Stochastic Resonance for the Superimposed Periodic Pulse Train

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Abstract

Stochastic Resonance in a coupled FitzHugh-Nagumo equation is investigated. The optimal noise intensity and the optimal input frequency, which maximize the signal to noise ratio of the output signal, are studied numerically, and their dependence on system parameters and connection coefficients is examined. It is found that a network composed of six elements can separate a superimposed periodic pulse train by controlling the noise intensity.

Keywords: Stochastic Resonance; FitzHugh-Nagumo equation; signal to noise ratio; eigenfrequency
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1 Introduction

In the last decade, there have been a considerable attraction of attention to Stochastic Resonance (SR), which is a phenomenon where a weak periodic signal is enhanced by its background noise [1,2] and observed in many nonlinear systems, such as bistable ring lasers, semiconductor devices, chemical reactions, and neural systems. When a periodic signal and a noise are injected to such systems simultaneously, the signal to noise ratio (SNR) of the output signal is maximized at an optimal noise intensity.

In particular, the neural system is essentially noisy [16,17], so SR may play a significant role [3–15]. The theoretical works on SR in a single neuron are performed on the integrate-and-fire neuron model [5–7] and the FitzHugh-Nagumo equation [8,9] with periodic stimulation and additive noise. In those works, it is observed that the output SNR [8] or the peak height of the interspike interval distribution [5–7,9] takes a maximum as a function of noise intensity. Some physiological experiments reinforce the hypothesis that the neural system uses SR to detect weak signals [3,4]. In Ref. [3], sinusoidally stimulated mechanoreceptor cells of a crayfish with additive noise show the property of SR, namely, the existence of the optimal noise intensity which maximizes the output SNR. In Ref. [4], SR is observed in caudal photoreceptor interneurons of a crayfish by the intrinsic and not external noise.

Recently SR in spatially extended systems is investigated and some new features are demonstrated [10–14]. In a model of neural system, the noise intensity dependence of the normalized power norm, which measures the correlation between the aperiodic input and the output of the system, by increasing the number of elements composing the system is investigated [10,11], and the frequency dependence of SR in a coupled system is suggested to be important for the neural information processing [14].

In the present paper, we shall consider SR in a coupled FitzHugh-Nagumo equation with a superimposed periodic pulse train, namely, a sum of three periodic pulse trains with mutually irrational frequencies, and show the system can separate three periodic pulse trains by controlling its background noise.

In Section 2, the FitzHugh-Nagumo equation with a periodic pulse train and a Gaussian white noise is investigated numerically and SR is observed. In Section 3, the eigenfrequency of the system is defined from the frequency dependence of SR and three sets of parameter values are adjusted to realize three mutually irrational eigenfrequencies. In Section 4, a coupled FitzHugh-Nagumo equation is treated and the dependence of the optimal noise intensity on the connection coefficient and the number of elements is considered. In Section 5, a superimposed periodic pulse train (SPPT) is defined and the noise intensity dependence in the coupled FitzHugh-Nagumo system with SPPT is considered. Conclusions and discussions are given in the last section.

2 SR in the FitzHugh-Nagumo Equation

In this work, as a model of a neuron, we use the FitzHugh-Nagumo equation written as

$$\dot{u} = c(-v + u - u^3/3 + S(f;t) + \xi(t)), \tag{1}$$

$$\dot{v} = u - bv + a, \tag{2}$$

$$S(f;t) = \begin{cases} I & \text{if } n/f \le t \le n/f + d \ \ (n = 0, 1, 2, \cdots) \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

$$\langle \xi(t)\xi(t')\rangle = D\delta(t-t'), \tag{4}$$

where u is the fast variable which denotes the internal state of a neuron, v is the slow variable which represents the refractory period, $S(f;t)$ is the periodic pulse train with the height I, the width d , and the frequency f, and $\xi(t)$ is the Gaussian white noise with the intensity D. Note that all the variables and constants are dimensionless in the above equations. In the following, $a = 0.7$, $b = 0.8$, $c = 10.0$, and $d = 0.3$ are mainly used, and the pulse height I is set small so that the system does not generate a pulse without a certain amount of noise, namely, the input pulse is sub-threshold.

Under the above conditions, a typical time series of u is shown in Fig. 1. When

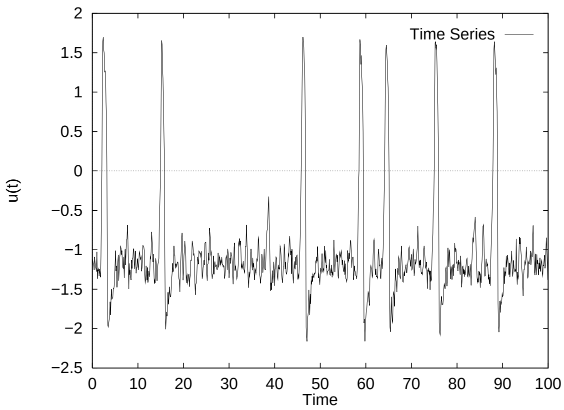

Fig. 1. A time series of u for $I = 0.1$, $f = 0.55$ and $D = 0.002$.

u takes a larger value than 0, we call that the system *fires*. The system can not fire without the noise because of the smallness of I, but Fig. 1 indicates that it can fire with the help of the noise. Let us assume that only the firing of the system can be observed, and regard

$$z(t) = \theta(u(t)) \equiv \begin{cases} u(t) & \text{if } u(t) > 0 \\ u_{eq} = -1.2 & \text{otherwise} \end{cases} \tag{5}$$

as the output of the system, where u_{eq} is the equilibrium value of $u(t)$ for $S(f;t) = \xi(t) = 0$. In Fig. 2, time series of $z(t)$ is shown for $f = 0.55$ and $f = 0.3$. Because all the peak heights of $z(t)$ are almost identical for each frequency, only the timing of firing is essential in $z(t)$. The time series of $z(t)$ seems to fire randomly in Fig. 2, but its power spectrum has a sharp peak at the frequency f of the input pulse train. Let us define the signal to noise ratio (SNR) as the ratio of the peak value S and the power N of the background noise, namely

$$\mathrm{SNR} = S/N. \tag{6}$$

In Fig. 3, SNR is plotted against the noise intensity D for $f = 0.3$ and 0.55

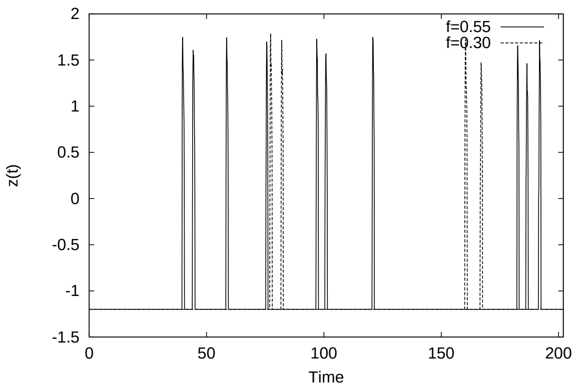

Fig. 2. Time series of z for $f = 0.55$ and $f = 0.3$ with $I = 0.1$ and $D = 0.002$.

with $I = 0.1$, showing the existence of the optimal intensity $D_0 \sim 0.003$. This

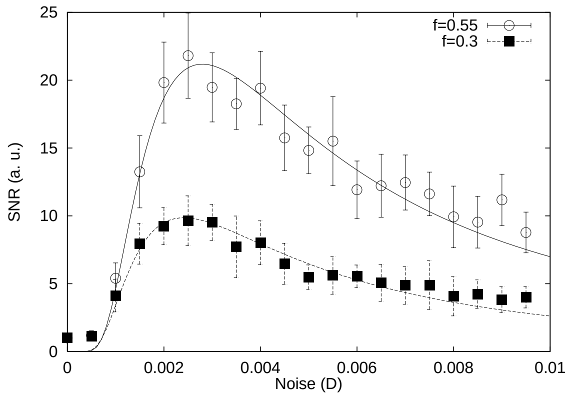

Fig. 3. SNR against the noise intensity D with $I = 0.1$. The lower data is for $f = 0.3$ and the upper is for $f = 0.55$. The lines are fitting curves by eq. (7). Each error bar denotes the standard deviation for 10 samples.

phenomenon is called Stochastic Resonance (SR) and widely seen in many bistable and excitable systems.

In many systems, it is reported that the noise intensity dependence of SNR obeys the relation

$$\mathrm{SNR} = \frac{A}{D^2} \exp\left(-\frac{B}{D}\right), \tag{7}$$

where A and B are constants which depend on system parameters [1]. SNR takes its maximum at $D = D_0$ given by

$$D_0 = \frac{B}{2}. \tag{8}$$

The fitting curves in Fig. 3 indicate that the eq. (7) is also valid for the FitzHugh-Nagumo equation [8].

3 The Frequency Dependence of SR

Figure 3 indicates that the peak value of SNR depends on the input frequency f, so in this section we examine the frequency dependence of SR for fixed $I = 0.1$ and $D = 0.002$. As shown in Fig. 4, SNR takes the maximum value

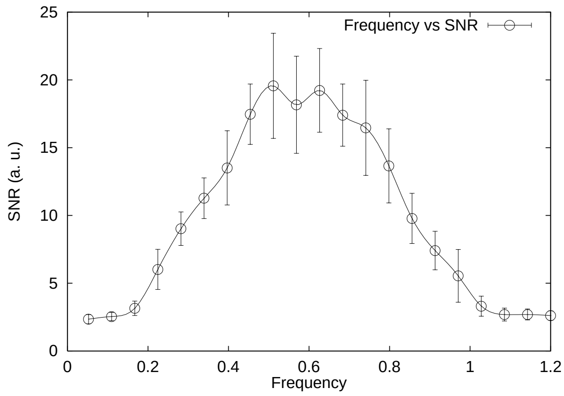

Fig. 4. SNR against the input frequency f for $I = 0.1$ and $D = 0.002$. Each error bar denotes the standard deviation for 10 samples.

at the optimal frequency $f_e \sim 0.55$, and we call f_e the eigenfrequency of the system on the analogy of classical resonance phenomena. The eigenfrequency f_e depends on the system parameters, a, b and c, and the wave form of the input S. For fixed input wave form, by changing the values of a, b, and c the eigenfrequency f_e can be adjusted by a series of numerical experiments and the system is denoted by the value of its eigenfrequency as element f_e. We prepare the three kinds of elements f_1, f_2 and f_3, which suffice the relationships $f_1 \sim 0.55$, $f_2 = f_1/\sqrt{2} \sim 0.39$, and $f_3 = f_1/\sqrt{5} \sim 0.25$ for convenience in following sections. The frequency dependence of SNR of each element f_1, f_2 and f_3 is shown in Fig. 5. It is observed that the peak of each SNR is located at each eigenfrequency. The noise intensity dependence of SNR of each element f_i with the input frequency f_i equal to its eigenfrequency, is plotted in Fig. 6. We observe that each element has the same optimal noise intensity $D_0 \sim 0.003$. Note that, in general, the optimal noise intensity depends on the frequency of the input signal [6,7], but in our range of input frequencies its deviation can be neglected.

4 SR in a Coupled System

In this section, we treat a coupled FitzHugh-Nagumo equation, written as

$$\dot{u_i} = c_i(-v_i + u_i - u_i^3/3 + S(f;t) + \xi_i(t) + \sum_{j=1}^{N} w_{ij}(u_j - u_i)), \tag{9}$$

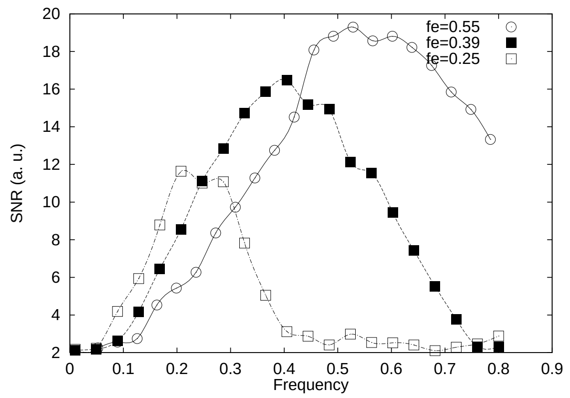

Fig. 5. SNR against the input frequency f for the three kinds of elements with the eigenfrequencies f_e=0.25, 0.39, and 0.55, where $I = 0.1$ and $D = 0.002$.

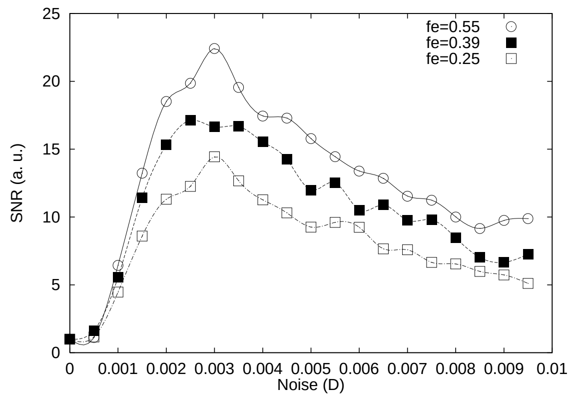

Fig. 6. SNR against the noise intensity D for the three kinds of elements with $I = 0.1$. Each input frequency to the elements is set for each eigenfrequency.

$$\dot{v}_i = u_i - b_i v_i + a_i, \tag{10}$$

$$\langle \xi_i(t)\xi_j(t') \rangle = D\delta_{ij}\delta(t - t'), \tag{11}$$

for $i = 1, 2, \cdots, N$, where a_i, b_i, and c_i are system parameters of the i-th element and δ_{ij} denotes Kronecker's delta. Note that the connection of each element is diffusive, the periodic pulse train $S(f;t)$ is applied to all the elements, and the noises for different elements are statistically independent.

Firstly, let us examine the effect of the connection coefficient w_{ij}. For $N = 2$, $a_1 = a_2 = 0.7$, $b_1 = b_2 = 0.8$, $c_1 = c_2 = 10.0$ and $w_{12} = w_{21} = w$, since the behaviors of the two elements may be statistically identical by the symmetry of the system, only $z_1 \equiv \theta(u_1)$ is observed as the output.

In Fig. 7, SNR from $z_1(t)$ is plotted against D for $w = 0$, 0.1, and 0.5, with $I = 0.1$ and $f = 0.55$. Similarly to the one element case, each SNR has a maximum, but the optimal noise intensities D_0's take different values depending on w. In Fig. 8, the optimal noise intensity $D_0(w)$ for the connection coefficient w is plotted against w. It is observed that $D_0(w)$ is an increasing function of w

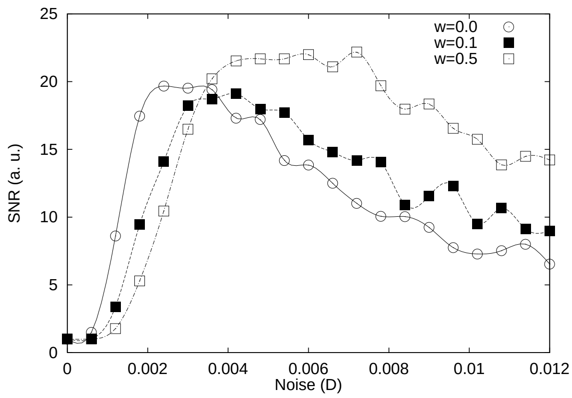

Fig. 7. SNR against D for $w = 0$, 0.1, and 0.5, with $I = 0.1$ and $f = 0.55$.

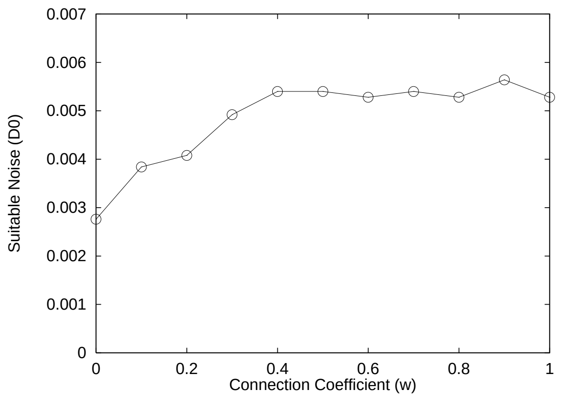

Fig. 8. D_0 against w for $I = 0.1$ and $f = 0.55$.

and converges to $D_0(\infty) \sim 0.005$.

To analyze the N dependence of $D_0(\infty)$ generally, let us consider a coupled system composed of N oscillating elements. For $i = 1, 2, \cdots, N$, by introducing $\boldsymbol{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, \cdots, x_d^{(i)})^t$, $\boldsymbol{\xi}^{(i)} = (\xi_1^{(i)}, \xi_2^{(i)}, \cdots, \xi_d^{(i)})^t$, and a d-dimensional diagonal matrix C with positive diagonal components $C_1, \cdots, C_d$, a coupled system is written as

$$\dot{\boldsymbol{x}}^{(i)} = \boldsymbol{F}(\boldsymbol{x}^{(i)}) + wC\left(\frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}^{(j)} - \boldsymbol{x}^{(i)}\right) + \boldsymbol{\xi}^{(i)}, \tag{12}$$

$$\langle \xi_k^{(i)}(t)\xi_l^{(j)}(t')\rangle = D_k\delta_{ij}\delta_{kl}\delta(t - t'), \tag{13}$$
$$i, j = 1, 2, \cdots, N, \quad \text{and} \quad k, l = 1, 2, \cdots, d,$$

where $\boldsymbol{F}(\boldsymbol{x}^{(i)})$ generates the internal motion of the i-th element and w denotes the connection strength. Let us define the mean value $\boldsymbol{X}$ and the deviation $\delta\boldsymbol{x}^{(i)}$ from $\boldsymbol{X}$ as

$$\boldsymbol{X} = \frac{1}{N}\sum_i \boldsymbol{x}^{(i)}, \tag{14}$$

$$\delta \boldsymbol{x}^{(i)} = \boldsymbol{x}^{(i)} - \boldsymbol{X}, \tag{15}$$

then $\boldsymbol{X}$ and $\delta \boldsymbol{x}^{(i)}$ obeys

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + \sum_{i=1}^{N} \frac{\boldsymbol{\xi}^{(i)}}{N} + \mathcal{O}(|\delta \boldsymbol{x}^{(i)}|^2), \tag{16}$$

$$\dot{\delta \boldsymbol{x}}^{(i)} = (D\boldsymbol{F}(\boldsymbol{X}) - wC)\, \delta \boldsymbol{x}^{(i)} + \boldsymbol{\xi}^{(i)} - \sum_{j=1}^{N} \frac{\boldsymbol{\xi}^{(j)}}{N} + \mathcal{O}(|\delta \boldsymbol{x}^{(i)}|^2), \tag{17}$$

where $D\boldsymbol{F}(\boldsymbol{x})$ is the Jacobian matrix of $\boldsymbol{F}(\boldsymbol{x})$. In the large w limit, eq. (17) becomes

$$\dot{\delta \boldsymbol{x}}^{(i)} = -wC\delta \boldsymbol{x}^{(i)} + \boldsymbol{\xi}^{(i)} - \sum_{j=1}^{N} \frac{\boldsymbol{\xi}^{(j)}}{N}, \tag{18}$$

so the variance of $\delta \boldsymbol{x}^{(i)}$ is estimated to be

$$\langle (\delta x_k^{(i)})^2 \rangle \simeq \frac{(1 - N^{-1}) D_k}{2wC_k}. \tag{19}$$

Equations (15), (16), and (19) indicate that the dynamics of each element for large w approaches to the dynamics of one element with the scaled noise intensity D_k/N ($k = 1, 2, \cdots, d$).

In the case of the coupled FitzHugh-Nagumo equation with uniform coupling, $d = 2$, $C_1 = 1$, $C_2 = 0$, $D_1 = D$, $D_2 = 0$, and eq. (18) is modified to

$$\dot{\delta x}_1^{(i)} = -w\delta x_1^{(i)} + \xi_1^{(i)} - \sum_{j=1}^{N} \frac{\xi_1^{(j)}}{N}, \tag{20}$$

$$\dot{\delta x}_2^{(i)} = \delta x_1^{(i)} - b\delta x_2^{(i)}. \tag{21}$$

Thus the variance of $\delta x_2^{(i)}$ can be estimated to be

$$\langle (\delta x_2^{(i)})^2 \rangle \simeq \frac{\langle (\delta x_1^{(i)})^2 \rangle}{b(b+w)}, \tag{22}$$

$$\sim \frac{1}{w^2}, \tag{23}$$

and it can be concluded that the dynamics of each element for large w is governed by eq. (16). So, between the optimal noise intensity $D_0^{(N)}(\infty)$ for N elements and $D_0^{(1)}(\infty)$ for one element, the relation

$$D_0^{(N)}(\infty) = N D_0^{(1)}(\infty) \tag{24}$$

holds. As shown in Fig. 9, numerically observed $D_0^{(N)}(\infty)$ shows a good agreement with eq. (24). In Fig. 9, the asymptotic value $D_0^{(N)}(\infty)$ is estimated by

$D_0^{(N)}(w)$ with $w = 1.0$, which is large enough for the saturation of $D_0(w)$ (see Fig. 8).

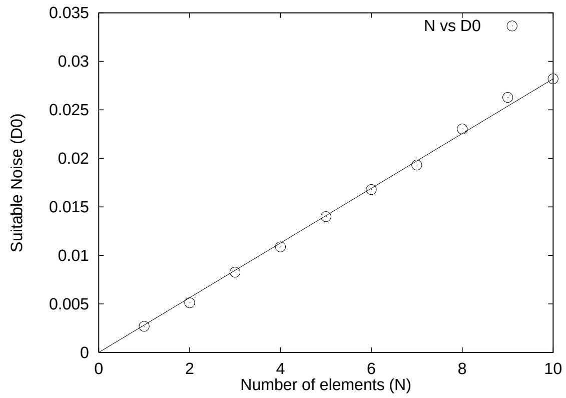

Fig. 9. $D_0^{(N)}(\infty)$ against N.

As shown in eq. (16), for finite w, the intensity of the effective noise on the mean motion depends on $\delta x^{(i)}$. Since the magnitude of $\delta x^{(i)}$ is thought to be a decreasing function of w, the effective noise intensity is also a decreasing function of w leading to a conclusion that the optimal noise intensity $D_0(w)$ is an increasing function of w as shown in Fig. 8. Further analysis on $D_0(w)$ is a future work.

5 Separation of a Superimposed Periodic Pulse Train

Using the preceding properties of SR, we construct a network which can separate a superimposed periodic pulse train (SPPT) by tuning the noise intensity.

Let us define SPPT as

$$T(t) = \max_{1 \le i \le m} S(f_i; t), \tag{25}$$

where m is the number of periodic components and f_i is the frequency of each component. In the following, we set $m = 3$, $f_1 = 0.55$, $f_2 = f_1/\sqrt{2}$, and $f_3 = f_1/\sqrt{5}$. In the power spectrum of $T(t)$, there are peaks with the same intensity at f_1, f_2, and f_3. Now let us apply $T(t)$ to a network, shown in Fig. 10, composed of three subnetworks, where each subnetwork contains a pair of elements with the eigenfrequency f_i and the connection coefficient w_i $(i = 1, 2, 3)$, that is, eq. (9) with $N = 6$, $w_{ij} = 0$ except $w_{12} = w_{21} = w_1$, $w_{34} = w_{43} = w_2$, and $w_{56} = w_{65} = w_3$, and $T(t)$ instead of $S(f; t)$. Note that $T(t)$ and the statistically independent noises with the same intensity D are applied to all the six elements, and the output $Z(t)$ of the network is defined

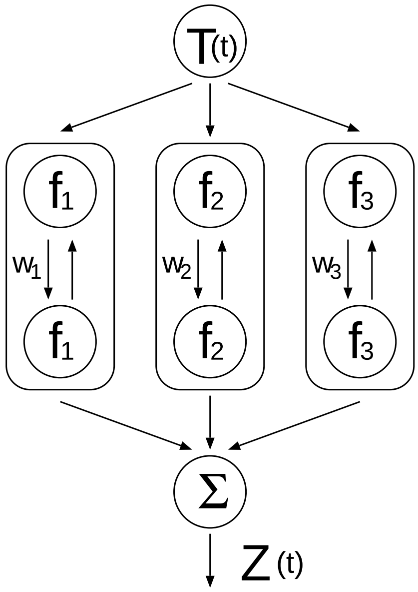

Fig. 10. A network composed of six elements.

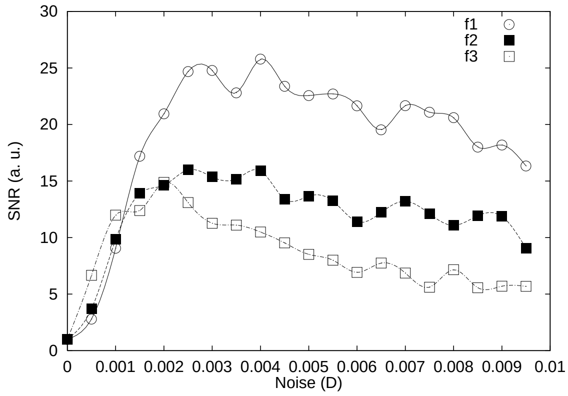

Fig. 11. SNR of each frequency for $I = 0.1$ and $w_1 = w_2 = w_3 = 0$.

as

$$Z(t) = \sum_{i=1}^{6} z_i(t) = \sum_{i=1}^{6} \theta(u_i(t)). \tag{26}$$

The power spectrum of $Z(t)$ has sharp peaks at f_1, f_2, f_3 and their linear combinations. SNR of each frequency is plotted against the noise intensity D for $w_1 = w_2 = w_3 = 0$ in Fig. 11, and we observe that the optimal noise intensities are almost identical for the three frequencies. It is because the optimal noise intensity of each element does not change for $w_1 = w_2 = w_3 = 0$. On the other hand, as shown in Fig. 12, each SNR has different optimal noise intensity for $w_1 = 0.5$, $w_2 = 0.1$, and $w_3 = 0.01$. If the noise intensity D is set around 0.0015, 0.003, or 0.007, then the output signal $Z(t)$ is dominated by a periodic motion with the frequency f_1, f_2, or f_3, respectively, since, at each noise intensity, SNR at the corresponding frequency is superior compared with those at the other frequencies. Thus a separation of SPPT by controlling the noise intensity is realized.

This result implies that noise can not only enhance weak signals, but also control the response of the system.

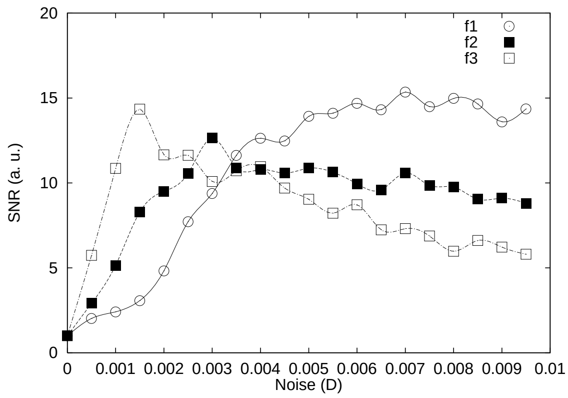

Fig. 12. SNR of each frequency for $I = 0.1$, $w_1 = 0.5, w_2 = 0.1$ and $w_3 = 0.01$.

6 Conclusions and Discussions

Using the properties of SR in a coupled system, a new feature in a noisy neural network is reported. In a single element case, the existence of the optimal noise intensity and the optimal input frequency is observed in the FitzHugh-Nagumo equation. The former is a common characteristic of the conventional SR phenomena, and the latter is only seen in excitable systems. The optimal frequency can be controlled by parameters of the system. In the two elements case, the optimal noise intensity is found to be an increasing function of the connection coefficient and the relationship between its saturation value and the number of coupled elements is derived analytically. Using this property, a network composed of six elements, which can separate a superimposed periodic pulse train by controlling the noise intensity, is constructed. In Ref.[10], Collins *et al.* considered an ensemble of the FitzHugh-Nagumo equations, and examined the noise intensity dependence of the normalized power norm, which measures the correlation between the aperiodic input and the output of the system. They found the flattening of the normalized power norm for sufficiently large number of elements in the ensemble, and suggested that sensory systems could detect weak signals without tuning the noise intensity. Similar results, that is, the increase of the optimal noise intensity with the magnitude of connection coefficient (Fig. 8) and with the number of elements (Fig. 9), are observed in our numerical experiments, but the separation of SPPT suggests that the noise intensity might control the response of the system and play a similar role as a parameter of dynamical systems.

In the problem of the information processing in the brain, the question of what carries the information in the brain is controversial. From physiological experiments, a single neuron is known to operate under a very noisy environment and its response seems to be stochastic [16,17], so it might be natural to assume that the information is coded in the firing rate of a single neuron or an ensemble of neurons. On the other hand, there is a hypothesis called temporal coding, which claims that the information is coded spatio-temporally by the temporal formation of cell assemblies whose elements are spiking correlatively [18]. Because the exact timing of spiking is important for the temporal coding, it seems to be difficult to construct a network composed of physiological stochastic neurons, which communicate using the temporal coding. But the noise induced enhancement of the coherence between the input and the output, which is one of the properties of SR, may make the temporal coding possible. Although we have treated only periodic inputs in this paper, SR is observed for aperiodic inputs [1,15], so there is a possibility that SR plays an important role in the neural information processing. The analysis of noisy coupled systems with aperiodic inputs is the future work.

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