研 究 速 報
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On the Constitutive Modeling of Ceramics Using a Stress-History Dependent Internal State Variable (Part 1)
——Microcracking——

応力履歴に依存する内部状態変数を用いたセラミックスの構成関係のモデル化について(その1)
——マイクロクラッキング——

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1. Introduction

The nonlinearity observed in the material behavior of ceramics is due to the following three different physical phenomena:

i) microcracking
ii) transformation plasticity
iii) creep

The microcracking plays an important role especially in single-phase ceramics such as polycrystalline alumina (Al_2O_3), which exhibit little plasticity, as well as in DZC (Dispersed Zirconia Ceramics) and ceramic composites. The transformation plasticity is essential in transforming ceramics such as PSZ (Partially Stabilized Zirconia). The creep behavior can be observed in each ceramic.

The purpose of the present study is to seek and develop the most appropriate form of constitutive equations in numerical computations, which represents the above materially nonlinear behaviors, based on the continuum damage mechanics approach[1] in dealing with the effect of microcracking.

There are two possible ways of formulating the combined material nonlinearities as shown above, one of which is to employ the so-called 'unified approach'[2], and the other is to formulate each nonlinearity independently and make the total strain as the summation of each component. The former way seems appropriate, when more than two nonlinearities are caused by the similar physical mechanism, as in the case of plasticity and creep in metals. However, in ceramics, the above three nonlinearities are due to the different physical mechanisms. Threfore the latter way, 'non-unified approach', is adopted here.

In dealing with the nonlinear behavior of materials subjected to the complicated stress-history, the following features, as possessed by the viscoplastic constitutive theory of Bodner[3] in which a load-history dependent internal state variable is introduced, seem effective in numerical computations:

i) non-zero microcracking (or plastic) deformation
ii) no microcracking (or plastic) condition
iii) no saturation (or critical transformation strain) criterion
iv) no unloading condition,

because a great number of judgement processes concerning the above conditions can be removed by having these features. The constitutive modeling for microcracking and transformation plasticity, to be proposed in the present study, will be given the same features by he use of an internal state variable depending on the stress-history.

This (Part 1) of the present study deals with the constitutive modeling for microcracking.

2. Constitutive Modeling for Microcracking

The rate-dependent constitutive modeling for brittle microcracking solids given in Ref. 4) is summarized in Subsection 2.1 and transformed into a new form in Subsection 2.2.

2.1 Modeling based on the continuum damage mechanics

The self-consistent theory of Budiansky and O'Connell[5] gives the incremental form of constitutive relation expressed as

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$$\dot{\varepsilon}_{ij}=(f+\nu)/E\cdot\dot{\sigma}_{ij}-\nu/E\cdot\dot{\sigma}_{kk}\delta_{ij}+\dot{f}/E\cdot\sigma_{ij} \qquad (1)$$

where

$$f=9/(9-16\xi) \qquad (2)$$

$$\dot{f}=144/(9-16\xi)^2\cdot\dot{\xi} \qquad (3)$$

in which f is an internal state variable related to the microcrack density ξ. For rate-dependent materials, the relation between the microcrack density and the stress is given by the following equations, depending on the stress level[4]:

$$\dot{\xi}=0 \qquad \text{when } \sigma_e<\sigma_c \qquad (4\text{a})$$

$$\dot{\xi}=(1/\eta)\{\sigma_e/(\sigma_c+\xi/\lambda)-1\} \qquad \text{when } \sigma_e\geqq\sigma_c+\xi/\lambda \qquad (4\text{b})$$

$$\dot{\xi}=0 \qquad \text{when } \xi_s\leqq\xi \qquad (4\text{c})$$

where σ_e is the equivalent stress, which is defined as

$$\sigma_e=(\sigma_{ij}\sigma_{ij})^{1/2} \qquad (5\text{a})$$

or

$$\sigma_e=(J_1^2+2J_2)^{1/2} \qquad (5\text{b})$$

where

$$J_1=\sigma_x+\sigma_y+\sigma_z$$

$$J_2=-(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x)+\tau_{xy}{}^2+\tau_{yz}{}^2+\tau_{zx}{}^2$$

Eqs. (4) are applicable only in the region where the maximum principal stress is tensile and increases monotonically. It is assumed that no microcracking occurs in a fully compressive region.

Substituting eqs. (2), (3), (4) and (5) into eq. (1), the following, final form of constitutive equations can be obtained:

$$\dot{\varepsilon}_{ij}=C_{ijkl}(\xi)\dot{\sigma}_{kl}=(1+\nu)/E\cdot\dot{\sigma}_{ij}-\nu/E\cdot\dot{\sigma}_{kk}\delta_{ij} \qquad \text{when } \sigma_e<\sigma_c \qquad (6\text{a})$$

$$\dot{\varepsilon}_{ij}=C_{ijkl}(\xi)\dot{\sigma}_{kl}+\dot{\varepsilon}_{ij}{}^{ve}(\sigma_{ij},\ \xi)=\{9/(9-16\xi)+\nu\}/E\cdot\dot{\sigma}_{ij}-\nu/E\cdot\dot{\sigma}_{kk}\delta_{ij}+\{144/(9-16\xi)^2\cdot(1/\eta)[\sqrt{\sigma_{ij}\sigma_{ij}}/(\sigma_c+\xi/\lambda)-1]\}/E\cdot\sigma_{ij} \qquad \text{when } \sigma_e\geqq\sigma_c+\xi/\lambda \qquad (6\text{b})$$

$$\dot{\varepsilon}_{ij}=C_{ijkl}(\xi_s)\dot{\sigma}_{kl}=\{9/(9-16\xi_s)+\nu\}/E\cdot\dot{\sigma}_{ij}-\nu/E\cdot\dot{\sigma}_{kk}\delta_{ij} \qquad \text{when } \xi_s\leqq\xi \qquad (6\text{c})$$

The unloading is assumed to take place in a microcracking region, when the following condition is satisfied:

$$\sigma_e<\sigma_c+\xi/\lambda \qquad (7)$$

and the microcrack density remains the current value during unloading.

2.2 Modeling with a stress-history dependent internal state variable

In One-dimensional case of Bodner's modeling, the viscoplastic strain rate is given as[3]

$$\dot{\varepsilon}_{11}{}^p=(\sigma_{11}/|\sigma_{11}|)D_0\exp\{(-1/2)[Z/\sigma_{11}]^{2n}\} \qquad (8)$$

where

$$Z=Z_1+(Z_0-Z_1)\exp(-mW_p) \qquad (9)$$

$$W_p=\int\sigma_{11}\dot{\varepsilon}_{11}{}^p dt \qquad (10)$$

These equations include five material constants; D_0, n, Z_0, Z_1 and m. Z is an internal state variable depending on the plastic work W_p taken as the measure of material hardening.

Because of the similarity between the stress-plastic strain behavior in viscoplastic solids and the stress-microcrack density behavior in micrcocracking solids, it is suggested that the main features of this modeling will remain if the similar form of equations are employed for the relation between the microcracking rate and the quivalent stress, instead of eqs. (4a)～(4c).

This suggestion leads to the following equation as the relation between the rate of microcrack density and the equivalent stress:

$$\dot{\xi}=D^M\exp\{(-1/2)[Z^M/\sigma_e]^{2n'}\} \qquad (11)$$

This equation is essentially similar to eq. (8). The exponential function gives the necessary characteristics in this relation both at low stress levels and at high microcrack-density rates.

The internal state variable Z^M is defined here as follows:

$$Z^M=A^M+B^M\exp\{[W^M/W_1^M]^{m'}\} \qquad (12\text{a})$$

where

$$A^M=Z_0^M-(Z_1^M-Z_0^M)/(e-1) \qquad (12\text{b})$$

$$B^M=(Z_1^M-Z_0^M)/(e-1) \qquad (12\text{c})$$

And W^M in eq. (12a) is a parameter defined as

$$W^M=\int\sigma_e\dot{\xi}dt \qquad (13)$$

The difference in form between eqs. (9) and (12a) is due to the saturation of microcrack density which occurs in microcracking materials. After the saturation takes place, the microcrack density remains the same value (ξ_s), therefore $\dot{\xi}$ must become zero for large W^M. This requires that the parameter Z^M for microcracking materials has no upper bound value unlike Z in eq. (9) (Z_1 is the upper bound value of Z in eq. (9)). It is noted that $Z^M=Z_0^M$ when $W^M=0$ ($\xi=0$) and $Z^M=Z_1^M$ when $W^M=W_1^M$ in eqs. (12).

When the rate of microcrack density is constant

$(\dot{\xi}=\dot{\xi}_o)$, eq. (11) can be written as

$$\dot{\xi}_o=D^M\exp\{(-1/2)[Z^M/\sigma_e]^{2n'}\} \quad (14)$$

From eq. (14), it can be seen that Z^M/σ_e is constant in the case of constant rate of the microcrack density, that is,

$$\sigma_e/Z^M=K^M=\text{const.} \quad (15)$$

Under this assumption, eq. (14) leads to the following expression for K^M:

$$K^M=(2\ell n[D^M/\dot{\xi}_o])^{-1/2n'} \quad (16)$$

2.3 Determination of material constants

The material constants contained in the present modeling can be determined by the procedure described below, assuming that we have the following two sets of material test data ($\sigma_e-\xi$ curves including the values of σ_c, σ_s and ξ_s) for different constant values of the microcrack-density rate, which are denoted by $\dot{\xi}_o'$ and $\dot{\xi}_o''$ respectively:

$$\sigma_c=\sigma_c',\ \sigma_s=\sigma_s',\ \xi_s=\xi_s' \quad \text{when } \dot{\xi}=\dot{\xi}_o'\ (\fallingdotseq 0) \quad (17\text{a})$$

$$\sigma_c=\sigma_c'',\ \sigma_s=\sigma_s'',\ \xi_s=\xi_s'' \quad \text{when } \dot{\xi}=\dot{\xi}_o''\ (\neq 0) \quad (17\text{b})$$

i) We assume that

$$D^M=10^4,\ m'=10 \quad (18)$$

ii) The value of n' can be determined by using the following equation, which is derived from eq. (16):

$$\sigma_c''/\sigma_c'=K^{M''}Z_0^M/K^{M'}Z_0^M =(\ell n[D^M/\dot{\xi}_o'']/\ell n[D^M/\dot{\xi}_o'])^{-1/2n'} \quad (19\text{a})$$

or

$$\sigma_s''/\sigma_s'=K^{M''}Z_1^M/K^{M'}Z_1^M =(\ell n[D^M/\dot{\xi}_o'']/\ell n[D^M/\dot{\xi}_o'])^{-1/2n'} \quad (19\text{b})$$

iii) Z_0^M and Z_1^M can be determined as follows:

$$Z_0^M=\sigma_c'/K^{M'} \quad (20\text{a})$$
$$=\sigma_c''/K^{M''} \quad (20\text{b})$$
$$Z_1^M=\sigma_s'/K^{M'} \quad (21\text{a})$$
$$=\sigma_s''/K^{M''} \quad (21\text{b})$$

iv) W_1^M can be determined as follows:

$$W_1^M=(1/2)(\sigma_c'+\sigma_s')\xi_s' \quad (22\text{a})$$
$$=(1/2)(\sigma_c''+\sigma_s'')\xi_s'' \quad (22\text{b})$$

Eqs. (19a), (20a), (21a), (22a) and eqs. (19b), (20b), (21b), (22b) might give different values with respect to the material constants of n', Z_0^M, Z_1^M and W_1^M. In such a case, the average values can be used.

Although this procedure is unpractical because of the use of material test data under the condition of constant rates of microcracking which is hard to realize, it has been described here to make it easier to understand the meanings of material constants. More complicated way with trial and error must be employed in practice.

3. Numerical Examples

3.1 Material constants

One-dimensional numerical calculations for an elastic-microcracking material have been conducted, by using the following material constants:

$D^M=10^4$
$m'=10$
$E=0.4\times10^{12}(\text{N/m}^2)$
$n'=4.36$
$Z_0^M=1.1920\times10^8(\text{N/m}^2)$
$Z_1^M=1.2510\times10^8(\text{N/m}^2)$
$W_1^M=0.3034\times10^8(\text{N/m}^2)$

The values of n', Z_0^M, Z_1^M and W_1^M have been determined according to the procedure described in Subsection 2.3, based on the following fictitious material test results:

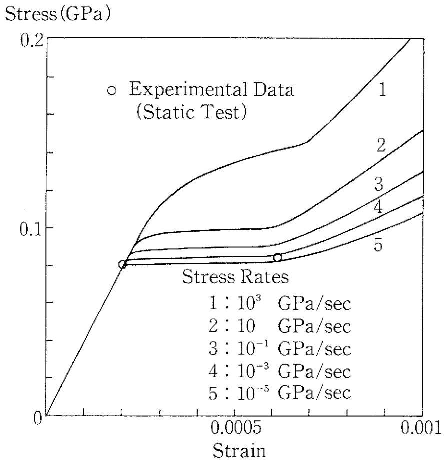
$\sigma_c'=0.80\times10^8(\text{N/m}^2)$　$[\dot{\xi}_o'=10^{-3}]$
$\sigma_s'=0.84\times10^8(\text{N/m}^2)$　$[\dot{\xi}_o'=10^{-3}]$
$\xi_s'=0.370$　$[\dot{\xi}_o'=10^{-3}]$
$\sigma_c''=1.00\times10^8(\text{N/m}^2)$　$[\dot{\xi}_o''=10^3]$
$\sigma_s''=1.05\times10^8(\text{N/m}^2)$　$[\dot{\xi}_o''=10^3]$
$\xi_s''=0.296$　$[\dot{\xi}_o''=10^3]$

3.2 Numerical results

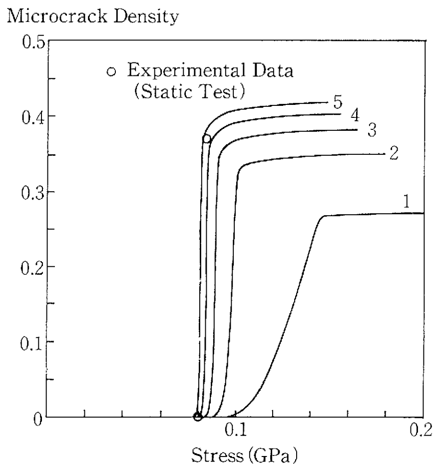
Figs. 1 and 2 show calculated results for constant stress rates, in which stress-strain relations as well as microcrack density-stress relations are plotted. The effect of high stress rates and the unloading behaviors can be understood from Fig. 1 and Fig. 2, respectively.

4. Concluding Remarks

The constitutive relation for microcracking in ceramics has been formulated by using an internal state variable depending on a stress-history, referring to the viscoplastic constitutive theory of Bodner. The proposed modeling has no microcracking, saturation and unloading conditions. All of these can be automatically treated in the same equations. The proposed type of constitutive equations can be conveniently used in numerical computations for the behav-

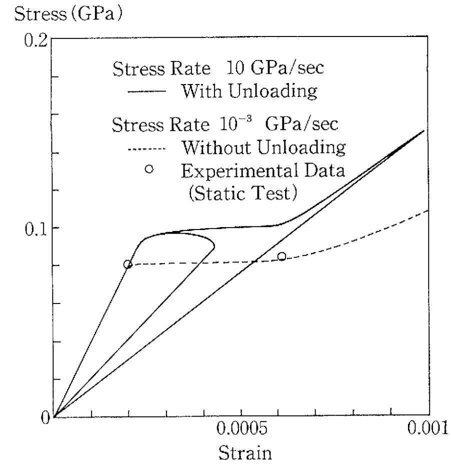


(a) stress-strain relations

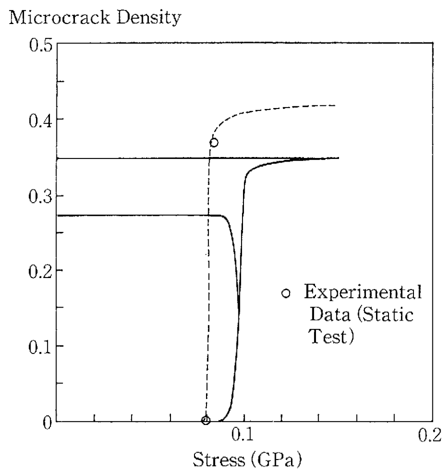


(b) microcrack density-stress relations

Fig. 1 Effect of stress rates in microcracking



(a) stress-strain relations



(b) microcrack density-stress relations

Fig. 2 Unloading in microcracking at a high stress rate

ior of ceramics under a complicated stress-history.

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