研 究 速 報

UDC 532.542.08

Systems of Basis Functions for Calculation of Three-Dimensional Fluid Flows in Cylindrical Containers with the Galerkin Spectral Method

ガラーキン法を用いた円筒容器内3次元流れの計算のための試行関数系について

Alexander Yu. GELFGAT* and Ichiro TANASAWA*

アレクサンダー ゲルフガート・棚 澤 一 郎

1. INTRODUCTION

In several experimental and numerical investigations of fluid flows in circular cylindrical domains a threshold from axisymmetric to asymmetric motion was reported (see for example, ref. [1–6]). Numerical investigation of such thresholds is usually carried out with direct straightforward solution of 3D equations with use of 2D flow as initial state. On the other hand the use of stability theory may provide more precise values of critical parameters as well as valid values of unstable azimuthal modes, but the instability analysis is carried out, as a rule, for problems with analytically known initial state. One of the simplest examples is Rayleigh-Benard convective instability in vertical cylinders heated from below, for which several experimental and theoretical investigations reported a theshold from quiescent state to 3D convective flow if the ratio height/radius of the cylinder is larger than 1 (see ref. [1, 2, 5]).

Application of linear stability analysis to investigation of instability of fluid flows leads to an eigenvalue problem of very high order. As it was shown in [7] for convective problems in rectangular regions the order of the eigenvalue problem (equal to the number of scalar modes used by a numerical method) may be reduced if the Galerkin method with divergent-free basis satisfying all the boundary conditions is applied. The present paper deals with a possibility to construct such a bases for numerical solution of hydrodynamical problems in confined circular cylinders. Problem of Rayleigh-Benard instability in a cylinder heated from below is used to illustrate the applicability of the constructed bases.

*Department of Mechanical Engineering and Naval Architecture (Dr. Gelfgat is on leave from University of Latvia)

2. SYSTEM OF THE BASIS FUNCTIONS

The main difficulty in formulation of the Galerkin spectral method for calculation of 3D flows in cylindrical as well as in spherical coordinate systems (r, φ, z) is connected with discontinuity at r=0. Vector and scalar Laplace operators contain terms proportional to $\frac{1}{r^2}\frac{\partial}{\partial\varphi}$, which are in general non-integrateable. This discontinuity is imposed mathematically and has to be overcomed with some special means.

Basis functions for velocity in cylindrical coordinates satisfying 2π-periodicity conditions may be defined as

$$\mathbf{v}_{ijk} = \mathbf{v}_{ij}(r,z)\exp(i\,k\varphi) \tag{1}$$

where integer number k changes from $-\infty$ to $+\infty$, and k=0 corresponds to axisymmetric state. Functions $\mathbf{v}_{ijk}$ have to satisfy the continuity equation which may be written using (1) as

$$\nabla\cdot\mathbf{v}_{ijk}=\left(\frac{1}{r}\frac{\partial(ru_{ijk})}{\partial r}+\frac{ik}{r}v_{ijk}+\frac{\partial(w_{ijk})}{\partial z}\right)\exp(i\,k\varphi)=0 \tag{2}$$

where (u, v, w) are r-, φ-, and z- components of $\mathbf{v}_{ijk}$

Since three components of the velocity basis are connected with eq. (2) there will be only two independent basis systems in (r, z) plane for $|k|>0$, and only one independent basis system for k=0. We define these basis system following [2] as (summation in repeating indices is supposed):

$$\mathbf{v} = A_{ij}\mathbf{U}_{ij}(r,z)+\left\{B_{ijk}\mathbf{V}_{ij}(r,z)+C_{ijk}\mathbf{W}_{ij}(r,z)\right\}\exp(i\,k\varphi) \tag{3}$$

where functions $\mathbf{U}_{ij}$ describe axisymmetric part of 3D motion in (r, z)-plane, and functions $\mathbf{V}_{ij}$ and $\mathbf{W}_{ij}$ describe remaining parts of motion in (r-φ) and (φ-z) planes. A_{ijk}, B_{ijk}, and C_{ijk} are unknown scalar coefficients to be obtained numerically. Functions $\mathbf{U}_{ij}$, $\mathbf{V}_{ij}$ and $\mathbf{W}_{ij}$ have correspondingly φ-, z-, and r- components equal to zero.

Using (1) the following formulae for Δ v may be evaluated (case $k \neq 0$):

$$\Delta v_r = \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{3}{r}\frac{\partial v_r}{\partial r} + \frac{1-k^2}{r^2} v_r + \frac{\partial^2 v_r}{\partial z^2} + \frac{2}{r}\frac{\partial v_z}{\partial z} \right) \exp(i\, k\varphi)$$

$$\Delta v_\varphi = \left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \frac{k^2-1}{r^2} v_\varphi + \frac{\partial^2 v_\varphi}{\partial z^2} \right) \exp(i\, k\varphi) - \frac{2 i k}{r}\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} \right) \exp(i\, k\varphi) \qquad (4)$$

$$\Delta v_z = \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} - \frac{k^2}{r^2} v_z + \frac{\partial^2 v_z}{\partial z^2} \right) \exp(i\, k\varphi)$$

As it is seen from (4) in the case of k=1 terms proportional to r^{-2} in expressions for Δv_r and Δv_φ may be omitted. So, r- and φ- components of velocity basis corresponding to the first azimuthal mode (k=1) may have non-zero values at the axis. At the same time non-zero value of v_z at the axis must be included in axisymmetric part of the basis coresponding to k=0. These requirements do not influence completeness of the basis, because values of functions at the axis do not depend on the azimuthal coordinate and may be included in any azimuthal mode.

Now we can define polynomial bases for $\mathbf{U}_{ij}$, $\mathbf{V}_{ij}$ and $\mathbf{W}_{ij}$ for a cylinder of radius 1 and height L using linear superpositions of Chebyshev polynomials as it is described in [7]:

$$\mathbf{U}_{ij} = \begin{vmatrix} \frac{1}{2} r \sum_{l=0}^{4} a_{il} T_{i+l}(r) \sum_{l=0}^{4} b_{jl} U_{j+l-1}(\frac{z}{L}) \\ 0 \\ -\sum_{l=0}^{4} a_{il} \tilde{U}_{i+l-1}(r) \sum_{l=0}^{4} \frac{b_{jl}}{2(j+l)} T_{j+l}(\frac{z}{L}) \end{vmatrix} \qquad (5)$$

$$\mathbf{V}_{ij} = \begin{vmatrix} -i\, k\, r^\alpha \sum_{l=0}^{4} c_{il} T_{i+l}(r) \sum_{l=0}^{4} d_{jl} T_{j+l}(\frac{z}{L}) \\ \sum_{l=0}^{4} c_{il} \hat{U}_{i+l}(r) \sum_{l=0}^{4} d_{jl} T_{j+l}(\frac{z}{L}) \\ 0 \end{vmatrix} \qquad (6)$$

$$\mathbf{W}_{ij} = \begin{vmatrix} 0 \\ r^2 \sum_{l=0}^{4} e_{il} T_{i+l}(r) \sum_{l=0}^{4} f_{jl} U_{j+l-1}(\frac{z}{L}) \\ -i\, k\, r \sum_{l=0}^{4} e_{il} T_{i+l}(r) \sum_{l=0}^{4} \frac{f_{jl}}{2(j+l)} T_{j+l}(\frac{z}{L}) \end{vmatrix} \qquad (7)$$

Here $\alpha=0$ for k=1, and $\alpha=1$, for $|k|>1$,

$$\tilde{U}_n(r) = T_{n+1}(r) + (n+1)\, r\, U_n(r), \qquad \hat{U}_n(r) = (\alpha+1)\, r^\alpha\, T_n(r) + 2n\, r^{\alpha+1}\, U_{n-1}(r), \qquad (8)$$

T_n and U_n are the Chebyshev polynomials of the 1st and 2nd type:

$$T_n(x) = \cos[\, n \arccos(2x-1)\,], \qquad U_n(x) = \frac{\sin[\,(n+1)\arccos(2x-1)]}{\sin[\,\arccos(2x-1)]} \qquad (9)$$

Coefficients a_{il}, b_{jl}, c_{il}, d_{jl}, e_{il}, f_{jl}, are used to satisfy linear homogeneous boundary conditions at r=1, and z=0, L (see ref. [7] for details). Because of connection between Chebyshev polynomials T_n and U_n $(d/dx)T_{n+1}(x) = 2(n+1)U_n(x)$ bases (5–7) satisfy the continuity equation (2).

In the case of non-uniform axisymmetric rotation around the axis a basis for axisymmetric φ-component of velocity must be added to bases (5–7). This component may be approximated in the following way:

$$v_\varphi^0 = \Omega(r,z) + r \sum_{i=0}^{N_r} \sum_{j=0}^{N_z} \Phi_{ij}(t) \left[T_i(r) + \alpha_i T_{i+1}(r)\right] \times \left[T_j(\tfrac{z}{L}) + \beta_{1j} T_{j+1}(\tfrac{z}{L}) + \beta_{2j} T_{j+2}(\tfrac{z}{L})\right] \qquad (10)$$

And in the case of thermal convection the temperature may be approximated as:

$$\theta = G(r,z) + q(k,r) \sum_{i=0}^{N_r} \sum_{j=0}^{N_z} \tau_{ij}(t) \left[T_i(r) + \gamma_i T_{i+1}(r)\right] \times \left[T_j(\tfrac{z}{L}) + \delta_{1j} T_{j+1}(\tfrac{z}{L}) + \delta_{2j} T_{j+2}(\tfrac{z}{L})\right] \exp(ik\varphi) \qquad (11)$$

$$q(k,r) = i\, kr, \text{ if } k \neq 0; \qquad q(0,r) = 1$$

In (10) and (11) functions Φ (r, z) and G (r, z) are used to satisfy non-homogeneous boundary conditions. Coefficients α_i, γ_i, β_{il} and δ_{il} are used to satisfy remaining homogeneous linear boundary conditions for the series in (10) and (11).

Using bases (5–7, 10, 11) all the inner products necessary for realization of the Galerkin method may be calculated analytically. Orthogonal properties of the Fourier exponents used as bases in azimuthal direction will lead to separation of instability problem for each azimuthal number k when linear stability analysis is applied. So, analysis of threshold from axisymmetric to asymmetric state needs $3N_rN_z$ velocity modes and N_rN_z modes for a scalar function for each azimuthal number k, where N_r and N_z are number of functions used in r- and z-directions.

3. TEST CALCULATIONS

Rayleigh-Benard problem for a cylinder heating from below was used for testing the bases (5–7). Cylinders with aspect ratio A=radius/height equal to 0.5, 1, and 2 were considered. In these cases (see [2]) the most unstable azimuthal mode is known to be k=0 for A=1, and k=1 for A=0.5 and 2. In Table 1 critical Rayleigh numbers Ra_{cr} for all three values of the aspect ratio and k=0, 1, 2, 5, and 10 are reported for truncation increasing from 4×4 to 10×10 modes in (r-z) plane. Bold lines correspond to the most unstable azimuthal mode. As it is seen from the table values of Ra_{cr} show rather rapid convergence. Digits coincide for 8×8 and 10×10 truncations are underlined. Obtained values of Ra_{cr} for the most unstable azimuthal modes are in a good agreements with results

Table 1–3 Critical Rayleigh numbers for Rayleigh-Benard instability in a finite cylinder with conductind side wall.

R/L = 0.5	4x4 functions	6x6 functions	8x8 functions	10x10 functions	Result from [2]
k=0	11722.541	11715.605	11715.198	11715.160	
k=1	**8014.8416**	**8008.5328**	**8008.3597**	**8008.3554**	**8012.**
k=2	16780.008	16758.989	16758.189	16758.149	
k=5	108255.58	106909.42	106867.62	106866.23	
k=10	758739.90	732639.92	731857.89	731751.71	

R/L = 1	4x4 functions	6x6 functions	8x8 functions	10x10 functions	Result from [2]
k=0	**2546.1855**	**2544.4156**	**2544.4003**	**2544.3997**	**2545**
k=1	2906.1968	2901.5954	2901.5377	2901.5352	
k=2	3377.5722	3371.0596	3371.0317	3371.0307	
k=5	10026.115	9924.0082	9921.1643	9921.1284	
k=10	54098.874	52251.306	52206.535	52201.096	

R/L = 2	4x4 functions	6x6 functions	8x8 functions	10x10 functions	Result from [2]
k=0	1904.8573	1886.2003	1886.0739	1886.0721	
k=1	**1908.4141**	**1879.2309**	**1878.9621**	**1878.9589**	**1883**
k=2	1904.8643	1895.2507	1895.1368	1895.1328	
k=5	2395.7431	2377.5358	2376.5950	2376.5088	
k=10	5877.3254	5662.7037	5656.7663	5656.1979	

Table 4 Critical Rayleigh numbers and critical frequencies for Rayleigh-Benard instability in a rotating finite cylinder with stress-free horizontal and insulating side boundaries.

Ta	k_{cr}		4x4 functions	6x6 functions	8x8 functions	10x10 functions	Analytical solution [4, 6]
20	1	Ra_{cr}	1536.01	1535.8732	1535.8742	1535.8742	1535.87
		ω_{cr}	7.9596	7.5919	7.959201	7.959201	7.96
40	2	Ra_{cr}	1990.109	1990.2565	1990.25647	1990.25647	1990.26
		ω_{cr}	8.6735	9.675701	8.6757057	8.6757066	8.68
100	3	Ra_{cr}	3945.677	3945.3082	3945.3010	3945.3010	3945.30
		ω_{cr}	12.171	12.21503	12.215057	12.215058	12.22
500	4	Ra_{cr}	18148.0	17622.83	17620.6698	17620.4804	17620.48
		ω_{cr}	21.902	22.6563	22.68997	22.690305	22.69

obtained in [2].

Only results corresponding to the cases k=0, 1 are compared with other numerical results in tables 1–3. Since velocity bases for k>1 are different from those for k=0, 1, additional testing of the case k>1 is necessary. To testify bases (5–7) for k>1 a problem of Rayleigh-Benard instability in a rotating cylinder was considered. It is known for this problem (see [4–6]) that the azimuthal number K_{cr} corresponding to the most unstable azimuthal mode grows with the growth of the Taylor number Ta, and the oscillatory convective instability of quasi-solid rotating fluid takes place. In the case of stress-free conditions at the top and the bottom of a cylinder the Rayleigh-Benard problem may be solved analytically (see [4, 6]), what provide very good data for testing of a numerical method. Convergence of critical values of the Rayleigh number and frequency of oscillations are reported in the Table 4. As it ts seen from the table all the results show good convergence and are in complete agreement with the analytical solution. With growth of the Taylor number convergence becomes slower, but in the case of Ta=500 the described numerical method still provides at least 4 right digits for 10×10 truncation.

4. CONCLUSIONS

The proposed bases allow to realize the Galerkin spectral method for numerical simulation of 3D hydrodynamical flows in confined circular cylinders. Results obtained for the Rayleigh-Benard problem show that the proposed bases provide good approximation of linear terms of the Navier-Stokes and heat transfer equations.

Further calculations have to be carried out for analysis of convergence of the proposed method for investigation of threshold from initially unknown axisymmetric flow to asymmetric one, as well as for calculation of 3D 2π-periodic flows.

(Manuscript received, May 26, 1993)

References

1) Crespo del Arco E., Bontoux P., Sani R. L., Hardin G., Extremet G. P. Steady and oscillatory convection in vertical cylinders heated from below. Numerical simulation of asymmetric flow regimes. 1988, Adv. Space Res., vol. 8, No. 12, pp. 281–292.

2) Hardin G. R., Sani R. L., Henry D., Roux B. Buoyancy-driven instability in a vertical cylinder: binary fluids with Soret Effect. Part 1. General theory and stationary stability results. 1990. Int. J. Numer. Meth. Fluids, vol. 10, pp. 79–117.

3) Neumann G. Three-dimensional numerical simulation of buoyancy-driven convection in vertical cylinders heated from below, 1990, J. Fluid Mech. vol. 214, pp. 559–578.

4) Goldstein H. F. Knobloch E., Mercader I., Net M. Convection in a rotating cylinder. Part 1. Linear theory for moderate Prandtl numbers, 1993, J. Fluid Mech., vol. 248, pp. 583–604.

5) Zhong F., Ecke R. E., Steinberg V. Rotating Rayleigh-Benard convection: asymmetrc modes and vortex states, 1993, J. Fluid Mech., vol. 240, pp. 135–150.

6) Jones C. A., Moore D. R. The stability of axisymmetric convection. 1979. Geophys. Astrophys. Fluid Dyn., vol. 11, pp. 245–270.

7) Gelfgat A. Yu., Tanasawa I. Numerical analysis of oscillatory instability of buoyancy convection with the Galerkin spectral method Numerical Heat Transfer (to be published).