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New Methods in Strongly-Coupled Field Theories towards Quantum Gravity
(強結合場の理論の新手法と量子重力理論)

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Abstract

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by

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We study strongly-coupled quantum field theories and their universalities aiming to constrain the theory-space of low-energy physics, characterising the Theory of Everything. In doing so, we develop an innovative and systematic method for analysing strongly-coupled field theories, called the large-charge expansion. Applying this new method to various systems, both non-supersymmetric and supersymmetric, we make various universal predictions about higher-dimensional CFTS. The main result includes determining the entire chiral ring OPE data of $D = 4$, $\mathcal{N} = 2$, rank-one SCFTS, exactly to all orders perturbatively in the inverse R-charge expansion, which turned out to be a universal expression only dependent on each theory's a-anomaly.

Acknowledgements

I cannot and will not spell out my deepest gratitude towards my family here; first of all, mere words would never be enough. Second of all, when I do spell out, I would like to express it in Japanese so we family all understand. Last of all, I would like to do so directly, not in a form in which the rest of the world knows what I have to say.

As I recollect, I indeed had a happy life as a graduate student at Kavli IPMU, and this is all thanks to the faculties, postdocs, students, and staffs here. The knowledge and wisdom towards theoretical physics of every faculty member and postdoc at Kavli IPMU are extraordinary and I learned many things from each of them. I would have fallen much shorter without them generously having enlightened me. I especially thank Taizan Watari for discussion and guidance as my formal advisor.

I also feel grateful to all of my collaborators for helping me understand interesting physics. I possibly could not have finished writing half the number of papers as I did without them. I thank Hiroyuki Fujita, Simeon Hellerman, Nozomu Kobayashi, Shunsuke Maeda, Yuya Nakagawa, Domenico Orlando, Susanne Reffert and Sho Sugiura for discussion.

Also, as a first class research and educational institute, students here were in a class of their own. I feel genuinely ashamed, looking back how much I learned from them and how little they were able to learn from me. I would like to thank the older members of the institute, Shunsuke Maeda (also one of my collaborators), Ryo Matsuda, and Hiroyuki Shimizu for long-term friendship. I am especially grateful to Shunsuke for patiently answering my questions about physics. I would also like to thank the younger members of the institute, Yuichi Enoki, Keita Kanno, Nozomu Kobayashi (also one of my collaborators), Yasunori Lee, and Takemasa Yamaura for friendship and various discussions, which more often than not converted themselves into and ended with genuinely fun conversation! On top of that, I thank Keita's generosity of often giving me a ride to the station from Kashiwa campus.

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Olympic weightlifting was an integral part of my life as a doctoral student. Sincere gratitude goes to all my training partners, especially Tom Snelling Kaneko, Kenta Koshiishi, and Kentaro Nishida. We spent most of the time rather eating and having all sorts of fun than training, and admittedly some of us, including myself, gained dirty bulk since we first met.

There are many things in life one cannot say for sure, but I can firmly say the best choice I made in my life is to have contacted Simeon Hellerman to be my thesis advisor. I am extremely indebted to him for patiently guiding me into the forest of physics; his answers to my questions were usually concise but extremely profound and I learned many things from them. I really admire his deep understanding of physics, which I hopefully stole a bit while working with him. Although I always bugged him with many questions, he sometimes also retaliated by asking me one question when I see him; "Did you solve everything?" I always answered "No.", but let me add one sentence today – "But *we* solved something!"

Contents

Introduction

The most unexplored subject in the theory of high-energy physics is unarguably quantum gravity. The evil but attractive face of quantum gravity appears most when one considers Black Holes. Considerations of Black Holes using semi-classical gravity (wrongly) immediately clashes with the fundamental law of nature which is quantum mechanics [6]. General relativity for example suggests the loss of information behind the horizon innocuously, but this is in stark contrast with unitarity. This seeming conflict between general relativity and quantum mechanics, called *the Black Hole information paradox*, ends with the victory of the latter, and an important takeaway is that one needs Planck-scale description of gravity to solve such a problem.

Now, the correct description of the theory of quantum gravity is string theory; put it more modestly, it is at least a way of quantising gravity without any known inconsistencies. Aside from its mathematical beauty, the indication of how it should be the correct Theory of Everything came from studying Black Holes, especially computing their entropy; The entropy of Black Holes can be correctly computed in string theory, sometimes up to and sometimes modulo numerical coefficients [7].

AdS/CFT correspondence was first devised in considering specific configurations of extended objects in string theory [8], but it offers a more precise way of analysing quantum gravity and Black Holes. It is, in short, a statement that each UV-complete quantum gravity on AdS_{d+1} (times some compact manifold) is dual to a corresponding CFT_d. The power of AdS/CFT is so extraordinary that one can compute the Black Hole entropy precisely, including its numerical factor [9]. One can even think of CFTs, which we somewhat know how to deal with, as the *definition* of quantum gravity, which we hardly know anything about.

Experimentally so hopelessly impaired is our current understanding of *the* universe that we have virtually no UV data which discerns what precise CFT corresponds to the Theory of Everything. The number of different models of quantum gravity roughly corresponds to that of various ways of compactifications which is humongous, and we are yet to know which is realised in the real world. However, we must not be left totally perplexed; rather at this point we should study general or model-independent aspects of quantum gravity. This, using AdS/CFT, reduces to studying the universalities of various CFTs. We can then set up a general formalism to study quantum gravity at least in AdS spacetime. Especially, studying strongly-coupled CFTs will be of great importance in this context because they correspond to weakly-coupled gravity theories, which we are interested in.

Strongly-coupled field theories are interesting in its own right aside from the context above. Although the Standard Model itself, for example, is not strongly-coupled at high energies, strongly-coupled field theories appear ubiquitously below certain energy scales, because of renormalisation group flows. One can find a lot of such examples in condensed matter or hadronic systems. One pity is that the fact that they are ubiquitous does not mean that they are well understood; on the contrary, they are hardly understood. Since most of the time we can only solve free theories and perturbations thereof, studying strongly-coupled theories is almost tantamount to trying to solve theories which we cannot solve. Maybe one can still extrapolate the perturbative series outside of its regime of validity and hope that the error would not come out *too* big. But that wouldn't either be applicable to non-Lagrangian theories, which, intrinsically strongly-coupled, lie exiled in the middle of the coupling constant space, without ay simplifying limits whatsoever.

Attitudes of people towards strongly-coupled theories can vary, but there are roughly three major ones and combinations thereof.[1] One way is to come up with a nice new parameter ϵ in terms of which one can expand, leading to a simplifying limit as $\epsilon \to 0$. Examples are the ϵ expansion in the ϕ^4 theory [10], the large-N expansion in gauge theories [11] and the Regge limit (large-angular-momentum limit) in string theory [12–14]. The last two are especially peculiar in that both limits are semi-classical, where the loop counting parameters are suppressed.

The second is to focus on the consistency conditions of a theory. The idea dates back as old as the S-matrix formulation of quantum field theory, and has recently revived itself as a method called conformal bootstrap [15]. The virtue of this method is that it is an intrinsic way of studying strongly-coupled theories (especially it does not need Lagrangians at all), and that it only uses physical observables like the n-point func-

[1] Exact methods like the SUSY localization and various techniques of integrability are also important.

tions. Recent developments on quantum information theory can also be classified in this category, considering how various quantum information theoretic inequalities produce similar constraints on strongly-coupled field theories [16, 17].

The last is to develop or use various dualities. Dualities map strongly-coupled theories into weakly-coupled ones, and they have been especially useful when the theory has supersymmetry [18]. There also has been a recent interest in non-supersymmetric counterpart in three dimensions [19]. Qualitative understanding of non-supersymmetric strongly-coupled theories has expanded because of this method [20].

The first option – finding a small parameter – has been abandoned for a long time, partly because it is indeed difficult to find such a parameter. However, this is the easiest and the most systematic way of studying any theories, because one can use Lagrangians and actually perturbatively compute physical quantities; if one can find such a parameter, it would be almost entirely solving the theory *quantitatively*, which could only be partially hoped for in the other methods.

The underlying theme of this thesis, "the large-charge expansion," is a new attempt of pursuing the first attitude of finding a small parameter. When a theory has a global symmetry, there is an intrinsic parameter in the theory, the global charge, J, and I, together with my collaborators, found that going to a sector of large charge, one can write down effective Lagrangians for such theories, weakly-coupled in terms of $1/J$ [1–5]. Simple as it is, it turned out to be a systematic and universal method for analysing strongly-coupled theories in different dimensions, with various different global symmetries, with or without supersymmetry, Lagrangian or non-Lagrangian. I hereby proclaim that, if what future is lying ahead of this thesis, it will be a new understanding of field theories and quantum gravity!

The rest of this thesis is organized as follows. In Chapter 1, I review the idea of the large-charge expansion, especially focusing on its non-supersymmetric aspects (or more precisely, theories without moduli space). This will make all the basic methodologies clear. This part of the thesis is based on [1, 2, 5] written by myself and collaborators, except the numerical parts which are based on [21] by Banerjee, *et.al.*, and on [22] by de la Fuente. In Chapter 2, I apply the method of the large-charge expansion to $D = 4$, $\mathcal{N} = 2$, rank-one SCFTs, deriving exactly to all orders in $1/n$ expansion the two-point functions of chiral-ring elements $\mathcal{O}^n$. I will also numerically check the formula for one of the theories using exact localisation. This part is based on [3, 4] written by myself and collaborators with a special emphasis on the latter. I will conclude this thesis with an outlook, where I explain how studying the large-charge expansion will lead to a new understanding of quantum gravity, although still a wild dream.

Chapter 1

Review: The Large-Charge Expansion and its Applications

1.1 The method of the large-charge expansion

Global symmetries can give conformal field theories (CFTs) interesting and useful simplifications. In spite of the common knowledge that most CFTs have no weakly-coupled Lagrangian, by taking a limit of large quantum number, $J \gg 1$, it is sometimes possible to write down a weakly-coupled effective Lagrangian for such a theory, expanded in terms of $1/J \ll 1$. Such an effective Lagrangian is useful in computing physical quantities (operator dimensions or operator product expansions (OPEs)) containing operators of large quantum number/dimension, which is complimentary to the region of low quantum number/dimension linear programming (of conformal bootstrap) has access to.

Examples of such simplification at large quantum numbers used to be known in various examples. The Regge theory is exactly the study of this kind, where one studies the effective string theory in the large-angular-momentum limit. The famous leading order behaviour of the spectrum of rotating strings was already known in [23] by Regge himself; the revival of the idea and the computation of sub-leading terms were given in [12–14].

Those papers pointed to a nice and general framework to describe the simplification that occurs at large quantum numbers, J, which is that there will be an effective theory description of physics whose Lagrangian is expanded in terms of $1/J$. Such a general framework was then indeed found out by myself and collaborators in [1] for CFTs with a global symmetry, followed by applications to various systems [2–5, 21, 22, 24–33].

The general mechanism in which the theory simplifies in the large-charge limit is its semi-classical nature. In order to understand why so, let us put the theory on a cylinder, $S^{D-1} \times \mathbb{R}$, with radius R and fix the charge density to ρ, where $J \propto \rho R^{D-1} \gg 1$. Then the effective Lagrangian at large charge has its UV scale at $\Lambda_{\text{UV}} = \rho^{1/(D-1)}$ and IR at $\Lambda_{\text{IR}} = 1/R$, so that the large separation in scales, $\Lambda_{\text{IR}}/\Lambda_{\text{UV}} \propto J^{-1/(D-1)}$ renders the theory semi-classical.[1] In other words, by taking the Wilsonian cut-off Λ so that $1/R = \Lambda_{\text{IR}} \ll \Lambda \ll \Lambda_{\text{UV}} = \rho^{1/(D-1)}$, we suppress both quantum effects and higher-derivative terms by a factor of $1/J^{\alpha}$. Because quantum effects are suppressed, the leading order Lagrangian in terms of $1/J$ expansion should just be classically conformally invariant. This can be easily and systematically done by writing down all terms in the effective action according to the J-scaling.

The simplest example of the theory in which the large-charge expansion works is the $O(2)$ Wilson-Fisher fixed point in three dimensions. I will use this pedagogical example to explain how the method works, by computing the lowest operator dimension at large charge, J, whose result becomes

$$\Delta_J = c_{3/2} J^{3/2} + c_{1/2} J^{1/2} - 0.094 \cdots + O(J^{-1/4}), \tag{1.1.1}$$

where $c_{3/2,1/2}$ are parameters that were undetermined by our method. I will then introduce the idea of the large-charge universality class, in which, although the actual IR fixed-point is different than the $O(2)$ Wilson-Fisher fixed point, the physical quantities of the theory at large charge are the same modulo undetermined coefficients like $c_{3/2,1/2}$. In other words, it means that $c_{3/2,1/2}$ were the non-universal parameters at large charge. I will also present two numerical results that checked our formula and fitted for the values of $c_{3/2,1/2}$.

This part of the thesis is mostly based on [1] written by myself and collaborators, except the numerical parts which are based on [21] by Banerjee, *et.al.*, and on [22] by de la Fuente.

[1] This is only when the theory has no moduli space of vacua. Also note that we can make artificial counterexamples to this statement, *e.g.*, two decoupled CFTs one of which the symmetry acts trivially.

1.2 Leading-order behaviour of models at large charge

1.2.1 $O(2)$ model at large charge

Spontaneous symmetry breaking at large charge

Let us start from the simplest model that has the $O(2)$ symmetry, which is called the ϕ^4 model, or simply the $O(2)$ model,

$$\mathcal{L}_{\mathrm{UV}} = -\partial\bar{\phi}\partial\phi + m^2 |\phi|^2 + g^2 |\phi|^4 \tag{1.2.1}$$

where ϕ is the complex bosonic field and m^2 is fine-tuned in order to get to the Wilson-Fisher conformal fixed point. For the sake of convenience, let us parametrise ϕ as

$$\phi = a \exp(i\chi). \tag{1.2.2}$$

Giving state a large charge density is the same as adding a chemical potential term $\omega \times \rho$ to the Lagrangian and taking ω big (in units of $1/R$, of course). Here ρ is the usual charge density, or the time-like component of the Noether current, so that

$$\rho \equiv i\left(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi}\right). \tag{1.2.3}$$

Although this looks as if an explicit symmetry breaking of the $O(2)$ symmetry, it isn't. By redefining $\phi_{\mathrm{new}} \equiv e^{i\omega t}\phi$, we recover the Lagrangian with the $O(2)$ symmetry, which is a combination of the original $O(2)$ and time translation symmetry,

$$\mathcal{L}_{\mathrm{UV}} + \omega \times \rho = -\partial\bar{\phi}\partial\phi + (m^2 - \omega^2)|\phi|^2 + g^2|\phi|^4 \tag{1.2.4}$$

So taking ω large is tantamount to moving to the symmetry breaking phase, in which the a-field gains a dimensionful vacuum expectation value (VEV), proportional to $\sqrt{\rho}$, so that $\langle a^2 \rangle \equiv \left\langle |\phi|^2 \right\rangle \propto \sqrt{\rho}$.

Renormalization group at large charge

The UV theory presented above was defined at the scale $\Lambda \sim g^2 \sim m$, and ω was taken large in units of $1/R$ but much smaller than m. Therefore the renormalisation group flow is essentially the same as the usual $O(2)$ model until the scale is of order ω.

Here the coupling constants quickly reach the Wilson-Fisher fixed point where $m[\Lambda]/\Lambda$ and $g^2[\Lambda]/\Lambda$ are both of $O(1)$, thus strongly-coupled. In other words,

$$g^2[\Lambda] = h\Lambda \qquad (\text{when } \Lambda \gtrsim \omega). \tag{1.2.5}$$

Because the mass parameter of the Lagrangian becomes negative, the theory now is in the symmetry breaking phase, where I denote the VEV of the a-field simply as $a \sim \sqrt{\omega}$.

Because of this VEV, the a-field gets mass

$$M_a^2 \sim g^2[\omega]a^2 \sim \omega^2, \tag{1.2.6}$$

and below M_a the renormalisation group flow comes to a halt and the running of $g^2[\Lambda]$ stops. Therefore the final value of $g^2[\Lambda]$ becomes

$$g^2[\Lambda] = g^2[\omega] \sim \frac{\omega^2}{a^2} \qquad (\text{when } \Lambda \lesssim \omega) \tag{1.2.7}$$

by using (1.2.6). Matching the coupling constant computed from both sides of $\Lambda \sim \omega$, *i.e.*, (1.2.5) and (1.2.6), we get

$$h\omega = \frac{\omega^2}{a^2} \Longleftrightarrow \omega = ha^2, \tag{1.2.8}$$

so that

$$g^2[\Lambda] = h^2a^2 \qquad (\text{when } \Lambda \lesssim \omega) \tag{1.2.9}$$

The potential generated from the renormalization group flow therefore becomes

$$V(a) = \frac{h^2}{12}a^6, \tag{1.2.10}$$

where I have chosen some convenient normalization so that the coefficient becomes $h^2/12$. Note that this computation should be possible also using the ϵ-expansion and setting $\epsilon = 1$ (which is of course an uncontrolled approximation), although I will not do this anywhere (I neither think there are any literatures yet that did this for this system).

Classical scale invariance

It was actually not necessary to work out the effective potential at large charge using renormalization group analysis we did above. As the Lagrangian near the IR fixed point is approximately classically scale invariant, one can just write down such a Lagrangian using a and χ. If we use a renomalization condition that the kinetic term for a is canonically normalized, the effective Lagrangian at leading order in the derivative expansion is uniquely determined,

$$\mathcal{L}_{\mathrm{IR}} = -\frac{1}{2}(\partial a)^2 - \frac{\kappa}{2}a^2(\partial\chi)^2 - \frac{h^2}{12}a^6 + (\text{higher derivative terms}), \tag{1.2.11}$$

where κ and h^2 are some numerical constants, which in principle can be calculated using, say, the ϵ-expansion.

It is now easy to determine the equilibrium value for a using the charge density, *via* Noether theorem and the Euler-Lagrange equation,

$$\rho = \sqrt{\frac{\kappa h^2}{2}}a^4. \tag{1.2.12}$$

We can additionally see that $\chi = \omega t$, where $\omega \propto \sqrt{\rho}$. Note that the analysis so far, again has been correct in the regime where

$$\frac{1}{R} \ll \Lambda \ll \sqrt{\rho}, \tag{1.2.13}$$

so that we need

$$J = 4\pi R^2 \rho \gg 1, \tag{1.2.14}$$

which is why this is called the large charge expansion. Schematic picture of this leading-order (classical) analysis is shown in Figure 1.2.1.

1.2.2 The supersymmetric $W = \Phi^3/3$ model

The $W = \Phi^3/3$ model at large R-charge

Consider the $\mathcal{N} = 2$ supersymmetric field theory in three dimensions, with a single chiral superfield, Φ, Kähler potential $K = \Phi^\dagger\Phi$ and superpotential $W = \Phi^3/3$. This theory can be shown to flow to an interacting conformal fixed point using the extremization principle [34, 35], where the R-charge and the dimension of Φ becomes equal to $2/3$, in the usual convention where the R-charge of Q is -1.

In this strongly-coupled theory, which has no marginal deformations nor any small parameters, we wish to understand the spectrum of operators which have large R-

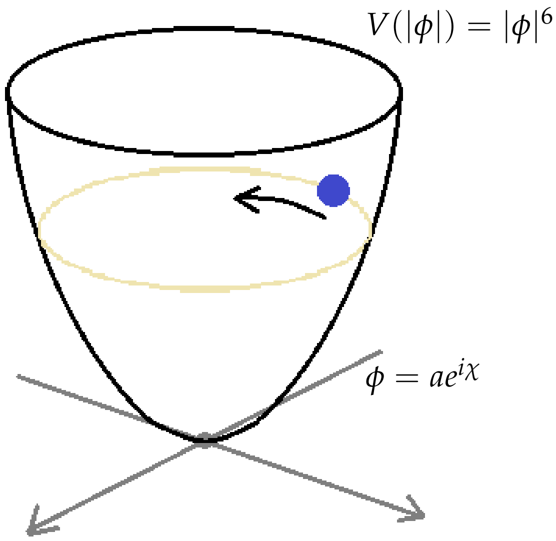


Figure 1.2.1: Schematic picture of our leading-order analysis. Because the ground state is homogeneous, the computation reduces to a classical mechanics problem, where we compute the energy of a particle in terms of its angular momentum. The "cup" represents the potential $V(|\phi|) \propto |\phi|^6$ and the blob represents the field value $\phi = ae^{i\chi}$.

charges. For the sake of convenience, we introduce the ϕ-charge, which is related to the R-charge by a multiple of $3/2$, so that ϕ has ϕ-charge 1. Hereafter when I write "charge" unspecified, it will mean the ϕ-charge.

IR *Lagrangian of the* $W = \Phi^3/3$ *model*

Because of the holomorphicity, the superpotential is not renormalized. The Kähler potential, however, gets renormalized, and because the Lagrangian has to be classically conformal invariant in the IR, it becomes

$$K = \frac{16 b_K}{9} |\Phi|^{3/2}, \tag{1.2.15}$$

where b_χ is again some undetermined proportionality constant. The component Lagrangian, therefore, includes the kinetic term

$$\mathcal{L}_{\text{kin}} = b_K \frac{\partial\phi\partial\bar{\phi}}{\sqrt{|\phi|}} \tag{1.2.16}$$

as well as the potential term,

$$V = \frac{1}{b_K} |\phi|^{9/2} . \qquad (1.2.17)$$

It also includes the Yukawa coupling of the form

$$i\phi\psi_\alpha\psi^\alpha + (\text{h.c.}), \qquad (1.2.18)$$

which will become important later on.

Taking the large R-charge limit

Taking the large R-charge limit is equivalent to taking the VEV of $|\phi|$ large. By applying the similar method we used in the above subsection, we arrive at the IR Lagrangian at large R-charge,

$$\mathcal{L}_{\text{IR}} = b_K \frac{(\partial A)^2}{A^{1/2}} + b_K A^{3/2}(\partial\chi)^2 + \frac{1}{b_K} A^{9/2} + (\text{higher-derivative}) + (\text{fermions}) \qquad (1.2.19)$$

where we have set $\phi \equiv A \exp(i\chi)$. We can see that this Lagrangian is essentially the same as (1.2.11), by setting $A \propto a^{4/3}$.

Fermions and the massive Goldstini

The action above contains terms including fermions, but Yukawa coupling makes fermions massive. This is in spite of the fact that we started from the supersymmetric Lagrangian – supersymmetry is spontaneously broken and the Goldstini's become massive. This fact can be explicitly seen by looking at the form of the Yukawa coupling, and the fermions get mass of order $O(\sqrt{\rho})$.

This is again consistent with supersymmetry, and we have massive Goldstini's in the system [36]. The lowest state at fixed large charge J is described by a Bose condensate, whose (classical) leading order energy is proportional to $E_J \propto \rho^{3/2}$. The supercharges remove one ϕ quanta from the Bose condensate and replace it with a fermion almost at rest. Because of supersymmetry, the energy of both states must be equal, and this is compensated by a heavy mass M_ψ for the fermion, which then becomes

$$E_J = E_{J-1} + M_\psi \Longrightarrow M_\psi \propto \sqrt{\rho}. \qquad (1.2.20)$$

This means that the fermions can safely be integrated out because they are massive.

1.3 The large-charge universality class

1.3.1 Classification of operators at large global charge

The leading order Lagrangian with the angular field

In both of the models presented in the last section, after integrating all the massive modes out, we are only left with the angular field, χ. In the $O(2)$ model the massive mode was the a-field, and in the cubic superpotential model, the massive modes were $|\phi|$ and all the fermions.

The leading order Lagrangian after integrating them out becomes

$$\mathcal{L}_{\mathrm{IR}} = b_\chi \left|\partial\chi\right|^3 + \cdots , \tag{1.3.1}$$

where again $|\partial\chi| \equiv \sqrt{-\partial\chi\partial\chi} \propto \sqrt{\rho}$. Note that we are only meant to use this Lagrangian around the classical configuration, $\chi_0 = \omega t$, such that $\omega \propto \sqrt{\rho}$. The singular form of this Lagrangian, therefore, is not a problem.

The fluctuations around the vacuum configuration can also be computed by separating χ into VEV and fluctuations,

$$\chi \equiv \chi_0 + \chi_{\mathrm{fluc}}. \tag{1.3.2}$$

Since the leading order Lagrangian becomes

$$\mathcal{L}_{\mathrm{leading}} = b_\chi|\partial\chi_0|^3 + \frac{3b_\chi|\partial\chi_0|}{2}\chi_{\mathrm{fluc}}\left(\partial_t^2 - \frac{1}{2}\triangle_{S^2}\right)\chi_{\mathrm{fluc}} + \cdots , \tag{1.3.3}$$

the fluctuation scales as

$$\chi_{\mathrm{fluc}} \propto \frac{1}{\sqrt{|\partial\chi|}} \propto \rho^{-1/4}, \tag{1.3.4}$$

because the canonical fluctuation of $O(1)$ has to have a unit coefficient in front of the quadratic fluctuation term.

Few remarks are in order. First, we can easily observe that this leading order term has all the correct features to be in an effective Lagrangian, that is, it is classically Weyl-invariant. Second, the form of the fluctuation indicates that the Goldstone boson of the theory has a speed of $1/\sqrt{2}$ times the speed of light. This clearly indicates that the conformal symmetry is spontaneously broken. We will see in later sections that this speed of the Goldstone represents the fact that there is a conformal symmetry in the underlying theory, or specifically because of the existence of the descendent of the lowest operator at charge J.

Sorting operators in terms of the J-scaling

One then can just sort out all the terms in the effective Lagrangian according to the following rules.

- The term must have Weyl weight 3.
- The term must be $O(2)$ invariant (*i.e.*, it must respect the shift symmetry of χ).
- The term must be parity invariant, $\chi \leftrightarrow -\chi$.
- Only $|\partial\chi|$ can appear in the denominator, because it is the mass for the a-field.

We also saw that the scaling of each operator follows the rules below

- $\partial\chi \propto \rho^{1/2}$
- $\partial \cdots \partial\chi \propto \rho^{-1/4}$
- The leading order equation of motion, $\partial_\mu \left(|\partial\chi|\partial^\mu\chi\right) = 0$, can be used.

The last rule is because whenever such a combination appears, it can be replaced by something of the lower ρ-scaling.

Order $\rho^{3/2}$ The only operator at this order is

$$|\partial\chi|^3 \tag{1.3.5}$$

which can also be seen from (1.3.1).

Order $\rho^{1/2}$ The only operator at this order is

$$\mathtt{Ric}_3\, |\partial\chi| \,. \tag{1.3.6}$$

This clearly has to be supplemented by a Weyl-completion, $(\partial\,|\partial\chi|)^2/\,|\partial\chi|$, but this term goes as $O(1/J)$. There can also be terms of the form $R_{\mu\nu}\partial^\mu\chi\partial^\nu\chi$, but this is vanishing when the background metric is non-warped.

Order $\rho^{1/4}$ There are no operators; One can naively write two terms that go as $O(\rho^{1/4})$, but one of them vanish upon using the equation of motion and the other cannot appear in the effective Lagrangian because it is parity odd.

Order ρ^0 There are again no operators. This is an important fact so let me establish it formally. First, there are no geometric invariants of dimension 3 including the background metric only (aside from the gravitational Chern-Simons term which is topological). Therefore, what we have to show now is that there are no operators of dimension 3 and scaling as $O(\rho^0)$, including just $\partial\chi$ and $\partial\cdots\partial\chi$.

This is indeed impossible. As the only thing that can appear in the denominator is $\partial\chi$, we can schematically only allow for the form

$$\frac{\partial^n\left[(\partial\chi)^m\right]}{|\partial\chi|^{n+m-3}} \tag{1.3.7}$$

The ρ-scaling of the operator of this form is

$$\frac{3-n}{2}-\frac{3\ell}{4}, \tag{1.3.8}$$

where ℓ indicates how many $\partial\cdots\partial\chi$ there are in the numerator, and $1\leqslant\ell\leqslant\min(n,m)$ when $n\geq 1$ (ℓ can only be 0 when $n=0$, trivially). Also in order to bring the ρ-scaling to an half-integer (this time, 0), we need to have ℓ even.

Hence, when $n=0$, we can only have $\ell=0$, and we cannot have operators of $O(\rho^0)$. Likewise, when $n=1$, we simply cannot take ℓ even, so we cannot realise such a possibility either. Finally, when $n\geqslant 2$, the ρ-scaling of operators is bounded above when $n=2$ and $\ell=2$ by $-1/2$. To sum up. we have proven the non-existence of operators with ρ-scaling 0.

1.3.2 The effective Lagrangian and universality at large charge

The effective Lagrangian at large charge

We have therefore determined the effective Lagrangian at large charge, including terms of order $O(\rho^0)$ or higher,

$$\mathcal{L}_{\text{IR}}=b_{3/2}|\partial\chi|^{3/2}+b_{1/2}\texttt{Ric}_3|\partial\chi|+O(J^{-1/4}) \tag{1.3.9}$$

on the unit sphere, $R = 1$. Here, $b_{3/2,1/2}$ were undetermined by our method, but it is possible to compute these constants for each IR fixed points using some other methods. The computation, however, could be cumbersome, and we leave them as undetermined $O(1)$ constants here. We can also determine them using numerical data, whose result I will show in later sections.

There also is a more important excuse to leave them undetermined here, which is the large-charge universality class; As one can see, the $O(2)$ model and the cubic superpotential model share the same effctive Lagrangian at large charge (with possibly different $b_{3/2,1/2}$) although they do not flow to the same fixed-point in the IR. This is not a coincidence and rather a generic feature; If taking a sector of large charge leaves only the χ-field as a massless mode, the only Lagrangian one can write down is of the form (1.3.9).

Operator dimensions at large charge

From this Lagrangian, one can compute, *e.g.*, the lowest operator dimension at large charge. Because of the state-operator map, it amounts to computing the energy of the lowest state at large charge on the unit sphere. This can be done using the textbook method of separating the field into VEV and fluctuations $\chi = \omega t + \chi_{\text{fluc}}$ and summing all the loop corrections.

Most importantly, the loop expansion parameter is $J^{3/2}$, which is an overall coefficient of the Lagrangian. This means that the first quantum correction is of $O(J^0)$, which comes from the leading term in the Lagrangian, $b_{3/2} |\partial\chi|^{3/2}$. This is a one-loop effect, and can be computed using the familiar Coleman-Weinberg formula for the effective action,

$$\frac{1}{2T} \log\det\left(-\partial_\tau^2 - \frac{1}{2}\triangle_{S^2}\right) = \frac{1}{2\sqrt{2}} \sum_{\ell=0}^{\infty} (2\ell+1)\sqrt{\ell(\ell+1)}, \tag{1.3.10}$$

where T is the total time. This sum, of course, is divergent. One needs to regulate and renormalise the sum. The counterterm that subtracts off the divergence is an explicit cut-off dependent term in the effective action; I didn't explicitly write down such a term, but the one-loop renormalisation procedure on spatial slice S^2 can be carried out using the ζ-function regularization, which happens to be a diffeomorphism invariant way of the renormalisation. The result becomes

$$\frac{1}{2\sqrt{2}} \sum_{\ell=0}^{\infty} (2\ell+1)\sqrt{\ell(\ell+1)} \xrightarrow{\text{renormlization}} -0.094, \tag{1.3.11}$$

which actually is a universal (in the sense of the large-charge universality class) number, because there are no cut-off independent counterterms at $O(\rho^0)$ in the effective action (1.3.9).

So the result for the lowest operator dimension $\Delta(J)$ at large charge, J, becomes

$$\Delta(J) = c_{3/2}J^{3/2} + c_{1/2}J^{1/2} - 0.094 + O(J^{-1/4}), \tag{1.3.12}$$

where $c_{3/2,1/2}$ are theory-dependent constants. This formula, again, holds for any theories in this large-charge universality class, which both the $O(2)$ model and the cubic superpotential model belong to. One bit surprising corollary from this asymptotic formula is that there are no scalar BPS operators in the $\mathcal{N} = 2$ supersymmetric model with $W = \Phi^3/3$, contrary to what people tend to think at first glance.

1.3.3 Spectrum of operators at large charge

Energies of excited states

Let me restate what the lowest dimension of the operator at charge $J \gg 1$,

$$\Delta(J) = c_{3/2}J^{3/2} + c_{1/2}J^{1/2} - 0.094 + O(J^{-1/4}). \tag{1.3.13}$$

Now I would like to compute the energies of $O(1)$ excited states from this large-charge ground state. Because the leading order action for the fluctuation $\hat{\chi} = \frac{\chi_{\text{fluctuation}}}{\sqrt{|\partial\chi|}}$ is given in (1.3.3) as

$$\hat{\chi}\left(\partial_t^2 - \frac{1}{2}\triangle_{S^2}\right)\hat{\chi}, \tag{1.3.14}$$

its equation of motion becomes

$$\ddot{\chi} = \frac{1}{2}\triangle_{S^2}\chi. \tag{1.3.15}$$

The dispersion relation for χ, therefore, becomes

$$\omega_\ell = \sqrt{\frac{\ell(\ell+1)}{2}}. \tag{1.3.16}$$

Note especially that $\omega_1 = 1$, which must be equivalent to acting with ∂ in the operator language, or taking the descendent. Therefore, one can see that the speed of the Goldstone boson, $1/\sqrt{2}$ times the speed of light, is necessary in order to respect the conformal symmetry (or specifically, the existence of the descendent). Other operators including modes with $\ell > 1$ are all primaries, which can be seen by the fact that the increase in energy cannot be integers.

Regge trajectory and the bootstrap

In [28], the authors reproduced that this excitation spectrum is the only one allowed by using conformal bootstrap, when you have only one Regge trajectory *i.e.,* when the number of Goldstone bosons in the system is 1. This is just as expected from our effective field theory approach. They also computed the spectrum of excited states when there are multiple Goldstone bosons in the large-charge effective action, which may or may not be reproduced using the Lagrangian method.

Energies of high-spin states

Apparently, in order for the effective action to be valid, the spin of the Goldstone mode, ℓ, cannot be larger than $O(\sqrt{J})$; otherwise it would contribute $O(\sqrt{\rho})$ or more to the operator dimension, which falls out of the regime of validity of the effective theory.

This criterion is actually proven to be the same as having no vortex in the superfluid description [29]. In [29], the effective field theory including vortices was given, using which the lowest operator dimension at large charge with spin $0 \leqslant \ell \lesssim J^{3/2}$ was computed.

1.4 Numerical simulations

1.4.1 Monte-Carlo simulation of the $O(2)$ model

$O(2)$ sigma model regularized on a cubic lattice

In [21], the authors computed the lowest operator dimension in the $O(2)$ model by using the lattice regularization with spacing a and taking $a \to 0$. This model is defined by phases, $\exp(i\theta_{\vec{x}})$ on (spatial) three-dimensional lattice sites (Remember that this is a statistical system), $\vec{x} \equiv (an_x, an_y, an_z)$ with the following Hamiltonian,

$$H = -\beta \sum_{\vec{x},\vec{\alpha}} \cos(\theta_{\vec{x}} - \theta_{\vec{x}+a\vec{\alpha}}), \tag{1.4.1}$$

where $\vec{\alpha}$ runs through all the unit lattice vectors and $\theta \sim \theta + 2\pi$. The Wilson-Fisher fixed-point is reached by tuning β to a critical value, $\beta_c = 0.4541652$ [37, 38].

Methods and the main numerical result

The authors computed the lowest operator dimension at fixed charge J by computing the two-point functions of the form

$$C_J(r) = \langle \exp(iJ\theta_{\vec{r}}) \exp(iJ\theta_{\vec{0}}) \rangle \sim \frac{A(J)}{|\vec{r}|^{2\Delta(J)}}, \quad (|\vec{r}| \to \infty) \tag{1.4.2}$$

extracting $\Delta(J)$ from its large-distance behaviour. Note that their actual algorithm measured not the conformal dimensions themselves, but the differences of them, $\Delta(J+1) - \Delta(J)$, in order to avoid severe signal-to-noise problems. The result from the Monte-Carlo simulation, as well as previous known results, are shown in Table 1.1.

J	$\Delta(J)$	ϵ-expansion	Monte-Carlo	bootstrap
1	0.516(3)	0.518(1)	0.5190(1)	0.5190(1)
2	1.238(5)	1.23(2)	1.236(1)	1.236(3)
3	2.116(6)	2.10(1)	2.108(2)	-
4	3.128(6)	3.103(8)	3.108(6)	-
5	4.265(6)			
6	5.509(7)			
7	6.841(8)			
8	8.278(9)	NO	KNOWN	RESULTS
9	9.796(9)			
10	11.399(10)			
11	13.077(11)			
12	14.825(12)			

Table 1.1: Results for the conformal dimensions $\Delta(J)$ given in [21]. Also compared with previous results using ϵ-expanison to 6 loops (5 loops for $Q = 1$) [39–41], the Monte-Carlo simulation [42], and the conformal bootstrap [43]. No known previous results to compare for $Q > 4$.

The result can be fitted using our formula for $\Delta(J)$,

$$\Delta(J) = 4\pi \left(\tilde{c}_{3/2} \left(\frac{J}{4\pi} \right)^{3/2} + \tilde{c}_{1/2} \left(\frac{J}{4\pi} \right)^{1/2} - \frac{0.094}{4\pi} + (\text{subleading}) \right). \tag{1.4.3}$$

and the result for the fit was

$$\tilde{c}_{3/2} = 1.195, \quad \tilde{c}_{1/2} = 0.075. \tag{1.4.4}$$

The plot of $\Delta(J)$ against J is shown in Figure 1.4.1, and one can see that not only is it perfectly fitted, but the fit works pretty well even down to $J = 1$.

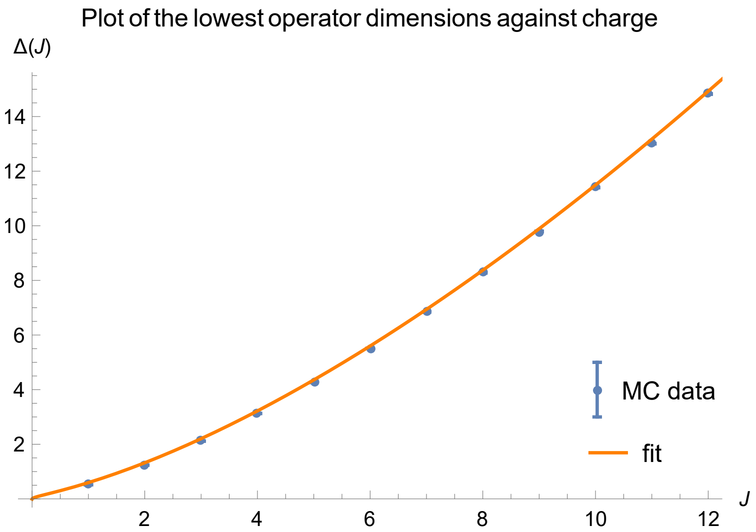


Figure 1.4.1: The plot of the values of $\Delta(J)$ from the Monte-Carlo simulations of the $O(2)$ model [21]. The solid line is the graph (1.4.3), with the estimated values of $\tilde{c}_{3/2} = 1.195$ and $\tilde{c}_{1/2} = 0.075$. Quite surprisingly, the fit works very well even down to $J = 1$. Note that the error bars in this graph are too tiny to be visible.

1.4.2 The large-N behaviour of the $\mathbb{C}P^{N-1}$ model at large monopole number

Dualitites at large charge and the $\mathbb{C}P^{N-1}$ model

Systems of compact bosons have a duality transformation to the Abelian gauge theory in three dimensions. The duality map transforms the Noether current $\mathcal{J}^\mu$ to the monopole current,

$$\mathcal{J}^\mu \mapsto \frac{1}{4\pi} \epsilon_{\mu\nu\rho} F^{\nu\rho}. \tag{1.4.5}$$

Note that this is the correct coefficient because of the Dirac quantization.

Let us now analyse the $\mathbb{C}P^{N-1}$ model at large monopole number. This model is the non-linear sigma model with target space $\mathbb{C}P^{N-1}$, and its action is given by

$$S = \frac{N}{g}\int dx^3 \left[\left| \left(\partial_\mu - iA_\mu\right)\phi_a \right|^2 + i\lambda\left(|\phi_a|^2 - 1\right) \right], \quad (a = 1, \ldots, N) \tag{1.4.6}$$

where g is tuned to criticality. By taking the monopole number large, the mass of the bosons gets heavy, and we are indeed left with just the gauge field, A_μ in the IR. The duality map relating $F_{\mu\nu}$ and $|\partial\chi|$ is the following

$$F_{\mu\nu} = \frac{1}{\sqrt{2}}\epsilon_{\mu\nu\rho}\sqrt{|g|}\,|\partial\chi|\,\partial^\rho\chi, \tag{1.4.7}$$

because this is the only relation respecting the Weyl and diffeomorphism invariance. The numerical factor was chosen so that $|F|^2 = |\partial\chi|^4$.

The effective action for the $\mathbb{C}P^{N-1}$ model at large monopole number is therefore given to be

$$\mathcal{L}_{\text{IR}} = b_\chi\,|F|^{3/2} + (\text{subleading}), \tag{1.4.8}$$

and we can see that the $\mathbb{C}P^{N-1}$ model also lies in the same large-charge universality class as in the $O(2)$ model.

The large-N behaviour of the $\mathbb{C}P^{N-1}$ model

The $\mathbb{C}P^{N-1}$ model can also be solved perturbatively in $1/N$-expansion (the large-N expansion). In [22], based on the method first presented in [44], the author computed the lowest operator dimension at large monopole number using the large-N expansion. The method is to write down the saddle point which gives contributions of $O(N)$, and then compute the one-loop determinant around it for contributions of $O(1)$.

Schematically the result of the large-N computation at large monopole number J must be written as a infinite double sum,

$$\Delta_N(J) = N\Delta^{[1]}(J) + \Delta^{[0]}(J) + \frac{1}{N}\Delta^{[-1]}(J) + \cdots \tag{1.4.9}$$

where

$$\Delta^{[i]}(J) = c^{[i]}_{3/2}J^{3/2} + c^{[i]}_{1/2}J^{1/2} + c^{[i]}_{0} + \cdots . \tag{1.4.10}$$

Now, because the model, for any N, falls into the same universality class at large monopole number, it should also give the universal prediction for the $O(J^0)$, which was

-0.094. This predicts

$$c_0^{[i]} = 0 \quad (i \neq 0), \quad c_0^{[0]} = -0.094. \tag{1.4.11}$$

Indeed this can be checked analytically for $i = 1$ too.

In [22], the author computed the one-loop determinant to compute $\Delta_0(J)$. The computation was done for $J = 1, \cdots, 100$, which was fitted with the function

$$\Delta^{[0]}(J) = c_{3/2}^{[0]} J^{3/2} + c_{1/2}^{[0]} J^{1/2} + c_0^{[0]} + c_{-1/2}^{[0]} J^{-1/2} + (\text{error}). \tag{1.4.12}$$

The result of the fit given in [22] is the following

$$\Delta_0^{\text{fit}} = 0.2182275\, J^{3/2} + 0.23764\, J^{1/2} - 0.0935(3) + 0.025\, J^{-1/2} + \cdots . \tag{1.4.13}$$

The fitted value of $c_0^{[0]} = -0.0935(3)$, therefore, comes out consistent with the universal prediction of the effective field theory, $-0.0937256\ldots$, within one percent.

The author of [22] used a machine-learning like method to do the above fit, because of the signal-to-noise ratio problem due to the rapidly increasing function, $J^{3/2}$. However, because the actual dataset was given in the paper, we can independently check this universal prediction using a standard fit method of the least square. In order to do this, notice that the following expression is independent of $c_{3/2}^{[0]}$ or $c_{1/2}^{[0]}$ at $O(J^0)$:

$$\mathcal{I}(J) \equiv J^2 \times \Delta^{[0]}(J) - \left(\frac{J^2}{2} + \frac{J}{4} + \frac{3}{16}\right) \Delta^{[0]}(J-1) - \left(\frac{J^2}{2} - \frac{J}{4} + \frac{3}{16}\right) \Delta^{[0]}(J+1), \tag{1.4.14}$$

which becomes, using (1.4.12),

$$\mathcal{I}(J) = \frac{3c_0^{[0]}}{8} + O\left(J^{-1/2}\right). \tag{1.4.15}$$

So we can easily fit $\mathcal{I}(J)$ using the fit function,

$$\mathcal{I}^{\text{fit}}(J) \equiv \frac{3c_0^{[0]}}{8} + pJ^{-1/2} + qJ^{-1} + rJ^{-3/2}. \tag{1.4.16}$$

Here I have assumed the operator dimension to be fitted with a function

$$\Delta^{[0]}(J) = c_{3/2}^{[0]} J^{3/2} + c_{1/2}^{[0]} J^{1/2} + c_0^{[0]} + c_{-1/2}^{[0]} J^{-1/2} + c_{-1}^{[0]} J^{-1} + (\text{error}) \tag{1.4.17}$$

where the term coloured in red turned out to be crucial for the precision of the fit. Even without knowing the existence of such a term in the classical effective action, we expect that the contribution of that order is present due to the one-loop correction to the Ricci curvature term in the Lagrangian. Now the result of the fit becomes

parameter	fitted value	standard error
$c_0^{[0]}$	-0.0942538	0.000222076
p	0.00463937	0.00152049
q	0.0318169	0.00223002
r	-0.0729223	0.000857753

(1.4.18)

and the resulting function along with the plot of the numerical data is shown in Figure 1.4.2.

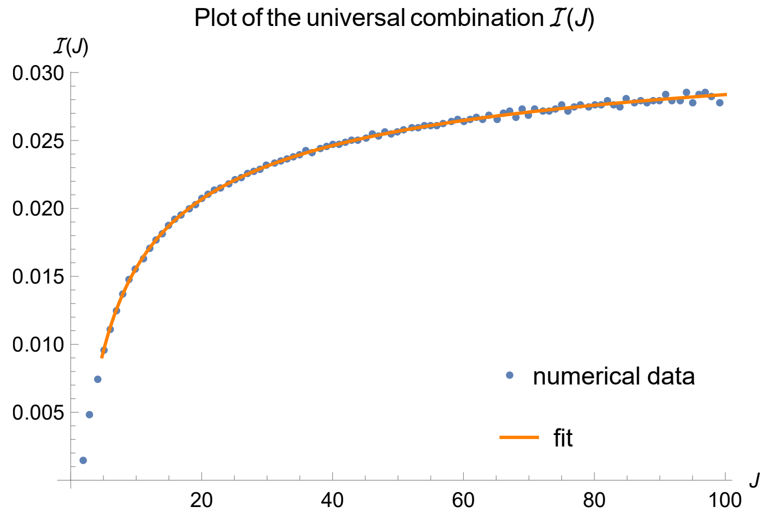


Figure 1.4.2: Plot of the universal combination $\mathcal{I}(J)$ constructed from $\Delta(J)$ in (1.4.14). The dot represents the numerical data calculated from the actual data presented in [22], while the line represents the result of the fit using the standard least square method. The result of the fit implies the universal contribution to be -0.094, consistent with [22] as well as with our universal prediction. The seeming randomness of data points starting roughly at $J \sim 40$ is because of the numerical error entirely from stopping at $\sim 100^{\text{th}}$ Landau level in computing the one-loop determinant.

1.5 Miscellanea

1.5.1 Higher-rank symmetries

Inhomogeneous ground states at large-charge

Throughout this part, I have assumed that the ground state at large charge is homogeneous, which is correct for the large-charge universality including the $O(2)$ model. For higher-rank symmetries, however, this is not always true, and the inhomogeneity happens when we turn on more than one Cartans of the charge matrix [24] (They actually proved the contrapositive statement). In [2, 5], I computed such an example of inhomogeneous ground states using the $O(4)$ model, making the above statement clearer. The result was that the inhomogeneity of the ground state at large charge is inevitable when we turn on more than one Cartans, but the inhomogeneity happens at the IR scale so that the effective field theory does not break down.

Goldstone counting and the inhomogeneity

In [5], I found that this inhomogeneity can be understood by counting the number of modes in the effective action and comparing it with the dimension of the coset of the spontaneously symmetry breaking. It was found there that the inhomogeneity, or the spontaneous symmetry breaking of the translational symmetry, can be related to the Goldstone bosons predicted from the low-energy effective action.

For example, in the $O(4)$ model on $T^2 \times \mathbb{R}$, the low-energy effective action at large charge is just the non-linear sigma model on S^3. This means that the number of available independent degrees of freedoms (DOFs) is three. Now the $O(4)$ symmetry is explicitly broken down to $U(1) \times U(1)$ by fixing two eigenvalues of the charge density matrix to $\rho_1 \neq 0$ and $\rho_2 \neq 0$. Because of the inhomogeneity, the solution to the classical equation of motion breaks the whole $U(1) \times U(1)$ spontaneously. But this breaking cannot be in two spatial directions, because if so, the coset dimension is four, but there are not enough available modes in the effective action. We were therefore able to prove that the inhomogeneity can only be in one spatial dimension.

1.5.2 Chern-Simons-matter duality at large charge

$SU(2)_k$ Chern-Simons-matter theory and the homogeneous-inhomogeneous transition

The Chern-Simon-matter theories at large k reduces to the usual $O(4)$ theory with the singlet condition, $\rho_1 = \rho_2$. It is therefore apparent that the ground state for the theory is inhomogeneous for large enough k. It is also analytically possible to solve the classical equation of motion analytically for low values of k, which gives homogeneous ground states, on the contrary.

These two facts, when combined, predicts the existence of a phase-transition for some value of k (which I already worked out but will not write here), where the translational symmetry is spontaneously broken. Note that because $SU(N)_k$ Chern-Simons-matter theory for $N < 12$ has no interacting fixed-point, the analysis of, say, the ground state energy has to be supplemented by a subleading one-loop effect that goes as $O(\log \rho)$. For $N \geqslant 12$, there is an interacting fixed-point, and thus the classical analysis is applicable around it.

Free Dirac fermion

Dealing with fermions is difficult in this method, because we do not know how to renormalise the fermi surface at finite volume. This difficulty might be solved using three-dimensional particle-vortex duality, as some of them map a boson into a fermion. For example, a free Dirac fermion is believed to be dual to the Chern-Simons-matter theory at level $k = 1$. Note that the lowest energy of the free fermion at charge J also scales as $J^{3/2}$, as in the case of the $O(2)$ model, but this of course lies in a different universality class than that.

Chapter 2

Chiral Ring from Moduli Space at Large-R-Charge

2.1 Moduli space and the effective action at large-R-charge

As we learned in Chapter 1, a theory with a global symmetry has a weakly-coupled effective Lagrangian at large charge, containing one or more Goldstone bosons which come from breaking the symmetries of the theory due to the VEVs of charged operators. This procedure is obviously dependent on the vacuum structure of the original theory. In Chapter 1, I have only used examples where the theories have no moduli space of vacua, but what happens if the original theory itself has one? Such a possibility is often realised in theories with supersymmetry, and for example the dimension of the Bogomol'nyi–Prasad–Sommerfield (BPS) operators at large charge should behave differently ($\Delta \propto J$) than the result we got in Chapter 1 ($\Delta \propto J^{3/2} + \cdots$).

One important fact to recoup here is that supersymmetry is not at all a sufficient condition for having a moduli space. Indeed as we saw in Section 1.2.2, the supersymmetric theory with superpotential $W = \Phi^3$ has no moduli spaces of vacua. As a result, the lowest operator dimension at large-R-charge does not saturate the BPS bound, but instead exceeds it parametrically;

$$\Delta(J) \propto J^{3/2} + \cdots \gg J \qquad (2.1.1)$$

This should not happen when the theory has moduli spaces of vacua, and hence has the BPS states, protected from quantum corrections. What I will write about here is this type of different behaviour at large charge, when the theory has moduli spaces of vacua.

Supersymmetry and moduli space accompanied by it makes a lot of things easier, and this also is such a case in which it does. First of all, the leading order action is just a free Lagrangian, because the Goldstone boson is just the entire multiplet on the moduli space, as opposed to just the angular field in the case without moduli. Also the supersymmetry is not broken spontaneously by the VEVs of charged operators when there are moduli spaces, so there are more constraints on the form of subleading operators. Sometimes there can be cases where there are no subleading operators at all of a certain class, which possibility is actually realised in four dimensional $\mathcal{N} = 2$, rank-one superconformal field theories (SCFTs), which is the main topic of this thesis.

This part of the thesis is mostly based on [4], written by myself and collaborators. This is also based on [45], where Hellerman and Maeda (also among the collaborators of [4]) computed the two-point functions of BPS operators of the form $\langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)\rangle$ of four dimensional $\mathcal{N} = 2$, rank-one SCFTs, up to $O(\log(n))$. New results from [4] include noticing the nonexistence of subleading F-terms in the effective action, and the computation of $\langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)\rangle$ exactly to all orders in $1/n$-expansion, based on this fact. This computation showed that the Coulomb branch chiral ring data is universal up to the a-anomaly in four dimensional $\mathcal{N} = 2$, rank-one SCFTs, modulo non-universal effects of $O(-\sqrt{n})$.

2.2 The effective action of four-dimensional $\mathcal{N} = 2$, rank-one SCFTs at large-R-charge

2.2.1 The effective Lagrangian at large-R-charge

Four-dimensional $\mathcal{N} = 2$, rank-one SCFTs and their Coulomb branch

What we will be interested in is four-dimensional $\mathcal{N} = 2$, rank-one SCFTs, whose Coulomb branch is complex one-dimensional. The moduli space of vacua is parametrised by the VEV of an operator, $\mathcal{O}$, which is the generator of the Coulomb branch chiral ring.[1] The theories are symmetric under rotating two supercharges, Q^1_α and Q^2_α, which is called

[1] The chiral ring of rank-one SCFTs is conjectured in [46–50] (and proven in [51]) to be freely generated, so the number of generators of the Coulomb branch chiral ring and the dimension of the moduli space match, as there are no algebraic relations between generators. For rank higher than one, non-freely generated chiral rings can be constructed *via* discrete gauging [52].

the $U(2)_R$-symmetry. The Coulomb branch is their moduli space of vacua, which contains a vectormultiplet charged under diagonal $U(1)_R \subset U(2)_R$ and is neutral under the remaining $SU(2)_R$ symmetry. Let us now denote $\mathcal{O}$ the generator of the Coulomb branch chiral ring, whose VEV parametrises the Coulomb branch moduli space.

The origin of the Coulomb branch is typically singular, in which the theory has more symmetry than the generic points on the moduli. However, for the rest of the analysis we take the large-R-charge limit, so we will be only looking at the classical saddle-points far from the origin, where it has flat sections.

Leading-order action on the moduli space

Because the moduli space is flat, the low-energy effective Lagrangian at large-R-charge is given by just the free Lagrangian, with the Goldstone boson denoted ϕ_{hol} inside a vectormultiplet, which is defined by[2]

$$\Phi_{\text{hol}} \equiv \mathcal{O}^{1/\Delta_{\mathcal{O}}}, \tag{2.2.1}$$

where $\Delta_{\mathcal{O}}$ is the dimension of $\mathcal{O}$, where ϕ is the bottom component of Φ. The definition for ϕ_{hol} is that it has both R-charge and dimension 1. Also note that this procedure is well-defined far from the origin of the Coulomb branch. Therefore, the leading order low-energy Lagrangian of four-dimensional $\mathcal{N} = 2$, rank-one SCFTs at large-R-charge is given by the usual free one,

$$\mathcal{L}_{\text{free}} = C \times \text{Im}\left[\tau \int d\theta^2 d\tilde{\theta}^2 \Phi_{\text{hol}}^2\right] \ni C \times \text{Im}(\tau) \times |\partial\phi_{\text{hol}}|^2, \tag{2.2.2}$$

where C is a holomorphic constant in terms of the background gauge fields like $\tau \equiv 1/g_{\text{YM}}^2$.

For the sake of the later analysis, it is convenient to newly define

$$\Phi \equiv \Phi_{\text{unit}} \equiv (N_{\mathcal{O}})^{-1} \Phi_{\text{hol}} \tag{2.2.3}$$

so that

$$\mathcal{L}_{\text{free}} \ni |\partial\phi_{\text{hol}}|^2, \tag{2.2.4}$$

[2] $\Delta_{\mathcal{O}}$, the dimension of $\mathcal{O}$, is shown to be a rational number in [53], so we can understand (2.2.1) as $(\Phi_{\text{hol}})^n \equiv \mathcal{O}^m$.

where $N_{\mathcal{O}} \equiv 1 \Big/ \sqrt{C \times \text{Im}(\tau)}$. Note that $N_{\mathcal{O}}$ is non-holomorphic in background gauge fields. Also note that $N_{\mathcal{O}}$ should not enter in the computation of actual physical quantities, because it is dependent on how we define $\mathcal{O}$; linear transformation of $\mathcal{O}$ can change it arbitrarily. To make this fact noticeable, I have coloured $N_{\mathcal{O}}$ in gray.

The structure of the effective Lagrangian at large-R-charge

Because the low-energy effective Lagrangian on the moduli space respects the $\mathcal{N} = 2$ supersymmetry, it can be broken up into three pieces,

$$\mathcal{L}_{\text{IR}} = \mathcal{L}_{\text{free}} + (F\text{-term}) + (D\text{-term}). \tag{2.2.5}$$

Surprisingly, on backgrounds conformally equivalent to $\mathbb{R}^4$ (like S^4 or $S^3 \times \mathbb{R}$), there are no subleading F-terms aside from the $\mathcal{N} = 2$ supersymmetric Wess-Zumino term (This will be proven in section 2.5.1.), so

$$\mathcal{L}_{\text{IR}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{WZ} + (D\text{-term}). \tag{2.2.6}$$

The first two terms scale as $O(J)$ and $O(\log J)$ (the latter will be explained in Section 2.2.2), respectively, evaluated on the saddle point we consider later, which go as $\langle|\phi_0|\rangle \sim \sqrt{J}$.

Almost UV-completeness of the action and regularization

The IR action at large charge in general must be supplemented by regulator dependent terms, which become counter terms in order to compute physical quantities and get a finite result. However, in (2.2.6), because of the absence of subleading F-terms, there are no superconformally invariant regulator dependent terms. In other words, the F-term action, $\mathcal{L}_{\text{IR}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{WZ}$, already has the desired (anomalous) superconformal invariance, there must not be any counterterms if we use a superconformally invariant regulator.

If we use a non-superconformal invariant regulator, there certainly exist terms that depend on it. In computing physical quantities, ultraviolet (UV) divergences proportional to powers of $|\Lambda| / |\phi|$ are subtracted in a canonical way by those counterterms to restore superconformal invariance.

For practical purposes, we can write down Feynmann diagrams with some regulator in mind, and use known results to actually know what values they are. For example, in [3], we are able to, in theory, compute the energy of the lowest state at large-R-charge, J, by computing and summing up all the diagrams. It is cumbersome to conduct such a calculation using a specific choice of a regulator, but we need not to. Because the state must saturate the BPS bound, we trivially know that the diagrams, after renormalizing, must give exactly 0 at each order in J^{-n} because of superconformal invariance.

2.2.2 The Weyl-anomaly and the WZ term

The axiodilaton for the WZ *term*

Because the Weyl-symmetry and the $U(1)_R$-symmetry are anomalous in theories we consider, the anomaly mismatch $\Delta a \equiv a_{\text{CFT}} - a_{\text{EFT}}$ between the original CFT and the effective field theory (EFT) must be compensated by the Wess-Zumino term [54, 55].[3] In rank-one theories, the a-anomaly for the EFT is that of a $U(1)$ vectormultiplet, so

$$a_{\text{EFT}} = a_{U(1)\text{-vector}}. \tag{2.2.7}$$

In this case, moving onto a generic point of the moduli space breaks conformal symmetry spontaneously, and one can write such a term using the dilaton, or the Nambu-Goldstone boson from spontaneous conformal symmetry breaking [56]. The transformation law for the axiodilaton $\tau_{\text{hol}} + i\beta_{\text{hol}}$ (τ_{hol} for the dilaton and β_{hol} for the axion associated to spontaneous $U(1)$-symmetry breaking) is

$$g_{\mu\nu} \mapsto e^{\sigma_{\text{hol}} + i\omega_{\text{hol}}} g_{\mu\nu} \qquad \tau_{\text{hol}} + i\beta_{\text{hol}} \mapsto \tau_{\text{hol}} + \sigma_{\text{hol}} + i(\beta_{\text{hol}} + \omega_{\text{hol}}), \tag{2.2.8}$$

so in four-dimensional $\mathcal{N} = 2$, rank-one SCFTs, the only option for the realisation of the dilaton is

$$\tau_{\text{hol}} + i\beta_{\text{hol}} \equiv -\log\left(\frac{\phi_{\text{hol}}}{\mu}\right), \tag{2.2.9}$$

where for later purposes we define

$$\tau + i\beta \equiv \tau_{\text{unit}} + i\beta_{\text{unit}} \equiv -\log\left(\frac{\phi_{\text{unit}}}{\mu}\right) = \tau_{\text{hol}} + i\beta_{\text{hol}} + \log N_{\mathcal{O}}, \tag{2.2.10}$$

throughout which μ was an arbitrary mass scale.

[3] This is specific in even-dimensional spacetimes. For example, one needed not worry about such a term in three-dimensional $\mathcal{N} = 2$ supersymemtric model with superpotential $W = gXYZ$ [3].

$\mathcal{N}=2$ supersymmetrization of the WZ *term*

Now we need to write down the Wess–Zumino (WZ) term using this axiodilaton field, which is given by the *the* $\mathcal{N}=2$ supersymmetrisation[4] [57] of the bosonic WZ term [56],

$$\mathcal{L}_{WZ}^{\mathcal{N}=2} = (\text{constant}) \times \int d^4\theta d^4\bar{\theta} \log\left(\frac{\Phi}{\mu}\right) \log\left(\frac{\Phi^\dagger}{\mu}\right) . \tag{2.2.11}$$

The overall coefficient is fixed by the requirement that the bosonic part of this supersymmetrization has the correct coefficient, $\Delta a^{[\mathrm{KS}]}$ in [56], where the superscript [KS] (which will be dropped unless one needs to be specifically careful) denotes the convention used in [56] where the Euler density $E_4^{[\mathrm{KS}]}$ is given by

$$E_4^{[\mathrm{KS}]} \equiv R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2, \tag{2.2.12}$$

related to the Euler number by the relation $\chi_{\mathcal{M}} \equiv \frac{1}{32\pi^2}\int_{\mathcal{M}} E_4^{[\mathrm{KS}]} \in \mathbb{Z}$. In this normalisation, the bosonic part of the $\mathcal{N}=2$ WZ term contains the term

$$-\left(\Delta a^{[\mathrm{KS}]}\right) \times \mathbb{t} E_4^{[\mathrm{KS}]}, \tag{2.2.13}$$

which goes as $\log J$. This is indeed the biggest subleading effect in the effective action. For later reference, we also write down the full form of the bosonic WZ action, which will be explicitly computed from (2.2.11) in Section 2.5.2,

$$\mathcal{L}_{WZ}^{[\mathrm{bosonic}]} = \mathcal{L}_{\mathbb{t}^1} + \mathcal{L}_{\mathbb{t}^2} + \mathcal{L}_{\mathbb{t}^3} + \mathcal{L}_{\mathbb{t}^4} + \mathcal{L}_{\mathbb{b}^2} + \mathcal{L}_{\mathbb{t}^1\mathbb{b}^2} + \mathcal{L}_{\mathbb{t}^2\mathbb{b}^2} + \mathcal{L}_{\mathbb{b}^4} , \tag{2.2.14}$$

4 The emphasis on *"the"* was because there is only one unique supersymmetrization of such a term [57], as opposed to the case of $\mathcal{N}=1$ supersymmetrization [58] where alternate supersymmetrizations of the term can be obtained by adding $\mathcal{N}=1$ superconformally-invariant terms. For $\mathcal{N}=2$, there are no such terms.

where

$$
\begin{aligned}
\mathcal{L}_{\tau^1} &= -\left(\Delta a^{[\mathrm{KS}]}\right) \tau\, E_4, \\
\mathcal{L}_{\tau^2} &= -4\left(\Delta a^{[\mathrm{KS}]}\right) \left[R^{\mu\nu} - \frac{1}{2}\,\mathtt{Ric}_4\, g^{\mu\nu}\right] \nabla_\mu \tau\, \nabla_\nu \tau, \\
\mathcal{L}_{\tau^3} &= +4\left(\Delta a^{[\mathrm{KS}]}\right) (\nabla\tau)^2\, (\nabla^2\tau), \\
\mathcal{L}_{\tau^4} &= -2\left(\Delta a^{[\mathrm{KS}]}\right) (\nabla\tau)^4, \\
\mathcal{L}_{\beta^2} &= -4\left(\Delta a^{[\mathrm{KS}]}\right) \left[R^{\mu\nu} - \frac{1}{6}\, R\, g^{\mu\nu}\right] (\nabla_\mu\beta)(\nabla_\nu\beta), \\
\mathcal{L}_{\tau^1\beta^2} &= -8\left(\Delta a^{[\mathrm{KS}]}\right) (\nabla^\mu\nabla^\nu\tau)\, \nabla_\mu\beta\, \nabla_\nu\beta, \\
\mathcal{L}_{\tau^2\beta^2} &= -4\left(\Delta a^{[\mathrm{KS}]}\right) \left[2\,(\nabla\tau\cdot\nabla\beta)^2 - (\nabla\tau)^2(\nabla\beta)^2\right], \\
\mathcal{L}_{\beta^4} &= -2\left(\Delta a^{[\mathrm{KS}]}\right) (\nabla\beta)^4
\end{aligned}
\tag{2.2.15}
$$

modulo terms involving the Weyl tensor, which vanish on conformally flat backgrounds we are interested in.

This Δa again is dependent on the convention of E_4, by a multiplicative factor, but a combination $(\Delta a) \times E_4$ should be convention independent and can enter the final result. As a convention independent alternative for Δa, we use

$$
\alpha \equiv \frac{\Delta a}{2} \int_{S^4} E_4 \quad \left(= 32\pi^2 \times \Delta a^{[\mathrm{KS}]}\right), \tag{2.2.16}
$$

which can also be represented as

$$
\alpha \equiv \frac{5}{12}\, \frac{a_{\mathrm{CFT}} - a_{U(1)\text{-vector}}}{a_{U(1)\text{-vector}}} \tag{2.2.17}
$$

by cancelling the convention dependent numerical factor.

2.3 Chiral ring data from EFT at large R-charge

2.3.1 What we compute

Chiral ring data, OPE and normalisations for two-point functions

In a CFT, it is customary to normalise every two-point function to have a unit normalised coefficient,

$$
\left\langle \mathcal{O}_i(x)\mathcal{O}_j(y) \right\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_{\mathcal{O}}}}. \tag{2.3.1}
$$

Concerning the chiral ring, this is not the most natural way of normalising; in fact, because of its ring structure, the most natural way of defining $\mathcal{O}^n$ is from

$$\mathcal{O}^n(x) \equiv \lim_{y\to x} \mathcal{O}^{n-m}(x)\mathcal{O}^m(y), \quad \text{(independent of } m\text{)} \tag{2.3.2}$$

so that the two-point functions of $\mathcal{O}^n$ are not unit normalised and rather becomes

$$\left\langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)\right\rangle = \frac{C(m,n-m;n)}{|x-y|^{2n\Delta_{\mathcal{O}}}}. \quad \text{(independent of } m\text{)} \tag{2.3.3}$$

Here $C(m,n-m;n)$ are the OPE coefficients, defined by the relation

$$\left\langle \mathcal{O}^m(x_1)\mathcal{O}^{n-m}(x_2)\bar{\mathcal{O}}^n(y)\right\rangle \equiv \frac{C(m,n-m;n)}{|x_1-y|^{2m\Delta_{\mathcal{O}}}\,|x_2-y|^{2(n-m)\Delta_{\mathcal{O}}}}. \tag{2.3.4}$$

This is independent of m, so we redefine

$$\mathcal{Y}_n \equiv C(m,n-m;n) \tag{2.3.5}$$

so that

$$\left\langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)\right\rangle = \frac{\mathcal{Y}_n}{|x-y|^{2n\Delta_{\mathcal{O}}}}. \tag{2.3.6}$$

Now go back to the usual CFT definition of operators where all of them are unit normalized,

$$\left\langle \mathcal{O}_n^{(\mathrm{CFT})}(x)\bar{\mathcal{O}}_n^{(\mathrm{CFT})}(y)\right\rangle = \frac{1}{|x-y|^{2n\Delta_{\mathcal{O}}}}, \quad \mathcal{O}_n^{(\mathrm{CFT})} \equiv \frac{\mathcal{O}^n}{\sqrt{\mathcal{Y}_n}}, \tag{2.3.7}$$

and then we have the OPE coefficients

$$\left\langle \mathcal{O}_m^{(\mathrm{CFT})}(x_1)\mathcal{O}_{n-m}^{(\mathrm{CFT})}(x_2)\bar{\mathcal{O}}_n^{(\mathrm{CFT})}(y)\right\rangle \equiv \sqrt{\frac{\mathcal{Y}_n}{\mathcal{Y}_m\mathcal{Y}_{n-m}}} \times \frac{1}{|x_1-y|^{2m\Delta_{\mathcal{O}}}\,|x_2-y|^{2(n-m)\Delta_{\mathcal{O}}}}. \tag{2.3.8}$$

So computing two-point functions in our normalization is equivalent to computing all the Coulomb branch chiral ring data.

Two-point functions of Coulomb branch chiral ring operators

What we are interested in computing here is the two-point function

$$Y_n \equiv \langle \mathcal{O}^n(x)\bar{\mathcal{O}}^n(y)\rangle = \left\langle \phi^J_{\text{hol}}(x)\bar{\phi}^J_{\text{hol}}(y)\right\rangle, \quad \mathcal{Y}_n \equiv |x-y|^{2n\Delta_{\mathcal{O}}}\, Y_n, \tag{2.3.9}$$

where

$$J \equiv n\Delta_{\mathcal{O}}. \tag{2.3.10}$$

By using the path integrals $Z_n^{(\text{hol})} \equiv \exp(q_n^{(\text{hol})})$ with source insertions $S^{(\text{hol})}_{\text{source}} = -J\log\phi_{\text{hol}}(x) - J\log\bar{\phi}_{\text{hol}}(y)$, the coefficient $\mathcal{Y}_n$ can be written as

$$\mathcal{Y}_n = |x-y|^{2n\Delta_{\mathcal{O}}} \times \frac{Z_n^{(\text{hol})}}{Z_0^{(\text{hol})}} \tag{2.3.11}$$

Let us rewrite the above formula in terms of ϕ_{unit} for convenience,

$$Y_n = (N_{\mathcal{O}})^{2J} \times \frac{Z_n^{(\text{unit})}}{Z_0^{(\text{unit})}} = (N_{\mathcal{O}})^{2J} \times \left\langle \phi^J_{\text{unit}}(x)\bar{\phi}^J_{\text{unit}}(y)\right\rangle, \tag{2.3.12}$$

where $Z_n^{(\text{unit})}$ is likewise the path integral with source insertions $S^{(\text{unit})}_{J\text{ sources}} = -J\log\phi_{\text{unit}}(x) - J\log\bar{\phi}_{\text{unit}}(y)$. What we will compute in the next subsection is the object $Z_n^{(\text{unit})}$. Notice that we already know such a correlator when $\Delta a = 0$, because

$$\left\langle \phi^J_{\text{unit}}(x)\bar{\phi}^J_{\text{unit}}(y)\right\rangle_{\Delta a=0} = \Gamma(J+1) = J!, \tag{2.3.13}$$

from Wick contraction. Let us abbreviate the superscript (unit) for simplicity hereafter, since we will only actually compute things with it.

2.3.2 The structure and the diagrammatics of q_n from EFT

The structure of the path integral

The whole action we are going to consider is the following,

$$S = S_{\text{free}} + S_{\text{WZ}} + S_{J\text{ sources}}, \tag{2.3.14}$$

where S_{WZ} was proportional to α. Now, using this action, $q_n \equiv \log Z_n$ is just the sum of all the connected vacuum diagrams, computed by separating ϕ into VEV and fluctuations.

A particularly simple way of organizing diagrams is to use the saddle point value of ϕ when $\Delta a = 0$ for the VEV even when $\Delta a \neq 0$. Such a saddle point value has already been worked out in [45] on $\mathbb{R}^4$. The result was then conformally transformed onto a cylinder $S^3 \times \mathbb{R}$ with radius R, whose result was

$$\phi_{\text{cl}}^{[\alpha=0]}(x) = \frac{e^{t/R}}{2\pi R} \times \sqrt{J}, \quad \bar{\phi}_{\text{cl}}^{[\alpha=0]}(x) = \frac{e^{-t/R}}{2\pi R} \times \sqrt{J}, \tag{2.3.15}$$

which is the homogeneous helical solution as expected. In terms of the axiodilaton, we have

$$\mathbb{T}_{\text{cl}}^{[\alpha=0]} = \log(2\pi\mu R) - \frac{1}{2}\log(J), \quad \mathbb{B}_{\text{cl}}^{[\alpha=0]} = i\frac{t}{R}\,, \tag{2.3.16}$$

on the cylinder frame. Let us now use this helical solution to compute q_n.

Loop contributions to q_n

When we expand ϕ around this VEV (with fluctuation ϕ_{fluc}), building blocks of the Feynmann diagram include a one-point vertex $(\Delta a)\,\phi_{\text{fluc}}$ as well as ordinary j-point vertices, $(\phi_{\text{fluc}})^j$. Term in the effective action, evaluated on this VEV again scale as

$$S_{\text{free}} = O(J), \quad S_{\text{WZ}} = \alpha \times O(\log J), \tag{2.3.17}$$

which means that a α-vertex comes with a subleading factor of $1/J$. Organizing diagrams in the $1/J$-expansion, at order $1/J^m$, we have diagrams that have the following numbers of loops and α-vertices,

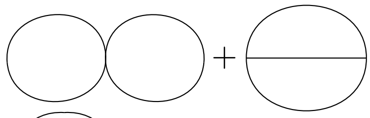
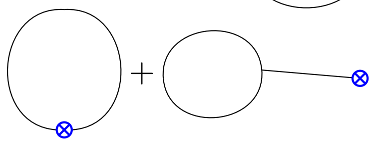
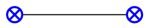
# of loops	# of α-vertices
$m+1$	1
m	2
$\vdots$	$\vdots$
1	m
0	$m+1$

(2.3.18)

so that when summed up, give a contribution of

$$\frac{P_{m+1}(\alpha)}{J^m}, \quad P_{m+1}(\alpha) \equiv \sum_{n=0}^{m+1} K_{m,n}\alpha^n \tag{2.3.19}$$

at order $O(J^{-m})$. For example, we have diagrams at $O(1/J)$,

term	diagrams
$\frac{K_{1,0}}{J}$	
$\frac{K_{1,1}\alpha}{J}$	
$\frac{K_{1,2}\alpha^2}{J}$	

(2.3.20)

Classical contributions to q_n

The classical contribution of this VEV can be computed by evaluating the action on this configuration of $\phi_{\rm cl}^{[\alpha=0]}(x)$. We already know how much $S_{\rm free}$ contributes to this classical configuration, because at $\alpha = 0$,

$$q_n(\alpha = 0) = A(\tau, \bar\tau)n + B(\tau, \bar\tau) + \log\Gamma(J + 1), \qquad (2.3.21)$$

which gives the all-loop sum of diagrams without α-vertices. Here $A(\tau, \bar\tau)$ is dependent on the convention of linear multiplication of ϕ, while $B(\tau, \bar\tau)$ is the sphere partition function which is scheme dependent. Note that both can depend on τ and $\bar\tau$, but that these are the only places they can appear. In order to make these facts noticeable, I have also coloured them in gray along with $N_{\mathcal{O}}$. Note again that because this is an all-loop sum, and the coefficient of each term in the loop-expansion can also inferred from the Stirling series,

$$q_n(0) = J\log J + \frac{1}{2}\log J + An + B + \sum_{m=0}^{\infty}\frac{K_{m,0}}{J^m}, \quad K_{m,0} = \frac{(-)^m B_{m+1}}{m(m+1)}, \qquad (2.3.22)$$

where B_{m+1} is the Bernoulli number. which not only determines the classical contribution, but also infinite coefficients of quantum corrections. To be more precise, $J\log J$ is the classical contribution, $\log J/2$ is the one-loop determinant, and the rest is higher-loop corrections.

We also need to take care of the WZ term which was given in (2.2.14). Evaluated on the classical configuration (2.3.16), this will give

$$-\int_{S^4}\left(\Delta a^{[\mathrm{KS}]}\right)\times \mathbb{T}_{\mathrm{cl}}^{[\alpha=0]}E_4^{[\mathrm{KS}]}=\alpha\log J \tag{2.3.23}$$

plus various constants of $O(J^0)$, which will be absorbed into the scheme dependent constant, B.

The whole structure of q_n All in all, the whole structure of q_n becomes

$$q_n = J\log J+\left(\alpha+\frac{1}{2}\right)\log J+A(\tau,\bar{\tau})n+B(\tau,\bar{\tau})+\sum_{m=0}^{\infty}\frac{P_{m+1}(\alpha)}{J^m}+O\left(e^{-\sqrt{n}}\right), \tag{2.3.24}$$

where again $P_{m+1}(\alpha)$ is a polynomial of order $m+1$. The error term $O\left(e^{-\sqrt{n}}\right)$ included in the final result should come from massive BPS dyons, ignored in our EFT analysis as parametrically heavy. In this case, because of the non-existence of subleading F-terms, it will be of great importance to keep track of such a term.

2.3.3 Computation of coefficients $K_{m,n}$

Computation of $K_{m,0}$

This was already done in the last subsection by using the result of Wick contraction and expanding in $1/J$. The results are

$$K_{m,0}=\frac{(-)^m B_{m+1}}{m(m+1)}, \tag{2.3.25}$$

so for example,

$$K_{1,0}=\frac{1}{12}. \tag{2.3.26}$$

Computation of $K_{m,m+1}$

The contributions are actually classical pieces in disguise, in that these are due to the shift of the classical saddle-point for finite α. The calculation can be done by finding the shifted saddle point of the action, and then plugging it into back into the action. Let us now find the shifted saddle point. The Lagrangian here is

$$\mathcal{L}=\mathcal{L}_{\text{free}}+\mathcal{L}_{WZ} \tag{2.3.27}$$

Here $\mathcal{L}_{\text{free}}$ can be written in terms of axiodilaton as

$$\mathcal{L}_{\text{free}} = -\mu \exp(-2\tau) \left[(\partial \mathbb{\tau})^2 + (\partial \mathbb{\beta})^2 - \frac{1}{6} \mathtt{Ric}_4 \right], \tag{2.3.28}$$

where $\mathtt{Ric}_4 = 6/R^2$ on $S^3 \times \mathbb{R}$, and $\mathcal{L}_{WZ}$ was given in (2.2.14).

Because of the BPS property if the helical frequency of the lowest solution [3], we exactly have

$$\mathbb{\beta}_{\text{cl}}^{[\alpha]} = \pm \frac{t}{R}. \tag{2.3.29}$$

in the Lorentzian signature, independent of α.

We are now left to determine the value of $\mathbb{\tau}_{\text{cl}}^{[\alpha]}$. This can be done by computing the R-charge using Noether theorem,

$$J = 2\pi^2 R^3 \times \frac{\delta \mathcal{L}}{\delta \dot{\mathbb{\beta}}} = (2\pi R)^2 \times \left[\left| \phi_{\text{cl}}^{[\alpha]} \right|^2 - \frac{8 \left(\Delta a^{[\text{KS}]} \right)}{R^2} \right] \tag{2.3.30}$$

where $2\pi^2 R^3$ is the area of S^3. Solving the above equation for $\left| \phi_{\text{cl}}^{[\alpha]} \right|^2$, we get

$$\left| \phi_{\text{cl}}^{[\alpha]} \right|^2 = \frac{J}{4\pi^2 R^2} \times \left(1 + \frac{\alpha}{J} \right). \tag{2.3.31}$$

Note that we recover (2.3.15) by setting $\alpha = 0$.

By plugging this classical configuration into the action, we get

$$S_{\text{saddle}}^{[\alpha]} = (J + \alpha) \log(J + \alpha) = J \log J + \alpha \log J \tag{2.3.32}$$

so modulo terms of $O(J)$ and $O(1)$ (which can be absorbed into A and B), the value of the action classical configuration is shifted by

$$\sum_{m=1}^{\infty} \frac{K_{m,m+1}}{J^m} = \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m(m+1)} \frac{\alpha^{m+1}}{J^m}. \tag{2.3.33}$$

For example, we have

$$K_{1,2} = \frac{1}{2}. \tag{2.3.34}$$

Determining $K_{1,1}$

Determining $K_{m,n}$ for generic values of m and n by brute force computation is difficult and includes regularizing and renormalizing the multi-loop diagrams, but we need not do so. Let us show how we determine all the coefficients at $O(1/J)$ explicitly here. Since the contribution at $O(1/J)$ has a quadratic polynomial dependence on α, and we already know

$$K_{1,0} = \frac{1}{12}, \quad K_{1,2} = \frac{1}{2}, \tag{2.3.35}$$

if by some other method we know the contribution at $O(J)$ for some theory, we can determine all the coefficients of the polynomial.

We take this theory to be $\mathcal{N} = 4$ super Yang–Mills (SYM) with gauge group $SU(2)$; this theory has $\alpha = 1$ (A list of values of α for known theories are given in [45].), and q_n are already computed in [59, 60],

$$q_n = An + B + \log \Gamma(J+2), \tag{2.3.36}$$

where $J = n\Delta_{\mathcal{O}} \equiv 2n$ in this case. By expanding q_n in terms of $1/J$, we get the coefficient of $O(1/J)$ contribution, and by matching with the EFT prediction we get

$$\frac{13}{12} = P_1(\alpha = 1) = K_{1,0} + K_{1,1} + K_{1,2}. \tag{2.3.37}$$

By combining with (2.3.35), we get

$$K_{1,1} = \frac{1}{2}. \tag{2.3.38}$$

This determines the $O(1/J)$ contribution to q_n, universally for all rank-one theories to be

$$\frac{1}{J}\left(\frac{1}{12} + \frac{\alpha}{2} + \frac{\alpha^2}{2}\right), \tag{2.3.39}$$

not just for $\mathcal{N} = 4$ SYM.

Determining other values of $K_{m,n}$

Likewise, we can determine all values of $K_{m,n}$ universally for any rank-one SCFTs. This requires a trick of preparing theories in which we know how to compute q_n. This can be done with the help of tt^*-equations [59, 60] for four-dimensional Lagrangian gauge theories with an exact marginal coupling (so that $\Delta_{\mathcal{O}} = 2$). There actually are countably infinite number of them, if we allow the theory to be non-unitary, whose construction will be given later in Section 2.5.3.

For such theories, it is known that q_n obeys a differential recursion equation called the Toda equation [59,60],

$$\partial_\tau \bar{\partial}_{\bar{\tau}} q_n = \exp\left(q_{n+1} - q_n\right) - \exp\left(q_n - q_{n-1}\right), \quad \partial_\tau \bar{\partial}_{\bar{\tau}} q_0 = \exp\left(q_1 - q_0\right). \tag{2.3.40}$$

Although there are many known solutions to the infinite Toda equation, in this case it has to satisfy the boundary condition at $n = 0$ so that $q_0 = Z_{S^4}(\tau, \bar{\tau})$. Now, any physical solution should be cast into the form of (2.3.24) because the EFT analysis should apply in those cases too. Having no factors of $\operatorname{Im}\tau$ in the final result aside from the overall multiplicative constant $A(\tau, \bar{\tau})$ and the error term, we can reduce this differential recursion relation to an algebraic recursion relation, whose solution is

$$q_n^{[\text{Toda}]} = A(\tau, \bar{\tau})n + B(\tau, \bar{\tau}) + \log\Gamma\left[\left(n - n_+ + 1\right)\left(n - n_- + 1\right)\right] + O(e^{-\sqrt{n}}), \tag{2.3.41}$$

We show how to derive this formula later in Section 2.5.4.

We have already partially determined the form of q_n using EFT and the exact localization result from SYM, so if the solution $q_n^{[\text{Toda}]}$ dare be a physical one at all, it should have the expansion like

$$q_n = \cdots + \left(\alpha + \frac{1}{2}\right)\log J + \frac{1}{J}\left(\frac{1}{12} + \frac{\alpha}{2} + \frac{\alpha^2}{2}\right) + \cdots \tag{2.3.42}$$

where $J = n\Delta_{\mathcal{O}} = 2n$ in this case. This information is enough to constrain n_+ and n_-, because by matching two coefficients in front of $\log J$ and $1/J$, we get two equations,

$$1 - n_+ - n_- = \alpha + \frac{1}{2}, \quad n_+(n_+ + 1) + n_-(n_- + 1) + \frac{1}{3} = \frac{1}{12} + \frac{\alpha}{2} + \frac{\alpha^2}{2} \tag{2.3.43}$$

whose solutions for $n_\pm$ are

$$n_+ = \frac{1}{2} - \frac{\alpha}{2}, \quad n_- = -\frac{\alpha}{2}. \tag{2.3.44}$$

Therefore we got

$$q_n^{[\text{Toda}]} = An + B + \log\Gamma(J + \alpha + 1) + O(e^{-\sqrt{n}}), \tag{2.3.45}$$

for countably infinite values of α, where SCFTs with a marginal coupling lie.

This is yet too early to conclude that this formula is universal for any values of

α, until we notice that the number of values for which the above formula holds are countable infinite. Let us hypothetically assume that the number of α were finite, or even just imagine an extreme possibility that we could not find any values of α where Toda equation holds, aside from $\alpha = 1$, or SYM. In this case, we could have safely add terms like $\alpha(\alpha-1)/J^2$ to q_n and still not contradict with all we had known. This possibility is not realised exactly because we have infinitely many data points of α to compare to.

2.3.4 The final result

The final result of the computation of q_n therefore becomes

$$q_n = An + B + \log\Gamma(J+\alpha+1) + O\left(e^{-\sqrt{n}}\right) \tag{2.3.46}$$

for any four-dimensional, $\mathcal{N}=2$, rank-one SCFTs. For $\mathcal{N}=2$ Lagrangian gauge theories with gauge coupling τ, the error term scales as $O\left(e^{\sqrt{n/\operatorname{Im}\tau}}\right)$. This is because the mass of the BPS dyon scales as $\sqrt{n/\operatorname{Im}\tau} \propto g_{\mathrm{YM}}\sqrt{n}$.

2.4 Comparison to known data

2.4.1 $\mathcal{N}=2$, $SU(2)$ SQCD with four fundamental hypermultiplets

The second difference of q_n

In order to do numerical checks of the result, what is most important is to choose the actual data which is independent of theory and scheme dependent coefficients, A and B. The easiest thing to compute here is the second difference of q_n,

$$\triangle_n^2 q_n = q_{n+2} - 2q_{n+1} + q_n. \tag{2.4.1}$$

The EFT prediction for such a value is

$$\triangle_n^2 q_n = \log\left[\frac{(2n+\alpha+3)\,(2n+\alpha+4)}{(2n+\alpha+1)\,(2n+\alpha+2)}\right] + O\left(\frac{e^{\sqrt{n/\operatorname{Im}\tau}}}{(\operatorname{Im}\tau)^\alpha}\right) \equiv \triangle_n^2 q_n^{[\mathrm{EFT}]} + O\left(\frac{e^{\sqrt{n/\operatorname{Im}\tau}}}{(\operatorname{Im}\tau)^\alpha}\right) \tag{2.4.2}$$

which is independent of $\operatorname{Im}\tau$ perturbatively in $1/n$-expansion.

In Figure 2.4.1, we compare this prediction of EFT with that of exact localization. The procedure of the numerical computation is given in the next subsection, whereas the theory we use is the $\mathcal{N} = 2$, $SU(2)$ supersymmetric quantum chromodynamics (SQCD) with four fundamental hypermultiplets (which is the only case where we have easily accessible exact localisation results while the sphere partition function having nontrivial $\operatorname{Im}\tau$ dependence). This theory has $\alpha = 3/2$. Note that it is correct to have bigger errors as $\operatorname{Im}\tau$ gets bigger, because the mass of the BPS dyon gets lighter as one goes to weaker-coupling.

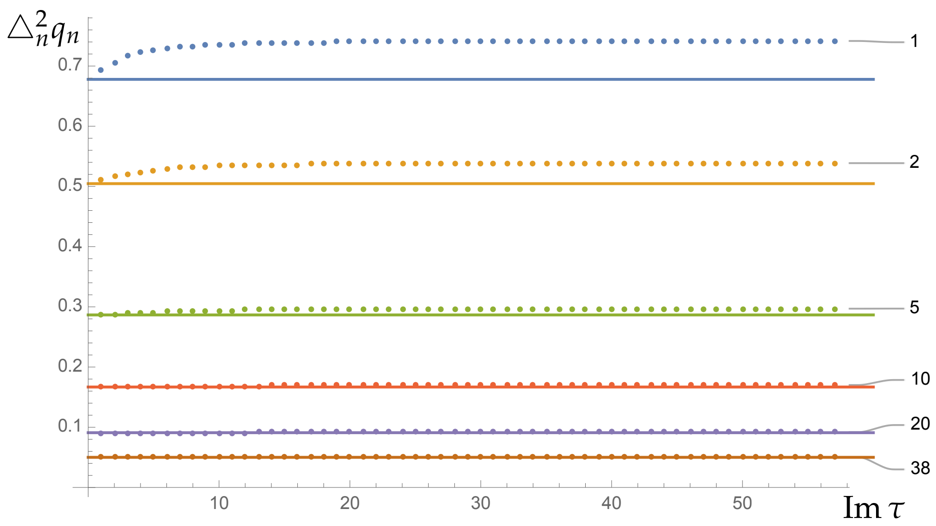


Figure 2.4.1: Second difference in n for $\triangle_n^2 q_n^{[\mathrm{loc}]}$ (dots) and for $\triangle_n^2 q_n^{[\mathrm{EFT}]}$ (continuous lines) as function of $\operatorname{Im}\tau$ at fixed values of n. The numerical results quickly reach τ-independent values that are well approximated by the asymptotic formula when n is larger than $n \gtrsim 5$.

The double scaling limit

As one can see above, weaker-coupling leads to bigger errors in EFT results. The double-scaling limit [61] is a way to avoid this problem by taking the weak-coupling limit while fixing

$$\lambda \equiv \frac{2\pi J}{\operatorname{Im}\tau}. \tag{2.4.3}$$

In this limit, the mass of the BPS dyon is fixed in terms of λ, so that the effect from the macroscopic massive propagation is supressed exponentially as $e^{-(\mathrm{const.})\times\sqrt{\lambda}}$.

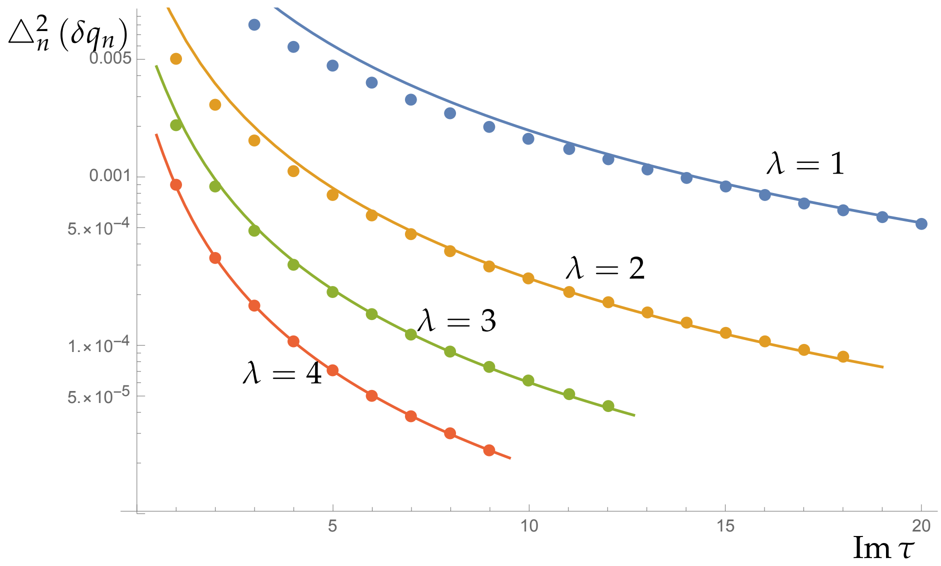


Figure 2.4.2: Second difference in n for the discrepancy between localization and EFT results $\triangle_n^2\,(\delta q_n)$ (dots) compared to $\triangle_n^2(1.6\,e^{-\sqrt{\pi\lambda}/2})$ (continuous lines) as functions of $\mathrm{Im}\,\tau$ at fixed values of $n/\,\mathrm{Im}\,\tau = \lambda/(4\pi)$. The agreement is quite good already for $\lambda = 3$.

In Figure 2.4.2, we plotted the error between the second differences of q_n, denoted $\delta q_n \equiv q_n^{[\mathrm{loc}]} - q_n^{[\mathrm{EFT}]}$, computed from EFT and exact localization, for the same theory, SQCD. The plot was fitted with a fit function for the error, based on the above prediction

$$\triangle_n^2\,(\delta q_n) \equiv \triangle_n^2\left(P \times e^{-\sqrt{Q\pi\lambda}/2}\right). \tag{2.4.4}$$

The result for the fit became

$$P \sim 1.6, \quad Q \sim 1.0, \tag{2.4.5}$$

where the agreement with the numerical data looked good for $\lambda \gtrsim 3$. The comparison of exact localization results and the EFT data plus the fit function for the error is also shown in Figure 2.4.3.

2.4.2 Numerical methods and error analysis

Computing two-point functions from exact localization.

In the case of $\mathcal{N} = 2$ SQCD with 4 flavors the correlators that we discuss can be computed via localization [62]. The two-point function between $\mathcal{O}^n$ and $\bar{\mathcal{O}}$, called G_{2n},

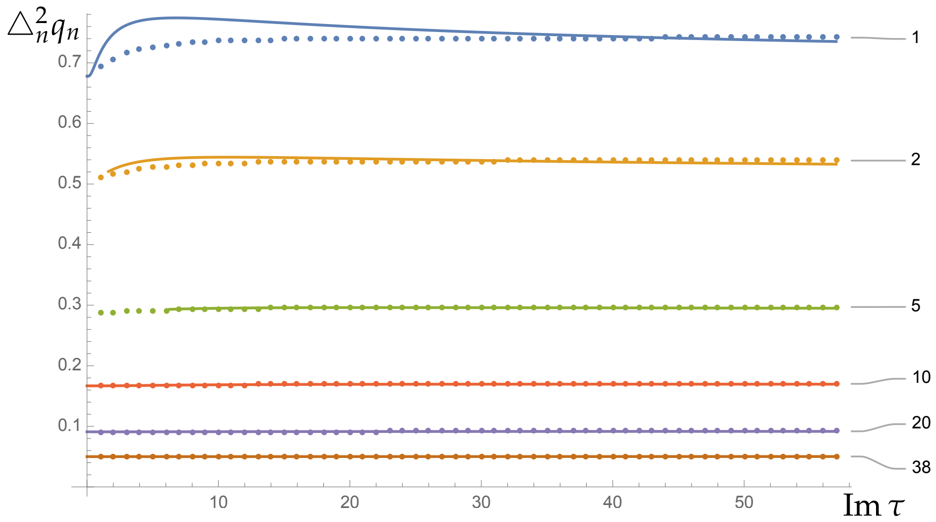


Figure 2.4.3: Comparing $\triangle_n^2 q_n^{[\text{loc}]}$ (dots) and for $\triangle_n^2 \left(q_n^{[\text{EFT}]} + 1.6\, e^{-\pi\sqrt{n/\operatorname{Im}\tau}}\right)$ (continuous lines) as function of $\operatorname{Im}\tau$ at fixed values of n. The exponential term seems to account for most of the discrepancy at small values of n (compare with Figure 2.4.1).

is the ratio of two determinants:

$$G_{2n} = 4^{2n} \frac{\det(M_n)}{\det(M_{n-1})}, \tag{2.4.6}$$

where M_n is the upper-left $(n-1) \times (n-1)$ submatrix of the (normalized) matrix of derivatives M of the partition function Z_0:

$$M|_{m,n} = \frac{1}{Z_0} \partial^n \bar{\partial}^m Z_0. \tag{2.4.7}$$

The partition function for $\mathcal{N} = 2$ SQCD is written in terms of the Barnes G-function :

$$Z_0 = Z_{S^4}^{\text{SQCD}}(\tau, \bar{\tau}) = \int_{-\infty}^{\infty} da\, a^2 e^{-4a \operatorname{Im}\tau} \frac{|G(1+2ia)|^4}{|G(1+ia)|^{16}} |Z_{\text{inst}}(ia, \tau)|^2, \tag{2.4.8}$$

where Z_{inst} is the instanton partition function [63, 64]:

$$Z_{\text{inst}}(a, \tau) = 1 + \frac{1}{2}\left(a^2 - 3\right) e^{2\pi i \tau} + O\left(e^{4\pi i \tau}\right). \tag{2.4.9}$$

For simplicity we will consider the regime $\operatorname{Im}\tau > 1$ and ignore the instanton corrections (this only produces the error of order $e^{-2\pi} \sim 1\%$). Note that in this approximation the partition function $Z_{\text{inst}}(a, \tau)$ is independent of $\text{Re}(\tau)$.

Sensitivity to initial conditions

Since we want to evolve the recursion relations numerically starting from an approximate initial condition for the S^4 partition function, we need to estimate the sensitivity of large-$\mathcal{J}$ correlation functions to imprecise initial conditions.

We may wish to start at some initial value n_i greater than 0. The recursion relation is second order, so in order to define initial conditions, we need to define both q_{n_i} and q_{n_i+1}. These initial conditions are of course functions of τ and $\bar{\tau}$, but we will suppress in this section the dependence on the arguments $\tau, \bar{\tau}$ in our notation.

It is useful to write the rank-one recursion relations in their "deterministic" form. Given any initial conditions at $n_i, n_i + 1$, there is always a unique solution to the recursion relations for $n \geq n_i + 2$. One can consider two nearby solutions, separated by a small amount δ_n, and analyze how the linearized deviation propagates to larger values of n. The deviation propagation equation is:

$$\begin{aligned} & \delta_{n+2} + \delta_n - 2\delta_{n+1} \\ = \; & \frac{16\,(\text{Im}(\tau))^2}{(2n+3+\alpha)(2n+4+\alpha)}\,\partial_\tau \partial_{\bar{\tau}}\,\delta_{n+1} - \frac{2\,(4n+2\alpha+5)}{(2n+\alpha+3)(2n+\alpha+4)}\,(\delta_{n+1} - \delta_n)\,. \end{aligned} \tag{2.4.10}$$

Even at the linearized level, this equation is nontrivial, and depends on the decomposition of the error into eigenvalues of the Laplacian on the upper half plane or its quotient under the modular group. We do not analyze the propagation of errors for general perturbations. Instead, we use the fact that the perturbative piece of $Z_0(\tau, \bar{\tau})$ is a good approximation at weak coupling. As pointed out in [61], the clash between weak coupling and large $\mathcal{J}$ can be avoided if one considers the limit $\mathcal{J} \to \infty$ while taking $\lambda \equiv 2\pi\,\mathcal{J}\,/\,\text{Im}(\tau)$ fixed. Since our formula for the power-law corrections is τ-independent for rank-one theories, these two limits coincide for the power-law piece $\log(\Gamma(\mathcal{J} + \alpha + 1))$, differing only in the behavior of the nonuniversal exponential correction. We can therefore isolate this correction easily in the fixed-λ limit, in which the instanton contributions to $Z_0(\tau, \bar{\tau})$ go to zero exponentially in n.

One might expect the exponentially small corrections to be associated with the breakdown of the EFT altogether, capturing the leading effects of massive states propagating over distances on the infrared scale, as discussed in (2.3.24). One would therefore anticipate exponentially small corrections proportional to $\propto \exp[-\kappa\lambda^{1/2}]$, with $\tilde{\kappa}$ some fixed number depending on the geometry of the virtual propagation, but not on n or $\tau, \bar{\tau}$. Numerically, we find a remarkably accurate match to such an exponential, with $\tilde{\kappa} = \sqrt{\pi}/2$, as shown below.

Consideration of the second difference

In (2.3.24) we have seen that only the coefficients of n^0 and n^1 in the asymptotic expansion of $q_n(\tau, \bar{\tau})$ are expected to depend on τ. This means that the second variation in n of $q_n(\tau, \bar{\tau})$ is τ-independent. Let $\triangle$ be the difference operator $\Delta_n q_n = q_{n+1} - q_n$. We want to compute the second difference

$$\triangle_n^2 q_n^{[\mathrm{loc}]}(\tau, \bar{\tau}) = q_{n+2}^{[\mathrm{loc}]}(\tau, \bar{\tau}) - 2q_{n+1}^{[\mathrm{loc}]}(\tau, \bar{\tau}) + q_n^{[\mathrm{loc}]}(\tau, \bar{\tau}) \tag{2.4.11}$$

and compare it with the result in (2.4.2),

$$\triangle_n^2 q_n^{[\mathrm{EFT}]} = \log\left[\frac{(2n+\alpha+3)\,(2n+\alpha+4)}{(2n+\alpha+1)\,(2n+\alpha+2)}\right]. \tag{2.4.12}$$

Figure 2.4.1 shows the results of a numerical computation for imaginary values of τ between 1 and 60 and for n between 1 and 40, representing the values of $\triangle_n^2 q^{[\mathrm{loc}]}$ as function of τ at fixed values of n. We see that quite rapidly, already for $\tau \simeq 4i$, the τ-dependence drops for all values of n. The asymptotic value is well approximated by $\triangle_n^2 q_n^{\mathrm{EFT}}$ for n larger that $n \gtrsim 5$, where the discrepancy is of order

$$1 - \left.\frac{\triangle_n^2 q_n^{[\mathrm{EFT}]}}{\triangle_n^2 q_n^{[\mathrm{loc}]}}\right|_{n=5,\tau\gg 1} \approx 1\%. \tag{2.4.13}$$

At $n = 1$, the discrepancy is of order

$$1 - \left.\frac{\triangle_n^2 q_n^{[\mathrm{EFT}]}}{\triangle_n^2 q_n^{[\mathrm{loc}]}}\right|_{n=1,\tau\gg 1} \approx 8\%. \tag{2.4.14}$$

Estimation of the error

The numerical data can help us estimate the τ and n dependence of the difference $\triangle_n^2(q_n^{[\mathrm{loc}]} - q_n^{[\mathrm{EFT}]})$. As discussed in the last subsection, we expect the leading contribution to the difference to have the form

$$q_n^{[\text{loc}]} - q_n^{[\text{EFT}]} \sim f_n(\tau, \bar{\tau}) e^{-\kappa \sqrt{n/\operatorname{Im}\tau}} = f_n(\tau, \bar{\tau}) e^{-\kappa \sqrt{\lambda/(4\pi)}}, \tag{2.4.15}$$

where $\lambda = ng^2 = 4\pi n/\operatorname{Im}\tau$. To verify this conjecture and estimate the proportionality factor $f_n(\tau, \bar{\tau})$ and the coefficient κ we have computed the difference as a function of τ, keeping the ratio $n/\operatorname{Im}\tau = \lambda/(4\pi)$ fixed (see Figure 2.4.2). The numerical data is consistent with $f_n(\tau, \bar{\tau})$ being a constant approximately equal to $f_n(\tau, \bar{\tau}) \approx 1.6$ and $\kappa \approx \pi$. Already for $\tau \approx 3$ our conjecture seems to reproduce the localization data to high accuracy. Interestingly, this single exponential term seems to account for the discrepancy $\triangle_n^2 \left(q_n^{(\text{loc})} - q_n^{[\text{EFT}]} \right)$ both in the small-τ, large-n (*i.e.* large-λ) regime and in the large-τ regime (see Figure 2.4.3).

2.5 Technical details

2.5.1 Nonexistence of higher-derivative *F*-terms on conformally flat space

Higher-derivative F-terms on the Coulomb branch

In general $\mathcal{N} = 2$ supersymmetric gauge theories, the effective action on the Coulomb branch has higher derivative F-terms, of which those with few derivatives have been partially classified by [65–68]. In the case of superconformal gauge theories with rank one, the remarkable simplifications of the dynamics of the Coulomb branch have to do with the absence of such terms.

More precisely, the only half-superspace integrands consistent with superconformal symmetry on a general curved background, are the tree-level kinetic term proportional to Φ^2, and terms involving the background Weyl multiplet, which contains the graviphoton background and the self-dual part of the Weyl tensor.

Coupling to background supergravity

Consider the effective action of a single Abelian vector multiplet in a superconformally invariant theory. The symmetries of a $\mathcal{N} = 2$ superconformal theory include dilatation and $U(1)$ R-symmetry, which act on a vector multiplet ϕ by rescalings and complex phase rotations respectively, both in the underlying microscopic CFT and in the

EFT of the Coulomb branch. The Weyl symmetry acts as

$$\phi \to e^{\rho}\phi \ , \tag{2.5.1}$$

and the $U(1)_R$ acts as

$$\phi \to e^{i\gamma}\phi \ . \tag{2.5.2}$$

One can combine the Weyl and $U(1)_R$ parameters into a single complex parameter $\sigma \equiv -\rho - i\gamma$, which acts as

$$\phi \to e^{-\sigma}\phi \ , \qquad\qquad \bar{\phi} \to e^{-\bar{\sigma}}\bar{\phi} \ . \tag{2.5.3}$$

As long as we only consider vector multiplets, it is possible to promote σ to a local function of superspace, not only just of the x coordinates. Such a superspace formalism in $\mathcal{N} = 2$ supergravity of vector multiplets has already been worked out in [69]. In order to preserve the chirality constraint $\bar{Q}^i_{\dot{\alpha}} \cdot \phi = 0$ we can require σ to obey the same chirality constraint $\bar{Q}^i_{\dot{\alpha}} \cdot \sigma = 0$.

The chiral superfield Weyl parameter σ consists of a complex scalar, fermions, and a vector parameter $\hat{\lambda}_\mu$, and other components which act only on the auxiliary fields. The scalar and fermionic members of the parameter superfield implement Weyl, $U(1)_R$, and local supersymmetry transformations, respectively, whereas the $\hat{\lambda}_\mu$-transformations shift the gauge field as

$$\delta A_\mu = \phi\,\hat{\lambda}_\mu \ . \tag{2.5.4}$$

Local transformations are not themselves symmetries of the dynamical fields alone, but can be understood as "spurionic" symmetries, that preserve the action for dynamical variables together with a set of transformations onto background fields. In the case of local dilatation and local $U(1)_R$ transformations, the corresponding background fields are the metric and the $U(1)_R$ gauge field, which transform by Weyl transformations and local $U(1)_R$ gauge transformations, respectively. The $\hat{\lambda}_\mu$-transformations can be thought of as shifting a background antisymmetric tensor field $B_{\mu\nu}$ by a gauge transformation

$$B_{\mu\nu} \mapsto B_{\mu\nu} + (d\hat{\lambda})_{\mu\nu} \qquad\qquad (d\hat{\lambda})_{\mu\nu} = \partial_\mu\hat{\lambda}_\nu - (\mu \leftrightarrow \nu) \ . \tag{2.5.5}$$

The EFT inherits the super-Weyl invariance of the underlying CFT, so we can now consider what possible terms one might write in a supersymmetric $\mathcal{N} = 2$ EFT consistent with super-Weyl invariance. For a single $U(1)_{\text{gauge}}$ vector multiplet, the Weyl and local $U(1)_R$ transformations give enough freedom to set the complex scalar ϕ equal to some fixed nonzero value, say μ, everywhere that it is nonvanishing: By choosing $\sigma = +\log(\phi/\mu)$ we can fix the "gauge," $\phi = \mu$.

The fermions ψ^i_α in the Abelian vector multiplet are superpartners of ϕ, and supersymmetry implies that if $\phi - \mu$ can be made to vanish with a local transformation, then ψ^i_α can be made to vanish as well at the cost of turning on a nonzero but flat background for the (spurionic) gravitini. Also, the freedom to make $\hat{\lambda}_\mu$-transformations allows us to set the gauge field to zero as well, and so the entire vector multiplet in the EFT can be gauged away.

We are now left with various background fields from which to construct super-Weyl invariant action, if it ever exists. As we are considering only the maximally supersymmetric background $\mathbb{R}^4$ (and backgrounds equivalent to it such as the sphere S^4 and the cylinder $S^3 \times \mathbb{R}$), we need only consider couplings involving the Ricci curvature and its derivatives, since the Weyl curvature and R-symmetry gauge flux vanish in the backgrounds we consider.[5]

For D-terms there are many such terms one can construct: The dressed metric $\hat{g}_{\mu\nu} \equiv |\phi|^2 g_{\mu\nu}$ is Weyl-invariant and its superspace extension is super-Weyl-invariant by construction. So any term constructed from these has Weyl weight zero and is suitable for addition to the action as a D-term (*i.e.*, full $\mathcal{N} = 2$ superspace $d^4\theta\, d^4\bar{\theta}$ integrand) consistent with super-Weyl invariance.

Two-point functions of elements of the chiral ring, correlators we considered in the main body of the thesis, are computable by localization and insensitive to D-terms however. We need therefore consider only super-Weyl-invariant F-term contributions to the effective action.[6]

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5 Remember that in four dimensions, the Riemann tensor can be decomposed into the Ricci and the Weyl tensors.

6 In addition to the familiar D-terms and F-terms, $\mathcal{N} = 2$ supersymmetric effective theories with hypermultiplets may have terms that can be represented as $\frac{3}{4}$-superspace integrals but not true F-terms. Some such terms have been worked out in [65–68]. However we can restrict our attention to theories with only a pure Coulomb branch and no massless neutral hypers, rather than an enhanced Coulomb branch. For theories with no hypermultiplets we may consider only the usual F-terms and D-terms.

Such terms must be of the form

$$\Delta\mathcal{L} = \int d^4\theta\, \phi^2\, \mathcal{I}_0 \,, \tag{2.5.6}$$

where $\mathcal{I}_0$ is a super-Weyl-invariant term that is also a chiral primary field, *i.e.*, annihilated by all the $\bar{D}^i_{\dot{\alpha}}$ superderivatives. As we have pointed out above, such terms must be constructed from Ricci curvatures $\hat{R}_{\mu\nu}$ of the hatted metric $\hat{g}_{\mu\nu}$ and the R-gauge field α with a condition $d\alpha \equiv 0$ imposed (pure gauge). It is rather immediate however that the latter ingredient, α, cannot really be used because when we demand gauge invariance, the only gauge invariant building block is $d\alpha$, which vanishes on our background. Of course you can also construct topological terms inside $\mathcal{I}_0$, but multiplying ϕ^2 will again break gauge invariance.

Now, let's turn to $\hat{g}_{\mu\nu}$. What is important is that this is not a chiral field, nor is the Ricci curvature or any of its derivatives;

$$\bar{D}^i_{\dot{\alpha}}(\hat{g}_{\mu\nu}) = \omega\, \hat{g}_{\mu\nu} \,, \qquad\qquad \omega \equiv \frac{\bar{\Psi}^i_{\dot{\alpha}}}{\bar{\phi}} \,. \tag{2.5.7}$$

In other words, even though $\hat{g}_{\mu\nu}$ is Weyl-invariant (*unhatted* Weyl invariance), acting with the antichiral supersapce derivative on $\hat{g}_{\mu\nu}$ is equivalent to infinitesimally Weyl-transforming the *hatted* metric by a Weyl parameter proportional to $\bar{\Psi}^i_{\dot{\alpha}}/\bar{\phi}$ (*hatted* Weyl transformation), which does not vanish identically, obviously.

Therefore only quantities that can be constructed from $\hat{g}_{\mu\nu}$ are the *hatted* Weyl-invariant quantities. But since these quantities are Weyl-invariant, *hatted* nor *unhatted* does exactly make a difference. They are exactly the same as the ones constructed from $g_{\mu\nu}$, that is, the *unhatted* Weyl curvature and various powers of it and its Weyl-covariantized derivatives.

Again, many such terms can be constructed, and would contribute to F-terms on a non-conformally-flat background; however for a background with vanishing Weyl curvature, all such terms vanish. We therefore conclude that all higher-derivative F-terms vanish identically on a conformally flat background, in the effective theory of a single Abelian vector multiplet.

More comments on the currents

The action of the super-Weyl transformations on the physical fields is generated by currents with protected integer operator dimensions living in a single current multiplet; for the case of dilatations the generating operator is the trace of the stress tensor with dimension 4 and for $U(1)$ transformations the generating current is the $U(1)_R$-current with dimension 3. The current generating the λ-transformations is an antisymmetric tensor of weight 3 (see for instance [70, 71]).

Since these currents are local, they can be integrated against arbitrary functions to generate well-defined local transformations of the fields. This is the physical basis of the super-Weyl transformation: An infinitesimal change of the SUGRA background is equivalent to an infinitesimal transformation of the physical degrees of freedom, which in turn is equivalent to inserting integrated currents into the path integral. For instance an infinitesimal change in the background metric is equivalent to

$$S \to S + \int \sqrt{|g|}\,(\delta g_{\mu\nu})\,T^{\mu\nu}\,; \tag{2.5.8}$$

an infinitesimal change in the R-symmetry gauge connection is equivalent to

$$S \to S + \int \sqrt{|g|}\,(\delta \mathcal{A}_\mu^{U(1)_R})\,J^\mu_{U(1)_R}\,; \tag{2.5.9}$$

and an infinitesimal change in the antisymmetric tensor background is equivalent to

$$S \to S + \int \sqrt{|g|}\,(\delta \mathcal{B}_{\mu\nu})\,\mathcal{Z}^{\mu\nu} + (\text{c.c.})\,. \tag{2.5.10}$$

Diffeomorphism and Weyl invariance are equivalent to the statements that $T_{\mu\nu}$ is divergenceless and traceless, respectively; $U(1)_R$ invariance is equivalent to the statement that $J^\mu_{U(1)_R}$ is divergenceless. There is no simple analogous statement about the $\mathcal{Z}$-current, which sits in the (short) stress tensor multiplet as an anti-self-dual tensor with conformal dimension 3. At the free-field level, the $\mathcal{Z}$-current is proportional to $\phi F^{(-)}_{\mu\nu}$, where $F^{(-)}$ is the anti-self-dual part of the gauge field strength. Its complex conjugate generates $\hat{\lambda}$-transformations on the vector multiplet when integrated against $\hat{\lambda}$.

Unlike the R-current and stress tensor, its divergence does not vanish. Correspondingly, the coupling of $\mathcal{Z}^{\mu\nu}$ to the background $\mathcal{B}_{\mu\nu}$-field is somewhat subtle; the $\hat{\lambda}$ one-form transformations must act on other background fields in addition to the $\mathcal{B}_{\mu\nu}$-field. The coupling of the $\mathcal{Z}$-current to the SUGRA background is formalism-dependent, as the $\mathcal{B}$-field is not part of the minimal $\mathcal{N} = 2$ SUGRA multiplet and the details have not been worked out in the SUGRA literature. One can infer the physically relevant properties of the coupling by considering the current directly, whose properties are formalism-independent.

The $\mathcal{Z}_{\mu\nu}$ current, which generates the $\hat{\lambda}_\mu$-transformations which shift the gauge field in the vector multiplet, is less well-studied than the other members of its multiplet, the stress tensor and R-current. Since the super-Weyl transformation generated in part by $\mathcal{Z}_{\mu\nu}$ plays a role in forbidding higher-derivative F-terms for one-dimensional Coulomb-branch EFTs, we comment briefly on properties of this current for the sake of context [70–72].

The $\mathcal{Z}_{\mu\nu}$ current is similar to the line-charge symmetry that shifts the photon in a weakly-coupled Maxwell gauge theory [73, 74], but it is a different sort of current. The line-charge current in four-dimensional Abelian gauge theory has dimension approximately two at weak coupling rather than three, and cannot be exactly conserved unless the dimension is exactly two and Maxwell field is exactly free, in analogy with the parallel Sugawara theorem for spin-one currents in two dimensions [75].

By contrast the $\mathcal{Z}$-current has dimension three and is not divergenceless. Indeed, the divergence of the $\mathcal{Z}$-current contributes to the central charge in the $\mathcal{N}=2$ supersymmetry algebra. That is,

$$\{Q^i_\alpha, Q^j_\beta\} = 2\,\epsilon^{ij}\,\epsilon_{\alpha\beta}\,Z\,,$$
$$Z \ni \int d^3\,\mathcal{N}^\mu\,\nabla^\nu\,\mathcal{Z}_{\mu\nu}\,, \tag{2.5.11}$$
$$\mathcal{Z}_{\mu\nu} \propto \epsilon_{ij}\left([\sigma_\mu,\sigma_\nu]\right)^{\alpha\beta}\epsilon_{ij}\,Q^i_\alpha\cdot Q^j_\beta\cdot \mathbf{J}_{\text{scalar}}\,,$$

where $\mathbf{J}_{\text{scalar}}$ is the lowest component of the stress tensor multiplet, a scalar primary of dimension $\Delta = 2$ transforming trivially under the R symmetry and equal to $\phi\bar{\phi}$ in the Coulomb-branch EFT [70, 71].

In a superconformal $\mathcal{N}=2$ theory without marginal operators, this current is the *only* contribution to the central charge; there are no other currents of dimension three and the correct quantum numbers to appear in the supersymmetry (SUSY) algebra. The normalization of the central charge Z is therefore determined by the three-point function of the current multiplet in such theories, which means its value is fixed entirely by the anomaly coefficients a and c. This has interesting implications for the BPS dyon spectrum on the Coulomb branch of non-Lagrangian $\mathcal{N} = 2$ SCFT.

In a superconformal $\mathcal{N} = 2$ theory with marginal operators, there is a second independent component of the central charge, also a total derivative, of a current which we shall call $\mathcal{Y}_{\mu\nu}$:

$$Z \ni \int d^3 \mathcal{N}^\mu \nabla^\nu \mathcal{Y}_{\mu\nu} \,,$$

$$\mathcal{Y}_{\mu\nu} = \sum_A y_A \, \epsilon_{ij} \left([\sigma_\mu, \sigma_\nu]\right)^{\alpha\beta} \epsilon_{ij} \, Q^i_\alpha \cdot Q^j_\beta \cdot \mathcal{O}^A \,, \tag{2.5.12}$$

where A runs over all marginal operators $\mathcal{O}^A$ and y_A can vary over the conformal manifold. All the dependence of the central charge on the marginal directions is through the $\mathcal{Y}$-current contribution.

2.5.2 Weyl anomaly action in 4d $\mathcal{N} = 2$, rank-1 scfts

$\mathcal{N} = 2$ super-Wess-Zumino term for the Weyl anomaly

When a conformal field theory has a moduli space, moving to a generic point of it breaks the conformal symmetry spontaneously. Because of the argument of t'Hooft anomaly matching, in flowing from UV to IR, there must be a term that cancels the change in a- and c-anomalies. This is accomplished by writing down the effective action of the dilaton, which is the Nambu-Goldstone boson of conformal symmetry breaking. This term, the WZ term for the Weyl anomaly, is given in [76].

An $\mathcal{N} = 1$ supersymmetrization of this term was given in [58]. This term is not the unique supersymmetrization preserving $\mathcal{N} = 1$ superconformal symmetry. Alternate supersymmetrizations of the term can be obtained by adding $\mathcal{N} = 1$ superconformally-invariant terms to the action, for instance involving a $\mathcal{N} = 1$ superconformal action for the gauge fields.

Because of peculiarity of the full superspace integrand in $\mathcal{N} = 2$ that it is dimensionless, this non-uniqueness is absent in $D = 4$, $\mathcal{N} = 2$, rank-1 theories [57]. In $\mathcal{N} = 2$ superspace, the term can be written formally as a full-superspace integral

$$\mathcal{L}_{\mathcal{N}=2 \text{ super}-WZ} = (\text{constant}) \times \int d^4\theta d^4\bar{\theta} \log(\Phi/\mu) \, \log(\Phi^\dagger/\mu) \,. \tag{2.5.13}$$

We wish to write this in components, particularly the terms involving the scalar $\phi, \bar{\phi}$

and its derivatives. The constant can also be determined by matching with the non-supersymmetric wz term.

$\mathcal{N}=1$ superfield decomposition of the $\mathcal{N}=2$ super-wz term

The full form of the $\mathcal{N}=2$ super-wz term is easiest to write in terms of $\mathcal{N}=1$ superfields, as expressed in [77, 78]. The $\mathcal{N}=2$ vector multiplet Φ decomposes into an $\mathcal{N}=1$ superfield $\phi_{\mathcal{N}=1}$ and an $\mathcal{N}=1$ vector multiplet V whose gauge-invariant super-field strength is $\mathcal{W}_\alpha$. In $\mathcal{N}=1$ superspace, the form of the term is

$$\mathcal{L}_{\mathcal{N}=2\ \mathrm{super-}WZ} = \int (d^4\theta)_{\mathcal{N}=1} \left[C_1\, \mathcal{I}_1^{[\mathcal{N}=1]} + C_2\, \mathcal{I}_2^{[\mathcal{N}=1]} \right] + (\text{terms involving } W_\alpha)\ , \tag{2.5.14}$$

where C_1 and C_2 are constants and

$$\mathcal{I}_1 \equiv \frac{1}{\phi_{\mathcal{N}=1}\,\bar{\phi}_{\mathcal{N}=1}} (\partial_\mu \phi_{\mathcal{N}=1})(\partial^\mu \bar{\phi}_{\mathcal{N}=1})\ , \tag{2.5.15}$$

$$\mathcal{I}_2 \equiv \frac{1}{\phi_{\mathcal{N}=1}\,\bar{\phi}_{\mathcal{N}=1}} (\epsilon^{\alpha\beta} D_\alpha D_\beta\, \phi_{\mathcal{N}=1})\, (\epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\alpha}} \bar{D}_{\dot{\beta}}\, \bar{\phi}_{\mathcal{N}=1}). \tag{2.5.16}$$

The D_α and $\bar{D}_{\dot{\alpha}}$ are the spinorial superspace covariant derivatives.

Reducing the $\mathcal{N}=2$ super-wz action into scalar components on flat space

First for our purposes, we only need to write down the purely scalar part of the whole action. To this goal, one just drops all terms including the auxiliary field $F \equiv \phi_{\mathcal{N}=1}|_{\theta^2}$, as well as fermions and superfield strength. The reason for this is the following; Because $\mathrm{Re}\, F$ and $\mathrm{Im}\, F$ form a triplet under $SU(2)_R$ together with D, the real auxiliary field for the vector multiplet, the coupling linear in those auxiliary fields will necessarily include two or more fermions. Especially, there is no way in which one can write down purely scalar terms after integrating out those auxiliary fields. So we need only consider the superspace integrals of the two terms $\mathcal{I}_{1,2}$, and in particular only the component terms containing no fermions or auxiliary fields.

With attention restricted to such component terms, the superspace integral of $\mathcal{I}_2$ is easiest to compute. In order to obtain a term involving only scalars, we must take the $\bar{\theta}\bar{\theta}$ component of $D^2\Phi_{\mathcal{N}=1}$ and the $\theta\theta$ component of $\bar{D}^2\Phi^\dagger_{\mathcal{N}=1}$, which are proportional to $\partial^2\phi$ and $\partial^2\bar{\phi}$, respectively. So we have

$$\int (d^4\theta)_{\mathcal{N}=1}\, \mathcal{I}_2 \simeq (\text{const.}) \times \frac{(\partial^2\phi)(\partial^2\bar{\phi})}{|\phi|^2}, \tag{2.5.17}$$

where the $\simeq$ denotes the omission of terms involving fermions and auxiliary fields.

The superspace integral of $\mathcal{I}_1$ can be evaluated easily using a trick: Treat $\partial_\mu \phi_{\mathcal{N}=1}$ and its conjugate as independent superfields $G_\mu, G_\mu^\dagger$, and write the superspace integrand as a Kähler potential for the five superfields $\chi^A \in \{\phi_{\mathcal{N}=1}, G_\mu\}$ and their conjugates. So

$$\mathcal{I}_1 = \mathcal{K}(\chi, \chi^\dagger) = |\phi_{\mathcal{N}=1}|^{-2}\, \eta^{\mu\nu} G_\mu G_\nu^\dagger \ . \tag{2.5.18}$$

Then the superspace integral is given by the usual formula written in terms of the Kahler potential,

$$\begin{aligned} \int (d^4\theta)_{\mathcal{N}=1} \mathcal{I}_1 &\simeq \mathcal{K}_{,\chi^A \chi^{B\dagger}} \partial_\nu \chi^A \partial^\nu \chi^{B\dagger} \\ &= |\phi|^{-4}\,(\partial\phi\partial\bar{\phi})^2 - \phi^{-2}\bar{\phi}^{-1}\,(\partial^\mu\partial^\nu\bar{\phi})(\partial_\mu\phi)(\partial_\nu\phi) \\ &\qquad - \bar{\phi}^{-2}\phi^{-1}\,(\partial^\mu\partial^\nu\phi)(\partial_\mu\bar{\phi})(\partial_\nu\bar{\phi}) + |\phi|^{-2}\,(\partial^\mu\partial^\nu\phi)(\partial_\mu\partial_\nu\bar{\phi}) \ . \end{aligned} \tag{2.5.19}$$

Rewriting the two terms with the substitution (black τ is preserved for the gauge coupling parameter),

$$\phi = \mu\, e^{-\uptau - i\upbeta} \ , \qquad \bar{\phi} = \mu\, e^{-\uptau + i\upbeta} \ , \tag{2.5.20}$$

we get

$$\int (d^4\theta)_{\mathcal{N}=1}\, \mathcal{I}_1 = (\text{const.}) \times (\partial^\mu\partial^\nu\uptau)(\partial_\mu\partial_\nu\uptau) + (\text{fermions and auxiliary}) \ , \tag{2.5.21}$$

and

$$\begin{aligned} \int (d^4\theta)_{\mathcal{N}=1}\, \mathcal{I}_2 = (\text{const.}) \times \Big[&(\partial^\mu\partial^\nu\uptau)(\partial_\mu\partial_\nu\uptau) - 2\,(\partial^\mu\partial^\nu\uptau)(\partial_\mu\uptau)(\partial_\nu\uptau) + (\partial\uptau)^4 \Big] \\ &+ (\text{fermions and auxiliary}) \ , \end{aligned} \tag{2.5.22}$$

where we call $\uptau$ as the dilaton and the $\upbeta$ as the axion and have set $\upbeta$ to be constant to simplify results.

Modulo total derivatives, this is

$$\int (d^4\theta)_{\mathcal{N}=1}\, \mathcal{I}_1 = A \times (\partial^2\uptau)^2 + (\text{fermions and auxiliary}) \ , \tag{2.5.23}$$

and

$$\int (d^4\theta)_{\mathcal{N}=1}\,\mathcal{I}_2 \;=\; B \times \left[(\partial^2\tau)^2 - 2\,(\partial\tau)^2(\partial^2\tau) + (\partial\tau)^4\right] + (\text{fermions and auxiliary}) \;. \tag{2.5.24}$$

The coefficients are given in [77, 78], and it can be determined as follows. First of all, because of the absence of the term proportional to $(\partial^2\tau)^2$ in the bosonic wz term in [76], the relative coefficient of the above two must be -1, *i.e.*, $A = -B$,

$$\mathcal{L}_{WZ}^{(\text{Euclidean})} = B \times \left[\, (\partial\tau)^4 - 2(\partial^2\tau)(\partial\tau)^2\right] \tag{2.5.25}$$

The absolute coefficient can hence be matched with [76] to be $B = 2 \times (\Delta a)^{[\text{KS}]}$,

$$\mathcal{L}_{WZ}^{(\text{Euclidean})} = 2(\Delta a)^{[\text{KS}]} \times \left[\, (\partial\tau)^4 - 2(\partial^2\tau)(\partial\tau)^2\right] \tag{2.5.26}$$

We therefore arrive at the dilaton and axion part of the super-wz term,

$$\begin{aligned}\mathcal{L}_{\text{super}-WZ}^{(\text{Euclidean})} = +2\,(\Delta a)^{[\text{KS}]} \times \Big[&(\partial\tau)^4 - 2(\partial^2\tau)(\partial\tau)^2 + 2\,(\partial^2\tau)(\partial\beta)^2 \\ &- 4(\partial\tau\cdot\partial\beta)\,(\partial^2\beta) - 2(\partial\tau)^2(\partial\beta)^2 + 4(\partial\tau\cdot\partial\beta)^2 + (\partial\beta)^4 \Big] \;,\end{aligned} \tag{2.5.27}$$

where we have evaluated the term in flat space, and dropped terms involving the gauge field, fermions, and auxiliary fields. Note that this agrees with the flat space expression given in [58].

Expression for other conformal frames

The term above is not conformally invariant – it is, after all, the wz term for the Weyl anomaly, and conformally transforming the above expression to S^4 or $S^3 \times \mathbb{R}$, legitimately, looks complicated. This impression, however, is delusional; the whole expression can be decomposed into the Komargodski–Schwimmer (ks) action (2.5.26) plus a reminder, and the former is solely responsible for the anomalous Weyl transformation. Especially, the reminder terms are invariant under Weyl transformations.

On flat space, the wz Lagrangian breaks up as:

$$\mathcal{L}[\tau, \beta, g] = \mathcal{L}_{[\text{KS}]}[\tau, g] + \mathcal{L}_{[\text{remainder}]}[\tau, \beta, g] \;, \tag{2.5.28}$$

where

$$\mathcal{L}_{[\mathrm{KS}]}[\tau, g] \equiv 2(\Delta a)^{[\mathrm{KS}]}\Big[(\partial\tau)^4 - 2(\partial^2\tau)(\partial\tau)^2\Big]$$

$$\mathcal{L}_{[\mathrm{remainder}]}[\tau, \beta, g] \equiv 2(\Delta a)^{[\mathrm{KS}]}\Big[2(\partial^2\tau)(\partial\beta)^2 - 4(\partial\tau\cdot\partial\beta)\,(\partial^2\beta) - 2(\partial\tau)^2(\partial\beta)^2 + 4(\partial\tau\cdot\partial\beta)^2 + (\partial\beta)^4\Big]\,. \tag{2.5.29}$$

The covariantization of the two terms gives [58]

$$\begin{aligned} &S^{(\mathrm{lorentzian})}_{\mathrm{super-WZ,\ curved}} \\ = &-\Delta a^{[\mathrm{KS}]}\int d^4x\sqrt{-g}\Big[\tau\, E_4^{[\mathrm{KS}]} + \Big(4\left(R^{\mu\nu} - \frac{1}{2}R\, g^{\mu\nu}\right)\nabla_\mu\tau\,\nabla_\nu\tau - 2\,(\nabla\tau)^2\Big(2\,\Box\tau - (\nabla\tau)^2\Big)\Big)\Big] \\ &+ 4\sqrt{-\hat{g}}\left[\left(\hat{R}^{\mu\nu} - \frac{1}{6}\hat{R}\,\hat{g}^{\mu\nu}\right)\nabla_\mu\beta\nabla_\nu\beta + \frac{1}{2}\left(\hat{g}^{\mu\nu}\,\nabla_\mu\beta\nabla_\nu\beta\right)^2\right] \\ &+ (\text{covariant terms invoving the Weyl tensor}) \end{aligned} \tag{2.5.30}$$

where the first term matches the one given in [76], while the rest can be interpreted as the supersymmetric completion of it. We also dropped all terms involving the Weyl tensor because we are only interested in conformally flat spaces.

It is convenient to separate the dependence on the powers of dilaton and axion:

$$\mathcal{L}^{(\mathrm{Lorentzian})}_{\mathrm{super-WZ}} = \mathcal{L}_{\tau^1} + \mathcal{L}_{\tau^2} + \mathcal{L}_{\tau^3} + \mathcal{L}_{\tau^4} + \mathcal{L}_{\beta^2} + \mathcal{L}_{\tau^1\beta^2} + \mathcal{L}_{\tau^2\beta^2} + \mathcal{L}_{\beta^4}\,, \tag{2.5.31}$$

where

$$\mathcal{L}_{\tau^1} = -(\Delta a)\,\tau\, E_4\,, \tag{2.5.32}$$

$$\mathcal{L}_{\tau^2} = -4\,(\Delta a)\left[R^{\mu\nu} - \frac{1}{2}\mathtt{Ric}_4\, g^{\mu\nu}\right]\nabla_\mu\tau\,\nabla_\nu\tau\,, \tag{2.5.33}$$

$$\mathcal{L}_{\tau^3} = +4(\Delta a)\,(\nabla\tau)^2\,(\nabla^2\tau) \tag{2.5.34}$$

$$\mathcal{L}_{\tau^4} = -2(\Delta a)\,(\nabla\tau)^4 \tag{2.5.35}$$

$$\mathcal{L}_{\beta^2} = -4(\Delta a)\left[R^{\mu\nu} - \frac{1}{6}\,R\, g^{\mu\nu}\right](\nabla_\mu\beta)(\nabla_\nu\beta) \tag{2.5.36}$$

$$\mathcal{L}_{\tau^1\beta^2} = -8\,(\Delta a)\,(\nabla^\mu\nabla^\nu\tau)\,\nabla_\mu\beta\,\nabla_\nu\beta\,, \tag{2.5.37}$$

$$\mathcal{L}_{\mathbb{v}^2\mathbb{\beta}^2} = -4\,(\Delta a)\left[2\,(\nabla\mathbb{v}\cdot\nabla\mathbb{\beta})^2 - (\nabla\mathbb{v})^2(\nabla\mathbb{\beta})^2\right], \qquad (2.5.38)$$

$$\mathcal{L}_{\mathbb{\beta}^4} = -2\,(\Delta a)\,(\nabla\mathbb{\beta})^4\,. \qquad (2.5.39)$$

2.5.3 $\mathcal{N} = 2$ superconformal gauge dynamics with ghost hypermultiplets and rank-1 scfts

We construct countably infinite number of super-Weyl invariant theories with a one-dimensional Coulomb branch.

Weyl anomalies and β-functions for $\mathcal{N} = 2$ gauge theory with $G = SU(2)$

Consider for instance the case of an $\mathcal{N} = 2$ gauge theory with $G = SU(2)$ and ordinary hypermultiplets. A hypermultiplet in a representation $\mathbf{R}$ of $SU(2)$ contributes to the β-function as

$$\beta_{\text{ordinary hypermultiplet in } \mathbf{R}} = +\frac{{g_{\text{YM}}}^3}{16\pi^2}\text{Tr}_{\mathbf{R}}(t^A\,t^A)\,, \qquad (2.5.40)$$

where $A = 1, 2, 3$ and the representation matrices t^A are taken to be Hermitean and normalized so that the level spacing of $t^{A=3}$ is differences of 1. In this normalization, if $\mathbf{R}$ is the k-dimensional representation then t^3 has eigenvalues $\{-\frac{k-1}{2}, -\frac{k-3}{2}, \cdots, +\frac{k-3}{2}, +\frac{k-1}{2}\}$, so

$$\texttt{Tr}_{\mathbf{R}}(t^A\,t^A) = 3\times \texttt{tr}_{\mathbf{R}}((t^{A=3})^2) = \frac{k(k^2-1)}{4}\,. \qquad (2.5.41)$$

In terms of the largest eigenvalue $\ell \equiv k - \frac{1}{2}$ of $t^{A=3}$, this is just the dimension $k = 2\ell + 1$ of the representation, times the quadratic Casimir $\ell(\ell+1) = \frac{1}{4}(k^2-1)$.

So the β-function of an ordinary hypermultiplet is

$$\beta_{\text{ordinary hypermultiplet in } \mathbf{R}_\text{k}} = +\frac{{g_{\text{YM}}}^3}{16\pi^2}\frac{k(k^2-1)}{4}\,. \qquad (2.5.42)$$

The β-function in $\mathcal{N} = 2$ theories comes entirely from one loop.

Ghost hypermultiplets

If we were to couple hypermultiplets in representation $\mathbf{R}$ with spin-statistics opposite to the usual ones, then the β function would be of the same magnitude and opposite sign as for ordinary matter. Such opposite-statistics "ghost matter" in supersymmetric gauge theory as been considered elsewhere in a similar spirit [79–82]. So

$$\beta_{\text{ghost hypermultiplet in } \mathbf{R}_\mathrm{k}} = -\frac{{g_{\mathrm{YM}}}^3}{16\pi^2}\frac{k(k^2-1)}{4} \,. \tag{2.5.43}$$

The $SU(2)$ vector multiplet contribution to the β-function is

$$\beta_{SU(2)\text{ vector multiplet}} = -6 \,, \tag{2.5.44}$$

so the condition for the cancellation of the β-function is

$$\sum_k \frac{k(k^2-1)}{4} \times (n_k^{(\text{hyper})} - n_k^{(\text{ghost hyper})}) = +6 \,, \tag{2.5.45}$$

where $n_k^{(\text{hyper})}$ and $n_k^{(\text{ghost hyper})}$ are the numbers of ordinary hypermultiplets and ghost hypermultiplets, respectively, in the k-dimensional representation of $SU(2)$.

The β-function depends only on the differences $\tilde{n}_k^{(\text{hyper})} \equiv n_k^{(\text{hyper})} - n_k^{(\text{ghost hyper})}$, and so we can write formula (2.5.45) as

$$\sum_k \frac{k(k^2-1)}{4} \times \tilde{n}_k^{(\text{hyper})} = +6 \,. \tag{2.5.46}$$

This is just the generalization of the usual β-function formula to negative numbers of hypermultiplets; the path integral with ghost hypers gives this generalized formula a concrete physical interpretation, at least in terms of a superconformal statistical system in four euclidean dimensions, if not a quantum theory in $3+1$ spacetime dimensions.

Ghost hypermultiplets as regulators

Our only intended use for this system is to serve as a nonunitary regulator for the effective vector multiplet action with various values of the α-coefficient of the super-WZ term. Since we only wish to define the effective theory up to the scale $\Lambda \ll |\phi|$, the nonunitary nature of the ghost hypers is irrelevant since all hypermultiplet degrees of freedom are massive at the scale set by $|\phi|$: So long as the ghost hypers satisfy this condition, then they just serve as a nice regulator for the WZ action that has the useful property of preserving the spontaneously broken $\mathcal{N}=2$ superconformal symmetry. Similar regulators for $\mathcal{N}=4$ theories involving ghost matter have been considered elsewhere [79–82].

The present ghost regulators are similar to those of [82], which are simpler than those of [79–81], in that the latter theories considered there involved nonunitary degrees of freedom in the gauge sector as well as in the matter sector, necessarily so in order to preserve the full $\mathcal{N} = 4$ supersymmetry. Our regulating theories, like those of [82], have nonunitarity only in the matter sector.

We therefore need to engineer a vacuum manifold consisting solely of an Abelian vector multiplet, with no additional massless degrees of freedom from the hypers when the vector multiplet scalar has a nonzero vev. That is, we wish to exclude the case of an "enhanced" Coulomb branch or its ghost generalization. To achieve this, it is necessary and sufficient to choose all the representations to be even-dimensional. Then the mass matrix for the hypers, $t^A_{\mathbf{R}}\phi^A$, has no vanishing eigenvalues for nonzero ϕ^A, and the vacuum manifold is a pure Coulomb branch. So we will restrict our representation content to k even. With this criterion, all ghost degrees of freedom have masses of order $|\phi|$ and are above the cutoff Λ.

Now let us write an expression for the a-anomaly of the underlying CFT. So long as the β-function vanishes, the gauge coupling τ is marginal and the anomaly is τ-independent, and we can compute the Weyl anomaly accurately in free field theory. Just as for the gauge anomaly, the ghost hypermultiplets contribute to the Weyl anomaly oppositely to the ordinary hypermultiplets in the same representation. Thus we have the total a- coefficient

$$a^{[\text{AEFJ}]}_{\text{CFT}} = a^{[\text{AEFJ}]}_{SU(2)\text{ vector multiplet}} + a^{[\text{AEFJ}]}_{\text{matter}} = \frac{5}{8} + \frac{1}{24} \times \sum_k k\, \tilde{n}^{\text{hyper}}_k \,, \qquad (2.5.47)$$

where Anselmi–Freedman–Grisaru–Johansen [83] (AEFJ) is a name for the different kind of normalization of the a-coefficient, related to $a^{[\text{ks}]}$ by

$$a^{[\text{AEFJ}]} \equiv 16\pi^2 a^{[\text{KS}]}. \qquad (2.5.48)$$

If we have chosen all the k to be even, then there are no massless degrees of freedom on the Coulomb branch other than the vector multiplet, and so the Coulomb branch EFT has

$$a^{[\text{AEFJ}]}_{\text{EFT}} = a^{[\text{AEFJ}]}_{\text{U}(1)\text{ vector multiplet}} = +\frac{5}{24} \,. \qquad (2.5.49)$$

Then the anomaly mismatch in AEFJ units [83] is

$$(\Delta a)^{[\mathrm{AEFJ}]} \equiv a_{\mathrm{EFT}}^{[\mathrm{AEFJ}]} - a_{\mathrm{EFT}}^{[\mathrm{AEFJ}]} = \frac{5}{12} + \frac{1}{24} \times \sum_k k\, \tilde{n}_k^{(\mathrm{hyper})} \tag{2.5.50}$$

and the α-coefficient then comes out to

$$\alpha \equiv 2 \times (\Delta a)^{[\mathrm{AEFJ}]} = \frac{5}{6} + \frac{1}{12} \sum_k k\, \tilde{n}_k^{(\mathrm{hyper})} \tag{2.5.51}$$

We include only even k in the sum, but other than that there is no restriction on the $\tilde{n}_k$ other than the requirement (2.5.46) that the β-function vanishes.

Conformal combinations of matter and ghost matter

Since the $\tilde{n}_k$ can be positive or negative, there are many ways to satisfy equation (2.5.46) while giving different values for α as determined by equation (2.5.51). For instance, for $\tilde{n}_4^{(\mathrm{hyper})}$ any integer, we can take

$$\begin{cases} \tilde{n}_2^{(\mathrm{hyper})} = 4 - 10\, \tilde{n}_4^{(\mathrm{hyper})} , & \\ \tilde{n}_k = 0 & \forall k \neq 2, 4 . \end{cases} \tag{2.5.52}$$

Then the β-function cancellation equation (2.5.46) is satisfied, and the value of α is

$$\alpha = \frac{3}{2} - \frac{4}{3}\, \tilde{n}_4^{(\mathrm{hyper})} . \tag{2.5.53}$$

Super-Weyl invariance of the ghost-hyper theories

The vanishing of the β-function means that these theories are scale-invariant. However we can see that they are not only scale-invariant, they are Weyl-invariant on curved space and therefore super-Weyl-invariant on curved superspace [84, 85].

The action for ghost hypers is Weyl-invariant at the Lagrangian level exactly as it is for ordinary hypers: For both types of multiplet, the action is exactly quadratic in hypermultiplet degrees of freedom, and the ghost hypers are taken to have exactly the same super-Weyl transformation laws as the ordinary hypers. So even though nonunitary scale-invariant theories are not Weyl-invariant in general, the ghost-hyper SCFTs are special cases which are in fact super-Weyl invariant. This is important to emphasize, because we will use super-Weyl invariance, not just scale invariance, as a symmetry to eliminate higher-derivative F-terms in the Coulomb branch EFT of the ghost-hyper theories.

For vector multiplet actions, $\mathcal{N} = 2$ super-Weyl invariance follows automatically from Weyl-invariance and $\mathcal{N} = 2$ SUSY because the $\mathcal{N} = 2$ supergravity background has a superspace formalism which couples naturally to half-superspace F-terms for vector multiplets as well as full-superspace terms. For hypermultiplet F-terms, maintaining manifest supersymmetry off-shell is more subtle, requiring more sophisticated superspace formalisms such as harmonic superspace or projective superspace, to which we know of no currently developed formalism for coupling to a curved superbackground.

However it is possible to see directly that the action for ghost hypermultiplets must be super-Weyl-invariant, if the action for ordinary hypermultiplets is super-Weyl-invariant. There are two more or less equivalent ways to see this, one "on-shell" and one "off-shell". Both forms of the proof use the fact that the action for hypermultiplets, both ghost type and ordinary type, is exactly quadratic in the hypermultiplet fields.

The on-shell, operator argument is as follows. Since the action is exactly quadratic in the hypermultiplet degrees of freedom, so must be the stress tensor, supersymmetry generators, and other currents. In particular, the virial current would have to be quadratic in hypermultiplet degrees of freedom, and there is no candidate virial current that is quadratic in hypermultiplet degrees of freedom.

This operator proof translates into an off-shell argument in component fields, as follows:

Given an $\mathcal{N} = 2$ supergravity background and a fixed (not necessarily supersymmetric or on-shell) background for the vector multiplet, we can perform a super-Weyl transformation on the metric and vector-multiplet degrees of freedom.

The full action for vector and hypermultiplets is super-Weyl invariant, and thus for an arbitrary super-Weyl transformation of the background metric and dynamical $SU(2)$ vector multiplet, there must exist a corresponding transformation on the components of the hypermultiplet that leaves the Lagrangian invariant, not just up to a total derivative or local SUSY transformation, but invariant exactly, since the virial current must vanish. The action is exactly quadratic, and the transformation of the off-shell hypermultiplet component fields under the super-Weyl transformation is linear.

The exact same super-Weyl transformation can be applied as a linear transformation to the off-shell ghost hypermultiplet component fields, and the action will necessarily still be invariant: For a quadratic action for a complex field, a linear transformation on a bosonic field leaves the action invariant if and only if the corresponding action for a fermionic field also does so: For a quadratic action for a complex field, the statistics of the field are irrelevant to the invariance of the action so long as the transformation is linear.

We therefore conclude that the fixed points with ghost-hypermultiplets are invariant under the same super-Weyl transformations as the SCFTs with ordinary unitary hypermultiplets.

2.5.4 The Chiral ring of 4d $\mathcal{N} = 2$ rank-1 scfts and recursion relations

The superconformal algebra on S^4

The superconformal group of $D = 4, \mathcal{N} = 2$ SCFTs on flat space is $SU(2,2|2)$. In doing exact localisation computation, whose result I used in the main body of the thesis, one needs to put the theory on S^4. The superconformal group is then broken explicitely and becomes $OSp(2|4)$ [60]. The R-symmetry group is especially broken to $SO(2)$, which is the Cartan of $SU(2)_R$, so that the usual $U(1)_R$-charge is explicitly broken. Therefore, one has to consider mixing between operators with operators with lower dimensions multiplied by the curvature of the sphere, because there are no superselection rules that prohibit this.

Let us restrict to the case where there is a chiral primary $\mathcal{O}$ of dimension 2. Then the two-point fucntions of $\mathcal{O}^n$ can be computed using the deformed partition functions $Z[S^4]$ on the sphere,

$$\langle \mathcal{O}^n(N)\overline{\mathcal{O}^m}(S)\rangle_{S^4} = \frac{1}{Z[S^4]}\partial_\tau^n \partial_{\bar{\tau}}^m Z[S^4]\ . \qquad (2.5.54)$$

where τ is a parameter associated to the exactly-marginal deformation. The correlators on $\mathbb{R}^4$ can be obtained by diagonalising the mixing between operators.

Solving the recurrence equation

The set of two-point functions can especially be cast into a form of recursion relations when there is only one exactly marginal operator. The diagonalisation is done with the usual Gram-Schmidt algorithm, which prodeces the Toda lattice equation as a final form, [60, 86–89]

$$\partial\bar{\partial} q_n(\tau,\bar{\tau}) = \exp[q_{n+1}(\tau,\bar{\tau}) - q_n(\tau,\bar{\tau})] - \exp[q_n(\tau,\bar{\tau}) - q_{n-1}(\tau,\bar{\tau})], \qquad (2.5.55)$$

where $q_n \equiv \log Z_n$, as defined in the main body of the text. Here we want to show how to solve this equation using the extra information coming from the EFT about the τ dependence of the asymptotic expansion of q_n for large n.

First, it is convenient to rewrite the second-order equation as a system of two first-order equations:

$$\begin{cases} \partial P_n(\tau,\bar{\tau}) = P_n(\tau,\bar{\tau})\left(Q_n(\tau,\bar{\tau}) - Q_{n-1}(\tau,\bar{\tau})\right) \\ \bar{\partial} Q_n(\tau,\bar{\tau}) = P_{n+1}(\tau,\bar{\tau}) - P_n(\tau,\bar{\tau}), \end{cases} \tag{2.5.56}$$

where

$$Q_n(\tau,\bar{\tau}) = \partial q_n(\tau,\bar{\tau}), \qquad P_n(\tau,\bar{\tau}) = \exp[q_n(\tau,\bar{\tau}) - q_{n-1}(\tau,\bar{\tau})]. \tag{2.5.57}$$

In (2.3.24) we saw that the dependence of $q_n(\tau,\bar{\tau})$ on τ is at most affine (*i.e.* only the constant and linear in n terms depend on τ). We can separate this by writing

$$q_n(\tau,\bar{\tau}) = nf(\tau,\bar{\tau}) + k_0(\tau,\bar{\tau}) + M_n. \tag{2.5.58}$$

The variables Q_n and P_n then read

$$Q_n(\tau,\bar{\tau}) = n\,\partial f(\tau,\bar{\tau}) + \partial k_0(\tau,\bar{\tau}), \tag{2.5.59}$$

$$P_n(\tau,\bar{\tau}) = e^{f(\tau,\bar{\tau})}\exp[M_n - M_{n-1}] = e^{f(\tau,\bar{\tau})}\Lambda_n. \tag{2.5.60}$$

With this ansatz the first equation in Eq. (2.5.56) is identically satisfied and we only need to solve

$$n\,\partial\bar{\partial} f(\tau,\bar{\tau}) + \partial\bar{\partial} k_0(\tau,\bar{\tau}) = e^{f(\tau,\bar{\tau})}\left(\Lambda_{n+1} - \Lambda_n\right). \tag{2.5.61}$$

If we isolate the terms that do not depend on $\tau, \bar{\tau}$ we can rewrite the equation as the system

$$\partial\bar{\partial} f(\tau,\bar{\tau}) = 2Ae^{f(\tau,\bar{\tau})}, \tag{2.5.62}$$

$$\partial\bar{\partial} k_0(\tau,\bar{\tau}) = Be^{f(\tau,\bar{\tau})}, \tag{2.5.63}$$

$$\Lambda_{n+1} - \Lambda_n = 2An + B, \tag{2.5.64}$$

where A and B are constants. We see that $f(\tau,\bar{\tau})$ obeys the Liouville equation (2.5.62) on a hyperbolic plane of Gaussian curvature $-4A$, and it sources the Poisson equation (2.5.63) satisfied by $k_0(\tau, \bar{\tau})$.

The equation for Λ_n is easily solved and gives

$$\Lambda_n = An(n-1) + Bn + C' = A(n - n_+)(n - n_-), \tag{2.5.65}$$

where C' is an integration constant and $n_\pm$ are two numbers that satisfy

$$n_+ + n_- = 1 - \frac{B}{A}, \qquad n_+ n_- = \frac{C'}{A}. \tag{2.5.66}$$

Using this expression we can solve for M_n:

$$e^{M_n - M_{n-1}} = \Lambda_n \tag{2.5.67}$$

and find

$$M(n) = D + n \log A + \log[\Gamma(n - n_- + 1)\Gamma(n - n_+ + 1)], \tag{2.5.68}$$

where D is an integration constant.

Let us now consider the τ-dependent equations. The Liouville equation (2.5.62) for $f(\tau, \bar{\tau})$ admits the general solution

$$e^{f(\tau,\bar{\tau})} = \frac{1}{A} \frac{|\partial\phi(\tau)|^2}{\left(1 - |\phi(\tau)|^2\right)^2}, \tag{2.5.69}$$

where $\phi(\tau)$ is a meromorphic function. Now that we have solved for $f(\tau, \bar{\tau})$, we can recast the equation for $k_0(\tau, \bar{\tau})$ as a Laplace equation:

$$\partial\bar{\partial}\left(k_0(\tau, \bar{\tau}) + \frac{B}{2A} f(\tau, \bar{\tau})\right) = 0 \tag{2.5.70}$$

so that $k_0(\tau, \bar{\tau})$ is given by

$$k_0(\tau, \bar{\tau}) = -\frac{B}{2A} f(\tau, \bar{\tau}) + \psi(\tau) + \bar{\psi}(\bar{\tau}). \tag{2.5.71}$$

We can now collect our results and write the final expression for $q_n(\tau, \bar{\tau})$:

$$\begin{aligned} q_n(\tau,\bar{\tau}) &= nf(\tau,\bar{\tau}) + k_0(\tau,\bar{\tau}) + M_n \\ &= n\left(f(\tau,\bar{\tau}) + \log A\right) + k_0(\tau,\bar{\tau}) + D \\ &\quad + \log\left[\Gamma(n - n_+ + 1)\Gamma(n - n_- + 1)\right]. \end{aligned} \tag{2.5.72}$$

Our solution depends on the constants, n_+, n_-. They can be fixed in terms of the anomaly coefficient α by comparing the large-n expansion of $q_n(\tau, \bar{\tau})$ with the results of the EFT. Expanding the gamma function in the expression in Eq. (2.5.72), we get

$$\begin{aligned} q_n(\tau,\bar{\tau}) = n\left(f(\tau,\bar{\tau}) + \log A - 2\right) + k_0(\tau,\bar{\tau}) + D + \log(2\pi) - (n_+ + n_- - 1)\log n + \\ + \frac{1 + 3n_+\left(n_+ - 1\right) + 3n_-\left(n_- - 1\right)}{6n} + \dots . \end{aligned} \tag{2.5.73}$$

2.5.5 Values of α for three Lagrangian rank-one theories

We list the values of α for three rank-one Lagrangian theories we know in $\mathcal{N} = 2$.

Theory	α
Free theory	0
$\mathcal{N} = 4$ SYM with $G = SU(2)$	1
$SU(2)$ SQCD with $N_f = 4$	3/2

(2.5.74)

Conclusions

Conclusions

In this thesis, I have used the large-J (large-charge) expansion to derive various universal formula for strongly-coupled theories. The large charge expansion, invented by myself and collaborators, is a way of analysing a strongly-coupled theory like a weakly-coupled one, where the loop counting parameter is $1/J$. As we have seen, this method turned out to be very systematic and thus powerful. One just needs to write down the effective action according to the symmetries of the system, and each allowed term can be classified using the J-scaling, because we can evaluate it around the classical saddle point far from the origin of the field space. Unlike the usual treatments where we need individual care depending on the dimension or the number of supersymmetries, this method treats SUSY and non-SUSY theories on an equal footing.

In Chapter 1, we find that the behaviour of the lowest operator dimension at large charge is the same for various theories with a global $U(1)$ symmetry. This universality, which we named the large-charge universality class, included both non-SUSY (the $O(2)$ model) as well SUSY models ($W = \Phi^3$ model). This fact was somewhat counterintuitive because naively we expect that there exist BPS operators of arbitrary high R-charge in SUSY theories (there actually are BPS operators, but those are of high-spin and their dimensions go as J^2); As we have correctly explained, the non-existence of the BPS operators should be associated with the absence of moduli space, so that the supersymmetry is spontaneously broken in the IR.

The distinction between explicit and spontaneous breaking was also made precise. Especially important was the fact that the chemical potential term is not an explicit breaking of symmetry in the usual sense, and that through redefinition of fields, we recover the same symmetry we started with. This was particularly important in higher-rank theories, where some of the original symmetries are explicitly broken by fixing the charge, while some are then broken spontaneously by the VEV of charged fields. By tracking what spontaneous breaking the system undergoes, we were able to prove the existence and the non-existence of translational Goldstone bosons in a large-charge EFT. The attempts in solving for such inhomogeneous ground states at large charge showed us one of the realizations of peculiar quantum phases of matter.

In Chapter 2, I applied the method of the large-charge expansion to systems with moduli space of vacua. This is particularly convenient for testing the method of the large charge expansion, because supersymmetry heavily constrains the EFT in the infrared (IR). For 4d $\mathcal{N} = 2$, rank-one SCFTs we considered in the chapter, the constraints from SUSY is so strong that there were no subleading F-terms in the EFT. This fact was quite useful because the EFT just reduced to be a sum of the free kinetic term and the WZ term for the Weyl anomaly. Using this Lagrangian, we computed the Coulomb branch chiral ring data exactly to all orders in the large-R-charge expansion, and as expected, the result is universal (perturbatively) only up to the theory's a-anomaly.

We also commented on the possible non-universal nonperturbative corrections to the universal perturbative formula. It made us possible to make predictions about chiral ring data between operators of low dimensions. This made us possible to verify our formula against known results for some of the theories using exact localization. In summary, the final formula we derived using the large charge expansion almost completely solves the chiral ring structure of *any* rank-one SCFTs, while exact localization can only determine it for Lagrangian theories with an exact marginal coupling and conformal bootstrap can for OPE data among operators of low dimension.

Future directions

As our understanding of the large charge expansion has become somewhat theoretically firm, what would be interesting in the future is to apply the method to various interesting systems. The most interesting direction would be to use holography to understand quantum gravity for finite N. For example, what is the holographic dual of the state at large charge for the $O(N)$ model? It cannot for sure be a Black Hole, because such a highly-charged Black Hole would be unstable *via* Schwinger mechanism. All the more, because the ground state at large charge is unique, it cannot account for the large entropy Black Holes must have. We are still yet to know the description of such an object in gravity, partly because we need to take $J \gg N$ in understanding the large-charge limit. However, in understanding such an object would directly contribute to our understanding of genuinely quantum regime of gravity theories, which we ultimately would like to understand.

The first step in this direction would be to consider higher-rank theories in four-dimensional $\mathcal{N} = 2$ SCFTs. The theories we considered were of rank one, so in the language of the gravity dual, these should correspond to the uninteresting theories without weakly-couple description of gravity. Even if we take the rank N to be large, the analysis is only for $J \gg N$, but because of the particularly simple structure of the effective action, one might hope to analyse the regime which intersects with the BMN limit [90] where $N \sim J^2$.

There are various interesting future directions aside from quantum gravity, which hopefully are more tractable. For example, understanding systems including fermi seas is theoretically important, but it is still yet to be understood. This is because of the technical difficulty in renormalising fermi seas at finite volume, but this needs to be solved in order to even understand a simple theory of a free fermion. Because the particle-vortex duality includes dualities of the bosonization type, combining it with the method of large charge is an interesting future problem to consider.

There are also various interesting directions for supersymmetric theories too. For example, making an exact bound for the non-universal corrections in (2.3.24) would be interesting, because this will make our predictions for low R-charge an exact bound, not an approximate computation. Related to this, it would also be interesting to compare the result to the bootstrap result given in [91] (preliminary attempts at matching these two predictions have been done and were successful). It could also be interesting to extend the analysis to Higgs branches. Because we have the $SU(2)_R$ symmetry at low-energies, even at leading order, the inhomogeneous ground states will appear in the theory. It would be very interesting to understand the consequence of such ground states in supersymmetric theories with a moduli space of vacua.