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Large Eddy Simulation of Turbulent Flows and MHD Subgrid Modeling

乱流のラージ・エディ・シミュレーションとMHDサブグリッド・モデリング

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1. Introduction

MHD flows are attracting much attention in close relation to nuclear fusion physics, especially, the reversed field pinches of plasma (RFP)[1]. In the research of plasma confinement, RFP provides a promising mechanism of plasma confinement, in addition to the tokamak confinement. The reversed-field configuration is considered to be sustained by turbulent dynamo that is familiar in the sustainment of earth magnetic field[2,3].

Phenomena related to RFP are so complicated that purely analytical approaches fail to give a clear detailed understanding. This situation also holds in the study of non-MHD turbulent shear flows. In it, numerical simulation has been an indispensable method for understanding complicated turbulence structures. In such simulations, however, some kind of turbulence model must be incorporated, because the dissipation scale of turbulent flows at high Reynolds numbers (R) is of $O(R^{-3/4}L)$ (L is a reference length), and cannot be treated numerically even by existing and prospective supercomputers.

Of a variety of simulation methods, large eddy simulation (LES) is the most promising in the accuracy and detail of computed results[4,5]. In LES, the energy dissipation mechanism, which is lost because of coarse grid resolution, is compensated for with the aid of subgrid-scale (SGS) eddy viscosity. LES was first applied to channel flows by Deardorff[6] using the Smagorinsky model for the SGS eddy viscosity. With the progress of supercomputers, LES has achieved much progress in the improvement of models and the imposition of exact boundary conditions on solid walls.

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In numerical simulation of RFP, their direct simulation with no turbulence model is also very difficult at high kinetic and magnetic Reynolds numbers. Because, all scales from torus radius to magnetic reconnection scales cannot be resolved simultaneously even by any existing supercomputers. This situation suggests to us that LES also will become a useful method in the research of various MHD turbulent shear flows.

In this paper, we first make an introduction to the subgrid modeling in LES of non-MHD turbulent flows, and remark about its application to some types of flows. Then, we give a MHD subgrid model, which can be derived using a two-scale direct-interaction approximation (TSDIA)[7]. For the details, refer to Ref. 8.

2. Non-MHD flows

A. Fundamental equations

The Navier-Stokes equation for a viscous incompressible fluid is written as

$$\frac{\partial u^\alpha}{\partial t}+\frac{\partial}{\partial x^a}u^\alpha u^a=-\frac{\partial p}{\partial x^\alpha}+\nu\Delta u^\alpha , \qquad (1)$$

with the solenoidal condition

$$\frac{\partial u^a}{\partial x^a}=0. \qquad (2)$$

Here, $\boldsymbol{u}$ is the velocity, p is the pressure divided by fluid density, ν is the kinematic viscosity, and the summation convention is applied to repeated Roman superscripts.

B. Filtering

In LES, we introduce the concept of filtering, i. e., filtering out the fluctuations smaller than grid intervals. The energy dissipation mechanism borne by those filtered-out fluctuations must be compensated for by what are called subgrid models. For the filtering, we introduce a filter $G(\mathbf{x}, \mathbf{y})$ to define the filtering of $f(\mathbf{x})$ as

$$\bar{f} = \int G(\mathbf{x},\mathbf{y})\ f(\mathbf{y})\,d\mathbf{y}. \tag{3}$$

As a representative filter, use is often made of the Gaussian filter

$$G(\mathbf{x},\mathbf{y}) = (\pi\Delta^2)^{-3/2}\exp[-(\mathbf{x}-\mathbf{y})^2/\Delta^2], \tag{4}$$

where Δ is the filter width related to computational grid sizes.

On applying (3) to (1) and (2), we have

$$\frac{\partial \bar{u}^\alpha}{\partial t} + \frac{\partial}{\partial x^a}\bar{u}^\alpha\,\bar{u}^a = -\frac{\partial \bar{p}}{\partial x^\alpha} + \frac{\partial}{\partial x^a}(R^{\alpha a} + L^{\alpha a} + C^{\alpha a} + \nu\frac{\partial \bar{u}^\alpha}{\partial x^a}), \tag{5}$$

with

$$\frac{\partial \bar{u}^a}{\partial x^a} = 0, \tag{6}$$

where

$$R^{\alpha\beta} = -\overline{u'^\alpha\,u'^\beta}, \tag{7}$$

$$L^{\alpha\beta} = -(\overline{\bar{u}^\alpha\,\bar{u}^\beta} - \bar{u}^\alpha\,\bar{u}^\beta) \tag{8}$$

$$C^{\alpha\beta} = -(\overline{\bar{u}^\alpha\,u'^\beta} + \overline{u'^\alpha\,\bar{u}^\beta}) \tag{9}$$

$(\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}})$. The most remarkable feature of filtering procedure is that the Leonard term (8) and the cross term (9) appear, whereas only the Reynolds stress corresponding to (7) appears in the ensemble-mean procedure.

C. Smagorinsky model

As the most fundamental of subgrid models, let us mention the Smagorinsky model. In the model, we neglect (8) and (9), and approximate $R^{\alpha\beta}$ by using the familiar eddy-viscosity concept as

$$R^{\alpha\beta} = -\frac{2}{3}K_G\delta^{\alpha\beta} + \nu_G(\frac{\partial \bar{u}^\alpha}{\partial x^\beta} + \frac{\partial \bar{u}^\beta}{\partial x^\alpha}), \tag{10}$$

where K_G is the SGS kinetic energy, and ν_G is the SGS eddy viscosity.

Next, we assume that the filter width Δ is approximately of the inertial-range scale, and that the SGS energy production rate balances the SGS energy dissipation rate. As a result, we have

$$\nu_G = (C_S\Delta)^2[(\frac{\partial \bar{u}^a}{\partial x^b} + \frac{\partial \bar{u}^b}{\partial x^a})^2/2]^{1/2}, \tag{11}$$

$$K_G = \nu_G^2/(C_K\Delta)^2. \tag{12}$$

The model constants C_S and C_K are optimized as

$$C_S = 0.1,\ C_K = 0.1. \tag{13}$$

Especially, C_S is called the Smagorinsky constant, which plays a crucially important role in estimating the SGS energy dissipation.

D. Applications

LES was first applied by Deardorff[6] using the Smagorinsky model to channel flows. In recent LES, the Leonard term (8) is often taken into account since it can be calculated directly by a filter function. LES accompanies a large amount of computing time by a supercomputer, and is restricted to flows with simple geometry at least at present. A comprehensive review of LES is given in Refs. 4 and 5.

3. MHD subgrid modeling

The Smagorinsky model was originally derived from purely dimensional analysis incorporated with some physical assumptions. The model can also be derived from the statistical viewpoint[9]. At this time, the usual filter such as the Gaussian one is not convenient. So, we introduce a statistical filter

$$\bar{f} = \int_{k<k_M} f(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{x})\,d\mathbf{k} + \left\langle \int_{k>k_M} f(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{x})\,d\mathbf{k} \right\rangle, \tag{14}$$

by using the Fourier component $f(\mathbf{k})$ of $f(\mathbf{x})$. Here, $\langle\ \rangle$ denotes the ensemble mean, and k_M is the wavenumber of the largest spatial scale of fluctuations filtered out, which is written using the filter width Δ as

$$k_M = \pi/\Delta. \tag{15}$$

We can combine (14) with a two-scale direct-interaction approximation (TSDIA) to derive the Smagorinsky model (10)-(12). At this time, the Smagorinsky constant C_S is estimated as 0.14.

A MHD subgrid model of Smagorinsky type can be derived similarly with the aid of TSDIA. The model is summarized as follows:

Navier-Stokes equation

$$\frac{\partial \bar{u}^\alpha}{\partial t} + \frac{\partial}{\partial x^a}(\bar{u}^\alpha\,\bar{u}^a - \bar{b}^\alpha\,\bar{b}^a)$$

$$=\frac{\partial \bar{p}}{\partial x^{\alpha}}+\frac{\partial}{\partial x^{a}}\left(R_{VG}^{\alpha\beta}+\nu\frac{\partial \bar{u}^{\alpha}}{\partial x^{a}}\right), \tag{16}$$

Magnetic induction equation

$$\frac{\partial \bar{b}^{\alpha}}{\partial t}+\frac{\partial}{\partial x^{a}}\left(\bar{u}^{a}\bar{b}^{\alpha}-\bar{u}^{\alpha}\bar{b}^{a}\right)$$

$$=\frac{\partial}{\partial x^{a}}\left(R_{BG}^{\alpha a}+\lambda\frac{\partial \bar{b}^{\alpha}}{\partial x^{a}}\right), \tag{17}$$

Solenoidal condition

$$\frac{\partial \bar{u}^{a}}{\partial x^{a}}=\frac{\partial \bar{b}^{a}}{\partial x^{a}}=0, \tag{18}$$

SGS MHD stresses

$$R_{VG}^{\alpha\beta}\equiv-\left(\overline{u'^{\alpha}u'^{\beta}}-\overline{b'^{\alpha}b'^{\beta}}\right)=-(C_K\Delta)^2D^2\delta^{\alpha\beta}$$

$$+(C_S\Delta)^2D\left(\frac{\partial \bar{u}^{\alpha}}{\partial x^{\beta}}+\frac{\partial \bar{u}^{\beta}}{\partial x^{\alpha}}\right), \tag{19}$$

$$R_{BG}^{\alpha\beta}\equiv-\left(\overline{u'^{\beta}b'^{\alpha}}-\overline{u'^{\alpha}b'^{\beta}}\right)=(C_B\Delta)^2$$

$$\times D\left(\frac{\partial \bar{b}^{\alpha}}{\partial x^{\beta}}-\frac{\partial \bar{b}^{\beta}}{\partial x^{\alpha}}\right)-(C_A\Delta)^2$$

$$\times\left(\bar{b}^{\alpha}\frac{\partial D}{\partial x^{\beta}}-\bar{b}^{\beta}\frac{\partial D}{\partial x^{\alpha}}\right), \tag{20}$$

with

$$D=\left[\left(\frac{\partial u^{a}}{\partial x^{b}}+\frac{\partial u^{b}}{\partial x^{a}}\right)^2/2\right]^{1/2}. \tag{21}$$

The model constants are estimated as

$$C_K=0.22,\ C_S=0.14,\ C_B=0.23,\ C_A=0.27. \tag{22}$$

4. Discussion

The filtered magnetic induction equation (17) is also written in vector form as

$$\frac{\partial \bar{\boldsymbol{b}}}{\partial t}+\nabla\times(\bar{\boldsymbol{b}}\times\bar{\boldsymbol{u}})=\nabla\times(\overline{\boldsymbol{u}'\times\boldsymbol{b}'})+\lambda\Delta\bar{\boldsymbol{b}}. \tag{23}$$

Here, the term $\overline{\boldsymbol{u}'\times\boldsymbol{b}'}$ is called the SGS electromotive force. When the filtering is replaced by the ensemble averaging, the importance of the term is very familiar in the study of earth magnetic dynamo[2,3].

In the case of earth magnetic dynamo, the electromotive force is modeled as

$$(\overline{\boldsymbol{u}'\times\boldsymbol{b}'})^{\alpha}=C^{\alpha a}\bar{b}^{a}+C^{\alpha ab}\frac{\partial \bar{\boldsymbol{b}}^{a}}{\partial \boldsymbol{x}^{b}}+\cdots. \tag{24}$$

The two terms in (24) are named the α (alpha) and β (beta) terms, respectively. Their feature is that the former is linearly dependent on the magnetic field itself and the latter on the magnetic shear. Those two terms stem from the isotropic non-mirrorsymmetric and mirrorsymmetric properties of the velocity field, respectively. In reality, the kinematical homogeneous turbulent dynamo theories can assume the isotropic non-mirrorsymmetric velocity field to show that

$$C^{\alpha\beta}=C_{\alpha}\delta^{\alpha\beta}, \tag{25}$$

$$C^{\alpha\beta\gamma}=C_{\beta}\varepsilon^{\alpha\beta\gamma}, \tag{26}$$

using the Kronecker delta symbol $\delta^{\alpha\beta}$ and the alternating tensor $\varepsilon^{\alpha\beta\gamma}$, where C_{α} and C_{β} are scalar quantities. At this time, C_{α} in the α term can be written in terms of the helicity

$$\overline{\boldsymbol{u}'\cdot\boldsymbol{\omega}'} \tag{27}$$

characterizing the non-mirrorsymmetry of the velocity field, where ω' $(=\nabla\times\boldsymbol{u}')$ is the vorticity. Within the framework of homogeneous turbulence theory, however, we have no way to relate C_{α} and C_{β} to the resolvable field $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{b}}$.

In the present result, we have

$$C^{\alpha\beta}=-(C_A\Delta)^2\varepsilon^{\alpha\beta a}\frac{\partial D}{\partial x^{a}}, \tag{28}$$

$$C^{\alpha\beta\gamma}=(C_B\Delta)^2D\varepsilon^{\alpha\beta\gamma}. \tag{29}$$

On comparing (28) and (29) with (25) and (26), (28) corresponds to the α term, and (29) to the β term. From the latter correspondence, C_{β} of (26) is written as

$$C_{\beta}=(C_B\Delta)^2D, \tag{30}$$

in our filtering procedure.

On the other hand, the α term in (28) is different essentially from (25). The cause of the present α term is the inhomogeneity of the fluctuating field, as was discussed in Ref. 7. The inhomogeneity generated by the resolvable-field shear (gradient) violates the isotropy of the field, and gives rise to nonvanishing helicity leading to the α term.

Finally, let us compare the present subgrid model with the corresponding MHD ensemble-mean model to investigate some differences between them. We denote the ensemble mean parts of $\boldsymbol{u}$, $\boldsymbol{b}$, and p by $\boldsymbol{U}$, $\boldsymbol{B}$, and $\hat{P}$, respectively. Moreover, the MHD turbulent energy or the sum of kinetic and magnetic turbulent energy and its dissipation rate are denoted by k_M and ε_M. Then, the ensemble-mean model, which is derived from TSDIA, is given as follows[10]:

Navier-Stokes equation

$$\frac{\partial U^{\alpha}}{\partial t}+\frac{\partial}{\partial x^{a}}\left(U^{\alpha}U^{a}-B^{\alpha}B^{a}\right)$$

$$=-\frac{\partial \hat{P}}{\partial x^{\alpha}}+\frac{\partial}{\partial x^{a}}\left(R_{V}^{\alpha a}+\nu\frac{\partial u^{\alpha}}{\partial x^{a}}\right), \tag{31}$$

Magnetic induction equation

$$\frac{\partial B^{\alpha}}{\partial t}+\frac{\partial}{\partial x^{a}}(U^{\alpha}B^{a}-U^{\alpha}B^{a})=\frac{\partial}{\partial x^{a}}\left(R_{B}^{\alpha a}+\lambda\frac{\partial B^{\alpha}}{\partial x^{a}}\right), \qquad (32)$$

Solenoidal condition

$$\frac{\partial U^{a}}{\partial x^{a}}=\frac{\partial B^{a}}{\partial x^{a}}=0, \qquad (33)$$

MHD stresses

$$R_{V}^{\alpha\beta}=-\left(M_{V1}k_{M}-M_{V2}\frac{k_{M}}{\varepsilon_{M}}\frac{Dk_{M}}{Dt}+M_{V3}\frac{k_{M}^{2}}{\varepsilon_{M}^{2}}\frac{D\varepsilon_{M}}{Dt}\right)\delta^{\alpha\beta}+M_{V4}\frac{k_{M}^{2}}{\varepsilon_{M}}\left(\frac{\partial U^{\alpha}}{\partial x^{\beta}}+\frac{\partial U^{\beta}}{\partial x^{\alpha}}\right), \qquad (34)$$

$$R_{B}^{\alpha\beta}=M_{B1}\frac{k_{M}^{2}}{\varepsilon_{M}}\left(\frac{\partial B^{\alpha}}{\partial x^{\beta}}-\frac{\partial B^{\beta}}{\partial x^{\alpha}}\right)-M_{B2}\frac{k_{M}}{\varepsilon_{M}}\left(B^{\alpha}\frac{\partial k_{M}}{\partial x^{\beta}}-B^{\beta}\frac{\partial k_{M}}{\partial x^{\alpha}}\right)+M_{B3}\frac{k_{M}^{2}}{\varepsilon_{M}^{2}}\left(B^{\alpha}\frac{\partial \varepsilon_{M}}{\partial x^{\beta}}-B^{\beta}\frac{\partial \varepsilon_{M}}{\partial x^{\alpha}}\right), \qquad (35)$$

k_M equation

$$\frac{Dk_{M}}{Dt}\equiv\left(\frac{\partial}{\partial t}+U^{a}\frac{\partial}{\partial x^{a}}\right)k_{M}=P_{M}-\varepsilon_{M}+D_{M}, \qquad (36)$$

with

$$P_{M}=R_{v}^{ab}\frac{\partial U^{b}}{\partial x^{a}}+R_{B}^{ba}\frac{\partial B^{b}}{\partial x^{a}}, \qquad (37)$$

ε_M equation

$$\frac{D\varepsilon_{M}}{Dt}=M_{\varepsilon 1}\frac{\varepsilon_{M}}{k_{M}}P_{M}-M_{\varepsilon 2}\frac{\varepsilon_{M}^{2}}{k_{M}}+B^{a}\left(M_{\varepsilon 3}\frac{\varepsilon_{M}}{k_{M}}\frac{\partial k_{M}}{\partial x^{a}}-M_{\varepsilon 4}\frac{\partial \varepsilon_{M}}{\partial x^{a}}\right)+D_{\varepsilon_{M}}, \qquad (38)$$

Here, M_{V1}, M_{V2}, etc. are model constants, and D_M and D_{ε_M} are the so-called diffusion terms (for the details, see Ref. 10).

The major differences between the above two models are that

(a) the ensemble-mean model is of two-equation type, wheres the subgrid model is of zero-equation type;

(b) in the MHD stresses $R_B^{\alpha\beta}$ and $R_{BG}^{\alpha\beta}$ that are crucial in dynamo effects, the alpha terms in the former consist of two different terms, unlike the latter.

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