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Recent Developments in the Field of Bifurcation Analysis
分岐解析における最近の研究動向

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This is a review paper about research works in the field of bifurcation analysis of geometrically nonlinear single-parameter conservative elastic structures. Only problems involving simple bifurcation point are treated. After a brief introduction, general theory of elastic stablity based on energy concept is first described. Direct and indirect methods for detecting bifurcation points and various path-switching strategies are then explained in the following two sections respectively. Lastly, a concluding remark about the present status of research activities regarding bifurcation analysis is given.

1. Introduction

An engineering structure under the action of external load might lose its stability at particular critical load level due to significant change in its structural geometry. Of particular interest to structural analysts is the possible occurence of the so-called elastic bifurcation buckling. Furthermore information about structural behaviour after bifurcation, represented by the so-called bifurcation path is important when investigating the sensitivity of load carrying capacity towards loading or geometrical imperfection (Fig. 1). For the above investigation, bifurcation analysis is necessary. The two essential processes involved in a bifurcation analysis are the detection of bifurcation point and the subsequent switching from primary to bifurcation path (Fig. 2). The first stage of the analysis is usually treated as part of the process of detecting critical points including both limit and bifurcation point with the help of the so called detecting parameter while equilibrium path is being traced. Recently, a new approach called direct method[8~11] has been used to compute critical point and the associated buckling mode of engineering structures simultaneously. Regarding the formulation of path–switching strategies, two main trends exist: the first one is where higher order terms are used in the incremental equations[5~7),12),13] and the second one where only linear terms are involved[14)~23] for obtaining an initial approximation for a point on bifurcation path. Riks[2),3),34] has presented proposal based on both trends. In the later category, very often additional constraints have to be introduced either during the construction of approximate bifurcation mode[2),3),15),18)~21] or during interation process[16),17].

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In view of the vast number of methods being proposed, this paper is written with the aim of providing an overall view about recent research works concerning bifurcation analysis. Only problems involving geometrically nonlinear conservative perfect elastic structures subjected to single proportional loading with simple bifurcation point will be treated. In section 2, a general theory of elastic stability will be first described. Methods for detecting bifurcation points and path-switching will be explained in Section 3 and 4 respectively. Lastly, a concluding remark about the present status of research activities will be given.

2. Theory of elastic stability

A structural system could lose its stability due either to the change in its geometry or material properties or both. Two phenomena associated with this lost of stability are caused by snap-through (limit point) and bifurcation buckling (bifurcation point) as shown in Fig. 3. Here, only the theory about elastic stability problems arising from geometrical changes will be described. According to the energy concept of stability[1], *a complete relative minimum of the total potential energy with respect to the generalized coordinates for a conservative system is necessary and sufficient for the stability of an equilibrium state of the system.* Assuming that a system at equilibrium configuration $\boldsymbol{P}_o = \{v_o, \lambda_o\}^T$ is displaced to an adjacent configuration $\boldsymbol{P}_{adj} = \{v_o+\varepsilon, \lambda_o\}^T$ under the action of a very small perturbation $\varepsilon = \{\varepsilon_i\}^T$, then the change in potential could be written as

$$\Delta\Pi = \Pi_{adj}-\Pi_o=\Pi,_{i}\varepsilon_i+1/2\Pi,_{ij}\varepsilon_i\varepsilon_j+1/6\Pi,_{ijk}\varepsilon_i\varepsilon_j\varepsilon_k +..,\ i, j, k = 1\sim N \quad (1)$$

where Π_{adj}, Π_o = *potential energy at state* $\boldsymbol{P}_{adj}$ and $\boldsymbol{P}_o$ respectively, $(\ ),_i$ = partial derivative with respect to generalized coordinates, $\boldsymbol{v}$ = vector of finite generalized coordinates, λ = dimensionless loading parameter and N = total number of system degrees of freedom. Hereafter superscript T will denote transposition. Summation convention is used here for repeated indices and they will run from 1 to N unless otherwise specified. Since $\boldsymbol{P}_o$ is in equilibrium, $\Pi,_i$ in eq. (1) should vanish. Therefore the important equation for the stability of conservative system could be stated as

$$1/2\Pi,_{ij}\varepsilon_i\varepsilon_j > 0 \quad (2)$$

where the contribution from the terms related to $\varepsilon_i\varepsilon_j\varepsilon_k$, $\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l$ etc have been neglected since ε is assumed to be infinitesimally small. The stationary condition of Π with respect to generalized coordinates will yield a set of N equilibrium equations

$$\Pi,_i = r_i(v, \lambda) = f_i(v)-\lambda p_{oi} = 0 \quad (3)$$

for a discretized system with N degrees of freedom, where $\boldsymbol{f} = \{f_i\}^T$ = internal force vector and $\boldsymbol{p}_o = \{p_{oi}\}^T$ = generalized loading mode. Making use of eq. (3), the stability condition could be written in matrix notation as

$$1/2\varepsilon^{\mathrm{T}}\boldsymbol{K}\varepsilon > 0 \quad (4)$$

where $\boldsymbol{K}$ = *a* $N\times N$ tangent stiffness matrix ($K_{ij} = r_{i,j}$). Thus, the stability of an equilibrium configuration could be judged by investigating the positive definiteness of tangent stiffness matrix. At critical point where $\boldsymbol{K}$ is semi-positive definite, the following relationship holds,

$$\boldsymbol{Kx} = \boldsymbol{0} \quad (5)$$

where $\boldsymbol{x}$ is an arbitrary non-zero vector. For $\boldsymbol{x}$ to be a non-zero solution, determinant of tangent stiffness matrix, denoted as $|\boldsymbol{K}|$ must vanish. This vanishing of $|\boldsymbol{K}|$ is equivalent to the condition represented by the following eigenvalue problem with $\omega_j = 0$

$$[\boldsymbol{K} - \omega_j\boldsymbol{I}]\,\phi_j = 0 \quad (6)$$

where ω_j, ϕ_j = j^{th} eigenvalue and the corresponding eigenvector of $\boldsymbol{K}$ respectively and $\boldsymbol{I}$ = identity matrix. Differentiating eq. (3) repeatedly with respect to a path parameter η, the following series of simultaneous differential equations

$$K_{ij}\dot{v}_j-p_{oi}\dot{\lambda} = 0 \quad (7)$$

$$K_{ij}\ddot{v}_j-p_{oi}\ddot{\lambda}+r_{i,jk}\dot{v}_j\dot{v}_k+2r_{i,j\lambda}\dot{v}_j\dot{\lambda}+r_{i,\lambda\lambda}\dot{\lambda}^2 = 0 \quad (8)$$

could be obtained, where $(\ ),_\lambda$ = partial differentiation with respect to λ, $(\dot{\ }) = d(\ )/d$ and $\boldsymbol{p}_{oi} = -\boldsymbol{r}_{i,\lambda}$. If eq. (7) is premultiplied with eigenvector ϕ_1 corresponding to the distinct smallest eigenvalue ω_1 and making use of eq. (5), the following will hold at critical point,

$$\dot{\lambda}\phi_1{}^T\boldsymbol{p}_o = 0 \quad (9)$$

Two cases are possible from eq. (9), namely

$$\phi_1{}^T\boldsymbol{p}_o \neq 0, \quad \dot{\lambda} = 0 \quad (10)$$

$$\text{and} \quad \phi_1{}^T\boldsymbol{p}_o = 0 \quad (11)$$

which represent the occurence of limit point and bifurcation point respectively.

3. Bifurcation point detection

3.1 Indirect method

This is the most commonly used method in which the encounter of bifurcation point is judged with the help of a detecting parameter while equilibrium path is being traced. As detecting parameter, the determinant $|\boldsymbol{K}|$ or the smallest eigenvalue ω_1 of tangent stiffness matrix which would vanish at critical point are the ones most frequently used. In order to differentiate between limit point and bifurcation point, eq. (11) is often used[8),9)]. Detecting parameters that show response only towards bifurcation point have also been proposed[2)~7),26)]. With indirect method, equilibrium path has to be traced until the vicinity of bifurcation point if accuracy is required. This requires that the increment size of path parameter when approaching bifurcation point be appropriately reduced. Linear extrapolation schemes[3),4),10),15)] in which the increment size is adjusted according to the changes in detecting parameter have been proposed. By using extrapolation scheme, the intermediate equilibrium points needed to be computed before reaching the desired bifurcation point could be reduced thus saving unnecessary computational time. A method involving the solution of a generalized eigenproblem formulated by using the difference between two tangent stiffness matrix at points corresponding to η_{i+1} and η_i which straddle a bifurcation

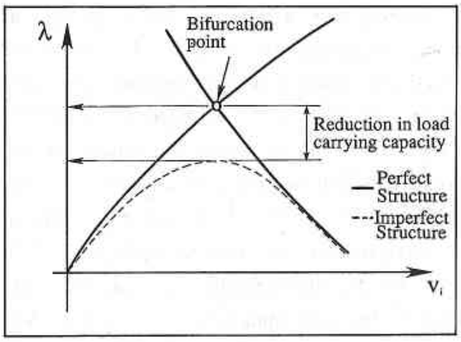

Fig. 1 Long-deflection path-Perfect and imperfect structure

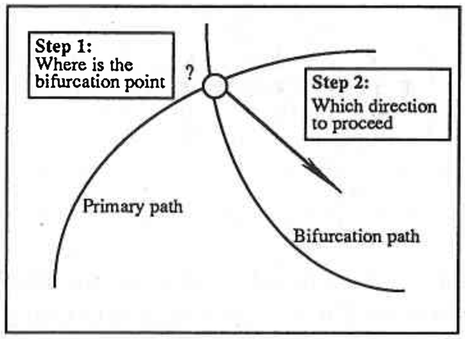

Fig. 2 Bifurcation analysis: detection and path-switching

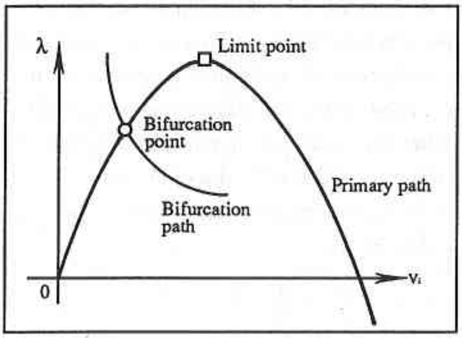

Fig. 3 Critical points: limit and bifurcation point

Original equilibrium equation:
$$\mathbf{r}(\mathbf{v},\lambda)=\mathbf{0}$$
+
Singularity defining equations and Normalizing equation

Extended system

$$\begin{cases} \mathbf{r}=\mathbf{0} \\ \mathbf{K}\boldsymbol{\phi}_1=\mathbf{0} \\ l(\boldsymbol{\phi}_1)=0 \end{cases} \qquad (12a)$$

$$\begin{cases} \mathbf{r}=\mathbf{0} \\ \mathbf{K}\boldsymbol{\phi}_1=\mathbf{0} \\ l(\boldsymbol{\phi}_1)=0 \\ \boldsymbol{\phi}_1^T\mathbf{p}_o=0 \end{cases} \qquad (12b)$$

$$\begin{cases} \mathbf{r}=\mathbf{0} \\ |\mathbf{K}|=0 \end{cases} \qquad (12c)$$

The normalizing equations:

$$l(\boldsymbol{\phi}_1)=\|\boldsymbol{\phi}_1\|-1=0 \qquad (13a)$$

$$l(\boldsymbol{\phi}_1)=\mathbf{e}_i^T\boldsymbol{\phi}_1-\phi_{1o}=0 \qquad (13b)$$

Fig. 4 Concept of extended system

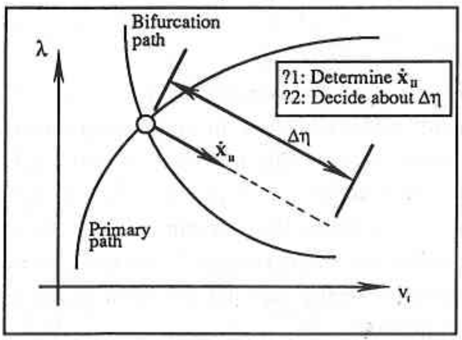

Fig. 5 The two process in path-switching

point has been proposed by Fujikake[25] to compute the critical path parameter η_{cr}. It is claimed that this method is applicable even to problems with closely spaced critical points.

3.2 Direct method[8)~11),24),29)]

In contrast with indirect method, constraint equations which are composed of a set of singularity defining equations and a normalizing equation ensuring non-triviality of solution vectors are included in the system of equations to be solved in direct method (Fig. 4). This set of new extended equations is then solved to yield both the position of critical point and its associated eigenmode directly (Fig. 4). In this way, the number of intermediate points needed to be computed via the tracing of equilibrium path before reaching the desired critical points could be reduced. In eq. (13b) (Fig. 4), e_i is the unit base vector and ϕ_{1o} ($o = 1 \sim N$) is a specified critical component of eigenvector ϕ_1. Wriggers and Simo[9] has proposed a criteria for the selection of index i. Addition of the constraint $\phi_1{}^T p_o = 0$ will ensure that only bifurcation point is converged to. The linearized form of the extended system of eq. (12c) which has been considered by Abbot[29] is not suited for incorporation into the Newton scheme[8),9)]. Linearizing the extended system of eq. (12a) with eq. (13a) as the normalizing equation will yield

$$\begin{bmatrix} K & 0 & p_o \\ \nabla_v (K\phi_1) & K & \nabla_\lambda (K\phi_1) \\ 0^T & \phi_1{}^T/|\phi_1| & 0 \end{bmatrix} \begin{Bmatrix} \Delta v \\ \Delta\phi_1 \\ \Delta\lambda \end{Bmatrix}$$

$$= - \begin{Bmatrix} r(v, \lambda) \\ K(v, \lambda)\phi_1 \\ |\phi_1| - 1 \end{Bmatrix} \tag{14}$$

where $\nabla_v (K\phi_1)\Delta v$ and $\nabla_\lambda (K\phi_1)\Delta\lambda$ are directional derivatives of K in the direction of Δv and $\Delta\lambda$ respectively. An elimination algorithm for evaluating the iterative changes of $\Delta w = \{\Delta v\ \Delta\phi_1\ \Delta\lambda\}^T$ has been proposed by Wriggers et al[8] by taking into consideration the sparsity of the coefficient matrix of eq. (14). This algorithm could be easily incorporated into any conventional finite element analysis code provided that an extra routine for evaluating the derivatives $\nabla_v (K\phi_1)$ and $\nabla_\lambda (K\phi_1)$ is included. Wriggers and Simo[9] have proposed a similar algorithm whereby the near-singular condition of tangent stiffness matrix K when iterating near critical point could be removed. Also, instead of evaluating the derivatives of tangent stiffnes matrix analytically, a numerical approximation for these derivatives by using finite difference equation has been proposed by Wriggers and Simo[9]. Criteria indicating the approaching or encounter of critical point is necessary in order to switch the computational procedures from that of usual path-tracing to that involving extended system. Initial guess of the starting eigenvector ϕ_1 could be computed by using say inverse interation procedures[9] or an alternative method proposed by Seydel[24),30] and used by Skeie and Felippa[10].

4. Path switching

An initial approximation $\mathbf{X}_{11}$ for a point lying on the bifurcation path could be obtained by adding a suitably chosen approximate branching direction $\dot{\mathbf{x}}_{11}$ with appropriate magnitude $\Delta\eta$ to the detected bifurcation point X_B as follows,

$$X_{11} = X_B + \Delta\eta\dot{x}_{11} \tag{15a}$$

$$\dot{x}_{11} = (\alpha_1\dot{x} + \alpha_2 b) \,/\, \| \alpha_1\dot{x}_1 + \alpha_2 b \|, \| \dot{x}_{11} \| = 1 \tag{15b}$$

where $\dot{x}_1$ = normalized tangent vector to the primary path in R^{N+1}, b = approximate bifurcation direction in R^{N+1} and α_1, α_2 = magnitude of $\dot{x}_1$ and $\mathbf{b}$ respectively. From eq. (15b) and as indicated in Fig. 5, two major tasks involved in path–switching are the determination of appropriate branching direction $\dot{x}_{11}$ and of specification of suitable magnitude $\Delta\eta$. The simplest and most commonly used method is to adopt the vector $\{\phi_1, 0\}^T$ as the approximate bifurcation direction b with a suitably specified $\Delta\eta$. Alternative strategies whereby no eigenproblem is involved have also been proposed[16),18)~21]. Methods where higher order derivatives of residual vectors are used in the determination of $\dot{x}_{11}$ are comparatively fewer[5)~7),12),13),27] than those that make use of only linear incremental stiffness equations[14),15),16)~21),28]. As for the determination of $\Delta\eta$, trial-and-error method is the most commonly used method. Methods whereby the magnitude of branching direction could be computed directly have also been proposed[16),20]. Hartono et al[13] obtained the incremental solution at bifurcation point by solving system of equations involving higher order derivatives of equilibrium equations. In order to attain a point on the yet unknown bifurcation path from the initial approximation, iterative process with various special constraints introduced is often used[2),10),16),17),28].

Riks[2),3),34] for example has proposed the following expression for $\dot{x}_{11}$:

$$\dot{x}_{11} = \dot{x}_1 + \alpha_2\{\phi_1, 0\}^T, \| \dot{x}_{11} \| = 1 \tag{16}$$

By requiring that $\dot{x}_{11}$ is perpendicular to $\dot{x}_1$ (Fig. 6), and noting that $\| \dot{x}_1 \| = 1$, α_2 will be given by

$$\alpha_2 = 1/\dot{x}_1{}^T\, \phi_1 \tag{17}$$

An iteration process according to Riks's arc-length method[2] is then carried out by using $\dot{x}_{11}$ as the normal to the constraint plane with $\Delta\eta$ selected on a trial-and-error basis. This iteration process will force the iterative corrections to lie on the constraint plane of Fig. 6 thus preventing any convergence back onto the known primary path. A similar idea has also been mentioned by Waszczysztn[15].

Skeie and Felippa[10] used the rather simple expression

$$\{\dot{v}_{11}, \dot{\lambda}_{11}\}^T = \{\Delta\eta\phi_1, 0\}^T \tag{18}$$

where only eigenvector is involved as the first approximation for $\dot{x}_{11} = \{\dot{v}_{11}, 0\}^T$. For the determination of ϕ_1, the approximation method due to Seydel[24),30] is used. In order to compensate for this rather simple approximation which might lead to possible failure especially in the case of asymmetric bifurcation point, special locally defined cylindrical constraint[10),27] (Fig. 7) represented as

$$\{\dot{v} - \dot{\lambda} q\}^T \{\dot{v} - \dot{\lambda} q\} - \Delta\eta^2 \,/\, q^T q = 0 \tag{19}$$

where $q = K^{-1} p_o$, has been introduced into the iteration process. Regarding the magnitude $\Delta\eta$, Skeie and

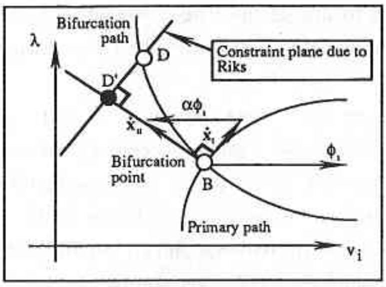

Fig. 6 Riks's proposal for locating a point on the bifurcation path

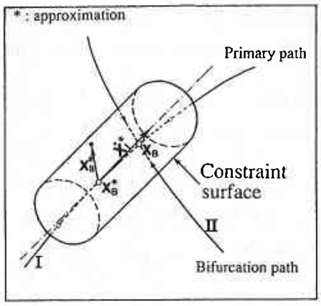

Fig. 7 A constraint in the form of cylindrical surface used during iteration process

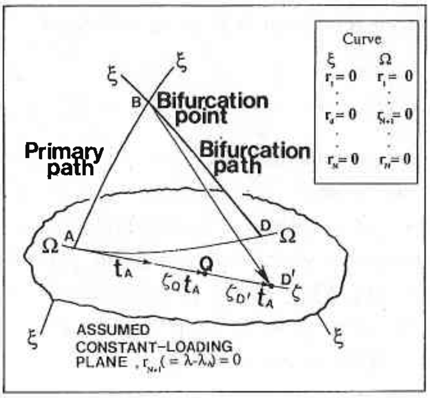

Fig. 8 Concept of line search

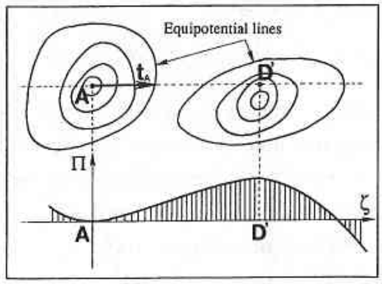

Fig. 9 Stationary point of potential Π along search direction t_A

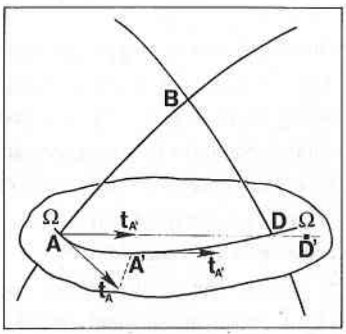

Fig. 10 Improved tangent vector $t_{A'}$ used as search direction

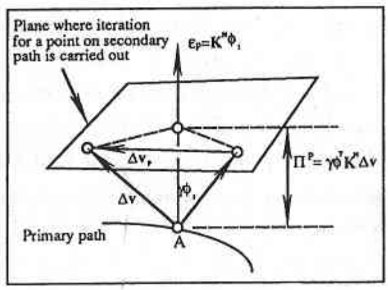

Fig. 11 Energy perturbation method by Kroplin

Felippa[10] stated that it could be chosen to be the same as the increment of path parameter used when tracing the primary equilibrium path in the vicinity of bifurcation point.

The existence condition of solutions for eq. (12) could be expressed by using generalized inverse[5)~7)] as

$$[\boldsymbol{I} - \boldsymbol{K}\boldsymbol{K}^-]^T \dot{\lambda} \boldsymbol{p}_o = \boldsymbol{0} \tag{20}$$

where $\boldsymbol{I}$ = identity matrix and $\boldsymbol{K}^-$ = Moore–Penrose generalized inverse matrix of $\boldsymbol{K}$. Incremental solution at bifurcation point where $[\boldsymbol{I} - \boldsymbol{K}\boldsymbol{K}^-]^T \boldsymbol{p}_o = \boldsymbol{0}$ holds could be written as

$$\dot{\boldsymbol{v}} = \dot{\lambda} \boldsymbol{K}^- \boldsymbol{p}_o + [\boldsymbol{I} - \boldsymbol{K}^-\boldsymbol{K}]\, \dot{\boldsymbol{\alpha}} \tag{21}$$

where $\dot{\boldsymbol{\alpha}}$ = arbitrary vector composed of the scaling factors for the linearly independent column vectors of matrix $[\boldsymbol{I} - \boldsymbol{K}^-\boldsymbol{K}]$. Since rank of $\boldsymbol{K}$ is $N-1$ at simple bifurcation point, rank $[\boldsymbol{I} - \boldsymbol{K}^-\boldsymbol{K}]$will be equal to 1. Denoting this single independent vector of $[\boldsymbol{I} - \boldsymbol{K}^-\boldsymbol{K}]$ as $\boldsymbol{a}$, eq. (21) will become

$$\dot{\boldsymbol{v}} = \dot{\lambda} \boldsymbol{K}^- \boldsymbol{p}_o + \dot{\alpha} \boldsymbol{a} \tag{22}$$

which contains two unknown scaling factor $\dot{\lambda}$ and $\dot{\alpha}$. In order to obtain a relationship between $\dot{\lambda}$ and $\dot{\alpha}$, existence condition of solutions for eq. (8), namely,

$$[\boldsymbol{I} - \boldsymbol{K}\boldsymbol{K}^-]^T \{\ddot{\lambda} \boldsymbol{p}_o - \boldsymbol{h}(\dot{\boldsymbol{v}}, \dot{\lambda})\} = \boldsymbol{0} \tag{23}$$

and the condition for bifurcation point, namely $[\boldsymbol{I}-\boldsymbol{K}\boldsymbol{K}^{-}]^{T}\boldsymbol{p}_o = \boldsymbol{0}$ are used together to yield the following equation

$$\boldsymbol{a}^T\boldsymbol{h}\ (\dot{\boldsymbol{v}}, \dot{\lambda}) = 0 \tag{24}$$

where $\boldsymbol{h} = \{h_i\}^T$ and $h_i = r_{i,jk}\dot{v}_j\dot{v}_k+2r_{i,j\lambda}\dot{v}_i\dot{\lambda}+r_{i,\lambda\lambda}\dot{\lambda}^2$. Substitution of eq. (22) into (24) will result in a quadratic equation as follow

$$A\dot{\alpha}^2 + 2B\dot{\alpha}\dot{\lambda} + C\dot{\lambda}^2 = 0 \tag{25}$$

where A, B, C = computed constants and $\boldsymbol{q} = \boldsymbol{K}^{-}\boldsymbol{p}_o$. By using the ratio $\dot{\kappa}_i$ $(= \dot{\lambda}_i \ / \ \dot{\alpha}_i)$ obtained from eq. (25), eq. (22) could be simplified to

$$\{\dot{\boldsymbol{v}}, \dot{\lambda}\}^T = (\{\boldsymbol{a}, 0\}^T + \{\boldsymbol{K}^{-}\boldsymbol{p}_o, 1\}^T \dot{\kappa}_i)\ \dot{\alpha}_i \tag{26}$$

$\dot{\alpha}$ in eq. (26) is determined based on a trial-and-error basis until convergence onto bifurcation path is achieved.

In line search[18)~21)], a search is carried out in the vicinity of bifurcation point on a constant loading plane along the direction of a tangent vector $\boldsymbol{t}_A$, in order to locate a point D' which lies sufficently close to point D as shown in Fig. 8. Search direction vector $\boldsymbol{t}_A$ is the tangent vector to curve Ω at point A. Curve Ω is defined by replacing one of the equilibrium equation say $r_d = 0$ $(d = 1\sim N)$ with the equation r_{N+1} $(= \lambda-\lambda_A) = 0$. Position of point D' corresponds to the stationary point of potential Π along search direction $\boldsymbol{t}_A$ (Fig. 9). With point D' determined, iteration at constant loading level λ_A to obtain point D could then be started by using

$$\boldsymbol{X}_{D'} = \boldsymbol{X}_A + \zeta_{D'} \cdot \boldsymbol{t}_A \tag{27}$$

as the first approximation, where $\boldsymbol{X}_{D'}$, $\boldsymbol{X}_A$ = position vector of point D' and A respectively, $\zeta_{D'}$ = arc-length along AD' measuring from point A. In this approach, by using the proposed criteria of stationary condition of potential $d\,\Pi \ / \ d\zeta = 0$ along search direction $\boldsymbol{t}_A$, the magnitude $\zeta_{D'}$ of approximate branching mode $\boldsymbol{t}_A$ could be determined through computation and not assigned on trial-and-error basis. The proposed criteria could be proved to be equivalent to

$$\boldsymbol{r}^T\boldsymbol{t}_A = 0 \tag{28}$$

which is used in the actual numerical computation as the search termination criteria[20)] where $\boldsymbol{r}$ is the original nonlinear equations given by eq. (3). The proposed line search scheme might fail to detect any stationary point in the direction of $\boldsymbol{t}_A$. In such cases, one possible remedial measure is to update the tangent vector of curve Ω by using Euler or Runge–Kutta method as shown in Fig. (10).

Kroplin et al[16)] have proposed a path–switching strategy similar to line search concept which is composed of the following three steps: i) determination of a perturbation pattern ψ ii) computation of the critical amplitude γ of perturbation ψ and iii) iteration to the new solution point (Fig. 11). This approach could be also applied to determine the range of stability of an equilibrium point. Kroplin introduced a quantity termed perturbation energy π_{p} which is being defined as

$$\pi_p = \varepsilon_p{}^T\Delta\boldsymbol{v} \tag{29a}$$

$$\Delta\boldsymbol{v} = \gamma\psi + \Delta\boldsymbol{v}_p \tag{29b}$$

where ε_p = perturbation force vector, $\Delta\boldsymbol{v}$ = incremental displacement vector, γ = amplitude of perturbation, $\Delta\boldsymbol{v}_p$ = part of $\Delta\boldsymbol{v}$ which is perpendicular to ε_p and ψ = displacement vector representing the perturbation pattern. Using $\boldsymbol{K}^N\ \psi$ as the perturbation vector and normalizing $\psi^T\boldsymbol{K}^N\psi$ to 1, equation for perturbation energy could be rewritten as

$$\pi_p = \gamma\psi^T\boldsymbol{K}^N\psi + \varepsilon_p{}^T\Delta\boldsymbol{v}_p \tag{30}$$

$$\pi_p = \gamma \tag{31}$$

where $\boldsymbol{K}^N$ = nonlinear part of tangent stiffness matrix $\boldsymbol{K}$ which is a linear function of $\boldsymbol{v}$ and $\boldsymbol{v}$ = displacement vector at known point A (Fig. 11). As perturbation pattern, Kroplin proposed the use of eigenvector ϕ_1. The eigenvector ϕ_1 and its critical magnitude corresponding to γ in eq. (31) are each computed by solving a similar variational problem where the incremental energy is constrained to be zero. The variational problem will each lead to a similar set of linear simultaneous equations where ϕ_1 and γ could be computed through iteration. A perturbation is imposed upon the structure in order to arrive at a new equilibrium point with the same energy level as follow

$$\delta\pi = \delta\{1/2\ \Delta\boldsymbol{v}^{iT}\,[\boldsymbol{K}^L+\boldsymbol{K}^N{}_{(v+\Delta v)}]\ \Delta\boldsymbol{v}^i + \Delta\lambda \\ [\phi_1{}^T\,[\boldsymbol{K}^N{}_{(v)}\Delta\boldsymbol{v}^i-\gamma]\} = 0 \tag{32}$$

In eq. (32), the perturbation energy is constratint to be of size γ. A point on new secondary equilibrium path is obtained by carrying out the following iteration

$$\begin{bmatrix} \boldsymbol{K}^L+\boldsymbol{K}^N{}_{(v+\Delta v)} & \boldsymbol{K}^N{}_{(v)}\phi_1 \\ \phi_1{}^T\boldsymbol{K}^N{}_{(v)} & 0 \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{v}^i \\ \Delta\lambda^i \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \gamma \end{Bmatrix} \tag{33}$$

In Kroplin's method, the iterative points will be forced to lie on a plane at a distance γ from the known point A. Since the secondary path searched for might not lie in the vicinity of the known point A where the perturbation is imposed, very stable iteration scheme has to be used. After some iterations, the constraint will be removed and usual unconstrained iteration will be continued until convergence is achieved.

Kouhia and Mikkola[17] used eigenvector ϕ_1 as the first approximation for bifurcation mode with its initial magnitude $\Delta\eta_o$ evaluated as follow

$$\Delta\eta_o = \Delta\lambda_o\sqrt{q^{1T}C^v q + \alpha^2} \tag{34}$$

where C^v and α are scaling matrix for displacement component and scaling factor for loading parameter respectively and $K^{i-1}\, q^i = p_o$. Here supercript is used to denote iteration step. $\Delta\lambda_o$ corresponds to the first load increment specified during path tracing. This first approximation is then updated by using the following corrector

$$\delta v^i = \delta\xi^i\, \phi_1 + \delta\lambda^i\, q^i + d^i \tag{35}$$

where δv^i = correction to the first approximate bifurcation mode $\dot{v}_{11}$, $d^i = K^{-1} r^{i-1}$, $q^{iT} C^v \phi_1 = 0$, $d^{iT} C^v \phi_1 = 0$, $\phi_1^T C^v \phi_1 = 1$ and $\delta\xi^i$, $\delta\lambda^i$ = computed coefficients. An orthogonal constraint in the form of

$$v^T r(v, \xi, \lambda) = 0 \tag{36}$$

is introduced in order to establish a linear relationship between $\delta\xi^i$ and $\delta\lambda^i$ (Fig. 12) as follow

$$\delta\xi^i = \varepsilon^i + \gamma^i\, \delta\lambda^i \tag{37}$$

where $\varepsilon^i = -(r^{i-1T} d^i)\,/\,(2r^{i-1T}\phi_1)$ and $\gamma^i = -(r^{i-1T} q^i)\,/\,(2r^{i-1T}\phi_1)$. The coefficients ε^i and λ^i are updated at each iteration step i. A modified elliptical constraint based on Crisfield's idea[33] where the arc-length is updated at each iteration step is then used to obtain a quadratic equation in $\delta\lambda^i$. With $\delta\lambda^i$ obtained, approximation for a point on bifurcation path could then be updated until the required accuracy is achieved.

5. Concluding remark

Occurence of bifurcation point could be judged easily with the help of certain detecting parameter obtained during the path tracing process or alternatively direct method could be used. Further classification could be made based on eq. (11). On the contrary, further works are necessary before a generally applicable path-switching procedures could be established. For all the methods covered here, it is difficult to establish a common ground which we can use to make comparison among them. Methods that make use of higher order terms are comparatively simple to implement and any source of possible numerical trouble could be easily identified. On the contrary, although computational time could be reduced if only linear incremental equations are used, the implementation of the solution algorithm involved might not be a simple matter. Furthermore when numerical trouble occurs during computation, its source might not be easy to pinpoint. Based on this observation, the question of which method to be used will eventually depend on the complexity of problem to be solved and the availablility of resources. Any future proposed method should perhaps be formulated in such a way that computer-human interaction could be carried out easily. They should be simple in formulation with a clear physical interpretation of their process. Even if this means that extra amout of effort is needed, it is still worthwhile if we could be sure that the solution algorithm is a stable one

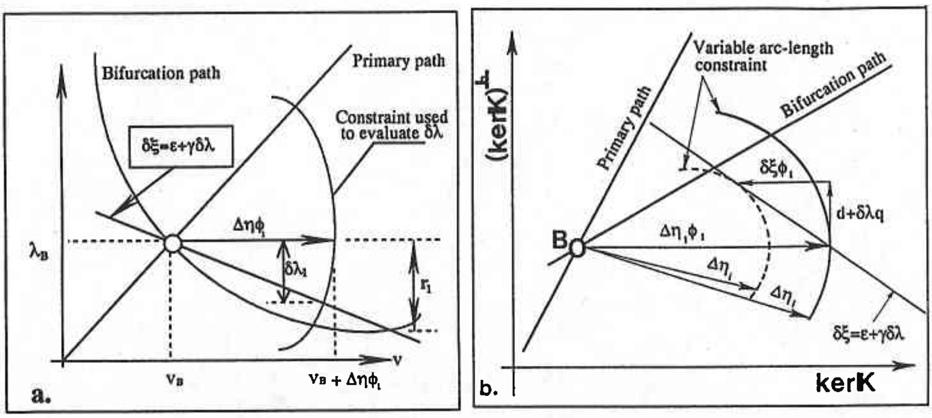

Fig. 12 Iterative procedure for obtaining a point on bifurcation path due to Kouhia and Mikkola

that will provide us with a significantly accurate result[32].

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