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Topological Vertex Formalism on Generalized Brane
Web via Ding-Iohara-Miki Algebra
(Ding-庵原-三木代数を用いた一般化されたブレー
ンウェブ上の Topological Vertex 定式化)

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A Dissertation for the degree of PhD

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Abstract

In this thesis, we deal with generalized (p, q) -brane web systems, which are motivated from the string theory realization of gauge theories beyond 5d $\mathcal{N} = 1$ gauge theories with A -type gauge group and quiver structure. We give a (generalized) topological vertex formalism on this kind of webs to reproduce the instanton partition function of the corresponding gauge theory. We use the underlying algebraic structure, the Ding-Iohara-Miki algebra, discovered in the original refined topological vertex formalism as the guiding principle in the generalization. The Ding-Iohara-Miki algebra is a quantum toroidal algebra deeply related to q -deformed \mathcal{W} -algebras. It provides an elegant framework to topics such as the Alday-Gaiotto-Tachikawa correspondence and the integrability of supersymmetric gauge theories. We check our generalized topological formalism by computing the double-quantized Seiberg-Witten curve, i.e. the qq-character, from the Ward identity of the associated algebra. In particular, we consider 5d $\mathcal{N} = 1$ gauge theories with A -type gauge group and D -type quivers and 6d $\mathcal{N} = (1, 0)$ theories in this article. We construct the former class of theories with orientifolds added in the (p, q) brane web and propose that an operator providing the reflection action in the transverse direction to the orientifold is the only new element we need to introduce in the corresponding topological vertex formalism. In the latter case, we find an elliptic version of the Ding-Iohara-Miki algebra as the associated algebraic structure and more strict constraints on the matter contents of the 6d theory from the Ward identity approach to the qq-characters. These constraints agree with the flux conservation condition discussed in the D6-D8 brane construction of 6d gauge theories.

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Notation

Let us declare the convention of notations used in this article.

One of the most frequently used notation is

$$q^\rho := \{q^{-i+\frac{1}{2}}\}_{i=1}^\infty, \quad (0.0.1)$$

as a set of variables in the Schur function. We also use a shorthand notation,

$$\mathcal{O}^{(i)} := 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\mathcal{O}}_{\text{i-th site}} \otimes 1 \otimes \cdots \otimes 1 \otimes \cdots \otimes 1, \quad (0.0.2)$$

when we consider the tensor product of vector spaces. A main object that appears in this article is the Ding-Iohara-Miki algebra, which is labeled by two parameters, q_1 and q_2 . Sometimes we will denote this algebra as DIM_{q_1, q_2} for short. It is also convenient in some cases to denote the class of gauge theories with Q_1 -type quiver structure and Q_2 type gauge groups as $\mathcal{T}_{(Q_1, Q_2)}$, where $Q_{1,2} = A, B, C, D, E, F, G$.

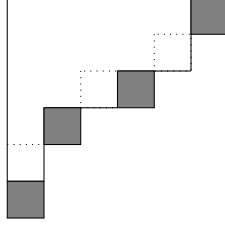


Figure 1: An example of Young diagram λ and boxes in $A(\lambda)$ (shaded) and in $R(\lambda)$ (dotted). (Boxes in $A(\lambda)$ do not belong to λ , and $R(\lambda) \subset \lambda$. We always have $|A(\lambda)| - |R(\lambda)| = 1$.)

Notations related to Young diagrams Given a Young diagram λ , we decompose it into a set of non-negative integer numbers $\{\lambda_i | i \in \mathbb{Z}_{\geq 0}\}$ satisfying, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$. These numbers correspond to the number of boxes in each row of the Young diagram. We denote λ^t as the transposed Young diagram of λ , and $(i, j) \in \lambda$ for the box in the i -th row and j -th column in λ . All notations related to Young diagrams used in this article are listed in the following table.

terminology	notation	meaning
arm length	$a(i, j)$	$a(i, j) = \lambda_i - j$
leg length	$\ell(i, j)$	$\ell(i, j) = \lambda_j^t - i$
coordinate of a box $x = (i, j) \in \lambda$	χ_x	$\chi_x = vq_1^{i-1}q_2^{j-1}$ with v the highest-weight parameter of λ
set of all boxes that can be added to λ	$A(\lambda)$	see Figure 1
set of all boxes that can be removed from λ	$R(\lambda)$	see Figure 1
squared sum of λ	$ \lambda ^2$	$ \lambda ^2 = \sum_i \lambda_i^2$
-	$n(\lambda)$	$n(\lambda) = \sum_i (i-1)\lambda_i$
-	$\kappa(\lambda)$	$\kappa(\lambda) = 2 \sum_{(i,j) \in \lambda} (j-i)$
size of λ	$ \lambda $	$ \lambda = \sum_i \lambda_i$

We also have the following useful identities.

$$n(\lambda) = \frac{1}{2}(|\lambda^t|^2 - |\lambda|), \quad n(\lambda^t) = \frac{1}{2}(|\lambda|^2 - |\lambda|), \quad (0.0.3)$$

$$n(\lambda) = n(\lambda^t) - \frac{1}{2}\kappa(\lambda^t) = n(\lambda^t) + \frac{1}{2}\kappa(\lambda), \quad (0.0.4)$$

$$\sum_{(i,j) \in \nu} a(i, j) = \frac{1}{2}(|\nu|^2 - |\nu|), \quad (0.0.5)$$

$$\sum_{(i,j) \in \nu} \ell(i, j) = \frac{1}{2}(|\nu^t|^2 - |\nu|). \quad (0.0.6)$$

The Frobenius coordinate $(\alpha_1, \alpha_2, \dots, \alpha_s | \beta_1, \beta_2, \dots, \beta_s)$ of a Young diagram λ is defined with

$$\alpha_i = a(i, i) + \frac{1}{2}, \quad \beta_i = \ell(i, i) + \frac{1}{2}, \quad \text{for } i = 1, 2, \dots, s, \quad (0.0.7)$$

where s is the number of boxes in the diagonal line of λ . Note that $\sum_i (\alpha_i + \beta_i) = |\lambda|$ and exchanging α_i and β_i transforms λ to λ^t .

Notations related to Ω -background When we consider the Ω -deformation of \mathbb{C}^2 , we use (q_1, q_2) to denote the deformation parameters, such that the $U(1) \times U(1)$ action on $(z_1, z_2) \in \mathbb{C}^2$ is given by

$$z_1 \rightarrow q_1 z_1, \quad z_2 \rightarrow q_2 z_2. \quad (0.0.8)$$

It is often convenient to introduce a third parameter q_3 to represent the following combination,

$$q_3 := q_1^{-1} q_2^{-1}, \quad (0.0.9)$$

and γ for its square root,

$$\gamma := q_3^{1/2} = (q_1 q_2)^{-1/2}. \quad (0.0.10)$$

Chapter 1

Introduction

Integrability is a very special and intriguing phenomenon in physics. It allows us to solve the underlying theory exactly by inducing infinite numbers of conserved charges. When the physical system has only a finite number of degree of freedoms, say N , the set of infinite charges will be expressed in terms of only N independent conserved charges. In a free theory or a one-particle system such as the harmonic oscillator, we are quite used to being able to solve the system exactly, but rather surprisingly this kind of special property was also discovered in interacting many-body systems, such as Heisenberg's XXX and XXZ models [1, 2, 3], around the 70's. After then, a huge number of works have been done on this topic, and a large class of integrable models were found to be characterized by a quantity called R -matrix, which is a solution to the scattering equation in two dimensions, i.e. the Yang-Baxter equation. Finally in the late 80's, a systematic way to build solutions to the Yang-Baxter equation was found in [4, 5, 6], where the R -matrix was found to be attached to an algebraic structure, the quantum group.

We work on integrable quantum field theories and string theory models in this thesis. Two dimensional conformal field theories go between the topic of integrability and string theory, as they show up on the one hand in the continuum limit of lattice models in condensed matter and on the other hand as the worldsheet description of the string theory. In this article, we focus on toy models of the super string theory: the topological string theory [7, 8, 9], which only captures a topological sector of the full string theory, and several supersymmetric Yang-Mills theories, that are low energy effective theories on D-branes. Historically, the integrability in this context was first discovered in [10], in which Seiberg and Witten solved four-dimensional $\mathcal{N} = 2$ super Yang-Mills theories with $SU(2)$ gauge group exactly in terms of an algebraic curve, now known as the Seiberg-Witten curve, and a holomorphic differential on it. It is later shown in a series of works starting from [11, 12, 13] that Seiberg-Witten's solution is related to classical integrable systems, which have one-to-one correspondence with 4d $\mathcal{N} = 2$ gauge theories. The quantization of these classical integrable systems turn out to correspond to the Ω -deformation of gauge theories [14, 15]. In dimensions larger than four, we can put the gauge theory on $\mathbb{C}^2 \times T^{d-4}$, and consider the Ω -deformation of \mathbb{C}^2 with

two parameters, q_1 and q_2 , so that the $U(1) \times U(1)$ rotational action on $(z_1, z_2) \in \mathbb{C}^2$ is given by

$$z_1 \rightarrow q_1 z_1, \quad z_2 \rightarrow q_2 z_2. \quad (1.0.1)$$

When both q_1 and q_2 are equal to one, the Ω -deformation is trivial. The usual quantization corresponds to turning on a non-trivial Ω -background with one non-trivial parameter, say $q_1 \neq 1$. This setup is now known as the Nekrasov-Shatashvili limit [14] of the full Ω -background. The most generic Ω -background with $q_1 \neq 1$, $q_2 \neq 1$ gives rise to a new concept, “double quantization”, of the classical integrable systems, and in [15] Nekrasov showed that the Seiberg-Witten curve is uplifted to an operator-valued quantity, named a qq-character, in this generic case.

The topological string theory, on the other hand, simplifies a lot when we consider a special class of geometries as the target space, the toric Calabi-Yau manifolds. Their special geometric structure allows us to compute the partition function of the topological string in a Feynman-diagram-like way [16], which is often referred to as the topological vertex formalism nowadays. What makes the story more interesting is that these topological string theories are related to five-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories on S^1 constructed from brane webs [17] via a series of string dualities [18]. Taking the radius of the S^1 to be zero, we obtain a large class of 4d $\mathcal{N} = 2$ gauge theories with known corresponding integrable models.

Another triumph in the study of 4d $\mathcal{N} = 2$ gauge theories is the discovery of the Alday-Gaiotto-Tachikawa (AGT) relation [19] between 4d $\mathcal{N} = 2$ gauge theories and 2d conformal field theories with \mathcal{W} -symmetry. It is often referred as a part of the so-called 4d/2d duality, based on the class \mathcal{S} construction of 4d $\mathcal{N} = 2$ gauge theories [20]. In the class \mathcal{S} construction, a 4d $\mathcal{N} = 2$ gauge theory is obtained by compactifying a 6d $\mathcal{N} = (2, 0)$ theory labeled by its gauge group G (of ADE type) on a punctured Riemann surface. It is then natural to have an alternative way to compute physical quantities in the 4d gauge theory via some 2d theory living on the corresponding Riemann surface. In particular, the S^4 partition function of a 4d $\mathcal{N} = 2$ theory obtained from the 6d theory with gauge group G is mapped to a correlator defined on the punctured Riemann surface, where each regular puncture is mapped to a primary field in the dual 2d CFT with \mathcal{W} -symmetry of type G [19, 21, 22] and irregular punctures correspond to irregular states such as the Whittaker vector [23] in the \mathcal{W} -algebra.

Interestingly, an affine Yangian algebra (of $\widehat{\mathfrak{gl}}(1)$) was discovered as a symmetry on the instanton moduli space relating solutions with different instanton numbers [24, 25], and it was identified to be a $\mathcal{W}_{1+\infty}$ -algebra with two free parameters, the central charge c and a truncation parameter μ . Let us denote this algebra by $\mathcal{W}_{1+\infty}[\mu]$. When μ is set to some positive integer N , $\mathcal{W}_{1+\infty}[\mu]$ reduces to the A -type \mathcal{W} -algebra, \mathcal{W}_N plus a free $u(1)$ boson sector. Therefore, we see that the value of μ corresponds to the gauge group label G (of A -type) on the gauge theory side. We note that there is also recent progress on the embedding of \mathcal{W} -algebras of BCD -type in the $\mathcal{W}_{1+\infty}[\mu]$ algebra [26]. One of the most important feature of this algebra is that (since it is an Yangian algebra) it is equipped with an R -matrix, and thus associated to a series of integrable systems. This result

inspires us to deal with the integrability and the 4d/2d duality in a unified algebraic framework to help understand the origin of these special properties of the 4d $\mathcal{N} = 2$ gauge theories.

This kind of algebraic description does not only exist for 4d $\mathcal{N} = 2$ gauge theories, and its sibling associated to the topological string theory on toric Calabi-Yaus is believed to be one of the most mathematically beautiful one appearing in this context. This algebra is known as the Ding-Iohara-Miki (DIM) algebra [27, 28] or the quantum toroidal algebra of $\widehat{\mathfrak{gl}}(1)$ [29] in mathematics. The DIM algebra is labeled by two parameters $q_{1,2}$, which correspond to the Ω -background parameters in the instanton counting calculation, but this time, as it is dual to 5d $\mathcal{N} = 1$ gauge theories on $\mathbb{C}^2 \times S^1$, $q_{1,2}$ are not only determined by the “amplitude” of the Ω -deformation, they also depend on the radius of the S^1 circle. In the limit where S^1 shrinks to zero size, $q_{1,2}$ go to 1, while $\beta := \log_{q_1} q_2$ is fixed to be finite. It is this same limit, which will be called the 4d limit in this article, that the DIM algebra reduces to the affine Yangian of $\widehat{\mathfrak{gl}}(1)$ (see [30]). The DIM algebra has two central elements, which will be denoted as $\hat{\gamma}$ and ψ_0^+/ψ_0^- in this article, and we consider representations of this algebra in which these two central elements are mapped to constants,

$$\hat{\gamma} \mapsto q_3^{\ell_1/2}, \quad \psi_0^+/\psi_0^- \mapsto q_3^{-\ell_2}. \quad (1.0.2)$$

In general, ℓ_1 and ℓ_2 can be any complex numbers, but in the context of the topological string, a special class of representations, with $(\ell_1, \ell_2) \in \mathbb{Z}^2$, are particularly useful, as these two labels are mapped to the axio-dilaton charge in the language of web of 5-branes, or the label of degenerate cycles in the toric diagram. In [31], it was shown that the topological vertex formalism perfectly fits into the DIM algebra, where each leg (corresponding to the degenerate locus of (p, q) cycle) of the topological vertex is mapped to a state in the representation $(\ell_1, \ell_2) = (q, p)$ of the DIM algebra and the vertex itself is given by the most natural three point function intertwining three representations associated to its three legs.

The above formulation of the topological vertex formalism in the algebraic language is nothing but an equivalent rewriting, however, it has already brought us several interesting consequences due to the nice properties of the algebra. First one would naturally expect, from the fact that the affine Yangian algebra is isomorphic to the $\mathcal{W}_{1+\infty}[\mu]$ algebra, that the DIM algebra is a one-parameter deformed $\mathcal{W}_{1+\infty}[\mu]$ algebra. Indeed, it was shown in [32] that all A -type q -deformed \mathcal{W}_N algebras are embedded in the DIM algebra. It reveals an interesting relation between 5d $\mathcal{N} = 1$ gauge theories and q -deformed \mathcal{W} -algebras, which was first proposed by [33] as the q -deformed version of the original AGT relation. Secondly, there is an $\mathrm{SL}(2, \mathbb{Z})$ automorphism in the DIM algebra, which acts as a matrix on the representation label (ℓ_1, ℓ_2) [28]. This $\mathrm{SL}(2, \mathbb{Z})$ automorphism can be identified with the $\mathrm{SL}(2, \mathbb{Z})$ S-duality, which acts exactly in the same way on the label (p, q) of 5-branes, in the type IIB super string. In the context of the DIM algebra, we thus expect that there exist different $\mathrm{SL}(2, \mathbb{Z})$ versions of the 5d AGT relations. This kind of S-dual AGT relation was first discovered in [34]. In particular, the S-duality that maps a D5-brane into an NS5-brane, i.e. that can be realized as a 90° rotation on the toric diagram plane, is equivalent to the fiber-base duality

[35] in the geometric approach to gauge theories, and it exchanges the information of the gauge group and the quiver structure in the corresponding gauge theory. In the original AGT relation, the gauge group information of the gauge theory is encoded in the type of the dual \mathcal{W} -algebra [21, 22], and in the fiber-base dual version of the AGT duality, the dual \mathcal{W} -algebra is of the quiver type, which is known as Kimura-Pestun's quiver \mathcal{W} -algebra nowadays. Last but not least, the DIM algebra, as it is known as the lowest rank quantum toroidal algebra (an class of algebras generalized from quantum groups) in mathematics, is equipped with a universal R -matrix [36]. The underlying integrability is known as the Ruijsenaars-Schneider model [37], and in the 4d limit, it reduces [38] to the Calogero-Sutherland integrable model [39, 40] found by Maulik & Okounkov on the 4d $\mathcal{N} = 2$ instanton moduli space [25].

The topological vertex formalism deeply connected to the DIM algebra, however, can only be applied to (refined) topological strings with toric Calabi-Yau manifolds as the target space, whose dual gauge theories are restricted to both A -type gauge groups and A -type quiver structures. It is widely believed, however, that the AGT relation holds for a more general class of 5d $\mathcal{N} = 1$ gauge theories with $ABCDEFG$ -type gauge groups and (at least) for ADE -type quiver structures (c.f. [22] in 4d)¹ and even for supersymmetric gauge theories in 6d [41, 42]. Since the DIM algebra provides a powerful tool to analyze all AGT-like dualities in one gulp, we would like to generalize the topological vertex formalism so that it can also be applied to brane webs with new ingredients such as orientifolds, and see how the algebraic structure is modified in these generalized brane web systems. By doing so, we expect to be able to test the limitation of the algebraic framework for gauge theories and sharpen our understanding on the origin of the AGT-like dualities and integrability of supersymmetric gauge theories. We note that the idea to generalize the topological vertex formalism is not an eccentric one at all, as this kind of effort has already been made in [43] for the jumpings appearing in the context of the Hanany-Witten transition [44] and in recent progress on generalizing the gauge group in the topological vertex formalism to BCD - and G -type [45, 46].

In this thesis, we focus on the generalization of the topological vertex formalism to 5d $\mathcal{N} = 1$ D -type quiver gauge theories with A -type gauge group (in our shorthand notation, theories of class $\mathcal{T}_{(D,A)}$) on the unrefined Ω -background, i.e. $q_1 q_2 = 1$, and 6d $\mathcal{N} = (1, 0)$ gauge theories with both A -type quiver and gauge group (theories of class $\mathcal{T}_{(A,A)}$), based on [47, 48, 49]. The article is organized as follows: We first give a brief review on the topological string theory and explain that the topological vertex formalism can be used to compute the A-model partition function when the target space is a toric Calabi-Yau manifold. In the second part, we present the Ding-Iohara-Miki algebra as the underlying algebraic structure in the topological vertex formalism and review its relation with q -deformed \mathcal{W} -algebras and qq-characters. Then we generalize the topological vertex formalism to 5d $\mathcal{N} = 1$ gauge theories of type $\mathcal{T}_{(D,A)}$ and 6d $\mathcal{N} = (1, 0)$ gauge theories of type $\mathcal{T}_{(A,A)}$ by introducing new vertices into the formulation, and discuss the related algebraic structure and

¹We remark that the fiber-base duality will exchange the gauge group type and quiver type once we establish certain duality relation on one side.

qq-characters in these systems. As the results, we found that a reflection operator can be assigned to the orientifold in the D -type quiver construction and an elliptic version of the Ding-Iohara-Miki algebra is behind the 6d theories considered in this article.

The original part that supports this article as a PhD dissertation is provided in chapter 4 and chapter 5 (more precisely section 4.1, 4.2, 5.2 and 5.3) based on the following published papers of the author: [47, 48, 49].

Chapter 2

Topological String and Topological Vertex

This is a chapter where we review some basic but famous results in the literature. We start from a brief description on what a topological string theory is and then derive the topological vertex formalism to compute the partition function when the target space is specified to be a toric Calabi-Yau manifold, by using the duality between a topological string theory and a Chern-Simons theory. A refined version of the topological vertex will be further introduced based on the melting crystal picture of the topological vertex. At the end of this chapter, we will see a correspondence between the partition function computed from the refined topological vertex and the Nekrasov instanton partition function of 5d $\mathcal{N} = 1$ gauge theory, which is the manifestation of the string duality between these two theories.

2.1 Topological String Theory

In this section, we briefly review what a topological string theory is, following the description used when it was discovered in the literature [7, 8, 9]. The topological string theory introduced in this section will be called the unrefined theory in this article and one of the best reviews on the unrefined theory can be found in [50].

The topological string theory is a topologically twisted $\mathcal{N} = 2$ sigma model coupled to the gravity. We start with two-dimensional sigma models defined on the locally flat Euclidean metric

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad (2.1.1)$$

with the two dimensional gamma matrices given by

$$(\gamma^1)_\alpha{}^\beta = \sigma^1, \quad (\gamma^2)_\alpha{}^\beta = \sigma^2, \quad (2.1.2)$$

and by lowering the index with $C_{\alpha\beta} = \sigma^1$, we obtain

$$(\gamma^1)_{\alpha\beta} = 1, \quad (\gamma^2)_{\alpha\beta} = -i\sigma^3. \quad (2.1.3)$$

The $\mathcal{N} = 2$ supersymmetry in 2d, which can be dimensionally reduced from the 4d $\mathcal{N} = 1$ SUSY, is generated by $Q_{\alpha a}$ ($\alpha, a = \pm$), satisfying

$$\{Q_{\alpha+}, Q_{\beta-}\} = \gamma_{\alpha\beta}^\mu P_\mu, \quad \{Q_{\alpha\pm}, Q_{\beta\pm}\} = 0. \quad (2.1.4)$$

The $\text{SO}(2) \simeq \text{U}(1)$ Lorentz symmetry, generated by J , acts on the super charge $Q_{\alpha a}$ as

$$[J, Q_{\pm a}] = \pm \frac{1}{2} Q_{\pm a}. \quad (2.1.5)$$

and there also exist left and right $\text{U}(1)$ R-symmetries in the system,

$$[F_L, Q_{\pm\pm}] = \pm \frac{1}{2} Q_{\pm\pm}, \quad [F_L, Q_{-\pm}] = 0, \quad (2.1.6)$$

$$[F_R, Q_{\pm\pm}] = 0, \quad [F_R, Q_{-\pm}] = \pm \frac{1}{2} Q_{-\pm}. \quad (2.1.7)$$

These two $\text{U}(1)$'s combine into a vectorial and an axial current, $F_V = F_L + F_R$ and $F_A = F_L - F_R$.

The $\mathcal{N} = 2$ superspace derivatives $D_{\alpha a}$ satisfy

$$\{D_{++}, D_{+-}\} = 2\partial_z, \quad \{D_{--}, D_{-+}\} = 2\partial_{\bar{z}}. \quad (2.1.8)$$

The basic ingredients we are going to use are chiral and anti-chiral multiplets. The chiral multiplet is defined by

$$D_{+-}\Phi = D_{--}\Phi = 0, \quad (2.1.9)$$

and the anti-chiral multiplet is defined by

$$D_{++}\bar{\Phi} = D_{-+}\bar{\Phi} = 0. \quad (2.1.10)$$

We denote the bottom component of chiral multiplets Φ^I and anti-chiral multiplets $\Phi^{\bar{I}}$ as x^I and $x^{\bar{I}}$ respectively. Their fermionic partners will be denoted as $D_{\alpha+}\Phi^I|_{\theta=0} = \psi_{\alpha+}^I$ and $D_{\alpha-}\Phi^{\bar{I}}|_{\theta=0} = \psi_{\alpha-}^{\bar{I}}$.

An $\mathcal{N} = 2$ sigma model is given by the action

$$S = \int d^2z d^4\theta K(\Phi^I, \Phi^{\bar{I}}), \quad (2.1.11)$$

with K the Kähler potential.

The topological twist requires the redefinition of the spin current. In the A-twist, we shift the spin current with the vectorial current,

$$\tilde{J}^A = J - F_V, \quad (2.1.12)$$

and in the B-twist we use the axial current,

$$\tilde{J}^B = J + F_A. \quad (2.1.13)$$

Under the topological twist, the super charges split into a scalar charge and a vector charge. The scalar charge in the A -twist and B -twist are respectively given by

$$\mathcal{Q}^A = Q_{++} + Q_{--}, \quad \mathcal{Q}^B = Q_{+-} + Q_{-+}, \quad (2.1.14)$$

and the vector super charge reads

$$G_z^A = Q_{+-}, \quad G_{\bar{z}}^A = Q_{-+}, \quad (2.1.15)$$

$$G_z^B = Q_{++}, \quad G_{\bar{z}}^B = Q_{--}. \quad (2.1.16)$$

Thanks to the diagonal property of $\gamma_{\alpha\beta}^\mu$, we can confirm the nilpotency of $\mathcal{Q} = \mathcal{Q}^{A,B}$,

$$\mathcal{Q}^2 = 0, \quad (2.1.17)$$

and thus we can consider the cohomology of this operator. We also note that there is an optional choice, $\tilde{\mathcal{Q}}^A = Q_{++} - Q_{--}$ and $\tilde{\mathcal{Q}}^B = Q_{+-} - Q_{-+}$, to be the cohomology operator, but we can always redefine the sign of spinors to map this alternative choice to the canonical one stated above.

From the explicit expression, which will be presented in the next section, the action of an $\mathcal{N} = 2$ sigma model after topological twisting is closed under the nilpotent charge \mathcal{Q} , i.e. $\exists V$,

$$S = \{\mathcal{Q}, V\}, \quad (2.1.18)$$

and we see that this theory is indeed topological.

As there is quantum anomaly associated with the axial current F_A , given by the form

$$\int_{\Sigma_g} x^*(c_1(X)), \quad (2.1.19)$$

where $x : \Sigma_g \rightarrow X$ maps from the worldsheet Σ_g with genus g to the target Kähler space X , the B-model might be problematic to define. However, for X being Calabi-Yau, i.e. $c_1(X) = 0$, there is no such a problem, so we would like to focus on Calabi-Yau target spaces here.

Once we have a nilpotent charge, we can write down a series of descent equations,

$$d\phi^{(n)} = [\mathcal{Q}, \phi^{(n+1)}], \quad (2.1.20)$$

starting from an operator in the cohomology of \mathcal{Q} , ϕ^0 . d here is the exterior derivative on the world sheet Σ_g , and we adopt a shorten notation to use $[\dots, \dots]$ to stand for either commutator or anti-commutator according to the bosonic or fermionic nature of the operators inside the parenthesis

until the end of this discussion. By using the closedness of the action (2.1.18), we can write the stress tensor in the form,

$$T_{\mu\nu} = [\mathcal{Q}, G_{\mu\nu}], \quad (2.1.21)$$

with $G_{\mu\nu} := \frac{\delta V}{\delta g^{\mu\nu}}$, and thus the momentum P_μ of the system is given by

$$P_\mu = [\mathcal{Q}, G_\mu], \quad (2.1.22)$$

where we further introduced the notation

$$G_\mu := \int dx G_{\mu 0}(t, x), \quad (2.1.23)$$

as the spatial integral of the current $G_{\mu 0}$. We can find the following solution to the descent equations,

$$\phi^{(n)}(x) = \frac{i^n}{n!} \phi_{\mu_1 \dots \mu_n}^{(n)}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} = \frac{i^n}{n!} [G_{\mu_n}, \dots [G_{\mu_1}, \phi^{(0)}(x)] \dots] dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}, \quad (2.1.24)$$

associated to each $\phi^{(0)}$, since

$$\begin{aligned} [\mathcal{Q}, \phi^{(n+1)}] &= \frac{i}{n+1} [\mathcal{Q}, [G_\mu, \phi^{(n)}]] \wedge dx^\mu \\ &= \frac{i}{n+1} [[\mathcal{Q}, G_\mu], \phi^{(n)}] \wedge dx^\mu + \frac{i}{n+1} [G_\mu, [\mathcal{Q}, \phi^{(n)}]] \wedge dx^\mu \\ &= \frac{1}{n+1} (d\phi^{(n)} + id([G_\mu, \phi^{(n-1)}] \wedge dx^\mu)) \\ &= \frac{1}{n+1} (d\phi^{(n)} + nd\phi^{(n)}) = d\phi^{(n)}, \end{aligned} \quad (2.1.25)$$

where we used $d \dots = [P_\mu, \dots] \wedge dx^\mu = [[\mathcal{Q}, G_\mu], \dots] \wedge dx^\mu$. We note that G_μ 's are all anti-commuting to each other, so $\phi^{(n)}(z)$'s are indeed n -forms on Σ_g , and that the above solution can be constructed for the descent equations in any dimensions. Paired with an element of the homology of Σ_g , $\gamma_n \in H_n(\Sigma_g)$, we define the quantity

$$W_{\phi^{(0)}}^{\gamma_n} = \int_{\gamma_n} \frac{i^n}{n!} \phi_{\mu_1 \dots \mu_n}^{(n)}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}, \quad (2.1.26)$$

which is by construction closed under \mathcal{Q} , as

$$\{\mathcal{Q}, W_{\phi^{(0)}}^{\gamma_n}\} = \int_{\gamma_n} d\phi^{(n-1)} = 0. \quad (2.1.27)$$

These operators are a family of natural observables in topological sigma models.

Topological A Model As we have already seen that the observables in the topological sigma model at the classical level are deeply related to elements in the de Rham cohomology of the world sheet. In a quantum theory, a similar correspondence can be found between \mathcal{Q} -exact observables and cohomology of the target space for the A-twisted model (A-model). In the A-model, we split the fermions into a set of scalar fields and a set of vector fields,

$$\chi^I = \psi_{++}^I, \quad \chi^{\bar{I}} = \psi_{--}^{\bar{I}}, \quad \rho_z^I = \psi_{-+}^I, \quad \rho_z^{\bar{I}} = \psi_{+-}^{\bar{I}}, \quad (2.1.28)$$

and the supersymmetry transformation related to χ^i is given by

$$[\mathcal{Q}, x^i] = \chi^i, \quad \{\mathcal{Q}, \chi^i\} = 0, \quad (2.1.29)$$

where $\{i\} = \{I\} \cup \{\bar{I}\}$. Taking an element of the de Rham cohomology $\phi = \phi_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$, we can construct a \mathcal{Q} -exact observable

$$\mathcal{O}_\phi = \phi_{i_1 \dots i_n} \chi^{i_1} \dots \chi^{i_n}, \quad (2.1.30)$$

as $[\mathcal{Q}, \phi_{i_1 \dots i_n}] = \chi^i \frac{\partial \phi_{i_1 \dots i_n}}{\partial x^i}$. There is a selection rule among these observables, originating from the anomalous axial U(1) symmetry: the VEV $\langle \mathcal{O}_{\phi_1} \dots \mathcal{O}_{\phi_l} \rangle$ vanishes unless the following equation holds,

$$\sum_{k=1}^l \deg(\mathcal{O}_{\phi_k}) = 2d(1-g) + 2 \int_{\Sigma_g} x^*(c_1(X)). \quad (2.1.31)$$

We are interested in the case that X is a three-dimensional Calabi-Yau manifold, i.e. $d = 3$ and $c_1(X) = 0$. For $g = 1$, the free energy itself is a non-vanishing observable. However, the problem for such sigma models is that the VEV for all observables vanishes when $g \geq 2$, as there is no operator with negative degree. Historically it was motivated to solve this problem that people started to consider the string theory version of the topologically twisted sigma models.

Topological B Model The supersymmetry transformation in the topologically B-twisted model (B-model) is different from the A model. Defining

$$\rho_z^I = 2\psi_{++}^I, \quad \rho_z^{\bar{I}} = 2\psi_{-+}^{\bar{I}}, \quad \chi^{\bar{I}} = \psi_{+-}^{\bar{I}}, \quad \bar{\chi}^{\bar{I}} = \psi_{--}^{\bar{I}}, \quad (2.1.32)$$

$$\eta^{\bar{I}} = \chi^{\bar{I}} + \bar{\chi}^{\bar{I}}, \quad \theta_I = G_{I\bar{J}}(\chi^{\bar{J}} - \bar{\chi}^{\bar{J}}), \quad (2.1.33)$$

we have

$$[\mathcal{Q}, x^I] = 0, \quad [\mathcal{Q}, x^{\bar{I}}] = \eta^{\bar{I}}, \quad \{\mathcal{Q}, \eta^{\bar{I}}\} = 0. \quad (2.1.34)$$

We see an analogy between $\mathcal{Q} \leftrightarrow \bar{\partial}$, $\eta^{\bar{I}} \leftrightarrow dx^{\bar{I}}$ in the B-model. The transformation rule for θ_I is given by

$$\{\mathcal{Q}, \theta_I\} = (\text{auxiliary field}), \quad (2.1.35)$$

and goes to zero on-shell. The observables in the B model takes the form

$$\mathcal{O}_\phi = \phi_{\bar{I}_1 \dots \bar{I}_p}^{J_1 \dots J_q} \eta^{\bar{I}_1} \dots \eta^{\bar{I}_p} \theta_{J_1} \dots \theta_{J_q}, \quad (2.1.36)$$

in one-to-one correspondence to $\phi = \phi_{\bar{I}_1 \dots \bar{I}_p}^{J_1 \dots J_q} dx^{\bar{I}_1} \wedge \dots \wedge dx^{\bar{I}_p} \frac{\partial}{\partial x^{J_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{J_q}} \in H_{\bar{\partial}}^p(X, \wedge^q TX)$. For a Calabi-Yau target space, the selection rule forces,

$$\sum_i p_i = \sum_i q_i = d(1 - g), \quad (2.1.37)$$

in a non-vanishing VEV $\langle \prod_i \mathcal{O}_{\phi_i} \rangle$. Again, for $g \geq 2$, there is no non-trivial observables in the theory.

To avoid this conundrum, one can couple the sigma model to gravity, and then G plays the role of anti-ghost fields in the string theory. The zero modes of anti-ghost fields will be introduced into the partition function through the Faddeev-Popov determinant, which can be potentially used to satisfy the selection rule. The genus- g free energy is defined by

$$F_g = \int_{\bar{M}_g: \text{ moduli space}} \left\langle \prod_{k=1}^{6g-6} G(\mu_k) \right\rangle, \quad (2.1.38)$$

where

$$G(\mu_k) = \int_{\Sigma_g} d^2 z [G_{zz}(\mu_k)_z^z + G_{\bar{z}\bar{z}}(\bar{\mu}_k)_z^{\bar{z}}], \quad (2.1.39)$$

and μ_k is the Beltrami differential. We will have non-vanishing free energies for $g \geq 2$ only if we take the target space to be a three-dimensional Calabi-Yau manifold. The partition function of this theory, i.e. the topological string theory,

$$Z = \exp[\mathcal{F}], \quad (2.1.40)$$

with

$$\mathcal{F} = \sum_{g=0}^{\infty} g_s^{2-2g} F_g, \quad (2.1.41)$$

where g_s is the string coupling, is the main object we study in this article.

2.2 Topological String Theories as Chern-Simons Theories

In this section, we follow the discussion of Witten [8] to reformulate the topological string theory as a string field theory, and at the end we will see that it is equivalent to a Chern-Simons theory. In section 2.4, we will use this fact to derive the expression of the topological vertex.

Topological A Model

The explicit expression for the action of the A-model is given by

$$S_A = \int_{\Sigma_g} d^2z \sqrt{g} \left[G_{I\bar{J}} (g^{\mu\nu} \partial_\mu x^I \partial_\nu x^{\bar{J}} + \frac{i\epsilon^{\mu\nu}}{\sqrt{g}} \partial_\mu x^I \partial_\nu x^{\bar{J}} - g^{\mu\nu} \rho_\mu^I D_\nu \chi^{\bar{J}} - g^{\mu\nu} \rho_\mu^{\bar{J}} D_\nu \chi^I) \right. \\ \left. - \frac{1}{2} G_{I\bar{J}} g^{\mu\nu} \tilde{F}_\mu^I \tilde{F}_\nu^{\bar{J}} + \frac{1}{2} R_{\bar{I}J\bar{K}L} \rho_\mu^{\bar{I}} \rho_\nu^J \chi^{\bar{K}} \chi^L \right], \quad (2.2.1)$$

with

$$V = \frac{1}{2} \int_{\Sigma_g} d^2z \sqrt{g} g^{\mu\nu} G_{I\bar{J}} \left[\frac{1}{2} \rho_\mu^I \tilde{F}_\nu^{\bar{J}} + \frac{1}{2} \rho_\mu^{\bar{J}} \tilde{F}_\nu^I + \rho_\mu^I \partial_\nu x^{\bar{J}} + \rho_\mu^{\bar{J}} \partial_\nu x^I \right], \quad (2.2.2)$$

under the BRST transformation

$$[\mathcal{Q}, x^i] = \chi^i, \quad \{\mathcal{Q}, \chi^i\} = 0, \quad \{\mathcal{Q}, \rho_\mu^i\} = \partial_\mu x^i - \tilde{F}_\mu^i - \Gamma_{jk}^i \chi^j \rho_\mu^k, \quad (2.2.3)$$

$$[\mathcal{Q}, \tilde{F}_\mu^i] = 2D_\mu \chi^i \mp \left(\Gamma_{jk}^i \chi^j \tilde{F}_\mu^k - R^i{}_{kjl} \chi^k \chi^j \rho_\mu^l \right). \quad (2.2.4)$$

The role of the ghost number when we uplift it to a string theory is played by the anomalous axial $U(1)_A$ charge, where we have

$$2Q_{U(1)_A}(\mathcal{Q}) = 1, \quad 2Q_{U(1)_A}(G) = -1. \quad (2.2.5)$$

As this is a topological theory, we can rescale the action to tS_A for any constant t , while physical quantities still have no dependence on t . We need to find the string field for this model, which is a solution annihilated by the Hamiltonian of the system, to go the string field theory formulation. To convert to the Hamiltonian formalism, we quantize the theory with the following canonical commutation relations,

$$\left[\frac{dx^j}{d\tau}(\sigma), x^k(\sigma') \right] = -\frac{i}{t} G^{jk} \delta(\sigma - \sigma'), \quad (2.2.6)$$

$$\{\rho_\tau^j(\sigma), \chi^k(\sigma')\} = \frac{1}{t} G^{jk} \delta(\sigma - \sigma'). \quad (2.2.7)$$

The Hamiltonian then can be found as

$$H = \int_0^\pi d\sigma \left(-\frac{1}{t} G^{ij} \frac{\delta^2}{\delta x^i(\sigma) \delta x^j(\sigma)} + t G_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \right) + (\text{fermion terms}), \quad (2.2.8)$$

with terms involving fermions of order $\mathcal{O}(t^0)$. In the large t limit, this subspace annihilated by H is spanned by solutions satisfying $\frac{dx^i}{d\sigma} = 0$ and fermion zero modes, denoted as χ_0^a and $\frac{\delta}{\delta \chi_0^a}$ (from the canonical quantization).

The string field, \mathcal{A} , can thus be expanded as

$$\mathcal{A} = \sum_p \chi_0^{a_1} \dots \chi_0^{a_p} A_{a_1 \dots a_p}^{(p)} \quad (2.2.9)$$

with its coefficients as functions of $x^i = q^i$. However, constrained from the ghost number of the string field, the string field can only take the form

$$\mathcal{A} = \chi_0^a A_a(q). \quad (2.2.10)$$

When we consider the open string version of the theory, as discussed in [8], the boundary of the spatial interval $I = [0, \pi]$, ∂I , must be mapped to a Lagrangian submanifold of the target space X . This boundary condition restricts the bosonic mode q to lie on some Lagrangian submanifold. In particular when we consider $X = T^*M$, ∂I is restricted on M .

The string field action given by [51]

$$S = \frac{1}{g_s} \int \mathcal{A} \star \mathcal{Q} \mathcal{A} + \frac{2}{3} \mathcal{A} \star \mathcal{A} \star \mathcal{A}, \quad (2.2.11)$$

as is integrated over zero modes of the fields, reduces to an integration of 3-form fields over M ,

$$S = \frac{1}{g_s} \int_M \text{Ad}A + \frac{2}{3} A \wedge A \wedge A, \quad (2.2.12)$$

where we used the correspondence between $\mathcal{Q} \leftrightarrow d$, $\chi^i \leftrightarrow dx^i$ in the target space discussed in the previous section. We can further attach a Chan-Paton factor to the string field, and the final result is simply a $U(N)$ Chern-Simons theory.

Topological B Model

A very similar structure can be found in the B-model. Its action is given by

$$S_B = \int_{\Sigma_g} d^2z \sqrt{g} \left[G_{I\bar{J}} \left(g^{\mu\nu} (\partial_\mu x^I \partial_\nu x^{\bar{J}} - \rho_\mu^I D_\nu \eta^{\bar{J}}) - \rho_z^I D_{\bar{z}} \theta_I + \rho_{\bar{z}}^I D_z \theta_I + F^I F^{\bar{J}} \right) - R^I{}_{J\bar{L}K} \eta^{\bar{L}} \rho_z^J \rho_{\bar{z}}^K \theta_I \right], \quad (2.2.13)$$

with

$$V = \int_{\Sigma_g} d^2z \sqrt{g} (G_{I\bar{J}} g^{\mu\nu} \rho_\mu^I \partial_\nu x^{\bar{J}} - F^I \theta_I), \quad (2.2.14)$$

under the transformation

$$[\mathcal{Q}, x^I] = 0, \quad [\mathcal{Q}, x^{\bar{I}}] = \eta^{\bar{I}}, \quad \{\mathcal{Q}, \eta^{\bar{I}}\} = 0, \quad \{\mathcal{Q}, \theta_I\} = G_{I\bar{J}} F^{\bar{J}}, \quad (2.2.15)$$

$$\{\mathcal{Q}, \rho_\mu^I\} = \partial_\mu x^I, \quad \{\mathcal{Q}, F^I\} = D_z \rho_z^I - D_{\bar{z}} \rho_{\bar{z}}^I + R^I{}_{J\bar{L}K} \eta^{\bar{L}} \rho_z^J \rho_{\bar{z}}^K, \quad \{\mathcal{Q}, F^{\bar{I}}\} = -\Gamma_{J\bar{K}}^{\bar{I}} \eta^{\bar{J}} F^{\bar{K}}. \quad (2.2.16)$$

The string field formulation can be obtained in a similar way. As discussed before, \mathcal{Q} can be identified as $\bar{\partial}$, which has nothing to do with θ_I 's in the field, and the selection rule is imposed

separately on the holomorphic and anti-holomorphic part, we thus can factorize the string field Lagrangian into two parts,

$$S = \frac{1}{g_s} \int_X \Omega \wedge \text{tr} \left(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.2.17)$$

where this time A is an anti-holomorphic $(0, 1)$ -form and Ω is a $(d = 3, 0)$ -form.

We will mainly focus on the A-model in this thesis, while it is well-known that there is a mirror symmetry [52] (see [53] for a review) between the A-model and B-model.

2.3 Toric Variety and Mirror Curve

Among various 3-dimensional Calabi-Yau manifolds, there are a class of geometries of special simplicity for topological strings of A-model, the toric Calabi-Yau varieties. The geometry is always $T^2 \times \mathbb{R}$ fibered over \mathbb{R}^3 , which can be locally embedded into a \mathbb{C}^3 patch. The toric variety is given by the following defining equation,

$$\left\{ \sum_{i=1}^{d+r} Q_i^a |z_i|^2 = t^a \right\} / \text{U}(1)^r, \quad (2.3.1)$$

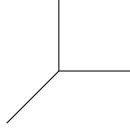
where t^a 's are Kähler parameters ($a = 1, \dots, r$), and r $\text{U}(1)$'s are identified as the transformation $z_i \rightarrow e^{iQ_i^a \alpha_a} z_i$ with transformation parameter α_a . The Calabi-Yau condition is equivalent to

$$\sum_i Q_i^a = 0, \quad \text{for } \forall a. \quad (2.3.2)$$

Locally we can take a \mathbb{C}^3 patch, say with coordinates (z_1, z_2, z_3) to describe the geometry. The torus action is generated by $r_\alpha = |z_1|^2 - |z_3|^2$ and $r_\beta = |z_2|^2 - |z_3|^2$, which via the standard symplectic form $\omega = i \sum_i dz_i \wedge d\bar{z}_i$ give rise to two $\text{U}(1)$ transformations

$$(z_1, z_2, z_3) \rightarrow (e^{i\alpha} z_1, e^{i\beta} z_2, e^{-i(\alpha+\beta)} z_3). \quad (2.3.3)$$

Let us denote the circle generated by r_α and r_β respectively to be the $(0, 1)$ and $(1, 0)$ circle of the torus. The base \mathbb{R}^3 is spanned by the coordinates r_α , r_β and $r_\gamma := \text{Im}(z_1 z_2 z_3)$. We want to know the location where the torus degenerates. In this \mathbb{C}^3 patch, the $(0, 1)$ circle r_α degenerates at $z_1 = 0 = z_3$, for which $r_\alpha = r_\gamma = 0$ and $r_\beta \geq 0$. The $(1, 0)$ circle r_β degenerates at $z_2 = 0 = z_3$, for which $r_\beta = r_\gamma = 0$ and $r_\alpha \geq 0$. In general, the $(-q, p)$ circle is generated by $pr_\alpha + qr_\beta$, and for a generic $(-q, p)$ circle, it degenerates at only one point $z_1 = z_2 = z_3 = 0$, but there is one exception: $(1, 1)$ circle, generated by $r_\alpha - r_\beta = |z_1|^2 - |z_2|^2$, degenerates at $z_1 = 0 = z_2$, which is described by $r_\alpha - r_\beta = 0$ and $r_\gamma = 0$. It is known that the degenerating loci always lie on a two-dimensional plane in the base space, with $r_\gamma = 0$, and therefore we can draw a one-dimensional diagram on the r_α - r_β plane to specify the location of the degenerate loci. This diagram is usually referred to as the toric diagram. In the current case of a single \mathbb{C}^3 -patch, the toric diagram is given by



General toric varieties can be obtained by gluing several patches together, among which the transformation rule is given by the defining equation of the toric Calabi-Yau. The toric diagram for a general toric Calabi-Yau is also obtained by gluing the diagrams for each patch together.

Let us see some examples. The first example is called $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ with $\vec{Q} = (-1, -1, 1, 1)$ and thus the identification of the coordinates (x_1, x_2, z_1, z_2) reads

$$(x_1, x_2, z_1, z_2) \sim (\lambda^{-1}x_1, \lambda^{-1}x_2, \lambda z_1, \lambda z_2). \quad (2.3.4)$$

We choose (x_1, x_2, z_1) to describe the first \mathbb{C}^3 patch. The second patch is related to the first one via

$$-|x_1|^2 - |x_2|^2 + |z_1|^2 + |z_2|^2 = t. \quad (2.3.5)$$

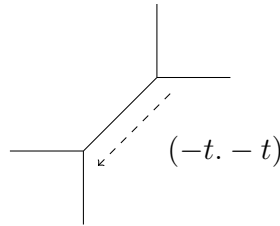
The $(0, 1)$ circle is generated by $r_\alpha = |x_1|^2 - |z_1|^2$ and $(1, 0)$ circle generated by $r_\beta = |x_2|^2 - |z_1|^2$. Switching to the (x_1, x_2, z_2) patch, we have

$$r_\alpha = -t + |z_2|^2 - |x_2|^2, \quad r_\beta = -t + |z_2|^2 - |x_1|^2, \quad (2.3.6)$$

which lead to the transformation

$$(x_1, x_2, z_2) \sim (e^{-i\beta}x_1, e^{-i\alpha}x_2, e^{i(\alpha+\beta)}z_2). \quad (2.3.7)$$

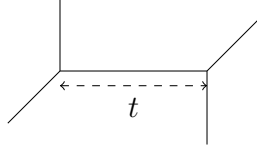
$(0, 1)$ circle degenerates at $x_2 = 0 = z_2$, with $r_\alpha = -t$, $r_\beta = -t - |x_1|^2$, and $(1, 0)$ circle degenerates at $r_\beta = -t$, $r_\alpha = -t - |x_2|^2$. The rest one dimensional degenerating loci is for $r_\alpha - r_\beta$ circle stretching from $(r_\alpha, r_\beta) = (-t, -t)$ described by $(r_\alpha, r_\beta) = (-t + |z_2|^2, -t + |z_2|^2)$. The toric diagram can therefore be depicted as



There are equivalent ways to draw the toric diagram by choosing a different patch to be the canonical \mathbb{C}^3 patch, or equivalently changing the coordinates of the torus, and all different choices are related by $\text{SL}(2, \mathbb{Z})$ transformations. Let us draw an equivalent figure for this geometry that will be frequently used later. The patches are taken to be the same, but we set instead

$$r_\alpha = |x_1|^2 - |x_2|^2, \quad r_\beta = |z_1|^2 - |x_2|^2. \quad (2.3.8)$$

The toric diagram in this choice is given by



We remark that this geometry is also known as the resolved conifold.

Another frequently considered example is local $\mathbb{CP}^1 \times \mathbb{CP}^1$, or $\mathcal{O}(-2, -2) \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$, with two $U(1)$ identifications,

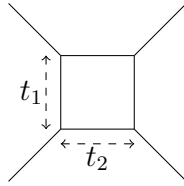
$$(x, y_1, y_2, z_1, z_2) \sim (\lambda^{-2} \mu^{-2} x, \lambda y_1, \lambda y_2, \mu z_1, \mu z_2). \quad (2.3.9)$$

Equations relating different patches read

$$-2|x|^2 + |y_1|^2 + |y_2|^2 = t_1, \quad (2.3.10)$$

$$-2|x|^2 + |z_1|^2 + |z_2|^2 = t_2. \quad (2.3.11)$$

By choosing $r_\alpha = |y_1|^2 - |x|^2$ and $r_\beta = |z_1|^2 - |x|^2$, we obtain the following toric diagram.



2.4 Topological Vertex

The A-model amplitude, as discussed in [16], can be divided into pairs of pants, i.e. the topological vertex, by inserting pairs of Lagrangian D-branes and anti-D-branes. The Lagrangian D-brane is lower dimensional compared to the Calabi-Yau target space. Its topology is $\mathbb{C} \times S^1$, and is projected to a line in the base space \mathbb{R}^3 , and wrap a circle of the T^2 fiber. In the toric diagram, the Lagrangian brane we consider here is point-like on a degenerate line, truncating the graph into two parts, and it wraps the remaining non-degenerate circle in the fiber. It extends in the r_γ direction, $r_\gamma \geq 0$.

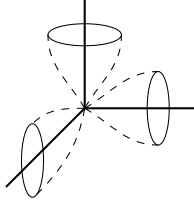
We integrate out the d.o.f.'s in other directions and formulate the calculation on the toric diagram and the torus fiber. Therefore, we can cut the toric diagram into vertices with three legs by inserting Lagrangian D-branes. Let us consider the following Lagrangian branes locating at

$$L_1 : \quad r_\alpha = 0, \quad r_\beta = r_1, \quad r_\gamma \geq 0,$$

$$L_2 : \quad r_\beta = 0, \quad r_\alpha = r_2, \quad r_\gamma \geq 0,$$

$$L_3 : \quad r_\alpha - r_\beta = 0, \quad r_\beta = r_3, \quad r_\gamma \geq 0,$$

in the base space. The truncated toric diagram is of the shape of a pair of pants.



Its partition function can be expressed as

$$Z = \sum_{\vec{k}^{(1)}, \vec{k}^{(2)}, \vec{k}^{(3)}} C_{\vec{k}^{(1)} \vec{k}^{(2)} \vec{k}^{(3)}} \prod_{i=1}^3 \frac{1}{z_{\vec{k}^{(i)}}} \text{tr}_{\vec{k}^{(i)}} V_i, \quad (2.4.1)$$

for some three-point vertex $C_{\vec{k}^{(1)} \vec{k}^{(2)} \vec{k}^{(3)}}$ we want to solve in this section, where V_i stands for the exponentiated Wilson line on the i -th stack of inserted D-branes, and

$$\text{tr}_{\vec{k}} V = \prod_{j=1}^{\infty} (\text{tr} V^j)^{k_j}, \quad (2.4.2)$$

$$z_{\vec{k}} = \prod_j k_j! j^{k_j}. \quad (2.4.3)$$

The D-brane wrapping the Lagrangian cycle has to be kept compact to perform the calculation (keeping track of the boundary condition at infinity), and one can deform the geometry without modifying the A-model amplitudes by allowing the $f_i = (p_i, q_i)$ cycle of the fiber torus to degenerate at a finite location along the Lagrangian brane L_i , if we pick an f_i satisfying [16]

$$f_i \wedge v_i = 1, \quad (2.4.4)$$

where v_i is the degenerating cycle of the line L_i attached to. We note that by choosing one specific $f_i^{(0)}$ to be the canonical choice, then we have an ambiguity $f_i \rightarrow f_i - n v_i$ in this calculation. Let us denote the shifted f_i as $f_i^{(n)}$, then it satisfies

$$f_i^{(n)} \wedge f_i^{(0)} = n. \quad (2.4.5)$$

Without losing any generality, we can choose $v_1 = (-1, -1)$, $v_2 = (1, 0)$ and $v_3 = (0, 1)$ by using the $\text{SL}(2, \mathbb{Z})$ transformation in each patch, and we denote the framed vertex as

$$C_{\vec{k}^{(1)}, \vec{k}^{(2)}, \vec{k}^{(3)}}^{(f_1, f_2, f_3)}. \quad (2.4.6)$$

The trace can be decomposed into a different and more convenient representation basis with the Frobenius formula

$$\text{tr}_{\vec{k}} V = \sum_{\lambda: \text{Young tableau}} \chi_{\lambda}(c(\vec{k})) \text{tr}_{\lambda} V, \quad (2.4.7)$$

where $\chi_\lambda(c(\vec{k}))$ is the character of symmetric group S_ℓ with $\ell = \sum_j j k_j$ for the conjugacy class $c(\vec{k})$ in representation λ . We define

$$C_{\lambda_1, \lambda_2, \lambda_3}^{f_1, f_2, f_3} = \sum_{\vec{k}^{(1)}, \vec{k}^{(2)}, \vec{k}^{(3)}} C_{\vec{k}^{(1)}, \vec{k}^{(2)}, \vec{k}^{(3)}}^{f_1, f_2, f_3} \prod_i \frac{\chi_{\lambda_i}(c(\vec{k}^{(i)}))}{z_{\vec{k}^{(i)}}}, \quad (2.4.8)$$

and then the partition function is rewritten into

$$Z = \sum_{\lambda_1, \lambda_2, \lambda_3} C_{\lambda_1, \lambda_2, \lambda_3}^{f_1, f_2, f_3} \prod_{i=1}^3 \text{tr}_{\lambda_i} V_i. \quad (2.4.9)$$

The framing dependence of the vertex is known to be [54]

$$C_{\lambda_1, \lambda_2, \lambda_3}^{f_1 - n_1 v_1, f_2 - n_2 v_2, f_3 - n_3 v_3} = (-1)^{\sum_i n_i |\lambda_i|} q^{\frac{1}{2} \sum_i n_i \kappa(\lambda_i)} C_{\lambda_1, \lambda_2, \lambda_3}^{f_1, f_2, f_3}, \quad (2.4.10)$$

where $\kappa(\lambda) = \sum_i \lambda^{(i)}(\lambda^{(i)} - 2i + 1)$ with $\lambda^{(i)}$ the number of boxes in the i -th row of λ .

To glue two patches of toric variety, we cut the shared degenerate locus with a pair of Lagrangian D-branes/anti-D-branes, and the gluing of two subgraph Γ_L and Γ_R reads [16]

$$\sum_{\vec{k}} Z(\Gamma_L)_{\vec{k}} \frac{\exp(-\ell(\vec{k})t)}{\prod_j k_j! j^{k_j}} Z(\Gamma_R)_{\vec{k}}, \quad (2.4.11)$$

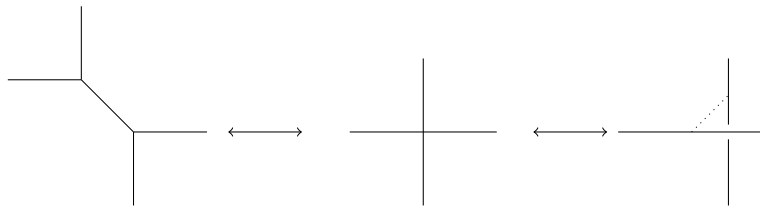
where $\ell(\vec{k}) = \sum_j j k_j$ is essentially $|\lambda|$ in the representation basis and t is the Kähler modulus of the degenerate locus. In the representation basis, we have

$$Z(\Gamma_L \times \Gamma_R) = \sum_{\lambda} Z(\Gamma_L)_{\lambda} (-e^{-t})^{|\lambda|} Z(\Gamma_R)_{\lambda^t}, \quad (2.4.12)$$

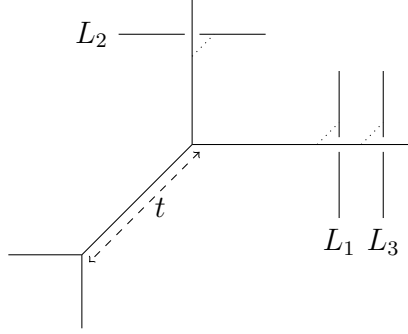
which follows directly from the orthogonality of the character,

$$\sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_{\lambda}(c(\vec{k})) \chi_{\lambda'}(c(\vec{k})) = \delta_{\lambda\lambda'}. \quad (2.4.13)$$

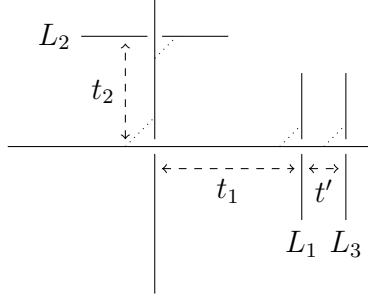
To derive the explicit expression of the vertex, we make use of the geometric transition, which can be interpreted as a transition between the local geometry of a resolved conifold and a deformed conifold. In the toric diagram, this transition is usually depicted as



We consider the following setup.



with three Lagrangian submanifold, $L_{1,2,3}$. Applying the geometric transition, and then taking t to be infinity, we obtain a configuration with four \mathbb{S}^3 's.



As suggested from the discussion in section 2.2, the above amplitude can be computed from the dual open string, the Chern-Simons theory, in which each pair of \mathbb{S}^3 along a degenerate line is mapped to the bi-fundamental operator [55]

$$\mathcal{O}(U, V; t) = \exp \left(- \sum_n \frac{Q^n}{n} \text{tr} U^n \text{tr} V^n \right), \quad (2.4.14)$$

where $Q = e^{-t}$ is the Kähler parameter of the corresponding degenerate line and U, V are holonomy operators. The above operator can also be rewritten to

$$\mathcal{O}(U, V; t) = \sum_{\lambda: \text{Young diagram}} (-Q)^{|\lambda|} \text{tr}_{\lambda} U \text{tr}_{\lambda^t} V. \quad (2.4.15)$$

The open string amplitude is given by the evaluation of

$$\sum_{\lambda_1, \lambda_2, \lambda', \lambda_3} (-1)^{|\lambda_1| + |\lambda_3| + |\lambda'|} Q_1^{|\lambda_1|} Q_2^{|\lambda_2|} Q'^{|\lambda'|} Q_3^{|\lambda_3|} \text{tr}_{\lambda_1^t} U_1 \text{tr}_{\lambda_2} U_2 \text{tr}_{\lambda_3^t} U_1 \text{tr}_{\lambda_1} \hat{V}_1 \text{tr}_{\lambda'^t} V_1 \text{tr}_{\lambda_2} V_2 \text{tr}_{\lambda' \otimes \lambda_3} V_3, \quad (2.4.16)$$

where $Q_3 = Q_2 Q'$ and \hat{V} represents an orientation-reversed holonomy operator. By absorbing the Kähler parameters into the source, we renormalize the amplitude to

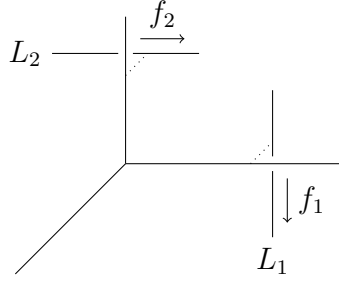
$$Z(V_1, V_2, V_3) = \frac{1}{S_{00}} \sum_{\lambda_1, \lambda_2, \lambda', \lambda_3} (-1)^{|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda'|} \langle \text{tr}_{\lambda_1^t} U_1 \text{tr}_{\lambda_3^t} U_1 \text{tr}_{\lambda_2} U_2 \rangle \text{tr}_{\lambda_1} \hat{V}_1 \text{tr}_{\lambda'^t} V_1 \text{tr}_{\lambda_2} V_2 \text{tr}_{\lambda' \otimes \lambda_3} V_3,$$

and by using (A.3.11) and (A.3.12),

$$Z(V_1, V_2, V_3) = \sum_{\lambda_1, \lambda_2, \lambda', \lambda_3} (-1)^{|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda'|} \frac{\mathcal{W}_{\lambda_3^t \lambda_2} \mathcal{W}_{\lambda_1^t \lambda_2}}{\mathcal{W}_{\lambda_2}} \text{tr}_{\lambda_1} \hat{V}_1 \text{tr}_{\lambda'^t} V_1 \text{tr}_{\lambda_2} V_2 \text{tr}_{\lambda' \otimes \lambda_3} V_3. \quad (2.4.17)$$

We have to be careful that we absorbed Kähler parameters into the source, so the partition functions of knots, $\mathcal{W}_{\lambda\lambda'}$'s in Appendix A.3, have to be rescaled when we use. Note that the dictionary reads $Q \leftrightarrow q^N$.

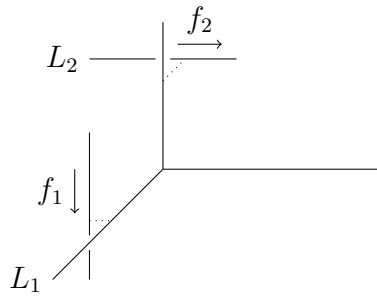
To work out the topological vertex, we need to somehow convert \hat{V}_1 in the partition function to V_1 . How to modify the above amplitude can be understood in two equivalent ways of the computation of $C_{\emptyset\lambda_1\lambda_2}$, i.e. the same setup without L_3 :



Repeating the similar evaluation via the open string, we obtain

$$Z(V_1, V_2) = \sum_{\lambda_1, \lambda_2} (-1)^{|\lambda_1|} \mathcal{W}_{\lambda_1^t \lambda_2} \text{tr}_{\lambda_1} \hat{V}_1 \text{tr}_{\lambda_2} V_2. \quad (2.4.18)$$

Permutating two of the three legs of the vertex, a framing dependence, $(-1)^{|\lambda_1|} q^{\kappa(\lambda_1)/2}$, is introduced into the expression of the partition function.



$$Z(V_1, V_2) = \sum_{\lambda_1, \lambda_2} q^{\kappa(\lambda_1)/2} \mathcal{W}_{\lambda_1 \lambda_2^t} \text{tr}_{\lambda_1} V_1 \text{tr}_{\lambda_2} V_2. \quad (2.4.19)$$

Applying the above rule to convert \hat{V}_1 to V_1 , we finally reach to the expression In the canonical frame we chose,

$$C_{\lambda_1, \lambda_2, \lambda_3} = (-1)^{|\lambda_2|} q^{-\frac{\kappa(\lambda_1)}{2}} \sum_{\nu_1, \nu_3, \nu} (-1)^{|\nu|+|\nu_3|} N_{\nu^t \nu_1}^{\lambda_1} N_{\nu \nu_3}^{\lambda_3} \frac{\mathcal{W}_{\nu_1 \lambda_2^t} \mathcal{W}_{\nu_3^t \lambda_2}}{\mathcal{W}_{\lambda_2}}. \quad (2.4.20)$$

As noted in Appendix A.3, coefficients $N_{\nu\mu}^\lambda$ coincide with the fusion coefficients $c_{\nu\mu}^\lambda$ of Schur functions, the topological vertex can thus be written in terms of the skew Schur functions as

$$C_{\lambda_1, \lambda_2, \lambda_3} = (-1)^{|\lambda_2|+|\lambda_3|} q^{-\frac{\kappa(\lambda_1)}{2}} s_{\lambda_2^t}(q^\rho) \sum_{\nu} s_{\lambda_1/\nu^t}(q^{\rho+\lambda_2^t}) s_{\lambda_3^t/\nu^t}(q^{\rho+\lambda_2}), \quad (2.4.21)$$

where we used $c_{\nu\mu^t}^\lambda = (-1)^{|\lambda|-|\nu|-|\mu|} c_{\nu^t\mu}^{\lambda^t}$ and

$$\mathcal{W}_{\lambda_1\lambda_2} = s_{\lambda_1}(q^{\rho+\lambda_2}) s_{\lambda_2}(q^\rho). \quad (2.4.22)$$

We can also use the identity,

$$s_{\lambda/\mu}(q^{\rho+\nu}) = (-1)^{|\lambda|-|\mu|} s_{\lambda^t/\mu^t}(q^{-\nu-\rho}), \quad (2.4.23)$$

to rewrite the topological vertex to

$$C_{\lambda_1, \lambda_2, \lambda_3} = (-1)^{|\lambda_1|} q^{-\frac{\kappa(\lambda_1)}{2}} s_{\lambda_2}(q^{-\rho}) \sum_{\nu} s_{\lambda_1^t/\nu}(q^{-\rho-\lambda_2^t}) s_{\lambda_3/\nu}(q^{-\rho-\lambda_2}). \quad (2.4.24)$$

The overall minus sign is completely conventional, as we can see that it cancels in the gluing of two vertices. We will simply omit it in the following sections.

Melting crystal model

The surprising fact that the topological vertex has an alternative interpretation as the partition function of the melting crystal model was first pointed out in [56], and it turned out to be an important step towards the refinement of the vertex, which will be discussed in the next section.

The partition function of the melting crystal model is given by

$$Z = \sum_{\pi} q^{|\pi|}, \quad (2.4.25)$$

where π is the partition of the molten part (disappearing part) of the corner of a crystal mapped to a plane partition. The plane partition is usually characterized by its asymptotes in three directions, i.e. with three Young diagrams. These Young diagrams correspond to the labels of the topological vertex.

Let us set the asymptotes in x , y and z directions respectively to be μ , ν , and λ . This time the amplitude is given by the number of boxes in the plane partition without counting the asymptotes. We slice the plane partition in the z direction and along the 45° line in the x - y plane, and denote the obtained Young diagram which does not contain the z -asymptote as $\sigma(t)$, $t \in (-N_1, N_2)$, with $\sigma(-N_1) = \mu^t$ and $\sigma(N_2) = \nu$. Then

$$Z(\mu, \nu, \lambda) = \lim_{N_1, N_2 \rightarrow \infty} q^{-N_1|\mu|-N_2|\nu|} \left(\sum_{\sigma(t)} \sum_{t=-N_1}^{N_2} q^{|\sigma(t)|} \right) q^{-\sum_i \frac{\mu_i(\mu_i-1)}{2} - \sum_i \frac{\nu_i^t(\nu_i^t-1)}{2}}. \quad (2.4.26)$$

The last factor compensates additional boxes counted in the slice along the 45° line. The above expression can in fact be reorganized into an expression in the form of an expectation value of the vertex operators,

$$V_{\pm}(\vec{x}) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_i x_i^n J_{\pm n} \right). \quad (2.4.27)$$

The detailed properties of these vertex operators and their relation with the Schur function can be found in Appendix A.2. The key formulae we will use are

$$V_{-}(1) |\mu\rangle = \sum_{\nu \succeq \mu} |\nu\rangle, \quad V_{+}(1) |\mu\rangle = \sum_{\nu \preceq \mu} |\nu\rangle. \quad (2.4.28)$$

They obey from the rewriting of (A.2.4),

$$V_{+}(\vec{x}) |Y\rangle = \sum_{R \preceq Y} s_{Y/R}(\vec{x}) |R\rangle, \quad V_{-}(\vec{x}) |R\rangle = \sum_{Y \succeq R} s_{Y/R}(\vec{x}) |Y\rangle, \quad (2.4.29)$$

and $s_{Y/R}(1) = 1$ for $Y \succeq R$. We consider first the case $\lambda = \emptyset$ for simplicity. We have

$$\sum_{\sigma(t)} \sum_{t=-N_1}^{N_2} q^{|\sigma(t)|} = \langle \nu | q^{L_0} V_{+}(1) q^{L_0} \dots q^{L_0} V_{+}(1) q^{L_0} V_{-}(1) q^{L_0} \dots q^{L_0} V_{-}(1) q^{L_0} | \mu^t \rangle. \quad (2.4.30)$$

By using

$$q^{aL_0} V_{\pm}(1) = V_{\pm}(q^{\mp a}) q^{aL_0}, \quad (2.4.31)$$

the above equation reduces to

$$\sum_{\sigma(t)} \sum_{t=-N_1}^{N_2} q^{|\sigma(t)|} = q^{(N_1-1/2)|\mu| + (N_2-1/2)|\nu|} \langle \nu | V_{+}(q^{-\rho(N_1-1)}) V_{-}(q^{-\rho(N_2-1)}) | \mu^t \rangle, \quad (2.4.32)$$

where $q^{-\rho(n)} := \{q^{1/2}, q^{3/2}, \dots, q^{n-1/2}\}$. Taking the large N_1, N_2 limit, we finally obtain

$$Z(\mu, \nu, \emptyset) = q^{-\sum_i \frac{\mu_i^t(\mu_i^t-2)}{2} - \sum_i \frac{\nu_i(\nu_i-2)}{2}} \sum_{\eta'} s_{\eta'/\mu^t}(q^{-\rho}) s_{\eta'/\nu}(q^{-\rho}), \quad (2.4.33)$$

where $q^{-\rho} = q^{-\rho(\infty)}$. Especially, when $\mu = \nu = \emptyset$,

$$Z(\emptyset, \emptyset, \emptyset) = \sum_{\eta'} s_{\eta'}(q^{-\rho}) s_{\eta'}(q^{-\rho}) = \prod_{n=1}^{\infty} (1 - q^n)^{-n}, \quad (2.4.34)$$

is the MacMahon function, well known as the partition function of the plane partition with empty asymptotes. We need to divide out this overall factor, as $C_{\emptyset, \emptyset, \emptyset} = 1$. We can easily see from (A.2.12) that

$$Z(\mu, \nu, \emptyset) / Z(\emptyset, \emptyset, \emptyset) = q^{-\sum_i \frac{\mu_i^t(\mu_i^t-2)}{2} - \sum_i \frac{\nu_i(\nu_i-2)}{2}} \sum_{\eta} s_{\mu^t/\eta}(q^{-\rho}) s_{\nu/\eta}(q^{-\rho}). \quad (2.4.35)$$

The second point to be careful with is that $C_{\mu,\nu,\lambda}$ is symmetric under permutations among μ , ν , and λ . We set

$$C_{\mu,\nu,\emptyset} = q^{\sum_i \frac{\mu_i(\mu_i-2)}{2} + \sum_i \frac{\nu_i(\nu_i-2)}{2}} Z(\mu, \nu, \emptyset) / Z(\emptyset, \emptyset, \emptyset) = q^{-\frac{\kappa(\mu)}{2}} \sum_{\eta} s_{\mu^t/\eta}(q^{-\rho}) s_{\nu/\eta}(q^{-\rho}). \quad (2.4.36)$$

For non-trivial λ , it is easy to see that whenever we encounter a corner in the base frame, we have to switch V_{\pm} to V_{\mp} in analogy of the evaluation in (2.4.30). Let us give two concrete examples here. The first is the simplest one, $\lambda = (1)$, for which we have

$$\begin{aligned} \sum_{\sigma(t)} \sum_{t=-N_1}^{N_2} q^{|\sigma(t)|} &= \langle \nu | \underbrace{q^{L_0} V_+(1) q^{L_0} \dots q^{L_0} V_+(1)}_{N_2-2} q^{L_0} V_-(1) q^{L_0} V_+(1) q^{L_0} \underbrace{V_-(1) q^{L_0} \dots q^{L_0} V_-(1) q^{L_0}}_{N_1-2} | \mu^t \rangle \\ &= q^{(N_1-1/2)|\mu| + (N_2-1/2)|\nu|} \langle \nu | V_+(q^{-\rho_2^{N_1-1}}) V_-(q^{-1/2}) V_+(q^{-1/2}) V_-(q^{-\rho_2^{N_2-1}}) | \mu^t \rangle, \end{aligned}$$

where we introduced a new notation, $q^{\rho_n} = \{q^{m-1/2}, q^{m+1/2}, \dots, q^{n-1/2}\}$. We can further deform the above equation to

$$\sum_{\sigma(t)} \sum_{t=-N_1}^{N_2} q^{|\sigma(t)|} = q^{(N_1-1/2)|\mu| + (N_2-1/2)|\nu|} (1 - q^{-1}) \langle \nu | V_+(q^{-\rho - \{1,0,0,\dots\}}) V_-(q^{-\rho - \{1,0,0,\dots\}}) | \mu^t \rangle$$

Now we switch gear to consider the next example $\lambda = (2)$. It is straightforward to write down

$$\begin{aligned} \sum_{\sigma(t)} \sum_{t=-N_1}^{N_2} q^{|\sigma(t)|} &= \langle \nu | \underbrace{q^{L_0} V_+(1) q^{L_0} \dots q^{L_0} V_+(1)}_{N_2-2} q^{L_0} V_-(1) q^{L_0} V_-(1) q^{L_0} V_+(1) q^{L_0} \underbrace{V_-(1) q^{L_0} \dots q^{L_0} V_-(1) q^{L_0}}_{N_1-3} | \mu^t \rangle \\ &\propto \langle \nu | V_+(q^{-\rho - \{2,0,0,\dots\}}) V_-(q^{-\rho - \{1,1,0,0,\dots\}}) | \mu^t \rangle. \end{aligned}$$

We can easily generalize this computation to conclude

$$C_{\mu,\nu,\lambda} \propto q^{-\frac{\kappa(\mu)}{2}} \sum_{\eta} s_{\mu^t/\eta}(q^{-\lambda-\rho}) s_{\nu/\eta}(q^{-\lambda^t-\rho}), \quad (2.4.37)$$

where the proportional prefactor only depends on λ . By using the symmetric property, $C_{\emptyset,\emptyset,\lambda} = C_{\emptyset,\lambda,\emptyset}$, we finally arrive at

$$C_{\mu,\nu,\lambda} = q^{-\frac{\kappa(\mu)}{2}} s_{\lambda}(q^{-\rho}) \sum_{\eta} s_{\mu^t/\eta}(q^{-\lambda-\rho}) s_{\nu/\eta}(q^{-\lambda^t-\rho}). \quad (2.4.38)$$

2.5 Refinement

In this section, we derive the expression of the refined topological vertex based on the relation with the melting crystal presented in the previous section. The word refinement means to add one

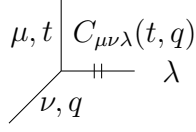


Figure 2.1: The refined topological vertex with labels. We use two short parallel lines to denote the preferred direction.

more parameter t in addition to the string coupling parameter $q = e^{-g_s}$. We build the vertex so that when $t \rightarrow q$, it reduces to the original topological vertex, (2.4.38), which will be called the unrefined topological vertex in the remaining part of this article. As we will see from the discussion in section 2.6, the refined topological string is dual to 5d $\mathcal{N} = 1$ gauge theories of type $\mathcal{T}_{(A,A)}$ on the Ω -background with $q_1 = t$ and $q_2 = q^{-1}$. The refinement of the topological string allows us to consider 5d gauge theories on a generic Ω -background, while the original unrefined topological string only gives us an approach to the self-dual Ω -background satisfying $q_1 q_2 = 1$.

The refinement of the topological vertex given in [57] does not have worldsheet description, but in stead, it is established in the melting crystal picture. The modification of (2.4.30) is given by

$$\langle \nu | q^{L_0} V_+(1) q^{L_0} \dots q^{L_0} V_+(1) t^{L_0} V_-(1) t^{L_0} \dots t^{L_0} V_-(1) t^{L_0} | \mu \rangle \propto \prod_{i,j=1}^{\infty} (1 - t^i q^{j-1})^{-1}. \quad (2.5.1)$$

The proper prescription is that whenever we switch $V_-(1)$ to $V_+(1)$, t^{L_0} has to be replaced by q^{L_0} . With this prescription, we can immediately see that the refined topological vertex is proportional to

$$C_{\mu,\nu,\lambda}(t, q) \propto \sum_{\eta} \left(\frac{q}{t} \right)^{\frac{|\eta| + |\mu| - |\nu|}{2}} s_{\mu/\eta}(q^{-\lambda} t^{-\rho}) s_{\nu/\eta}(t^{-\lambda} q^{-\rho}). \quad (2.5.2)$$

The prefactor depends on our precise definition of the vertex. We have lost the symmetry among three legs, so the only obvious guiding principle is that $C_{\mu,\nu,\lambda}(t, q)$ reduces back to $C_{\mu,\nu,\lambda}$ when we set $t = q$. The leg corresponding to λ is called the preferred direction of the refined topological vertex, which is also the slicing direction of the melting crystal. For convenience, let us name the legs of the vertex with Young diagram μ , ν , and λ respectively by t -direction, q -direction and the preferred direction as in Figure 2.1. The t - and q -direction exchange when we replace the parameter $t \leftrightarrow q$ in the refined topological vertex.

We set the gluing rule that only pairs of legs that are both in the preferred direction, or one is in the t -direction and another is in the q -direction can be glued together.

Let us consider the configurations shown in Figure 2.2, which are two equivalent but different ways to resolve the conifold singularity, i.e. connected by the geometric transition. We have to



Figure 2.2: Two ways of resolution of the conifold.

impose

$$\sum_{\lambda} (-Q)^{|\lambda|} C_{\emptyset\lambda\emptyset}(t, q) C_{\emptyset\lambda^t\emptyset}(q, t) = \sum_{\lambda} (-Q)^{|\lambda|} C_{\lambda\emptyset\emptyset}(q, t) C_{\lambda^t\emptyset\emptyset}(t, q). \quad (2.5.3)$$

We require that the prefactor, as in the unrefined case, has no dependence on the second Young diagram, ν , and factorizes into two parts of μ and λ . Let us denote the prefactor as

$$C_{\mu,\nu,\lambda}(t, q) = \tilde{h}_{\mu}(t, q) \tilde{Z}_{\lambda}(t, q) \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta|+|\mu|-|\nu|}{2}} s_{\mu^t/\eta}(q^{-\lambda} t^{-\rho}) s_{\nu/\eta}(t^{-\lambda^t} q^{-\rho}). \quad (2.5.4)$$

By writing down explicit expressions, (2.5.3) becomes

$$\sum_{\lambda} (-Q)^{|\lambda|} s_{\lambda}(q^{-\rho}) s_{\lambda^t}(t^{-\rho}) = \sum_{\lambda} (-Q)^{|\lambda|} \tilde{h}_{\mu}(q, t) \tilde{h}_{\mu^t}(t, q) s_{\lambda}(q^{-\rho}) s_{\lambda^t}(t^{-\rho}), \quad (2.5.5)$$

and therefore

$$\tilde{h}_{\mu^t}(q, t) \tilde{h}_{\mu}(t, q) = 1. \quad (2.5.6)$$

Setting

$$\tilde{h}_{\mu}(t, q) = h_{\mu}(q/t) q^{-\frac{\kappa(\mu)}{2}},$$

we obtain

$$h_{\mu^t}(t/q) h_{\mu}(q/t) (t/q)^{\frac{\kappa(\mu)}{2}} = 1. \quad (2.5.7)$$

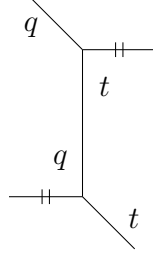
This reminds us of the identity,

$$||\mu||^2 + \kappa(\mu) = ||\mu^t||^2. \quad (2.5.8)$$

We note that the d.o.f. of $(q/t)^{|\mu|}$ in $h_{\mu}(q/t)$ can be absorbed into the framing factor $f_{\mu}(t, q)$. We can thus choose

$$\tilde{h}_{\mu}(t, q) = q^{\frac{||\mu^t||^2}{2}} t^{-\frac{||\mu||^2}{2}}. \quad (2.5.9)$$

\tilde{Z}_{λ} can be fixed from yet one more equivalent toric diagram for the resolved conifold.

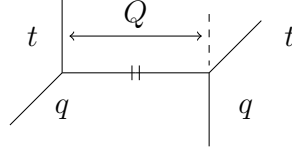


Its partition function is given by

$$\sum_{\lambda} (-Q)^{|\lambda|} s_{\lambda}(q^{-\rho}) s_{\lambda^t}(t^{-\rho}) = \prod_{i,j=1}^{\infty} (1 - Q q^{i-1/2} t^{j-1/2}) = \exp \left(- \sum_{n=1}^{\infty} \frac{(Q \sqrt{t/q})^n}{n(1 - q^{-n})(1 - t^n)} \right). \quad (2.5.10)$$

This is exactly the Cauchy identity of the Schur function, and a natural candidate for refinement is to realize the same factor with Macdonald function. That is to say, we would like to further impose the constraint that the partition function does not depend on the choice of the preferred direction in the toric diagram.

From (A.4.7), we see that the natural candidate for $\tilde{Z}_{\lambda}(q, t)$ is $P_{\lambda}(t^{-\rho}, q, t)$, with which the calculation of



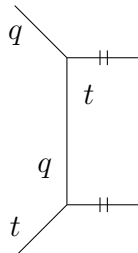
can be carried out as

$$\sum_{\lambda} (-Q)^{|\lambda|} C_{\emptyset, \emptyset, \lambda}(t, q) C_{\emptyset, \emptyset, \lambda^t}(q, t) = \sum_{\lambda} (-Q)^{|\lambda|} P_{\lambda}(t^{-\rho}, q, t) P_{\lambda^t}(q^{-\rho}, t, q) = \prod_{i,j=1}^{\infty} (1 - Q q^{i-1/2} t^{j-1/2}). \quad (2.5.11)$$

There is some ambiguity in the particular choice of $\tilde{Z}_{\lambda}(q, t)$, but it is only conventional. Finally we obtained the full expression for the refined topological vertex as

$$\boxed{C_{\mu, \nu, \lambda}(t, q) = q^{\frac{\|\mu^t\|^2}{2}} t^{-\frac{\|\mu\|^2}{2}} P_{\lambda}(t^{-\rho}, q, t) \sum_{\eta} \left(\frac{q}{t} \right)^{\frac{|\eta| + |\mu| - |\nu|}{2}} s_{\mu^t/\eta}(q^{-\lambda} t^{-\rho}) s_{\nu/\eta}(t^{-\lambda^t} q^{-\rho}).}$$

So far, we have only considered the toric geometry of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$, let us examine the example $\mathcal{O}(0) \oplus \mathcal{O}(-2) \rightarrow \mathbb{CP}^1$ for the independence of choice of preferred direction.



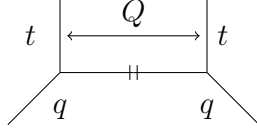
The partition function can be computed as

$$\sum_{\nu} (-Q)^{|\nu|} s_{\nu}(t^{-\rho}) s_{\nu}(q^{-\rho}) q^{-\frac{\|\nu^t\|^2}{2}} t^{\frac{\|\nu\|^2}{2}} f_{\nu}(q, t)^{-1} = \prod_{i,j=1}^{\infty} \frac{1}{1 - Q t^i q^{j-1}}, \quad (2.5.12)$$

where we used the expression for the framing factor [58]

$$f_{\lambda}(t, q) = (-1)^{|\lambda|} t^{-n(\lambda)} q^{n(\lambda^t)}. \quad (2.5.13)$$

We note that this framing factor reduces to the framing discussed in the previous section in the $t \rightarrow q$ limit. Changing the preferred direction,



the partition function is given by

$$\sum_{\lambda} (-Q)^{|\lambda|} P_{\lambda}(t^{-\rho}, q, t) P_{\lambda^t}(q^{-\rho}, t, q) f_{\lambda}(t, q)^{-1}. \quad (2.5.14)$$

Note that from (A.4.7),

$$P_{\lambda}(t^{-\rho}, q, t) = t^{\frac{|\lambda|}{2} + n(\lambda)} \prod_{(i,j) \in \lambda} \frac{1}{1 - q^{a(i,j)} t^{\ell(i,j)+1}}, \quad (2.5.15)$$

and from (A.4.11), we have

$$\begin{aligned} & \sum_{\lambda} (-Q)^{|\lambda|} P_{\lambda}(t^{-\rho}, q, t) P_{\lambda^t}(q^{-\rho}, t, q) f_{\lambda}(t, q)^{-1} \\ &= \sum_{\lambda} Q^{|\lambda|} (t/q)^{\frac{|\lambda|}{2}} P_{\lambda}(t^{-\rho}, q, t) Q_{\lambda}(t^{-\rho}, q, t) \\ &= \prod_{i,j=1}^{\infty} \frac{1}{1 - Q q^{j-1-1/2} t^{i+1/2}}. \end{aligned} \quad (2.5.16)$$

We see that the above two ways of computing the partition function of $\mathcal{O}(0) \oplus \mathcal{O}(-2) \rightarrow \mathbb{CP}^1$ does not depend on the choice of preferred direction up to a rescaling of the Kähler parameter $Q \rightarrow Q \sqrt{t/q}$. Therefore, we have to loosen the constraint on independence of the preferred direction.

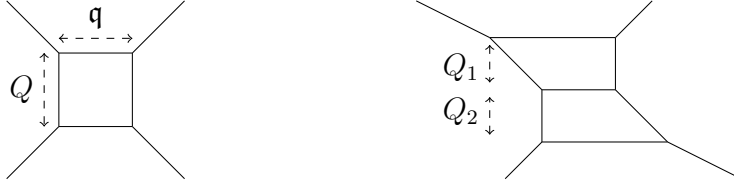


Figure 2.3: (a) toric diagram corresponding to $\mathcal{O}(-2, -2) \rightarrow \mathbb{CP}^1$ with one rhombus in it. (b) configuration with two rhombi.

2.6 Building blocks and dualities

Let us evaluate the partition function of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ as

$$\begin{aligned} \sum_{\lambda} (-Q)^{|\lambda|} P_{\lambda}(t^{-\rho}, q, t) P_{\lambda^t}(q^{-\rho}, t, q) &= \sum_{\lambda} (Q\sqrt{q/t})^{|\lambda|} \prod_{(i,j) \in \lambda} \frac{1}{(1 - q^{-a(i,j)} t^{-\ell(i,j)-1})(1 - t^{\ell(i,j)} q^{a(i,j)+1})} \\ &= \sum_{\lambda} (Q\sqrt{q/t})^{|\lambda|} \frac{1}{N_{\lambda\lambda}(1; t, q^{-1})}. \end{aligned} \quad (2.6.1)$$

This is exactly the $U(1)$ instanton partition function of 5d gauge theory on the Omega-background, $\Omega_{t, q^{-1}} \times \mathbb{S}^1$ [59]. $q_1 = t =: e^{\epsilon_1 R}$ and $q_2 = q^{-1} =: e^{\epsilon_2 R}$ are two Omega-background parameters. In the remaining of this article, we will stick to this convention and use $q_{1,2}$ alternatively to represent t and q^{-1} .

Similarly, the partition function of $\mathcal{O}(0) \oplus \mathcal{O}(-2) \rightarrow \mathbb{CP}^1$ is given by,

$$\begin{aligned} &\sum_{\lambda} Q^{|\lambda|} (tq)^{\frac{|\lambda|}{2}} t^{2n(\lambda)} \prod_{(i,j) \in \lambda} \frac{1}{(1 - q^{a(i,j)} t^{\ell(i,j)+1})(1 - t^{\ell(i,j)} q^{a(i,j)+1})} \\ &= \sum_{\lambda} (-Q)^{|\lambda|} (q/t)^{\frac{|\lambda|}{2}} \prod_{(i,j) \in \lambda} \frac{t^{i-1} q^{-j+1}}{(1 - q^{-a(i,j)} t^{-\ell(i,j)-1})(1 - t^{\ell(i,j)} q^{a(i,j)+1})} \\ &= \sum_{\lambda} (-Q)^{|\lambda|} (q/t)^{\frac{|\lambda|}{2}} \frac{\prod_{(i,j) \in \lambda} t^{i-1} q^{-j+1}}{N_{\lambda\lambda}(1; t, q^{-1})}. \end{aligned} \quad (2.6.2)$$

It matches with the 5d $U(1)$ instanton partition function with non-trivial Chern-Simons level $\kappa_{cs} = 1$.

Let us further consider two fundamental classes of toric diagrams. The first one is with the shape of a pile of $n - 1$ rhombi (see Figure 2.3).

The partition function for $n = 2$ in the first class can be divided into four parts. We first consider

the gluing of vertices on the left vertical internal line.

$$\begin{aligned} \sum_{\nu} (-Q)^{|\nu|} q^{\frac{||\nu||^2}{2}} t^{-\frac{||\nu^t||^2}{2}} f_{\nu}(t, q)^{-1} s_{\nu}(t^{-\lambda_1^t} q^{-\rho}) s_{\nu}(q^{-\lambda_2} t^{-\rho}) &= \sum_{\nu} (Q \sqrt{q/t})^{|\nu|} s_{\nu}(t^{-\lambda_1^t} q^{-\rho}) s_{\nu}(q^{-\lambda_2} t^{-\rho}) \\ &= \prod_{i,j=1}^{\infty} \frac{1}{1 - Q t^{-\lambda_{1,i}^t + j - 1} q^{-\lambda_{2,j} + i}} = \frac{1}{\mathcal{G}(Q; t, q^{-1}) N_{\lambda_1 \lambda_2}(Q; t, q^{-1})}, \end{aligned} \quad (2.6.3)$$

where we defined the \mathcal{G} factor as

$$\mathcal{G}(Q; q_1, q_2) = \prod_{i,j=1}^{\infty} (1 - Q q_1^{i-1} q_2^{-j}). \quad (2.6.4)$$

We note that $\mathcal{G}(Q \sqrt{q_1 q_2}; q_1, q_2)$ is exactly the pure $U(1)$ instanton partition function. The gluing of vertices on the right vertical internal line reads

$$\begin{aligned} \sum_{\mu} (-Q)^{|\mu|} q^{-\frac{||\mu||^2}{2}} t^{\frac{||\mu^t||^2}{2}} f_{\mu}(t, q) s_{\mu^t}(t^{-\lambda_1^t} q^{-\rho}) s_{\mu^t}(q^{-\lambda_2} t^{-\rho}) &= \sum_{\mu} (Q \sqrt{t/q})^{|\mu|} s_{\mu^t}(t^{-\lambda_1^t} q^{-\rho}) s_{\mu^t}(q^{-\lambda_2} t^{-\rho}) \\ &= \prod_{i,j=1}^{\infty} \frac{1}{1 - Q t^{-\lambda_{1,i}^t + j} q^{-\lambda_{2,j} + i - 1}} = \frac{1}{\mathcal{G}(Qt/q; t, q^{-1}) N_{\lambda_1 \lambda_2}(Qt/q; t, q^{-1})}. \end{aligned} \quad (2.6.5)$$

The remaining factors are from the gluing of horizontal lines, which is locally of the geometry $\mathcal{O}(0) \otimes \mathcal{O}(-2) \rightarrow \mathbb{CP}^1$, and have already been computed. The total partition function is given by¹

$$\begin{aligned} Z_{\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1}(Q, \mathbf{q}) &= \frac{1}{\mathcal{G}(Q; t, q^{-1}) \mathcal{G}(Qq/t; t, q^{-1})} \sum_{\lambda_{1,2}} (\mathbf{q})^{|\lambda_1| + |\lambda_2|} t^{n(\lambda_1) - n(\lambda_2)} q^{-n(\lambda_1^t) + n(\lambda_2^t)} \\ &\quad \times (\sqrt{q/t})^{|\lambda_1| + |\lambda_2|} \frac{1}{N_{\lambda_1 \lambda_1}(1; t, q^{-1}) N_{\lambda_2 \lambda_2}(1; t, q^{-1}) N_{\lambda_1 \lambda_2}(Qq/t; t, q^{-1}) N_{\lambda_1 \lambda_2}(Q; t, q^{-1})} \\ &= \frac{1}{\mathcal{G}(Q; t, q^{-1}) \mathcal{G}(Qq/t; t, q^{-1})} \\ &\quad \times \sum_{\lambda_{1,2}} \frac{(\mathbf{q} Q^{-1} q_3^{-1/2})^{|\lambda_1| + |\lambda_2|}}{N_{\lambda_1 \lambda_1}(1; t, q^{-1}) N_{\lambda_2 \lambda_2}(1; t, q^{-1}) N_{\lambda_1 \lambda_2}(Q; t, q^{-1}) N_{\lambda_2 \lambda_1}(Q^{-1}; t, q^{-1})}, \end{aligned} \quad (2.6.6)$$

where we used the identity (A.1.7),

$$N_{\lambda_1 \lambda_2}(Qq_3; q_1, q_2) = (-Qq_3)^{|\lambda_1| + |\lambda_2|} q_1^{n(\lambda_1) - n(\lambda_2)} q_2^{n(\lambda_1^t) - n(\lambda_2^t)} N_{\lambda_2 \lambda_1}(Q^{-1}; q_1, q_2), \quad (2.6.7)$$

with $q_3 := q_1^{-1} q_2^{-1} = q/t$. The above partition function is exactly the 5d $SU(2)$ instanton partition function on the Omega-background, up to \mathcal{G} -factors, which contain part of the perturbative contributions.

¹For our convenience, we shifted the Kähler parameter as $Q \rightarrow Qq/t$.

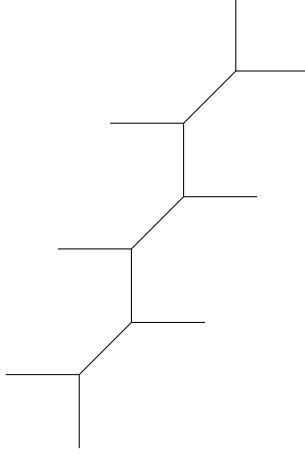


Figure 2.4: A strip with $2N = 6$ preferred legs.

The exact same computation can be done for Figure 2.3-(b). The gluing of left internal lines reads

$$\begin{aligned}
& \sum_{\nu, \eta, \tau} (-Q_1)^{|\nu|} q^{\frac{\|\nu\|^2}{2}} t^{-\frac{\|\nu^t\|^2}{2}} f_\nu(t, q)^{-1} s_\nu(t^{-\lambda_1^t} q^{-\rho}) s_{\nu/\eta}(q^{-\lambda_2} t^{-\rho}) (q/t)^{\frac{|\eta|}{2}} s_{\tau/\eta}(t^{-\lambda_2^t} q^{-\rho}) \\
& \quad \times (-Q_2)^{|\tau|} q^{\frac{\|\tau\|^2}{2}} t^{-\frac{\|\tau^t\|^2}{2}} f_\tau(t, q)^{-1} s_\tau(q^{-\lambda_3} t^{-\rho}) \\
& = \frac{1}{\mathcal{G}(Q_1 q/t; t, q^{-1}) N_{\lambda_1 \lambda_2}(Q_1 q/t; t, q^{-1})} \frac{1}{\mathcal{G}(Q_2 q/t; t, q^{-1}) N_{\lambda_2 \lambda_3}(Q_2 q/t; t, q^{-1})} \\
& \quad \times \sum_{\eta} (q/t)^{\frac{|\eta|}{2}} s_\eta(Q_1 t^{-\lambda_1^t} q^{-\rho}) s_\tau(Q_2 q^{-\lambda_3} t^{-\rho}) \\
& = \frac{1}{\mathcal{G}(Q_1 q/t; t, q^{-1}) N_{\lambda_1 \lambda_2}(Q_1 q/t; t, q^{-1})} \frac{1}{\mathcal{G}(Q_2 q/t; t, q^{-1}) N_{\lambda_2 \lambda_3}(Q_2 q/t; t, q^{-1})} \\
& \quad \times \frac{1}{\mathcal{G}(Q_1 Q_2 q/t; t, q^{-1}) N_{\lambda_1 \lambda_3}(Q_1 Q_2 q/t; t, q^{-1})}. \tag{2.6.8}
\end{aligned}$$

We obtain Nekrasov factors from the contraction of three vertices, where the contraction of the most upper and lowest vertices has Kähler parameter $Q_1 Q_2$. The same generalization is applied for the right gluing. We need to (A.1.7) for three times now to match with $SU(3)$ instanton partition function. The factor $q_1^{2n(\lambda_1)-2n(\lambda_3)} q_2^{2n(\lambda_1^t)-2n(\lambda_3^t)}$ from this conversion exactly cancels the framing factor $f_{\lambda_1}^2(t, q) f_{\lambda_3}^{-2}(t, q)$ in the gluing of the horizontal direction. Indeed, we see that the resulting partition function up to \mathcal{G} -factors, matches with pure 5d $\mathcal{N} = 1$ pure $SU(3)$ instanton partition function.

Another class of toric diagrams we want to consider is shown in Figure 2.4. In general we can add arbitrary number of vertices to have $2N$ legs in the preferred direction. We denote the Young diagram label for legs on the right of the strip as σ_i , and those of the legs on the left as λ_i^t .

Let us perform the calculation for $N = 2$ explicitly. We omit the factors that will be used in

	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—	—	—	•	•	•	•
NS5	—	—	—	—	—	•	—	•	•	•
7-brane	—	—	—	—	—	•	•	—	—	—

Table 2.1: Configuration of branes in the brane web construction. Bar represents the direction branes stretch along, and dot means the point-like direction for branes.

the horizontal gluing and absorbed into Nekrasov factors from the preferred direction.

$$\begin{aligned}
Z_{N=2} &= \sum_{\eta_{1,2}} (q/t)^{\frac{-|\eta_1|}{2} + \frac{|\eta_2| - |\epsilon_2|}{2}} \sum_{\nu_1} (-Q_1)^{|\nu_1|} s_{\nu_1}(t^{-\sigma_1^t} q^{-\rho}) s_{\nu_1^t/\eta_1}(q^{-\lambda_1} t^{-\rho}) \\
&\times \sum_{\nu_2} (-Q_2)^{|\nu_2|} s_{\nu_2^t/\eta_1}(t^{-\lambda_1^t} q^{-\rho}) s_{\nu_2/\eta_2}(q^{-\sigma_2} t^{-\rho}) \\
&\times \sum_{\nu_3} (-Q_3)^{|\nu_3|} s_{\nu_3/\eta_2}(t^{-\sigma_2^t} q^{-\rho}) s_{\nu_3^t}(q^{-\lambda_2} t^{-\rho}) \\
&= \prod_{i,j=1}^{\infty} \left(1 - Q_1 \gamma q^{-\lambda_{1,i}+j} t^{-\sigma_{1,j}^t+i-1}\right) \left(1 - Q_2 \gamma q^{-\sigma_{2,i}+j} t^{-\lambda_{1,j}^t+i-1}\right) \left(1 - Q_3 \gamma q^{-\lambda_{2,i}+j} t^{-\sigma_{2,j}^t+i-1}\right) \\
&\times \prod_{i,j=1}^{\infty} \frac{\left(1 - Q_1 Q_2 Q_3 \gamma q^{-\lambda_{2,i}+j} t^{-\sigma_{1,j}^t+i-1}\right)}{\left(1 - Q_2 Q_3 q^{-\lambda_{2,i}+j} t^{-\lambda_{1,j}^t+i-1}\right) \left(1 - Q_1 Q_2 q_3 q^{-\sigma_{2,i}+j} t^{-\sigma_{1,j}^t+i-1}\right)} \\
&= \mathcal{G}(Q_1 q/t; t, q^{-1}) N_{\sigma_1 \lambda_1}(Q_1 q/t; t, q^{-1}) \mathcal{G}(Q_2 q/t; t, q^{-1}) N_{\lambda_1 \sigma_2}(Q_2 q/t; t, q^{-1}) \\
&\times \mathcal{G}(Q_3 q/t; t, q^{-1}) N_{\sigma_2 \lambda_2}(Q_3 q/t; t, q^{-1}) \mathcal{G}(Q_1 Q_2 Q_3 q/t; t, q^{-1}) N_{\sigma_1 \lambda_2}(Q_1 Q_2 Q_3 q/t; t, q^{-1}) \\
&\times \frac{1}{\mathcal{G}(Q_2 Q_3; t, q^{-1}) N_{\lambda_1 \lambda_2}(Q_2 Q_3; t, q^{-1}) \mathcal{G}(Q_1 Q_2; t, q^{-1}) N_{\sigma_1 \sigma_2}(Q_1 Q_2; t, q^{-1})}. \tag{2.6.9}
\end{aligned}$$

We see the Nekrasov factors showing up in the calculation which correspond to contributions of bifundamental hypermultiplets between two gauge groups. Gluing strips like this together, we will have the partition function corresponding to 5d gauge theories with $\dots \text{SU}(N) \times \text{SU}(N) \dots$ gauge groups.

From the above explicit calculations, we see the connection between topological strings and 5d $\mathcal{N} = 1$ gauge theories. In general, it was argued in [18] that the topological string on Calabi-Yau X captures the BPS spectrum of M-theory on the same Calabi-Yau X . Especially, when X is a toric Calabi-Yau, we have M-theory on a torus fibered over some base space, and it can be dualized to type IIB string theory on some geometry with \mathbb{S}^1 circle. This geometry is a Taub-NUT space, and it is further dual to a (p, q) -brane web introduced in [17]. The brane configuration is given in Table 2.1 with D5-branes separated along the 6-direction and NS5-branes separated along the 5-direction. It is conventional to add a brane diagram in addition to the table 2.1 to specify the non-trivial information of brane configurations on the 5-6 plane (see Figure 2.5). Due to the conservation of the axio-dilaton charge and the balance of tension, there in fact exist various kind of 5-branes with

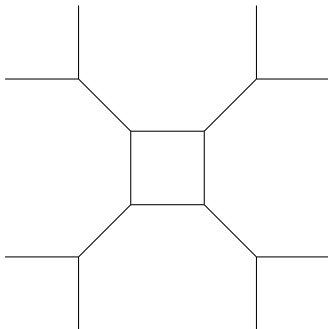


Figure 2.5: An example of the (p, q) brane web, where D5-branes are represented as horizontal lines and NS5-branes are depicted as vertical lines. When there are N D5 branes sandwiched between two NS5-branes, we will have an $SU(N)$ gauge field in the 5d effective low energy theory.

(p, q) axio-dilaton charges stretching along the (p, q) direction of the 5-6 plane in this setup. This is why the brane diagram here is also called a (p, q) -brane web. One can end a 5-brane in the (p, q) -brane web at a finite position in the 5-6 plane with a 7-brane², but we only focus on the cases dual to M-theory on toric Calabi-Yau, where external 5-branes extend to the infinity, in this article. The (p, q) -brane web is mapped to the toric diagram of the toric Calabi-Yau on which M-theory is compactified in this duality. In the field theory limit, the low energy effective theory on the brane web reduces to 5d gauge theories with $\mathcal{N} = 1$ supersymmetry. The relatively surprising fact here is that the refinement we have performed in the previous section corresponds exactly to putting the dual 5d gauge theory on an Ω -background with parameters $q_1 = t$ and $q_2 = q^{-1}$. The computation of the partition function on a generic Ω -background was first done in [59].

In this way, we expect to engineer certain class of 5d theories with $\mathcal{N} = 1$ supersymmetry by using the refined topological vertex formalism on toric diagrams.

²We note that the information of the position of 7-branes are not encoded in the 5d low energy effective field theory.

Chapter 3

Algebraic Framework of Topological Vertex Formalism and \mathcal{W} -symmetry

In this chapter, we present an algebraic structure associated to the topological vertex formalism, which was first discovered in [31]. We will take a rather heuristic way to see the connection between the underlying algebra, i.e. the Ding-Iohara-Miki algebra, and the refined topological vertex. At the end of this chapter, we will establish the correspondence between the brane web and the web of representations in the Ding-Iohara-Miki algebra glued together with the refined topological vertex acting as a three-point structure in the web. An important class of physical observables, qq-characters, will also be considered in this chapter, and we propose a systematic way to compute them based on the earlier effort done in [60, 61].

3.1 Awata-Feigin-Shiraishi Rewriting and the Algebraic Structure

As we have already noted before in the derivation of the melting crystal picture, the (skew) Schur function can be expressed as a matrix element of a particular vertex operator, and one will thus expect that the topological vertex itself can also be expressed as a matrix element of some operator. In this section, we focus on the factor of the “unpreferred” part of the refined topological vertex, and rewrite it into a matrix element of a vertex operator that has clear connection with the Ding-Iohara-Miki algebra.

The skew Schur functions appearing in the refined topological vertex can be assembled as follows,

$$\begin{aligned} C_{\mu,\nu,\lambda} &\propto \sum_{\eta} s_{\mu^t/\eta}(q^{-\lambda}t^{-\rho+\{1/2\}}) s_{\nu/\eta}(t^{-\lambda^t}q^{-\rho-\{1/2\}}) \\ &= \langle \mu^t | V_{-}(q^{-\lambda}t^{-\rho+\{1/2\}}) V_{+}(t^{-\lambda^t}q^{-\rho-\{1/2\}}) | \nu \rangle. \end{aligned} \quad (3.1.1)$$

Note that when $\lambda = \emptyset$, the vertex operator becomes extremely simple,

$$\begin{aligned} V_-(t^{-\rho+\{1/2\}})V_+(q^{-\rho-\{1/2\}}) &= \exp\left(\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}t^{in}J_{-n}\right)\exp\left(\sum_{i=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}q^{in}J_n\right) \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{1-t^{-n}}J_{-n}\right)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{1-q^n}J_n\right). \end{aligned} \quad (3.1.2)$$

We define the above quantity as Φ_{\emptyset} for later convenience. We go to the next simplest case, $\lambda = (1)$.

$$\begin{aligned} V_-(q^{-(1)}t^{-\rho+\{1/2\}})V_+(t^{-(1)}q^{-\rho-\{1/2\}}) &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}(t^n/q^n + \sum_{i=2}^{\infty}t^{in})J_{-n}\right)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}(t^{-n} + \sum_{i=1}^{\infty}q^{in})J_n\right) \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}t^n(1-q^{-n})J_{-n}\right)\Phi_{\emptyset}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}(1-t^{-n})J_n\right). \end{aligned} \quad (3.1.3)$$

It is straightforward to repeat the computation for $\lambda = (2)$ and $\lambda = (1, 1)$,

$$\begin{aligned} &V_-(q^{-(2)}t^{-\rho+\{1/2\}})V_+(t^{-(1,1)}q^{-\rho-\{1/2\}}) \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}t^n(1-q^{-n})(1+q^{-n})J_{-n}\right)\Phi_{\emptyset}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}(1-t^{-n})(1+q^n)J_n\right), \end{aligned} \quad (3.1.4)$$

$$\begin{aligned} &V_-(q^{-(1,1)}t^{-\rho+\{1/2\}})V_+(t^{-(2)}q^{-\rho-\{1/2\}}) \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}t^n(1-q^{-n})(1+t^n)J_{-n}\right)\Phi_{\emptyset}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}(1-t^{-n})(1+t^{-n})J_n\right), \end{aligned} \quad (3.1.5)$$

and we observe that the vertex operator always takes the form

$$V_-(q^{-\lambda}t^{-\rho+\{1/2\}})V_+(t^{-\lambda^t}q^{-\rho-\{1/2\}}) =: \Phi_{\emptyset}\prod_{x \in \lambda}\eta(\chi_x), \quad (3.1.6)$$

where

$$\eta(\chi_x) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}t^n(1-q^{-n})\chi_x^n J_{-n}\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}(1-t^{-n})\chi_x^{-n} J_n\right), \quad (3.1.7)$$

with

$$\chi_x = t^{i-1}q^{-j+1}, \quad \text{for } x = (i, j) \in \lambda. \quad (3.1.8)$$

It is easy to prove this rewriting formula backwards in the most general case by computing

$$\begin{aligned} : \prod_{x \in \lambda} \eta(\chi_x) : &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}t^n(1-q^{-n})\sum_i \frac{t^{in-n}(1-q^{-n\lambda_i})}{1-q^{-n}}J_{-n}\right) \\ &\quad \times \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}(1-t^{-n})\sum_j \frac{q^{jn-n}(1-t^{-n\lambda_j^t})}{1-t^{-n}}J_n\right) \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_i t^{in}(1-q^{-n\lambda_i})J_{-n}\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_j q^{jn-n}(1-t^{-n\lambda_j^t})J_n\right), \end{aligned} \quad (3.1.9)$$

and we see that inserting the above expression back we obtain the original vertex operator. We call this vertex operator

$$\Phi_\lambda := V_-(q^{-\lambda}t^{-\rho+\{1/2\}})V_+(t^{-\lambda^t}q^{-\rho-\{1/2\}}) =: \Phi_\emptyset \prod_{x \in \lambda} \eta(\chi_x) :, \quad (3.1.10)$$

the Awata-Feigin-Shiraishi (AFS) vertex as this rewriting formula was first developed in [31].

The above rewriting was for the refined topological vertex $C_{\mu,\nu,\lambda}(t, q)$, and there is another dual refined topological vertex $C_{\mu^t,\nu^t,\lambda^t}(q, t)$ we need to glue with to obtain the full partition function. It seems straightforward to repeat the calculation with $t \leftrightarrow q$, but we want to at the end rewrite the partition function (with \emptyset external legs) in the form of a correlator of vertex operators, so we have to take into account the gluing rules.

The first point we have to be careful with is that we glue the first (or the second resp.) leg of $C_{\mu^t,\nu^t,\lambda^t}(q, t)$ with the first (or the second resp.) leg of $C_{\mu,\nu,\lambda}(t, q)$ together, thus we need a different expression for the skew Schur functions,

$$\begin{aligned} & \sum_{\eta} s_{\mu/\eta}(t^{-\lambda^t}q^{-\rho+\{1/2\}})s_{\nu^t/\eta}(q^{-\lambda}t^{-\rho-\{1/2\}}) \\ &= \langle \nu^t | V_+(q^{-\lambda}t^{-\rho-\{1/2\}})V_-(t^{-\lambda^t}q^{-\rho+\{1/2\}}) | \mu \rangle \\ &= (-1)^{|\mu|+|\nu|} \langle \nu | V_+^{-1}(q^{-\lambda}t^{-\rho-\{1/2\}})V_-^{-1}(t^{-\lambda^t}q^{-\rho+\{1/2\}}) | \mu^t \rangle. \end{aligned} \quad (3.1.11)$$

The vertex operator is almost the same. The remaining prefactors (excluding the ones corresponding to non-trivial framing) can be tracked as

$$\left(\frac{q}{t}\right)^{\frac{|\mu|}{2}} t^{\frac{|\nu|-|\mu|}{2}}. \quad (3.1.12)$$

When the framing is trivial in the gluing of a vertex $C_{\mu,\nu,\lambda}(t, q)$ and a dual vertex, we absorb this remaining factor

$$(tq)^{\frac{|\mu|}{2}} t^{\frac{|\nu|-|\mu|}{2}}, \quad (3.1.13)$$

into the dual vertex to obtain

$$(-1)^{|\mu|+|\nu|} \langle \nu | V_+^{-1}(q^{-\lambda}t^{-\rho+\{1/2\}}\gamma)V_-^{-1}(t^{-\lambda^t}q^{-\rho+\{1/2\}}\gamma) | \mu^t \rangle, \quad (3.1.14)$$

where for later convenience we introduced the parameter to measure the degree of refinement, $\gamma = q_3^{1/2} = \sqrt{q/t}$. We see that the vertex operators related to the dual vertex are

$$\Phi_\emptyset^* = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{\gamma^n}{1-t^{-n}} J_{-n}\right) \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{\gamma^n}{1-q^n} J_n\right), \quad (3.1.15)$$

$$\xi(\chi_x) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} t^n (1-q^{-n}) \chi_x^n \gamma^n J_{-n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} (1-t^{-n}) \chi_x^{-n} \gamma^n J_n\right), \quad (3.1.16)$$

and the dual vertex associated with $C_{\mu^t, \nu^t, \lambda^t}(q, t)$ is defined as

$$\Phi_\lambda^* := \Phi_\emptyset^* \prod_{x \in \lambda} \xi(\chi_x) : . \quad (3.1.17)$$

We are allowed to absorb all these factors into the dual vertex, as we can see that the framing factor cancels the remaining factor in the gluing of two vertices $C_{\mu, \nu, \lambda}(t, q)$ even with non-trivial framing.

Now we turn to deal with the Kähler parameter. Let us split the Kähler parameter Q of an internal line not in the preferred direction into $Q = v_1/v_2$. v_1 and v_2 are respectively assigned to the top and bottom vertices sandwiching the internal line as the position parameter of each vertex¹. We fix these parameters for all the vertices and decompose all Kähler parameters in a consistent way. Under this decomposition, the position parameter v can be completely absorbed into the vertex operator, and we define the vertex and dual vertex after this shift as

$$\Phi_\emptyset[v] := \exp \left(- \sum_{n=1}^{\infty} \frac{v^n}{n} \frac{1}{1-t^{-n}} J_{-n} \right) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{v^{-n}}{1-q^n} J_n \right), \quad (3.1.18)$$

$$\Phi_\emptyset^*[v] := \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{\gamma^n v^n}{1-t^{-n}} J_{-n} \right) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{\gamma^n v^{-n}}{1-q^n} J_n \right), \quad (3.1.19)$$

$$\eta(\chi_x) = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} t^n (1-q^{-n}) \chi_x^n J_{-n} \right) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} (1-t^{-n}) \chi_x^{-n} J_n \right), \quad (3.1.20)$$

$$\xi(\chi_x) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} t^n (1-q^{-n}) \chi_x^n \gamma^n J_{-n} \right) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} (1-t^{-n}) \chi_x^{-n} \gamma^n J_n \right), \quad (3.1.21)$$

where we redefined

$$\chi_x := vt^{i-1} q^{-j+1}, \quad \text{for } x = (i, j) \in \lambda. \quad (3.1.22)$$

Multiplying these vertex operators together and taking the expectation value of the corresponding correlator, we obtain the factors in the unpreferred direction of the refined topological string partition function. We will give the prescription for preferred directions in a later section, and before that we want to know the algebraic structure of the above vertex operators, especially that of $\eta(z)$ and $\xi(z)$.

¹We see from the dictionary with string theory/brane web that they parameterize the position of D5-branes.

From explicit computation, we have

$$\eta(z)\eta(w) = \frac{(1-w/z)(1-q_3^{-1}w/z)}{(1-q_1w/z)(1-q_2w/z)} : \eta(z)\eta(w) :, \quad (3.1.23)$$

$$\begin{aligned} \xi(z)\xi(w) &= \frac{(1-q_3w/z)(1-w/z)}{(1-q_1^{-1}w/z)(1-q_2^{-1}w/z)} : \eta(z)\eta(w) : \\ &= \frac{(1-q_3^{-1}z/w)(1-z/w)}{(1-q_1z/w)(1-q_2z/w)} : \eta(z)\eta(w) :, \end{aligned} \quad (3.1.24)$$

$$\eta(z)\xi(w) = \frac{(1-q_1\gamma w/z)(1-q_2\gamma w/z)}{(1-\gamma w/z)(1-q_3^{-1}\gamma w/z)} : \eta(z)\xi(w) :, \quad (3.1.25)$$

$$\xi(z)\eta(w) = \frac{(1-q_1\gamma w/z)(1-q_2\gamma w/z)}{(1-\gamma w/z)(1-q_3^{-1}\gamma w/z)} : \eta(z)\xi(w) :, \quad (3.1.26)$$

where we repeatedly used $q_1q_2q_3 = 1$. We thus have the following algebraic relations,

$$\eta(z)\eta(w) = g(z/w)\eta(w)\eta(z), \quad (3.1.27)$$

$$\xi(z)\xi(w) = g(z/w)^{-1}\xi(w)\xi(z), \quad (3.1.28)$$

with

$$g(z) = \frac{(1-q_1z)(1-q_2z)(1-q_3z)}{(1-q_1^{-1}z)(1-q_2^{-1}z)(1-q_3^{-1}z)}, \quad (3.1.29)$$

satisfying $g(z^{-1}) = g(z)^{-1}$. $\eta(z)\xi(w)$ and $\xi(w)\eta(z)$ are almost the same, but we have to be careful because

$$\frac{\prod_i(1-a_iz)}{\prod_j(1-b_jz)} - z^{|\{i\}|-|\{j\}|} \frac{\prod_i(z^{-1}-a_i)}{\prod_j(z^{-1}-b_j)} = \sum_k \frac{\prod_i(1-a_i/b_k)}{\prod_{j \neq k}(1-b_j/b_k)} \delta(b_k z). \quad (3.1.30)$$

As a consequence,

$$\begin{aligned} \eta(z)\xi(w) - \xi(w)\eta(z) &= \frac{(1-q_1)(1-q_2)}{(1-q_3^{-1})} (\delta(\gamma w/z) - \delta(\gamma^{-1}w/z)) : \eta(z)\xi(w) : \\ &= \frac{(1-q_1)(1-q_2)}{(1-q_3^{-1})} (\delta(\gamma w/z)\varphi^+(\gamma^{1/2}w) - \delta(\gamma^{-1}w/z)\varphi^-(\gamma^{-1/2}w)), \end{aligned} \quad (3.1.31)$$

where we introduced two new vertex operators,

$$\varphi^+(z) =: \eta(z\gamma^{1/2})\xi(z\gamma^{-1/2}) :, \quad \varphi^-(z) =: \eta(z\gamma^{-1/2})\xi(z\gamma^{1/2}) :. \quad (3.1.32)$$

We can compute their algebraic relations to find

$$\eta(z)\varphi^\pm(w) = g(\gamma^{\mp 1/2}w/z)\varphi^\pm(w)\eta(z), \quad (3.1.33)$$

$$\xi(z)\varphi^\pm(w) = g(\gamma^{\mp 1/2}z/w)\varphi^\pm(w)\xi(z). \quad (3.1.34)$$

Interestingly, the above algebraic relations are exactly those of Ding-Iohara type [27] with $g(z)$ specified by (3.1.29).

This algebra of Ding-Iohara type more generally is given by

$$[\psi^\pm(z), \psi^\pm(w)] = 0, \quad (3.1.35)$$

$$\psi^+(z) \psi^-(w) = \frac{g(\hat{\gamma}z/w)}{g(\hat{\gamma}^{-1}z/w)} \psi^-(w) \psi^+(z) \quad (3.1.36)$$

$$\psi^\pm(z) x^+(w) = g\left(\hat{\gamma}^{\pm\frac{1}{2}}z/w\right) x^+(w) \psi^\pm(z) \quad (3.1.37)$$

$$\psi^\pm(z) x^-(w) = g\left(\hat{\gamma}^{\mp\frac{1}{2}}z/w\right)^{-1} x^-(w) \psi^\pm(z) \quad (3.1.38)$$

$$x^\pm(z) x^\pm(w) = g(z/w)^{\pm 1} x^\pm(w) x^\pm(z) \quad (3.1.39)$$

$$[x^+(z), x^-(w)] = \frac{(1-q_1)(1-q_2)}{(1-q_3^{-1})} \left(\delta(\hat{\gamma}w/z) \psi^+\left(\hat{\gamma}^{\frac{1}{2}}w\right) - \delta(\hat{\gamma}^{-1}w/z) \psi^-\left(\hat{\gamma}^{-\frac{1}{2}}w\right) \right), \quad (3.1.40)$$

where $\hat{\gamma}$ is a central element of the algebra. We can see that the realization discussed above,

$$x^+(z) \mapsto \eta(z), \quad x^-(z) \mapsto \xi(z), \quad \psi^\pm(z) \mapsto \varphi^\pm(z), \quad \hat{\gamma} \mapsto \gamma, \quad (3.1.41)$$

gives a particular representation of the algebra for the choice (3.1.29). With the property $g(z^{-1}) = g(z)^{-1}$, this algebra admits the standard Drinfeld coproduct structure,

$$\begin{aligned} \Delta(x^+(z)) &= x^+(z) \otimes 1 + \psi^-(\hat{\gamma}_{(1)}^{1/2}z) \otimes x^+(\hat{\gamma}_{(1)}z), \\ \Delta(x^-(z)) &= x^-(\hat{\gamma}_{(2)}z) \otimes \psi^+(\hat{\gamma}_{(2)}^{1/2}z) + 1 \otimes x^-(z), \\ \Delta(\psi^+(z)) &= \psi^+(\hat{\gamma}_{(2)}^{1/2}z) \otimes \psi^+(\hat{\gamma}_{(1)}^{-1/2}z), \\ \Delta(\psi^-(z)) &= \psi^-(\hat{\gamma}_{(2)}^{-1/2}z) \otimes \psi^-(\hat{\gamma}_{(1)}^{1/2}z), \end{aligned} \quad (3.1.42)$$

where $\hat{\gamma}_{(1)} = \hat{\gamma} \otimes 1$ and $\hat{\gamma}_{(2)} = 1 \otimes \hat{\gamma}$. This fact can be easily confirmed from explicit computation.

We have already seen the connection between this algebra and the vertex operators though, this relation is not artificial and a more precise statement can be made once we constructed a representation of it along the preferred direction. This argument was first given in [31] and we will review it in section 3.4.

Before we conclude this section, we remark that

$$x^+(z) \mapsto u\gamma^n z^{-n} \eta(z), \quad x^-(z) \mapsto u^{-1} \gamma^{-n} z^n \xi(z), \quad \psi^\pm(z) \mapsto \gamma^{\mp n} \varphi^\pm(z), \quad \hat{\gamma} \mapsto \gamma, \quad (3.1.43)$$

is also a representation of the algebra related to refined topological vertex. We denote the map as $\rho_u^{(1,n)}$, as u is a free weight parameter of the representation, and (ℓ_1, ℓ_2) labels the value of central elements,

$$(\hat{\gamma}, \psi_0^+/\psi_0^-) \mapsto (\gamma^{\ell_1}, \gamma^{-2\ell_2}), \quad (3.1.44)$$

where ψ_0^+ and ψ_0^- are zero modes of $\psi^\pm(z)$ and also central elements of the algebra. This class of representations will be referred to as the $(1, n)$ representations in this article.

3.2 \mathcal{W} -symmetry in Ding-Iohara-Miki Algebra

It was only after 10 years of the work by Ding and Iohara [27] that in [28] Miki noticed that the algebra of Ding-Iohara type with the special choice (3.1.29) has very close relation with a particular q -deformed $\mathcal{W}_{1+\infty}$ algebra. This algebra is now called the Ding-Iohara-Miki (DIM) algebra.

The q -deformation of (A -type) \mathcal{W}_N -algebras, which are expected to be subalgebras of Miki's q -deformed $\mathcal{W}_{1+\infty}$ algebra, are constructed in [62] via the q -deformation of quantum Miura transformation. A key function in this construction is given by

$$f_k(z) := \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^{-(k-1)n})}{1-q_3^{-kn}} z^n \right). \quad (3.2.1)$$

The building constituents $\Lambda_i(z)$'s satisfy

$$f_N(w/z)\Lambda_i(z)\Lambda_j(w) =: \Lambda_i(z)\Lambda_j(w) : \times \begin{cases} 1 & i = j, \\ \frac{(z-q_1^{-1}w)(z-q_2^{-1}w)}{(z-w)(z-q_3w)} & i < j, \\ \frac{(z-q_1w)(z-q_2w)}{(z-w)(z-q_3^{-1}w)} & i < j, \end{cases} \quad (3.2.2)$$

and

$$: \Lambda_1(z)\Lambda_2(q_3^{-1}z) \dots \Lambda_N(q_3^{-N+1}z) := 1, \quad (3.2.3)$$

and the generators of the algebra,

$$T_1(z) = \sum_{i=1}^N u_i \Lambda_i(z), \quad (3.2.4)$$

$$T_i(z) = \sum_{1 \leq j_1 < \dots < j_i \leq N} u_{j_1} u_{j_2} \dots u_{j_i} : \Lambda_{j_1}(z) \Lambda_{j_2}(z q_3^{-1}) \dots \Lambda_{j_i}(z q_3^{-i+1}) :, \quad (3.2.5)$$

obey the following OPE relation,

$$f_N(w/z)T_1(z)T_i(w) - f_N(q_3^{i-1}z/w)T_i(w)T_1(z) = \frac{(1-q_1)(1-q_2)}{1-q_3^{-1}} \left(\delta(q_3w/z)T_{i+1}(z) - \delta(q_3^{-i}w/z)T_{i+1}(w) \right). \quad (3.2.6)$$

A nice review on these \mathcal{W}_N -algebras is available in [63]. We already see the similarity with the DIM algebra here. It is straightforward to confirm that the coproduct of $\hat{\gamma}$ is given by

$$\Delta(\hat{\gamma}) = \hat{\gamma} \otimes \hat{\gamma}, \quad (3.2.7)$$

and therefore the coproduct of an (ℓ_1, ℓ_2) representation and an (ℓ'_1, ℓ'_2) representation gives an $(\ell_1 + \ell'_1, \ell_2 + \ell'_2)$ representation of the DIM algebra. For $N = 2$, the relevant representation seems

to be $\hat{\gamma} \mapsto q_3$, mostly naively we choose the coproduct of two $(1, 0)$ representations to embed the q -deformed Virasoro algebra.

We define the expansion modes of ψ^\pm as

$$\psi^\pm(z) = \psi_0^\pm \exp \left(\pm \sum_{n>0} b_{\pm n} \hat{\gamma}^{n/2} z^{\mp n} \right). \quad (3.2.8)$$

We note that $\psi^\pm(z)$ respectively contains only positive/negative modes. In the $(1, 0)$ representation, since

$$\varphi^+(z) = \exp \left(\sum_{n>0} \frac{1}{n} (1 - t^{-n}) \gamma^{-\frac{n}{2}} (q_3^n - 1) z^{-n} J_n \right), \quad (3.2.9)$$

$$\varphi^-(z) = \exp \left(\sum_{n>0} \frac{1}{n} t^n (1 - q^{-n}) \gamma^{-\frac{n}{2}} (q_3^n - 1) z^n J_{-n} \right), \quad (3.2.10)$$

we have (for $n > 0$)

$$b_n = \frac{1}{n} (1 - t^{-n}) \gamma^{-n} (q_3^n - 1) J_n, \quad b_{-n} = -\frac{1}{n} t^n (1 - q^{-n}) \gamma^{-n} (q_3^n - 1) J_{-n}, \quad (3.2.11)$$

and

$$[b_n, b_m] = \frac{1}{n} (1 - t^{|n|}) (1 - q^{-|n|}) q_3^{-|n|} (1 - q_3^{|n|})^2 \delta_{n+m,0}. \quad (3.2.12)$$

The coproduct of b_n in $(1, 0) \otimes (1, 0)$ representation is given by (for $n > 0$)

$$\Delta(b_n) = \frac{1}{n} (1 - t^{-n}) \gamma^{-n} (1 - q_3^{-n}) J_n \otimes 1 + \frac{1}{n} (1 - t^{-n}) (q_3^n - 1) 1 \otimes J_n, \quad (3.2.13)$$

$$\Delta(b_{-n}) = -\frac{1}{n} t^n (1 - q^{-n}) (1 - q_3^{-n}) J_{-n} \otimes 1 - \frac{1}{n} t^n (1 - q^{-n}) \gamma^{-n} (q_3^n - 1) 1 \otimes J_{-n}. \quad (3.2.14)$$

Let us define the current

$$t(z) := \alpha(z) x^+(z) \beta(z), \quad (3.2.15)$$

where

$$\alpha(z) = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{\hat{\gamma}^n - \hat{\gamma}^{-n}} b_{-n} z^n \right), \quad \beta(z) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{\hat{\gamma}^n - \hat{\gamma}^{-n}} b_n z^{-n} \right). \quad (3.2.16)$$

From

$$\frac{1}{\hat{\gamma}^n - \hat{\gamma}^{-n}} = \sum_{i=0}^{\infty} \hat{\gamma}^{-(2i+1)n}, \quad (3.2.17)$$

we can write

$$\alpha(z) = \prod_{i=0}^{\infty} \psi^{-}(\hat{\gamma}^{-(2i+\frac{3}{2})}z)/\psi_0^{-}, \quad \beta(z) = \prod_{i=0}^{\infty} \psi^{+}(\hat{\gamma}^{(2i+\frac{3}{2})}z)/\psi_0^{+}, \quad (3.2.18)$$

and thus compute the commutation relation between $\alpha(z)$ or $\beta(z)$ with other elements in the DIM algebra. It is then easy to compute that

$$\beta(z)\alpha(w) = \prod_{i=0}^{\infty} g(\hat{\gamma}^{-2(i+1)}w/z)\alpha(w)\beta(z), \quad (3.2.19)$$

$$\beta(z)x^{+}(w) = \prod_{i=0}^{\infty} g(\hat{\gamma}^{-2(i+1)}w/z)^{-1}x^{+}(w)\beta(z), \quad (3.2.20)$$

$$\alpha(z)x^{+}(w) = \prod_{i=0}^{\infty} g(\hat{\gamma}^{-2(i+1)}z/w)x^{+}(w)\alpha(z). \quad (3.2.21)$$

The first equality (3.2.19) is computed from the commutator of bosonic modes b_n 's directly under the assumption that $[b_n, b_m]$ is central in the algebra,

$$[b_n, b_m] = \frac{1}{n}(1 - q_1^{|n|})(1 - q_2^{|n|})(1 - q_3^{|n|})(\hat{\gamma}^{-2|n|} - 1)\delta_{n+m,0}. \quad (3.2.22)$$

Thus

$$t(z)t(w) = g(z/w) \prod_{i=0}^{\infty} \frac{g(\hat{\gamma}^{-2(i+1)}w/z)g(\hat{\gamma}^{-2(i+1)}z/w)g(\hat{\gamma}^{-2(i+1)}z/w)}{g(\hat{\gamma}^{-2(i+1)}z/w)g(\hat{\gamma}^{-2(i+1)}w/z)g(\hat{\gamma}^{-2(i+1)}w/z)}t(w)t(z). \quad (3.2.23)$$

Let us put

$$A(z) := \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1 - q_1^n)(1 - q_2^n)(1 - q_3^n \hat{\gamma}^{-2n})}{1 - \hat{\gamma}^{-2n}} z^n \right), \quad (3.2.24)$$

we can rewrite it as

$$\begin{aligned} A(z) &= \prod_{i=0}^{\infty} \frac{(1 - q_1 \hat{\gamma}^{-2i}z)(1 - q_2 \hat{\gamma}^{-2i}z)}{(1 - \hat{\gamma}^{-2i}z)(1 - q_3^{-1} \hat{\gamma}^{-2i}z)} \frac{(1 - q_3 \hat{\gamma}^{-2i-2}z)(1 - \hat{\gamma}^{-2i-2}z)}{(1 - q_1^{-1} \hat{\gamma}^{-2i-2}z)(1 - q_2^{-1} \hat{\gamma}^{-2i-2}z)} \\ &= \frac{(1 - q_1 z)(1 - q_2 z)}{(1 - z)(1 - q_3^{-1} z)} \prod_{i=1}^{\infty} g(\hat{\gamma}^{-2i}z), \end{aligned} \quad (3.2.25)$$

and thus

$$\frac{A(w/z)}{A(z/w)} = g(w/z) \prod_{i=0}^{\infty} \frac{g(\hat{\gamma}^{-2(i+1)}w/z)}{g(\hat{\gamma}^{-2(i+1)}z/w)}. \quad (3.2.26)$$

At the first sight, we tend to conclude that $A(w/z)t(z)t(w) - A(z/w)t(w)t(z) = 0$, but in fact to reach the above equation, we converted $\frac{1 - q_3^{-1}z/w}{1 - q_3 z/w} = \frac{1 - q_3 w/z}{1 - q_3^{-1}w/z}$ by multiplying $q_3 w/z$ both to the

numerator and the denominator. This operation can often causes problems due to the identity (3.1.30). Taking this fact into account, we conclude,

$$A(w/z)t(z)t(w) - A(z/w)t(w)t(z) = \frac{(1-q_1)(1-q_2)}{1-q_3^{-1}} (\delta(q_3w/z)t^{(2)}(z) - \delta(q_3^{-1}z/w)t^{(2)}(w)) \quad (3.2.27)$$

where

$$t^{(2)}(z) = \alpha(q_3^{-1}z)\alpha(z)x^+(q_3^{-1}z)x^+(z)\beta(q_3^{-1}z)\beta(z). \quad (3.2.28)$$

We note that the representation $(N, 0)$ maps $A(z)$ to $f_N(z)$,

$$\rho_u^{(N,0)}(A(z)) = f_N(z), \quad (3.2.29)$$

and the q -Virasoro part of a general $q\mathcal{W}_N$ algebra can be embedded into the $(N, 0)$ representation of the DIM algebra. We can even modify the definition of α and β to include proper powers of ψ_0^\pm , so that all $(N, {}^\forall m)$ representations contain a q -Virasoro subalgebra. In the following, we focus on the $(N, 0)$ -representation realization.

In general, the coproduct for b_n can be found to

$$\Delta(b_n) = b_n \otimes \hat{\gamma}^{-|n|} + 1 \otimes b_n. \quad (3.2.30)$$

Let us construct the $(N, 0)$ -representation from N $(1, 0)$ -representations with the coproduct, then we can formally write

$$\rho_{\{u\}}^{(N,0)}(t(z)) =: \sum_{i=1}^N u_i \Lambda_i(z), \quad (3.2.31)$$

as in (3.2.4). It is easy to find

$$\Lambda_i(z) = \rho_{\{u\}}^{(N,0)}(\alpha(z)) (\varphi^-(\gamma^{1/2}z) \otimes \varphi^-(\gamma^{3/2}z) \otimes \dots \varphi^-(\gamma^{i-3/2}z) \otimes \eta(\gamma^{i-1}z) \otimes 1 \otimes \dots \otimes 1) \rho_{\{u\}}^{(N,0)}(\beta(z)), \quad (3.2.32)$$

and the commutation relation,

$$f_N(w/z)\Lambda_i(z)\Lambda_j(w) =: \Lambda_i(z)\Lambda_j(w) : \times \begin{cases} 1 & i = j, \\ \frac{(z-q_1^{-1}w)(z-q_2^{-1}w)}{(z-w)(z-q_3w)} & i < j, \\ \frac{(z-q_1w)(z-q_2w)}{(z-w)(z-q_3^{-1}w)} & i < j, \end{cases} \quad (3.2.33)$$

where we used

$$\rho_{\{u\}}^{(N,0)}(\beta(z)) = \prod_{i=1}^N \beta^{(i)}(\gamma^{N-i}z)|_{\hat{\gamma}=\gamma^N}, \quad (3.2.34)$$

$$\rho_{\{u\}}^{(N,0)}(\alpha(z)) = \prod_{i=1}^N \alpha^{(i)}(\gamma^{i-N}z)|_{\hat{\gamma}=\gamma^N}, \quad (3.2.35)$$

$$\eta(z) = \exp\left(-\sum_{n>0} \frac{\gamma^n}{1-q_3^n} b_{-n} z^n\right) \exp\left(\sum_{n>0} \frac{\gamma^n}{1-q_3^n} b_n z^{-n}\right). \quad (3.2.36)$$

Detailed calculations can be found in Appendix B.2 together with the elliptic generalization of the same computation.

Once we can further check that (3.2.3) holds, we will be able to claim that the q -deformed \mathcal{W}_N algebra is embedded in the coproduct of N copies of Ding-Iohara-Miki algebra. The check is a straightforward summation of all modes b_n 's at each site:

$$\underbrace{-\frac{\gamma^{in-2Nn}}{1-\gamma^{-2n}}b_{-n}z^n}_{\alpha} + \underbrace{\frac{\gamma^{-in}}{1-\gamma^{-2n}}b_{-n}z^n}_{\eta} - \underbrace{\gamma^{in}\frac{\gamma^{-2in}-\gamma^{-2Nn}}{1-\gamma^{-2n}}b_{-n}z^n}_{\varphi^-} = 0, \quad (3.2.37)$$

$$\underbrace{\frac{\gamma^{in}}{1-\gamma^{2n}}b_n z^{-n}}_{\eta} - \underbrace{\frac{\gamma^{in}}{1-\gamma^{2n}}}_{\beta} = 0. \quad (3.2.38)$$

Therefore, we see that the q -deformed \mathcal{W}_N -algebra can be embedded in the $(N, 0)$ representation of the DIM algebra.

3.3 S-duality and Automorphism in Ding-Iohara-Miki Algebra

When constructing the refined version of the topological vertex, we required that the partition function does not depend on the choice of the preferred direction in the toric diagram up to some potential rescaling (depending only on q/t) of Kähler parameters. The changing of the preferred direction corresponds to re-identify lines that represented D5-branes to some other type of branes, say, NS5-branes. This invariance is exactly the well-known $\text{SL}(2, \mathbb{Z})$ duality acting on the axio-dilaton charge in the type IIB superstring theory. Especially, the duality transformation which interchanges the D5- and NS5-brane is the most interesting one for us. It can be represented as a 90° rotation on the 5-6 plane of the 5-brane system (refer to Table 2.1), and we will simply use the terminology “S-duality” for this transformation in this article.

On the other hand, one of the most astonishing results in [28] is that the Ding-Iohara-Miki algebra was found to possess an $\text{SL}(2, \mathbb{Z})$ automorphism. It is tempting to identify this $\text{SL}(2, \mathbb{Z})$ symmetry as the same one in the type IIB superstring, and the most natural identification here is between the label of the central elements (ℓ_1, ℓ_2) of the algebra and the axio-dilaton charge labeling 5-branes. Then the candidate transformation for S-duality in the Ding-Iohara-Miki algebra is expected to map $\hat{\gamma}$ to ψ_0^\pm .

Note that there are two gradings in the Ding-Iohara-Miki algebra, under which

$$\deg(x_n^+) = (1, n), \quad \deg(x_n^-) = (-1, n), \quad \deg(b_n) = (0, n), \quad \deg(\hat{\gamma}, \psi_0^\pm) = (0, 0), \quad (3.3.1)$$

where we expand $x^\pm(z) = \sum_n x_n^\pm z^{-n}$. Following [64], we introduce a more symmetric free boson modes,

$$h_n := \text{sgn}(n) \frac{1}{(1-q_1^n)(1-q_2^n)(1-q_3^n)} \hat{\gamma}^{\frac{n}{2}} b_n, \quad (3.3.2)$$

and set $\psi_0^+ = (\psi_0^-)^{-1}$, then the S-duality transformation acts as [64]

$$\begin{aligned} \mathcal{S} : x_0^+ &\mapsto h_{-1}, & h_{-1} &\mapsto x_0^-, & x_0^- &\mapsto h_1, & h_1 &\mapsto x_0^+, \\ & & \psi_0^- &\mapsto \hat{\gamma}, & \hat{\gamma} &\mapsto \psi_0^+. \end{aligned} \quad (3.3.3)$$

Especially a $(0, 1)$ representation is sent to $(1, 0)$ representation under \mathcal{S} .

Let us investigate some basic properties of this $(0, 1)$ representation from the explicit expression we have for $(1, 0)$ representation. We have simple commutation relations between h_n and x_n^\pm ,

$$[h_n, x_m^+] = -\frac{1}{n} x_{n+m}^+ \hat{\gamma}^{-\frac{n+|n|}{2}}, \quad (3.3.4)$$

$$[h_n, x_m^-] = \frac{1}{n} x_{n+m}^- \hat{\gamma}^{-\frac{n-|n|}{2}}. \quad (3.3.5)$$

In $(1, 0)$ representation, by explicit computation we find

$$\rho_u^{(1,0)}(x_0^+) = u \sum_{\substack{\lambda_1, \lambda_2 \\ |\lambda_1|=|\lambda_2|}} \prod_{i,j=1}^{\infty} \frac{(-1)^{\lambda_{1,i}+\lambda_{2,j}}}{\lambda_{1,i}! \lambda_{2,j}!} \frac{1}{i^{\lambda_{1,i}} j^{\lambda_{2,j}}} t^{i\lambda_{1,i}} (1-q^{-i})^{\lambda_{1,i}} (1-t^{-j})^{\lambda_{2,j}} J_{-\lambda_1} J_{\lambda_2}, \quad (3.3.6)$$

where

$$J_{\pm\lambda} := \prod_{i=1}^{\infty} J_{\pm i}^{\lambda_i}. \quad (3.3.7)$$

This operator converts some state $|\lambda_1\rangle$ to other states $|\lambda_2\rangle$ with the same size $|\lambda_1| = |\lambda_2|$. It is natural to find a basis (labeled by one Young diagram) in which $\rho_u^{(1,0)}(x_0^+)$ is diagonal, as x_0^+ is mapped to h_1 in $(0, 1)$ representation and all h_n 's commute with each other in this representation. From

$$\rho^{(0,1)}(x_0^+) = \rho^{(1,0)}(h_{-1}) \propto J_{-1}, \quad \rho^{(0,1)}(x_0^-) = \rho^{(1,0)}(h_1) \propto J_1. \quad (3.3.8)$$

and the commutation relations (3.3.4) and (3.3.5), we see that in the diagonal basis, x^+ adds one box to the Young-diagram label of the basis and x^- removes one box from it. In the next section, we construct the concrete expression of the $(0, 1)$ -representation based on this observation and the defining commutation relations (3.1.35) to (3.1.40) of the DIM algebra.

3.4 Representation in Preferred Direction

So far, we have seen that the vertex operator in the unpreferred direction is deeply related to the DIM algebra, and thus the A -type q -deformed \mathcal{W}_N algebras. From the expectation of S-duality discussed in the previous section, we would like to construct a $(0, N)$ -representation, which is S-dual to $(N, 0)$ representation. We expect to be able to construct a $(0, N)$ -representation from the coproduct of N $(0, 1)$ -representations as in the case of $(N, 0)$ representation, so we focus on the $(0, 1)$ -representation first. This representation of the DIM algebra was first explicitly written down in [29]. However, as the gluing of vertices in unpreferred directions can be thought as a propagator Q^{L_0} of a free boson, we would also like to build the $(0, 1)$ -representation in such a way that the gluing in the preferred direction also has the interpretation as a propagator,

$$\sum_{\lambda} Q^{|\lambda|} |\lambda\rangle a_{\lambda} \langle \lambda|, \quad (3.4.1)$$

where a_{λ} is the normalization of the inner product,

$$\langle \mu | \lambda \rangle = a_{\lambda}^{-1} \delta_{\lambda, \mu}, \quad (3.4.2)$$

given by

$$a_{\lambda} \sim P_{\lambda}(t^{-\rho}, q, t) P_{\lambda^t}(q^{-\rho}, t, q). \quad (3.4.3)$$

Let us first construct the action of elements of the DIM algebra on ket states. The result obtained from the previous section can be summarized as

- The basis of the representation space is labeled by one Young diagram. This fact can also be naturally understood from the view point of the topological vertex.
- $\psi^{\pm}(z)$ are diagonal in the above basis.
- $x^+(z)$ adds one box to the basis and $x^-(z)$ removes one box.

Let us denote

$$\psi^{\pm}(z) |\lambda\rangle = \psi_0^{\pm} \psi_{\lambda}^{\pm}(z) |\lambda\rangle, \quad (3.4.4)$$

then from the commutation relation,

$$\frac{\psi_{\lambda+x}^{\pm}(z)}{\psi_{\lambda+x}^{\pm}(w)} = g(z/w), \quad (3.4.5)$$

where w is the coordinate of $x^{\pm}(w)$. The result cannot depend on w , so we need some factor in x^{\pm} to force w to be some number depending on x , the box added to λ . Inspired from the commutation relation between x^+ and x^- , we put the ansatz

$$x^{\pm}(z) |\lambda\rangle = \sum_{\substack{x \in A(\lambda) \\ (\text{resp. } R(\lambda))}} \delta(z/X_x) \Lambda_{\lambda}^{\pm}(x) |\lambda \pm x\rangle, \quad (3.4.6)$$

with X_x and $\Lambda_\lambda^\pm(x)$ unknown. By introducing

$$S(z) := \frac{(1 - q_1 z)(1 - q_2 z)}{(1 - z)(1 - q_1 q_2 z)}, \quad (3.4.7)$$

we can rewrite

$$g(z) = \frac{S(z)}{S(q_3 z)}, \quad (3.4.8)$$

and thus by using $g(z^{-1}) = g(z)^{-1}$, we have

$$\psi_\lambda^\pm(z) \propto \prod_{x \in \lambda} \frac{S(q_3 X_x / z)}{S(X_x / z)}. \quad (3.4.9)$$

We note that even though $\psi_\lambda^\pm(z)$ have the same expression², as $\psi^\pm(z)$ contain on negative and positive power of z respectively, they are series of z^{-1} and z , and converge in different regions. Similarly, from the commutation relation between x^\pm and x^\pm (with the same superscript), we obtain (for $x \neq y$, $x, y \in A(\lambda)$ resp. $R(\lambda)$)

$$\Lambda_{\lambda \pm x}^\pm(y) \Lambda_\lambda^\pm(x) = g(X_y / X_x)^{\pm 1} \Lambda_{\lambda \pm y}^\pm(x) \Lambda_\lambda^\pm(y). \quad (3.4.10)$$

Using the property,

$$S(q_3 z) = S(z^{-1}), \quad (3.4.11)$$

the above equation for $\Lambda_\lambda^\pm(x)$ reduces to the following recursive relation,

$$\frac{\Lambda_{\lambda+x}^+(y)}{\Lambda_\lambda^+(y)} = S(X_x / X_y)^{-1}, \quad (3.4.12)$$

and

$$\frac{\Lambda_{\lambda-x}^-(y)}{\Lambda_\lambda^-(y)} = S(q_3 X_x / X_y)^{-1}. \quad (3.4.13)$$

Note that $\Lambda_\lambda^+(y)$ is always paired with the delta-function $\delta(z / X_y)$, we are then allowed to solve the recursive relations up to an overall normalization,

$$\Lambda_\lambda^+(y) \big|_{X_y \rightarrow z} = \Lambda_\emptyset^+[z] \prod_{x \in \lambda} \frac{1}{S(X_x / z)}, \quad (3.4.14)$$

and

$$\Lambda_\lambda^-(y) \big|_{X_y \rightarrow z} = \Lambda_\emptyset^-[z] \prod_{x \in \lambda} S(q_3 X_x / z). \quad (3.4.15)$$

²We will see later that indeed they have to be the same function.

However, the above solution can be wrong if for some x , we have $X_y = q_1^{\pm 1} X_x$ or $X_y = q_2^{\pm 1} X_x$. Substituting $X_y = q_1 X_x$ or $X_y = q_2 X_x$ into (3.4.10) for Λ_λ^+ , we see that the equality can only hold when

$$\Lambda_{\lambda+y}^+(x) \Lambda_\lambda^+(y) = 0. \quad (3.4.16)$$

Similarly setting $X_y = q_1^{-1} X_x$ or $X_y = q_2^{-1} X_x$ in (3.4.10) for Λ_λ^- , we need to require

$$\Lambda_{\lambda-y}^-(x) \Lambda_\lambda^-(y) = 0. \quad (3.4.17)$$

We apparently cannot achieve this with the solution (3.4.14) and (3.4.15), especially as Λ_λ^\pm diverges at these specific values. One way out of this difficult situation is to make it that box y can only be added or removed after we added to or removed from λ the box x . It can be realized with the choice

$$X_{(i,j)} = v q_1^{i-1} q_2^{j-1}, \quad (3.4.18)$$

i.e. $X_x = \chi_x$, with some weight parameter v in the representation. The solution (3.4.14) and (3.4.15) will be singular when used in $x^\pm(z)$ under the above choice of X_x , but we note that

$$\Lambda_\lambda^+(y) = \bar{\Lambda}_\emptyset^+[\chi_y] \operatorname{Res}_{z \rightarrow \chi_y} \frac{1}{1-v/z} \prod_{x \in \lambda} \frac{1}{S(\chi_x/z)}, \quad (3.4.19)$$

and

$$\Lambda_\lambda^-(y) = \bar{\Lambda}_\emptyset^-[\chi_y] \operatorname{Res}_{z \rightarrow \chi_y} (1 - v q_3/z) \prod_{x \in \lambda} S(q_3 \chi_x/z), \quad (3.4.20)$$

still satisfies the recursive relation as we do not need to consider the case $y = x \pm (1, 0)$ or $y = x \pm (0, 1)$ for Λ_λ^\pm anymore. The pole at $z = v$ when $\lambda = \emptyset$ is compensated into Λ^+ by hand and we put a similar factor in Λ^- for convenience. We can also see this solution indeed works from the commutation relation of $[x^+(z), x^-(w)]$. First note that by assuming that ψ^\pm are different expansions of the same rational function $\Psi_\lambda(z)$, with the help of (3.1.30), we can rewrite

$$(\delta(w/z) \psi^+(w) - \delta(w/z) \psi^-(w)) |\lambda\rangle = \delta(w/z) \sum_{x_i: \text{pole}} \delta(x_i/w) \operatorname{Res}_{w \rightarrow x_i} w^{-1} \Psi_\lambda(w) |\lambda\rangle. \quad (3.4.21)$$

In the commutator $[x^+(z), x^-(w)]$, when the added box x by x^+ , and the removed box y by x^- are not the same, the coefficient before the basis $|\lambda + x - y\rangle$ up to delta functions is

$$\Lambda_{\lambda-y}^+[\chi_x] \Lambda_\lambda^-[\chi_y] - \Lambda_{\lambda+x}^-[\chi_y] \Lambda_\lambda^+[\chi_x] = S(\chi_y/\chi_x) \Lambda_\lambda^+[\chi_x] \Lambda_\lambda^-[\chi_y] - S(q_3 \chi_x/\chi_y) \Lambda_\lambda^-[\chi_y] \Lambda_\lambda^+[\chi_x] = 0. \quad (3.4.22)$$

The last equality again follows from the property $S(q_3 z) = S(z^{-1})$. The remaining contributions are only from those with $x = y$, therefore

$$\begin{aligned} & [x^+(z), x^-(w)] |\lambda\rangle \\ &= \left[\frac{(1-q_1)(1-q_2)}{1-q_3^{-1}} \sum_{x \in R(\lambda)} \delta(\chi_x/z) \delta(\chi_x/w) \bar{\Lambda}_\emptyset^+[\chi_x] \bar{\Lambda}_\emptyset^-[\chi_x] \operatorname{Res}_{w \rightarrow \chi_x} w \frac{1-vq_3^{-1}/w}{1-v/w} \prod_{y \in \lambda} \frac{S(q_3 \chi_y/w)}{S(\chi_y/w)} \right. \\ & \quad \left. + \frac{(1-q_1)(1-q_2)}{1-q_3^{-1}} \sum_{x \in A(\lambda)} \delta(\chi_x/z) \delta(\chi_x/w) \bar{\Lambda}_\emptyset^+[\chi_x] \bar{\Lambda}_\emptyset^-[\chi_x] \operatorname{Res}_{w \rightarrow \chi_x} w \frac{1-vq_3^{-1}/w}{1-v/w} \prod_{y \in \lambda} \frac{S(q_3 \chi_y/w)}{S(\chi_y/w)} \right] |\lambda\rangle, \end{aligned}$$

where the prefactor $\frac{(1-q_1)(1-q_2)}{1-q_3^{-1}}$ comes from the compensation of the factor corresponding to box x in $\Lambda_{\lambda-x}^+[\chi_x]$ and $\Lambda_{\lambda+x}^-[\chi_x]$ respectively. Let us now define the combination

$$\mathcal{Y}_\lambda(z) := (1-v/z) \prod_{x \in \lambda} S(\chi_x/z), \quad (3.4.23)$$

as the \mathcal{Y} -function, then it is easy to find several equivalent expressions for it:

$$\mathcal{Y}_\lambda(z) = \frac{\prod_{x \in A(\lambda)} (1 - \chi_x/z)}{\prod_{y \in R(\lambda)} (1 - \chi_y q_3^{-1}/z)}, \quad (3.4.24)$$

and

$$\mathcal{Y}_\lambda(z) = \prod_{x' \in \mathcal{X}_\lambda} \frac{1 - x'/z}{1 - q_1 x'/z}, \quad (3.4.25)$$

where $\mathcal{X}_\lambda = \{vq_1^{k-1}q_2^{\lambda_k}\}_{k=1}^\infty$. The second expression can be checked from the recursive relation and the overall normalization. A good candidate for $\Psi(z)$ is then given by

$$\Psi_\lambda(z) = \frac{\mathcal{Y}_\lambda(zq_3^{-1})}{\mathcal{Y}_\lambda(z)} = \prod_{x \in A(\lambda)} \frac{1 - q_3 \chi_x/z}{1 - \chi_x/z} \prod_{x \in R(\lambda)} \frac{1 - q_3^{-1} \chi_x/z}{1 - \chi_x/z}, \quad (3.4.26)$$

and it satisfies

$$\Psi_\lambda(z) \rightarrow \begin{cases} 1 & z \rightarrow \infty, \\ q_3 & z \rightarrow 0, \end{cases} \quad (3.4.27)$$

which is exactly the property needed for the $(0, 1)$ representation, i.e. $\psi_0^-/\psi_0^+ = q_3$. It is clear now that the following expression gives rise to a $(0, 1)$ representation of the DIM algebra,

$$x^+(z) |v, \lambda\rangle = b \sum_{x \in A(\lambda)} \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{z \mathcal{Y}_\lambda(z)} |v, \lambda + x\rangle, \quad (3.4.28)$$

$$x^-(z) |v, \lambda\rangle = \gamma^{-1} b^{-1} \sum_{x \in R(\lambda)} \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} z^{-1} \mathcal{Y}_\lambda(zq_3^{-1}) |v, \lambda - x\rangle, \quad (3.4.29)$$

$$\psi^\pm(z) |v, \lambda\rangle = \gamma^{-1} \Psi_\lambda(z) |v, \lambda\rangle, \quad (3.4.30)$$

where we symmetrized ψ_0^\pm by multiplying γ^{-1} to $\Psi_\lambda(z)$. We note that only the product $\bar{\Lambda}_\emptyset^+[\chi_x]\bar{\Lambda}_\emptyset^-[\chi_x]$ can be fixed from the commutation relation and there will always be an automorphism to change the relative normalization b of x^\pm . It is also allowed to multiply to $\Psi_\lambda(z)$ with a constant, when we change the value of $\bar{\Lambda}_\emptyset^+[\chi_x]\bar{\Lambda}_\emptyset^-[\chi_x]$ at the same time.

Now we want to construct the dual representation acting on a bra state. The easiest way to work it out is to consider the following trivial identity,

$$\langle v, \mu | (g(z) | v, \lambda \rangle) = (\langle v, \mu | g(z)) | v, \lambda \rangle. \quad (3.4.31)$$

We adapt the normalization described at the beginning of this section,

$$a_\lambda = \frac{c^{|\lambda|} \left(\prod_{(i,j) \in \lambda} \chi_{(i,j)} \right)^d}{N_{\lambda\lambda}(1; q_1, q_2)}, \quad (3.4.32)$$

with potential additional contributions from the framing factor and Kähler parameter (c and d are some unknown numbers). The action of ψ^\pm is diagonal, so it acts in the same on ket states and bra states. To write down the action of x^\pm on bras, we need to know the behavior of $a_{\lambda+s}/a_\lambda$ and $a_{\lambda-s}/a_\lambda$, as these generators add or remove boxes in λ . Quoting from the recursive formulas derived from an equivalent expression of the Nekrasov factor, (A.1.5) and (A.1.6),

$$\frac{N_{(\lambda+s)\nu}(Q; q_1, q_2)}{N_{\lambda\nu}(Q; q_1, q_2)} = \prod_{x' \in \mathcal{X}_\nu} \frac{1 - x'/\chi_s}{1 - q_1 x'/\chi_s} = \mathcal{Y}_\nu(\chi_s), \quad (3.4.33)$$

$$\frac{N_{\lambda(\nu+s)}(Q; q_1, q_2)}{N_{\lambda\nu}(Q; q_1, q_2)} = \prod_{x' \in \mathcal{X}_\lambda} \frac{1 - q_1 q_2 \chi_s/x'}{1 - q_2 \chi_s/x'} = \tilde{\mathcal{Y}}_\lambda(\chi_s q_1 q_2), \quad (3.4.34)$$

where

$$\tilde{\mathcal{Y}}_\nu(z) := \prod_{x' \in \mathcal{X}_\nu} \frac{1 - z/x'}{1 - q_1^{-1} z/x'}, \quad (3.4.35)$$

and it is related to \mathcal{Y}_ν through

$$\tilde{\mathcal{Y}}_\nu(z) = (-z/v_2) \mathcal{Y}_\nu(z). \quad (3.4.36)$$

We thus have

$$\frac{N_{(\lambda+s)(\lambda+s)}(1; q_1, q_2)}{N_{\lambda\lambda}(1; q_1, q_2)} = -\frac{\chi_s q_1 q_2}{v} \mathcal{Y}_{\lambda+s}(\chi_s q_1 q_2) \mathcal{Y}_\lambda(\chi_s), \quad (3.4.37)$$

where $\mathcal{Y}_{\lambda+s}(\chi_s q_1 q_2)$ is divergent and $\mathcal{Y}_\lambda(\chi_s)$ is zero individually, but their product is well-defined and finite. We can write it in a more elegant form,

$$\frac{N_{(\lambda+s)(\lambda+s)}(1; q_1, q_2)}{N_{\lambda\lambda}(1; q_1, q_2)} = -\frac{\chi_s q_1 q_2 (1 - q_1^{-1})(1 - q_2^{-1})}{v(1 - q_1^{-1} q_2^{-1})} \lim_{z \rightarrow \chi_s} \frac{1}{1 - \chi_s/z} \mathcal{Y}_\lambda(z q_1 q_2) \mathcal{Y}_\lambda(z). \quad (3.4.38)$$

Now consider the computation,

$$\langle v, \lambda | x^+(z) | v, \lambda - s \rangle = b\delta(\chi_s/z) \operatorname{Res}_{z \rightarrow \chi_s} \frac{1}{z\mathcal{Y}_{\lambda-s}(z)} a_\lambda^{-1}, \quad (3.4.39)$$

and use the recursive formula for a_λ , we rewrite it to

$$\langle v, \lambda | x^+(z) | v, \lambda - s \rangle = -\frac{b\chi_s q_1 q_2}{v} \delta(\chi_s/z) \operatorname{Res}_{z \rightarrow \chi_s} z^{-1} \mathcal{Y}_\lambda(z q_1 q_2) c^{-1} \chi_s^{-d} a_{\lambda-s}^{-1}. \quad (3.4.40)$$

Similarly, for

$$\langle v, \lambda | x^-(z) | v, \lambda + s \rangle = b^{-1} \gamma^{-1} \delta(\chi_s/z) \operatorname{Res}_{z \rightarrow \chi_s} z^{-1} \mathcal{Y}_{\lambda+s}(z q_1 q_2) a_\lambda^{-1}, \quad (3.4.41)$$

we have an equivalent expression,

$$\langle v, \lambda | x^-(z) | v, \lambda + s \rangle = -\frac{v b^{-1} \gamma^{-1}}{\chi_s q_1 q_2} \delta(\chi_s/z) \operatorname{Res}_{z \rightarrow \chi_s} z^{-1} \frac{1}{\mathcal{Y}_\lambda(z)} c \chi_s^d a_{\lambda+s}^{-1}. \quad (3.4.42)$$

By choosing $c = v^{-1} \gamma^{-1}$ (for later convenience), $d = 1$, we have the following expression for the dual $(0, 1)$ representation of the DIM algebra:

$$\langle v, \lambda | \psi^\pm(z) = \gamma^{-1} \Psi_\lambda(z) \langle v, \lambda |, \quad (3.4.43)$$

$$\langle v, \lambda | x^+(z) = -b\gamma^{-1} \sum_{x \in R(\lambda)} \langle v, \lambda - x | \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \mathcal{Y}_\lambda(z q_3^{-1}), \quad (3.4.44)$$

$$\langle v, \lambda | x^-(z) = -b^{-1} \sum_{x \in A(\lambda)} \langle v, \lambda + x | \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{z\mathcal{Y}_\lambda(z)}. \quad (3.4.45)$$

3.5 Topological Vertex as Awata-Feigin-Shiraishi Intertwiner

Now we would like to define the refined topological vertex in terms of the DIM algebra as a three-leg object. As we want to identify the weight parameter v in the $(0, 1)$ representation as the position of D5-branes, we decompose the Kähler parameter as $Q = u_1/u_2$ and absorb u_i 's into two vertices sandwiching the degenerate locus line whose Kähler parameter is Q . We have already defined the shifted vertex in (3.1.18), (3.1.19), (3.1.20) and (3.1.21), and roughly speaking, the three-leg vertex we would like to define takes the form,

$$\Phi | v, \lambda \rangle \sim \Phi_\lambda[v] := \Phi_\emptyset[v] \prod_{x \in \lambda} \eta(\chi_x), \quad (3.5.1)$$

$$\langle v, \lambda | \Phi^* \sim \Phi_\lambda^*[v] := \Phi_\emptyset^*[v] \prod_{x \in \lambda} \xi(\chi_x). \quad (3.5.2)$$

To complete the prescription in terms of the DIM algebra, we need to introduce the framing dependence in the preferred direction. We add an integer label (n) to the vertex, $\Phi^{(n)}$ and $\Phi^{*(n)}$. See

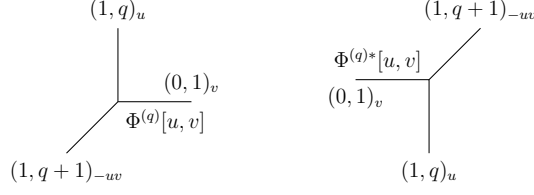


Figure 3.1: Correspondence between $\Phi^{(q)}[u, v]$, $\Phi^{(q)*}[u, v]$ and graphic vertices.

Figure 3.1 for the way to assign vertices with respect to (p, q) brane and we note that a (p, q) brane is mapped to a $(\ell_1 = q, \ell_2 = p)$ representation of the DIM algebra.

Taking the convenient normalization we chose in the previous section for the $(0, 1)$ representation,

$$a_\lambda = (v\gamma)^{-|\lambda|} \left(\prod_{(i,j) \in \lambda} \chi_{(i,j)} \right) N_{\lambda\lambda}^{-1}(1; q_1, q_2), \quad (3.5.3)$$

the following definition of the topological vertices

$$\Phi_\lambda^{(n)}[u, v] := \Phi^{(n)}[u, v] |v, \lambda\rangle := (-uv)^{|\lambda|} \prod_{x \in \lambda} (\gamma/\chi_x)^{n+1} : \Phi_\emptyset[v] \prod_{x \in \lambda} \eta(\chi_x) :, \quad (3.5.4)$$

$$\Phi_\lambda^{*(n)}[u^*, v] := \langle v, \lambda | \Phi^{*(n)}[u^*, v] := (u^*\gamma)^{-|\lambda|} \prod_{x \in \lambda} (\chi_x/\gamma)^n : \Phi_\emptyset^*[v] \prod_{x \in \lambda} \xi(\chi_x) :, \quad (3.5.5)$$

reproduce the calculation of refined topological string partition function.

Interestingly, Awata, Feigin and Shiraishi in [31] proved that the refined topological vertices defined above behave as intertwiners of representation in the DIM algebra.

$$\Phi^{(n)}[u, v] : (1, n)_u \otimes (0, 1)_v \rightarrow (1, n+1)_{-uv}, \quad (3.5.6)$$

$$\Phi^{(n)*}[u, v] : (1, n+1)_{-uv} \rightarrow (1, n)_u \otimes (0, 1)_v, \quad (3.5.7)$$

i.e. they satisfy

$$(\rho_u^{(1,n)} \otimes \rho_v^{(0,1)}) \Delta(g(z)) \Phi^{*(n)}[u, v] = \Phi^{*(n)}[u, v] \rho_{-uv}^{(1,n+1)}(g(z)), \quad (3.5.8)$$

$$\Phi^{(n)}[u, v] (\rho_v^{(0,1)} \otimes \rho_u^{(1,n)}) \Delta(g(z)) = \rho_{-uv}^{(1,n+1)}(g(z)) \Phi^{(n)}[u, v], \quad (3.5.9)$$

where the coproduct Δ is given by (4.2.3). We refer to this relation as the Awata-Feigin-Shiraishi (AFS) property in this article.

The above property can be checked by explicit computation as follows. We need to compute

the contraction of vertex operators, and the result is listed below.

$$\overline{\eta(z)\Phi_\lambda[u, v]} = \frac{1}{\mathcal{Y}_\lambda(z)} : \eta(z)\Phi_\lambda[u, v] :, \quad (3.5.10)$$

$$\overline{\Phi_\lambda[u, v]\eta(z)} = -\frac{vq_3}{z} \frac{1}{\mathcal{Y}_\lambda(zq_3^{-1})} : \Phi_\lambda[u, v]\eta(z) :, \quad (3.5.11)$$

$$\overline{\xi(z)\Phi_\lambda[u, v]} = \mathcal{Y}_\lambda(\gamma^{-1}z) : \xi(z)\Phi_\lambda[u, v] :, \quad (3.5.12)$$

$$\overline{\Phi_\lambda[u, v]\xi(z)} = -\gamma^{-1} \frac{z}{v} \mathcal{Y}_\lambda(z\gamma^{-1}) : \Phi_\lambda[u, v]\xi(z) :, \quad (3.5.13)$$

$$\overline{\varphi^-(z)\Phi_\lambda[u, v]} =: \varphi^-(z)\Phi_\lambda[u, v] :, \quad (3.5.14)$$

$$\overline{\Phi_\lambda[u, v]\varphi^-(z)} = \gamma^2 \frac{\mathcal{Y}_\lambda(z\gamma^{-1/2})}{\mathcal{Y}_\lambda(z\gamma^{-1/2}q_3^{-1})} : \Phi_\lambda[u, v]\varphi^-(z) :, \quad (3.5.15)$$

$$\overline{\varphi^+(z)\Phi_\lambda[u, v]} = \frac{\mathcal{Y}_\lambda(z\gamma^{1/2}q_3^{-1})}{\mathcal{Y}_\lambda(z\gamma^{1/2})} : \varphi^+(z)\Phi_\lambda[u, v] :, \quad (3.5.16)$$

$$\overline{\Phi_\lambda[u, v]\varphi^+(z)} =: \Phi_\lambda[u, v]\varphi^+(z) :, \quad (3.5.17)$$

$$\overline{\eta(z)\Phi_\lambda^*[u, v]} = \mathcal{Y}_\lambda(z\gamma^{-1}) : \eta(z)\Phi_\lambda^*[u, v] :, \quad (3.5.18)$$

$$\overline{\Phi_\lambda^*[u, v]\eta(z)} = -\gamma^{-1}z/v \mathcal{Y}_\lambda(z\gamma^{-1}) : \Phi_\lambda^*[u, v]\eta(z) :, \quad (3.5.19)$$

$$\overline{\xi(z)\Phi_\lambda^*[u, v]} = \frac{1}{\mathcal{Y}_\lambda(zq_3^{-1})} : \xi(z)\Phi_\lambda^*[u, v] :, \quad (3.5.20)$$

$$\overline{\Phi_\lambda^*[u, v]\xi(z)} = -\frac{v}{z} \frac{1}{\mathcal{Y}_\lambda(z)} : \Phi_\lambda^*[u, v]\xi(z) :, \quad (3.5.21)$$

$$\overline{\varphi^-(z)\Phi_\lambda^*[u, v]} =: \varphi^-(z)\Phi_\lambda^*[u, v] :, \quad (3.5.22)$$

$$\overline{\Phi_\lambda^*[u, v]\varphi^-(z)} = \gamma^{-2} \frac{\mathcal{Y}_\lambda(z\gamma^{-3/2})}{\mathcal{Y}_\lambda(z\gamma^{1/2})} : \Phi_\lambda^*[u, v]\varphi^-(z) :, \quad (3.5.23)$$

$$\overline{\varphi^+(z)\Phi_\lambda^*[u, v]} = \frac{\mathcal{Y}_\lambda(z\gamma^{-1/2})}{\mathcal{Y}_\lambda(z\gamma^{-5/2})} : \varphi^+(z)\Phi_\lambda^*[u, v] :, \quad (3.5.24)$$

$$\overline{\Phi_\lambda^*[u, v]\varphi^+(z)} =: \Phi_\lambda^*[u, v]\varphi^+(z) :. \quad (3.5.25)$$

For $g = x^+(z)$ in the AFS property of $\Phi^{(n)}$, we have

$$\begin{aligned} & \rho_{-uv}^{(1, n+1)}(x^+(z))\Phi^{(n)}[u, v] - \Phi^{(n)}[u, v](\rho_v^{(0, 1)} \otimes \rho_u^{(1, n)}) (\psi^-(z) \otimes x^+(z)) \\ &= -\sum_\lambda a_\lambda uv\gamma^{n+1}z^{-(n+1)}\overline{\eta(z)\Phi_\lambda^{(n)}} \langle v, \lambda | - \sum_\lambda a_\lambda u\gamma^n z^{-n}\gamma^{-1}\overline{\Psi_\lambda\Phi_\lambda^{(n)}} \langle v, \lambda | \\ &= -uv\gamma^{n+1}z^{-(n+1)}\sum_\lambda a_\lambda \left(\frac{1}{\mathcal{Y}_\lambda(z)} \Big|_{z \sim \infty} - \frac{1}{\mathcal{Y}_\lambda(z)} \Big|_{z \sim 0} \right) : \eta(z)\Phi_\lambda^{(n)}[u, v] : \langle v, \lambda |, \end{aligned}$$

where $\dots|_{z \sim \infty}$ stands for the expansion of \dots in terms of z^{-1} and similarly $\dots|_{z \sim 0}$ represents the

expansion over z . By using the identity (3.1.30), we find the above expression matching with

$$\begin{aligned}
\Phi^{(n)}[u, v] \rho_v^{(0,1)}(x^+(z)) &= \sum_{\lambda} a_{\lambda} \Phi_{\lambda}^{(n)}[u, v] \langle v, \lambda | x^+(z) \\
&= -\gamma^{-1} b \sum_{\lambda} \sum_{x \in R(\lambda)} a_{\lambda} \Phi_{\lambda}^{(n)} \langle v, \lambda - x | \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} z^{-1} \mathcal{Y}_{\lambda}(z q_3^{-1}) \\
&= -\gamma^{-1} b \sum_{\lambda'} \sum_{x \in A(\lambda')} a_{\lambda'} \frac{a_{\lambda'+x}}{a_{\lambda'}} \Phi_{\lambda'+x}^{(n)} \langle v, \lambda' | \delta(z/\chi_x) \frac{(1-q_1)(1-q_2)}{(1-q_3)} \mathcal{Y}_{\lambda'}(\chi_x q_3^{-1}) \\
&= -uv \gamma^{n+1} b \sum_{\lambda'} \sum_{x \in A(\lambda')} a_{\lambda'} \chi_x^{-(n+1)} : \eta(\chi_x) \Phi_{\lambda'}^{(n)} : \langle v, \lambda' | \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{z \mathcal{Y}_{\lambda'}(z)},
\end{aligned}$$

if we set $b = 1$. For $g = x^-$, we find

$$\begin{aligned}
&\rho_{-uv}^{(1,n+1)}(x^-(z)) \Phi^{(n)}[u, v] - \Phi^{(n)}[u, v] (\rho_v^{(0,1)} \otimes \rho_u^{(1,n)}) (1 \otimes x^-(z)) \\
&= - \sum_{\lambda} a_{\lambda} u^{-1} v^{-1} \gamma^{-(n+1)} z^{n+1} \overline{\xi(z)} \Phi_{\lambda}^{(n)} \langle v, \lambda | - \sum_{\lambda} a_{\lambda} u^{-1} \gamma^{-n} z^n \overline{\Phi_{\lambda}^{(n)} \xi(z)} \langle v, \lambda | \\
&= - \sum_{\lambda} a_{\lambda} u^{-1} v^{-1} \gamma^{-(n+1)} z^{n+1} (\mathcal{Y}_{\lambda}(\gamma^{-1} z)|_{z \sim \infty} - \mathcal{Y}_{\lambda}(\gamma^{-1} z)|_{z \sim 0}) : \xi(z) \Phi_{\lambda}^{(n)}[u, v] : \langle v, \lambda |, \quad (3.5.26)
\end{aligned}$$

matching with

$$\begin{aligned}
&\Phi^{(n)}[u, v] (\rho_v^{(0,1)} \otimes \rho_u^{(1,n)}) (x^-(\gamma z) \otimes \psi^+(\gamma^{1/2} z)) \\
&= b^{-1} \gamma^{-n} \sum_{\lambda} \sum_{x \in A(\lambda)} a_{\lambda} : \Phi_{\lambda}^{(n)}[u, v] \varphi^+(\gamma^{-1/2} \chi_x) : \langle v, \lambda + x | \delta(\gamma z / \chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{z \mathcal{Y}_{\lambda}(z)} \\
&= -b^{-1} u^{-1} v^{-1} \gamma^{-(n+1)} \sum_{\lambda'} \sum_{x \in R(\lambda')} a_{\lambda'} (\chi_x / \gamma)^{n+1} : \Phi_{\lambda'}^{(n)}[u, v] \xi(\gamma^{-1} \chi_x) : \langle v, \lambda' | \delta(\gamma z / \chi_x) \operatorname{Res}_{z \rightarrow \chi_x} z^{-1} \mathcal{Y}_{\lambda}(z q_3^{-1}),
\end{aligned}$$

where we used

$$\varphi^+(z) =: \eta(\gamma^{1/2} z) \xi(\gamma^{-1/2} z) :, \quad (3.5.27)$$

and

$$\operatorname{Res}_{z \rightarrow \chi_x / \gamma} z^{-1} \mathcal{Y}_{\lambda}(z \gamma^{-1}) = \operatorname{Res}_{z \rightarrow \chi_x} z^{-1} \mathcal{Y}_{\lambda}(z q_3^{-1}). \quad (3.5.28)$$

For $g = \psi^{\pm}$,

$$\rho_{-uv}^{(1,n+1)}(\psi^+(z)) \Phi^{(n)}[u, v] = \sum_{\lambda} a_{\lambda} \gamma^{-(n+1)} \Psi_{\lambda}(z \gamma^{1/2}) : \varphi^+(z) \Phi_{\lambda}^{(n)}[u, v] : \langle v, \lambda |, \quad (3.5.29)$$

$$\rho_{-uv}^{(1,n+1)}(\psi^-(z)) \Phi^{(n)}[u, v] = \sum_{\lambda} a_{\lambda} \gamma^{n+1} : \varphi^-(z) \Phi_{\lambda}^{(n)}[u, v] : \langle v, \lambda |, \quad (3.5.30)$$

and

$$\begin{aligned} & \Phi^{(n)}[u, v](\rho_v^{(0,1)} \otimes \rho_u^{(1,n)}) (\psi^+(\gamma^{1/2}z) \otimes \psi^+(z)) \\ &= \sum_{\lambda} a_{\lambda} \gamma^{-n} \gamma^{-1} \Psi_{\lambda}(z\gamma^{1/2}) : \Phi_{\lambda}^{(n)}[u, v] \varphi^+(z) : \langle v, \lambda |, \end{aligned} \quad (3.5.31)$$

$$\begin{aligned} & \Phi^{(n)}[u, v](\rho_v^{(0,1)} \otimes \rho_u^{(1,n)}) (\psi^-(\gamma^{-1/2}z) \otimes \psi^-(z)) \\ &= \sum_{\lambda} a_{\lambda} \gamma^n \gamma^{-1} \Psi_{\lambda}(z\gamma^{-1/2}) \gamma^2 \Psi_{\lambda}^{-1}(z\gamma^{-1/2}) : \Phi_{\lambda}^{(n)}[u, v] \varphi^-(z) : \langle v, \lambda |. \end{aligned} \quad (3.5.32)$$

They agree with each other again, and we see that the AFS property holds for $\Phi^{(n)}[u, v]$.

Now we turn to check the AFS property for Φ^* . The confirmation for $g = x^+$ goes as,

$$\begin{aligned} & (\rho_u^{(1,n)} \otimes \rho_v^{(0,1)})(x^+(z) \otimes 1) \Phi^{*(n)}[u, v] - \Phi^{*(n)}[u, v] \rho_{-uv}^{(1,n+1)}(x^+(z)) \\ &= \sum_{\lambda} a_{\lambda} |v, \lambda\rangle \left(u \gamma^n z^{-n} \mathcal{Y}_{\lambda}(z\gamma^{-1}) \Big|_{z \sim \infty} : \eta(z) \Phi_{\lambda}^{*(n)}[u, v] : \right. \\ & \quad \left. + uv \gamma^{n+1} z^{-(n+1)} (-\gamma^{-1}z/v) \mathcal{Y}_{\lambda}(z\gamma^{-1}) \Big|_{z \sim 0} : \Phi_{\lambda}^{*(n)}[u, v] \eta(z) : \right) \\ &= \sum_{\lambda} a_{\lambda} |v, \lambda\rangle u \gamma^n z^{-n} (\mathcal{Y}_{\lambda}(z\gamma^{-1}) \Big|_{z \sim \infty} - \mathcal{Y}_{\lambda}(z\gamma^{-1}) \Big|_{z \sim 0}) : \eta(z) \Phi_{\lambda}^{*(n)}[u, v] :, \end{aligned} \quad (3.5.33)$$

and

$$\begin{aligned} & (\rho_u^{(1,n)} \otimes \rho_v^{(0,1)})(\psi^-(\gamma^{1/2}z) \otimes x^+(\gamma z)) \Phi^{*(n)}[u, v] \\ &= \gamma^n b \sum_{\lambda} a_{\lambda} \sum_{x \in A(\lambda)} \delta(z\gamma/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{z \mathcal{Y}_{\lambda}(z)} |v, \lambda + x\rangle : \varphi^-(z\gamma^{1/2}) \Phi_{\lambda}^{*(n)}[u, v] : \\ &= -\gamma^n b \sum_{\lambda'} \sum_{x \in R(\lambda')} a_{\lambda'} \gamma^{-1} \chi_x^{-1} \delta(z\gamma/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \mathcal{Y}_{\lambda'}(zq_3^{-1}) |v, \lambda'\rangle \gamma u (\gamma/\chi_x)^n : \eta(\chi_x \gamma^{-1}) \Phi_{\lambda'}^{*(n)}[u, v] : \\ &= -u \gamma^n b \sum_{\lambda'} \sum_{x \in R(\lambda')} a_{\lambda'} \delta(z\gamma/\chi_x) z^{-n} \operatorname{Res}_{z \rightarrow \chi_x} z^{-1} \mathcal{Y}_{\lambda'}(zq_3^{-1}) |v, \lambda'\rangle : \eta(z) \Phi_{\lambda'}^{*(n)}[u, v] :, \end{aligned} \quad (3.5.34)$$

where we used

$$\varphi^-(z) =: \eta(z\gamma^{-1/2}) \xi(z\gamma^{1/2}), \quad (3.5.35)$$

and (3.5.28) again.

For $g = x^-$, we have

$$\begin{aligned} & (\rho_u^{(1,n)} \otimes \rho_v^{(0,1)})(x^-(z) \otimes \psi^+(z)) \Phi^{*(n)}[u, v] - \Phi^{*(n)}[u, v] \rho_{-uv}^{(1,n+1)}(x^-(z)) \\ &= u^{-1} \gamma^{-n} z^n \sum_{\lambda} a_{\lambda} \gamma^{-1} \Psi_{\lambda}(z) |v, \lambda\rangle \frac{1}{\mathcal{Y}_{\lambda}(zq_3^{-1})} : \xi(z) \Phi_{\lambda}^{*(n)}[u, v] : \\ & \quad - u^{-1} v^{-1} \gamma^{-(n+1)} z^{n+1} \sum_{\lambda} a_{\lambda} |v, \lambda\rangle \frac{v}{z} \frac{1}{\mathcal{Y}_{\lambda}(z)} : \xi(z) \Phi_{\lambda}^{*(n)}[u, v] : \\ &= u^{-1} \gamma^{-(n+1)} z^n \sum_{\lambda} a_{\lambda} \left(\frac{1}{\mathcal{Y}_{\lambda}(z)} \Big|_{z \sim \infty} - \frac{1}{\mathcal{Y}_{\lambda}(z)} \Big|_{z \sim 0} \right) |v, \lambda\rangle : \xi(z) \Phi_{\lambda}^{*(n)}[u, v] :, \end{aligned} \quad (3.5.36)$$

matching with

$$\begin{aligned}
& (\rho_u^{(1,n)} \otimes \rho_v^{(0,1)})(1 \otimes x^-(z)) \Phi^{*(n)}[u, v] \\
&= b^{-1} \gamma^{-1} \sum_{\lambda} a_{\lambda} \sum_{x \in R(\lambda)} \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} z^{-1} \mathcal{Y}_{\lambda}(z q_3^{-1}) |v, \lambda - x\rangle \Phi_{\lambda}^{*(n)}[u, v] \\
&= -b^{-1} \gamma^{-1} \sum_{\lambda'} \sum_{x \in A(\lambda')} a_{\lambda'} \delta(z/\chi_x) \gamma \chi_x^{-1} \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{\mathcal{Y}_{\lambda'}(z)} |v, \lambda'\rangle \gamma^{-1} u^{-1} \chi_x^n / \gamma^n : \xi(\chi_x) \Phi_{\lambda'}^{*(n)}[u, v] : \\
&= -b^{-1} \gamma^{-(n+1)} \sum_{\lambda'} \sum_{x \in A(\lambda')} a_{\lambda'} \delta(z/\chi_x) z^n \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{z \mathcal{Y}_{\lambda'}(z)} |v, \lambda'\rangle : \xi(\chi_x) \Phi_{\lambda'}^{*(n)}[u, v] : . \tag{3.5.37}
\end{aligned}$$

For $g = \psi^{\pm}$, we have

$$\begin{aligned}
& (\rho_u^{(1,n)} \otimes \rho_v^{(0,1)})(\psi^+(z) \otimes \psi^+(\gamma^{-1/2} z)) \Phi^{*(n)}[u, v] \\
&= \sum_{\lambda} a_{\lambda} \gamma^{-1} \Psi_{\lambda}(z \gamma^{-1/2}) |v, \lambda\rangle \gamma^{-n} \frac{\mathcal{Y}_{\lambda}(z \gamma^{-1/2})}{\mathcal{Y}_{\lambda}(z q_3^{-1} \gamma^{-1/2})} : \varphi^+(z) \Phi_{\lambda}^{*(n)}[u, v] : , \tag{3.5.38}
\end{aligned}$$

$$\begin{aligned}
& (\rho_u^{(1,n)} \otimes \rho_v^{(0,1)})(\psi^-(z) \otimes \psi^-(\gamma^{1/2} z)) \Phi^{*(n)}[u, v] \\
&= \sum_{\lambda} a_{\lambda} \gamma^{-1} \Psi_{\lambda}(z \gamma^{1/2}) |v, \lambda\rangle \gamma^n : \varphi^-(z) \Phi_{\lambda}^{*(n)}[u, v] : , \tag{3.5.39}
\end{aligned}$$

respectively equal to

$$\Phi^{*(n)}[u, v] \rho_{-uv}^{(1,n+1)}(\psi^+(z)) = \gamma^{-(n+1)} \sum_{\lambda} a_{\lambda} |v, \lambda\rangle : \Phi_{\lambda}^{*(n)}[u, v] \varphi^+(z) : , \tag{3.5.40}$$

$$\Phi^{*(n)}[u, v] \rho_{-uv}^{(1,n+1)}(\psi^-(z)) = \gamma^{n+1} \sum_{\lambda} a_{\lambda} |v, \lambda\rangle \gamma^{-2} \Psi_{\lambda}(z \gamma^{1/2}) : \Phi_{\lambda}^{*(n)}[u, v] \varphi^-(z) : . \tag{3.5.41}$$

The AFS property allows us to convert the action of elements of the DIM algebra in the preferred direction to non-preferred directions on the topological vertex. We will explain this idea later. As an alternative usage of it, we note that the refined topological vertex can be defined with this property if $\Phi_{\emptyset}[u, v]$ and $\Phi_{\emptyset}^*[u, v]$ are specified. For example, by applying the AFS property for $g = x^+$ to Φ , when $\lambda = \emptyset$, we find

$$\delta(z/v) \operatorname{Res}_{z \rightarrow v} \frac{1}{z - v} \Phi_{\{1\}}^{(n)}[u, v] = -uv \gamma^{n+1} z^{-n-1} \left(\frac{1}{1 - v/z} - \frac{-z/v}{1 - z/v} \right) : \eta(v) \Phi_{\emptyset}^{(n)}[u, v] : . \tag{3.5.42}$$

In this way, it is easy to find

$$\Phi_{\lambda}^{(n)}[u, v] \propto: \prod_{x \in \lambda} \eta(\chi_x) : , \tag{3.5.43}$$

and also the proportional coefficient given in (3.5.4).

3.6 Brane Web as Representation Web

With all preparations done so far, we managed to translate the topological formalism to a web of representations of the DIM algebra, where each vertex is defined as an intertwiner mapping two representations into one. For a given web diagram, the partition function is given by the product of corresponding intertwiners schematically as

$$Z = \begin{array}{ccccccc} \widehat{0} & & \widehat{0} & & \widehat{0} & & \widehat{0} \\ \Phi^{(n_{11})}[u_{11}, v_1] & \cdot & \Phi^{*(n_{21})}[u_{21}, v_1] & & & & \\ \Phi^{(n_{12})}[u_{12}, v_2] & \cdot & \Phi^{*(n_{22})}[u_{22}, v_2] & & \vdots & & \vdots \\ \dots & \cdot & \dots & & & & \\ & & \Phi^{(n'_{21})}[u'_{21}, v'_1] & \cdot & \Phi^{*(n_{31})}[u_{31}, v'_1] & \dots & \vdots \\ \dots & \cdot & \dots & & & & \\ \Phi^{(n_{1N})}[u_{1N}, v_N] & \cdot & \Phi^{*(n_{2N})}[u_{2N}, v_N] & & \vdots & & \vdots \\ & & \Phi^{(n'_{2(N'-1)})}[u'_{2(N'-1)}, v'_{N'-1}] & \cdot & \Phi^{*(n_{3(N'-1)})}[u_{3(N'-1)}, v'_{N'-1}] & & \\ & & \Phi^{(n'_{2N'})}[u'_{2N'}, v'_{N'}] & \cdot & \Phi^{*(n_{3N'})}[u_{3N'}, v'_{N'}] & & \vdots \\ \overline{0} & & \overline{0} & & \overline{0} & & \overline{0} \end{array}, \quad (3.6.1)$$

where we introduced a two-dimensional way to denote the vertex computation to simplify the notation. The multiplication in the Fock space is written in the vertical direction, and we used $\widehat{0}$ and $\overline{0}$ to represent the bra and ket vacuum states in the Fock space associated to the unpreferred direction. We remark that each intertwiner put a constraint on the weight parameters of the three representation spaces it connects, and therefore at the end only parameters that can be interpreted as the exponentiated position of D5-branes and NS5-branes in the context of the gauge theory. To evaluate the above vacuum expectation value of the product of intertwiners, we insert an identity operator,

$$\text{id} = \sum_{\lambda} a_{\lambda} |v_i, \lambda\rangle \langle v_i, \lambda|, \quad (3.6.2)$$

into the representation with weight v_i in the preferred direction, and the ket state $|v_i, \lambda\rangle$ and the bra state $\langle v_i, \lambda|$ respectively project $\Phi[u_i, v_i]$ and $\Phi^*[u'_i, v_i]$ into vertex operators $\Phi_{\lambda}[u_i, v_i]$ and $\Phi_{\lambda}^*[u'_i, v_i]$ defined in (3.5.4) and (3.5.5). The r.h.s. of (3.6.1) is then reduced to the sum (over Young diagrams λ_i 's attached to each internal line in the preferred direction) of products of correlation functions of vertex operators in the Fock spaces of the unpreferred direction, which can be calculated with the following contraction rules in the Fock space of the unpreferred direction³:

$$\overbrace{\Phi_{\mu}[u_2, v_2] \Phi_{\lambda}[u_1, v_1]} = \mathcal{G}^{-1}(v_1/v_2 q_3) N_{\lambda\mu}(v_1/v_2 q_3; q_1, q_2)^{-1} : \Phi_{\mu}[u_2, v_2] \Phi_{\lambda}[u_1, v_1] :, \quad (3.6.3)$$

$$\overbrace{\Phi_{\mu}^*[u_2, v_2] \Phi_{\lambda}^*[u_1, v_1]} = \mathcal{G}^{-1}(v_1/v_2) N_{\lambda\mu}(v_1/v_2; q_1, q_2)^{-1} : \Phi_{\mu}^*[u_2, v_2] \Phi_{\lambda}^*[u_1, v_1] :, \quad (3.6.4)$$

³The reason that the Nekrasov factors obtained from the contractions are different by shifting the Kähler parameter from the Cauchy identity of the Schur function is that we are in fact organizing the vertex operator in an opposite direction along the toric diagram.

$$\overline{\Phi_\mu[u_2, v_2] \Phi_\lambda^*[u_1, v_1]} = \mathcal{G}(v_1/v_2\gamma) N_{\lambda\mu}(v_1/v_2\gamma; q_1, q_2) : \Phi_\mu[u_2, v_2] \Phi_\lambda^*[u_1, v_1] :, \quad (3.6.5)$$

$$\overline{\Phi_\mu^*[u_2, v_2] \Phi_\lambda[u_1, v_1]} = \mathcal{G}(v_1/v_2\gamma) N_{\lambda\mu}(v_1/v_2\gamma; q_1, q_2) \Phi_\mu^*[u_2, v_2] \Phi_\lambda[u_1, v_1] :. \quad (3.6.6)$$

where

$$\mathcal{G}(v_1/v_2) = \exp \left(- \sum_{n>0} \frac{1}{n} \frac{(v_1/v_2)^n}{(1-q_1^n)(1-q_2^n)} \right) = \mathcal{G}(v_1/v_2; q_1, q_2), \quad (3.6.7)$$

is the same factor we defined in (2.6.4). For example the U(1) partition function is calculated as

$$\begin{aligned} Z_{U(1)} &= \frac{\widehat{0}}{\widehat{0}} \cdot \frac{\widehat{0}}{\widehat{0}} = \sum_{\lambda} a_{\lambda} \frac{\widehat{0}}{\widehat{0}} |v, \lambda\rangle \langle v, \lambda| \frac{\widehat{0}}{\widehat{0}} \\ &= \sum_{\lambda} a_{\lambda} (-uv\gamma^{n-n^*}/u^*)^{|\lambda|} \prod_{x \in \lambda} \chi_x^{n+1-n^*} = \sum_{\lambda} \mathfrak{q}^{|\lambda|} \frac{\prod_{y \in \lambda} \chi_y^{\kappa}}{N_{\lambda\lambda}(1; q_1, q_2)}, \end{aligned} \quad (3.6.8)$$

where $\mathfrak{q} = -\frac{u\gamma^{n-n^*-1}}{u^*}$ and $\kappa = n^* - n$.

Let us briefly explain how the above results are obtained. Recall the recursive relations, (3.4.33) and (3.4.34), and the defining expression for $\mathcal{Y}_{\lambda}(z)$,

$$\mathcal{Y}_{\lambda}(z) = (1 - v/z) \prod_{x \in \lambda} S(\chi_x/z), \quad (3.6.9)$$

we can derive an alternative but equivalent expression for the Nekrasov factor,

$$N_{\lambda\mu}(v_1/v_2; q_1, q_2) = \prod_{x \in \lambda} (1 - \chi_x/v_2) \prod_{y \in \mu} (1 - v_1 q_1 q_2 / \chi_y) \prod_{x \in \lambda, y \in \mu} S(\chi_y/\chi_x). \quad (3.6.10)$$

The product of S -functions is reproduced from the contraction of $\eta(z)$ (resp. $\xi(z)$) and $\eta(w)$ (resp. $\xi(w)$), e.g.

$$\overline{\eta(z)\eta(w)} = S(w/z)^{-1} : \eta(z)\eta(w) :, \quad (3.6.11)$$

and the prefactors come from the contraction of Φ_{\emptyset} (or Φ_{\emptyset}^*) and η (or ξ).

We notice that after projecting the topological vertex to certain state $|v, \lambda\rangle$ in the preferred direction, it is merely a vertex operator defined in the Fock space of free boson. Each Fock space associated to different degenerate loci with different charge (p, q) though (and the intertwiner preserves this charge, which is), in the practical calculation (especially of the partition function and related quantities) they are all identical Fock spaces. It is thus convenient to introduce a simplified web diagram which only distinguishes the Fock spaces from the representation space of the preferred direction. We represent the representation space in the preferred direction with a horizontal

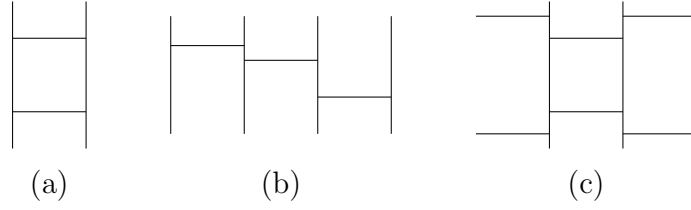


Figure 3.2: Examples of simplified webs: (a) $SU(2)$ gauge theory, (b) A_3 quiver $U(1)$ gauge theory, (c) $SU(2)$ gauge theory with four flavors.

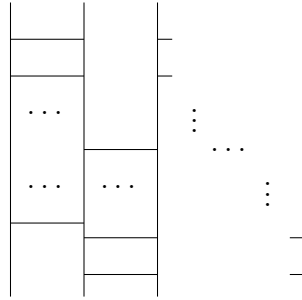


Figure 3.3: A (schematic) simplified brane web corresponds to the schematic expression of the partition function presented in (3.6.1).

line again, and all Fock spaces in non-preferred directions with vertical lines. Several simple examples are presented in Figure 3.2, and the corresponding simplified web diagram for the schematic expression of the partition function (3.6.1) is given in Figure 3.3.

We cannot read off the Chern-Simons level (or framing in the preferred direction) from the simplified web diagram without specifying which topological vertex is assigned to a concrete vertex. The simplified web diagram, however, makes the structure of bifundamental contributions very clear, i.e., the quiver structure of the underlying gauge theory. The gauge theories dual to M-string on toric Calabi-Yau always have A-type quiver structure, as we may have already seen in various examples presented in previous sections. In the next chapter, we will try to generalize the quiver structure of guage theories based on this simplified brane web by further introducing new type of vertices into the calculation.

3.7 qq-characters and Ward Identity

The qq-character is a quantity introduced by Nekrasov [15] as an operator which accounts for the integrability structure of the gauge theory with eight supercharges. It can be viewed a double quantized version of the Seiberg-Witten curve (see [65] for details, we will not discuss this aspect

in this article).

The expression of the qq-character only depends on the quiver structure of the gauge theory, and its formula (for 5d theories) is given by (a 5d version of Nekrasov's integral formula in [15])

$$\chi_{\vec{w}, \vec{\nu}}(z) = \sum_{\vec{a}} \prod_{i \in \text{Vert}(\Gamma')} \frac{1}{a_i!} \left(\frac{\mathbf{q}_i(1 - q_3^{-1})}{2\pi i(1 - q_1)(1 - q_2)} \right)^{a_i} \prod_{j=1}^{w_i} \mathcal{Y}_i(z q_3^{-1} \nu_{i,j}) \oint_{C_{w, \nu, a}} \Upsilon_{w, \nu, a}(z), \quad (3.7.1)$$

where

$$\Upsilon_{w, \nu, a}(z) = \prod_{e \in \text{Edge}(\Gamma')} \Upsilon_{w, \nu, a}^e(z) \prod_{i \in \text{Vert}(\Gamma')} \Upsilon_{w, \nu, a}^i(z), \quad (3.7.2)$$

$$\Upsilon_{w, \nu, a}^e(z) = \prod_{k=1}^{a_{t(e)}} \mathcal{Y}_{s(e)}(z q_3^{-1} \mu_e^{-1} \phi_k^{(t(e))}) \prod_{l=1}^{a_{s(e)}} \mathcal{Y}_{t(e)}(z \mu_e \phi_l^{(s(e))}) S(\phi_k^{(t(e))} / \phi_l^{(s(e))} \mu_e), \quad (3.7.3)$$

$$\Upsilon_{w, \nu, a}^i(z) = \prod_{k=1}^{a_i} \frac{d\phi_k^{(i)} P_i(z \phi_k^{(i)})}{\phi_k^{(i)} \mathcal{Y}_i(z q_3^{-1} \phi_k^{(i)}) \mathcal{Y}_i(z \phi_k^{(i)})} \prod_{l \neq k} S(\phi_l^{(i)} / \phi_k^{(i)})^{-1} \prod_{j=1}^{w_i} S(\nu_{i,j} / \phi_k^{(i)}), \quad (3.7.4)$$

and the integral contour is taken s.t. poles of the form $\frac{1}{1-z}$ in factor $S(z)$'s are picked up. $z_{i,j} := z \nu_{i,j}$ are independent parameters in the qq-character. Γ' denotes the quiver of the gauge theory, that is composed of (gauge) nodes $\text{Vert}(\Gamma')$ and edges $\text{Edge}(\Gamma')$ connecting nodes, and \vec{w} is the highest weight, which specifies the representation of the quiver Lie algebra that classifies the qq-character. $P_i(z)$ in the above formula is some fractional function which contains the information of matter contents and Chern-Simons levels. The information of the gauge group is decoded in the expression of \mathcal{Y}_i , in the case of U(1) gauge theory,

$$\langle v_i, \lambda | \mathcal{Y}_i(z) | v_i, \lambda \rangle = \mathcal{Y}_\lambda(z) \langle v_i, \lambda | v_i, \lambda \rangle, \quad (3.7.5)$$

and for $\text{SU}(N)$, it is given by N products of U(1) \mathcal{Y} -functions.

The general expression for A_1 quiver qq-character (after absorbing the gauge coupling \mathbf{q}_i and $P_i(z)$ into \mathcal{Y}_i to symmetrize the expression) can be calculated to

$$\chi_{n, \vec{\nu}}(z) = \sum_{I \cup J = \{1, \dots, n\}} \prod_{i \in I, j \in J} S(\nu_j / \nu_i) \prod_{i \in I} Y(\nu_i z)^{-1} \prod_{j \in J} Y(\nu_j q_3^{-1} z). \quad (3.7.6)$$

In [60, 61] (similar proposal was also suggested in [66]), how to derive the qq-character associated to the fundamental representation of A_n quiver is proposed. Let us review the computation in A_1 quiver U(1) gauge theory. We consider the insertion of positive-mode part of $x^-(z)$ in the preferred direction⁴, as shown in the following diagram,

⁴It is equivalent to consider the insertion of x^+ . The qq-character obtained is completely the same.

$$\left| \begin{array}{c} x_{>}^- \\ \hline \end{array} \right|$$

where $x_{>}^- = \sum_{k>0} x_k^- z^{-k}$.

In the two-dimensional notation of the multiplication of vertices, the insertion we want to consider is

$$\frac{\widehat{0}}{\Phi} \cdot x_{>}^- \cdot \frac{\widehat{0}}{\Phi^*} . \quad (3.7.7)$$

Then we have a trivial identity that the action of $x_{>}^-$ on Φ and Φ^* is equal to each other, i.e.

$$\sum_{\lambda} a_{\lambda} \frac{\widehat{0}}{\Phi} x_{>}^-(z) |v, \lambda\rangle \langle v, \lambda| \frac{\widehat{0}}{\Phi^*} = \sum_{\lambda} a_{\lambda} \frac{\widehat{0}}{\Phi} |v, \lambda\rangle \langle v, \lambda| x_{>}^-(z) \frac{\widehat{0}}{\Phi^*} . \quad (3.7.8)$$

We call this trivial identity the Ward identity of the DIM algebra, as it resembles the conformal Ward identity in the two-point function of 2d CFT,

$$\langle \varphi_1 | (L_n | \varphi_2 \rangle) = (\langle \varphi_1 | L_n) | \varphi_2 \rangle = \langle L_{-n} \varphi_1 | \varphi_2 \rangle . \quad (3.7.9)$$

It can be evaluated to

$$\gamma^{-1} \sum_{\lambda} a_{\lambda} \sum_{x \in R(\lambda)} \frac{\widehat{0}}{\Phi_{\lambda-x}} \frac{\widehat{0}}{\Phi_{\lambda}^*} \frac{1}{z - \chi_x} \text{Res}_{z \rightarrow \chi_x} \mathcal{Y}_{\lambda}(z q_3^{-1}) = - \sum_{\lambda} a_{\lambda} \sum_{x \in A(\lambda)} \frac{\widehat{0}}{\Phi_{\lambda}} \frac{\widehat{0}}{\Phi_{\lambda+x}^*} \frac{1}{z - \chi_x} \text{Res}_{z \rightarrow \chi_x} \frac{1}{\mathcal{Y}_{\lambda}(z)} ,$$

or equivalently,

$$z^{1-\kappa} u^* \sum_{\lambda} \mathfrak{q}^{|\lambda|} \frac{\prod_{y \in \lambda} \chi_y^{\kappa}}{N_{\lambda\lambda}(1; q_1, q_2)} \sum_{x \in R(\lambda)} \frac{1}{z - \chi_x} \text{Res}_{z \rightarrow \chi_x} \mathcal{Y}_{\lambda}(z q_3^{-1}) = uv\gamma \sum_{\lambda} \mathfrak{q}^{|\lambda|} \frac{\prod_{y \in \lambda} \chi_y^{\kappa}}{N_{\lambda\lambda}(1; q_1, q_2)} \sum_{x \in A(\lambda)} \frac{1}{z - \chi_x} \text{Res}_{z \rightarrow \chi_x} \frac{1}{\mathcal{Y}_{\lambda}(z)} ,$$

where the gauge coupling $\mathfrak{q} = -\frac{u\gamma^{n-n^*-1}}{u^*}$ and the Chern-Simons level $\kappa = n^* - n$ in the calculation with vertices $\Phi^{(n)}[u, v]$ and $\Phi^{*(n^*)}[u^*, v]$. We can further rewrite the above equation in the form of contour integral,

$$\sum_{\lambda} \mathfrak{q}^{|\lambda|} \frac{\prod_{y \in \lambda} \chi_y^{\kappa}}{N_{\lambda\lambda}(1; q_1, q_2)} \oint_{C_{\lambda}} dx \left(\frac{z}{z-x} \mathcal{Y}_{\lambda}(x q_3^{-1}) + \frac{\mathfrak{q} v q_3 z^{\kappa}}{(z-x) \mathcal{Y}_{\lambda}(x)} \right) = 0, \quad (3.7.10)$$

where the contour C_{λ} surrounds all the finite poles in $\mathcal{Y}_{\lambda}(x q_3^{-1})$ and $1/\mathcal{Y}_{\lambda}(x)$, i.e. $\{\chi_x | x \in A(\lambda) \cup R(\lambda)\}$. We can convert the integral to contour around $x \sim z$, $x \sim 0$ and $x \sim \infty$ and arrive at

$$\sum_{\lambda} \mathfrak{q}^{|\lambda|} \frac{\prod_{y \in \lambda} \chi_y^{\kappa}}{N_{\lambda\lambda}(1; q_1, q_2)} \left(z \mathcal{Y}_{\lambda}(z q_3^{-1}) + \frac{\mathfrak{q} P'(z)}{\mathcal{Y}_{\lambda}(z)} \right) = (\text{residue at } x \sim 0) + (\text{residue at } x \sim \infty), \quad (3.7.11)$$

where $P'(z) = vq_3z^\kappa$. We identify the left-hand side as the expectation value of the qq-character associated to the fundamental representation of the A_1 quiver, $\chi_1(z)$, i.e.

$$\langle z\chi_1(z) \rangle = \sum_{\lambda} \mathbf{q}^{|\lambda|} \frac{\prod_{y \in \lambda} \chi_y^\kappa}{N_{\lambda\lambda}(1; q_1, q_2)} \left(z\mathcal{Y}_\lambda(zq_3^{-1}) + \frac{\mathbf{q}^{P'(z)}}{\mathcal{Y}_\lambda(z)} \right), \quad (3.7.12)$$

where $\langle \mathcal{O} \rangle = \sum_{\lambda} Z_{\lambda} \frac{\langle v, \lambda | \mathcal{O} | v, \lambda \rangle}{\langle v, \lambda | v, \lambda \rangle}$ is the expectation value over all instanton configurations associated to the partition function $Z = \sum_{\lambda} \mathbf{q}^{|\lambda|} Z_{\lambda}$.

In [60], it was proposed that the restriction that there is no pole at the origin $x = 0$ in equation (3.7.11) and the pole at infinity gives rise to only polynomial of z up to degree n for $U(n)$ gauge theory reproduces the constraint on Chern-Simons level derived in [67]. In the current case of $U(1)$ gauge theory,

$$\mathcal{Y}_\lambda(z) \sim \begin{cases} 1/z & z \rightarrow 0, \\ 1 & z \rightarrow \infty, \end{cases} \quad (3.7.13)$$

therefore we see that $U(1)$ gauge theory with Chern-Simons level only is constrained to $-1 \leq \kappa \leq 1$. More generally, for $SU(n)$ gauge theory, this constraint is modified to $-n \leq \kappa \leq n$. When matter is added, a similar constraint can be derived. There are two kinds of matters that can be considered in the toric diagram approach, fundamental and anti-fundamental matter, whose contribution to the partition function is respectively given by

$$N_{\lambda\emptyset}(m = v_1/v_2; q_1, q_2) = \prod_{x \in \lambda} (1 - \chi_x q_1 q_2 / v_2), \quad N_{\emptyset\lambda}(m = v_1/v_2; q_1, q_2) = \prod_{y \in \lambda} (1 - v_1 / \chi_y). \quad (3.7.14)$$

The factors, $(1 - zq_1q_2/v)$ and $(1 - v/z)$, are multiplied to $P'(z)$ in the calculation with matter. We denote the number of fundamental matters to be n_f and the number of anti-fundamental matters to be n_{af} , in $U(1)$ gauge theory, the constraint can be read off as

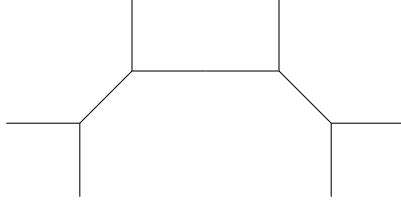
$$0 \leq n_f \leq 1 - \kappa, \quad 0 \leq n_{af} \leq \kappa + 1. \quad (3.7.15)$$

This implies the constraint on Chern-Simons level, $-1 \leq \kappa \leq 1$, and that on the flavor number $n_f + n_{af} =: N_f \leq 2$. For $SU(n)$, the constraint becomes,

$$0 \leq n_f \leq n - \kappa, \quad 0 \leq n_{af} \leq \kappa + n. \quad (3.7.16)$$

We can easily see that these constraints agrees with the condition on no intersections for external branes discussed in [68] (It is known that this is no longer the criterion for allowed gauge theories, as discussed in [43]. See also the remark at the end of this section). For example, the maximal $U(1)$ theory that saturates all the constraints is given by the following toric diagram⁵,

⁵We note that a geometric transition can take the diagram to that of a two-flavor $U(1)$ theory with trivial Chern-Simons level.



The combination $\chi_1(z) = \mathcal{Y}(zq_3^{-1}) + \frac{\mathfrak{q}P(z)}{\mathcal{Y}(z)}$ with $P(z) = z^{-1}P'(z)$, whose expectation value is evaluated in (3.7.11), is the qq-character obtained from the web diagram. Now we give a prescription to compute the qq-character for a general quiver and a general representation of the quiver Lie algebra.

- First we write down an insertion, whose action on Φ gives the residue of the highest weight term in the qq-character.
- Using the Ward identity to convert the residue obtained from the previous step to residue of some other term in the qq-character.
- Consider all insertions with which all the residue of the new term obtained from the Ward identity so that we can rewrite the sum of residues into the contour integral around $x = z$, the origin and the infinity.
- Repeat from the second step to pick up all poles of all possible terms until we get a closed form of the qq-character in a contour integral form.

As an example, we derive the qq-character corresponding to the spin-1 (3-dim) representation of A_1 quiver. The highest weight term is $Y(z_1q_3^{-1})Y(z_2q_3^{-1})$ and its poles can be picked up from two insertions (for simplicity of notation, we omit the subscript $>$ in $x_{>}^-$ inserted) ,

$$\frac{|x^-(z_1)\mathcal{Y}(z_2q_3^{-1})|}{|}$$

and

$$\frac{|x^-(z_2)\mathcal{Y}(z_1q_3^{-1})|}{|}$$

They respectively lead to the following terms in the qq-character by using the Ward identity of the insertions,

$$\chi_{2,\nu_1,\nu_2}(z) \supset \oint_{x \sim z} \frac{dx}{2\pi i} \frac{1}{x - z} \left(\mathcal{Y}(x\nu_1q_3^{-1})\mathcal{Y}(x\nu_2q_3^{-1}) + \mathfrak{q}P(x\nu_1)S(\nu_2/\nu_1)\frac{\mathcal{Y}(x\nu_2q_3^{-1})}{\mathcal{Y}(x\nu_1)} \right), \quad (3.7.17)$$

and

$$\chi_{2,\nu_1,\nu_2}(z) \supset \oint_{x \sim z} \frac{dx}{2\pi i} \frac{1}{x-z} \left(\mathcal{Y}(x\nu_2 q_3^{-1}) \mathcal{Y}(x\nu_1 q_3^{-1}) + \mathfrak{q} P(x\nu_2) S(\nu_1/\nu_2) \frac{\mathcal{Y}(x\nu_1 q_3^{-1})}{\mathcal{Y}(x\nu_2)} \right), \quad (3.7.18)$$

where we used color blue to denote the part whose poles have not been picked up with the insertion considered. The contour integral form of the highest weight term $cY(z_1 q_3^{-1}) \mathcal{Y}(z_2 q_3^{-1})$ is completed by adding the contributions of these two insertions together. The factor $S(\nu_2/\nu_1) = S(z_2/z_1)$ comes from the commutation of $x^-(z_1)$ and $\mathcal{Y}(z_2 q_3^{-1})$. To complete the contour integral for new terms obtained from the contour integral, we need to further consider the insertion

$$\left| \frac{x^-(z_2) \mathcal{Y}^{-1}(z_1)}{1} \right|$$

whose contribution through the Ward identity to the qq-character is

$$\chi_{2,\nu_1,\nu_2}(z) \supset \oint_{x \sim z} \frac{dx}{2\pi i} \frac{1}{x-z} \left(\mathfrak{q} P(x\nu_1) S(\nu_2/\nu_1) \frac{\mathcal{Y}(x\nu_2 q_3^{-1})}{\mathcal{Y}(x\nu_1)} + \mathfrak{q}^2 P(x\nu_1) P(x\nu_2) \frac{1}{\mathcal{Y}(x\nu_1) \mathcal{Y}(x\nu_2)} \right), \quad (3.7.19)$$

and the insertion

$$\left| \frac{x^-(z_1) \mathcal{Y}^{-1}(z_2)}{1} \right|$$

whose contribution is

$$\chi_{2,\nu_1,\nu_2}(z) \supset \oint_{x \sim z} \frac{dx}{2\pi i} \frac{1}{x-z} \left(\mathfrak{q} P(x\nu_2) S(\nu_1/\nu_2) \frac{\mathcal{Y}(x\nu_1 q_3^{-1})}{\mathcal{Y}(x\nu_2)} + \mathfrak{q}^2 P(x\nu_1) P(x\nu_2) \frac{1}{\mathcal{Y}(x\nu_1) \mathcal{Y}(x\nu_2)} \right). \quad (3.7.20)$$

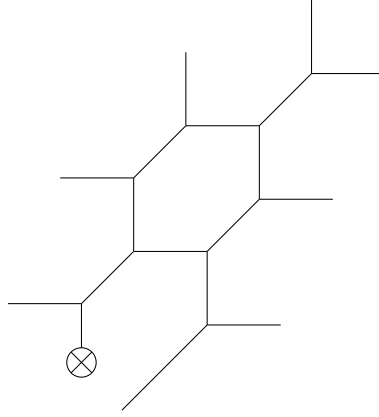
Interestingly, new terms derived from the Ward identity of these two insertions add up to a complete contour integral form of the lowest weight term, and thus the computation of the qq-character is terminated at this step. The final expression of the spin-1 qq-character is found to be (after symmetrizing the expression)

$$\begin{aligned} \chi_{2,\nu_1,\nu_2}(z) &= Y(z_1 q_3^{-1}) Y(z_2 q_3^{-1}) + S(z_2/z_1) \frac{Y(z_2 q_3^{-1})}{Y(z_1)} \\ &\quad + S(z_1/z_2) \frac{Y(z_1 q_3^{-1})}{Y(z_2)} + \frac{1}{Y(z_1) Y(z_2)}. \end{aligned} \quad (3.7.21)$$

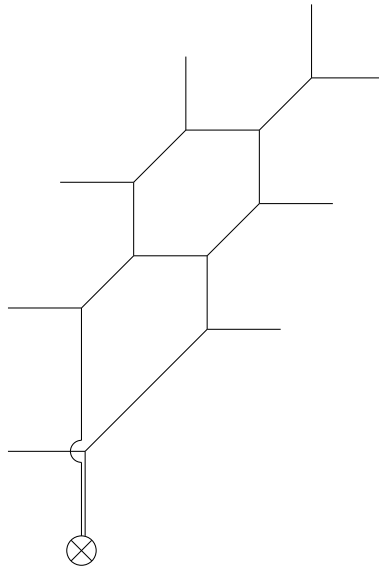
The qq-character is named so because it can be seen as a deformed q-character of the associated representation of the quiver Lie algebra. In the prescription we gave above to compute the qq-character, there is a clear map to the representation theory of Lie algebra: the Ward identity is

mapped to the Weyl reflection and searching remaining poles corresponds to finding positive weights that can be Weyl reflected. S -function appears whenever the Weyl reflection is performed with the existence of other positive weights, and we confirm that the prescription we give to compute the qq-character for the A -type quiver from the web diagram agrees with Nekrasov's integral formula. In later chapters, we will generalize the construction to D-type and finally arbitrary type of quivers, and examine that this prescription still gives the correct expression of the qq-characters.

Remark: It is for example possible to consider the $SU(2)$ gauge theory with five flavors as in the following toric diagram.



The circle with cross here represents the 7-brane, on which the attached NS5 brane ends. When we try to pull the 7-brane to the infinity, two external 5-branes will run into each other, and the correct way to deal with this process is to follow the Hanany-Witten transition [69]. The diagram obtained contains a jump in it as follows:



In [43], it was discussed how to deal with the jump in the calculation of partition function. We will also give a review and discussion in Chapter 5. As for the qq-character, the above construction introduce factors $(1 - m_1/z)$ and $(1 - m_2/z)^{-1}$ simultaneously into $P(z)$, so it is still allowed from the constraint on the qq-character. Adding more flavors make the diagram more and more complicated, and in [43], it was shown that $N_f = 8$ the maximal case one can consider, where the diagram extends to the infinity and makes sense as a 6d theory instead of the original 5d setup.

3.8 Physical Meaning of qq-characters

One key property of the qq-characters is that it has no pole at $z = \chi_x$ for $\forall x \in \lambda$. This allows us to convert the Ward identity of DIM algebra in the preferred direction to a contour integral around these points, $\{\chi_{x \in \lambda}\}$. The easiest way to see this property in the most general case is through the ADHM construction discussed in [70]. Let us briefly review the results together with the physical meaning of the qq-characters.

ADHM construction and LMNS integral form So far in this article, we dealt with the instanton partition function from the topological vertex formalism, but for the purpose in this section, we need to go over the localization approach, which was basically the method used in the original work of [71, 59], where the instanton partition function was first written down. The ADHM construction [72] provides a linear algebraic way to build instanton solutions systematically, and a more modern way to understand the ADHM construction had been provided in [73, 74] by using the brane picture (see also Tong's TASI lecture notes [75] for a nice summary). As instantons are realized as $D(p-4)$ branes embedded in Dp branes, the instanton moduli space is described by the vacuum configuration of the low energy effective theory on the $D(p-4)$ brane. 4d $\mathcal{N} = 2$ gauge theories can be obtained from the type IIA brane configuration shown in Table 3.1 and also see Figure 3.4. $x^{0,1,2,3}$ represents the spacetime directions and D4 branes are separated along the 6-direction to go to the Coulomb branch. The most convenient way to obtain the effective action on the $D(p-4)$ branes is to perform the dimensional reduction on the D3-D7 system (to a D0-D4 system in the current context), and the scalar potential is given by

$$V = \frac{1}{g^2} \sum_{m,n} [\hat{X}_m, \hat{X}_n]^2 + \sum_{m,\mu} [\hat{X}_m, X_\mu]^2 + \sum_m \left(I^\dagger \hat{X}_m^2 I + J \hat{X}_m^2 J^\dagger \right) + g^2 \text{tr} \left(I I^\dagger - J^\dagger J + [B_1, B_1^\dagger] + [B_2, B_2^\dagger] \right)^2 + g^2 \text{tr} |IJ + [B_1, B_2]|^2. \quad (3.8.1)$$

X 's in the above equation classically measure the transverse positions of $D(p-4)$ branes, and hence are $k \times k$ matrices when there are k $D(p-4)$ branes in the system. m is the label for directions transverse to both $D(p-4)$ branes and Dp branes, and μ is the label for directions parallel to Dp branes but transverse to $D(p-4)$ branes. I and J are respective a $k \times N$ matrix and a $N \times k$

	0	1	2	3	4	5	6	7	8	9
D4	—	—	—	—	—	•	•	•	•	•
NS5	—	—	—	—	•	—	—	•	•	•
D0	•	•	•	•	—	•	•	•	•	•
D4'	—	•	•	•	•	—	•	—	—	—

Table 3.1: Configuration of branes in the brane web construction. Bar represents the direction branes stretch along, and dot means the point-like direction for branes.

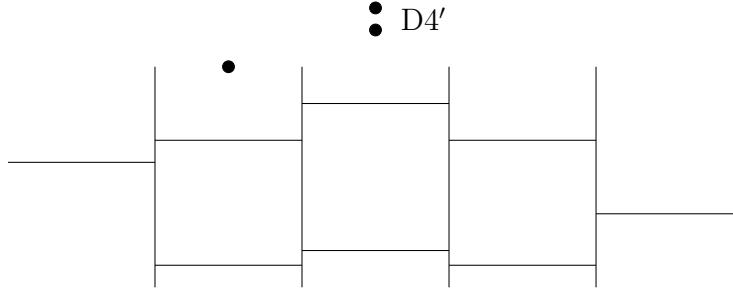


Figure 3.4: An example of the brane configuration of the D4-NS5-D4' system on the 4-6 plane, with D4-branes stand by horizontal lines and NS5-branes represented by horizontal lines. D4' branes are point-like on this plane, and they can be assigned to different chambers separated by Ns5-branes. This allocation of D4' branes corresponds to the representation of the quiver Lie algebra that labels the qq-character, although we only considered the case of A_1 quiver in this article.

matrix, where N represents the number of Dp branes, and they come from strings stretching between $D(p-4)$ branes and Dp branes, as we can see that separation between $D(p-4)$ branes and Dp branes in \hat{X}_m directions gives masses to these fields. Let $\mu = 1, 2, 3, 4$, then $B_{1,2}$ are special holomorphic coordinates in the space $\mathbb{R}^4 \simeq \mathbb{C}^2$ spanned by X_μ 's:

$$B_1 = X_1 + iX_2, \quad B_2 = X_3 + iX_4. \quad (3.8.2)$$

Since we are considering the configuration that $D(p-4)$ branes are embedded in Dp branes, we need to solve the equation $V = 0$ under the condition $\hat{X}_m = 0$. This implies

$$\begin{cases} II^\dagger - J^\dagger J + [B_1, B_1^\dagger] + [B_2, B_2^\dagger] = 0, \\ IJ + [B_1, B_2] = 0, \end{cases} \quad (3.8.3)$$

which is exactly the ADHM equation.

One can then apply the systematic way of the localization method to write down the expression of the partition function from the ADHM data and it turns out to be a finite-dimensional integral

[71], which is often referred to as the LMNS integral. The 5d version of this integral is given by

$$Z_k = \frac{1}{k!} \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^k \oint \left(\prod_{i=1}^k \frac{dx_i}{2\pi i} \right) \prod_{i=1}^k \prod_{a=1}^N \frac{1}{(1 - v_a/x_i)(1 - q_1 q_2 x_i/v_a)} \prod_{\substack{i,j=1 \\ i \neq j}}^k S^{-1}(x_i/x_j), \quad (3.8.4)$$

where v_a 's are the exponential of positions of D5 branes (uplifted from D4 branes) in the 6-direction and $S(z)$ is the S -function defined in (3.4.7). The full instanton partition function is a sum of the k -instanton partition function Z_k with the instanton number graded by the complexified coupling \mathfrak{q} ,

$$Z_{\text{instanton}} = \sum_k \mathfrak{q}^k Z_k. \quad (3.8.5)$$

The physical setup to obtain the qq-characters is to further add another set of D4 branes, which are denoted as D4' in Table 3.1, into the system. D4' branes are also separated in the 6-direction and we denote their (exponentiated) positions by M_i . These new D4' branes introduce a 1d fermionic degree of freedom into the effective field theory on D4 branes, and in terms of the gauge theory, it corresponds to a half-BPS line defect (or Wilson loop). From the view point of the effective theory on D0's, there will be a twisted hypermultiplet and a Fermi multiplet induced from strings between D0 and D4', and a Fermi multiplet from D4-D4' strings after introducing D4' branes into the system. These new multiplets introduce the following factor,

$$\prod_{a=1}^N \prod_{j=1}^w (1 - q_1 q_2 M_j/v_a) \prod_{j=1}^w \prod_{i=1}^k \frac{(1 - q_1 M_j/x_i)(1 - q_2 M_j/x_i)}{(1 - M_j/x_i)(1 - q_1 q_2 M_j/x_i)}, \quad (3.8.6)$$

into the integrand of the LMNS integral [70]. In addition to the usual poles to pick up in the Jeffrey-Kirwan residue prescription, we also need to consider the poles at $x_i = M_j$ in the evaluation of the integral. Let us define the expectation value for operator \mathcal{O} as

$$\langle \mathcal{O} \rangle = \sum_{\lambda} \mathfrak{q}^{|\lambda|} N_{\lambda\lambda}^{-1}(1; q_1, q_2) \mathcal{O}_{\lambda}, \quad (3.8.7)$$

where \mathcal{O}_{λ} is the value of \mathcal{O} on the $|\lambda|$ instanton background with $\{x_i\} = \{\chi_{x \in \lambda}\}$. The instanton partition function with one D4' inserted can then be evaluated to

$$\left(- \prod_{a=1}^N \frac{q_1 q_2 M}{v_a} \right) \langle \mathcal{Y}(M q_1 q_2) \rangle + \mathfrak{q} \langle \mathcal{Y}^{-1}(M) \rangle, \quad (3.8.8)$$

where in the second term we pick up the pole at $x_i = M$ and $\{x_j\}_{j \neq i} = \{\chi_{x \in \lambda}\}$ for all choices of i , and thus it is effectively the contribution from the background with one less instanton. Note that there exists a zero at $x_i = x_j$ which prevents the appearance of more \mathcal{Y}^{-1} in the final expression,

and the above result matches exactly with the qq-character of the fundamental representation of A_1 quiver.

The apparent set of poles in $\mathcal{Y}_\lambda(zq_1q_2)$ and $\mathcal{Y}_\lambda^{-1}(z)$ are $\{\chi_x\}_{x \in A(\lambda) \cup R(\lambda)}$, and we can easily examine that if we put $M = \chi_{x \in R(\lambda)}$ into the integral $k = |\lambda|$, there will be a double pole in the integrand and thus the qq-character is integrated to be zero, i.e. the pole is cancelled at the level of the qq-character. More precisely, the pole at $M = \chi_x$ in $\mathcal{Y}_{\lambda+x}(Mq_1q_2)$ is cancelled by the same pole in $\mathcal{Y}_\lambda^{-1}(M)$. This integral form is behind the pole cancellation mechanism that makes the Ward identity approach work.

The above fact motivates us to consider an algebraic approach to instanton counting whenever there exists an ADHM construction as a future work.

When there exist more than one D4' branes, we are allowed to pick the pole $x_i = M_j$ for each of these branes, and whenever we do this, a $\mathcal{Y}(M_jq_1q_2)$ is converted to $\mathcal{Y}^{-1}(M_j)$ in the expression of the qq-character. This corresponds exactly to the Weyl reflection in the representation theory of $SU(2)$, and we see that the instanton partition function with w D4' branes inserted gives the qq-character of weight- w representation of $SU(2)$.

We remark at the end of this section that the qq-characters are also argued in [76] to be realized as surface operators in 5d gauge theories.

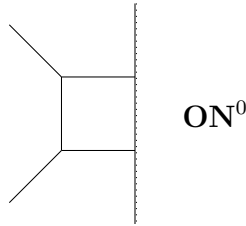
Chapter 4

D-type Quiver and Orientifolds in Topological Vertex Formalism

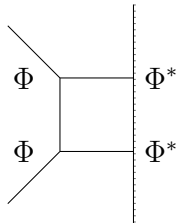
From this chapter, we turn to the generalization of the topological vertex formalism. The starting point is to include 5d $\mathcal{N} = 1$ gauge theories with D -type quiver structure into our framework.

4.1 Orientifold and D-type quivers

The construction of D-type quiver gauge theories in string theory was initiated in [77, 78] by adding an \mathbf{ON}^0 plane, which can be think of as a bound state from an \mathbf{ON}^- plane and an NS5-brane, to the brane web. The simplest case, D_2 -quiver, is given by the following web diagram ($D_2 \simeq A_1 \times A_1$, i.e. two decoupled gauge theories),



We represent the \mathbf{ON}^0 plane by a line merged from a normal line and a dotted line. The vertex assignment in the above web diagram is given by



With only these four vertices, we will only obtain the partition function of an $SU(2)$ gauge theory. To cancel the contraction between two Φ 's and two Φ^* 's, we need to insert a new vertex corresponding to the effect of the orientifold. We note that if we focus only on either the positive-mode half or the negative-mode half of Φ^* , it can be obtained from Φ^{-1} with a shift of the variable z . Therefore we can introduce a two-leg vertex $\Omega_{-1,\alpha}$ satisfying

$$J_n \Omega_{-1,\alpha} = -\alpha^n \Omega_{-1,\alpha} J_n, \quad (4.1.1)$$

and insert it between two Φ^* vertices in the above brane web to cancel the bifundamental contribution between two D5-branes. $\Omega_{-1,\alpha}$ can be realized in two steps: 1) reverse the sign of J_n ; 2) shift the argument by α . The first effect is achieved by

$$\Omega_{-1,1} = \prod_{n=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{i! n^i} J_{-n}^i |0\rangle \langle 0| (-J_n)^i, \quad (4.1.2)$$

where the product in the above expression does duplicate the vacuum states of the Fock space. The second step can be realized with

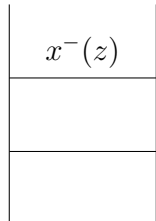
$$\alpha^{L_0} \text{ s.t. } \alpha^{L_0} J_n = \alpha^{-n} J_n \alpha^{L_0}, \quad (4.1.3)$$

and at the end we find the realization of the two-leg vertex as $\Omega_{-1,\alpha} = \Omega_{-1,1} \alpha^{L_0}$. By choosing $\alpha = q_3^{-1}$, we can see the decoupling of two D5-branes in the partition function that

$$Z_{D_2} = \frac{\widehat{0}}{\Phi^{(n_1)}[u, v_1]} \cdot \frac{\widehat{0}}{\Phi^{*(n_1^*)}[u_1^*, v_1]} \cdot \frac{\Omega_{-1,q_3^{-1}}}{\Phi^{(n_1+1)}[-uv_2, v_2]} \cdot \frac{\widehat{0}}{\Phi^{*(n_2^*)}[u_2^*, v_2]} = \left(\sum_{\lambda_1} \mathbf{q}_1^{|\lambda_1|} \frac{\prod_{x \in \lambda_1} \chi_x^{\kappa_1}}{N_{\lambda_1 \lambda_1}(1; q_1, q_2)} \right) \left(\sum_{\lambda_2} \mathbf{q}_2^{|\lambda_2|} \frac{\prod_{y \in \lambda_2} \chi_y^{\kappa_2}}{N_{\lambda_2 \lambda_2}(1; q_1, q_2)} \right), \quad (4.1.4)$$

where $\mathbf{q}_1 = -\frac{u\gamma^{n_1-n_1^*-1}}{u_1^*}$, $\kappa_1 = n_1^* - n_1$, $\mathbf{q}_2 = \frac{uv_2\gamma^{n_1-n_2^*}}{u_2^*}$ and $\kappa_2 = n_2^* - n_1 - 1$. As the new vertex $\Omega_{-1,\alpha}$ may carry a non-zero axio-dilaton charge, we keep n_1^* and n_2^* independent in the above calculation. Adding more D5-branes respectively above and below the new vertex $\Omega_{-1,\alpha}$ lifts up the rank of the gauge groups in two nodes of the D_2 quiver, and adding more branes on the left of the D_2 diagram gives higher-rank D_n quiver theories (see Figure 4.1 for examples of simplified diagrams).

Let us examine the qq-character from the Ward identity in the web diagram. We start from D_2 quiver again. The following two insertions,



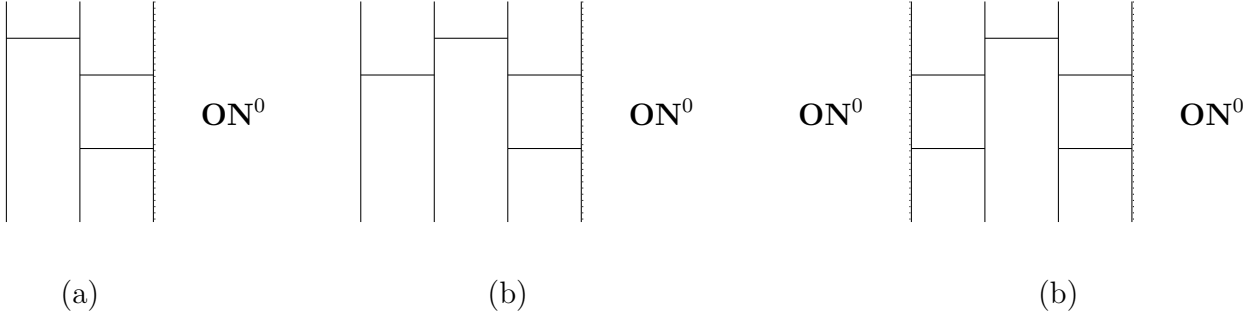
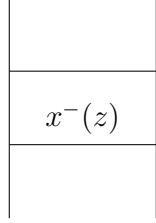


Figure 4.1: (a) $D_3 \simeq A_3$ quiver. (b) D_4 quiver, which is the simplest non-trivial example of D -type quiver. (c) affine D -type quiver, \hat{D}_4 .

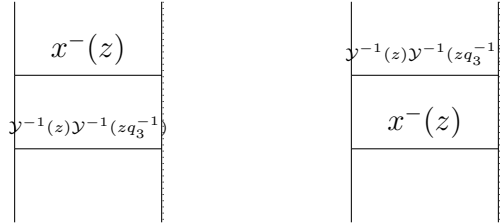
and



complete the contour integral for the term $Y_1(zq_3^{-1})Y_2(zq_3^{-1})$, which is the highest-weight term in the qq-character of the vector representation (representation with weight $(1, 1)$) of D_2 . Due to the existence of the orientifold, the new terms obtained from the Ward identity (Weyl reflection) take the form Y_i/Y_j ($i, j = 1, 2, i \neq j$),

$$\chi_{(1,1)}(z) \supset Y_1(zq_3^{-1})Y_2(zq_3^{-1}) + \frac{Y_2(zq_3^{-1})}{Y_1(z)} + \frac{Y_1(zq_3^{-1})}{Y_2(z)}. \quad (4.1.5)$$

The contour integral of the qq-character $\chi_{(1,1)}(z)$ can be completed by the following insertions:



and the final expression of the qq-character is given by

$$\begin{aligned} \chi_{(1,1)}(z) &= Y_1(zq_3^{-1})Y_2(zq_3^{-1}) + \frac{Y_2(zq_3^{-1})}{Y_1(z)} + \frac{Y_1(zq_3^{-1})}{Y_2(z)} + \frac{1}{Y_1(z)Y_2(z)} \\ &= \left(Y_1(zq_3^{-1}) + \frac{1}{Y_1(z)} \right) \left(Y_2(zq_3^{-1}) + \frac{1}{Y_2(z)} \right). \end{aligned} \quad (4.1.6)$$

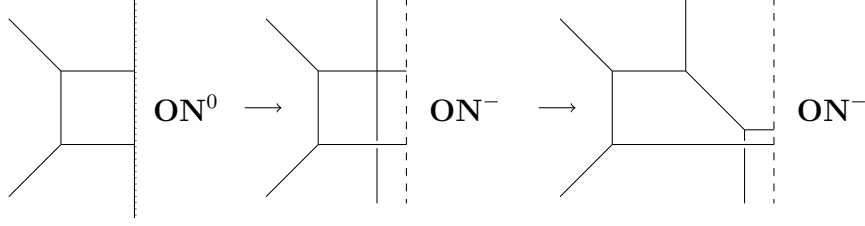
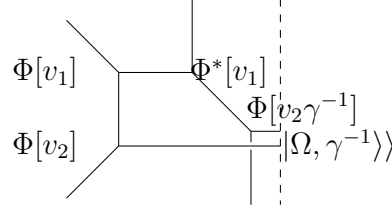


Figure 4.2: Resolution of the orientifold, $\mathbf{ON}^0 \rightarrow \mathbf{ON}^- + \text{NS5}$, and the conifold singularity give rise to the “resolved” brane web proposed in [79].

The d.o.f.’s in two stacks of D5-branes are factorized into two pieces, and this factorization in general construction of D-type quiver gauge theories helps to establish the map between the representation theory of the quiver Lie algebra and the qq-character.

When all gauge groups are restricted to $U(1)$, there will be an alternative way to formulate the D-type quiver construction, which was first reported in [47]. The idea is very simple. As first done in [79], we resolve the \mathbf{ON}^0 -plane into an NS5-brane and an \mathbf{ON}^- -plane, and then try to apply the topological vertex formalism in this diagram (see Figure 4.2). This alternative prescription works because, as we noted before, the positive-mode (or negative mode) half of the vertex $\Phi^*{}^{-1}$ can be realized by Φ with a shift of the argument.

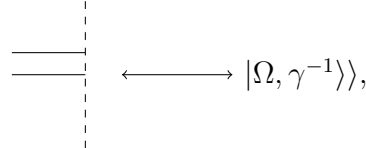
The vertex assignment is given as the following diagram.



where

$$|\Omega, \alpha\rangle\rangle = \sum_{\lambda} a_{\lambda} |v, \lambda\rangle \otimes |v\alpha, \lambda\rangle, \quad (4.1.7)$$

is the reflection state assigned to the orientifold \mathbf{ON}^- ,



and it is defined in the tensor product of two $(0, 1)$ representations satisfying

$$(x^{\pm}(z) \otimes 1 + 1 \otimes x^{\mp}(z\alpha)) |\Omega, \alpha\rangle\rangle = 0, \quad (4.1.8)$$

$$(\psi^{\pm}(z) \otimes 1 - 1 \otimes \psi^{\pm}(z\alpha)) |\Omega, \alpha\rangle\rangle = 0. \quad (4.1.9)$$

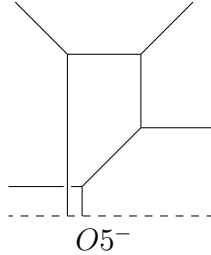
The partition function can be calculated as (for $t = q$)

$$\begin{aligned}
Z_{D_2} &= \sum_{\lambda_2} a_{\lambda} \begin{array}{ccc} \widehat{0} & & \widehat{0} \\ \Phi^{(n_1)}[u, v_1] & \cdot & \Phi^{*(n_1^*)}[u_1^*, v_1] \\ & \cdot & \Phi^{(n_1^*)}[u_1^*, v_2] \\ \Phi^{(n_1+1)}[-uv_2, v_2] & \cdot & \end{array} \begin{array}{c} |v_2, \lambda_2\rangle \\ |v_2, \lambda_2\rangle \end{array} \\
&= \left(\sum_{\lambda_1} \mathbf{q}_1^{|\lambda_1|} \frac{\prod_{x \in \lambda_1} \chi_x^{\kappa_1}}{N_{\lambda_1 \lambda_1}(1; q, q^{-1})} \right) \left(\sum_{\lambda_2} \mathbf{q}_2^{|\lambda_2|} \frac{\prod_{y \in \lambda_2} \chi_y^{\kappa_2}}{N_{\lambda_2 \lambda_2}(1; q, q^{-1})} \right), \tag{4.1.10}
\end{aligned}$$

where $\mathbf{q}_1 = -\frac{u}{u_1^*}$, $\kappa_1 = n_1^* - n_1$, $\mathbf{q}_2 = u_1 u_1^* v_2^2$ and $\kappa_2 = -n_1 - n_1^* - 2$. We remark that the expression of \mathbf{q}_2 does show that the orientifold acts as a mirror on the 5-6 plane in the above calculation formalism, as $u_1 v_2$ and $u_1^* v_2$ respectively encodes the (exponentiated) position of two NS5-branes that constitute the second copy the U(1) gauge theory.

We emphasize again that this construction works only for U(1) gauge theories unless we consider the unrefined case $t = q$, i.e. $q_3 = 1$, where $\Phi^* = \Phi^{-1}$ is genuinely true. In [47], it is argued that by replacing the usual topological vertex with the generalized vertex introduced in [61], one can also raise the rank of the gauge group in this construction. However, the generalized vertex does not behave as well as the topological vertex under the S-duality transformation, which we would like to focus in the remaining of this section.

S-duality The S-dual configuration of the $U(1) \otimes U(1)$ gauge theory,



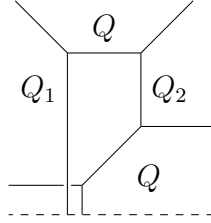
is argued in [79] to be an “Sp(0)” gauge theory (with 2 flavors). Let us compute its unrefined partition function by using the S-dual of the reflection state. As discussed in [47], the S-dual state of $|\Omega, 1\rangle$ is given by

$$|\Omega_s\rangle\rangle = \exp \left(- \sum_{n>0} J_{-n} \otimes J_{-n} \right) |\emptyset\rangle \otimes |\emptyset\rangle, \tag{4.1.11}$$

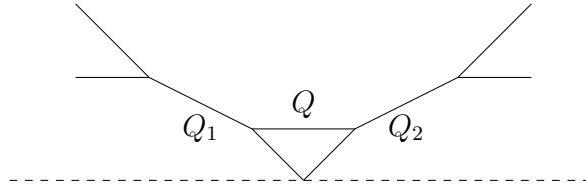
which plays the role of

$$(J_n \otimes 1) |\Omega_s\rangle\rangle = -(1 \otimes J_{-n}) |\Omega_s\rangle\rangle. \tag{4.1.12}$$

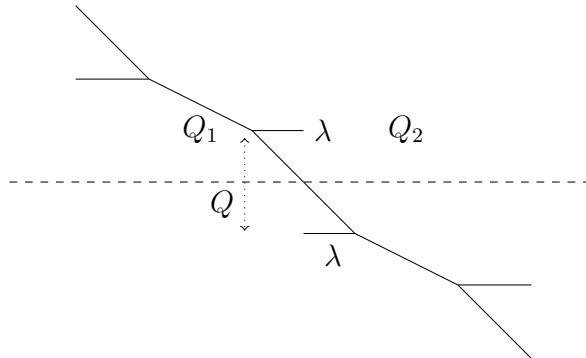
It is easy to understand the above expression as the original reflection state $|\Omega, 1\rangle\rangle$ provides a reflection in the 5-direction of the brane web, and therefore its S-dual should act as a reflection in the 6-direction, i.e. $z \rightarrow z^{-1}$ in $\eta(z)$ and $\Phi_\emptyset[z]$ in the refined topological vertex, and $J_n \rightarrow -J_{-n}$ serves as this reflection¹. It implies that this reflection can effectively be achieved by interchanging the arguments in the Schur functions associated to the first and second legs in the topological vertices in the right half of the web diagram, i.e. we replace λ with λ^t in the Schur function appearing in the topological vertex $C_{\mu,\nu,\lambda}$, as we can see that the first leg and the second leg of the vertex exchanges under the reflection map along the 6-direction. This is the prescription proposed in [45] for the topological formalism in the presence of an $O5^-$ plane. Let us consider the partition function of this “Sp(0)” theory with two flavor under the following assignment of the Kähler parameters,



It is possible to perform some proper geometric transitions to obtain the following equivalent configuration,



Following [45], it is more convenient to map the right half of the diagram to its mirror image with respect to the $O5^-$ orientifold to read off the expression of the partition function,



¹The minus sign here takes Φ^* to Φ and is a manifestation of the fact that the first and the second leg of the topological vertex are exchanged in the mirror image.

The partition function is given by

$$Z_{\text{Sp}(0), N_f=2} = \mathcal{G}(Q_1)\mathcal{G}(Q_2) \frac{\mathcal{G}(Q)\mathcal{G}(QQ_1Q_2)}{\mathcal{G}(QQ_1)\mathcal{G}(QQ_2)} \sum_{\lambda} f_{\lambda}^{-3} Q^{|\lambda|} \frac{N_{\lambda\lambda^t}(Q; q, q^{-1}) N_{\emptyset\lambda}(Q_1; q, q^{-1}) N_{\lambda^t\emptyset}(Q_2; q, q^{-1})}{N_{\lambda\lambda}(1; q, q^{-1}) N_{\emptyset\lambda^t}(QQ_1; q, q^{-1}) N_{\lambda\emptyset}(QQ_2; q, q^{-1})}, \quad (4.1.13)$$

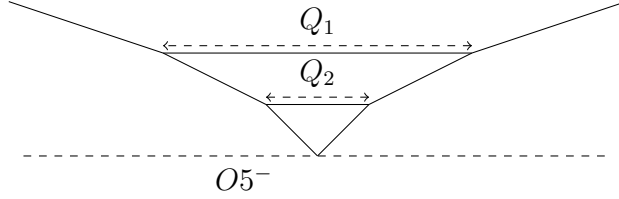
where we denoted the framing factor in the unrefined case as $f_{\nu} := f_{\nu}(q, q)$. We can check with *mathematica* to see that $Z_{\text{Sp}(0), N_f=2}$ does not depend on Q at all, and in particular

$$Z_{\text{Sp}(0), N_f=2} = \mathcal{G}(Q_1)\mathcal{G}(Q_2), \quad (4.1.14)$$

which is precisely the partition function of two decoupled $U(1)$ theories, $U(1) \otimes U(1)$. The cancellation of dependence on Q in the partition function does not rely on the number of flavors in the theory, especially the pure “ $\text{Sp}(0)$ ” theory is truly a trivial theory, whose partition function is

$$Z_{\text{Sp}(0)} = \mathcal{G}(Q) \sum_{\lambda} f_{\lambda}^{-3} Q^{|\lambda|} \frac{N_{\lambda\lambda^t}(Q; q, q^{-1})}{N_{\lambda\lambda}(1; q, q^{-1})} = 1. \quad (4.1.15)$$

Adding one more D5 brane gives the brane web for the pure $\text{Sp}(1)$ gauge theory,



Of course, as $\text{Sp}(1) \simeq \text{SU}(2)$, the instanton partition function of an $\text{Sp}(1)$ gauge theory should be equivalent to that of an $\text{SU}(2)$ theory after proper identification of parameters. Let us see how this happens at the level of one instanton.

$$\begin{aligned} Z_{\text{Sp}(1)} \propto \sum_{\lambda_1, \lambda_2} (Q_2 Q_F^4)^{|\lambda_1|} (Q_2)^{|\lambda_2|} f_{\lambda_1}^{-5} f_{\lambda_2}^{-3} \frac{N_{\lambda_1\lambda_1^t}(Q_2 Q_F^2; q, q^{-1}) N_{\lambda_2\lambda_2^t}(Q_2; q, q^{-1})}{N_{\lambda_1\lambda_1}(1; q, q^{-1}) N_{\lambda_2\lambda_2}(1; q, q^{-1})} \\ \times \frac{N_{\lambda_1\lambda_2^t}(Q_2 Q_F; q, q^{-1}) N_{\lambda_2\lambda_1^t}(Q_2 Q_F; q, q^{-1})}{N_{\lambda_1\lambda_2}(Q_F; q, q^{-1}) N_{\lambda_2^t\lambda_1^t}(Q_F; q, q^{-1})} \\ \times \mathcal{G}(Q_2 Q_F^4) \mathcal{G}(Q_2) \mathcal{G}(Q_2 Q_F) \mathcal{G}(Q_2 Q_F), \end{aligned} \quad (4.1.16)$$

where we defined $Q_F := (Q_1/Q_2)^{1/4}$ following the notation of [45]. One can see that the effective complexified gauge coupling is proportional to Q_2 in the gauge theory, and by extracting out the term of order Q_2 , we obtain

$$Z_{\text{Sp}(1) \text{ one instanton}} = - \frac{2Q_2 Q_F^2}{(1 - q^{-1})(1 - q)(1 - Q_F)^2}. \quad (4.1.17)$$

Indeed the complexified gauge coupling is given by $\mathfrak{q} = Q_2 Q_F$ and the Coulomb modulus corresponds to Q_F in this setup, as the one instanton partition function of $SU(2)$ is given by

$$Z_{SU(2) \text{ one instanton}} = \frac{2\mathfrak{q}}{(1 - q^{-1})(1 - q)(1 - Q_F)(1 - Q_F^{-1})}. \quad (4.1.18)$$

We note that the Young diagram labels $\lambda_{1,2}$ have almost nothing to do with the instanton number. Contributions from different numbers of instantons mix at the same level of $|\lambda_{1,2}|$. It is claimed in [45] that the above computation agrees with the result obtained from E-strings up to 10-instanton orders.

4.2 qq-characters and Quiver \mathcal{W} -algebra

We have been deriving the expressions of the qq-characters from the Ward identity of the DIM algebra in the preferred direction in this article so far. It is in fact possible to convert this computation to a formulation in the unpreferred direction by using the AFS property [61].

Let us consider the simplest case, a pure $U(1)$ gauge theory. For later convenience, let us write down the graphic representation of the AFS property,

$$(\rho_u^{(1,n)} \otimes \rho_v^{(0,1)})\Delta(g(z))\Phi^{*(n)}[u, v] = \Phi^{*(n)}[u, v]\rho_{-uv}^{(1,n+1)}(g(z)), \quad (4.2.1)$$

$$\Phi^{(n)}[u, v](\rho_v^{(0,1)} \otimes \rho_u^{(1,n)})\Delta(g(z)) = \rho_{-uv}^{(1,n+1)}(g(z))\Phi^{(n)}[u, v], \quad (4.2.2)$$

for $g = x^\pm, \psi^\pm$ as

$$\begin{array}{c} \text{---} x^+(z) \end{array} + \begin{array}{c} x^+(z) \\ \text{---} \psi^-(z) \end{array} = \begin{array}{c} \text{---} \\ x^+(z) \end{array}$$

$$\begin{array}{c} \psi^+(\gamma^{1/2}z) \\ \text{---} x^-(\gamma z) \end{array} + \begin{array}{c} x^-(z) \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ x^-(z) \end{array}$$

$$\begin{array}{c} \text{---} \\ x^+(z) \end{array} + \begin{array}{c} x^+(\gamma z) \text{---} \\ \psi^-(\gamma^{1/2}z) \end{array} = \begin{array}{c} x^+(z) \\ \text{---} \end{array}$$

$$\begin{array}{c} \psi^+(z) \text{ --- } | \\ | \\ x^-(z) \end{array} + \begin{array}{c} x^-(z) \text{ --- } | \\ | \end{array} = \begin{array}{c} x^-(z) \\ | \end{array}$$

and

$$\begin{array}{c} \psi^\pm(\gamma^{\pm 1/2}z) \\ | \text{ --- } \psi^\pm(z) \end{array} = \begin{array}{c} | \text{ --- } \\ | \\ \psi^\pm(z) \end{array}$$

$$\begin{array}{c} \psi^\pm(\gamma^{\mp 1/2}z) \text{ --- } | \\ | \\ \psi^\pm(z) \end{array} = \begin{array}{c} \psi^\pm(z) \\ | \end{array}$$

We recall that the coproduct structure in the DIM algebra is given by

$$\begin{aligned} \Delta(x^+(z)) &= x^+(z) \otimes 1 + \psi^-(\hat{\gamma}_{(1)}^{1/2}z) \otimes x^+(\hat{\gamma}_{(1)}z), \\ \Delta(x^-(z)) &= x^-(\hat{\gamma}_{(2)}z) \otimes \psi^+(\hat{\gamma}_{(2)}^{1/2}z) + 1 \otimes x^-(z), \\ \Delta(\psi^+(z)) &= \psi^+(\hat{\gamma}_{(2)}^{1/2}z) \otimes \psi^+(\hat{\gamma}_{(1)}^{-1/2}z), \\ \Delta(\psi^-(z)) &= \psi^-(\hat{\gamma}_{(2)}^{-1/2}z) \otimes \psi^-(\hat{\gamma}_{(1)}^{1/2}z), \end{aligned} \tag{4.2.3}$$

where $\hat{\gamma}_{(1)} = \hat{\gamma} \otimes 1$ and $\hat{\gamma}_{(2)} = 1 \otimes \hat{\gamma}$. These equalities allows us to rewrite the insertion considered in section 3.7 to derive the fundamental qq-character²

$$\begin{array}{c} | \\ | \text{ --- } x^+(z) \\ | \text{ --- } \\ | \\ \psi^-(\gamma^{-1/2}z) \end{array}$$

in two ways:

²For later convenience, we further multiplied a $\psi^-(\gamma^{-1/2}z)$ on the right bottom Fock space.

$$\begin{array}{ccccccc}
x^+(z) & & & & & & x^+(\gamma^{-1}z) \\
- \left| \begin{array}{c} \psi^-(z) \\ \hline \end{array} \right| & + & \left| \begin{array}{c} x^+(z) \\ \hline \end{array} \right| & = & - \left| \begin{array}{c} \psi^-(\gamma^{-1/2}z) \\ \hline \end{array} \right| & + & \left| \begin{array}{c} x^+(\gamma^{-1}z) \\ \hline \end{array} \right| \\
& & \psi^-(\gamma^{-1/2}z) & & & &
\end{array}$$

The above equation can be reorganized into

$$\begin{array}{ccccccc}
x^+(z) & \psi^-(\gamma^{-1/2}z) & & x^+(\gamma^{-1}z) & & & \\
\left| \begin{array}{c} \hline \end{array} \right| & + & \left| \begin{array}{c} \hline \end{array} \right| & = & \left| \begin{array}{c} \hline \end{array} \right| & + & \left| \begin{array}{c} \hline \end{array} \right| \\
& & & & x^+(z) & \psi^-(\gamma^{-1/2}z) & x^+(\gamma^{-1}z)
\end{array}$$

We note that the above equality takes exactly the form that $\Delta(x^+)(\gamma^{-1}z) = x^+(z) \otimes 1 + \psi^-(\gamma^{1/2}z) \otimes x^+(\gamma z)$ commutes with the brane web operator $\mathcal{S} := \Phi \cdot \Phi^*$. By using the contraction rules listed in (3.5.10) to (3.5.25), one can confirm that the vacuum expectation value of $\Delta(x^+)(\gamma^{-1}z)$ inserted from the unpreferred direction agrees with the vacuum expectation value of the fundamental qq-character for A_1 quiver,

$$\langle \chi_1(z) \rangle = \frac{\widehat{0}}{\widehat{\Phi}} \otimes \frac{\widehat{0}}{\widehat{\Phi^*}} + \frac{\widehat{0}}{\widehat{\Phi}} \otimes \frac{\widehat{0}}{\widehat{\Phi^*}}. \quad (4.2.4)$$

We further remark that the $U(1)$ boson part, $\alpha(z)$ and $\beta(z)$, in the q -deformed \mathcal{W} -algebra discussed in section 3.2 (see e.g. equation (3.2.15)) acts trivially on $\mathcal{S} = \Phi \cdot \Phi^*$, and thus the qq-character can also be obtained from the expectation value of the stress tensor $t(z\gamma^{-1})$ in (3.2.15),

$$\langle \chi_1(z) \rangle = \widehat{t(z\gamma^{-1})}. \quad (4.2.5)$$

This is the manifestation of Kimura-Pestun's quiver \mathcal{W} -algebra [34] for A_1 quiver in the topological vertex formalism. Since the stress tensor $t(z)$ also commutes with the brane web operator $\mathcal{S} = \Phi \cdot \Phi^*$, we see that \mathcal{S} plays the role of the screening charge in the quiver \mathcal{W} -algebra. We remark that from the result of [34], it is in fact an infinite product of the screening charges in the q -deformed Virasoro algebra in the usual sense.

We can repeat the same procedure for D -type quivers, although we note that it can only be performed in the unrefined case, i.e. with $q_3 = 1$, as the operator $\Omega_{-1,\alpha}$, introduced in the previous

section to represent the effect of \mathbf{ON}^0 orientifold, maps elements in the DIM algebra outside the algebra. This might suggest that the full construction for D -type quiver gauge theories on a generic refined Ω -background is still not available in our approach.

In the unrefined case, the AFS property simplifies a lot, as $\psi^\pm(z)$ are always equal to 1 in any representation we used. We can easily find by using (4.1.1) or (4.1.8) that for the D_2 quiver, which was intensively discussed in the previous section,

$$\tilde{t}_{D_2}(z) := x^+(z) \otimes 1 + x^-(z) \otimes 1 + 1 \otimes x^+(z) + 1 \otimes x^-(z), \quad (4.2.6)$$

commutes with the corresponding brane web operator, and its expectation value matches with the vector qq-character $\chi_{(1,1)}(z)$,

$$\langle \chi_{(1,1)}(z) \rangle = \widehat{\tilde{t}_{D_2}(z)}. \quad (4.2.7)$$

In general, for a D_n quiver, the same quantity takes the form,

$$\tilde{t}_{D_n}(z) := \sum_{i=1}^n (x^{+(i)}(z) + x^{-(i)}(z)). \quad (4.2.8)$$

It is reasonable to expect that by further attaching a $U(1)$ part to $\tilde{t}_{D_n}(z)$, it gives rise to the stress tensor in the q -deformed \mathcal{W} -algebra of D_n type defined in [62]. We leave this confirmation as a future task.

Chapter 5

Elliptic Topological Vertex and 6d $\mathcal{N} = (1, 0)$ theories

In the previous chapter, we introduced new elements into the brane web diagram to generalize the original framework of refined topological string on toric Calabi-Yau manifold and assign new vertices to the newly introduced elements. However, there are also physical contexts that one can keep the web diagram fixed but generalize the topological vertex itself applied to the web. We will see in this chapter that this kind of deformation of the vertex corresponds to the deformation from Schur function to symmetric functions with more parameters.

5.1 6d gauge theories and compactified brane web

Our starting point is the $D6$ - $D8$ brane construction for 6d $\mathcal{N} = (1, 0)$ gauge theories. The brane configuration is given in Table 5.1 (see also Figure 5.1) and all the possible choices of gauge theories constructed in this way (together with orientifolds) are classified in [80, 81, 82].

This class of theories can be dualized to 5d brane webs by compactifying the 6-direction on a circle, and performing the T-duality. There is, however, a flux conservation condition on each NS5 brane, as each D6 brane ending on NS5 generates a source of RR 7-form charge. This constraint makes the construction of 6d gauge theories very rigid, i.e. there has always to be the same number of D6 branes on the left and the right of each NS5 brane. Translated into the language of gauge theories, it becomes the anomaly cancellation condition. Let us give a brief explanation on it based on the discussion given in [83]. The anomaly 8-form coming from the gauge multiplet is given by

$$I_8 = \text{tr}_{adj} F^4 - \sum_R n_R \text{tr}_R F^4 =: \alpha \text{tr}_{fund} F^4 + c (\text{tr}_{fund} F^2)^2. \quad (5.1.1)$$

When there is no fourth order Casimir in the gauge group, α will always be trivially zero, and the only condition will be $c > 0$ so that the anomaly can be cancelled by adding tensor multiplets into

	0	1	2	3	4	5	6	7	8	9
D8	—	—	—	—	—	—	•	—	—	—
NS5	—	—	—	—	—	—	•	•	•	•
D6	—	—	—	—	—	—	—	•	•	•

Table 5.1: Brane configuration for the $D6$ - $D8$ system.



Figure 5.1: An example of the configuration of the $D6$ - $D8$ system along the 6-direction of the spacetime. In this figure, we represent a D6-brane, an NS5-brane and a D8-brane respectively with a horizontal line, a black dot and a small circle with cross. The flux conservation requires the number of D6-branes ending from left side and from the right side to be equal for each NS5-brane, but one can still end D6-branes on a D8-brane to change the number of D6-branes among chambers separated by NS5-branes.

the theory. For example, in the case of $SU(2)$, the matter contents are restricted to fundamental hypermultiplets and c is proportional to [83]

$$16 - n_{fund}. \quad (5.1.2)$$

Further in [84], the global anomaly is argued to put a strict constraint on n_{fund} and it results in that the only allowed matter content is $n_{fund} = 4$ or 10 for $SU(2)$ gauge group. In the case of gauge groups with the fourth order Casimir, for example $SU(N)$ with $N > 3$, there will be two constraints, $\alpha = 0$ and $c > 0$, which narrow down the possible combinations of different representations. For $SU(N)$, $N > 8$, there are only three possibilities: $2N$ copies of fundamental hypermultiplets; $N+8$ fundamental hypermultiplets plus one antisymmetric hypermultiplets; and $N-8$ fundamental hypermultiplets plus one symmetric hypermultiplets. Depending on the number of N , different choices of matter contents will be possible, but there is always the consistent possibility: $SU(N)$ theory with $2N$ fundamental matters, which is naturally realized in the brane construction.

The $D6$ - $D8$ system, after compactification and taking T-duality, becomes a (p, q) 5-brane system, compactified along the NS5 direction. Equivalently, in the language of toric diagram, the degenerate loci at the top are identified with those at the bottom (in the unpreferred direction). This set-up is argued in [85, 86] to be further dual to M2 branes suspended between parallel M5 brane on A -type ALE space, and its partition function can be again computed with the (refined) topological vertex applied to the toric diagram with its bottom and top identified. In this kind of computation, we always take the D5-direction in the brane diagram to be the preferred direction. Let us first compute the simplest example here, a strip with two external preferred legs (see Figure 5.2). Its

partition function is given by

$$\begin{aligned}
Z_2 &= P_\lambda(t^{-\rho}, q, t) P_{\sigma^t}(q^{-\rho}, t, q) \sum_{\mu, \nu} \sum_{\eta_1, \eta_2} (q/t)^{\frac{|\eta_1| - |\eta_2|}{2}} (-Q_1)^{|\nu|} (-Q_2)^{|\mu|} s_{\mu^t/\eta_1}(q^{-\lambda} t^{-\rho}) \\
&\quad \times s_{\nu/\eta_1}(t^{-\lambda^t} q^{-\rho}) s_{\mu/\eta_2}(t^{-\sigma^t} q^{-\rho}) s_{\nu^t/\eta_2}(q^{-\sigma} t^{-\rho}) \\
&= P_\lambda(t^{-\rho}, q, t) P_{\sigma^t}(q^{-\rho}, t, q) \prod_{i,j=1}^{\infty} (1 - Q_1 t^{-\lambda_i^t + j - \frac{1}{2}} q^{i - \sigma_j - \frac{1}{2}}) (1 - Q_2 t^{-\sigma_i^t + j - \frac{1}{2}} q^{i - \lambda_j - \frac{1}{2}}) \\
&\quad \times \sum_{\eta_1, \eta_2} (q/t)^{\frac{|\eta_1| - |\eta_2|}{2}} \sum_{\eta_3, \eta_4} (-Q_1)^{|\eta_1|} (-Q_2)^{|\eta_2|} s_{\eta_2^t/\eta_3}(-Q_2 q^{-\lambda} t^{-\rho}) s_{\eta_1^t/\eta_3^t}(t^{-\sigma^t} q^{-\rho}) \\
&\quad \times s_{\eta_2^t/\eta_4}(-Q_1 t^{-\lambda^t} q^{-\rho}) s_{\eta_1^t/\eta_4^t}(q^{-\sigma} t^{-\rho}),
\end{aligned}$$

where we used the Cauchy identity of the Schur function in the above calculation. Repeating this procedure for several times, we find

$$\begin{aligned}
Z_2 &= P_\lambda(t^{-\rho}, q, t) P_{\sigma^t}(q^{-\rho}, t, q) \prod_{i,j=1}^{\infty} (1 - Q_1 t^{-\lambda_i^t + j - \frac{1}{2}} q^{i - \sigma_j - \frac{1}{2}}) (1 - Q_2 t^{-\sigma_i^t + j - \frac{1}{2}} q^{i - \lambda_j - \frac{1}{2}}) \\
&\quad \times \prod_{i,j=1}^{\infty} \frac{1}{1 - Q_1 Q_2 t^{-\lambda_i^t + j} q^{-\lambda_j + i - 1}} \frac{1}{1 - Q_1 Q_2 t^{-\sigma_i^t + j - 1} q^{-\sigma_j + i}} \\
&\quad \times \sum_{\eta_3, \eta_4} (-Q_1)^{|\eta_3|} (-Q_2)^{|\eta_4|} \sum_{\eta_5, \eta_6} s_{\eta_4/\eta_5}(-Q_2 (t/q)^{1/2} q^{-\lambda} t^{-\rho}) s_{\eta_4^t/\eta_6}(-Q_1 (q/t)^{1/2} t^{-\sigma^t} q^{-\rho}) \\
&\quad \times s_{\eta_3/\eta_5}(-Q_1 t^{-\lambda^t} q^{-\rho}) s_{\eta_3^t/\eta_6}(-Q_2 q^{-\sigma} t^{-\rho}) \\
&= P_\lambda(t^{-\rho}, q, t) P_{\sigma^t}(q^{-\rho}, t, q) \prod_{i,j=1}^{\infty} (1 - Q_1 t^{-\lambda_i^t + j - \frac{1}{2}} q^{i - \sigma_j - \frac{1}{2}}) (1 - Q_2 t^{-\sigma_i^t + j - \frac{1}{2}} q^{i - \lambda_j - \frac{1}{2}}) \\
&\quad \times \prod_{i,j=1}^{\infty} \frac{1 - Q_1 Q_2^2 t^{-\sigma_i^t + j - \frac{1}{2}} q^{-\lambda_j + i - \frac{1}{2}}}{1 - Q_1 Q_2 t^{-\lambda_i^t + j} q^{-\lambda_j + i - 1}} \frac{1 - Q_1^2 Q_2 t^{-\lambda_i^t + j - \frac{1}{2}} q^{-\sigma_j + i - \frac{1}{2}}}{1 - Q_1 Q_2 t^{-\sigma_i^t + j - 1} q^{-\sigma_j + i}} \\
&\quad \times \sum_{\eta_5, \eta_6} (Q_1 Q_2)^{|\eta_6|} \sum_{\eta_7, \eta_8} s_{\eta_6^t/\eta_7}(-Q_2 (t/q)^{1/2} q^{-\lambda} t^{-\rho}) s_{\eta_5^t/\eta_7^t}(Q_1 Q_2 (q/t)^{1/2} t^{-\sigma^t} q^{-\rho}) \\
&\quad \times s_{\eta_6^t/\eta_8}(-Q_1 t^{-\lambda^t} q^{-\rho}) s_{\eta_5^t/\eta_8^t}(Q_1 Q_2 q^{-\sigma} t^{-\rho}) \\
&= P_\lambda(t^{-\rho}, q, t) P_{\sigma^t}(q^{-\rho}, t, q) \prod_{i,j=1}^{\infty} \frac{1 - Q_1 t^{-\lambda_i^t + j - \frac{1}{2}} q^{i - \sigma_j - \frac{1}{2}}}{1 - Q_1 Q_2 t^{-\lambda_i^t + j} q^{-\lambda_j + i - 1}} \frac{1 - Q_2 t^{-\sigma_i^t + j - \frac{1}{2}} q^{i - \lambda_j - \frac{1}{2}}}{1 - Q_1 Q_2 t^{-\sigma_i^t + j - 1} q^{-\sigma_j + i}} \\
&\quad \times \prod_{i,j=1}^{\infty} \frac{1 - Q_1 Q_2^2 t^{-\sigma_i^t + j - \frac{1}{2}} q^{-\lambda_j + i - \frac{1}{2}}}{1 - Q_1^2 Q_2^2 t^{-\lambda_i^t + j} q^{-\lambda_j + i - 1}} \frac{1 - Q_1^2 Q_2 t^{-\lambda_i^t + j - \frac{1}{2}} q^{-\sigma_j + i - \frac{1}{2}}}{1 - Q_1^2 Q_2^2 t^{-\sigma_i^t + j - 1} q^{-\sigma_j + i}} \\
&\quad \times \sum_{\eta_7, \eta_8} (Q_1 Q_2)^{|\eta_7|} \sum_{\eta_9, \eta_{10}} s_{\eta_8/\eta_9}(Q_1 Q_2^2 (t/q)^{1/2} q^{-\lambda} t^{-\rho}) s_{\eta_8^t/\eta_{10}}(Q_1 Q_2 (q/t)^{1/2} t^{-\sigma^t} q^{-\rho}) \\
&\quad \times s_{\eta_7/\eta_9}(-Q_1 t^{-\lambda^t} q^{-\rho}) s_{\eta_7^t/\eta_{10}}(Q_1 Q_2 q^{-\sigma} t^{-\rho}). \quad (5.1.3)
\end{aligned}$$

The above computation can be repeated for infinitely many times, and at the end, we obtain the partition function as a product of four series of infinite products of factors,

$$Z_2 = Z_2^{(1)} Z_2^{(2)} Z_2^{(3)} Z_2^{(4)}, \quad (5.1.4)$$

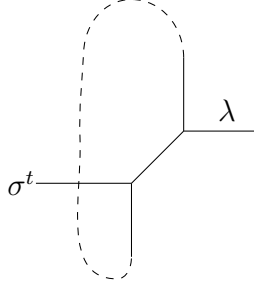


Figure 5.2: The compactified strip brane web with two external legs.

where each factor respectively takes the form,

$$\begin{aligned}
Z_2^{(1)} &= \prod_{n \geq 0} \prod_{i,j=1}^{\infty} (1 - Q_1(Q_1 Q_2)^n t^{-\lambda_i^t + j - \frac{1}{2}} q^{-\sigma_j + i - \frac{1}{2}}) \\
&= \prod_{n \geq 0} \prod_{i,j=1}^{\infty} (1 - Q_1(Q_1 Q_2)^n t^{j - \frac{1}{2}} q^{i - \frac{1}{2}}) \times N_{\lambda\sigma}(Q_1^{n+1} Q_2^n (t/q)^{1/2}; t, q^{-1}), \tag{5.1.5}
\end{aligned}$$

$$\begin{aligned}
Z_2^{(2)} &= \prod_{n \geq 0} \prod_{i,j=1}^{\infty} (1 - Q_2(Q_1 Q_2)^n t^{-\sigma_i^t + j - \frac{1}{2}} q^{-\lambda_j + i - \frac{1}{2}}) \\
&= \prod_{n \geq 0} \prod_{i,j=1}^{\infty} (1 - Q_2(Q_1 Q_2)^n t^{j - \frac{1}{2}} q^{i - \frac{1}{2}}) \times N_{\sigma\lambda}(Q_1^n Q_2^{n+1} (t/q)^{1/2}; t, q^{-1}), \tag{5.1.6}
\end{aligned}$$

$$\begin{aligned}
Z_2^{(3)} &= \prod_{n \geq 0} \prod_{i,j=1}^{\infty} \frac{1}{1 - (Q_1 Q_2)^{n+1} t^{-\lambda_i^t + j} q^{-\lambda_j + i - 1}} \\
&= \prod_{n \geq 0} \prod_{i,j=1}^{\infty} \frac{1}{1 - (Q_1 Q_2)^{n+1} t^j q^{i-1}} \times N_{\lambda\lambda}^{-1}(Q_1^{n+1} Q_2^{n+1} t/q; t, q^{-1}), \tag{5.1.7}
\end{aligned}$$

and

$$\begin{aligned}
Z_2^{(4)} &= \prod_{n \geq 0} \prod_{i,j=1}^{\infty} \frac{1}{1 - (Q_1 Q_2)^{n+1} t^{-\sigma_i^t + j - 1} q^{-\sigma_j + i}} \\
&= \prod_{n \geq 0} \prod_{i,j=1}^{\infty} \frac{1}{1 - (Q_1 Q_2)^{n+1} t^{j-1} q^i} \times N_{\sigma\sigma}^{-1}(Q_1^{n+1} Q_2^{n+1}; t, q^{-1}). \tag{5.1.8}
\end{aligned}$$

We may further use the identity (A.1.8) to convert the factors appearing in (5.1.6) and (5.1.7) to

$$Z_2^{(2)} = \prod_{n \geq 0} \prod_{i,j=1}^{\infty} (1 - Q_2(Q_1 Q_2)^n t^{j - \frac{1}{2}} q^{i - \frac{1}{2}}) \times N_{\lambda\sigma}(Q_1^n Q_2^{n+1} (t/q)^{1/2}; t^{-1}, q), \tag{5.1.9}$$

and

$$Z_2^{(3)} = \prod_{n \geq 0} \prod_{i,j=1}^{\infty} \frac{1}{1 - (Q_1 Q_2)^{n+1} t^j q^{i-1}} \times N_{\lambda\lambda}^{-1}(Q_1^{n+1} Q_2^{n+1}; t^{-1}, q). \quad (5.1.10)$$

This rewriting allows us to combine the Nekrasov factors depending on the same set of Young diagrams into an elliptic one,

$$\begin{aligned} N_{\lambda\sigma}^{ellip}(Q_1; t, q^{-1}, p) &:= N_{\lambda\sigma}(Q_1; t, q^{-1}) \prod_{n=1}^{\infty} N_{\lambda\sigma}(Q_1 p^n; t, q^{-1}) N_{\lambda\sigma}(Q_1^{-1} p^n; t^{-1}, q) \\ &= \prod_{n=0}^{\infty} \prod_{(i,j) \in \lambda} \left(1 - Q_1 p^n q^{\sigma_i - j + 1} t^{\lambda_j^t - i}\right) \prod_{(i,j) \in \sigma} \left(1 - Q_1 p^n q^{-\lambda_i + j} t^{-\sigma_j^t + i - 1}\right) \\ &\times \prod_{(i,j) \in \lambda} \left(1 - Q_1^{-1} p^{n+1} q^{-\sigma_i + j - 1} t^{-\lambda_j^t + i}\right) \prod_{(i,j) \in \sigma} \left(1 - Q_1^{-1} p^{n+1} q^{\lambda_i - j} t^{\sigma_j^t - i + 1}\right) \\ &= \prod_{(i,j) \in \lambda} \theta_p \left(Q_1 q^{\sigma_i - j + 1} t^{\lambda_j^t - i}\right) \prod_{(i,j) \in \sigma} \theta_p \left(Q_1 q^{-\lambda_i + j} t^{-\sigma_j^t + i - 1}\right), \end{aligned} \quad (5.1.11)$$

where we defined the θ -function, $\theta_p(x)$ as ¹

$$\theta_p(x) := \prod_{n=0}^{\infty} (1 - p^n x)(1 - p^{n+1} x^{-1}), \quad (5.1.12)$$

and in the current calculation, p is identified with $Q_1 Q_2$, i.e. the Kähler parameter of the compactification circle. When we have internal preferred lines, the remaining factors of the form $N_{\lambda\lambda}^{-1}$ can be absorbed into the contribution of the vector multiplet Nekrasov factor, and similarly, the combined contribution is given by the elliptic Nekrasov factor $N_{\lambda\sigma}^{ellip}$ defined above as

$$N_{\lambda\lambda}^{ellip}{}^{-1}(1; t, q^{-1}, p). \quad (5.1.13)$$

Overall factors as $\prod_{i,j=1}^{\infty} (1 - Q_1 (Q_1 Q_2)^n t^{j-\frac{1}{2}} q^{i-\frac{1}{2}})$ can be absorbed into the \mathcal{G} -factor introduced in (2.6.4),

$$\mathcal{G}(Q; t, q^{-1}) = \prod_{i,j=1}^{\infty} (1 - Q t^{i-1} q^j), \quad (5.1.14)$$

to define a new \mathcal{G} -factor,

$$\mathcal{G}^{ellip}(Q; q_1, q_2, p) := \exp \left(- \sum_{n \geq 0} \frac{1}{n(1 - q_1^n)(1 - q_2^n)} \left[Q^n + \frac{p^n}{1 - p^n} (Q^n + q_3^{-n} Q^{-n}) \right] \right). \quad (5.1.15)$$

¹It is more often in the literature to define the θ -function by further including a normalization factor $(p; p)_{\infty}$, but this factor is cancelled in the current calculation and thus it is more convenient to define the θ -function without this factor in this article.

With the above notations, one can express the strip partition function with two external legs,

$$Z_2 = Z_2^{(3)} Z_2^{(4)} \mathcal{G}^{ellip}(Q_1(t/q)^{1/2}; t, q^{-1}, p) N_{\lambda\sigma}^{ellip}(Q_1(t/q)^{1/2}; t, q^{-1}, p). \quad (5.1.16)$$

The partition function of strip with more preferred legs can be carried out in the same way, however, it is in general more convenient to convert the Schur functions in the topological vertex into vertex operators and calculate the torus correlator of them. Let us recall how to compute a torus correlator of harmonic oscillators, $[\alpha, \alpha^\dagger] = 1$, where we denote the torus modulus as τ . For a normal-ordered correlator $: V(\alpha, \alpha^\dagger) :$, the torus correlator, i.e. the trace of $: V(\alpha, \alpha^\dagger) :$ can be computed as

$$\begin{aligned} \text{tr}_\tau : V(\alpha, \alpha^\dagger) : &:= \text{tr} \left(e^{-\tau \alpha^\dagger \alpha} : V(\alpha, \alpha^\dagger) : \right) = \sum_{n \geq 0} \frac{1}{n!} e^{-n\tau} \langle 0 | \alpha^n : V(\alpha, \alpha^\dagger) : \alpha^{\dagger n} | 0 \rangle \\ &= \int dz d\bar{z} e^{-z\bar{z}} \langle 0 | e^{\alpha\bar{z}} : V(\alpha, \alpha^\dagger) : e^{e^{-\tau} \alpha^\dagger z} | 0 \rangle, \end{aligned} \quad (5.1.17)$$

where we used the orthogonality of the Gaussian integral,

$$\int dz d\bar{z} z^n \bar{z}^m e^{-z\bar{z}} = n! \delta_{n,m}. \quad (5.1.18)$$

By further using

$$\alpha e^{e^{-\tau} \alpha^\dagger z} | 0 \rangle = z e^{-\tau} e^{e^{-\tau} \alpha^\dagger z} | 0 \rangle, \quad \langle 0 | e^{\alpha\bar{z}} \alpha^\dagger = \langle 0 | e^{\alpha\bar{z}} \bar{z}, \quad (5.1.19)$$

we obtain

$$\text{tr}_\tau : V(\alpha, \alpha^\dagger) : = \int dz d\bar{z} e^{-(1-e^{-\tau})z\bar{z}} V(e^{-\tau} z, \bar{z}). \quad (5.1.20)$$

Especially what we are interested in is $: V :$ of the form

$$: V(\alpha, \alpha^\dagger) : = \exp(c\alpha^\dagger) \exp(d\alpha), \quad (5.1.21)$$

and its trace can be computed to

$$\text{tr}_\tau : \phi(\alpha, \alpha^\dagger) : = \frac{1}{1 - e^{-\tau}} \exp\left(\frac{e^{-\tau}}{1 - e^{-\tau}} cd\right). \quad (5.1.22)$$

In the computation of trace of free boson vertex operators, note that the propagator is given by $L_0 = \sum_{n>0} J_{-n} J_n$, we have

$$\text{tr}_\tau : \prod_{n>0} V_n(J_n, J_{-n}) := \prod_{n>0} \int dz_n d\bar{z}_n e^{-(1-e^{-n\tau})z_n \bar{z}_n} \phi_n(e^{-n\tau} z_n, n\bar{z}_n), \quad (5.1.23)$$

and with the notation

$$V_n(J_n, J_{-n}) =: \exp(c_n J_{-n}) \exp(d_n J_n) :, \quad (5.1.24)$$

the trace formula for vertex operators is obtained as

$$\mathrm{tr}_\tau : \prod_{n>0} V_n(J_n, J_{-n}) := \exp \left(\sum_{n>0} n \frac{e^{-n\tau}}{1 - e^{-n\tau}} c_n d_n - \log(1 - e^{-n\tau}) \right). \quad (5.1.25)$$

Comparison with the result from the topological vertex formalism can be done under the identification,

$$p = e^{-\tau}, \quad (5.1.26)$$

and we can see from the formula (5.1.25) that the θ -function structure is embedded in the factor $\frac{p^n}{1-p^n}$. It is now straightforward to check with (5.1.25) that the instanton partition function for 6d $\mathcal{N} = (1, 0)$ theory can be alternatively obtained from the partition function of its corresponding T-dual 5d theory under the replacement

$$N_{\lambda\nu}(Q; q_1, q_2) \rightarrow N_{\lambda\nu}^{ellip}(Q; q_1, q_2, p), \quad \mathcal{G}(Q; q_1, q_2) \rightarrow \mathcal{G}^{ellip}(Q; q_1, q_2, p). \quad (5.1.27)$$

We remark that in the limit when the compactification circle of the (p, q) 5-brane system is infinitely large, i.e. when the $D6$ - $D8$ system is truly T-dual to the 5-brane web,

$$N_{\lambda\nu}^{ellip}(Q; q_1, q_2, p) \rightarrow N_{\lambda\nu}(Q; q_1, q_2), \quad \text{for } p \rightarrow 0, \quad (5.1.28)$$

and thus the 6d instanton partition function reduces to the corresponding 5d partition function. These two observations allow us to view p as a deformation parameter, and motivate us to introduce a p -deformed topological vertex to reproduce the 6d instanton partition function.

5.2 Elliptic Topological Vertex

As the 6d instanton partition function can be written as a torus correlation function, we can also apply the thermo double trick to compute the trace. Let us again illustrate this idea with the example of the harmonic oscillators.

$$\mathrm{tr} \left(p^{\alpha^\dagger \alpha} V(\alpha, \alpha^\dagger) : \right) = \langle 0 | e^{b\alpha} p^{\alpha^\dagger \alpha} V(\alpha, \alpha^\dagger) e^{\alpha^\dagger b^\dagger} | 0 \rangle, \quad (5.2.1)$$

where b and b^\dagger form a new set of harmonic oscillators, $[b, b^\dagger] = 1$, and this time instead of the orthogonality of the Gaussian integral, we used the orthogonality of the basis spanned by b and b^\dagger . For vertex operators of the form,

$$V(\alpha, \alpha^\dagger) = \exp(c\alpha^\dagger) \exp(d\alpha), \quad (5.2.2)$$

we can verify by the explicit evaluation and comparison to (5.1.22) to see that

$$\mathrm{tr} \left(p^{\alpha^\dagger \alpha} V(\alpha, \alpha^\dagger) : \right) = \frac{1}{1-p} \langle 0 | V\left(\frac{\alpha}{1-p} + b^\dagger, \alpha^\dagger - \frac{b}{1-p^{-1}}\right) | 0 \rangle. \quad (5.2.3)$$

We note that $|0\rangle$ here is a shorthand notation for a tensor product of the vacuum state with respect to α and that of b . Therefore the above rewriting allows us to re-express the trace of vertex operators into the vacuum expectation value of two sets of vertex operators commuting with each other respectively depending on α and b only. This method is known as the Clavelli-Shapiro trick [87]. We see that the elliptic vertex $V(\frac{\alpha}{1-p} + b^\dagger, \alpha^\dagger - \frac{b}{1-p^{-1}})$ whose VEV gives the same results as the trace of the original vertex operator $V(\alpha, \alpha^\dagger)$ is constituted from two p -deformed Harmonic oscillators,

$$\alpha^{(p)} := \frac{1}{1-p}\alpha, \quad \alpha^{(p)\dagger} := \alpha^\dagger, \quad (5.2.4)$$

$$b^{(p)} = \frac{p}{1-p}b, \quad b^{(p)\dagger} := b^\dagger, \quad (5.2.5)$$

which satisfy

$$[\alpha^{(p)}, \alpha^{(p)\dagger}] = \frac{1}{1-p}, \quad [b^{(p)}, b^{(p)\dagger}] = \frac{p}{1-p}, \quad (5.2.6)$$

and certainly both $\alpha^{(p)}$ and $\alpha^{(p)\dagger}$ commute with $b^{(p)}$ and $b^{(p)\dagger}$. Note that we are left with a rescaling freedom, $b \rightarrow \Lambda b$ and $b^\dagger \rightarrow \Lambda^{-1}b^\dagger$, therefore only the product of the coefficients before these two oscillator, cd , matters in the final result. Similarly, when we go to the realistic model to consider the trace of vertex operators of a free boson J_n , we would like to replace it with a p -deformed boson $J_n^{(p)}$ satisfying

$$[J_n^{(p)}, J_m^{(p)\dagger}] = \frac{n}{1-p^n} \delta_{n+m,0}, \quad (5.2.7)$$

and introducing a new boson $K_n^{(p)}$ that commutes with $J_n^{(p)}$ satisfying

$$[K_n^{(p)}, K_m^{(p)\dagger}] = \frac{np^n}{1-p^n} \delta_{n+m,0}. \quad (5.2.8)$$

We need to duplicate the vertex operator and substitute $K_n^{(p)}$ into J_n in the new copy of the vertex to complete the Clavelli-Shapiro trick for the free boson. Applying a similar idea to the topological vertex, we obtain the following definitions for the elliptic topological vertex, $\Phi^{\text{ellip}(n)}[u, v]$,

$$\Phi_\lambda^{\text{ellip}(n)}[u, v] := \Phi^{\text{ellip}(n)}[u, v] |v, \lambda\rangle := (-uv)^{|\lambda|} \prod_{x \in \lambda} (\gamma/\chi_x)^{n+1} : \Phi_\emptyset^{\text{ellip}}[v] \prod_{x \in \lambda} \eta^{\text{ellip}}(\chi_x) :, \quad (5.2.9)$$

where

$$\begin{aligned} \Phi_\emptyset^{\text{ellip}}[z] := & \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1-t^{-n}} J_{-n}^{(p)} z^n \right) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1-q^n} J_n^{(p)} z^{-n} \right) \\ & \times \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1-t^{-n}} K_{-n}^{(p)} z^{-n} \right) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1-q^n} K_n^{(p)} z^n \right), \end{aligned} \quad (5.2.10)$$

$$\begin{aligned} \eta^{ellip}(z) = & \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} t^n (1 - q^{-n}) z^n J_{-n}^{(p)} \right) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} (1 - t^{-n}) z^{-n} J_n^{(p)} \right) \\ & \times \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} t^n (1 - q^{-n}) z^{-n} K_{-n}^{(p)} \right) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} (1 - t^{-n}) z^n K_n^{(p)} \right). \end{aligned} \quad (5.2.11)$$

The reason that z^n is replaced by z^{-n} in the K -boson part comes from the fact that b is attached to α^\dagger and b^\dagger is combined into α in the Clavelli-Shapiro trick (c.f. equation (5.2.3)) and we cannot rescale the variable z away by changing the normalization. In the context of the trace computation, the K -boson part corresponds to the contraction of a bottom vertex with a top vertex through the compactification circle. Similarly the dual elliptic topological vertex $\Phi^{ellip*(n)}[u^*, v]$ is given by

$$\Phi_\lambda^{ellip*(n)}[u^*, v] := \langle v, \lambda | \Phi^{ellip*(n)}[u^*, v] := (u^* \gamma)^{-|\lambda|} \prod_{x \in \lambda} (\chi_x / \gamma)^n : \Phi_\emptyset^{ellip*}[v] \prod_{x \in \lambda} \xi^{ellip}(\chi_x) :, \quad (5.2.12)$$

where

$$\begin{aligned} \Phi_\emptyset^{ellip*}[v] := & \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{\gamma^n v^n}{1 - t^{-n}} J_{-n}^{(p)} \right) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{\gamma^n v^{-n}}{1 - q^n} J_n^{(p)} \right) \\ & \times \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \frac{\gamma^n v^n}{1 - t^{-n}} K_{-n}^{(p)} \right) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{\gamma^n v^{-n}}{1 - q^n} K_n^{(p)} \right) \end{aligned} \quad (5.2.13)$$

and

$$\begin{aligned} \xi^{ellip}(z) = & \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} t^n (1 - q^{-n}) z^n \gamma^n J_{-n}^{(p)} \right) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} (1 - t^{-n}) z^{-n} \gamma^n J_n^{(p)} \right) \\ & \times \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} t^n (1 - q^{-n}) z^{-n} \gamma^{-n} K_{-n}^{(p)} \right) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} (1 - t^{-n}) z^n \gamma^{-n} K_n^{(p)} \right). \end{aligned} \quad (5.2.14)$$

One can compute the contraction rules of these elliptic vertex operators, and in particular, we have

$$\eta^{ellip}(z) \eta^{ellip}(w) = \frac{\theta_p(w/z) \theta_p(q_3^{-1} w/z)}{\theta_p(q_1 w/z) \theta_p(q_2 w/z)} : \eta^{ellip}(z) \eta^{ellip}(w) :. \quad (5.2.15)$$

The corresponding contraction factor in the original DIM case is $S^{-1}(w/z)$ and this S -function determines the commutation relation of the underlying DIM algebra through

$$g(z) = \frac{S(z)}{S(z^{-1})}. \quad (5.2.16)$$

Therefore we can interpret it in the current case that an elliptic version of the DIM algebra is obtained, whose S -function is given by

$$S^{ellip}(z) = \frac{\theta_p(q_1 z) \theta_p(q_2 z)}{\theta_p(z) \theta_p(q_1 q_2 z)}. \quad (5.2.17)$$

Interestingly this algebra of Ding-Iohara type specified by

$$g^{ellip}(z) = \frac{S^{ellip}(z)}{S^{ellip}(z^{-1})} = \frac{\theta_p(q_1 z) \theta_p(q_2 z) \theta_p(q_3 z)}{\theta_p(q_1^{-1} z) \theta_p(q_2^{-1} z) \theta_p(q_3^{-1} z)}, \quad (5.2.18)$$

where we used the property $\theta_p(z) = -z\theta_p(z^{-1})$ of the θ -function, is exactly the elliptic DIM algebra introduced by Saito in [88].

We list the contraction rules of elliptic topological vertices in the unpreferred direction to conclude this section,

$$\overbrace{\Phi_\mu^{ellip}[u_2, v_2] \Phi_\lambda^{ellip}[u_1, v_1]} = \mathcal{G}^{ellip -1}(v_1/v_2 q_3; q_1, q_2, p) N_{\lambda\mu}^{ellip}(v_1/v_2 q_3; q_1, q_2, p)^{-1} : \Phi_\mu^{ellip}[u_2, v_2] \Phi_\lambda^{ellip}[u_1, v_1] :, \quad (5.2.19)$$

$$\overbrace{\Phi_\mu^{ellip*}[u_2, v_2] \Phi_\lambda^{ellip*}[u_1, v_1]} = \mathcal{G}^{ellip -1}(v_1/v_2; q_1, q_2, p) N_{\lambda\mu}^{ellip}(v_1/v_2; q_1, q_2, p)^{-1} : \Phi_\mu^{ellip*}[u_2, v_2] \Phi_\lambda^{ellip*}[u_1, v_1] :, \quad (5.2.20)$$

$$\overbrace{\Phi_\mu^{ellip}[u_2, v_2] \Phi_\lambda^{ellip*}[u_1, v_1]} = \mathcal{G}^{ellip}(v_1/v_2 \gamma; q_1, q_2, p) N_{\lambda\mu}^{ellip}(v_1/v_2 \gamma; q_1, q_2, p) : \Phi_\mu^{ellip}[u_2, v_2] \Phi_\lambda^{ellip*}[u_1, v_1] :, \quad (5.2.21)$$

$$\overbrace{\Phi_\mu^{ellip*}[u_2, v_2] \Phi_\lambda^{ellip}[u_1, v_1]} = \mathcal{G}^{ellip}(v_1/v_2 \gamma; q_1, q_2, p) N_{\lambda\mu}^{ellip}(v_1/v_2 \gamma; q_1, q_2, p) \Phi_\mu^{ellip*}[u_2, v_2] \Phi_\lambda^{ellip}[u_1, v_1] : . \quad (5.2.22)$$

5.3 Elliptic Ding-Iohara-Miki Algebra, Elliptic \mathcal{W} -algebra and Representation Webs

Let us reproduce the algebraic relations in the elliptic Ding-Iohara-Miki algebra here.

$$[\psi^\pm(z), \psi^\pm(w)] = 0, \quad (5.3.1)$$

$$\psi^+(z) \psi^-(w) = \frac{g^{ellip}(\hat{\gamma} z/w)}{g^{ellip}(\hat{\gamma}^{-1} z/w)} \psi^-(w) \psi^+(z) \quad (5.3.2)$$

$$\psi^\pm(z) x^+(w) = g^{ellip}\left(\hat{\gamma}^{\pm \frac{1}{2}} z/w\right) x^+(w) \psi^\pm(z) \quad (5.3.3)$$

$$\psi^\pm(z) x^-(w) = g^{ellip}\left(\hat{\gamma}^{\mp \frac{1}{2}} z/w\right)^{-1} x^-(w) \psi^\pm(z) \quad (5.3.4)$$

$$x^\pm(z) x^\pm(w) = g^{ellip}(z/w)^{\pm 1} x^\pm(w) x^\pm(z) \quad (5.3.5)$$

$$[x^+(z), x^-(w)] = \frac{\theta_p(q_1) \theta_p(q_2)}{(p; p)_\infty^2 \theta_p(q_1 q_2)} \left(\delta(\hat{\gamma} w/z) \psi^+\left(\hat{\gamma}^{\frac{1}{2}} w\right) - \delta(\hat{\gamma}^{-1} w/z) \psi^-\left(\hat{\gamma}^{-\frac{1}{2}} w\right) \right), \quad (5.3.6)$$

where the factor $\frac{\theta_p(q_1) \theta_p(q_2)}{(p; p)_\infty^2 \theta_p(q_1 q_2)}$ can be expressed as $-\text{Res}_{z \rightarrow 1} S^{ellip}(z)$ and is also encoded by the $S(z)$ function. A big difference between the elliptic DIM algebra and the usual DIM algebra is that as the

θ -function contains both z and z^{-1} as an expansion about p , $\psi^\pm(z)$ have both positive and negative modes in them. $\hat{\gamma}$ is still by definition a central element, while zero modes ψ_0^\pm are no longer central elements in the algebra. We also expect that the $\text{SL}(2, \mathbb{Z})$ automorphism existed in the DIM algebra is lost in its elliptic sibling.

However, since the elliptic DIM algebra reduces to the DIM algebra in the $p \rightarrow 0$ limit (where $\theta_p(x) \rightarrow (1 - x)$), a class of representations of the elliptic DIM algebra should also exist so that they goes to the (p, q) -representations of the DIM algebra in the same limit. Representations corresponding to the $(1, n)$ representations, are constructed in [88], with a simple replacement of vertex operators by their elliptic versions,

$$x^+(z) \mapsto u\gamma^n z^{-n} \eta^{\text{ellip}}(z), \quad x^-(z) \mapsto u^{-1} \gamma^{-n} z^n \xi^{\text{ellip}}(z), \quad \psi^\pm(z) \mapsto \gamma^{\mp n} \varphi_{\text{ellip}}^\pm(z), \quad \hat{\gamma} \mapsto \gamma, \quad (5.3.7)$$

where

$$\varphi_{\text{ellip}}^+(z) =: \eta^{\text{ellip}}(z\gamma^{1/2}) \xi^{\text{ellip}}(z\gamma^{-1/2}) :, \quad \varphi_{\text{ellip}}^-(z) =: \eta^{\text{ellip}}(z\gamma^{-1/2}) \xi^{\text{ellip}}(z\gamma^{1/2}) :. \quad (5.3.8)$$

That the above representation makes sense, especially that it satisfies the commutation relation of $[x^+(z), x^-(w)]$, is guaranteed by the following Ramanujan's identity

$$\frac{1}{\theta_p(x)} + \frac{x^{-1}}{\theta_p(x^{-1})} = \frac{1}{(p; p)_\infty^2} \delta(x). \quad (5.3.9)$$

Its proof is particularly simple, as we can separate the θ -function into two parts,

$$\theta_p(x) = (1 - x) \prod_{n=1}^{\infty} (1 - p^n x)(1 - p^n x^{-1}), \quad (5.3.10)$$

where the second part, i.e. the infinite product is symmetric about $x \leftrightarrow x^{-1}$, and the left-hand side of (5.3.9) can be organized into

$$\frac{1}{\theta_p(x)} + \frac{x^{-1}}{\theta_p(x^{-1})} = \left(\frac{1}{1 - x} + \frac{x^{-1}}{1 - x^{-1}} \right) \frac{1}{\prod_{n=1}^{\infty} (1 - p^n x)(1 - p^n x^{-1})}, \quad (5.3.11)$$

and the identity follows from the definition of the δ -function,

$$\frac{1}{1 - x} + \frac{x^{-1}}{1 - x^{-1}} = \sum_{n \geq 0} x^n + \sum_{n \geq 1} x^{-n} = \sum_{n \in \mathbb{Z}} x^n = \delta(x). \quad (5.3.12)$$

More generally, corresponding to the general formula (3.1.30), we have the generalized Ramanujan's identity,

$$\frac{\prod_i \theta_p(a_i z)}{\prod_j \theta_p(b_j z)} - (-z)^{|\{i\}| - |\{j\}|} \frac{\prod_i a_i \theta_p(z^{-1} a_i^{-1})}{\prod_j b_j \theta_p(z^{-1} b_j^{-1})} = \frac{1}{(p; p)_\infty^2} \sum_k \frac{\prod_i \theta_p(a_i / b_k)}{\prod_{j \neq k} \theta_p(b_j / b_k)} \delta(b_k z). \quad (5.3.13)$$

This identity allows us to further uplift the $(0, 1)$ representation of the DIM algebra to a representation of the elliptic DIM algebra, by replacing the \mathcal{Y} -function defined in (3.4.23) to an elliptic version,

$$\mathcal{Y}_\lambda^{ellip}(z) := \theta_p(v/z) \prod_{x \in \lambda} S^{ellip}(\chi_x/z). \quad (5.3.14)$$

The explicit expression of the $(0, 1)$ representation of the elliptic DIM algebra is given by

$$x^+(z) |v, \lambda\rangle = \sum_{x \in A(\lambda)} \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{z \mathcal{Y}_\lambda^{ellip}(z)} |v, \lambda + x\rangle, \quad (5.3.15)$$

$$x^-(z) |v, \lambda\rangle = \gamma^{-1} \sum_{x \in R(\lambda)} \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} z^{-1} \mathcal{Y}_\lambda^{ellip}(z q_3^{-1}) |v, \lambda - x\rangle, \quad (5.3.16)$$

$$\psi^\pm(z) |v, \lambda\rangle = \gamma^{-1} \Psi_\lambda^{ellip}(z) |v, \lambda\rangle, \quad (5.3.17)$$

where similarly we have

$$\Psi_\lambda^{ellip}(z) = \frac{\mathcal{Y}_\lambda^{ellip}(z q_3^{-1})}{\mathcal{Y}_\lambda^{ellip}(z)}, \quad (5.3.18)$$

and again the matrix element of ψ_n^\pm is obtained respectively by expanding the above function in the form of $\theta_p(z)$ and $-z\theta_p(z^{-1})$. By setting the normalization to be $\langle v, \lambda | v, \mu \rangle = a_\lambda^{ellip}{}^{-1} \delta_{\lambda, \mu}$ for

$$a_\lambda^{ellip} = \frac{(v\gamma)^{-|\lambda|} \left(\prod_{(i,j) \in \lambda} \chi_{(i,j)} \right)}{N_{\lambda\lambda}^{ellip}(1; q_1, q_2, p)}, \quad (5.3.19)$$

the dual $(0, 1)$ representation of the elliptic DIM algebra can be found as

$$\langle v, \lambda | x^+(z) = -\gamma^{-1} \sum_{x \in R(\lambda)} \langle v, \lambda - x | \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} z^{-1} \mathcal{Y}_\lambda^{ellip}(z q_3^{-1}), \quad (5.3.20)$$

$$\langle v, \lambda | x^-(z) = - \sum_{x \in A(\lambda)} \langle v, \lambda + x | \delta(z/\chi_x) \operatorname{Res}_{z \rightarrow \chi_x} \frac{1}{z \mathcal{Y}_\lambda^{ellip}(z)}, \quad (5.3.21)$$

$$\langle v, \lambda | \psi^\pm(z) = \gamma^{-1} \Psi_\lambda^{ellip}(z) \langle v, \lambda|. \quad (5.3.22)$$

Inspired by the uplift of these representations, one may want to define a new expansion series for elements in the elliptic DIM algebra,

$$x^\pm(z) =: \sum_{n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}} x_{m,n}^\pm z^{-n} p^m, \quad \psi^\pm(z) =: \sum_{n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}} \psi_{m,n}^\pm z^{-n} p^m, \quad (5.3.23)$$

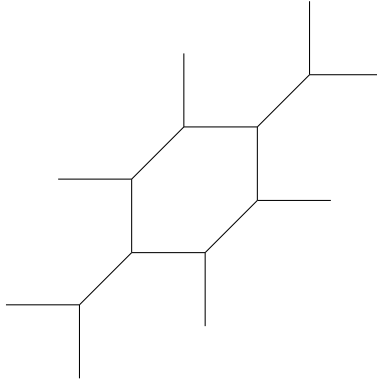
and it is tempting to claim that $\psi_{0,0}^\pm$ are central in the elliptic algebra. Unfortunately, we have not managed to show or disprove it so far, as the usual way to prove ψ_0^\pm are central elements in the DIM algebra does not work in this case.

Another statement that can be uplifted to the elliptic circumstance is the existence of the \mathcal{W} -symmetry. As explicitly shown in [41], the elliptic Virasoro symmetry can be embedded in the $(2, 0) \simeq (1, 0) \otimes (1, 0)$ representation of the elliptic DIM algebra. We mimic the work of [32] and [41] and show the embedding of a general elliptic \mathcal{W}_N algebra in the $(N, 0) \simeq \bigotimes^N (1, 0)$ representation in Appendix B.2. This fact suggests the existence of an elliptic version of the AGT relation for 6d $\mathcal{N} = 1$ gauge theories of type $\mathcal{T}_{(A,A)}$, which was first proposed in [41, 42].

With the uplifted $(1, n)$ and $(0, 1)$ representations built in the elliptic DIM algebra, we can show the AFS property for the elliptic topological vertex with a completely parallel computation to that shown in section 3.5 by replacing all quantities with their elliptic versions [48]. That is to say, once we are given a brane web diagram, we assign to each vertex in the diagram an elliptic topological vertex and compute the partition function by evaluating the VEV of these elliptic topological vertices, i.e. schematically we have

$$\begin{aligned}
Z_{\text{ellip}} = & \begin{array}{ccccccc}
\widehat{0} & & \widehat{0} & & \widehat{0} & & \widehat{0} \\
\Phi^{\text{ellip}}[u_{11}, v_1] & \cdot & \Phi^{\text{ellip}*}[u_{21}, v_1] & & & & \\
\Phi^{\text{ellip}}[u_{12}, v_2] & \cdot & \Phi^{\text{ellip}*}[u_{22}, v_2] & & \vdots & & \vdots \\
\vdots & \cdot & \vdots & & & & \\
& & \Phi^{\text{ellip}}[u'_{21}, v'_1] & \cdot & \Phi^{\text{ellip}*}[u_{31}, v'_1] & \dots & \vdots \\
\vdots & \cdot & \vdots & & & & \\
\Phi^{\text{ellip}}[u_{1N}, v_N] & \cdot & \Phi^{\text{ellip}}[u_{2N}, v_N] & & \vdots & & \vdots \\
& & \Phi^{\text{ellip}}[u'_{2(N'-1)}, v'_{N'-1}] & \cdot & \Phi^{\text{ellip}*}[u_{3(N'-1)}, v'_{N'-1}] & & \\
& & \Phi^{\text{ellip}}[u'_{2N'}, v'_{N'}] & \cdot & \Phi^{\text{ellip}*}[u_{3N'}, v'_{N'}] & & \vdots \\
\widehat{0} & & \widehat{0} & & \widehat{0} & & \widehat{0}
\end{array} \quad (5.3.24)
\end{aligned}$$

The difference from the topological vertex formalism for 5d gauge theories is that we are only allowed to consider the web diagrams corresponding to superconformal field theories, due to the flux conservation constraint on the field contents of 6d theories. Let us take the following diagram, which corresponds to the 6d $\text{SU}(2)$ gauge theory with four flavors, as an example.

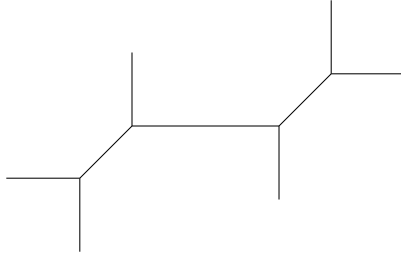


Its partition function can be calculated via

$$\begin{array}{ccc}
\widehat{0} & & \widehat{0} \\
\Phi^{ellip(n)}[u, v_1] & \cdot & \Phi^{ellip(n^*)}[u^*v_1/m_3, m_3] \quad |m_1, \emptyset\rangle \\
\Phi^{ellip*(n)}[uv_1/m_1, m_1] & & \Phi^{ellip*(n^*)}[u^*, v] \\
\Phi^{ellip(n)}[uv_1/m_1, v_2] & \cdot & \Phi^{ellip(n^*)}[u^*, m_4] \quad |m_2, \emptyset\rangle \\
\langle m_2, \emptyset| \Phi^{ellip*(n)}[uv_1v_2/(m_1m_2), m_2] & & \Phi^{ellip*(n^*)}[u^*m_4/v_2, v_2] \\
\widehat{0} & & \widehat{0}
\end{array} \quad (5.3.25)$$

In this way, we established the computation of the Nekrasov partition function for 6d $\mathcal{N} = (1, 0)$ gauge theories via a representation web of the elliptic DIM algebra as we did for 5d $\mathcal{N} = 1$ theories in section 3.6. One can see that this representation web takes the same form as the brane web of the corresponding 5d $\mathcal{N} = 1$ theory obtained from the underlying 6d $\mathcal{N} = (1, 0)$ theory after S^1 compactification. We can equivalently say that this is a brane web T-dual to the $D6$ - $D8$ system.

We can also perform the Ward identity trick discussed in section 3.7 to the representation web of elliptic topological vertices. Let us again look at the simplest example of $U(1)$ gauge theory with A_1 quiver, whose representation web is given by the following diagram.



The insertion of elements in the elliptic DIM algebra, say $x^-(z)$,

$$\begin{array}{ccc} \widehat{0} & & \widehat{0} \\ & \Phi^{ellip}[u^*, m_2] & |m_2, \emptyset\rangle \\ \Phi^{ellip}[u, v] & \cdot x_{>}^- \cdot & \Phi^{ellip*}[u^*, v] \\ \langle m_1, \emptyset | & \Phi^{ellip*}[u, m_1] & \\ \widehat{0} & & \widehat{0} \end{array}, \quad (5.3.26)$$

leads to the equality,

$$\begin{aligned}
& \gamma^{-1} \sum_{\lambda} a_{\lambda}^{ellip} \sum_{x \in R(\lambda)} \frac{\widehat{0}}{\Phi_{\emptyset}^{ellip*}} \frac{\Phi_{\emptyset}^{ellip}}{\Phi_{\lambda}^{ellip*}} \frac{1}{z - \chi_x} \text{Res}_{z \rightarrow \chi_x} \mathcal{Y}_{\lambda}^{ellip}(z q_3^{-1}) \\
&= - \sum_{\lambda} a_{\lambda}^{ellip} \sum_{x \in A(\lambda)} \frac{\widehat{0}}{\Phi_{\emptyset}^{ellip*}} \frac{\Phi_{\emptyset}^{ellip}}{\Phi_{\lambda+x}^{ellip*}} \frac{1}{z - \chi_x} \text{Res}_{z \rightarrow \chi_x} \frac{1}{\mathcal{Y}_{\lambda}^{ellip}(z)},
\end{aligned}$$

or equivalently

$$\begin{aligned}
& \sum_{\lambda} \mathfrak{q}^{|\lambda|} \frac{N_{\lambda \emptyset}^{ellip}(v/m_1; q_1, q_2, p) N_{\emptyset \lambda}^{ellip}(m_2/v; q_1, q_2, p)}{N_{\lambda \lambda}^{ellip}(1; q_1, q_2, p)} \\
& \times \oint_{C_{\lambda}} dx \left(\frac{z}{z-x} \frac{\mathcal{Y}_{\lambda}^{ellip}(x q_3^{-1})}{\theta_p(x/(m_1 q_3))} + \frac{\mathfrak{q} v q_3 \theta_p(m_2/x)}{(z-x) \mathcal{Y}_{\lambda}^{ellip}(x)} \right) = 0,
\end{aligned} \tag{5.3.27}$$

where $\mathfrak{q} = -\frac{u \gamma^{n-n^*-1}}{u^*}$ and same as the 5d case, C_{λ} surrounds all poles in $\{\chi_x | x \in A(\lambda) \cup R(\lambda)\}$. Converting to the standard form of the qq-character, the qq-character of 6d gauge theories that corresponds to the fundamental representation of A_1 quiver is given by the above contour integral around $x \sim z$ over elliptic Y -operators:

$$\chi_1^{ellip}(z) = \mathcal{Y}^{ellip}(z q_3^{-1}) + \frac{\mathfrak{q} P^{ellip}(z)}{\mathcal{Y}^{ellip}(z)}, \tag{5.3.28}$$

where $P^{ellip}(z) = v q_3 z^{-1} \theta_p\left(\frac{x}{m_1 q_3}\right) \theta_p\left(\frac{m_2}{x}\right)$. Recall that after we complete the contour integral form of the qq-character (i.e. (5.3.27)), we need to convert it to the integral around the remaining poles including $x \sim z$, the origin and the infinity, where we further require the pole at the origin to vanish to constrain the matter contents. The asymptotic behavior of the elliptic Y -function $\mathcal{Y}^{ellip}(z)$ near the origin, however, is drastically different from $\mathcal{Y}(z)$:

$$\mathcal{Y}_{\lambda}^{ellip}(z) \sim \theta_p(1/z), \quad z \rightarrow 0. \tag{5.3.29}$$

The essential singularity introduced by the θ -function can only be cancelled by another copy of θ -function, which is deduced from the matter multiplet. To cancel such essential singularities in both $\mathcal{Y}^{ellip}(z q_3^{-1})$ and $1/\mathcal{Y}^{ellip}(z)$ in the A_1 qq-character, we need two matter multiplets, i.e. $N_f = 2$, in the A_1 quiver gauge theory with $U(1)$ gauge group, as we considered in the above concrete example. In general for the gauge group $SU(N)$, this constraint asks us to consider the $N_f = 2N$ theory

as the only physical one in this context. This result matches exactly with the flux conservation condition discussed in section 5.1.

It is also possible to repeat what we did in section 4.2 to convert the Ward identity of the elliptic DIM algebra to the qq-character expressed as an expectation value of certain operator in the Fock space of the unpreferred direction. It is again related to the elliptic version of the quiver \mathcal{W} -algebra discussed in [89].

At last, we comment that one can follow the AFS rewriting, presented in section 3.1 in this article, in the backward direction to write down the matrix element of the elliptic topological vertex as a p -deformation of the IKV form of the refined topological vertex. This work was done in [49]. The elliptic version of the vertex operator (2.4.27) is given by

$$V_{\pm}^{ellip}(\vec{x}) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_i x_i^n J_{\pm n}^{(p)} \right) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \sum_i x_i^{-n} K_{\pm n}^{(p)} \right). \quad (5.3.30)$$

One has the freedom to choose the basis to write down the matrix element, as pointed out in [31] that the IKV form is equivalent to the Awata-Kanno form proposed in [98] as matrix elements of the same vertex operator in different basis. Let us first introduce the Macdonald basis written in terms of q -bosons,

$$[a_n^{qt}, a_m^{qt}] = n \frac{1 - q^{|n|}}{1 - t^{|n|}} \delta_{m+n, 0}. \quad (5.3.31)$$

The Macdonald basis can be abstractly expressed as $|P_{\lambda}\rangle$, which can be obtained by expanding the Macdonald function $P_{\lambda}(x, q, t)$ in terms of the power sum $p_n(x) = \sum_i x_i^n$, and replacing the power sum with the q -boson,

$$p_n(x) \rightarrow a_n^{qt}. \quad (5.3.32)$$

In the definition of the dual bra basis $\langle P_{\lambda}|$, we perform the dual replacement

$$p_n(x) \rightarrow a_{-n}^{qt}. \quad (5.3.33)$$

Similarly, one introduces the basis associated to the dual Macdonald function $Q_{\lambda}(x, q, t)$: $|Q_{\lambda}\rangle$ and $\langle Q_{\lambda}|$. We remark that this Macdonald basis reduces to the Frobenius basis used to express the Schur function (refer to (A.2.8)) through the bosonization of free fermions. The Macdonald version of (A.2.4) is given by

$$\langle P_{\lambda}| V_{-}^{qt}(x) |Q_{\mu}\rangle = P_{\lambda/\mu}(x, q, t), \quad \langle P_{\mu}| V_{+}^{qt}(x) |Q_{\lambda}\rangle = Q_{\lambda/\mu}(x, q, t), \quad (5.3.34)$$

where

$$V_{\pm}^{qt}(x) = \exp \left(\sum_i \sum_{n=1}^{\infty} \frac{x_i^n}{n} \frac{1 - t^n}{1 - q^n} a_{\pm n}^{qt} \right). \quad (5.3.35)$$

One observes that taking $t \rightarrow 0$ in the above vertex operator, it satisfies exactly the same algebraic relation as the $J^{(p)}$ -part of the vertex operator V_{\pm}^{ellip} , i.e. $\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_i x_i^n J_{\pm n}^{(p)}\right)$. Therefore, we see that the Macdonald basis in the $t \rightarrow 0$ limit is a convenient one to work with. Let us denote

$$|\lambda; q\rangle_J := \lim_{t \rightarrow 0} |Q\lambda\rangle, \quad \langle \lambda; q|_J := \lim_{t \rightarrow 0} \langle P\lambda|, \quad (5.3.36)$$

and identify $J_n^{(p)} = \lim_{t \rightarrow 0} \frac{1}{1-q^{|n|}} a_n^{qt}$, then the vertex operator can be expressed as a product of two skew q -Whittaker functions, as

$$\langle \mu, p|_J \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_i x_i^n J_n^{(p)}\right) |\lambda, p\rangle_J = W'_{p, \lambda/\mu}(x), \quad \langle \lambda, p|_J \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_i x_i^n J_{-n}^{(p)}\right) |\mu, p\rangle_J = W_{p, \lambda/\mu}(x), \quad (5.3.37)$$

where $W_{q, \lambda}(x) = \lim_{t \rightarrow 0} P_{\lambda}(x, q, t)$ is the q -Whittaker function defined as the $t \rightarrow 0$ limit of the Macdonald function, $W'_{q, \lambda}(x) = \lim_{t \rightarrow 0} Q_{\lambda}(x, q, t)$ is the dual version of the q -Whittaker function, and a similar expression holds for the $K^{(p)}$ -part of the elliptic topological vertex. In total, the matrix element of the elliptic topological vertex is given by²

$$C_{\mathbf{Y}_1, \mathbf{Y}_2, \lambda}^{ellip} = t^{n(\lambda)} \prod_{(i,j) \in \lambda} \frac{1}{\theta_p(q^{a(i,j)} t^{\ell(i,j)+1})} \sum_{\mathbf{Y}} \mathbf{W}_{\mathbf{Y}_1^t/\mathbf{Y}}(q^{-\lambda} t^{-\rho+\{1/2\}}) \mathbf{W}'_{\mathbf{Y}_2/\mathbf{Y}}(t^{-\lambda^t} q^{-\rho-\{1/2\}}), \quad (5.3.38)$$

where $\mathbf{Y} = (Y^{(1)}, Y^{(2)})$ is a set of Young diagrams,

$$\mathbf{W}_{\mathbf{Y}_1^t/\mathbf{Y}_2}(x) := W_{p, Y_1^{(1)}/Y_2^{(1)}}(x) W_{p^{-1}Y_1^{(2)}/Y_2^{(2)}}^*(-x^{-1}), \quad \mathbf{W}'_{\mathbf{Y}_1^t/\mathbf{Y}_2}(x) := W'_{p, Y_1^{(1)}/Y_2^{(1)}}(x) W'_{p^{-1}Y_1^{(2)}/Y_2^{(2)}}(x^{-1}), \quad (5.3.39)$$

with $*$ denoting the q -Whittaker function after the involution action,

$$\iota : p_n(x) \mapsto (-1)^{n-1} p_n(x), \quad (5.3.40)$$

i.e. $W_{p, Y}^*(x) := \iota \circ W_{p, Y}(x)$. We see that uplifting the gauge theory from 5d to 6d deforms the matrix element of the topological vertex in the same way as the Schur function deformed to the q -Whittaker function. This observation motivates us to further consider the full Macdonald deformation of the topological vertex analyzed in [90, 91]. Its physical meaning and the relation with algebras of Ding-Iohara type remain to be interesting topics to study in the future.

²We omitted the factor that corresponds to $q^{\frac{\|\mu^t\|^2}{2}} t^{-\frac{\|\mu\|^2}{2}}$ in the refined topological vertex, as the framing factor in the current framework is always trivial.

Chapter 6

Conclusion and Discussions

In this article, we considered the generalization of the topological vertex formalism to 5d $\mathcal{N} = 1$ gauge theories of type $\mathcal{T}_{(D,A)}$ and 6d $\mathcal{N} = (1,0)$ gauge theories of the class $\mathcal{T}_{(A,A)}$. In particular, in the first class of theories, we introduced reflection states defined in (4.1.7) and (4.1.2) to express the effect of the orientifold used in the brane construction. On the unrefined Ω -background, these reflection operators map x^\pm in the DIM algebra to x^\mp , which acts in the same way as the automorphism of the algebra that maps $x^5 \rightarrow -x^5$ on the 5-6 plane of the (p,q) 5-brane system (see Table 2.1). For 6d $\mathcal{N} = (1,0)$ theories, the algebraic structure of the topological vertex formalism is uplifted to the elliptic DIM algebra and an elliptic version of the refined topological vertex was found to be embedded in the elliptic DIM algebra in the same way as the usual refined topological vertex in the DIM algebra. This underlying elliptic DIM algebra behind the topological vertex formalism that can be used to compute the Nekrasov partition function of 6d theories provides a natural explanation to the discovery of the elliptic version of the AGT relation in [41, 42], and also a beautiful mathematical framework to compute physical quantities such as the qq-characters as discussed in section 5.3.

There are, however, still a lot of open problems remaining to be solved as future works. First of all, despite that we proposed a refined topological vertex formalism to compute the instanton partition function of D -type quiver gauge theories, as mentioned in section 4.2, the reflection state (4.1.2) maps elements outside the DIM algebra, and thus it is still not clear what is the algebraic structure appearing in the unpreferred direction of the brane web with orientifolds. We expect it to be an algebra that contains D -type q -deformed \mathcal{W} -algebras from Kimura-Pestun's work [34], and this might be related to recent progress in mathematics to embed BCD -type \mathcal{W} -algebras in the $\mathcal{W}_{1+\infty}$ -algebra [26]. A potential more systematic way to solve this problem is to observe from our uplift of 5d $\mathcal{N} = 1$ theories to 6d $\mathcal{N} = (1,0)$ theories in which we only replaced $S(z)$ that completely determines the DIM algebraic structure by its elliptic version, $S^{ellip}(z)$. The fact that the underlying algebra only depends on $S(z)$ originates from the expression (3.6.10) of the Nekrasov

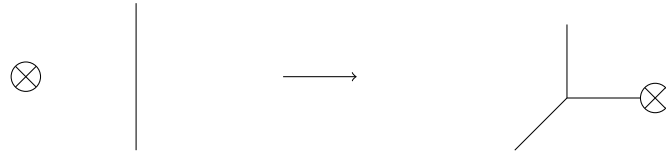
factor,

$$N_{\lambda\mu}(v_1/v_2; q_1, q_2) = \prod_{x \in \lambda} (1 - \chi_x/v_2) \prod_{y \in \mu} (1 - v_1 q_1 q_2 / \chi_y) \prod_{x \in \lambda, y \in \mu} S(\chi_y / \chi_x), \quad (6.0.1)$$

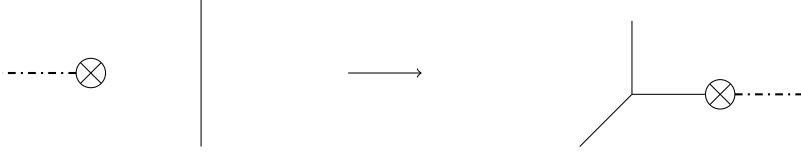
as the algebra connects instanton partition functions differing by one instanton number. As we can see from the LMNS integral form (3.8.4), the above structure that allows us to write the partition function as a product of S -functions is rather common when there exists an ADHM construction of instanton solutions. The ADHM constructions for $ABCD$ -type gauge groups are known in the literature [92], and we expect to discover an algebra of Ding-Iohara type with $S(z)$ replaced by a proper function appearing in the ADHM construction for $\mathrm{Sp}(N)$ gauge group. This algebra might be what we need in the unpreferred direction of the brane web with orientifolds, and we wish to establish a defining relation similar to the AFS property for a proper refined topological vertex for D -type quivers in the future.

There is also a recent work [93] that proposed the instanton partition function for gauge theories with non-simply-laced quivers, i.e. $BCFG$ -type quivers. These quiver structures are not approachable with the conventional techniques in the string theory, but one would expect to have such quivers from the fiber-base duality. [93] wrote down the partition function for these quivers by requiring the existence of the quiver \mathcal{W} -algebra of $BCFG$ -type in analogue to [34]. Roughly speaking, the instanton partition function is obtained by performing the instanton counting on an Ω -background with twisting parameters $q_{1,2}^{(i)}$ differing on each quiver node i : $q_1^{(i)} = q_1^{d_i}$, $q_2^{(i)} = q_2$, where $d_i = (\alpha_i, \alpha_i)$ and α_i is the simple root associated to the i -th node. One can also introduce a new type of vertex operators that intertwine between a $\mathrm{DIM}_{q_1^{d_i}, q_2}$ algebra and a $\mathrm{DIM}_{q_1^{d_{i+1}}, q_2}$ algebra and reproduces the instanton partition function with the generalized topological vertex formalism including the new vertices [94]. The new vertex appearing here, however, does not have a clear physical meaning as the orientifold used in the construction of D -type quiver, and it leaves us a task to do in the future.

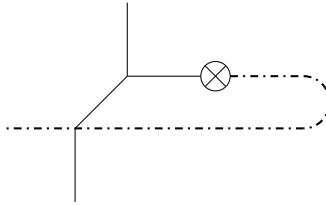
Another open problem comes from the naive question about that the Hanany-Witten transition can take a brane web, whose top and bottom branes can be identified, to a brane web that does not allow such an identification. The simplest example is given in the following diagram.



To solve this tension between the Hanany-Witten transition and the T-duality that takes the $D6$ - $D8$ system to a compactified (p, q) 5-brane web, we recall that there is a monodromy for the axio-dilaton field when we go around a 7-brane. That is to say, if we add the branch cut of the monodromy into the above diagram, it will look like



and in fact we rotated the monodromy cut to the other side. The compactified brane webs used to describe 6d theories, on the other hand, are defined on a cylinder, which means that the rotation of the cut cannot be done in this case. The remnant of the monodromy cut modifies the diagram and makes the identification possible again.



In the topological vertex formalism for 5d gauge theories, we can simply ignore this phenomenon, as the external legs in the toric diagram are mapped to vacuum states in the Fock space. In the compactified brane webs, however, we have to introduce a new vertex for the locations where 5-branes bump into the branch cuts. This new vertex must satisfy the property that the partition function remains to be the same (up to some overall factor given by the \mathcal{G} -factor) before and after the Hanany-Witten transition. A potential solution¹ is to require the 5-brane to keep its Young-diagram label untouched when crossing the cut, while it also introduces a proper framing factor to cancel the prefactor in the refined topological vertex. It still needs a lot of consistency tests for this proposal, but we remark that the invariance of the partition function before and after the Hanany-Witten transition is built in the generalized topological vertex formalism with the elliptic topological vertex. This might be one of the advantages to use the elliptic topological vertex.

In addition to the generalization and deformation of the existing topological vertex formalism associated to 5d $\mathcal{N} = 1$ gauge theories, or $D0$ - $D4$ instanton counting in the language of D-branes, one may also want to apply the same paradigm to systems such as the $D0$ - $D6$ [95] and $D0$ - $D8$ BPS counting [96, 97], as the ADHM construction is also given in these cases. We leave it as a future work to do.

¹Based on personal communication with Futoshi Yagi.

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Appendix A

Preliminaries

A.1 Nekrasov Factor

The Nekrasov factor has various equivalent expressions, which all look so differently. Its definition is given by

$$N_{\lambda\nu}(Q; q_1, q_2) := \prod_{(i,j) \in \lambda} \left(1 - Qq_2^{-\nu_i+j-1} q_1^{\lambda_j^t-i}\right) \prod_{(i,j) \in \nu} \left(1 - Qq_2^{\lambda_i-j} q_1^{-\nu_j^t+i-1}\right), \quad (\text{A.1.1})$$

and two of the equivalent expressions were derived in [98]:

$$N_{\lambda\nu}^{IKV}(Q; q_1, q_2) = \prod_{i,j=1}^{\infty} \frac{1 - Qq_2^{\nu_j-i} q_1^{-\lambda_i^t+j-1}}{1 - Qq_2^{-i} q_1^{j-1}}, \quad (\text{A.1.2})$$

which is one of the key identities in [57], and

$$N_{\lambda\nu}^{AK}(Q; q_1, q_2) = \prod_{i,j=1}^{\infty} \frac{(Qq_1^{i-j-1} q_2^{\nu_i-\lambda_j}; q_2)_{\infty}}{(Qq_1^{i-j} q_2^{\nu_i-\lambda_j}; q_2)_{\infty}}. \quad (\text{A.1.3})$$

The convention used here is that $(i, j) \in \lambda$ is the box in the i -th row and j -th column of the Young diagram λ . In some cases, it is more convenient to rescale Q to Qq_1q_2 , and rewrite (A.1.3) to

$$N_{\lambda\nu}^{loc}(Q = v_1/v_2; q_1, q_2) = \prod_{(x,x') \in \chi_{\lambda} \times \chi_{\nu}} \frac{(q_2x'/x; q_2)_{\infty}}{(q_1q_2x'/x; q_2)_{\infty}}, \quad (\text{A.1.4})$$

where $\mathcal{X}_{\lambda} = \{v_1q_1^{k-1}q_2^{\lambda_k}\}_{k=1}^{\infty}$ and $\mathcal{X}_{\nu} = \{v_2q_1^{k-1}q_2^{\nu_k}\}_{k=1}^{\infty}$.

It is straightforward to derive

$$\frac{N_{(\lambda+s)\nu}^{loc}(Q; q_1, q_2)}{N_{\lambda\nu}^{loc}(Q; q_1, q_2)} = \prod_{x' \in \mathcal{X}_{\nu}} \frac{1 - q_2x'/x_s}{1 - q_1q_2x'/x_s}, \quad (\text{A.1.5})$$

$$\frac{N_{\lambda(\nu+s)}^{loc}(Q; q_1, q_2)}{N_{\lambda\nu}^{loc}(Q; q_1, q_2)} = \prod_{x' \in \mathcal{X}_{\lambda}} \frac{1 - q_1x_s/x'}{1 - x_s/x'}, \quad (\text{A.1.6})$$

where $x_s = v_1 q_1^{k-1} q_2^{\lambda_k+1}$ for some $s = (k, \lambda_k + 1) \in A(\lambda)$ in the first recursive relation and a similar quantity in the second one for some box s in $A(\nu)$.

From (A.1.1), we can easily derive two useful identities,

$$N_{\lambda\nu}(Q; q_1, q_2) = (-Q)^{|\lambda|+|\nu|} q_1^{n(\lambda)-n(\nu)} q_2^{-n(\nu^t)+n(\lambda^t)} N_{\nu\lambda}((Qq_1q_2)^{-1}; q_1, q_2), \quad (\text{A.1.7})$$

and

$$N_{\lambda\nu}(Q; q_1, q_2) = N_{\nu\lambda}(Qq_1^{-1}q_2^{-1}; q_1^{-1}, q_2^{-1}). \quad (\text{A.1.8})$$

A.2 Schur functions

The definition of a Schur function $s_\lambda(\{x_i\}_{i=1}^n)$ is

$$s_\lambda(x_1, \dots, x_n) := \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})}, \quad (\text{A.2.1})$$

for some Young diagram $\lambda = \{\lambda_j\}$. It is a symmetric polynomial and the set of all Schur functions forms a complete basis of symmetric polynomials. Therefore, the product of two Schur functions can again be expressed in Schur functions,

$$s_\mu s_\nu = \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda. \quad (\text{A.2.2})$$

With this fusion coefficient $c_{\mu\nu}^\lambda$ of Schur functions, we define the skew Schur function,

$$s_{\lambda/\mu} := \sum_{\nu} c_{\mu\nu}^\lambda s_\nu. \quad (\text{A.2.3})$$

The most important fact about the skew Schur function we use in this article is that it can be expressed as a fermion correlation.

$$s_{\lambda/\mu}(\vec{x}) = \langle \mu | V_+(\vec{x}) | \lambda \rangle = \langle \lambda | V_-(\vec{x}) | \mu \rangle, \quad (\text{A.2.4})$$

where

$$V_\pm(\vec{x}) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_i x_i^n J_{\pm n} \right), \quad (\text{A.2.5})$$

with the commutation relation,

$$\{\psi_n, \psi_m\} = \{\psi_n^*, \psi_m^*\} = 0, \quad \{\psi_n, \psi_m^*\} = \delta_{n+m,0}, \quad J_n := \sum_{j \in \mathbb{Z}+1/2} \psi_{-j} \psi_{j+n}^*, \quad (\text{A.2.6})$$

$$[J_n, \psi_k] = \psi_{n+k}, \quad [J_n, \psi_k^*] = -\psi_{n+k}^*, \quad [J_n, J_m] = n\delta_{n+m,0}. \quad (\text{A.2.7})$$

$|\lambda\rangle$ is a fermion basis with the label of Frobenius coordinates $\lambda = (\alpha_1, \alpha_2, \dots | \beta_1, \beta_2, \dots)$ of the Young diagram λ ,

$$|\lambda\rangle = (-1)^{\beta_1 + \beta_2 + \dots + \beta_s + \frac{s}{2}} \psi_{-\beta_1}^* \psi_{-\beta_2}^* \dots \psi_{-\beta_s}^* \psi_{-\alpha_s} \psi_{-\alpha_{(s-1)}} \dots \psi_{-\alpha_1} |\text{vac}\rangle, \quad (\text{A.2.8})$$

where s is the number of diagonal boxes in λ .

There are two Cauchy identities known for the skew Schur functions.

$$\sum_{\lambda} s_{\lambda/\mu}(x) s_{\lambda/\nu}(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \sum_{\eta} s_{\nu/\eta}(x) s_{\mu/\eta}(y), \quad (\text{A.2.9})$$

$$\sum_{\lambda} s_{\lambda/\mu^t}(x) s_{\lambda^t/\nu}(y) = \prod_{i,j} (1 + x_i y_j) \sum_{\eta} s_{\nu^t/\eta}(x) s_{\mu/\eta^t}(y). \quad (\text{A.2.10})$$

The first identity can simply be derived from

$$\sum_{\lambda} \langle \mu | V_+(\vec{x}) | \lambda \rangle \langle \lambda | V_-(\vec{y}) | \nu \rangle = \prod_{i,j} (1 - x_i y_j)^{-1} \sum_{\eta} \langle \mu | V_-(\vec{y}) | \eta \rangle \langle \eta | V_+(\vec{x}) | \nu \rangle, \quad (\text{A.2.11})$$

which follows from the commutation

$$V_+(\vec{x}) V_-(\vec{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j} V_-(\vec{y}) V_+(\vec{x}). \quad (\text{A.2.12})$$

There exists an automorphism of the fermionic algebra, $\mathcal{U} : \psi \leftrightarrow \psi^*$, under which λ transforms to λ^t as \mathcal{U} exchanges $\{\alpha_i\}$ and $\{\beta_i\}$. In particular, we have

$$|\lambda^t\rangle = (-1)^{|\lambda|} \mathcal{U}(|\lambda\rangle). \quad (\text{A.2.13})$$

The second Cauchy identity follows directly from the fact that the transformation \mathcal{U} does not change the expectation value of a correlator and $\mathcal{U}^2 = \text{id}$. i.e.

$$\langle \lambda^t | V_-(\vec{y}) | \nu \rangle = (-1)^{|\lambda|} \mathcal{U}(\langle \lambda^t | V_-(\vec{y}) | \nu \rangle) = (-1)^{|\lambda| - |\nu|} \langle \lambda^t | \mathcal{U}(V_-(\vec{y})) | \nu^t \rangle = \langle \lambda | V_-^{-1}(-\vec{y}) | \nu^t \rangle \quad (\text{A.2.14})$$

where we used that $\mathcal{U}(J_n) = -J_n$, and therefore

$$\sum_{\lambda} \langle \mu^t | V_+(\vec{x}) | \lambda \rangle \langle \lambda^t | V_-(\vec{y}) | \nu \rangle = \prod_{i,j} (1 + x_i y_j) \sum_{\eta} \langle \mu | V_-(\vec{y}) | \eta^t \rangle \langle \eta | V_+(\vec{x}) | \nu^t \rangle. \quad (\text{A.2.15})$$

A.3 Chern-Simons Theory on \mathbb{S}^3

In this appendix, we review some basic facts on the Chern-Simons in \mathbb{S}^3 . A more detailed review can be found for example in [99].

Since \mathbb{S}^3 can be obtained by gluing two solid tori together through the S -transformation, the partition function on \mathbb{S}^3 can be expressed as an matrix element of S ,

$$Z_{RR'} = \langle R' | S | R \rangle =: S_{R'R}. \quad (\text{A.3.1})$$

$|R\rangle$ is a state defined on the Hilbert space of a torus corresponding to the insertion of a non-contractable Wilson loop in the solid torus in the representation R . The normalized partition function is defined by

$$\mathcal{W}_{RR'} = \frac{S_{\bar{R}'R}}{S_{00}}, \quad (\text{A.3.2})$$

and it is known that [100] for the gauge group $G = \text{U}(N)$, $\mathcal{W}_{RR'}$ can be written in terms of the Schur function,

$$\mathcal{W}_{RR'} = q^{\frac{|R_1|}{2}(N+1)} s_{R_1}(q^{R_{2,i}-i}) \dim_q R_2, \quad (\text{A.3.3})$$

where

$$q = \exp\left(\frac{2\pi i}{k+N}\right), \quad (\text{A.3.4})$$

and the quantum dimension is given by

$$\dim_q R = q^{\frac{|R|}{2}N} s_R(q^{-i+\frac{1}{2}}). \quad (\text{A.3.5})$$

The amplitude between a Wilson loop and a trivial configuration,

$$\langle 0 | S | R \rangle, \quad (\text{A.3.6})$$

is called an unknot. The partition function of an unknot is known to be given by the quantum dimension,

$$\mathcal{W}_R := \frac{S_{0R}}{S_{00}} = \dim_q R. \quad (\text{A.3.7})$$

From the state-operator correspondence, we can consider the insertion of more general link \mathcal{L} , which has n representational indices,

$$\mathcal{W}_{R_1 R_2 \dots R_n}(\mathcal{L}). \quad (\text{A.3.8})$$

It is obvious that if \mathcal{L} is constituted of n disjoint knots,

$$\mathcal{W}_{R_1 R_2 \dots R_n}(\mathcal{L}) = \prod_{i=1}^n \mathcal{W}_{R_i}. \quad (\text{A.3.9})$$

In general, we can fuse two holonomy operators into one,

$$\text{tr}_{R_1} U \text{tr}_{R_2} U = \sum_R N_{R_1 R_2}^R \text{tr}_R U, \quad (\text{A.3.10})$$

and if there are two parallel unknot components R_1 and R_2 ,

$$\mathcal{W}_{R_1 R_2 R_3}(\mathcal{L}) = \sum_{R'} N_{R_1 R_2}^{R'} \langle \text{tr}_{R'} U' \text{tr}_{R_3} U \rangle. \quad (\text{A.3.11})$$

Similar configuration with \mathcal{L}_1 and \mathcal{L}_2 respectively having two components $\mathcal{K}_1, \mathcal{K}$ and $\mathcal{K}_2, \mathcal{K}$ (\mathcal{K} is common in two links) was considered in [100], and it was proven that for this configuration

$$\mathcal{W}_{R_1 R_2 R_3}(\mathcal{L}_1 \oplus \mathcal{L}_2) = \frac{\mathcal{W}_{R_1 R_3}(\mathcal{L}_1) \mathcal{W}_{R_2 R_3}(\mathcal{L}_2)}{\mathcal{W}_{R_3}(\mathcal{K})}. \quad (\text{A.3.12})$$

Inversely, we can compute the fusion coefficient $N_{R_1 R_2}^R$ from the above result. This coefficient can actually also be computed from the fusion of two unknots,

$$\dim_q R_1 \dim_q R_2 = \sum_R N_{R_1 R_2}^R \dim_q R. \quad (\text{A.3.13})$$

Since the quantum dimension is expressed in terms of the Schur function, we see that $N_{R_1 R_2}^R$ coincides with the fusion coefficient of the Schur function, $c_{R_1 R_2}^R$.

A.4 Macdonald functions

We first define the A_n -type Macdonald polynomial as the eigenfunction of the Macdonald operator,

$$D = \sum_{i=1}^n \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,i}, \quad (\text{A.4.1})$$

where $T_{q,i}$ is the operator that shifts the i -th variable $x_i \rightarrow x_i q$.

For example, the A_1 Macdonald polynomial is the eigenfunction of

$$D = t \frac{1 - x_2/(tx_1)}{1 - x_2/x_1} T_{q,1} + \frac{1 - tx_2/x_1}{1 - x_2/x_1} T_{q,2}, \quad (\text{A.4.2})$$

and it can be solved with two integer parameters $\lambda_{1,2}$ with $\lambda_1 \geq \lambda_2$,

$$f(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2} \sum_{n=0}^{\lambda_1 - \lambda_2} c_n (x_2/x_1)^n, \quad (\text{A.4.3})$$

where

$$c_n = \frac{(t; q)_n (ts_2/s_1; q)_n}{(q; q)_n (qs_2/s_1; q)_n} (q/t)^n. \quad (\text{A.4.4})$$

We can view (λ_1, λ_2) as a two-row Young diagram, and in general, the A_n Macdonald polynomial is labeled by an $(n+1)$ -row Young diagram.

The Macdonald function is the origin of all A_n Macdonald polynomials. It has a Young-diagram-valued label λ , and is usually denoted as

$$P_\lambda(\{x_i\}, q, t). \quad (\text{A.4.5})$$

When λ is an $(n+1)$ -row Young diagram, it reduces to the A_n Macdonald polynomial.

Let us list several important facts about the Macdonald function. (For more details, refer to [101].)

- In the $t \rightarrow q$ limit, the Macdonald function P_λ reduces to the Schur function s_λ .
- Under the standard inner product, $\langle h_\mu, m_\nu \rangle = \delta_{\mu\nu}$ ($m_\lambda(x) = \sum_\sigma \prod_i x_{\sigma(i)}^{\lambda_i}$, $h_n = \sum_{|\lambda|=n} m_\lambda$ and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots$), there exists a dual Macdonald function, Q_λ , s.t. $\langle P_\mu, Q_\nu \rangle = \delta_{\mu\nu}$.

•

$$\sum_\lambda P_\lambda(x, q, t) Q_\lambda(y, q, t) = \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty}. \quad (\text{A.4.6})$$

•

$$\sum_\lambda P_\lambda(x, q, t) P_{\lambda^t}(y, t, q) = \sum_\lambda Q_\lambda(x, q, t) Q_{\lambda^t}(y, t, q) = \prod_{i,j} (1 + x_i y_j). \quad (\text{A.4.7})$$

•

$$P_\lambda(\{1, t, t^2, \dots, t^{n-1}\}, q, t) = t^{n(\lambda)} \prod_{(i,j) \in \lambda} \frac{1 - q^{j-1} t^{n-i+1}}{1 - q^{a(i,j)} t^{\ell(i,j)+1}}. \quad (\text{A.4.8})$$

•

$$Q_\lambda(x, q, t) = b_\lambda(q, t) P_\lambda(x, q, t), \quad (\text{A.4.9})$$

where

$$b_\lambda(q, t) = \prod_{x \in \lambda} \frac{1 - q^{a(x)} t^{\ell(x)+1}}{1 - q^{a(x)+1} t^{\ell(x)}}. \quad (\text{A.4.10})$$

- From the above two properties, we have

$$Q_{\lambda^t}(q^{-\rho}, t, q) = q^{\frac{|\lambda|}{2} + n(\lambda^t)} \prod_{x \in \lambda} \frac{1}{1 - q^{a(x)} t^{\ell(x)+1}} = (q/t)^{\frac{|\lambda|}{2}} q^{n(\lambda^t)} t^{-n(\lambda)} P_\lambda(t^{-\rho}, q, t). \quad (\text{A.4.11})$$

Appendix B

Computational details

B.1 Useful Rewritings

We list several useful rewriting formulas here for the computation in the next section.

$$S(z) = \frac{(1 - q_1 z)(1 - q_2 z)}{(1 - z)(1 - q_1 q_2 z)} = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} (1 - q_1^n)(1 - q_2^n) z^n \right), \quad (\text{B.1.1})$$

$$g(z) = \frac{S(z)}{S(q_3 z)} = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} (1 - q_1^n)(1 - q_2^n)(1 - q_3^n) z^n \right), \quad (\text{B.1.2})$$

$$S^{\text{ellip}}(z) = \frac{\theta_p(q_1 z) \theta_p(q_2 z)}{\theta_p(z) \theta_p(q_1 q_2 z)} = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1 - q_1^n)(1 - q_2^n)}{1 - p^n} (z^n + p^n z^{-n}) \right), \quad (\text{B.1.3})$$

$$g^{\text{ellip}}(z) = \frac{S^{\text{ellip}}(z)}{S^{\text{ellip}}(q_3 z)} = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1 - q_1^n)(1 - q_2^n)(1 - q_3^n)}{1 - p^n} (z^n + p^n z^{-n}) \right). \quad (\text{B.1.4})$$

B.2 Commutation Relations of $q\text{-}\mathcal{W}_N$ Algebra and its elliptic generalization

We want to show the normal ordering (3.2.33) in this Appendix by explicit computations. We first consider the case $i = j$:

$$\begin{aligned}
\Lambda_i(z)\Lambda_i(w) &= \rho_{\{u\}}^{(N,0)}(\alpha(z)) (\varphi^-(\gamma^{1/2}z) \otimes \varphi^-(\gamma^{3/2}z) \otimes \dots \varphi^-(\gamma^{i-3/2}z) \otimes \eta(\gamma^{i-1}z) \otimes 1 \otimes \dots \otimes 1) \rho_{\{u\}}^{(N,0)}(\beta(z)) \\
&\quad \times \rho_{\{u\}}^{(N,0)}(\alpha(w)) (\varphi^-(\gamma^{1/2}w) \otimes \varphi^-(\gamma^{3/2}w) \otimes \dots \varphi^-(\gamma^{i-3/2}w) \otimes \eta(\gamma^{i-1}w) \otimes 1 \otimes \dots \otimes 1) \rho_{\{u\}}^{(N,0)}(\beta(w)) \\
&= \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^{-nN})}{1-q_3^{-nN}} \frac{w^n}{z^n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^n)}{1-q_3^{-nN}} q_3^{-nN} \frac{w^n}{z^n}\right) \\
&\quad \times \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^n)}{1-q_3^{-nN}} (-q_3^{-nN} - q_3^{ni-n-nN} - q_3^{-nN} + q_3^{ni-n-nN}) \frac{w^n}{z^n}\right) : \Lambda_i(z)\Lambda_i(w) : \\
&= \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^{-n(N-1)})}{1-q_3^{-nN}} \frac{w^n}{z^n}\right) : \Lambda_i(z)\Lambda_i(w) : \\
&= f_N(w/z)^{-1} : \Lambda_i(z)\Lambda_i(w) :, \tag{B.2.1}
\end{aligned}$$

where we used

$$\eta^{(i)}(z)\alpha^{(i)}(w) = \exp\left(\sum_{n=1}^{\infty} \frac{\gamma^{n-nN}}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^{-n})}{(1-q_3^{-nN})} \left(\frac{w}{z}\right)^n\right) : \eta^{(i)}(z)\alpha^{(i)}(w) :, \tag{B.2.2}$$

$$\beta^{(i)}(z)\eta^{(i)}(w) = \exp\left(\sum_{n=1}^{\infty} \frac{\gamma^{n-nN}}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^{-n})}{(1-q_3^{-nN})} \left(\frac{w}{z}\right)^n\right) : \beta^{(i)}(z)\eta^{(i)}(w) :, \tag{B.2.3}$$

$$\prod_{i=1}^{\infty} g(\gamma^{-2N(i+1)}w/z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^n)}{1-q_3^{-nN}} \gamma^{-2nN} w^n/z^n\right), \tag{B.2.4}$$

$$\eta^{(i)}(z)\eta^{(i)}(w) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} (1-q_1^n)(1-q_2^n) w^n/z^n\right) : \eta^{(i)}(z)\eta^{(i)}(w) :, \tag{B.2.5}$$

$$\beta^{(i)}(z)\varphi^{-(i)}(w) = \exp\left(\sum_{n=1}^{\infty} \frac{\gamma^{\frac{n}{2}-nN}}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^n)(1-q_3^{-n})}{1-q_3^{-nN}} \left(\frac{w}{z}\right)^n\right) : \beta^{(i)}(z)\varphi^{-(i)}(w) :. \tag{B.2.6}$$

For $i < j$, by further using

$$\eta^{(i)}(z)\varphi^{-(i)}(w) = \exp\left(\sum_{n=1}^{\infty} \frac{\gamma^{\frac{3}{2}n}}{n} (1-q_1^n)(1-q_2^n)(1-q_3^{-n}) w^n/z^n\right) : \eta^{(i)}(z)\varphi^{-(i)}(w) :, \tag{B.2.7}$$

we have

$$\begin{aligned}
\Lambda_i(z)\Lambda_j(w) &= \rho_{\{u\}}^{(N,0)}(\alpha(z)) (\varphi^-(\gamma^{1/2}z) \otimes \varphi^-(\gamma^{3/2}z) \otimes \dots \varphi^-(\gamma^{i-3/2}z) \otimes \eta(\gamma^{i-1}z) \otimes 1 \otimes \dots \otimes 1) \rho_{\{u\}}^{(N,0)}(\beta(z)) \\
&\quad \times \rho_{\{u\}}^{(N,0)}(\alpha(w)) (\varphi^-(\gamma^{1/2}w) \otimes \varphi^-(\gamma^{3/2}w) \otimes \dots \varphi^-(\gamma^{j-3/2}w) \otimes \eta(\gamma^{j-1}w) \otimes 1 \otimes \dots \otimes 1) \rho_{\{u\}}^{(N,0)}(\beta(w)) \\
&= \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^n)}{1-q_3^{-nN}} q_3^{-nN} \frac{w^n}{z^n} \right) \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} (1-q_1^n)(1-q_2^n)(1-q_3^n) w^n / z^n \right) \\
&\quad \times \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^n)}{1-q_3^{-nN}} (-q_3^{-nN} - q_3^{ni-n-nN} - q_3^{-nN} + q_3^{ni-n-nN}) \frac{w^n}{z^n} \right) : \Lambda_i(z)\Lambda_j(w) : .
\end{aligned}$$

The prefactor appears in this calculation only differs from the case $i = j$ by $g^{-1}(w/z)S(w/z) = S(z/w)$ (obtained by using the rewriting formulas presented in the previous section), which matches exactly with the prefactor presented in (3.2.33).

In the elliptic case, we need to introduce two independent elliptic bosons¹ satisfying [41],

$$[b_n^{(p)}, b_m^{(p)}] = \frac{1}{n} \frac{(1-q_1^{|n|})(1-q_2^{|n|})(1-q_3^{|n|})(\hat{\gamma}^{-2|n|}-1)}{1-p^{|n|}} \delta_{n+m,0}, \quad (\text{B.2.8})$$

$$[d_n^{(p)}, d_m^{(p)}] = \frac{p^{|n|}}{n} \frac{(1-q_1^{|n|})(1-q_2^{|n|})(1-q_3^{|n|})(\hat{\gamma}^{-2|n|}-1)}{1-p^{|n|}} \delta_{n+m,0}, \quad (\text{B.2.9})$$

together with the elliptic version of the U(1) part,

$$\alpha^{(p)}(z) = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{\hat{\gamma}^n - \hat{\gamma}^{-n}} (b_{-n}^{(p)} z^n + d_{-n}^{(p)} z^{-n}) \right), \quad \beta^{(p)}(z) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{\hat{\gamma}^n - \hat{\gamma}^{-n}} (b_n^{(p)} z^{-n} + d_n^{(p)} z^n) \right). \quad (\text{B.2.10})$$

By defining

$$t^{(p)}(z) := \alpha^{(p)}(z) x^+(z) \beta^{(p)}(z), \quad (\text{B.2.11})$$

in the elliptic DIM algebra and

$$A^{(p)}(z) := \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^n \hat{\gamma}^{-2n})}{(1-\hat{\gamma}^{-2n})(1-p^n)} (z^n + p^n z^{-n}) \right), \quad (\text{B.2.12})$$

we find

$$A^{(p)}(w/z) t^{(p)}(z) t^{(p)}(w) - A^{(p)}(z/w) t^{(p)}(w) t^{(p)}(z) = \frac{\theta_p(q_1) \theta_p(q_2)}{(p; p)_{\infty}^2 \theta_p(q_3^{-1})} \left(\delta(q_3 w/z) t_{\text{ellip}}^{(2)}(z) - \delta(q_3^{-1} z/w) t_{\text{ellip}}^{(2)}(w) \right), \quad (\text{B.2.13})$$

where

$$t_{\text{ellip}}^{(2)}(z) = \alpha^{(p)}(q_3^{-1}z) \alpha^{(p)}(z) x^+(q_3^{-1}z) x^+(z) \beta^{(p)}(q_3^{-1}z) \beta^{(p)}(z). \quad (\text{B.2.14})$$

¹They are again the expansion modes of $\psi^{\pm}(z)$ in the elliptic DIM algebra.

This is exactly the elliptic Virasoro algebra defined in [41].

We further introduce

$$f_k^{(p)}(z) := \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)(1-q_3^{-(k-1)n})}{(1-q_3^{-kn})(1-p^n)} (z^n + p^n z^{-n}) \right), \quad (\text{B.2.15})$$

and

$$\Lambda_i^{(p)}(z) = \rho_{\{u\}}^{(N,0)}(\alpha^{(p)}(z)) (\varphi_{\text{ellip}}^-(\gamma^{1/2}z) \otimes \varphi_{\text{ellip}}^-(\gamma^{3/2}z) \otimes \dots \varphi_{\text{ellip}}^-(\gamma^{i-3/2}z) \otimes \eta^{\text{ellip}}(\gamma^{i-1}z) \otimes 1 \otimes \dots \otimes 1) \rho_{\{u\}}^{(N,0)}(\beta^{(p)}(z))$$

in the elliptic DIM algebra, then we find an elliptic version of the contraction rule (3.2.33) in the q -deformed \mathcal{W} -algebra,

$$f_N^{(p)}(w/z) \Lambda_i^{(p)}(z) \Lambda_j^{(p)}(w) =: \Lambda_i^{(p)}(z) \Lambda_j^{(p)}(w) : \times \begin{cases} 1 & i = j, \\ S^{\text{ellip}}(z/w) & i < j, \\ S^{\text{ellip}}(w/z) & i > j. \end{cases} \quad (\text{B.2.17})$$

One can check that generators in the elliptic \mathcal{W} -algebra defined here given by

$$T_i^{(p)}(z) = \sum_{1 \leq j_1 < \dots < j_i \leq N} u_{j_1} u_{j_2} \dots u_{j_i} : \Lambda_{j_1}^{(p)}(z) \Lambda_{j_2}^{(p)}(z q_3^{-1}) \dots \Lambda_{j_i}^{(p)}(z q_3^{-i+1}) :, \quad (\text{B.2.18})$$

reproduce the expressions of qq-characters for 6d gauge theories with A -type quiver derived in [89].

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