研 究 速 報
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A Note on Finite Element Synthesis of Structures (Part 6)
——Optimization Technique Using Circumscribed Hypersphere Based on Hessian Matrix——
有限要素法による構造シンセシスに関するノート（第6報）
——ヘッセ行列の外接超球に基づく最適化手法——

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1. Introduction

The objective function and constraint conditions employed in structural optimization are nonlinear by nature. Most of the optimization techniques developed so far make use of variability of the objective function and its first-order derivatives with respect to design variables only, while the second-order derivatives are set aside. Hessian matrix, which is formed by the second-order derivatives or sensitivities arranged in a square matrix, has been used in the variable metric method[1]. The entities of the Hessian matrix are not evaluated rigorously but are approximated on the basis of changes of the first-order derivatives, in usual, however. Such approximation of the Hessian matrix tends to lose useful information, which can be derived from the objective function, so that optimization design has been prone to be inert and laborious so far.

This note shows that constrained optimization problems can be made straightforward and efficient by the use of accurately evaluated Hessian matrix, which is applied to standardization of the quadratic approximation of the objective function into a circumscribed hyper-sphere. The circumscribed hypersphere enables us to search the minimal objective function systematically. A formulation is presented to find the minimal value of the objective function under linear inequality conditions and is discussed with a simple but typical numerical example.

2. Eigenvalue Analysis of Hessian Matrix

Suppose that we have a prototype design, N structural parameters of which are indicated by upper bars and to be changed to obtain the minimal objective by use of design variables α_n as given in Eq. (1).

$$x_n=\bar{x}_n(1+\alpha_n),\quad n=1\sim N \tag{1}$$

The second-order approximation of change of any objective function f near the prototype design is expressed by Eq. (2), where $\{F\}^T$ stands for a row vector consisting of the first-order sensitiveities, and $[H]$ the Hessian matrix of the second-order sensitivities. Superfix T denotes transpose hereafter.

$$f(x_n)=f(\bar{x}_n)+\{F\}^T\{\alpha_n\}+\{\alpha_n\}^T[H]\{\alpha_n\} \tag{2}$$

The first and second-order sensitivities of the finite element analysis are evaluated for the prototype design by means of perturbation technique[2]. The Hessian matrix furnishes us with N eigenvalues λ_n and modal matrix $[\Phi]$. The result of such eigenvalue analysis can be used for linear transformation between the basic design variables α_n and standardized design variables ε_n in the form of Eq. (3),

$$\{\alpha_n\}=[T]\{\varepsilon_n\} \tag{3}$$

where $[T]$ is a transformation matrix defined as follows,

$$[T]=[\Phi]^T\ diag\ [1/\sqrt{\lambda_n}][\Phi] \tag{4}$$

in cases that the Hessian matrix is positive-definite.

3. Hypersphere Derived from Quadratic Function

The transformation matrix $[T]$ enables us to rewrite the objective function in the form of Eq. (5),

$$f(x_n)=f(\bar{x}_n)+\{B\}^T\{\varepsilon_n\}+\{\varepsilon_n\}^T[I]\{\varepsilon_n\} \tag{5}$$

where $[I]$ is unit matrix, and $\{B\}^T$ is equal to

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$\{F\}^T[T]$. This procedure means that the original quadratic of the objective function is expressed as a hypersphere in the standardized design variable space. The hypersphere has a center and features that the magnitude of the objective function is related uniquely to the distance from the center. This means that the objective function which satisfies the constraint conditions takes minimal value on the surface of the feasible domain when the center is outside the domain. Then it can be devised to find a hypersphere which touches the hyperplanes from the outside for the search of the minimal objective funcuion. This hypersphere is called circumscribed hyperesphere hereafter. The transformation can be applied also to constraint conditions, so that J linear inequality constraint conditions are expressed as given below, where s_{jn} and d_j indicate the sensitivity of the j-th response with respect to the n-th design variable and the deviation between the prototype structural response and its limiting value, respectively.

$$\sum_{n=1}^{N} s_{jn}\varepsilon_n - d_j \leqq 0,\quad j=1\sim J \tag{6}$$

This expression implies that the feasible domain of the design variables is confined by the hyperplanes.

4. Minimization Using Circumscribed Hypersphere

Equation (5) simply determines the center of the hypersphere as $\{\varepsilon_n\}=-\{B\}/2$ in any case. In case if this center falls in the feasible domain defined by Eq. (6), the constraint minimization problem can be solved very easily and straiqhtforwardly, that is, the center gives rise to the basic disign variables that minimize the objective function after the inverse transformation of Eq. (3). Such trivial cases are omitted from the following discussion.

In cases that the center is outside the feasible domain, namely, the stationary point of the objective function does not result in the minimal point because of the inequality constraint conditions, we make use of the feature of the hypersphere. In these cases, the design variables corresponding to the minimal objective function are located on the hyperplane, and can be searched by means of the circumscribed hypersphere having the largest distance from the center.

4.1 Circumscribed hypersphere decided by a hyperplane

Figure 1 illustrates the circumscribed hypersphere touching a hyperplane in a simple case of $N=2$ and $J=3$, the feasible domain being expressed by matting. The point O denotes the aforementioned center of the hypersphere outside the feasible domain. The design variables corresponding to the center are indicated by asterisk. The radius of a hypersphere touching the j-th hyperplane R_j is given by the following formula (7).

$$R_j=\left(\sum_{n=1}^{N} s_{jn}\varepsilon_n^* - d_j\right) \Big/ \sqrt{\sum_{n=1}^{N} s_{jn}^2} \tag{7}$$

It is easy to find the circumscribed hypersphere to the feasible domain by means of combing out the largest radius R_m of Eq. (7). The point A in Fig. 1 gives rise to the largest radius. The design variables at the touch are obtained as the foot of the perpendicular from the center to the hyperplane as follows, being indicated by prime.

$$\varepsilon_l' = \varepsilon_l^* - R_m s_{ml} \Big/ \sqrt{\sum_{n=1}^{N} s_{mn}^2},\quad l=1\sim N \tag{8}$$

It is also easy to examine whether the design variables corresponding to the foot satisfy all the constraint conditions or not. It turns out that the design varibles that minimize the objective function can be

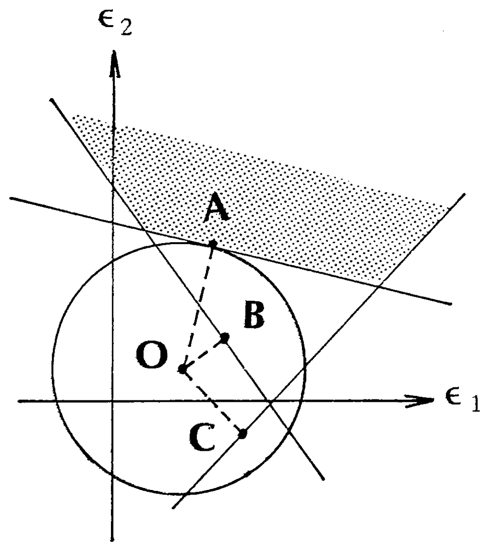

Fig. 1 Circumscribed circle in contact with a line

determined by the circumscribed hyperplane having the largest radius and the foot on the hyperplane. The radii of OA and OB are positive, while the radius of OC is negative. The center O falls in a domain where the inequality constraint condition relating to the point C is satisfied. If the foot of the circumscribed hypersphere is found to be outside the feasible domain, we should proceed to the search described in the following.

4.2 Circumscribed hypersphere decided by intersection of hyperplanes

The circumscribed hypersphere is likely to be searched on the intersection of the hyperplanes in the cases of more inequality constraint conditions than design variables. Figure 2 illustrates the situation in the case of $N=2$ and $J=3$, when an inequality condition is made idle. In this case of Fig. 2, the point A correspoinding to the largest radius described in the preceding section is found to be outside the feasible domain. Then we shift the point of interest to the point A and look for the foot on the other hyperplanes, to which the perpendicular from the point A becomes positive and longest. Suppose that the foot is found to be the point B, to which the point of interest is shifted further. When the point B is outside the feasible domain, the above mentioned procedure is to be repeated to trace the points C, D and so forth. It is not necessary to reiterate the procedure until the successive points are converged to an intersection of two hyperplanes, because we can know a set of the decisive hyperplanes specifying the intersection on which the circumscribed hypersphere should be searched, when the procedure is repeated J times at most. Then the circumscribed hypersphere can be determined as the hypersphere having the smallest radius from the center O, for wihich the dicisive constraint conditions are taken as equality conditions. The determination can be done by means of the sift synthesis solution[3] or the Moore-Penrose

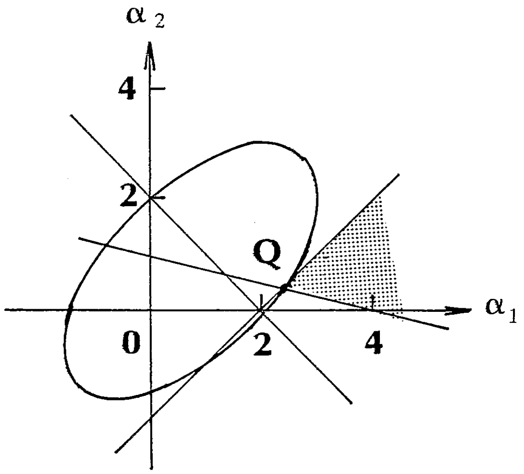

Fig. 3 Circumscribed ellipse in contact with an apex

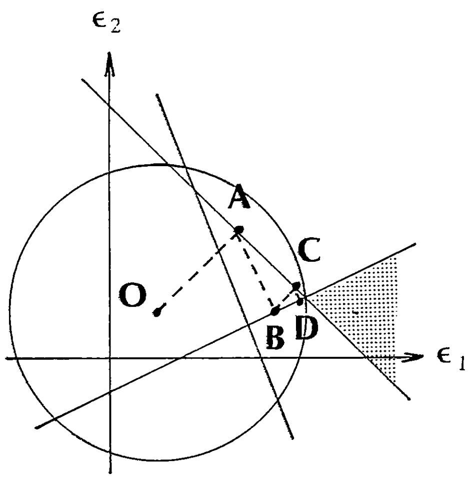

Fig. 2 Successive shift of points of interest

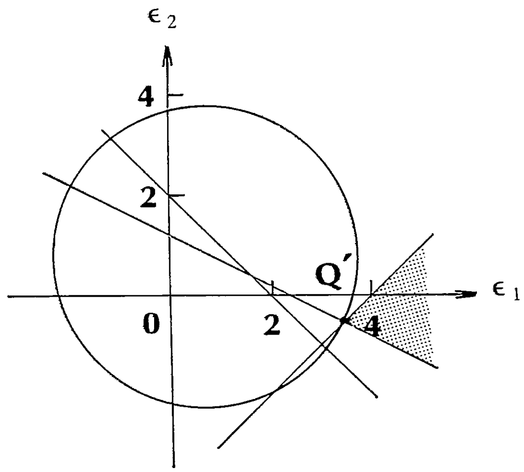

Fig. 4 Circumscribed circle in contact with an apex

generalized inverse solution[4].

5. Numerical Example

Equations (9) and (10) are the objective function to be minimized and associated linear inequality constraint condtions, as shown in Fig. 3.

$$f(\alpha_n) = -\sqrt{2}(\alpha_1+\alpha_2) + 2.5\alpha_1^2 - 3.0\alpha_1\alpha_2 + 2.5\alpha_2^2 \quad (9)$$

$$\left.\begin{array}{l} -\alpha_1-\alpha_2+2 \leqq 0 \\ -0.25\alpha_1-\alpha_2+1 \leqq 0 \\ -\alpha_1+\alpha_2+2 \leqq 0 \end{array}\right\} \quad (10)$$

The ellipse in Fig. 3 is transformed into the circle in Fig. 4 by virtue of the standardization due to the Hessian matrix. Figure 4 shows the circumscribed hypersphere searched on the intersection. The basic design variables of the point Q in Fig. 3 can be obtained easily by the inverse transformation of the point Q′ in Fig. 4.

6. Concluding Remarks

The present formulation is able to search the design variables that minimize the objective function inside the feasible domain in a straightforward way which requires simple arithmetic only, once the transformation matrix is generated on the basis of the Hessian matrix.

When the Hessian matrix is lack of definiteness, we had better omit some of the design variables so that all the eigenvalues are turned to positive. In case of such omission taken, it is meant that the minimization is done approximately by means of reduced number of the design variables. The present formulation should be sophisticated to compensate the approximation.

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