研究速報
UDC 624.04 : 539.3

A Note on Finite Element Synthesis of Structures (Part 4)
—A Formulation of Design Change under Inequality Constraint Conditions by Use of Generalized Inverse—

有限要素法による構造シンセシスに関するノート (第4報)
—不等式制約条件の下での一般逆行列による定式—

Shigeru NAKAGIRI* and Keiko SUZUKI*
中桐　滋・鈴木敬子

1. Introduction

The preceding parts of this note dealt with the shift synthesis under equality constraint conditions by the use of a finite elment formulation based on the notion of the minimum design change in order to determine the design variables able to attain the desired shift of structural responses. Inequality constraint conditions also play important role in structural synthesis as much as equality constraint conditions. According to the requisite of so-called elastic design of structures, the generated stresses should remain in elastic domain, that is, equal to or less than the yield stress of the material, but all the stresses are not requested to be equal to the yield stress. Such a requisite forms inequality constraint conditions.

Powell proposed a formulation to deal with inequality constraint conditions by means of introducing slack variables and employing the quadratic terms of the constraint conditions incorporated by multipliers used in the sense of penalty coefficients[1]. His iterative formulation requires initial guess of all the variables and multipliers employed, hinting that the resultant design variables are dependent on the initial values guessed. The multipliers are to be varied judiciously, since no rules are given to vary the multipliers in the iterative formulation.

This note presents a generalized inverse formulation for the structural synthesis under inequality constraint conditions which are modified into the equality constraint conditions with the aid of slack variables. The validity of the formulation is verified by the numerical example of mode shape change of beam vibration.

*Dept. of Applied Physics and Applied Mechanics, Institute of Industrial Science, University of Tokyo

2. Description of problem

Suppose that we have a baseline design whose structural responses do not satisfy J inequality constraint conditions given in the following linear form,

$$\bar{z}_j + \sum_{n=1}^{N} z_{jn}{}^1 \alpha_n - z_j{}^* = z_j + \sum_{n=1}^{N} z_{jn}{}^1 \alpha_n \leqq 0 \qquad (1)$$

in regard to N design variables α_n defined for the design parameters x_n as follows.

$$x_n = \bar{x}_n (1 + \alpha_n) \qquad (2)$$

The upper bar indicates the quantities corresponding to the baseline design, asterisk the allowable response limit, and $z_{jn}{}^1$ the response sensitivity calculated by the finite element analysis[2]. The problem in this note is to determine the design variables that satisfy Eq. (1).

3. Proposed formulation

The inequality constraint conditions are modified into the equality constraint conditions with the aid of slack variables β_j in the following form. All the slack variables should be non-negative.

$$z_j + \sum_{n=1}^{N} z_{jn}{}^1 \alpha_n + \beta_n = 0 \qquad (3)$$

Equation (3) is rewritten in a matrix form of Eq. (4) where $\{x\}$ denotes M unknowns consisting of the design variables and slack variables. M is equal to $N+J$. $\{D\}$ is the deviation vector of the response limits and baseline responses. The expression given in the following shows an example for three design variables and two constraint conditions.

$$[S]\{x\} = \{D\} \qquad (4)$$

$$\begin{bmatrix} z_{11}{}^{I} & z_{12}{}^{I} & z_{13}{}^{I} & 1 & 0 \\ z_{21}{}^{I} & z_{22}{}^{I} & z_{23}{}^{I} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \end{Bmatrix} = \begin{Bmatrix} z_1{}^{*} - z_1 \\ z_2{}^{*} - \bar{z}_2 \end{Bmatrix}$$

From the viewpoint of numerical treatment, it is recommended to normalize Eq. (2) adequately, for instance, by means of the largest sensitivity in absolute sense for each inequality condition in order to have numerals in Eq. (4) of approximately same order. The matrix $[S]$ is rectangular in size of $J \times M$.

The unknowns vector $\{x\}$ is determined by using the Moore and Penrose generalized inverse $[S^{-}]$ in size of $M \times J$. The necessary and sufficient condition for the existence of the solution $\{x\}$ of Eq. (4) is that the following equation (5) holds.

$$([S][S^{-}]-[I])[D]=\{0\} \tag{5}$$

The matrix $[I]$ denotes identity matrix in size of either $J \times J$ or $M \times M$. When Eq. (5) holds, the general solution of $\{x\}$ is obtained as the sum of the particular solution $\{x_p\}$ and the complimentary solution $\{x_c\}$ given in the following form[3].

$$\{x_p\}=\{x_p\}+\{x_c\} \tag{6}$$

$$\{x_p\}=[S^{-}]\{D\} \tag{7}$$

$$\{x_c\}=([I]-[S^{-}][S])\{h\} \tag{8}$$

The vector $\{h\}$ on the right hand side of Eq. (8) can be taken arbitrary. The Moore and Penrose generalized inverse $[S^{-}]$ is calculated by the Penrose method[4].

The aimed design variables α_n are determined simply as the upper part of the particular solution $\{x_p\}$, when all the slack variables β_j in the lower part of $\{x_p\}$ are non-negative, since all the inequality constraint conditions are satisfied. This case implies that the arbitrary vector $\{h\}$ is taken equal to $\{0\}$. If $\{x_p\}$ gives rise to many negative slack variables, the vector $\{h\}$ is chosen so that the general solution $\{x\}$ of Eq. (5) results in non-negative slack variables.

4. Inequality conditions in small number

Suppose that the number of the inequality constraint conditions is small, for instance, J is less than N. In this case, the rank of $[S]$ is equal to J so that Eq. (5) holds. Then the solution exists in the form of Eq. (6). A case is dealt with in this section that the number of the negative slack variables of the particular solution is L. L is less than or equal to J, that is, less than N. Under such circumstances, it is easy to make all the slack variables non-negative by adding the complimentary solution. At first, the following vector $\{b\}$ is made in the form of Eq. (9) in order to set L negative slack variables equal to zero. The governing equation of $\{h\}$ is then obtained as Eq. (10).

$$\{b\}=-\{\beta_j(x_p)\} \tag{9}$$

$$[C]\{h\}=\{b\} \tag{10}$$

In the above, $[C]$ is a $L \times M$ rectangular matrix generated by means of extracting L lines from $[I]-[S^{-}][S]$ corresponding to L ingredients of Eq. (9). Then $\{h\}$ can be determined as the particular solution of Eq. (10) again by making use of the generalized inverse $[C^{-}]$ as given below.

$$\{h\}=[C^{-}]\{b\} \tag{11}$$

In this manner, the complimentary solution $\{x_c\}$ is obtained, and the aimed design variables are determined corresponding to the non-negative slack variables.

It is worthy to note that some of the slack variables, the particular component of which is non-negative, might turn to negative when the complimentary component is added. In such a case, a simplified algorithm is devised by setting all of J slack variables equal to zero, instead of looking for the combination of the inequality constraint conditions which results in non-negative slack variables of the general solution. This means that a set of the design variables satisfying the inequality constraint conditions can be determined by taking the generalized inverse three times at most.

5. Numerical example

The validity of the proposed formulation is examined by the shape change of the first mode of an elastic beam vibration. A straight beam simply supported at two points as shown in Fig. 1 is taken as the example. The beam is modeled by ten finite elements of equal length, whose stiffness matrix is constituted by the assumption that the moment of inertia is distributed lineraly in an element. Eleven design variables are taken for the nodal values of the moment of inertia. The mass matrix is kept unchanged. The moment of inertia of 8.33×10^{-2}mm^4 is constant over the full length of D=20 mm as for the baseline design, giving rise to the eigenvalue of 4.194.

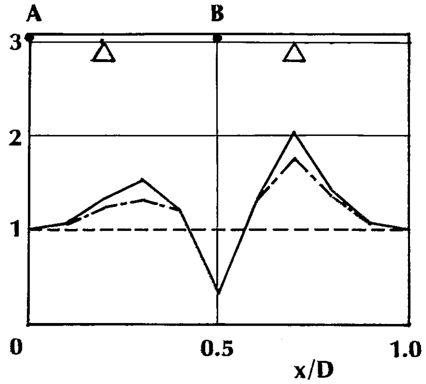

Fig. 1　Finite element beam model and distribution of moment of inertia I before and after structural modification

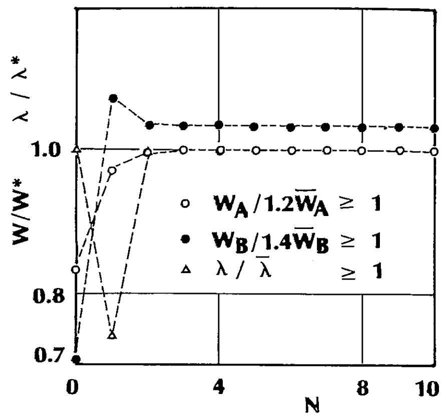

Fig. 3　Iteration history of structural modification

The normalization of the eigenvector is to set the right end deflection equal to unity. The sensitivities of the eigenpair are calculated by the finite element analysis of the beam vibration[5].

As the inequality constraint conditions of $J=3$, the eigenvalue and deflections of the eigenvector at two points A and B in Fig. 1 are taken into account. The variation of the eigenpair is not linear with respect to the change of the nodal moments of inertia so that the inequality constraint conditions in the form of Eq. (1) is none but their first-order approximation. It means that structural modification in this case should be iterated by renewing the baseline design and sensitivity analysis in order to cope with the deficiency caused by the first-order approximation. The eigenvalue and deflections at the points A and B in Fig. 2 of the eigenvector are set as follows.

$$W_A \geqq 1.2\overline{W}_A\ ,\ \ W_B \leqq 1.4\overline{W}_B\ ,\ \ \lambda \geqq \bar{\lambda}.$$

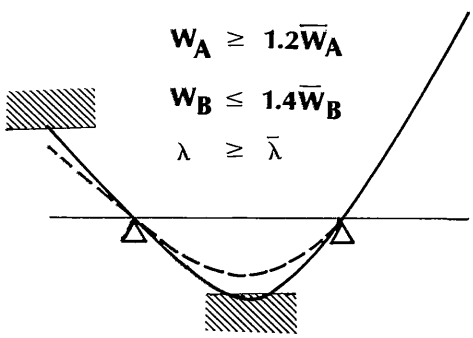

Fig. 2　Mode shapes before and after structural modification

When the defelections are changed by two equality constraint conditions, the eigenvalue is decreased to 2.791. Thus the condition is imposed on the eigenvalue not to take the smaller value than the inital value. In Fig. 1, the distribution of the moment of inertial is depicted, while a chain line indicates the result obtained by taking all the three conditions as mere equality constraint and based on the notion of design change minimum[2]. Figure 2 shows the mode shape, the hatched domain indicating that the inequality condition is satisfied. In the figures, the solid lines correspond to the design after three iterations, and the broken lines to the initial design. Figure 3 illustrates that the iteration history of the eigenvalue and deflections converges fast.

6. Discussion

When structural modification is tried by not so many design variables or under many but dependent inequality constraint conditions, the number of design variables is smaller than that of inequality constraint conditions. In such a case, the application of the generalized inverse formulation is not straightforward, because the slack variables, whose particular component is non-negative, might be made negative by adding the complimentary components. Two typical examples are discussed hereby with respect to two variables and three conditions, under the assumption that Eq. (1) is exact.

Figrue 4 illustrates the matted feasible domain and an idle inequality condition. The square means the

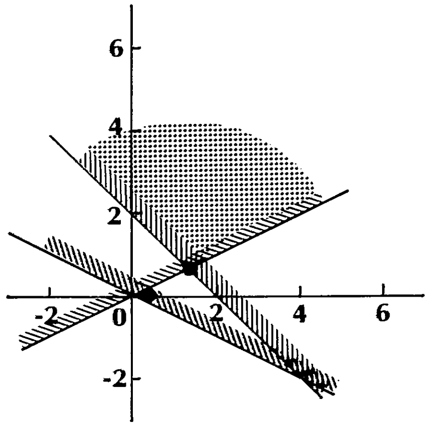

Fig. 4 Idle inequality constraint condition included

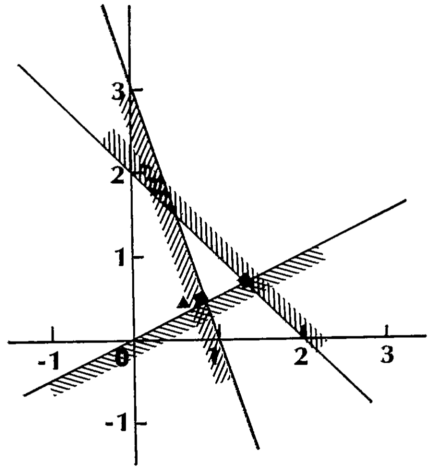

Fig. 5 No feasible domain existing

design variables obtained as the particular solution and outside the feasible domain. The circle indicates the design variables falling into the feasible domain after adding the complimentary solution. When the number of idle inequality constraint condition is not so large, the generalized inverse formulation can cope with more conditions than the design variables.

There is no feasible domain in Fig. 5, because of wrong combination of inequality constraint conditions. The triangle indicates the particular solution of the design variables outside the feasible domain with two negative slack variables. If these two variables are made positive, the positive third is made negative by addition of the complimentary solution. When the negative third is made positive further, the two positives are made negative again. This repetition is depicted as the two squares in the figure. This means that the periodic repetition of the design variables by the proposed formulation can be a clue to judge whether the structural synthesis under peculiar inequality constraint conditions is possible or not.

7. Concluding remarks

The generalized inverse formulation is proposed to determine the design variables which satisfy the inequality constraint conditions given in the linear form. The formulation needs the correction of the design variables by addition of the complimentary solution so that all the slack variables are positive as requested. Some difficulties are associated with the structural synthesis with peculiar inequality constraint conditions, as we cannot know which conditions are idle or there is no feasible domain. The discussion suggests that the formulation is still dependable in presence of the peculiarities.

(Manuscript received, June, 22, 1990)

References

1) Powell, M.J.D., A Method for Nonlinear Constraints in Minimization Problem, Fletcher, R. (ed.), Optimization, (1969), Academic Press.
2) Nakagiri, S. and Suzuki, K., A Note on Finite Element Synthesis (Part 1)-Shift Synthesis of Vibration Eigenpairs-, Seisan-Kenkyu, Vol. 39, No. 11 (1988), pp. 460-463.
3) Hangai, H., Mathematical Analysis of Shapes of Structural Spatial Structures (in Japanese), Seiken Seminar Text, (1989).
4) Penrose, R., On Best Approximate Solutions of Linear Matrix Equations, Camb. Philos., Vol. 52, No. 1 (1956), pp. 17-19.
5) Fox, R.L. and Kapoor, M.P., Rate of Change of Eigenvalues and Eigenvectors, AIAA J., Vol. 6, No. 12 (1968), pp. 2426-2429.