研 究 速 報
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Modelling of Non-Linear Stress-Strain Relations of Soils and Rocks
——Part 1, Discussion of Hyperbolic Equation——
土と岩の非線型応力～ひずみ関係のモデル化
——その 1，双曲線関数の検討——

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INTRODUCTION

For modelling a given highly non-linear stress-strain relation of soil or rock, the use of hyperbolic equation is very popular, because (1) it has a simple form using only two parameters having clear physical meanings, which are the initial stiffness and peak strength, (2) the determination of the parameters for a given stress-strain relation is rather straight-forward, and (3) it is considered that it can model most given stress-strain relations rather satisfactorily by selecting the appropriate parameters. According to Kondner (1963), the hyperbolic equation for monotonic loading triaxial compression tests with a constant confining pressure σ_3 is:

$$q=\sigma_1-\sigma_3=\frac{\varepsilon_1}{a+b\cdot\varepsilon_1} \qquad (1)$$

$$1/a=\lim_{\varepsilon_1\to 0}\{d(\sigma_1-\sigma_3)/d\varepsilon_1\}$$

$$1/b=\lim_{\varepsilon_1\to\infty}(\sigma_1-\sigma_3)$$

The parameter b is related to the measured strength $(\sigma_1-\sigma_3)_{max}$ by using the coefficient of correction for the strength, k as:

$$1/b=k\cdot(\sigma_1-\sigma_3)_{max} \qquad (2)$$

Since for actual soil or rock, the peak strength is mobilized at a finite strain, the value of k should be larger than unity so as to make Eq. (1) fit a given measured stress-strain relation up to the peak stress condition, but it is on the expense that the tangent modulus is still a non-zero positive value at the peak.

Kondner (1963) also showed a method to determine the parameters based on the following equation which is derived from Eq. (1):

$$\frac{\varepsilon_1}{\sigma_1-\sigma_3}=a+b\cdot\varepsilon_1 \qquad (3)$$

From a linear regression analysis of the $\varepsilon_1/(\sigma_1-\sigma_3)\sim\varepsilon_1$ plot of data as shown in Fig. 7 of Shibuya et al. (1991a), the fitted values of a and b are obtained. While this method has been widely in use (e. g., Duncan and Chang, 1970), Kondner (1963) has already noticed that the actual initial maximum modulus of the specimen be greater than 1/a. He also speculated that the initial portion at very small strains of the stress-strain curve be linear and therefore, the low strain plots of Eq. (3), which is shown in Fig. 5 of this paper, should be horizontal. This means that the fitting method based on Eq. (3) is not appropriate for modelling stress-strain relations at small strains, say less than 0.01%, as shown below. While he could not confirm these points, recently obtained data support these points.

In this and next parts, the characteristic features of the various forms of hyperbolic equation which have been proposed (the original one and other modified ones) are examined and their limitation is pointed out. Then, a new equation, which is very versatile and can model a wide range of non-linear stress-strain relations of soils and rocks from very small strains to the peak stress condition, will be proposed.

ORIGINAL HYPERBOLIC RELATION

When $a=1/E_{max}$ and $b=1/q_{max}$, Eq. (1) can be re-written by using the normalized stress and strain: $y=q/q_{max}$ and $x=\varepsilon_1/(\varepsilon_1)_r$, $(\varepsilon_1)_r=q_{max}/E_{max}$ as:

$$y=\frac{x}{1+x}=1-\frac{1}{1+x} \qquad (4a)$$

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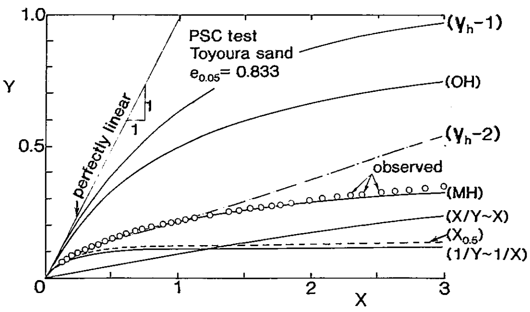

Fig. 1　Various forms of hyperbolic relations and observed relation in y-x relation at small strains (x=1~3)

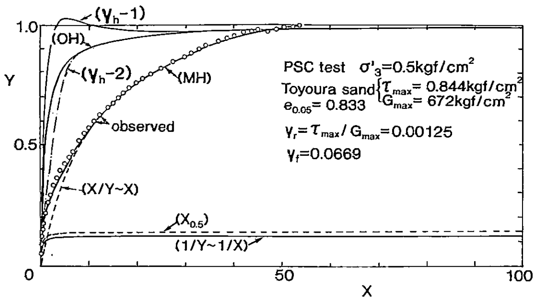

Fig. 2　Various forms of hyperbolic relations and observed relation in y-x relation at large strains (x=1~300)

$$\frac{y}{x}=E_{sec}/E_{max}=\frac{1}{1+x}=1-y \tag{4b}$$

$$x/y=1+x,\ 1/y=1/x+1 \tag{4c}$$

Eq. (4) will herein be called the original hyperbolic (OH) equation (see Figs. 1 through 6). In this paper, the data of one monotonic loading (ML) plane strain compression (PSC) test at a constant σ_3 with $q_0=0$, which are denoted as 'observed' in Figs. 1 through 6, will be used. E_{max} is defined as $d(q)/d(\varepsilon_1)$ at $q=0$. It may be seen that the OH equation does not fit at all the ML PSC test data for the whole range of strain; i. e., the OH equation overestimates the stiffness except the very initial part. With few exceptions, it is also the case for the ML stress-strain relations of a wide variety of geotechnical materials (Shibuya et al., 1991a, b, c)

On the other hand, Hardin and Drnevich (1972) showed that for cyclic torsional shear test data of a wide variety of soils, the relationship between the peak-to-peak equivalent shear modulus G_{eq} and the single amplitude shear strain $d(\gamma)_{SA}$ can be modelled by the following form of hyperbolic relation:

$$G_{eq}/G_{max}=1/(1+d(\gamma)_{SA}/\gamma_r) \tag{5}$$

in which G_{max} is the maximum shear modulus, and γ_r is the reference strain equal to τ_{max}/G_{max}. Eq. (5) is equivalent to Eq. (4), since G_{max} and τ_{max} used in Eq. (5) are the measured values, differently those obtained from the fitting method based on Eq. (3). Also Eq. (5) is obtained from Eq. (1) by replacing $(\sigma_1-\sigma_3)$ with $d(\tau)_{SA}$, ε_1 with $d(\gamma)_{SA}$, 'a' with $1/G_{max}$ and 'b' with $1/\tau_{max}$. Teachavorasinskun et al. (1991) also shows that Eq. (5) fits well the some stress-strain relations under cyclic loading conditions of sands, while Eq. (5) does not fit the corresponding ML stress-strain relations (see Figs. 11 and 15 of Teachavorasinskun et al., 1991).

In Figs. 1 through 6, other forms of hyperbolic equation are also shown, which have been proposed for a better degree of fitting than Eq. (4). They can be classified into the following two categories:

(1) While using the form of hyperbolic equation, the coefficients of correction for both the peak strength and the initial stiffness (or only for the peak strength) which are constant for the stress-strain relation to be fitted are introduced (e.g., Kondner's method).

(2) While using the form of hyperbolic equation, the coefficients of correction for the peak strength and initial stiffness (or only for the peak strength) which are a function of strain are introduced.

HYPERBOLIC MODELS USING CONSTANT PARAMETER(S)

(1-1) Method using the coefficients of correction for both strength and initial stiffness (x/y-x method):

This is the method proposed by Kondner (1963) (i.e., Eqs. 1, 2 and 3). Namely, by introducing the correction factors, $c_1=1/(a\cdot E_{max})$ and $c_2=1/(b\cdot q_{max})$, we obtain from Eq. (1):

$$y=\frac{x}{1/c_1+x/c_2} \tag{6}$$

Note that in the y/x-y plot shown in Fig. 4, the intercepts of the relation of Eq. (6) with the y/x-axis and the y-axis are c_1 and c_2, respectively. From

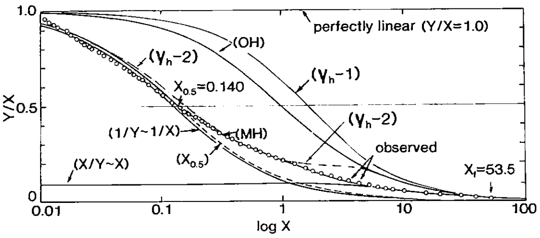

Fig. 3　Various forms of hyperbolic relations and observed relation in y/x-log(x) relation

the linear regression analysis of the x/y-x plot of the MC PSC test data shown in Fig. 5, $c_1=0.10$ and $c_2=1.24$ are obtained. Note that this value of c_1 is much smaller than unity, which means that the initial Young's modulus of the fitted hyperbolic equation, $(E_{max})_{fitting}=c_1 \cdot E_{max}$, is much smaller than the value of E_{max} actually measured at small strains at 0.001% or less. Namely, $(E_{max})_{fitting}$ has no clear physical background. In Figs. 1 through 6, Eq. (6) with these values of c_1 and c_2 is denoted as (x/y-x). It may be seen that this relation can simulate reasonably the observed relation only at large strains (see Figs. 2 and 5), while it cannot simulate utterly those at small strains (see Figs. 1, 3, 4 and 6).

(1-2) Method using only the coefficient of correction for strength (1/y-1/x method):

The OH equation Eq. (5) was originally proposed to model the stress-strain relations at small strains, say less than 1%, under cyclic loading conditions to be used in, for example, earthquake response analyses of geotechnical structures. In this case, usually the peak strength is not the major concern. Also for in some ML problems, only the stress-strain relations at small strains are needed. In this case, a model using the measured elastic shear modulus G_{max}, but not using the measured peak strength, may be employed, which is in the normalized form:

$$y=\frac{x}{1+x/c_2} \qquad (7)$$

The value of $c_2=0.125$ is obtained from the linear fitting of Eq. (6) (or $1/y=c_1(1/x)+1/c_2$) with $c_1=1.0$ to the 1/y-1/x plot (Fig. 6). Eq. (7) with $c_1=1.0$ and $c_2=0.125$ is denoted as (1/y-1/x) in Figs. 1 through 6. It may be seen that this relation can simulate only the observed stress-strain relation at small strains, but cannot utterly that at large strains.

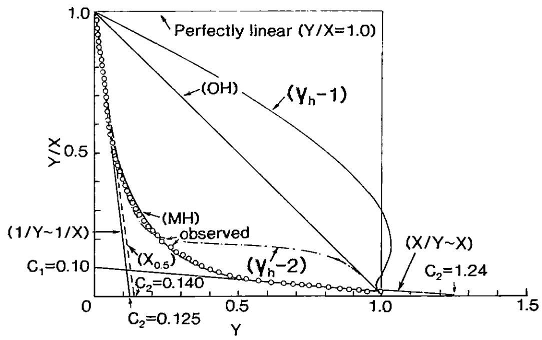

Fig. 4　Various forms of hyperbolic relations and observed relation in y/x-y relation

On the other hand, for the use in earthquake response analyses, another method for determining the value of c_2 has been proposed (Tatsuoka and Fukushima, 1978). Namely, by replacing γ_r in Eq. (5) with $\gamma_{0.5}(=d(\gamma)_{SA}$ when $G_{sec}/G_{max}=0.5)$, we obtain:

$$G_{eq}/G_{max}=1/(1+d(\gamma)_{SA}/\gamma_{0.5}) \qquad (8)$$

Its normalized form is obtained by using $G_{eq}/G_{max}=y/x$ and $d(\gamma)_{SA}/\gamma_{0.5}=(d(\gamma)_{SA}/\gamma_r)/(\gamma_{0.5}/\gamma_r)=x/x_{0.5}$:

$$y=\frac{x}{1+x/x_{0.5}} \qquad (9)$$

Namely, for Eq. (9), $c_1=1.0$ and $c_2=x_{0.5}=\gamma_{0.5}/\gamma_r$. In Figs. 1 through 6, Eq. (9) is denoted as $(x_{0.5})$. It may be seen that this relation can simulate to some extent the observed stress-strain relation at small strains, but cannot utterly that at large strains as the (1/y-1/x) method cannot.

SUMMARY

While many forms of hyperbolic equation have been proposed to model stress-strain relations of soils and rocks, when using constant parameters in each case, they are valid only for a limited range of strain. In particular, for sands, the original hyperbolic relation may not model utterly a given measured stress-strain relations during monotonic loading, while it may model those during cyclic loading.

In the next and last part, a new equation having the parameters of initial stiffness and strength which are a function of strain is proposed to model a given stress-strain relation from very small strain levels to

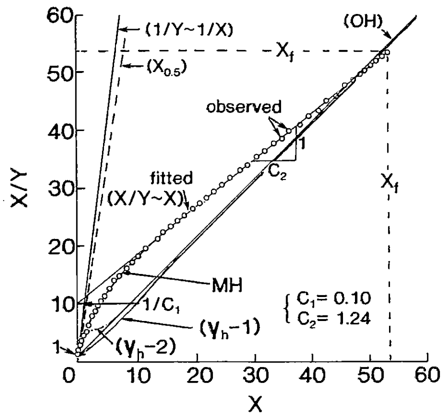

Fig. 5 Various forms of hyperbolic relations and observed relation in x/y-x relation

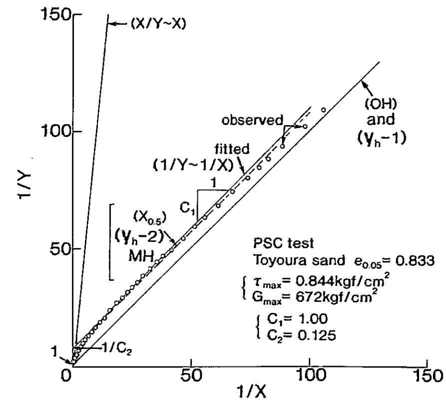

Fig. 6 Various forms of hyperbolic relations and observed relation in 1/y-1/x relation

the peak stress level.

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