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On the Physical Interpretation of the Cubic Finite Element for Beams and Axisymmetric Shells

はりおよび回転対称シェルのための3次有限要素の物理的解釈について

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1. Introduction

In the present study, the beam bending finite element using a cubic polynomial as a shape function, which is the most popular element in the bending problem, is physically explained, as previously conducted by the author for the linear Timoshenko beam element[1].

In Ref. 1), the strain energy approximation of the linear Timoshenko beam element[2], which is based on the reduced integration technique (one-point quadrature in this case), was compared with that for the beam bending element including the shear effect in the Rigid Bodies-Spring Models[3] (abbreviated as the RBSM hereafter), and it was shown that these two elements are equivalent to each other when

$$\rho_1 = -\xi_1 \quad \text{or} \quad \xi_1 = -\rho_1 \qquad (1)$$

where ξ_1 is the coordinate of the integration point in the finite element and ρ_1 indicates the location of a connecting point between rigid bars in the RBSM, as shown in Fig. 1. This comparison also suggests that the linear Timoshenko beam element is suitable, as well as the RBSM, to the plastic collapse analysis using the concept of plastic hinges and that the location of the occurence of a plastic hinge can be controlled by the movement of the numerical integration point. This variable location technique for numerical integration points can be conveniently used in the finite element collapse analysis of framed structures, as conducted in Ref. 4).

In the present report, the similar consideration is carried out for the cubic beam bending element, which can be expected useful in the collapse analysis of framed structures using this element. Numerical studies are carried out for the linear problem of axisymmetric shells as well as the plastic collapse problem of beams.

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2. Physical Interpretation of the Cubic Beam Element

2.1 Strain energy for the cubic finite element

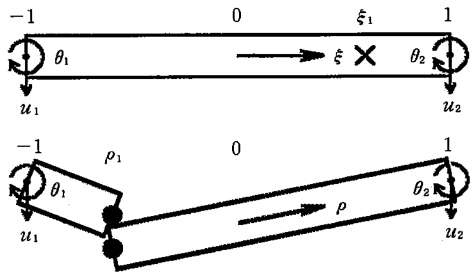
In the most frequently used beam bending finite element, the lateral deflection $u(z)$ is assumed as follows, using a cubic polynomial:

$$u(z) = H_{00}u_1 + H_{10}L\theta_1 + H_{01}u_2 + H_{11}L\theta_2$$

where

$$H_{00} = (1/8)(4 - 6\xi + 2\xi^3) \qquad (2)$$
$$H_{10} = (1/8)(1 - \xi - \xi^2 + \xi^3)$$
$$H_{01} = (1/8)(4 + 6\xi - 2\xi^3)$$
$$H_{11} = (1/8)(-1 - \xi + \xi^2 + \xi^3)$$

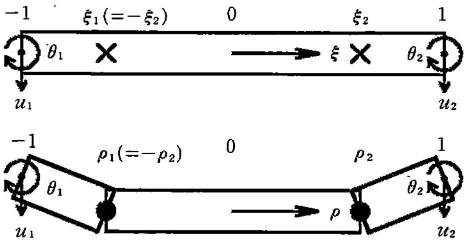
In eq. (2), $(u_1, \theta_1, u_2, \theta_2)$ are nodal displacements shown in Fig. 2. Z is the axial coordinate of a beam, and ξ is the nondimensional axial coordinate for each element, which has the value in the following range:



× numerical integration point

● rotational and shear springs connecting rigid bars (plastic hinge including the effect of shear force)

Fig. 1 Linear Timoshenko beam element and its physical equivalent



× numerical integration point

● rotational spring connecting rigid bars (plastic hinge)

Fig. 2 Cubic beam element and its physical equivalent

$$-1 \leqq \xi \leqq 1 \quad (3)$$

The second-order derivative of the lateral deflection is calculated as follows:

$$d^2u/dz^2 = (6\xi/L^2)u_1 + \{(3\xi-1)/L\}\theta_1 - (6\xi/L^2)u_2 + \{(3\xi+1)/L\}\theta_2 \quad (4)$$

which gives the curvature change during the deformation of the beam.

The strain energy stored in this element can be obtained as follows, by using the two-point numerical integration scheme:

$$V_{CFE} = (\mathrm{EIL}/4)\{[d^2u/dz^2(\xi_1)]^2 + [d^2u/dz^2(\xi_2)]^2\} \quad (5)$$

where the two numerical integration points are located at ξ_1 and ξ_2, respectively. EI and L in this equation are the bending rigidity and the length of the beam element, respectively.

2. 2 Strain energy for the RBSM

In order to obtain the physical interpretation of the cubic beam element, we consider the RBSM as shown in Fig. 2, which is composed of three rigid bars connected with two rotational springs. Writing the non-dimensional coordinates of the connection points by ρ_1 and ρ_2, the curvature change at ρ_1 is assumed as follows in the RBSM:

$$\kappa_1 = C_{10}u_1 + C_{11}\theta_1 + C_{12}u_2 + C_{13}\theta_2$$

where

$$C_{10} = -8/\{[2+(\rho_1+\rho_2)](\rho_2-\rho_1)L^2\} \quad (6)$$
$$C_{11} = -4/(1+\rho_2)/\{[2+(\rho_1+\rho_2)](\rho_2-\rho_1)L\}$$
$$C_{12} = 8/\{[2+(\rho_1+\rho_2)](\rho_2-\rho_1)L^2\}$$
$$C_{13} = -4(1-\rho_2)/\{[2+(\rho_1+\rho_2)](\rho_2-\rho_1)L\}$$

Eqs. (6) give the ratio of the relative rotational angle between adjacent rigid bars to the distance between the node 1 and the center of the middle rigid bar. Similarly, the curvature change at ρ_2 is given by the following equations:

$$\kappa_2 = C_{20}u_1 + C_{21}\theta_1 + C_{22}u_2 + C_{23}\theta_2$$

where

$$C_{20} = 8/\{[2-(\rho_1+\rho_2)](\rho_2-\rho_1)L^2\} \quad (7)$$
$$C_{21} = 4(1+\rho_1)/\{[2-(\rho_1+\rho_2)](\rho_2-\rho_1)L\}$$
$$C_{22} = -8/\{[2-(\rho_1+\rho_2)](\rho_2-\rho_1)L^2\}$$
$$C_{23} = 4(1-\rho_2)/\{[2-(\rho_1+\rho_2)](\rho_2-\rho_1)L\}$$

In the RBSM theory, the strain energy stored in the rotational springs is approximated as follows:

$$V_{RBSM} = (EIL/4)\{(\kappa_2)^2 + (\kappa_2)^2\} \quad (8)$$

2. 3 Equivalence between the cubic finite element and the RBSM

The equivalence between the cubic finite element and the rigid bars-spring model requires

$$V_{CFE} = V_{RBSM} \quad (9)$$

which is satisfied by

$$d^2u/dz^2(\xi_1) = \kappa_1 \quad (10a)$$

and

$$d^2u/dz^2(\xi_2) = \kappa_2 \quad (10b)$$

Eq. (10a) holds when the following conditions are satisfied:

$$C_{10} = 6\xi_1/L^2$$
$$C_{11} = (3\xi_1-1)/L \quad (11)$$
$$C_{12} = -6\xi_1/L^2$$
$$C_{13} = (3\xi_1+1)/L$$

It is noted that the first equation in eqs. (11) is the same as the third equation, because C_{10} is equal to $-C_{12}$, as seen from eqs. (6).

The following three variables are defined here for the convenience:

$$R_1 = \{(\rho_2)^2 - (\rho_1)^2\}/4$$
$$R_2 = (\rho_2-\rho_1)/2 \quad (12)$$
$$R_3 = \rho_2$$

Using these variables, eqs. (11) can be rewritten as

$$3\xi_1R_1 + 3\xi_1R_2 + 1 = 0$$
$$(3\xi_1-1)R_1 + (3\xi_1-1)R_2 + R_3 + 1 = 0 \quad (13)$$
$$(3\xi_1+1)R_1 + (3\xi_1+1)R_2 - R_3 + 1 = 0$$

It is clear that eqs. (13) can be replaced with the following set of equations:

$$3\xi_1R_1 + 3\xi_1R_2 + 1 = 0$$
$$R_1 + R_2 - R_3 = 0 \quad (14)$$

Eqs. (12) give the following relation between R_3 and (R_1, R_2):

$$R_3 = (R_1/R_2) + R_2 \quad (15)$$

Substituting eq. (15) into the second equation in eqs.

(14), the following set of equations for R_1 and R_2 can finally be obtained:

$$3\xi_1 R_1+3\xi_1 R_2+1=0 \quad (16a)$$

$$R_1-R_1/R_2=0 \quad (16b)$$

Eqs. (16) give the following two sets of solutions:

$$R_1=0 \quad \text{and} \quad R_2=-1/3\xi_1 \quad (17a)$$

or

$$R_1=-1-1/3\xi_1 \quad \text{and} \quad R_2=1 \quad (17b)$$

As for the values of ρ_1 and ρ_2, two equations in (17b) are inconsistent with each other. Therefore eqs. (17a) give required solutions for ρ_1 and ρ_2, which are

$$\rho_1=1/3\xi_1 \quad \text{and} \quad \rho_2=-1/3\xi_1 \quad (18)$$

From eqs. (10b), the following solutions for ρ_1 and ρ_2 can be similarly obtained:

$$\rho_1=-1/3\xi_2 \quad \text{and} \quad \rho_2=1/3\xi_2 \quad (19)$$

It can be seen that the following relation between ξ_1 and ξ_2 must hold in order to have the equivalence between the finite element and the RBSM:

$$\xi_1=-\xi_2 \quad (20)$$

It can also be said that ξ_1 and ξ_2 must have the following ranges:

$$-1\leqq\xi_1\leqq 1/3 \quad \text{and} \quad 1/3\leqq\xi_2\leqq 1 \quad (21)$$

It is interesting to note that when (ξ_1, ξ_2) are located on the Gaussian integration points, (ρ_1, ρ_2) are located on the same points, that is,

$$\xi_1=\rho_1=-1/\sqrt{3} \quad \text{and} \quad \xi_2=\rho_2=1/\sqrt{3} \quad (22)$$

3. Numerical Studies

3.1 Plastic collapse of a clamped beam

As mentioned in the introductory remarks, the relation between (ξ_1, ξ_2) and (ρ_1, ρ_2) can be conveniently used in the plastic collapse analysis of framed structures. The plastic collapse of a beam with both ends clamped and subjected to a centrally concentrated lateral loading is studied as a numerical example to show the validity of the variable location technique for numerical integration points. In the analysis, the concept of plastic hinges is introduced by assuming zero bending rigidity at the integration point where the value of bending moment reaches a fully-plastic value. This treatment is equivalent to the assumption of zero spring constant at the corresponding connection point in the RBSM.

In the present numerical study, the following two elements with different locations of integration points are used:

FC1: $\xi_i=\pm 1/\sqrt{3}$ ($\rho_i=\pm 1/\sqrt{3}$)

FC2: $\xi_i=\pm 1/3$ ($\rho_i=\pm 1.0$)

In FC1, the integration points are exactly located at Gaussian integration points, and the plastic hinges are formed at the same points. In FC2, the integration points are located at $\xi_i=\pm 1/3$, and the plastic hinges are formed at the end points ($\rho_i=\pm 1.0$) of the element.

The numerical results given by these elements are shown in Table 1, where the ratios of the calculated collapse loads and linear deflections to the exact solutions are presented. FC1, the standard form in the beam bending analysis, has the best accuracy for linear displacements, however, the convergence for the collapse load is slow, because in FC1 the plastic hinges are not formed exactly at the loaded point and clamped ends. On the other hand, the convergence of FC2 for the collapse load is extremely fast, because the locations of the plastic hinges formed are 'exact' in this element, while the accuracy for linear displacements are disturbed by the movement of integration points. (FC1+FC2) in Table 1 means the combination of FC1 and FC2, in which the elements contaning loaded or clamped nodes, at which plastic hinges are to be formed, are FC2 and others are FC1. This modeling has higher accuracy for linear solutions than FC2. Therefore it can be said that (FC1+FC2) is the most efficient and recommended modeling.

3.2 Linear analysis of a cantilever cylinder

The similar numerical studies have been conducted for the axisymmetric linear analysis of a cantilever circular cylinder, the free edge of which is subjected

Table 1 Convergence of beam bending elements with a cubic polynomial

Number of Elements	FC1 ($\xi_i=\pm 0.57735$) ($\rho_i=\pm 0.57735$) P_{max}	U_{lin}	FC2 ($\xi_i=\pm 1/3$) ($\rho_i=\pm 1.0$) P_{max}	U_{lin}	FC1+FC2 P_{max}	U_{lin}
1	1.732	1.000	1.000	3.000	—	—
2	1.268	1.000	1.000	1.500	—	—
3	1.164	1.000	1.000	1.222	1.000	1.148
4	1.118	1.000	1.000	1.125	1.000	1.062
5	1.092	1.000	1.000	1.080	1.000	1.032
6	1.076	1.000	1.000	1.056	1.000	1.019
7	1.064	1.000	1.000	1.041	1.000	1.012
8	1.056	1.000	1.000	1.031	1.000	1.008
9	1.049	1.000	1.000	1.025	1.000	1.005
10	1.044	1.000	1.000	1.020	1.000	1.004

Table 2 Convergence of axisymmetric shell elements

Number of Elements	FC1 (exact)	FC2 (Gaussian)	FC3 (trape-zoidal)	RE1 ($\xi_i=\pm 0.57735$) ($\rho_i=\pm 0.57735$)	RE2 ($\xi_i=\pm 2/3$) ($\rho_i=\pm 1/2$)	RE3 ($\xi_i=\pm 1/3$) ($\rho_i=\pm 1.0$)
9	0.995	0.995	0.700	1.027	1.100	0.733
14	1.000	1.000	0.896	1.007	1.022	0.921
24	1.000	1.000	0.971	1.002	1.005	0.980
34	1.000	1.000	0.987	1.001	1.002	0.991
44	1.000	1.000	0.992	1.001	1.001	0.995

to a ring load in the radial direction. For this analysis, the circumferential membrane stiffness has to be added to the beam stiffness described in Section 2. Two approaches to do this have been tested. In the first approach the circumferential membrane strain ($\varepsilon=-u/R$) is estimated, using lateral deflection expressed by a cubic polynomial, while in the second approach, the displacement field given by rigid bar elements (refer to the lower figure in Fig. 2) is used to calculate the membrane strain.

FC1, FC2 and FC3 in Table 2 belong to the first, conventional approach. The bending as well as the membrace stiffness are estimated exactly in FC1. In FC2 and FC3, Gaussian and trapezoidal numerical integration schemes are used, respectively, with two integration points. On the other hand, RE1, RE2 and RE3 in Table 2 are derived by the second approach. The locations of numerical integration points and connection springs are noted in the table. In all cases the five-sixth part of the cylinder from the clamped end is subdivided unequally using four elements and the rest is subdivided uniformly, as conducted in Ref. 5).

From the results for the FC family, it is seen that the trapezoidal rule, which was used in the pioneering work of Grafton and Strome[5], gives poor results in comparison with the other two solutions.

Among the RE family, RE1, in which Gaussian points are adopted as the connection points, is the most accurate. RE2, which is almost equivalent to the variational difference modeling given by Bushnell[6], gives a little less accurate results than RE1. RE3, in which plastic hinge circles can occur at ends of the element, gives poor solutions for elastic behaviors, however, this element can be expected useful in the plastic collapse analysis, like FC2 in the beam bending problem.

It should be mentioned that the cubic finite element with the variational location technique for numerical integration points includes the RBSM (RE1~RE3) and the variational difference model (RE2) in itself and it also offers the easiest way for the development of the RBSM and the variational difference programs, as previously pointed out for linear beam and axisymmetric shell elements[1].

4. Concluding Remarks

In this brief note, the cubic finite element for beams and axisymmetric shells has been physically explained through the comparison of its strain energy approximation with that of the rigid bars-spring element, and the relation between the locations of numerical integration points and those of connection springs (or plastic hinges) has been derived, which ensures the equivalence between these two discrete elements. The obtained relation is useful, not only in the development of the unified computer code for the FEM, the RBSM and the variational difference method, but also in the plastic collapse analysis of framed structures and axisymmetric shells, especially in the case when coarse mesh is required to reduce the computing cost.

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