcibly to vector, the consistent interval of the nodal displacement vector ranges from $\{1.11111, 2.22222\}^T$ to

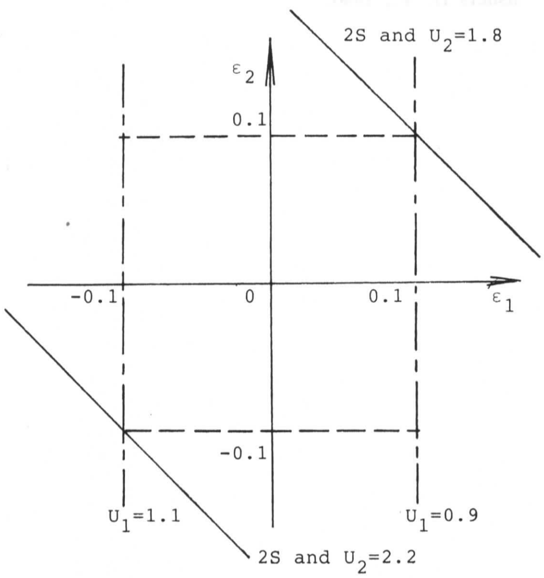

Figure 2 Convex hull and contours of interval index

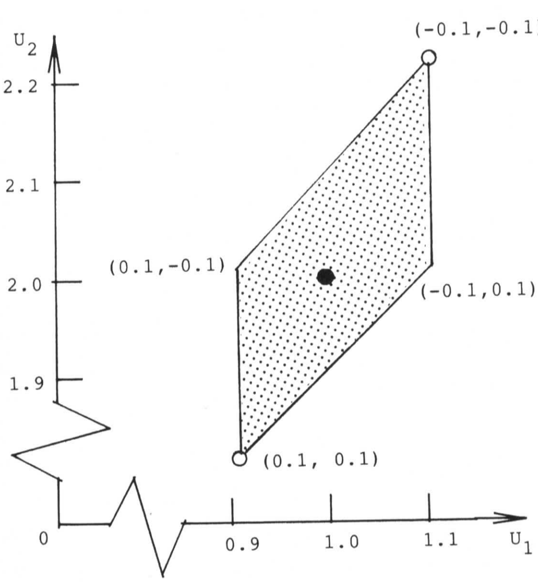

Figure 3 Intervals of displacements U_l and U_2

$\{0.90909, 1.81818\}^T$ corresponding to the value of the system variables of $(-0.1, -0.1)$ and $(0.1, 0.1)$ earmarked by the index of the strain energy. The superfix T denotes transpose of vector. The consistent interval mentioned above enables us to derive the individual interval of U_1 and U_2 and calculate that the interval of the strain term, that is, U_1–0 and U_2–U_1 in two elements ranges from [1.11111, 1.11111] to [0.90909, 0.90909]. On the other hand, when attention is paid to the interval of a particular displacement, for instance U_1 through the index picked up from Eq. (9), what we can know is that the interval of U_1 is [0.90909, 1.11111] from the earmark value of ε_1 of 0.1 and -0.1. The interval of U_2 cannot be specified in this example because the earmark value of ε_2 is not determined. Consequently, the estimation of interval of strain term is left impossible.

If interval of a particular displacement is needed, it can be estimated through the particular component picked up from Eq. (9), but is so local that the interval of stress may not be evaluated duly by the combination of such local intervals of displacements. Instead of tiresome choice of the interval of proper displacements under the influence of the system variables included, it is recommended to employ the strain energy as the index for the determination of the earmark value of the system variables that bounds the interval of structural responses, because the effect of all the system variables is taken into account in the computation of the global strain energy.

7. Conclusion

A formulation is presented to estimate the interval of global strain energy, each displacement and strains with respect to the imposed variability of the system variables taken. The finite element sensitivity analysis is carried out in order to constitute the first-order approximation of the variability of the strain energy. The variability of the system variables is expressed by a convex hull. The earmark value of system variables that bounds the strain energy is searched on the boundary of the convex hull by the technique of linear programing in the case of linear convex hull. The numerical example shows that the interval of the global strain energy, the largest displacement and strain terms in two elements can be estimated consistently from the earmark value of the system variables.

How to cope with the deficient first-order approximation of the structural variability and quadratic convex hull will be presented in the subsequent notes.

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