

Foundations of Algebraic Theories
and Higher Dimensional Categories
(代数学理論の基礎と高次元圏)

by

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ABSTRACT

Universal algebra uniformly captures various algebraic structures, by expressing them as equational theories or abstract clones. The ubiquity of algebraic structures in mathematics and related fields has given rise to several variants of universal algebra, such as symmetric operads, non-symmetric operads, generalised operads, and monads. These variants of universal algebra are called notions of algebraic theory. Although notions of algebraic theory share the basic aim of providing a background theory to describe algebraic structures, they use various techniques to achieve this goal and, to the best of our knowledge, no general framework for notions of algebraic theory which includes all of the examples above was known. Such a framework would lead to a better understanding of notions of algebraic theory by revealing their essential structure, and provide a uniform way to compare different notions of algebraic theory. In the first part of this thesis, we develop a unified framework for notions of algebraic theory which includes all of the above examples. Our key observation is that each notion of algebraic theory can be identified with a monoidal category, in such a way that theories correspond to monoid objects therein. We introduce a categorical structure called *metamodel*, which underlies the definition of models of theories. The notion of *metamodel* subsumes not only the standard definitions of models but also non-standard ones, such as graded algebras of symmetric operads and relative algebras of monads on **Set** introduced by Hino, Kobayashi, Hasuo and Jacobs. We also consider morphisms between notions of algebraic theory, which are a monoidal version of profunctors. Every strong monoidal functor gives rise to an adjoint pair of such morphisms, and provides a uniform way to establish isomorphisms between categories of models in different notions of algebraic theory. A general structure-semantics adjointness result and a double categorical universal property of categories of models are also shown.

In the second part of this thesis, we shift from the general study of algebraic structures, and focus on a particular algebraic structure: higher dimensional categories. Higher dimensional categories arise in such diverse fields as topology, mathematical physics and theoretical computer science. On the other hand, the structure of higher dimensional categories is quite complex and even their definition is known to be subtle. Among several existing definitions of higher dimensional categories, we choose to look at the one proposed by Batanin and later refined by Leinster. In Batanin and Leinster's approach, higher dimensional categories are defined as models of a certain generalised operad, hence it falls within the unified framework developed in the first part of this thesis. Batanin and Leinster's definition has also been used by van den Berg, Garner and Lumsdaine to describe the higher dimensional structures of types in Martin-Löf intensional type theory. We show that the notion of extensive category plays a central role in Batanin and Leinster's definition. Using this, we generalise their definition by allowing enrichment over any locally presentable extensive category.

論文要旨

普遍代数は様々な代数構造を、等式理論または抽象クローンとして表現することで統一的にとらえる理論である。数学やその関連分野においては至る所に代数構造が現れるが、それらに対応するためにいくつかの普遍代数の変種が提案されてきた。対称オペラド、非対称オペラド、一般化オペラド、モナドなどがその例である。これらの普遍代数の変種は、代数理論の概念と呼ばれる。代数理論の概念はどれも代数構造を記述するための基礎理論を提供するという基本目的を共有しているものの、その達成のためにはそれぞれの代数理論の概念によって多様な方法を用いており、我々の知る限り上で挙げた例を全て含むような代数構造の概念のための一般的な枠組みは知られていなかった。こうした枠組みは、代数理論の概念の本質的な構造を明らかにすることで、そのより深い理解を可能にすると同時に、異なる代数理論の概念を比較するための統一的な方法も与えらる。本論文の第一部では、上で挙げた例を全て含む代数理論の概念のための統一的な枠組みを展開する。そのために重要となるのは、それぞれの代数理論の概念はモノイダル圏と同一視でき、理論はモノイダル圏のモノイド対象と対応する、という観察である。さらに我々は、理論のモデルの定義の背景にある概念として、メタモデルと呼ばれる圏論的構造を導入する。メタモデルは標準的な理論のモデルの定義を含むだけでなく、非標準的な定義、例えば対称オペラドの次数付き代数や、日野、小林、蓮尾、Jacobsにより導入された **Set** 上のモナドの相対代数といったものも例として含んでいる。我々はさらに代数理論の概念の間の射を、モノイダル版の profunctor として導入する。任意の強モノイダル関手はそのような射の随伴対を生成し、異なる代数理論の概念におけるモデルの圏の間の同型を与える統一的な手法を提供する。その他、一般的な構造-意味論随伴に関する結果やモデルの圏の二重圏的な普遍性も示される。

本論文の第二部では、代数的構造の一般論の研究から特定の代数的構造である高次元圏についての研究へと話題を転じる。高次元圏はトポロジー、数理物理学や理論計算機科学といった様々な分野において現れる。一方で高次元圏の構造は非常に複雑であり、その定義ですら困難を伴うことが知られている。既知のいくつかの高次元圏の定義のうちで、我々は Batanin により提案され Leinster により改良された定義を研究の対象とする。Batanin と Leinster のアプローチは高次元圏をある一般化オペラドのモデルとして定義するものであり、従ってそれは本論文の第一部で展開される統一的な枠組みの範疇に属する。彼らの定義はまた van den Berg, Garner, Lumsdaine により Martin-Löf の内包的型理論における型を持つ高次元の構造を記述する目的にも用いられた。我々は Batanin と Leinster の定義において extensive 圏の概念が中心的な役割を果たしていることを示す。また、この結果を用いて彼らの定義を任意の局所表示可能 extensive 圏によって enrich できるように一般化する。

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Chapter 1

Introduction

1.1 Algebraic structures in mathematics and computer science

Algebras permeate both pure and applied mathematics. Important types of algebras, such as vector spaces, groups and rings, arise naturally in many branches of mathematical sciences and it would not be an exaggeration to say that algebraic structures are one of the most universal and fundamental structures in mathematics.

In computer science, too, concepts related to algebraic structures play an essential role. For example, in programming language theory, we can find relationship to algebraic structures via the study of *computational effects*. Let us start with an explanation of computational effects. Computer programs may roughly be thought of as mathematical functions, mapping an input to the result of computation. However, this understanding is too crude and in reality programs often show non-functional behaviours; for example, if a program interacts with the memory of the computer, then an input to the program alone might not suffice to determine its output (one has to know the initial state of the memory as well). Such non-functional behaviours of programs are called computational effects. It has been known since the work by Moggi [75] that computational effects can be modelled uniformly using the notion of *monad*. As we shall see later, a monad can be thought of as a specification of a type of algebras.

More recently, another approach to computational effects has been proposed by Plotkin and Power [76]. In this approach, computational effects are modelled by *Lawvere theories* [58] instead of monads; a Lawvere theory can also be thought of as a specification of a type of algebras, akin to *equational theory* in universal algebra. Constructions on Lawvere theories originally developed in the study of algebraic structures, such as tensors and sums of Lawvere theories [25], have been shown to be capable of modelling combinations of computational effects [40], and the resulting Lawvere theories can verify equivalences of programs which are crucial in program optimisation [45].

As another example of algebraic structures arising in computer science, one can point out a deep connection of higher dimensional categories and the Martin-Löf (intensional) type theory [72]. Higher dimensional categories may be thought of as particularly intricate types of algebras, defined by a number of complex operations and equations. Their importance was first recognised in homotopy theory [35], because they naturally arise as higher dimensional versions of the fundamental groupoids of topological spaces. It has been shown that equality types in the Martin-Löf type theory endow a weak ω -category structure to each type [38, 86, 69]. This observation has led the researchers to seek more profound connections of type theory and homotopy theory, bridged by higher category

theory, culminating in the introduction and recent intensive study of *homotopy type theory* [85].

This thesis studies foundational issues around algebraic structures. In the first part of the thesis, we investigate metamathematical aspects of algebraic structures, by developing a unified framework for notions of algebraic theory. In the second part, we focus on a particular type of algebras, weak n -dimensional categories for each natural number n , and generalise a known definition. We now turn to more detailed outlines of these parts of the thesis.

1.2 Unifying notions of algebraic theory

A type of algebras, such as groups, is normally specified by a family of operations and a family of equational axioms. We call such a specification of a type of algebras an *algebraic theory*, and call a background theory for a type of algebraic theories a *notion of algebraic theory*. In order to capture various types of algebras, a variety of notions of algebraic theory have been introduced. Examples include universal algebra [8], symmetric and non-symmetric operads [73], generalised operads (also called clubs) [11, 55, 36, 64], PROPs and PROs [70], and monads [22, 66]; we shall review these notions of algebraic theory in Chapter 2.

Notions of algebraic theory all aim to provide a means to define algebras, but they attain this goal in quite distinct manners. The diversity of the existing notions of algebraic theory leaves one wonder what, if any, is a formal core or essence shared by them. Our main aim in the first part of this thesis is to provide an answer to this question, by developing a unified framework for notions of algebraic theory.

The starting point of our approach is quite simple. We identify a notion of algebraic theory with an (arbitrary) monoidal category, and algebraic theories in a notion of algebraic theory with monoid objects in the corresponding monoidal category. As we shall review in Section 3.1.1, it has been observed that each type of algebraic theories we have listed above can be characterised as monoid objects in a suitable monoidal category. From now on let us adopt the terminology to be introduced in Chapter 3: we call a monoidal category a *metatheory* and a monoid object therein a *theory*, to remind ourselves of our intention.

In order to formalise the semantical aspect of notions of algebraic theory—by which we mean definitions of *models* (= algebras) of an algebraic theory, their homomorphisms, and so on—we introduce the concept of *metamodel*. Metamodels are a certain categorical structure defined relative to a metatheory \mathcal{M} and a category \mathcal{C} , and are meant to capture a *notion of model* of an algebraic theory, i.e., what it means to take a model of a theory in \mathcal{M} in the category \mathcal{C} . A model of an algebraic theory is always given relative to some notion of model, even though usually it is not recognised explicitly. We shall say more about the idea of notions of model at the beginning of Section 3.1.2. A metamodel of a metatheory \mathcal{M} in a category \mathcal{C} generalise both an \mathcal{M} -category (as in enriched category theory) having the same set of objects as \mathcal{C} , and a (left) oplax action of \mathcal{M} on \mathcal{C} . Indeed, as we shall see in Sections 3.1.2 and 3.1.3, it has been observed that enrichments (which we introduce as a slight generalisation of \mathcal{M} -categories) and oplax actions can account for the standard semantics of the known notions of algebraic theory. Our concept of metamodel provides a unified account of the semantical aspects of notions of algebraic theory.

Metamodels of a fixed metatheory \mathcal{M} naturally form a 2-category $\mathcal{M}\text{Mod}(\mathcal{M})$, and we shall see that theories in \mathcal{M} can be identified with certain metamodels

of \mathcal{M} in the terminal category $\mathbf{1}$. This way we obtain a fully faithful 2-functor from the category $\mathbf{Th}(\mathcal{M})$ of theories in \mathcal{M} (which is identical to the category of monoid objects in \mathcal{M}) to $\mathcal{M}\text{Mod}(\mathcal{M})$. A metamodel Φ of \mathcal{M} in \mathcal{C} provides a definition of model of a monoid object in \mathcal{M} as an object of \mathcal{C} with additional structure, hence if we fix a metamodel (\mathcal{C}, Φ) and a theory \mathbb{T} , we obtain the category of models $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi))$ equipped with the forgetful functor $U: \mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi)) \rightarrow \mathcal{C}$. By exploiting the 2-category $\mathcal{M}\text{Mod}(\mathcal{M})$, the construction $\mathbf{Mod}(-, -)$ of categories of models may be expressed as the following composition

$$\begin{array}{ccc}
\mathbf{Th}(\mathcal{M})^{\text{op}} \times \mathcal{M}\text{Mod}(\mathcal{M}) & & \\
\downarrow \text{inclusion} & & \\
\mathcal{M}\text{Mod}(\mathcal{M})^{\text{op}} \times \mathcal{M}\text{Mod}(\mathcal{M}) & & (1.1) \\
\downarrow \mathcal{M}\text{Mod}(\mathcal{M})(-, -) & & \\
\mathcal{CAT} & &
\end{array}$$

where $\mathcal{M}\text{Mod}(\mathcal{M})(-, -)$ is the hom-2-functor and \mathcal{CAT} is a 2-category of categories.

We also introduce morphisms (and 2-cells) between metatheories (Section 3.2.3). Such morphisms are a monoidal version of *profunctors*. The principal motivation of the introduction of morphisms of metatheories is to compare different notions of algebraic theory, and indeed our morphisms of metatheories induce 2-functors between the corresponding 2-categories of metamodells. Analogously to the well-known fact for profunctors that any functor induces an adjoint pair of profunctors, we see that any strong monoidal functor F induces an adjoint pair $F_* \dashv F^*$ of morphisms of metatheories. Therefore, whenever we have a strong monoidal functor $F: \mathcal{M} \rightarrow \mathcal{N}$ between metamodells, we obtain a 2-adjunction

$$\mathcal{M}\text{Mod}(\mathcal{M}) \begin{array}{c} \xrightarrow{\mathcal{M}\text{Mod}(F_*)} \\ \perp \\ \xleftarrow{\mathcal{M}\text{Mod}(F^*)} \end{array} \mathcal{M}\text{Mod}(\mathcal{N}). \quad (1.2)$$

Now, the strong monoidal F also induces a functor

$$\mathbf{Th}(F): \mathbf{Th}(\mathcal{M}) \rightarrow \mathbf{Th}(\mathcal{N}),$$

which is in fact a restriction of $\mathcal{M}\text{Mod}(F_*)$. This implies that, immediately from the description (1.1) of categories of models and the 2-adjointness (1.2), for any $\mathbb{T} \in \mathbf{Th}(\mathcal{M})$ and $(\mathcal{C}, \Phi) \in \mathcal{M}\text{Mod}(\mathcal{N})$, we have a canonical isomorphism of categories

$$\mathbf{Mod}(\mathbf{Th}(F)(\mathbb{T}), (\mathcal{C}, \Phi)) \cong \mathbf{Mod}(\mathbb{T}, \mathcal{M}\text{Mod}(F^*)(\mathcal{C}, \Phi)). \quad (1.3)$$

In fact, as we shall see, the action of $\mathcal{M}\text{Mod}(-)$ on morphisms of metatheories preserves the ‘‘underlying categories’’ of metamodells. So $\mathcal{M}\text{Mod}(F^*)(\mathcal{C}, \Phi)$ is also a metamodel of \mathcal{M} in \mathcal{C} , and we have an isomorphism of categories *over* \mathcal{C} (that is, the isomorphism (1.3) commutes with the forgetful functors).

The above argument gives a unified conceptual account for a range of known results on the compatibility of semantics of notions of algebraic theory. For example, it is known that any Lawvere theory \mathbb{T} induces a monad \mathbb{T}' on \mathbf{Set} in a way such that the models of \mathbb{T} and \mathbb{T}' in \mathbf{Set} (with respect to the standard notions of model) coincide; this result follows from the existence of a natural strong monoidal functor between the metatheories corresponding to Lawvere theories and monads on \mathbf{Set} , together with the simple observation that the induced

2-functor between the 2-categories of metamodels preserves the standard metamodel. This and other examples will be treated in Section 3.3.

In Chapter 4 we study *structure-semantic adjunctions* within our framework. If we fix a metatheory \mathcal{M} and a metamodel (\mathcal{C}, Φ) of \mathcal{M} , we obtain a functor

$$\mathbf{Th}(\mathcal{M})^{\text{op}} \longrightarrow \mathbf{CAT}/\mathcal{C} \quad (1.4)$$

by mapping a theory \mathbb{T} in \mathcal{M} to the category of models $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi))$ equipped with the forgetful functor into \mathcal{C} . The functor (1.4) is sometimes called the *semantic functor*, and it has been observed for many notions of algebraic theory that this functor (or an appropriate variant of it) admits a left adjoint called the *structure functor* [58, 66, 67, 20, 81, 3]. The idea behind the structure functor is as follows. One can understand a functor $V: \mathcal{A} \rightarrow \mathcal{C}$ into \mathcal{C} as specifying an additional structure (in a broad sense) on objects in \mathcal{C} , by viewing \mathcal{A} as the category of \mathcal{C} -objects equipped with that structure, and V as the forgetful functor. The structure functor then maps V to the best approximation of that structure by theories in \mathcal{M} . Indeed, if (1.4) is fully faithful (though this is not always the case), then the structure functor reconstructs the theory from its category of models.

We cannot get a left adjoint to the functor (1.4) for an arbitrary metatheory \mathcal{M} and its metamodel (\mathcal{C}, Φ) . In order to get general structure-semantic adjunctions, we extend the category $\mathbf{Th}(\mathcal{M})$ of theories in \mathcal{M} to the category $\mathbf{Th}(\widehat{\mathcal{M}})$ of theories in the metatheory $\widehat{\mathcal{M}} = [\mathcal{M}^{\text{op}}, \mathbf{SET}]$ equipped with the *convolution monoidal structure* [19]. We show in Theorem 4.1 that the structure-semantic adjunction

$$\mathbf{Th}(\widehat{\mathcal{M}})^{\text{op}} \begin{array}{c} \xleftarrow{\text{Str}} \\ \perp \\ \xrightarrow{\text{Sem}} \end{array} \mathbf{CAT}/\mathcal{C}$$

exists for any metatheory \mathcal{M} and its metamodel (\mathcal{C}, Φ) .

We conclude the first part of this thesis in Chapter 5, by giving a universal characterisation of categories of models in our framework. It is known that the Eilenberg–Moore categories (= categories of models) of monads can be characterised by a 2-categorical universal property in the 2-category \mathcal{CAT} of categories [81]. We show in Theorem 5.5 that our categories of models admit a similar universal characterisation, but instead of inside the 2-category \mathcal{CAT} , inside the *pseudo double category* \mathbf{PROF} of categories, functors, profunctors and natural transformations. The notion of pseudo double category, as well as \mathbf{PROF} itself, was introduced by Grandis and Paré [34]. In the same paper they also introduced the notion of *double limit*, a suitable limit notion in (pseudo) double categories. The double categorical universal property that our categories of models enjoy can also be formulated in terms of double limits; see Corollary 5.8.

1.3 Higher dimensional category theory

Higher dimensional category theory is a relatively young field. It studies higher dimensional generalisations of categories, such as *strict n -categories* and *weak n -categories* for $n \in \mathbb{N} \cup \{\omega\}$; in this thesis we shall only consider the case where $n \in \mathbb{N}$.

Let us start with the description of the simpler strict n -categories. A strict n -category has 0-cells, which we draw as

•,

1-cells lying between pairs of 0-cells

$$\bullet \longrightarrow \bullet,$$

2-cells lying between pairs of parallel 1-cells

$$\bullet \begin{array}{c} \curvearrowright \\ \downarrow \\ \curvearrowleft \end{array} \bullet,$$

and so on up to n -cells lying between pairs of parallel $(n - 1)$ -cells. There are also various identity cells and composition operations of cells, which are required to satisfy a number of equations. One way to make this informal description of strict n -category precise without too much complication is to define it by induction on n . That is, an $(n + 1)$ -category \mathcal{A} may be given by a set $\text{ob}(\mathcal{A})$ of 0-cells (or objects), and for each pair $A, B \in \text{ob}(\mathcal{A})$ of 0-cells, an n -category $\mathcal{A}(A, B)$, together with a family of operations (n -functors) $j_A: 1 \longrightarrow \mathcal{A}(A, A)$ and $M_{A,B,C}: \mathcal{A}(B, C) \times \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$, subject to the category axioms. Using the notion of enriched category [53], we may give a succinct inductive definition of the category $n\text{-Cat}$ of small strict n -categories and (strict) n -functors as follows:

$$0\text{-Cat} = \mathbf{Set}, \quad (n + 1)\text{-Cat} = (n\text{-Cat})\text{-Cat}. \quad (1.5)$$

Here, the construction $(-)\text{-Cat}$ maps any monoidal category \mathcal{V} to the category $\mathcal{V}\text{-Cat}$ of all small \mathcal{V} -categories and \mathcal{V} -functors. In the above definition, we always use the cartesian monoidal structure, the category $\mathcal{V}\text{-Cat}$ having all finite products whenever \mathcal{V} does.

The more general *weak* n -categories may be obtained by modifying the definition of strict n -category, replacing equational axioms by coherent equivalences. For $n = 0$ and 1 there is no difference between the strict and weak notions, 0-categories being sets and 1-categories being ordinary categories. Weak 2-categories are known as *bicategories* [6]. In a bicategory, the compositions $(h \circ g) \circ f$ and $h \circ (g \circ f)$ of 1-cells may not be equal; instead there must be a designated invertible 2-cell $\alpha_{f,g,h}: (h \circ g) \circ f \longrightarrow h \circ (g \circ f)$, and these 2-cells are required to satisfy some *coherence* axioms, such as the pentagon axiom asserting the commutativity of the diagram

$$\begin{array}{ccc} & ((k \circ h) \circ g) \circ f & \\ \alpha_{g,h,k} \circ f \swarrow & & \searrow \alpha_{f,g,k \circ h} \\ (k \circ (h \circ g)) \circ f & & (k \circ h) \circ (g \circ f) \\ \alpha_{f,h \circ g,k} \searrow & & \swarrow \alpha_{g \circ f,h,k} \\ k \circ ((h \circ g) \circ f) & \xrightarrow{k \circ \alpha_{f,g,h}} & k \circ (h \circ (g \circ f)). \end{array} \quad (1.6)$$

Weak 3-categories are known as *tricategories* [33]. In a tricategory we also have 2-cells like $\alpha_{f,g,h}$, which are now required to be only equivalences rather than isomorphisms; instead of the commutativity of the diagram (1.6) there is a designated invertible 3-cell (say, $\pi_{f,g,h,k}$) filling that diagram, and these 3-cells must satisfy their own coherence axioms.

Although weak n -categories are fundamental, arising in topology (as the fundamental n -groupoid of a topological space [35]) as well as in computer science (as the structure of a type in Martin-Löf intensional type theory [86, 69]), they are quite complex structure. Various authors have proposed definitions of weak

n -category (see e.g., [63]). Among them, we shall focus on the one proposed by Batanin [5] and later modified by Leinster [64]; we remark that it is their definition which is used in [86, 69] to capture the structure of a type in Martin-Löf type theory.

Let us describe Leinster’s approach, since that is what we shall consider in this thesis. Leinster defines weak n -categories as n -graphs with additional algebraic structure, where an n -graph consists of 0-cells, 1-cells lying between pairs of 0-cells, 2-cells lying between pairs of parallel 1-cells, and so on up to n -cells lying between pairs of parallel $(n - 1)$ -cells (and no operations). Using the notion of enriched graph [88], the category $n\text{-Gph}$ of n -graphs can be given inductively as follows:

$$0\text{-Gph} = \mathbf{Set}, \quad (n + 1)\text{-Gph} = (n\text{-Gph})\text{-Gph}. \quad (1.7)$$

It is easily shown by induction that the canonical forgetful functor $U^{(n)} : n\text{-Cat} \rightarrow n\text{-Gph}$ has a left adjoint $F^{(n)}$, and the adjunction $F^{(n)} \dashv U^{(n)}$ generates a monad $\mathbb{T}^{(n)}$ on $n\text{-Gph}$, the *free strict n -category monad*.¹ The monad $\mathbb{T}^{(n)}$ is in fact *cartesian*, and it is known that any cartesian monad \mathbf{S} on a category \mathcal{C} with finite limits defines a notion of algebraic theory (in the sense of the previous section), that of \mathbf{S} -operads. An \mathbf{S} -operad naturally takes models in the category \mathcal{C} ; thus in the current case, $\mathbb{T}^{(n)}$ -operads takes models in $n\text{-Gph}$. Leinster then introduces the notion of *contraction* on $\mathbb{T}^{(n)}$ -operads, and defines a $\mathbb{T}^{(n)}$ -operad $\mathbf{L}^{(n)}$ as the *initial operad with a contraction*. Finally, weak n -categories are defined to be models of $\mathbf{L}^{(n)}$.

Leinster’s definition of weak n -category starts from the category \mathbf{Set} of sets, in the sense that the key inductive definitions (1.5) and (1.7) have the base cases \mathbf{Set} . Necessarily, certain properties of \mathbf{Set} must be used to carry out the definition, but it has not been clear precisely which properties are used, because many propositions in [64] are proved by set-theoretic manipulation. Our main goal in the second part of this thesis is to clarify this. The conclusion we get is that, among many properties that the category \mathbf{Set} enjoys, *extensivity* [12, 14] and *local presentability* [27, 1] are enough to carry out the definition of weak n -category. We show this by generalising Leinster’s definition, starting from an arbitrary extensive and locally presentable category \mathcal{V} (again in the sense that we modify the base cases of (1.5) and (1.7), replacing \mathbf{Set} by \mathcal{V}). We call the resulting “enriched” weak n -categories *weak n -dimensional \mathcal{V} -categories*. Examples of categories \mathcal{V} of interest other than \mathbf{Set} satisfying both extensivity and local presentability include the category $\omega\text{-Cpo}$ of posets with sups of ω -chains, $\omega\text{-Cpo}$ -bicategories (weak 2-dimensional $\omega\text{-Cpo}$ -categories) being used in the work [78] axiomatising binders [24].

In Chapter 6, we prepare for our main development by showing several properties of extensive categories. In particular, we show that if \mathcal{V} is extensive, then so are $\mathcal{V}\text{-Gph}$ and $\mathcal{V}\text{-Cat}$ (for the latter category to make sense, we also have to assume that \mathcal{V} has finite products), thus illuminating the implicit induction in Leinster’s approach.

Using properties on extensive categories shown in Chapter 6, in Chapter 7 we prove that even when we start from an arbitrary extensive category \mathcal{V} with finite limits, we obtain an adjunction between the category $\mathcal{V}\text{-Gph}^{(n)}$ of n -dimensional \mathcal{V} -graphs (enriching $n\text{-Gph}$) and the category $\mathcal{V}\text{-Cat}^{(n)}$ of strict n -dimensional \mathcal{V} -categories (enriching $n\text{-Cat}$). We moreover show that the resulting monad $\mathbb{T}^{(n)}$ on $\mathcal{V}\text{-Gph}^{(n)}$ is cartesian. This allows us to consider $\mathbb{T}^{(n)}$ -operads.

¹The functor $U^{(n)}$ is in fact monadic, so Eilenberg–Moore algebras of $\mathbb{T}^{(n)}$ are precisely strict n -categories.

In Chapter 8, we first generalise Leinster’s notion of contraction to the enriched case. Leinster’s original definition of contraction was couched in purely set theoretic terms, so we adapt Garner’s conceptual reformulation [29] of it (with homotopy theoretic background [30]). This way we may give a meaning to the phrase $\mathbb{T}^{(n)}$ -operad with a contraction for an arbitrary extensive category \mathcal{V} with finite limits. Finally, to show the existence of the *initial* such, we assume that our \mathcal{V} is locally presentable as well. Under this additional assumption we prove that the initial $\mathbb{T}^{(n)}$ -operad with a contraction $\mathbb{L}^{(n)}$ exists, and we define weak n -dimensional \mathcal{V} -categories to be models of $\mathbb{L}^{(n)}$.

1.4 Set theoretic conventions

As is typical in category theory, in this thesis we will occasionally have to consider sets larger than those one usually encounters in other areas of mathematics and computer science. In order to deal with them, we shall assume the existence of a few universes. Roughly speaking, a universe \mathcal{U} is a set with a sufficiently strong closure property so that one can perform a range of set theoretic operations on elements in \mathcal{U} without having to worry about the resulting set popping out of \mathcal{U} . For example, if a group G is an element of \mathcal{U} (that is, the tuple consisting of the underlying set, the unit element, the inverse operation and the multiplication operation of G , is in \mathcal{U}), so are all subgroups of G , quotient groups of G , powers of G by elements of \mathcal{U} , etc. Note, however, that the set of all groups in \mathcal{U} is not in \mathcal{U} .

Although we will never refer to the details of the definition of universe in this thesis, we state it here for the sake of completeness.

Definition 1.1 ([46, Definition 1.1.1]). A set \mathcal{U} is called a **universe** if the following hold:

- if $x \in \mathcal{U}$ and $y \in x$, then $y \in \mathcal{U}$;
- if $x \in \mathcal{U}$, then $\{x\} \in \mathcal{U}$;
- if $x \in \mathcal{U}$, then $\mathcal{P}(x) = \{y \mid y \subseteq x\} \in \mathcal{U}$;
- if $I \in \mathcal{U}$ and $(x_i)_{i \in I}$ is an I -indexed family of elements of \mathcal{U} , then $\bigcup_{i \in I} x_i \in \mathcal{U}$;
- $\mathbb{N} \in \mathcal{U}$, where $\mathbb{N} = \{0, 1, \dots\}$ and for all $n \in \mathbb{N}$, $n = \{0, 1, \dots, n-1\}$. ■

The following axiom of universes is often assumed in addition to ZFC in the literature.

Axiom 1.2. For each set x , there exists a universe \mathcal{U} such that $x \in \mathcal{U}$. ■

In fact, in this thesis we will only need three universes \mathcal{U}_1 , \mathcal{U}_2 and \mathcal{U}_3 with $\mathcal{U}_1 \in \mathcal{U}_2 \in \mathcal{U}_3$. We now fix these universes once and for all.

Let \mathcal{U} be a universe. We define several size-regulating conditions on sets and other mathematical structures in reference to \mathcal{U} .

- A set is said to be **in** \mathcal{U} if it is an element of \mathcal{U} .

In this thesis, a category is always assumed to have sets of objects and of morphisms (rather than *proper classes* of them). We say that a category \mathcal{C} is

- **in \mathcal{U}** if the tuple $(\text{ob}(\mathcal{C}), (\mathcal{C}(A, B))_{A, B \in \text{ob}(\mathcal{C})}, (\text{id}_C \in \mathcal{C}(C, C))_{C \in \text{ob}(\mathcal{C})}, (\circ_{A, B, C} : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C))_{A, B, C \in \text{ob}(\mathcal{C})})$, consisting of the data for \mathcal{C} , is an element of \mathcal{U} ;
- **locally in \mathcal{U}** if for each $A, B \in \text{ob}(\mathcal{C})$, the hom-set $\mathcal{C}(A, B)$ is in \mathcal{U} .

We also write $C \in \mathcal{C}$ for $C \in \text{ob}(\mathcal{C})$.

We extend these definitions to other mathematical structures. For example, a group is said to be **in \mathcal{U}** if it is an element of \mathcal{U} , a 2-category is **locally in \mathcal{U}** if all its hom-categories are in \mathcal{U} , and so on.

Recall the universes $\mathcal{U}_1, \mathcal{U}_2$ and \mathcal{U}_3 we have fixed above.

Convention 1.3. A set or other mathematical structure (group, category, etc.) is said to be:

- **small** if it is in \mathcal{U}_1 ;
- **large** if it is in \mathcal{U}_2 ;
- **huge** if it is in \mathcal{U}_3 .

Sets and other mathematical structures are often assumed to be small by default, even when we do not say so explicitly.

A category (or a 2-category) is said to be:

- **locally small** if it is large and locally in \mathcal{U}_1 ;
- **locally large** if it is huge and locally in \mathcal{U}_2 . ■

In the following, we mainly talk about the size-regulating conditions using the terms *small*, *large* and *huge*, avoiding direct references to the universes $\mathcal{U}_1, \mathcal{U}_2$ and \mathcal{U}_3 .

We shall use the following basic (2-)categories throughout this thesis.

- **Set**, the (large) category of all small sets and functions.
- **SET**, the (huge) category of all large sets and functions.
- **Cat**, the (large) category of all small categories and functors.
- **CAT**, the (huge) category of all large categories and functors.
- **Cat**, the (large) 2-category of all small categories, functors and natural transformations.
- **CAI**, the (huge) 2-category of all large categories, functors and natural transformations.
- **2-CAI**, the 2-category of all huge 2-categories, 2-functors and 2-natural transformations.

1.5 2-categorical notions

In order to fix the terminology, we define various 2-categorical notions here.

A 2-functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is called:

- **fully faithful** iff for each $A, A' \in \mathcal{A}$, $F_{A,A'}: \mathcal{A}(A, A') \rightarrow \mathcal{B}(FA, FA')$ is an isomorphism of categories;
- **locally an equivalence** iff for each $A, A' \in \mathcal{A}$, $F_{A,A'}: \mathcal{A}(A, A') \rightarrow \mathcal{B}(FA, FA')$ is an equivalence of categories;
- **locally fully faithful** iff for each $A, A' \in \mathcal{A}$, $F_{A,A'}: \mathcal{A}(A, A') \rightarrow \mathcal{B}(FA, FA')$ is fully faithful;
- **locally faithful** iff for each $A, A' \in \mathcal{A}$, $F_{A,A'}: \mathcal{A}(A, A') \rightarrow \mathcal{B}(FA, FA')$ is faithful;
- **bijective on objects** iff $\text{ob}(F): \text{ob}(\mathcal{A}) \rightarrow \text{ob}(\mathcal{B})$ is a bijection;
- **essentially surjective (on objects)** iff for each $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ and an isomorphism $FA \cong B$ in \mathcal{B} ;
- an **isomorphism** iff it is bijective on objects and fully faithful;
- an **equivalence** iff it is essentially surjective and fully faithful.

For a 2-category \mathcal{B} , let

- \mathcal{B}^{op} be the 2-category obtained by reversing 1-cells: $\mathcal{B}^{\text{op}}(A, B) = \mathcal{B}(B, A)$;
- \mathcal{B}^{co} be the 2-category obtained by reversing 2-cells: $\mathcal{B}^{\text{co}}(A, B) = \mathcal{B}(A, B)^{\text{op}}$;
- $\mathcal{B}^{\text{coop}}$ be the 2-category obtained by reversing both 1-cells and 2-cells: $\mathcal{B}^{\text{coop}}(A, B) = \mathcal{B}(B, A)^{\text{op}}$.

We adopt the same notation for bicategories as well.

Part I

A unified framework for notions of algebraic theory

Chapter 2

Notions of algebraic theory

In almost every field of pure and applied mathematics, *algebras* (in a broad sense) arise quite naturally in one way or another. An algebra, typically, is a set equipped with a family of operations. So for example the symmetric group of order five \mathfrak{S}_5 and the ring of integers \mathbb{Z} are both algebras. Structural similarities between important algebras have led to the introduction and study of various *types of algebras*, such as monoids, groups, rings, vector spaces, lattices, Boolean algebras, and Heyting algebras. A type of algebras is normally specified by a family of operations and a family of equational axioms. We shall use the term *algebraic theory* to refer to a specification of a type of algebras.

Subsequently, various authors have set out to develop *notions of algebraic theory*. A notion of algebraic theory is a background theory for a certain type of algebraic theories. The most famous classical example of notions of algebraic theory is Birkhoff's *universal algebra* [8].

There are several motivations behind the introduction of notions of algebraic theory. First, by working at this level of generality, one can prove theorems for various types of algebras once and for all; for instance, the homomorphism theorems in universal algebra (see e.g., [10, Section II.6]) generalise the homomorphism theorems for groups to monoids, rings, lattices, etc. Second, novel notions of algebraic theory have sometimes been proposed in order to set up powerful languages expressive enough to capture interesting but intricate types of algebras. This applies to (the topological versions of) symmetric and non-symmetric operads, used to define up-to-homotopy topological commutative monoids and monoids [73], and to globular operads, by which a definition of weak ω -category is given [5, 64].

In this chapter we shall review several known notions of algebraic theory, in order to provide motivation and background knowledge for our unified framework for notions of algebraic theory developed from Chapter 3 on. The contents of this chapter are well-known to the specialists.

2.1 Universal algebra

Universal algebra [8] deals with types of algebras defined by finitary operations and equations between them. As a running example, let us consider *groups*. A group can be defined as a set G equipped with an element $e^G \in G$ (the unit), and two functions $i^G: G \rightarrow G$ (the inverse) and $m^G: G \times G \rightarrow G$ (the multiplication), satisfying the following axioms:

- for all $g_1 \in G$, $m^G(g_1, e^G) = g_1$ (the right unit axiom);
- for all $g_1 \in G$, $m^G(g_1, i^G(g_1)) = e^G$ (the right inverse axiom);

- for all $g_1, g_2, g_3 \in G$, $m^G(m^G(g_1, g_2), g_3) = m^G(g_1, m^G(g_2, g_3))$ (the associativity axiom).

(From these three axioms it follows that for all $g_1 \in G$, $m^G(e^G, g_1) = g_1$ (the left unit axiom) and $m^G(i^G(g_1), g_1) = e^G$ (the left inverse axiom).) This definition of group turns out to be an instance of the notion of *presentation of an equational theory*, one of the most fundamental notions in universal algebra introduced below.

First we introduce the notion of *graded set*, which provides a convenient language for clean development of universal algebra.

Definition 2.1. 1. An **(\mathbb{N} -)graded set** Γ is a family $\Gamma = (\Gamma_n)_{n \in \mathbb{N}}$ of sets indexed by natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. By an **element of Γ** we mean an element of the set $\coprod_{n \in \mathbb{N}} \Gamma_n = \{(n, \gamma) \mid n \in \mathbb{N}, \gamma \in \Gamma_n\}$. We write $x \in \Gamma$ if x is an element of Γ .

2. If $\Gamma = (\Gamma_n)_{n \in \mathbb{N}}$ and $\Gamma' = (\Gamma'_n)_{n \in \mathbb{N}}$ are graded sets, then a **morphism of graded sets** $f: \Gamma \rightarrow \Gamma'$ is a family of functions $f = (f_n: \Gamma_n \rightarrow \Gamma'_n)_{n \in \mathbb{N}}$. ■

We can routinely extend the basic notions of set theory to graded sets. For example, we say that a graded set Γ' is a **graded subset** of a graded set Γ (written as $\Gamma' \subseteq \Gamma$) if for each $n \in \mathbb{N}$, Γ'_n is a subset of Γ_n . Given arbitrary graded sets Γ and Γ' , their **cartesian product** (written as $\Gamma \times \Gamma'$) is defined by $(\Gamma \times \Gamma')_n = \Gamma_n \times \Gamma'_n$ for each $n \in \mathbb{N}$. An **equivalence relation** on a graded set Γ is a graded subset $R \subseteq \Gamma \times \Gamma$ such that each $R_n \subseteq \Gamma_n \times \Gamma_n$ is an equivalence relation on the set Γ_n . Given such an equivalence relation R on Γ , we can form the **quotient graded set** Γ/R by setting $(\Gamma/R)_n = \Gamma_n/R_n$, the quotient set of Γ_n with respect to R_n . These notions will be used below.

A graded set can be seen as a (*functional*) *signature*. That is, we can regard a graded set Σ as the signature whose set of n -ary functional symbols is given by Σ_n for each $n \in \mathbb{N}$. We often use the symbol Σ to denote a graded set when we want to emphasise this aspect of graded sets, as in the following definition.

Definition 2.2. Let Σ be a graded set.

1. A **Σ -algebra** is a set A equipped with, for each $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, a function $[\sigma]^A: A^n \rightarrow A$ called the **interpretation of σ** . We write such a Σ -algebra as $(A, ([\sigma]^A)_{n \in \mathbb{N}, \sigma \in \Sigma_n})$ or simply $(A, [-]^A)$. We often omit the superscript in $[-]^A$.
2. If $(A, [-]^A)$ and $(B, [-]^B)$ are Σ -algebras, then a **Σ -homomorphism** from $(A, [-]^A)$ to $(B, [-]^B)$ is a function $f: A \rightarrow B$ such that for any $n \in \mathbb{N}$, $\sigma \in \Sigma_n$ and $a_1, \dots, a_n \in A$,

$$f([\sigma]^A(a_1, \dots, a_n)) = [\sigma]^B(f(a_1), \dots, f(a_n))$$

holds (that is, the diagram

$$\begin{array}{ccc} A^n & \xrightarrow{f^n} & B^n \\ \downarrow [\sigma]^A & & \downarrow [\sigma]^B \\ A & \xrightarrow{f} & B \end{array}$$

commutes). ■

As an example, let us consider the graded set Σ^{Grp} defined as $\Sigma_0^{\text{Grp}} = \{e\}$, $\Sigma_1^{\text{Grp}} = \{i\}$, $\Sigma_2^{\text{Grp}} = \{m\}$ and $\Sigma_n^{\text{Grp}} = \emptyset$ for all $n \geq 3$. Then the structure of a group is given by that of a Σ^{Grp} -algebra. Note that to give an element $e^G \in G$ is equivalent to give a function $\llbracket e \rrbracket: 1 \rightarrow G$ where 1 is a singleton set, and that for any set G , G^0 is a singleton set. Also, between groups, the notions of group homomorphism and Σ^{Grp} -homomorphism coincide.

However, not all Σ^{Grp} -algebras are groups; for a Σ^{Grp} -algebra to be a group, the interpretations must satisfy the group axioms. Notice that all group axioms are equations between certain expressions built from variables and operations. This is the fundamental feature shared by all algebraic structures expressible in universal algebra. The following notion of Σ -term defines “expressions built from variables and operations” relative to arbitrary graded sets Σ .

Definition 2.3. Let Σ be a graded set. The graded set $T(\Sigma) = (T(\Sigma)_n)_{n \in \mathbb{N}}$ of Σ -terms is defined inductively as follows.

1. For each $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$,

$$x_i^{(n)} \in T(\Sigma)_n.$$

We sometimes omit the superscript and write x_i for $x_i^{(n)}$.

2. For each $n, k \in \mathbb{N}$, $\sigma \in \Sigma_k$ and $t_1, \dots, t_k \in T(\Sigma)_n$,

$$\sigma(t_1, \dots, t_k) \in T(\Sigma)_n.$$

When $k = 0$, we usually omit the parentheses in $\sigma()$ and write instead as σ . ■

An immediate application of the inductive nature of the above definition of Σ -terms is the canonical extension of the interpretation function $\llbracket - \rrbracket$ of a Σ -algebra from Σ to $T(\Sigma)$.

Definition 2.4. Let Σ be a graded set and $(A, \llbracket - \rrbracket)$ be a Σ -algebra. We define the **interpretation** $\llbracket - \rrbracket'$ of Σ -terms recursively as follows.

1. For each $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$,

$$\llbracket x_i^{(n)} \rrbracket': A^n \rightarrow A$$

is the i -th projection $(a_1, \dots, a_n) \mapsto a_i$.

2. For each $n, k \in \mathbb{N}$, $\sigma \in \Sigma_k$ and $t_1, \dots, t_k \in T(\Sigma)_n$,

$$\llbracket \sigma(t_1, \dots, t_k) \rrbracket': A^n \rightarrow A$$

maps $(a_1, \dots, a_n) \in A^n$ to $\llbracket \sigma \rrbracket'(\llbracket t_1 \rrbracket'(a_1, \dots, a_n), \dots, \llbracket t_k \rrbracket'(a_1, \dots, a_n))$; that is, $\llbracket \sigma(t_1, \dots, t_k) \rrbracket'$ is the following composite:

$$A^n \xrightarrow{\langle \llbracket t_1 \rrbracket', \dots, \llbracket t_k \rrbracket' \rangle} A^k \xrightarrow{\llbracket \sigma \rrbracket'} A.$$

Note that for any $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, $\llbracket \sigma \rrbracket = \llbracket \sigma(x_1^{(n)}, \dots, x_n^{(n)}) \rrbracket'$. Henceforth, for any Σ -term t we simply write $\llbracket t \rrbracket$ for $\llbracket t \rrbracket'$ defined above. ■

Definition 2.5. Let Σ be a graded set. An element of the graded set $T(\Sigma) \times T(\Sigma)$ is called a Σ -**equation**. We write a Σ -equation $(n, (t, s)) \in T(\Sigma) \times T(\Sigma)$ (that is, $n \in \mathbb{N}$ and $t, s \in T(\Sigma)_n$) as $t \approx_n s$ or $t \approx s$. ■

Definition 2.6. A **presentation of an equational theory** $\langle \Sigma \mid E \rangle$ is a pair consisting of:

- a graded set Σ of **basic operations**, and
- a graded set $E \subseteq T(\Sigma) \times T(\Sigma)$ of **equational axioms**. ■

Definition 2.7. Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory.

1. A **model** of $\langle \Sigma \mid E \rangle$ is a Σ -algebra $(A, \llbracket - \rrbracket)$ such that for any $t \approx_n s \in E$, $\llbracket t \rrbracket = \llbracket s \rrbracket$ holds.
2. A **homomorphism** between models of $\langle \Sigma \mid E \rangle$ is just a Σ -homomorphism between the corresponding Σ -algebras. ■

Consider the presentation of an equational theory $\langle \Sigma^{\text{Grp}} \mid E^{\text{Grp}} \rangle$, where

$$E_1^{\text{Grp}} = \{ m(x_1^{(1)}, e) \approx x_1^{(1)}, \quad m(x_1^{(1)}, i(x_1^{(1)})) \approx e \},$$

$$E_3^{\text{Grp}} = \{ m(m(x_1^{(3)}, x_2^{(3)}), x_3^{(3)}) \approx m(x_1^{(3)}, m(x_2^{(3)}, x_3^{(3)})) \}$$

and $E_n^{\text{Grp}} = \emptyset$ for all $n \in \mathbb{N} \setminus \{1, 3\}$. Clearly, groups are the same as models of $\langle \Sigma^{\text{Grp}} \mid E^{\text{Grp}} \rangle$. Many other types of algebras—indeed all examples we have mentioned in the first paragraph of this chapter—can be written as models of $\langle \Sigma \mid E \rangle$ for a suitable choice of the presentation of an equational theory $\langle \Sigma \mid E \rangle$; see any introduction to universal algebra (e.g., [10]) for details.

We conclude this section by reviewing the machinery of *equational logic*, which enables us to investigate consequences of equational axioms without referring to their models. We assume that the reader is familiar with the basics of mathematical logic, such as substitution of a term t for a variable x in a term s (written as $s[x \mapsto t]$), simultaneous substitutions (written as $s[x_1 \mapsto t_1, \dots, x_k \mapsto t_k]$), and the notion of proof (tree) and its definition by inference rules.

Definition 2.8. Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory.

1. Define the set of $\langle \Sigma \mid E \rangle$ -**proofs** inductively by the following inference rules. Every $\langle \Sigma \mid E \rangle$ -proof is a finite rooted tree whose vertices are labelled by Σ -equations.

$$\begin{array}{l} \text{(AX)} \quad \frac{}{t \approx_n s} \quad (\text{if } t \approx_n s \in E) \\ \text{(REFL)} \quad \frac{}{t \approx_n t} \quad \text{(SYM)} \quad \frac{t \approx_n s}{s \approx_n t} \quad \text{(TRANS)} \quad \frac{t \approx_n s \quad s \approx_n u}{t \approx_n u} \\ \text{(CONG)} \quad \frac{s \approx_k s' \quad t_1 \approx_n t'_1 \quad \cdots \quad t_k \approx_n t'_k}{s[x_1^{(k)} \mapsto t_1, \dots, x_k^{(k)} \mapsto t_k] \approx_n s'[x_1^{(k)} \mapsto t'_1, \dots, x_k^{(k)} \mapsto t'_k]} \end{array}$$

2. A Σ -equation $t \approx_n s \in T(\Sigma) \times T(\Sigma)$ is called an **equational theorem** of $\langle \Sigma \mid E \rangle$ if there exists a $\langle \Sigma \mid E \rangle$ -proof whose root is labelled by $t \approx_n s$. We write

$$\langle \Sigma \mid E \rangle \vdash t \approx_n s$$

to mean that $t \approx_n s$ is an equational theorem of $\langle \Sigma \mid E \rangle$, and denote by $\overline{E} \subseteq T(\Sigma) \times T(\Sigma)$ the graded set of all equational theorems of $\langle \Sigma \mid E \rangle$. ■

Equational logic is known to be both *sound* and *complete*, in the following sense.

Definition 2.9. 1. Let Σ be a graded set and $(A, \llbracket - \rrbracket)$ be a Σ -algebra. For any Σ -equation $t \approx_n s \in T(\Sigma) \times T(\Sigma)$, we write

$$(A, \llbracket - \rrbracket) \models t \approx_n s$$

to mean $\llbracket t \rrbracket = \llbracket s \rrbracket$.

2. Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory. For any Σ -equation $t \approx_n s \in T(\Sigma) \times T(\Sigma)$, we write

$$\langle \Sigma \mid E \rangle \models t \approx_n s$$

to mean that for any model $(A, \llbracket - \rrbracket)$ of $\langle \Sigma \mid E \rangle$, $(A, \llbracket - \rrbracket) \models t \approx_n s$. ■

Theorem 2.10. Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory.

1. (*Soundness*) Let $t \approx_n s \in T(\Sigma) \times T(\Sigma)$. If $\langle \Sigma \mid E \rangle \vdash t \approx_n s$ then $\langle \Sigma \mid E \rangle \models t \approx_n s$.
2. (*Completeness*) Let $t \approx_n s \in T(\Sigma) \times T(\Sigma)$. If $\langle \Sigma \mid E \rangle \models t \approx_n s$ then $\langle \Sigma \mid E \rangle \vdash t \approx_n s$.

Proof. The soundness theorem is proved by a straightforward induction over $\langle \Sigma \mid E \rangle$ -proofs. For the completeness theorem, see e.g., [44, Corollary 1.5] or [10, Section II.14]. □

2.2 Clones

The central notion we have introduced in the previous section is that of *presentation of an equational theory* (Definition 2.6), whose main purpose is to define its *models* (Definition 2.7). It can happen, however, that two different presentations of equational theories define the “same” models, sometimes in a quite superficial manner.

For example, consider the following presentation of an equational theory $\langle \Sigma^{\text{Grp}'} \mid E^{\text{Grp}'} \rangle$:

$$\Sigma^{\text{Grp}'} = \Sigma^{\text{Grp}},$$

$$E_1^{\text{Grp}'} = \{ m(x_1^{(1)}, e) \approx x_1^{(1)}, \quad m(e, x_1^{(1)}) \approx x_1^{(1)}, \\ m(x_1^{(1)}, i(x_1^{(1)})) \approx e, \quad m(i(x_1^{(1)}), x_1^{(1)}) \approx e \},$$

$$E_n^{\text{Grp}'} = E_n^{\text{Grp}} \quad \text{for all } n \in \mathbb{N} \setminus \{1\}.$$

It is a classical fact that a group can be defined either as a model of $\langle \Sigma^{\text{Grp}} \mid E^{\text{Grp}} \rangle$ or as a model of $\langle \Sigma^{\text{Grp}'} \mid E^{\text{Grp}'} \rangle$. Indeed, we may add arbitrary equational theorems of $\langle \Sigma^{\text{Grp}} \mid E^{\text{Grp}} \rangle$, such as $i(i(x_1)) \approx x_1$, $i(m(x_1, x_2)) \approx m(i(x_2), i(x_1))$ and $x_1 \approx x_1$, as additional equational axioms and still obtain the groups as the models.

As another example, let us consider the presentation of an equational theory $\langle \Sigma^{\text{Grp}''} \mid E^{\text{Grp}''} \rangle$ defined as:

$$\Sigma_0^{\text{Grp}''} = \{e, e'\}, \quad \Sigma_n^{\text{Grp}''} = \Sigma_n^{\text{Grp}} \quad \text{for all } n \in \mathbb{N} \setminus \{0\},$$

$$E_0^{\text{Grp}''} = \{e \approx e'\}, \quad E_n^{\text{Grp}''} = E_n^{\text{Grp}} \quad \text{for all } n \in \mathbb{N} \setminus \{0\}.$$

To make a set A into a model of $\langle \Sigma^{\text{Grp}''} \mid E^{\text{Grp}''} \rangle$, formally we have to specify two elements $\llbracket e \rrbracket$ and $\llbracket e' \rrbracket$ of A , albeit they are forced to be equal and play the role of unit with respect to the group structure determined by $\llbracket m \rrbracket$. We cannot quite say that models of $\langle \Sigma^{\text{Grp}''} \mid E^{\text{Grp}''} \rangle$ are *equal* to models of $\langle \Sigma^{\text{Grp}} \mid E^{\text{Grp}} \rangle$, since their data differ; however, it should be intuitively clear that there is no point in distinguishing them. (In precise mathematical terms, our claim of the “sameness” amounts to the existence of an isomorphism of categories between the categories of models of $\langle \Sigma^{\text{Grp}} \mid E^{\text{Grp}} \rangle$ and of models of $\langle \Sigma^{\text{Grp}''} \mid E^{\text{Grp}''} \rangle$ preserving the underlying sets of models, i.e., commuting with the forgetful functors into **Set**.)

A presentation of an equational theory has much freedom in choices both of basic operations and of equational axioms. It is really a *presentation*. In fact, there is a notion which may be thought of as an *equational theory* itself, something that a presentation of an equational theory presents; it is called an (*abstract*) *clone*.

Definition 2.11. A **clone** \mathbb{T} consists of:

(CD1) a graded set $T = (T_n)_{n \in \mathbb{N}}$;

(CD2) for each $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, an element

$$p_i^{(n)} \in T_n;$$

(CD3) for each $k, n \in \mathbb{N}$, a function

$$\circ_k^{(n)}: T_k \times (T_n)^k \longrightarrow T_n$$

whose action on an element $(\phi, \theta_1, \dots, \theta_k) \in T_k \times (T_n)^k$ we write as $\phi \circ_k^{(n)}(\theta_1, \dots, \theta_k)$ or simply as $\phi \circ (\theta_1, \dots, \theta_k)$;

satisfying the following equations:

(CA1) for each $k, n \in \mathbb{N}$, $j \in \{1, \dots, k\}$ and $\theta_1, \dots, \theta_k \in T_n$,

$$p_j^{(k)} \circ_k^{(n)}(\theta_1, \dots, \theta_k) = \theta_j;$$

(CA2) for each $n \in \mathbb{N}$, $\theta \in T_n$,

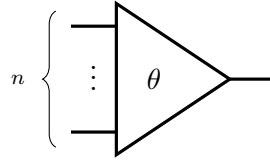
$$\theta \circ_n^{(n)}(p_1^{(n)}, \dots, p_n^{(n)}) = \theta;$$

(CA3) for each $l, k, n \in \mathbb{N}$, $\psi \in T_l$, $\phi_1, \dots, \phi_l \in T_k$, $\theta_1, \dots, \theta_k \in T_n$,

$$\begin{aligned} \psi \circ_l^{(k)}(\phi_1 \circ_k^{(n)}(\theta_1, \dots, \theta_k), \dots, \phi_l \circ_k^{(n)}(\theta_1, \dots, \theta_k)) \\ = (\psi \circ_l^{(k)}(\phi_1, \dots, \phi_l)) \circ_k^{(n)}(\theta_1, \dots, \theta_k). \end{aligned}$$

Such a clone is written as $\mathbb{T} = (T, (p_n^{(i)})_{n \in \mathbb{N}, i \in \{1, \dots, n\}}, (\circ_k^{(n)})_{k, n \in \mathbb{N}})$ or simply (T, p, \circ) . ■

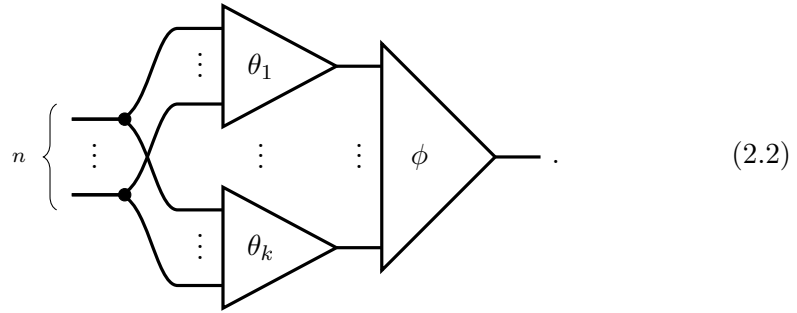
To understand the definition of clone, it is helpful to draw some pictures known as *string diagrams* (cf. [18, 64]). Given a clone $\mathbb{T} = (T, p, \circ)$, let us denote an element θ of T_n by a triangle with n “input wires” and a single “output wire”:



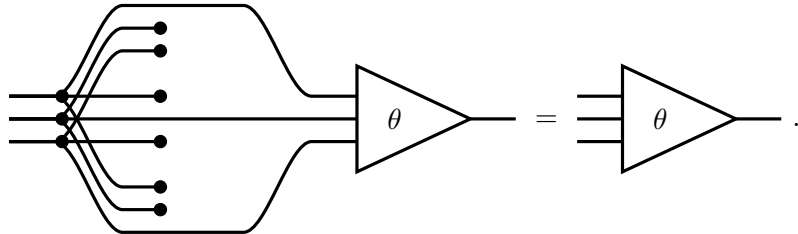
The element $p_i^{(n)}$ in (CD2) may also be denoted by



and $\phi \circ_k^{(n)}(\theta_1, \dots, \theta_k)$ in (CD3) by



Then the axioms (CA1)–(CA3) simply assert natural equations between the resulting “circuits”. For instance, (CA2) for $n = 3$ reads:



Next we define models of a clone. We first need a few preliminary definitions.

Definition 2.12. Let A be a set. Define the clone $\text{End}(A) = (\langle A, A \rangle, p, \circ)$ as follows:

(CD1) for each $n \in \mathbb{N}$, let $\langle A, A \rangle_n$ be the set of all functions from A^n to A ;

(CD2) for each $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, let $p_i^{(n)}$ be the i -th projection $A^n \rightarrow A$, $(a_1, \dots, a_n) \mapsto a_i$;

(CD3) for each $k, n \in \mathbb{N}$, $g: A^k \rightarrow A$ and $f_1, \dots, f_k: A^n \rightarrow A$, let $g \circ_k^{(n)}(f_1, \dots, f_k)$ be the function $(a_1, \dots, a_n) \mapsto g(f_1(a_1, \dots, a_n), \dots, f_k(a_1, \dots, a_n))$, that is, the following composite:

$$A^n \xrightarrow{\langle f_1, \dots, f_k \rangle} A^k \xrightarrow{g} A.$$

It is straightforward to check the axioms (CA1)–(CA3). ■

Definition 2.13. Let $\mathbb{T} = (T, p, \circ)$ and $\mathbb{T}' = (T', p', \circ')$ be clones. A **clone homomorphism** from \mathbb{T} to \mathbb{T}' is a morphism of graded sets (Definition 2.1) $h: T \rightarrow T'$ which preserves the structure of clones; precisely,

- for each $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, $h_n(p_i^{(n)}) = p_i'^{(n)}$;
- for each $k, n \in \mathbb{N}$, $\phi \in T_k$ and $\theta_1, \dots, \theta_k \in T_n$,

$$h_n(\phi \circ_k^{(n)} (\theta_1, \dots, \theta_k)) = h_k(\phi) \circ_k'^{(n)} (h_n(\theta_1), \dots, h_n(\theta_k)). \quad \blacksquare$$

Definition 2.14. Let \mathbb{T} be a clone. A **model of \mathbb{T}** consists of a set A and a clone homomorphism $\alpha: \mathbb{T} \rightarrow \text{End}(A)$. ■

Let us then define the notion of homomorphism between models. First we extend the definition of the graded set $\langle A, A \rangle$ introduced in Definition 2.12.

Definition 2.15. 1. Let A and B be sets. The graded set $\langle A, B \rangle$ is defined by setting, for each $n \in \mathbb{N}$, $\langle A, B \rangle_n$ be the set of all functions from A^n to B .

2. Let A, A' and B be sets and $f: A' \rightarrow A$ be a function. The morphism of graded sets $\langle f, B \rangle: \langle A, B \rangle \rightarrow \langle A', B \rangle$ is defined by setting, for each $n \in \mathbb{N}$, $\langle f, B \rangle_n: \langle A, B \rangle_n \rightarrow \langle A', B \rangle_n$ be the precomposition by $f^n: (A')^n \rightarrow A^n$; that is, $h \mapsto h \circ f^n$.

3. Let A, B and B' be sets and $g: B \rightarrow B'$ be a function. The morphism of graded sets $\langle A, g \rangle: \langle A, B \rangle \rightarrow \langle A, B' \rangle$ is defined by setting, for each $n \in \mathbb{N}$, $\langle A, g \rangle_n: \langle A, B \rangle_n \rightarrow \langle A, B' \rangle_n$ be the postcomposition by $g: B \rightarrow B'$; that is, $h \mapsto g \circ h$. ■

Definition 2.16. Let \mathbb{T} be a clone, and (A, α) and (B, β) be models of \mathbb{T} . A **homomorphism** from (A, α) to (B, β) is a function $f: A \rightarrow B$ making the following diagram of morphisms of graded sets commute:

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & \langle A, A \rangle \\ \beta \downarrow & & \downarrow \langle A, f \rangle \\ \langle B, B \rangle & \xrightarrow{\langle f, B \rangle} & \langle A, B \rangle. \end{array} \quad \blacksquare$$

Now let us turn to the relation between presentations of equational theories (Definition 2.6) and clones. We start with the observation that the graded set $T(\Sigma)$ of Σ -terms (Definition 2.3) has a canonical clone structure, given as follows:

(CD2) for each $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, let $p_i^{(n)}$ be $x_i^{(n)} \in T(\Sigma)_n$;

(CD3) for each $k, n \in \mathbb{N}$, $s \in T(\Sigma)_k$ and $t_1, \dots, t_k \in T(\Sigma)_n$, let $s \circ_k^{(n)} (t_1, \dots, t_k)$ be $s[x_1^{(k)} \mapsto t_1, \dots, x_k^{(k)} \mapsto t_k] \in T(\Sigma)_n$.

We denote the resulting clone by $\mathbb{T}(\Sigma)$. In fact, this clone is characterised by the following universal property.

Proposition 2.17. *Let Σ be a graded set. The clone $\mathbb{T}(\Sigma)$ is the free clone generated from the graded set Σ . That is, the morphism of graded sets $\eta_\Sigma: \Sigma \rightarrow T(\Sigma)$, defined by $(\eta_\Sigma)_n(\sigma) = \sigma(x_1^{(n)}, \dots, x_n^{(n)})$ for each $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, satisfies the following property: given any clone $\mathbb{S} = (S, p, \circ)$ and any morphism of graded sets $f: \Sigma \rightarrow S$, there exists a unique clone homomorphism $g: \mathbb{T}(\Sigma) \rightarrow \mathbb{S}$ such that $g \circ \eta_\Sigma = f$.*

$$\begin{array}{ccc}
 \Sigma & \xrightarrow{\eta_\Sigma} & T(\Sigma) & & \mathbb{T}(\Sigma) \\
 & \searrow f & \downarrow g & & \vdots g \\
 & & S & & S \\
 & & \text{(graded sets)} & & \text{(clones)}
 \end{array}$$

Proof. The clone homomorphism g may be defined by using the inductive nature of the definition of $T(\Sigma)$, as follows:

1. for each $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, let

$$g_n(x_i^{(n)}) = p_i^{(n)};$$

2. for each $k, n \in \mathbb{N}$, $\sigma \in \Sigma_k$ and $t_1, \dots, t_k \in T(\Sigma)_n$, let

$$g_n(\sigma(t_1, \dots, t_k)) = f_k(\sigma) \circ_k^{(n)} (g_n(t_1), \dots, g_n(t_k)).$$

To check that g is indeed a clone homomorphism, it suffices to show for each $s \in T(\Sigma)_k$ and $t_1, \dots, t_k \in T(\Sigma)_n$,

$$g_n(s[x_1^{(k)} \mapsto t_1, \dots, x_k^{(k)} \mapsto t_k]) = g_k(s) \circ_k^{(n)} (g_n(t_1), \dots, g_n(t_k));$$

this can be shown by induction on s . The uniqueness of g is clear. \square

The construction given in Definition 2.4 is a special case of the above; let \mathbb{S} be $\text{End}(A)$.

Recall from Definition 2.8 the graded set $\overline{E} \subseteq T(\Sigma) \times T(\Sigma)$ of equational theorems of a presentation of an equational theory $\langle \Sigma \mid E \rangle$. By the rules (REFL), (SYM) and (TRANS), \overline{E} is an equivalence relation on $T(\Sigma)$. Hence we may consider the quotient graded set $T(\Sigma)/\overline{E}$. By the rule (CONG), the clone operations on $T(\Sigma)$ induce well-defined operations on $T(\Sigma)/\overline{E}$; in particular, we can define $\circ_k^{(n)}$ on $T(\Sigma)/\overline{E}$ by

$$[\phi]_{\overline{E}} \circ_k^{(n)} ([\theta_1]_{\overline{E}}, \dots, [\theta_k]_{\overline{E}}) = [\phi(\theta_1, \dots, \theta_k)]_{\overline{E}}.$$

This makes the graded set $T(\Sigma)/\overline{E}$ into a clone; the clone axioms for $T(\Sigma)/\overline{E}$ may be immediately checked by noticing the existence of a surjective morphism of graded sets $q: T(\Sigma) \rightarrow T(\Sigma)/\overline{E}$ (given by $\theta \mapsto [\theta]_{\overline{E}}$) preserving the clone operations. The resulting clone is denoted by $\mathbb{T}^{\langle \Sigma \mid E \rangle}$. It is also characterised by a universal property.

Proposition 2.18. *Let $\langle \Sigma \mid E \rangle$ be a presentation of an equational theory. The clone homomorphism $q: \mathbb{T}(\Sigma) \rightarrow \mathbb{T}^{\langle \Sigma \mid E \rangle}$, defined by $q_n(\theta) = [\theta]_{\overline{E}}$ for each $n \in \mathbb{N}$ and $\theta \in T(\Sigma)_n$, satisfies the following property: given any clone $\mathbb{S} = (S, p, \circ)$ and a clone homomorphism $g: \mathbb{T}(\Sigma) \rightarrow \mathbb{S}$ such that for any $t \approx_n s \in E$, $g_n(t) =$*

$g_n(s)$, there exists a unique clone homomorphism $h: \mathbb{T}^{\langle \Sigma | E \rangle} \rightarrow S$ such that $h \circ q = g$.

$$\begin{array}{ccc} \mathbb{T}(\Sigma) & \xrightarrow{q} & \mathbb{T}^{\langle \Sigma | E \rangle} \\ & \searrow g & \downarrow h \\ & & S \end{array}$$

Proof. The clone homomorphism h is given by $h_n([\theta]_{\bar{E}}) = g_n(\theta)$; this is shown to be well-defined by induction on $\langle \Sigma | E \rangle$ -proofs (see Definition 2.8). The uniqueness of h is immediate from the surjectivity of q . \square

We can now show that for any presentation of an equational theory $\langle \Sigma | E \rangle$, to give a model of $\langle \Sigma | E \rangle$ is equivalent to give a model of the clone $\mathbb{T}^{\langle \Sigma | E \rangle}$. A model of the clone $\mathbb{T}^{\langle \Sigma | E \rangle}$ (Definition 2.14) can be—by Proposition 2.18—equivalently given as a suitable clone homomorphism out of $\mathbb{T}(\Sigma)$; this in turn is—by Proposition 2.17—equivalently given as a suitable morphism of graded sets out of Σ , which is nothing but a model of the presentation of an equational theory $\langle \Sigma | E \rangle$ (Definition 2.7).

We also remark that every clone is isomorphic to a clone of the form $\mathbb{T}^{\langle \Sigma | E \rangle}$ for some presentation of an equational theory $\langle \Sigma | E \rangle$.

The inference rules of equational logic we have given in Definition 2.8 can be understood as the inductive definition of the congruence relation $\bar{E} \subseteq T(\Sigma) \times T(\Sigma)$ on the clone $\mathbb{T}(\Sigma)$ generated by $E \subseteq T(\Sigma) \times T(\Sigma)$. The notion of clone therefore provides conceptual understanding of equational logic.

We conclude that the classical universal algebra based on presentations of equational theories may be replaced by the theory of clones to a certain extent. Given a presentation of an equational theory $\langle \Sigma | E \rangle$, the clone $\mathbb{T}^{\langle \Sigma | E \rangle}$ it presents can be obtained by letting $T^{\langle \Sigma | E \rangle}$ to be the graded set of Σ -terms modulo equational theorems of $\langle \Sigma | E \rangle$.

We remark that the well-known notion of *Lawvere theory* [58] is essentially equivalent to that of clones; see e.g., [84]. In this thesis, we shall only deal with clones, leaving the translation of the results to Lawvere theories or universal algebra to the interested reader.

2.3 Non-symmetric operads

Non-symmetric operads [73] may be seen as a variant of clones. Compared to clones, non-symmetric operads are less expressive (for example, the groups cannot be captured by non-symmetric operads), but their models can be taken in wider contexts than for clones (we will introduce a notion of model of a non-symmetric operad using abelian groups and their tensor products).

Before giving the definition of non-symmetric operad, we shall introduce the corresponding notion of presentation. Let Σ be a graded set. We say that a Σ -term (Definition 2.3) $t \in T(\Sigma)_n$ is **strongly regular** if in t each of the variables $x_1^{(n)}, \dots, x_n^{(n)}$ appears precisely once, and in this order (from left to right). For example, consider the graded set Σ^{Mon} defined by $\Sigma_0^{\text{Mon}} = \{e\}$, $\Sigma_2^{\text{Mon}} = \{m\}$ and $\Sigma_n^{\text{Mon}} = \emptyset$ for all $n \in \mathbb{N} \setminus \{0, 2\}$. Among the Σ^{Mon} -terms,

$$m(x_1^{(1)}, e) \in T(\Sigma)_1, \quad m(x_1^{(2)}, x_2^{(2)}) \in T(\Sigma)_2, \quad m(m(x_1^{(3)}, x_2^{(3)}), x_3^{(3)}) \in T(\Sigma)_3$$

are strongly regular, but

$$m(x_1^{(1)}, x_1^{(1)}) \in T(\Sigma)_1, \quad x_1^{(2)} \in T(\Sigma)_2, \quad m(x_2^{(2)}, x_1^{(2)}) \in T(\Sigma)_2$$

are not. The following definition introduces the same notion inductively.

Definition 2.19. Let Σ be a graded set. The graded set $T_{\text{SR}}(\Sigma) = (T_{\text{SR}}(\Sigma)_n)_{n \in \mathbb{N}}$ of **strongly regular Σ -terms** is defined inductively as follows.

1. $x_1^{(1)} \in T_{\text{SR}}(\Sigma)_1$.
2. For each $k, n_1, \dots, n_k \in \mathbb{N}$, $\sigma \in \Sigma_k$ and $t_1 \in T_{\text{SR}}(\Sigma)_{n_1}, \dots, t_k \in T_{\text{SR}}(\Sigma)_{n_k}$, writing $n_1 + \dots + n_k = n$,

$$\begin{aligned} & \sigma(t_1[x_1^{(n_1)} \mapsto x_1^{(n)}, \dots, x_{n_1}^{(n_1)} \mapsto x_{n_1}^{(n)}], \dots, \\ & t_k[x_1^{(n_k)} \mapsto x_{n_1+\dots+n_{k-1}+1}^{(n)}, \dots, x_{n_k}^{(n_k)} \mapsto x_{n_1+\dots+n_{k-1}+n_k}^{(n)}]) \in T_{\text{SR}}(\Sigma)_n. \end{aligned}$$

When $k = 0$, we usually write σ instead of $\sigma()$. ■

Definition 2.20 (cf. Definition 2.5). Let Σ be a graded set. An element of the graded set $T_{\text{SR}}(\Sigma) \times T_{\text{SR}}(\Sigma)$ is called a **strongly regular Σ -equation**. We write a strongly regular Σ -equation $(n, (t, s)) \in T_{\text{SR}}(\Sigma) \times T_{\text{SR}}(\Sigma)$ as $t \approx_n s$ or $t \approx s$. ■

Definition 2.21 (cf. Definition 2.6). A **strongly regular presentation of an equational theory** $\langle \Sigma \mid E \rangle$ is a pair consisting of:

- a graded set Σ of **basic operations**, and
- a graded set $E \subseteq T_{\text{SR}}(\Sigma) \times T_{\text{SR}}(\Sigma)$ of (strongly regular) **equational axioms**. ■

The notion of model of a strongly regular presentation of an equational theory may be defined just as in Definition 2.7, since any strongly regular presentation of an equational theory can be seen as a presentation of an equational theory.

As an example of strongly regular presentations of equational theories, consider $\langle \Sigma^{\text{Mon}} \mid E^{\text{Mon}} \rangle$ defined as follows:

$$E_1^{\text{Mon}} = \{ m(x_1^{(1)}, e) \approx x_1^{(1)}, \quad m(e, x_1^{(1)}) \approx x_1^{(1)} \},$$

$$E_3^{\text{Mon}} = \{ m(m(x_1^{(3)}, x_2^{(3)}), x_3^{(3)}) \approx m(x_1^{(3)}, m(x_2^{(3)}, x_3^{(3)})) \},$$

$E_n^{\text{Mon}} = \emptyset$ for all $n \in \mathbb{N} \setminus \{1, 3\}$. Models of $\langle \Sigma^{\text{Mon}} \mid E^{\text{Mon}} \rangle$ are precisely *monoids*.

In order to appreciate the value of strongly regular presentations of equational theories (and of non-symmetric operads), let us now introduce another notion of model. This notion of model is based on abelian groups, in contrast to the one introduced in Definition 2.7 based on sets.

Let $\langle \Sigma \mid E \rangle$ be a strongly regular presentation of an equational theory. Define an **interpretation** of Σ on an abelian group A to be a function $\llbracket - \rrbracket$ which for each $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, assigns a group homomorphism $\llbracket \sigma \rrbracket: A^{\otimes n} \longrightarrow A$, where $A^{\otimes n}$ is the tensor product of n -many copies of A ($A^{\otimes 0}$ is the additive abelian group \mathbb{Z} , the unit for tensor). Given such an interpretation $\llbracket - \rrbracket$ of Σ , we can extend it to $T_{\text{SR}}(\Sigma)$, following the inductive definition of $T_{\text{SR}}(\Sigma)$ in Definition 2.19 (cf. Definition 2.4):

1. Let $\llbracket x_1^{(1)} \rrbracket: A \longrightarrow A$ be the identity homomorphism.

2. For $k, n_1, \dots, n_k \in \mathbb{N}$, $\sigma \in \Sigma_k$ and $t_1 \in T_{\text{SR}}(\Sigma)_{n_1}, \dots, t_k \in T_{\text{SR}}(\Sigma)_{n_k}$, let

$$\llbracket \sigma(t_1, \dots, t_k) \rrbracket: A^{\otimes(n_1 + \dots + n_k)} \longrightarrow A$$

(we omit the substitutions in t_i) to be the composite:

$$A^{\otimes(n_1 + \dots + n_k)} \xrightarrow{\llbracket t_1 \rrbracket \otimes \dots \otimes \llbracket t_k \rrbracket} A^{\otimes k} \xrightarrow{\llbracket \sigma \rrbracket} A.$$

(Note that, in contrast, we cannot extend $\llbracket - \rrbracket$ to $T(\Sigma)$ in any natural way. For example, tensor products do not have an analogue of projections for cartesian products.) We define an **abelian group model of $\langle \Sigma \mid E \rangle$** to be an abelian group A together with an interpretation $\llbracket - \rrbracket$ of Σ on A such that for any $t \approx_n s \in E$, $\llbracket t \rrbracket = \llbracket s \rrbracket$ (cf. Definition 2.7, which may be called a *set model of $\langle \Sigma \mid E \rangle$*). The abelian group models of $\langle \Sigma^{\text{Mon}} \mid E^{\text{Mon}} \rangle$ are precisely the *rings (with 1)*.

We now turn to the definition of non-symmetric operad:

Definition 2.22 (cf. Definition 2.11). A **non-symmetric operad** \mathbb{T} consists of:

(ND1) a graded set $T = (T_n)_{n \in \mathbb{N}}$;

(ND2) an element $\text{id} \in T_1$;

(ND3) for each $k, n_1, \dots, n_k \in \mathbb{N}$, a function (we omit the sub- and super-scripts)

$$\circ: T_k \times T_{n_1} \times \dots \times T_{n_k} \longrightarrow T_{n_1 + \dots + n_k}$$

whose action we write as $(\phi, \theta_1, \dots, \theta_k) \mapsto \phi \circ (\theta_1, \dots, \theta_k)$

satisfying the following equations:

(NA1) for each $n \in \mathbb{N}$ and $\theta \in T_n$,

$$\text{id} \circ (\theta) = \theta;$$

(NA2) for each $n \in \mathbb{N}$ and $\theta \in T_n$,

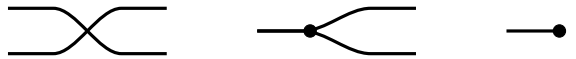
$$\theta \circ (\text{id}, \dots, \text{id}) = \theta;$$

(NA3) for each $l, k_1, \dots, k_l, n_{1,1}, \dots, n_{1,k_1}, \dots, n_{l,1}, \dots, n_{l,k_l} \in \mathbb{N}$, $\psi \in T_l$, $\phi_1 \in T_{k_1}, \dots, \phi_l \in T_{k_l}$, $\theta_{1,1} \in T_{n_{1,1}}, \dots, \theta_{1,k_1} \in T_{n_{1,k_1}}, \dots, \theta_{l,1} \in T_{n_{l,1}}, \dots, \theta_{l,k_l} \in T_{n_{l,k_l}}$,

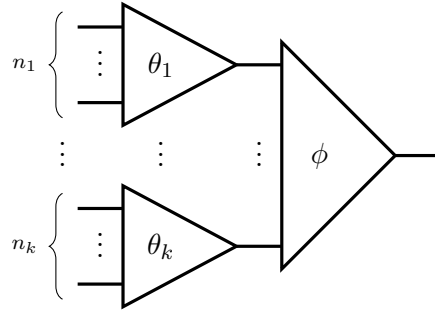
$$\begin{aligned} & \psi \circ (\phi_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \phi_l \circ (\theta_{l,1}, \dots, \theta_{l,k_l})) \\ &= (\psi \circ (\phi_1, \dots, \phi_l)) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \dots, \theta_{l,1}, \dots, \theta_{l,k_l}). \end{aligned}$$

Such a non-symmetric operad is written as $\mathbb{T} = (T, \text{id}, \circ)$. ■

We can understand the above definition by string diagrams. Compared to the case of clones, this time we use a rather restricted class of diagrams; we no longer allow the permuting, copying and discarding facilities, previously drawn as follows:



Without these components, we cannot draw the picture (2.2) for composition in clones. The natural alternative would be the picture



and this is our interpretation of (ND3). The element id in (ND2) is represented by the only diagram of the form (2.1) which we can still draw, namely,



The axioms (NA1)–(NA3) may be understood in the light of these diagrams.

Let us move on to the definition of models of a non-symmetric operad. As with strongly regular presentations of equational theories, non-symmetric operads also admits both notions of model, one based on sets and the other based on abelian groups (and a lot more, as we shall see later).

Definition 2.23 (cf. Definition 2.12). Let A be a set. Define the non-symmetric operad $\text{End}_{\text{Set}}(A) = (\langle A, A \rangle, \text{id}, \circ)$ as follows:

- (ND1) for each $n \in \mathbb{N}$, let $\langle A, A \rangle_n$ be the set of all functions from A^n to A ;
- (ND2) the element $\text{id} \in \langle A, A \rangle$ is the identity function on A ;
- (ND3) for each $k, n_1, \dots, n_k \in \mathbb{N}$, $g: A^k \rightarrow A$, $f_1: A^{n_1} \rightarrow A, \dots, f_k: A^{n_k} \rightarrow A$, let $g \circ (f_1, \dots, f_k)$ be the following composite:

$$A^{n_1 + \dots + n_k} \xrightarrow{f_1 \times \dots \times f_k} A^k \xrightarrow{g} A. \quad \blacksquare$$

Definition 2.24. Let A be an abelian group. Define the non-symmetric operad $\text{End}_{\text{Ab}}(A) = (\langle A, A \rangle, \text{id}, \circ)$ as follows:

- (ND1) for each $n \in \mathbb{N}$, let $\langle A, A \rangle_n$ be the set of all group homomorphisms from $A^{\otimes n}$ to A ;
- (ND2) the element $\text{id} \in \langle A, A \rangle$ is the identity homomorphism on A ;
- (ND3) for each $k, n_1, \dots, n_k \in \mathbb{N}$, $g: A^{\otimes k} \rightarrow A$, $f_1: A^{\otimes n_1} \rightarrow A, \dots, f_k: A^{\otimes n_k} \rightarrow A$, let $g \circ (f_1, \dots, f_k)$ be the following composite:

$$A^{\otimes (n_1 + \dots + n_k)} \xrightarrow{f_1 \otimes \dots \otimes f_k} A^{\otimes k} \xrightarrow{g} A. \quad \blacksquare$$

We define the notion of **non-symmetric operad homomorphism** between non-symmetric operads just in the same way as that of clone homomorphism (Definition 2.13).

Definition 2.25 (cf. Definition 2.14). Let T be a non-symmetric operad.

1. A **set model of \mathbb{T}** consists of a set A and a non-symmetric operad homomorphism $\mathbb{T} \rightarrow \text{End}_{\text{Set}}(A)$.
2. An **abelian group model of \mathbb{T}** consists of an abelian group A and a non-symmetric operad homomorphism $\mathbb{T} \rightarrow \text{End}_{\text{Ab}}(A)$.

Homomorphisms between set or abelian group models of \mathbb{T} are defined just as in Definition 2.16. ■

The relationship between strongly regular presentations of equational theories and non-symmetric operads is completely parallel to the one between presentations of equational theories and clones: for each graded set Σ , the graded set $T_{\text{SR}}(\Sigma)$ has the structure of non-symmetric operad, and is moreover the free such generated by Σ , there is a version of equational logic which can be seen as giving an inductive definition of the congruence relation for non-symmetric operad, and so on.

2.4 Symmetric operads

Symmetric operads [73] are an intermediate notion of algebraic theory which lie between clones and non-symmetric operads, in terms of expressive power as well as in terms of range of notions of models.

Let us first discuss the corresponding presentation. Given a graded set Σ , a Σ -term (Definition 2.3) $t \in T(\Sigma)_n$ is called **regular** if in t each of the variables $x_1^{(n)}, \dots, x_n^{(n)}$ appears precisely once. By way of illustration, consider the graded set Σ^{Mon} . Every strongly regular Σ^{Mon} -term is regular, and

$$m(x_2^{(2)}, x_1^{(2)}) \in T(\Sigma^{\text{Mon}})_2$$

is an example of Σ^{Mon} -terms which are regular but not strongly regular.

For any graded set Σ , let us denote by $T_{\text{R}}(\Sigma) \subseteq T(\Sigma)$ the graded set of all regular Σ -terms. The following definitions should now be straightforward.

Definition 2.26 (cf. Definition 2.5). Let Σ be a graded set. An element of the graded set $T_{\text{R}}(\Sigma) \times T_{\text{R}}(\Sigma)$ is called a **regular Σ -equation**. We write a regular Σ -equation $(n, (t, s)) \in T_{\text{R}}(\Sigma) \times T_{\text{R}}(\Sigma)$ as $t \approx_n s$ or $t \approx s$. ■

Definition 2.27 (cf. Definition 2.6). A **regular presentation of an equational theory** $\langle \Sigma \mid E \rangle$ is a pair consisting of:

- a graded set Σ of **basic operations**, and
- a graded set $E \subseteq T_{\text{R}}(\Sigma) \times T_{\text{R}}(\Sigma)$ of (regular) **equational axioms**. ■

The notion of model of a regular presentation of an equational theory (based on sets) may be defined just as in Definition 2.7, since any regular presentation of an equational theory can be seen as a presentation of an equational theory.

As an example of regular (but not strongly regular) presentations of equational theories, consider $\langle \Sigma^{\text{Mon}} \mid E^{\text{CMon}} \rangle$ defined as follows:

$$\begin{aligned} E_1^{\text{CMon}} &= \{ m(x_1^{(1)}, e) \approx x_1^{(1)}, \quad m(e, x_1^{(1)}) \approx x_1^{(1)} \}, \\ E_2^{\text{CMon}} &= \{ m(x_1^{(2)}, x_2^{(2)}) \approx m(x_2^{(2)}, x_1^{(2)}) \}, \\ E_3^{\text{CMon}} &= \{ m(m(x_1^{(3)}, x_2^{(3)}), x_3^{(3)}) \approx m(x_1^{(3)}, m(x_2^{(3)}, x_3^{(3)})) \}, \end{aligned}$$

$E_n^{\text{CMon}} = \emptyset$ for all $n \in \mathbb{N} \setminus \{1, 2, 3\}$. Models of $\langle \Sigma^{\text{Mon}} \mid E^{\text{CMon}} \rangle$ are precisely *commutative monoids*.

We can also define the notion of model of a regular presentation of an equational theory based on abelian groups; we omit the details here.

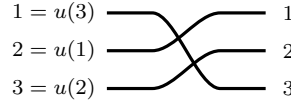
Let us turn to the definition of symmetric operad. In order to define symmetric operads, we have to give preliminary definitions concerning symmetric groups.

For each natural number n , the **symmetric group of order n** , written as \mathfrak{S}_n , is defined as the set of all bijections on the set $[n] = \{1, \dots, n\}$ together with the multiplication \cdot given by composition of functions: for $u, v: [n] \rightarrow [n]$, their multiplication $v \cdot u$ is the composite

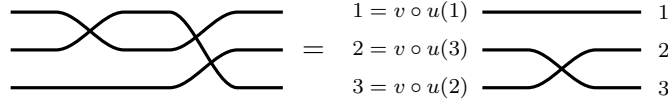
$$[n] \xrightarrow{u} [n] \xrightarrow{v} [n].$$

The identity function on $[n]$ is written as $e_n \in \mathfrak{S}_n$.

We may visualise elements of \mathfrak{S}_n by string diagrams. For example, the element $u \in \mathfrak{S}_3$ defined as $u(1) = 2$, $u(2) = 3$ and $u(3) = 1$ may be drawn as follows:



The composition $v \cdot u$ of u with $v \in \mathfrak{S}_3$ such that $v(1) = 2$, $v(2) = 1$ and $v(3) = 3$ is then drawn as:



For each $k, n_1, \dots, n_k \in \mathbb{N}$, there is a canonical group homomorphism

$$\oplus: \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_k} \longrightarrow \mathfrak{S}_{n_1 + \dots + n_k}$$

which, in terms of string diagrams, just “stacks the diagrams vertically”; we view the set $[n_1 + \dots + n_k]$ as consisting of k blocks, and perform permutation inside each block. Formally, \oplus maps $(u_1, \dots, u_k) \in \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_k}$ to $u_1 \oplus \dots \oplus u_k \in \mathfrak{S}_{n_1 + \dots + n_k}$, the bijection on $[n_1 + \dots + n_k]$ mapping an element $j \in [n_1 + \dots + n_k]$ with $j = n_1 + \dots + n_{i-1} + j'$ for $1 \leq i \leq k$ and $1 \leq j' \leq n_i$ (the j' -th element in the i -th block) to

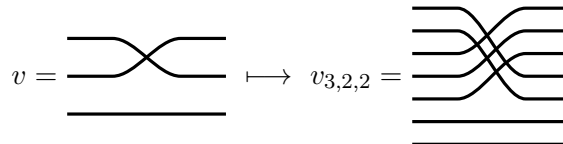
$$(u_1 \oplus \dots \oplus u_k)(j) = n_1 + \dots + n_{i-1} + u_i(j')$$

(the $u_i(j')$ -th element in the i -th block).

Still letting k, n_1, \dots, n_k be arbitrary natural numbers, we have another function (not a group homomorphism in general)

$$(-)_{n_1, \dots, n_k}: \mathfrak{S}_k \longrightarrow \mathfrak{S}_{n_1 + \dots + n_k}.$$

We again view the set $[n_1 + \dots + n_k]$ as consisting of k blocks, but this time we permute these blocks. As an example, take $k = 3$, $n_1 = 3$, $n_2 = 2$ and $n_3 = 2$, and consider $v \in \mathfrak{S}_3$ defined above:



Formally, given any $v \in \mathfrak{S}_k$, the bijection $v_{n_1, \dots, n_k} \in \mathfrak{S}_{n_1 + \dots + n_k}$ maps an element $j \in [n_1 + \dots + n_k]$ with $j = n_1 + \dots + n_{i-1} + j'$ for $1 \leq i \leq k$ and $1 \leq j' \leq n_i$ (the j' -th element in the i -th block) to

$$v_{n_1, \dots, n_k}(j) = n_{v^{-1}(1)} + \dots + n_{v^{-1}(v(i)-1)} + j'$$

(the j' -th element in the $v(i)$ -th block).

Definition 2.28 (cf. Definition 2.11). A **symmetric operad** \mathbb{T} consists of:

(SD1) a graded set $T = (T_n)_{n \in \mathbb{N}}$;

(SD2) an element $\text{id} \in T_1$;

(SD3) for each $k, n_1, \dots, n_k \in \mathbb{N}$, a function (we omit the sub- and superscripts)

$$\circ: T_k \times T_{n_1} \times \dots \times T_{n_k} \longrightarrow T_{n_1 + \dots + n_k}$$

whose action we write as $(\phi, \theta_1, \dots, \theta_k) \mapsto \phi \circ (\theta_1, \dots, \theta_k)$;

(SD4) for each $n \in \mathbb{N}$, a function

$$\bullet: \mathfrak{S}_n \times T_n \longrightarrow T_n$$

satisfying the following equations:

(SA1) for each $n \in \mathbb{N}$ and $\theta \in T_n$,

$$\text{id} \circ (\theta) = \theta;$$

(SA2) for each $n \in \mathbb{N}$ and $\theta \in T_n$,

$$\theta \circ (\text{id}, \dots, \text{id}) = \theta;$$

(SA3) for each $l, k_1, \dots, k_l, n_{1,1}, \dots, n_{1,k_1}, \dots, n_{l,1}, \dots, n_{l,k_l} \in \mathbb{N}$, $\psi \in T_l$, $\phi_1 \in T_{k_1}, \dots, \phi_l \in T_{k_l}$, $\theta_{1,1} \in T_{n_{1,1}}, \dots, \theta_{1,k_1} \in T_{n_{1,k_1}}, \dots, \theta_{l,1} \in T_{n_{l,1}}, \dots, \theta_{l,k_l} \in T_{n_{l,k_l}}$,

$$\begin{aligned} & \psi \circ (\phi_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \phi_l \circ (\theta_{l,1}, \dots, \theta_{l,k_l})) \\ &= (\psi \circ (\phi_1, \dots, \phi_l)) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \dots, \theta_{l,1}, \dots, \theta_{l,k_l}); \end{aligned}$$

(SA4) for each $n \in \mathbb{N}$, the function $\bullet: \mathfrak{S}_n \times T_n \longrightarrow T_n$ is a left group action, that is, for each $\theta \in T_n$ and $u, v \in \mathfrak{S}_n$,

$$e_n \bullet \theta = \theta, \quad (v \cdot u) \bullet \theta = v \bullet (u \bullet \theta);$$

(SA5) for each $k, n_1, \dots, n_k \in \mathbb{N}$, $\phi \in T_k$, $\theta_1 \in T_{n_1}, \dots, \theta_k \in T_{n_k}$ and $u_1 \in \mathfrak{S}_{n_1}, \dots, u_k \in \mathfrak{S}_{n_k}$,

$$\phi \circ (u_1 \bullet \theta_1, \dots, u_k \bullet \theta_k) = (u_1 \oplus \dots \oplus u_k) \bullet (\phi \circ (\theta_1, \dots, \theta_k));$$

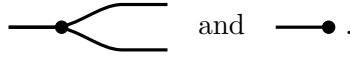
(SA6) for each $k, n_1, \dots, n_k \in \mathbb{N}$, $\phi \in T_k$, $\theta_1 \in T_{n_1}, \dots, \theta_k \in T_{n_k}$ and $v \in \mathfrak{S}_k$,

$$(v \bullet \phi) \circ (\theta_{v^{-1}(1)}, \dots, \theta_{v^{-1}(k)}) = v_{n_1, \dots, n_k} \bullet (\phi \circ (\theta_1, \dots, \theta_k)). \quad \blacksquare$$

In terms of string diagrams, symmetric operads correspond to the intermediate class of the diagrams in which we can use the component



for permutation, but not



Once again, there is a completely parallel story for symmetric operads as the ones for clones and non-symmetric operads. Indeed, Curien [18] and Hyland [39] have developed a unified framework for clones, symmetric operads and non-symmetric operads.

2.5 Monads

Monads are introduced in category theory, and the language of categories is the best way to present them. Hence from now on we shall assume the reader is familiar with the basics of category theory [71].

The definition of monad is quite simple, so we begin with it.

Definition 2.29. Let \mathcal{C} be a large category.

1. A **monad on \mathcal{C}** consists of:

- a functor $T: \mathcal{C} \rightarrow \mathcal{C}$;
- a natural transformation $\eta: \text{id}_{\mathcal{C}} \rightarrow T$ (called the **unit**);
- a natural transformation $\mu: T \circ T \rightarrow T$ (called the **multiplication**),

making the following diagrams commute:

$$\begin{array}{ccc}
 \text{id}_{\mathcal{C}} \circ T \xrightarrow{\eta \circ T} T \circ T & T \circ \text{id}_{\mathcal{C}} \xrightarrow{T \circ \eta} T \circ T & T \circ T \circ T \xrightarrow{\mu \circ T} T \circ T \\
 \searrow \text{id}_T \quad \downarrow \mu & \searrow \text{id}_T \quad \downarrow \mu & \downarrow T \circ \mu \quad \downarrow \mu \\
 T & T & T \circ T \xrightarrow{\mu} T
 \end{array}$$

2. Let $\mathbb{T} = (T, \eta, \mu)$ and $\mathbb{T}' = (T', \eta', \mu')$ be monads on \mathcal{C} . A **morphism of monads on \mathcal{C}** from \mathbb{T} to \mathbb{T}' is a natural transformation $\alpha: T \rightarrow T'$ which commutes with the units and multiplications.

We denote the category of monads on \mathcal{C} by $\mathbf{Mnd}(\mathcal{C})$. ■

Next we introduce models of a monad, usually called *Eilenberg–Moore algebras*.

Definition 2.30 ([22]). Let \mathcal{C} be a large category and $\mathbb{T} = (T, \eta, \mu)$ be a monad on \mathcal{C} .

1. An **Eilenberg–Moore algebra of \mathbb{T}** consists of:

- an object $C \in \mathcal{C}$;
- a morphism $\gamma: TC \rightarrow C$ in \mathcal{C} ,

making the following diagrams commute:

$$\begin{array}{ccc}
 C & \xrightarrow{\eta_C} & TC \\
 & \searrow \text{id}_C & \downarrow \gamma \\
 & & C
 \end{array}
 \quad
 \begin{array}{ccc}
 TTC & \xrightarrow{\mu_C} & TC \\
 \downarrow T\gamma & & \downarrow \gamma \\
 TC & \xrightarrow{\gamma} & C.
 \end{array}$$

2. Let (C, γ) and (C', γ') be Eilenberg–Moore algebras of T . A **homomorphism** from (C, γ) to (C', γ') is a morphism $f: C \rightarrow C'$ in \mathcal{C} making the following diagram commute:

$$\begin{array}{ccc}
 TC & \xrightarrow{Tf} & TC' \\
 \downarrow \gamma & & \downarrow \gamma' \\
 C & \xrightarrow{f} & C'.
 \end{array}$$

The category of all Eilenberg–Moore algebras of T and their homomorphism is called the **Eilenberg–Moore category of T** , and is denoted by \mathcal{C}^T . ■

Excellent introductions to monads abound (see e.g., [71, Chapter VI]). Here, we simply remark that monads typically arise from free constructions. For example, there is a monad T on the category **Set** of (small) sets which maps any (small) set X to the underlying set of the free group generated by X (with the canonical unit and multiplication), and the Eilenberg–Moore algebras of T are precisely groups.

We interpret that for each large category \mathcal{C} , the monads on \mathcal{C} form a single notion of algebraic theory; hence in this section we have actually introduced a family of notions of algebraic theory, one for each large category. This is in contrast to the previous sections (Sections 2.2–2.4), where a single notion of algebraic theory was introduced in each section.

2.6 Generalised operads

Just like monads are a family of notions of algebraic theory parameterised by a large category, the term *generalised operads* [11, 55, 36, 64] also refer to a family of notions of algebraic theory, this time parameterised by a large category with finite limits and a *cartesian monad* thereon. We start with the definition of cartesian monad.

Definition 2.31. 1. Let \mathcal{C} and \mathcal{D} be categories, and $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors. A natural transformation $\alpha: F \rightarrow G$ is called **cartesian** if and only if all naturality squares of α are pullback squares; that is, if and only if for any morphism $f: C \rightarrow C'$ in \mathcal{C} , the square

$$\begin{array}{ccc}
 FC & \xrightarrow{\alpha_C} & GC \\
 \downarrow Ff & & \downarrow Gf \\
 FC' & \xrightarrow{\alpha_{C'}} & GC'
 \end{array}$$

is a pullback of Gf and $\alpha_{C'}$.

2. Let \mathcal{C} be a category with all pullbacks. A monad $\mathbf{S} = (S, \eta, \mu)$ on \mathcal{C} is called **cartesian** if and only if the functor S preserves pullbacks, and η and μ are cartesian. \blacksquare

For each cartesian monad $\mathbf{S} = (S, \eta, \mu)$ on a large category \mathcal{C} with all finite limits we now introduce \mathbf{S} -operads, which form a single notion of algebraic theory.

The crucial observation is that under this assumption, the slice category $\mathcal{C}/S1$ (where 1 is the terminal object of \mathcal{C}) acquires a canonical monoidal structure (see [71, Chapter VII] or [53, Section 1.1] for the definition of monoidal category). We write an object of $\mathcal{C}/S1$ either as $p: P \rightarrow S1$ or (P, p) .

- The unit object is given by $I = (1, \eta_1: 1 \rightarrow S1)$.
- Given a pair of objects $p: P \rightarrow S1$ and $q: Q \rightarrow S1$ in $\mathcal{C}/S1$, first form the pullback

$$\begin{array}{ccc}
 (Q, q) * P & \xrightarrow{\pi_2} & SP \\
 \pi_1 \downarrow & \lrcorner & \downarrow S! \\
 Q & \xrightarrow{q} & S1,
 \end{array} \tag{2.3}$$

where $!: P \rightarrow 1$ is the unique morphism to the terminal object. The monoidal product $(Q, q) \otimes (P, p) \in \mathcal{C}/S1$ is $((Q, q) * P, \mu_1 \circ Sp \circ \pi_2)$:

$$(Q, q) * P \xrightarrow{\pi_2} SP \xrightarrow{Sp} SS1 \xrightarrow{\mu_1} S1.$$

We remark that this monoidal category arises as a restriction of Burroni's bicategory of \mathbf{S} -spans [11].

Definition 2.32. Let \mathcal{C} be a large category with all finite limits and $\mathbf{S} = (S, \eta, \mu)$ a cartesian monad on \mathcal{C} .

1. An **\mathbf{S} -operad** is a monoid object in the monoidal category $(\mathcal{C}/S1, I, \otimes)$ introduced above; see Definition 3.1 for the definition of monoid object in a monoidal category.
2. A **morphism of \mathbf{S} -operads** is a homomorphism of monoid objects; see Definition 3.1 again.

We denote the category of \mathbf{S} -operads by $\mathbf{S}\text{-Opd}$; by definition it is identical to the category $\mathbf{Mon}(\mathcal{C}/S1)$ of monoid objects in $\mathcal{C}/S1$. \blacksquare

We normally write an \mathbf{S} -operad as $\mathbf{T} = ((\text{ar}_T: T \rightarrow S1), e, m)$, where $e: 1 \rightarrow T$ and $m: (T, \text{ar}_T) * T \rightarrow T$ are morphisms in \mathcal{C} . The reason for the notation ar_T is that often the object $S1$ in \mathcal{C} may be interpreted as the object of arities, T as the object of all (derived) operations of the algebraic theory expressed by \mathbf{T} , and ar_T as assigning the arity to each operation. Sometimes we also write an \mathbf{S} -operad simply as $\mathbf{T} = (T, e, m)$, and in this case T refers to an object of $\mathcal{C}/S1$ (rather than \mathcal{C}).

Example 2.33 ([64, Example 4.2.7]). If we let $\mathcal{C} = \mathbf{Set}$ and \mathbf{S} be the free monoid monad (which is cartesian), then \mathbf{S} -operads are equivalent to non-symmetric operads. The arities are the natural numbers: $S1 \cong \mathbb{N}$.

In more detail, the data of an \mathbf{S} -operad in this case consist of a set T , and functions $\text{ar}_T: T \rightarrow \mathbb{N}$, $e: 1 \rightarrow T$ and $m: (T, \text{ar}_T) * T \rightarrow T$. Unravelling this,

we obtain a graded set $(T_n)_{n \in \mathbb{N}}$, an element $\text{id} \in T_1$ and a family of functions $(m_{k,n_1,\dots,n_k} : T_k \times T_{n_1} \times \dots \times T_{n_k} \longrightarrow T_{n_1+\dots+n_k})_{k,n_1,\dots,n_k \in \mathbb{N}}$, agreeing with Definition 2.22. Note that indeed T_n may be interpreted as the set of all (derived) operations of arity n . ■

Example 2.34. If we set $\mathcal{C} = n\text{-Gph}$, the category of n -graphs for $n \in \mathbb{N} \cup \{\omega\}$ and \mathbf{S} be the free strict n -category monad, then \mathbf{S} -operads are called *n -globular operads*; see [64, Chapter 8] for illustrations. These generalised operads have been used to give a definition of weak n -categories, and they (and their generalisations) will play a central role in the second part of this thesis. ■

Next we define models of an \mathbf{S} -operad. For this, we first show that the monoidal category $\mathcal{C}/\mathbf{S1}$ has a canonical pseudo action on \mathcal{C} . Pseudo actions of monoidal categories are a category version of actions of monoids. The precise definition of pseudo action is a variant of Definition 3.23, obtained by replacing the term “natural transformation” there by “natural isomorphism”. The functor

$$*: (\mathcal{C}/\mathbf{S1}) \times \mathcal{C} \longrightarrow \mathcal{C}$$

defining this pseudo action is given by mapping $((Q, q), P) \in (\mathcal{C}/\mathbf{S1}) \times \mathcal{C}$ to $(Q, q) * P \in \mathcal{C}$ defined as the pullback (2.3).

Definition 2.35. Let \mathcal{C} be a large category with finite limits, $\mathbf{S} = (S, \eta, \mu)$ a cartesian monad on \mathcal{C} , and $\mathbf{T} = (T, e, m)$ an \mathbf{S} -operad.

1. A **model of \mathbf{T}** consists of:

- an object $C \in \mathcal{C}$;
- a morphism $\gamma: T * C \longrightarrow C$ in \mathcal{C} ,

making the following diagrams commute:

$$\begin{array}{ccc} I * C & \xrightarrow{e * C} & T * C \\ & \searrow \cong & \downarrow \gamma \\ & & C \end{array} \quad \begin{array}{ccc} (T \otimes T) * C & \xrightarrow{m * C} & T * C \\ \cong \downarrow & & \downarrow \gamma \\ T * (T * C) & \xrightarrow{T * \gamma} & T * C \xrightarrow{\gamma} C, \end{array}$$

where the arrows labelled with \cong refer to the isomorphisms provided by the pseudo action.

2. Let (C, γ) and (C', γ') be models of \mathbf{T} . A **homomorphism from (C, γ) to (C', γ')** is a morphism $f: C \longrightarrow C'$ in \mathcal{C} making the following diagram commute:

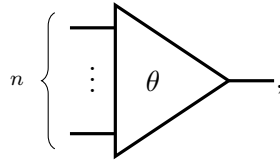
$$\begin{array}{ccc} T * C & \xrightarrow{T * f} & T * C' \\ \gamma \downarrow & & \downarrow \gamma' \\ C & \xrightarrow{f} & C'. \end{array} \quad \blacksquare$$

We remark that in the setting of Example 2.33, the models defined by the above definition coincide with the set models of Definition 2.25.

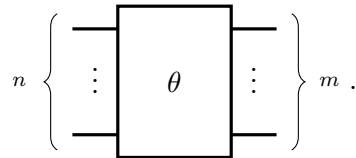
2.7 Other examples

As our principal aim in the first part of this thesis is to develop a formal framework and not to study a variety of concrete examples of notions of algebraic theory in detail, we briefly mention other examples of notions of algebraic theory and conclude the chapter.

First, there are PROPs and PROs [70], which are the “many-in, many-out” versions of symmetric operads and non-symmetric operads respectively. In contrast to operations in a symmetric or non-symmetric operad, which we have drawn in string diagrams as



operations in a PROP or PRO may be drawn as



Another class of examples would be the multi-sorted versions of clones, symmetric and non-symmetric operads, known as *multicategories*. They are included in the work by Curien [18] and Hyland [39].

Finally we mention enriched algebraic theories, such as enriched Lawvere theories [77], the enriched versions of symmetric and non-symmetric operads [73, 56], and enriched monads [20].

We expect that these examples can also be incorporated into our framework without much difficulty (for the enriched algebraic theories, we would have to develop the enriched version of our framework), but will not treat them further in this thesis.

Chapter 3

The framework

In the previous chapter we have seen several examples of notions of algebraic theory, in which the corresponding types of algebraic theories are called under various names, such as *clones*, *non-symmetric operads* and *monads (on \mathcal{C})*. Being a background theory for a type of algebraic theories, each notion of algebraic theory has definitions of algebraic theory, of model of an algebraic theory and of homomorphism between models. Nevertheless, different notions of algebraic theory take different approaches to define these concepts, and the resulting definitions (say, of algebraic theory) can look quite remote.

The aim of this chapter is to provide a *unified framework for notions of algebraic theory* which includes all of the notions of algebraic theory reviewed in the main body of the previous chapter (Sections 2.2–2.6) as instances. To the best of our knowledge, this is the first framework for notions of algebraic theory attaining such generality. Due to the diversity of notions of algebraic theory we aim to capture, we take a very simple approach. The basic idea is that we identify *notions of algebraic theory* with (large) *monoidal categories*, and *algebraic theories* with *monoid objects* therein. We also give definitions of models of an algebraic theory and of their homomorphisms (relative to a notion of model). Further consequences of this framework will be investigated in the subsequent chapters.

In Section 3.1, we motivate our framework by reformulating the notions of algebraic theories reviewed in the previous chapter using the structure of monoidal category. We expect that the contents of this section are mostly known to the specialists, and try to refer to related papers that have come to our attention. The main body of our framework, developed from Section 3.2 on, is our original contribution.

3.1 Prelude: monoidal categorical perspectives on notions of algebraic theory

In this section we motivate our framework by illuminating the key role that certain monoidal categories play in both syntax and semantics of various notions of algebraic theory.

3.1.1 Algebraic theories as monoid objects

We begin with the observation that algebraic theories in each of the notions of algebraic theory reviewed in Chapter 2 may be understood as monoid objects in a certain monoidal category.

See [71, Chapter VII] or [53, Section 1.1] for the definition of monoidal category. We normally write the unit object of a monoidal category as I and the

monoidal product as \otimes . We will denote the coherent structural isomorphisms (obtained from associativity, and left and right unit isomorphisms) by arrows labelled with \cong (see below).

Definition 3.1. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category.

1. A **monoid object in \mathcal{M}** (or simply a **monoid in \mathcal{M}**) is a triple $\mathbb{T} = (T, e, m)$ consisting of an object T in \mathcal{M} , and morphisms $e: I \rightarrow T$ and $m: T \otimes T \rightarrow T$ in \mathcal{M} , such that the following diagrams commute:

$$\begin{array}{ccc}
 I \otimes T & \xrightarrow{e \otimes T} & T \otimes T \\
 \searrow \cong & & \downarrow m \\
 & & T
 \end{array}
 \qquad
 \begin{array}{ccc}
 T \otimes I & \xrightarrow{T \otimes e} & T \otimes T \\
 \searrow \cong & & \downarrow m \\
 & & T
 \end{array}$$

$$\begin{array}{ccc}
 (T \otimes T) \otimes T & \xrightarrow{m \otimes T} & T \otimes T \\
 \downarrow \cong & & \downarrow m \\
 T \otimes (T \otimes T) & \xrightarrow{T \otimes m} & T \otimes T \xrightarrow{m} T
 \end{array}$$

(Recall that the arrows labelled with \cong are the suitable instances of structural isomorphisms of \mathcal{M} .)

2. Let $\mathbb{T} = (T, e, m)$ and $\mathbb{T}' = (T', e', m')$ be monoid objects in \mathcal{M} . A **homomorphism** from \mathbb{T} to \mathbb{T}' is a morphism $f: T \rightarrow T'$ in \mathcal{M} such that the following diagrams commute:

$$\begin{array}{ccc}
 & I & \\
 e \swarrow & & \searrow e' \\
 T & \xrightarrow{f} & T'
 \end{array}
 \qquad
 \begin{array}{ccc}
 T \otimes T & \xrightarrow{f \otimes f} & T' \otimes T' \\
 m \downarrow & & \downarrow m' \\
 T & \xrightarrow{f} & T'
 \end{array}$$

The category of all monoid objects in \mathcal{M} and homomorphisms is denoted by $\mathbf{Mon}(\mathcal{M})$. ■

Clones as monoid objects in $[\mathbf{F}, \mathbf{Set}]$

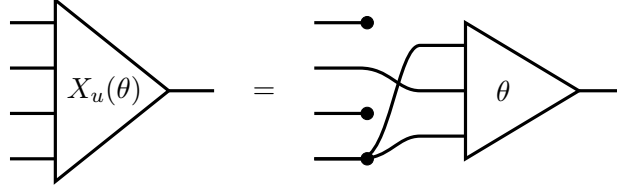
Clones (Definition 2.11) may be identified with monoid objects in a certain monoidal category. We first describe the underlying category.

Definition 3.2. Let \mathbf{F} be the category defined as follows:

- The set of objects is $\text{ob}(\mathbf{F}) = \{[n] \mid n \in \mathbb{N}\}$, where for each natural number $n \in \mathbb{N}$, $[n]$ is defined to be the n -element set $\{1, \dots, n\}$.
- A morphism is any function between these sets. ■

So the category \mathbf{F} is a skeleton of the category \mathbf{FinSet} of all (small) finite sets and functions. The underlying category of the monoidal category for clones is the category $[\mathbf{F}, \mathbf{Set}]$ of all functors from \mathbf{F} to \mathbf{Set} and natural transformations. For $X \in [\mathbf{F}, \mathbf{Set}]$ and $[n] \in \mathbf{F}$, we write the set $X([n])$ as X_n .

We view an object $X \in [\mathbf{F}, \mathbf{Set}]$ as a functional signature, just as we viewed a graded set as a functional signature in Section 2.1. However, objects in $[\mathbf{F}, \mathbf{Set}]$ have richer structure than graded sets, namely the action of morphisms in \mathbf{F} . We can understand this additional structure as certain basic operations on function symbols in the signature. For instance, given a morphism $u: [3] \rightarrow [4]$ in \mathbf{F} with $u(1) = 4, u(2) = 2, u(3) = 4$ and an element $\theta \in X_3$, the element $X_u(\theta) \in X_4$ may be drawn as



(we are using the string diagram notation introduced in Section 2.1). In the symbolic notation,

$$(X_u(\theta))(x_1, x_2, x_3, x_4) = \theta(x_4, x_2, x_4).$$

The monoidal structure on the category $[\mathbf{F}, \mathbf{Set}]$ we shall consider is known as the *substitution monoidal structure*.

Definition 3.3 ([49, 24]). The **substitution monoidal structure** on $[\mathbf{F}, \mathbf{Set}]$ is defined as follows:

- The unit object is $I = \mathbf{F}([1], -): \mathbf{F} \rightarrow \mathbf{Set}$.
- Given $X, Y: \mathbf{F} \rightarrow \mathbf{Set}$, their monoidal product $Y \otimes X: \mathbf{F} \rightarrow \mathbf{Set}$ maps $[n] \in \mathbf{F}$ to

$$(Y \otimes X)_n = \int^{[k] \in \mathbf{F}} Y_k \times (X_n)^k. \quad (3.1)$$

■

The integral sign with a superscript in (3.1) stands for a *coend* (dually, we will denote an *end* by the integral sign with a subscript); see [71, Section IX. 6]. By definition, this coend is a suitable quotient of the set

$$\coprod_{[k] \in \mathbf{F}} Y_k \times (X_n)^k,$$

whose element we may draw as (2.2), assuming $\phi \in Y_k$ and $\theta_1, \dots, \theta_k \in X_n$; the idea is that \otimes performs a “sequential composition” of signatures. Note that symbolically this indeed amounts to a (simultaneous) *substitution*.

We claim that clones are essentially the same as monoids in $[\mathbf{F}, \mathbf{Set}]$ (with respect to the substitution monoidal structure). A monoid in $[\mathbf{F}, \mathbf{Set}]$ consists of

- a functor $T: \mathbf{F} \rightarrow \mathbf{Set}$;
- a natural transformation $e: I \rightarrow T$;
- a natural transformation $m: T \otimes T \rightarrow T$

satisfying the monoid axioms. By the Yoneda lemma, e corresponds to an element $\bar{e} \in T_1$, and by the universality of coends, m corresponds to a natural transformation¹

$$(\bar{m}_{n,k}: T_k \times (T_n)^k \longrightarrow T_n)_{n,k \in \mathbb{N}}.$$

Hence given a monoid (T, e, m) in $[\mathbf{F}, \mathbf{Set}]$, we can construct a clone with the underlying graded set $(T_n)_{n \in \mathbb{N}}$ by setting $p_i^{(n)} = T_{[i]}(\bar{e})$ (here, $[i]: [1] \rightarrow [n]$ is the morphism in \mathbf{F} defined as $[i](1) = i$) and $\circ_k^{(n)} = \bar{m}_{n,k}$. Conversely, given a clone (T, p, \circ) , we can construct a monoid in $[\mathbf{F}, \mathbf{Set}]$ as follows. First we extend the graded set T to a functor $T: \mathbf{F} \rightarrow \mathbf{Set}$ by setting, for any $u: [m] \rightarrow [n]$ in \mathbf{F} ,

$$T_u(\theta) = \theta \circ_m^{(n)} (p_{u(1)}^{(m)}, \dots, p_{u(m)}^{(m)}).$$

Then we may set $\bar{e} = p_1^{(1)}$ and $\bar{m}_{n,k} = \circ_k^{(n)}$.

Proposition 3.4 (cf. [18, 39]). *The above constructions establish an isomorphism of categories between the category of clones and $\mathbf{Mon}([\mathbf{F}, \mathbf{Set}])$.*

Symmetric operads as monoid objects in $[\mathbf{P}, \mathbf{Set}]$

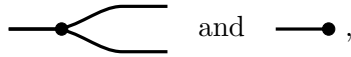
Symmetric operads (Definition 2.28) can be similarly seen as monoids. The main difference from the case of clones is that, instead of the category \mathbf{F} , we use the following category.

Definition 3.5. Let \mathbf{P} be the category defined as follows:

- The set of objects is the same as \mathbf{F} : $\text{ob}(\mathbf{P}) = \{[n] \mid n \in \mathbb{N}\}$ where $[n] = \{1, \dots, n\}$.
- A morphism is any *bijective* function. ■

So \mathbf{P} is the subcategory of \mathbf{F} consisting of all isomorphisms. For any $[n] \in \mathbf{P}$, the monoid $\mathbf{P}([n], [n])$ of endomorphisms on $[n]$ is isomorphic to the symmetric group \mathfrak{S}_n .

Symmetric operads are monoids in a monoidal category whose underlying category is the functor category $[\mathbf{P}, \mathbf{Set}]$. We again interpret $[\mathbf{P}, \mathbf{Set}]$ as a category of functional signatures, but this time a signature $X \in [\mathbf{P}, \mathbf{Set}]$ is only equipped with action of morphisms in \mathbf{P} . In terms of string diagrams, this amounts to restricting the class of diagrams by prohibiting the use of



but not



in terms of symbolic representations, we are restricting Σ -terms to regular Σ -terms.

The monoidal structure on $[\mathbf{P}, \mathbf{Set}]$ we shall use is also called the substitution monoidal structure.

Definition 3.6 ([56]). The **substitution monoidal structure** on $[\mathbf{P}, \mathbf{Set}]$ is defined as follows:

¹In more detail, the relevant naturality here may also be phrased as “natural in $[n] \in \mathbf{F}$ and *extranatural* in $[k] \in \mathbf{F}$ ”; see [71, Section IX. 4]. Following [53], in this thesis we shall not distinguish (terminologically) extranaturality from naturality, using the latter term for both.

- The unit object is $I = \mathbf{P}([1], -): \mathbf{P} \rightarrow \mathbf{Set}$.
- Given $X, Y: \mathbf{P} \rightarrow \mathbf{Set}$, their monoidal product $Y \otimes X: \mathbf{P} \rightarrow \mathbf{Set}$ maps $[n] \in \mathbf{P}$ to

$$(Y \otimes X)_n = \int^{[k] \in \mathbf{P}} Y_k \times (X^{\otimes k})_n,$$

where

$$(X^{\otimes k})_n = \int^{[n_1], \dots, [n_k] \in \mathbf{P}} \mathbf{P}([n_1 + \dots + n_k], [n]) \times X_{n_1} \times \dots \times X_{n_k}. \quad \blacksquare$$

Proposition 3.7 (cf. [18, 39]). *The category of symmetric operads is isomorphic to $\mathbf{Mon}([\mathbf{P}, \mathbf{Set}])$.*

Non-symmetric operads as monoid objects in $[\mathbf{N}, \mathbf{Set}]$

For non-symmetric operads (Definition 2.22), we use the following category.

Definition 3.8. Let \mathbf{N} be the category defined as follows:

- The set of objects is the same as \mathbf{F} and \mathbf{P} .
- There are only identity morphisms in \mathbf{N} . ■

\mathbf{N} is the discrete category with the same objects as \mathbf{F} and \mathbf{P} .

We consider the functor category $[\mathbf{N}, \mathbf{Set}]$, which is nothing but the category of graded sets and their morphisms (Definition 2.1).

Definition 3.9. The **substitution monoidal structure** on $[\mathbf{N}, \mathbf{Set}]$ is defined as follows:

- The unit object is $I = \mathbf{N}([1], -): \mathbf{N} \rightarrow \mathbf{Set}$.
- Given $X, Y: \mathbf{N} \rightarrow \mathbf{Set}$, their monoidal product $Y \otimes X: \mathbf{N} \rightarrow \mathbf{Set}$ maps $[n] \in \mathbf{N}$ to

$$(Y \otimes X)_n = \coprod_{[k] \in \mathbf{N}} Y_k \times \left(\coprod_{\substack{[n_1], \dots, [n_k] \in \mathbf{N} \\ n_1 + \dots + n_k = n}} X_{n_1} \times \dots \times X_{n_k} \right). \quad \blacksquare$$

Proposition 3.10 (cf. [18, 39]). *The category of non-symmetric operads is isomorphic to $\mathbf{Mon}([\mathbf{N}, \mathbf{Set}])$.*

For unified studies of various substitution monoidal structures, see [83, 23], as well as the aforementioned [18, 39].

Monads on \mathcal{C} as monoid objects in $[\mathcal{C}, \mathcal{C}]$

Monads on a large category \mathcal{C} (Definition 2.29) are also monoid objects, this time rather immediately from the definition.

Definition 3.11. Let \mathcal{C} be a large category. Define the monoidal category $[\mathcal{C}, \mathcal{C}] = ([\mathcal{C}, \mathcal{C}], \text{id}_{\mathcal{C}}, \circ)$ of endofunctors on \mathcal{C} as follows:

- The underlying category is the category $[\mathcal{C}, \mathcal{C}]$ of all functors $\mathcal{C} \rightarrow \mathcal{C}$ and natural transformations.
- The unit object is the identity functor $\text{id}_{\mathcal{C}}$ on \mathcal{C} .
- The monoidal product is given by composition of functors. ■

The category $\mathbf{Mnd}(\mathcal{C})$ of monads on \mathcal{C} is clearly identical to $\mathbf{Mon}([\mathcal{C}, \mathcal{C}])$.

S-operads as monoid objects in $\mathcal{C}/S1$

Finally, we recall that generalised operads (S-operads for a cartesian monad S on a large category \mathcal{C} with finite limits; Definition 2.32) were introduced as monoid objects in the first place.

3.1.2 Notions of model as enrichments

In this section and next, we shall rephrase definitions of *model* of an algebraic theory via monoidal categorical structures.

We start with a discussion on *notions of model*. An important feature of several notions of algebraic theory—most notably clones, symmetric operads and non-symmetric operads—is that we may consider models of an algebraic theory in more than one category. For example, it is known that models of a clone can be taken in any category with finite products [58] (or even with finite powers). We may phrase this fact by saying that clones admit many notions of model, one for each category with finite products.

Informally, a *notion of model* for a notion of algebraic theory is a definition of model of an algebraic theory in that notion of algebraic theory. Hence whenever we consider actual *models* of an algebraic theory, we must specify in advance a notion of model with respect to which the models are taken. Our framework emphasises the inevitable fact that *models are always relative to notions of model*, by treating notions of model as independent mathematical structures.

But how can we formalise such notions of model? Below we show that the standard notions of model for clones, symmetric operads and non-symmetric operads can be captured by a categorical structure which we call *enrichment*. Recall that we identify notions of algebraic theory with large monoidal categories.

Definition 3.12. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category. An **enrichment over \mathcal{M}** consists of:

- a large category \mathcal{C} ;
- a functor $\langle -, - \rangle: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{M}$;
- a natural transformation $(j_C: I \rightarrow \langle C, C \rangle)_{C \in \mathcal{C}}$;
- a natural transformation $(M_{A,B,C}: \langle B, C \rangle \otimes \langle A, B \rangle \rightarrow \langle A, C \rangle)_{A,B,C \in \mathcal{C}}$,

making the following diagrams commute for all $A, B, C, D \in \mathcal{C}$:

$$\begin{array}{ccc}
 I \otimes \langle A, B \rangle & \xrightarrow{j_B \otimes \langle A, B \rangle} & \langle B, B \rangle \otimes \langle A, B \rangle & \quad & \langle A, B \rangle \otimes I & \xrightarrow{\langle A, B \rangle \otimes j_A} & \langle A, B \rangle \otimes \langle A, A \rangle \\
 & \searrow \cong & \downarrow M_{A,B,B} & & \searrow \cong & & \downarrow M_{A,A,B} \\
 & & \langle A, B \rangle & & & & \langle A, B \rangle
 \end{array}$$

$$\begin{array}{ccc}
 (\langle C, D \rangle \otimes \langle B, C \rangle) \otimes \langle A, B \rangle & \xrightarrow{M_{B,C,D} \otimes \langle A, B \rangle} & \langle B, D \rangle \otimes \langle A, B \rangle \\
 \downarrow \cong & & \downarrow M_{A,B,D} \\
 \langle C, D \rangle \otimes (\langle B, C \rangle \otimes \langle A, B \rangle) & \xrightarrow{\langle C, D \rangle \otimes M_{A,B,C}} & \langle C, D \rangle \otimes \langle A, C \rangle & \xrightarrow{M_{A,C,D}} & \langle A, D \rangle.
 \end{array}$$

We say that $(\mathcal{C}, \langle -, - \rangle, j, M)$ is an enrichment over \mathcal{M} , or that $(\langle -, - \rangle, j, M)$ is an enrichment of \mathcal{C} over \mathcal{M} . ■

An enrichment over \mathcal{M} is not the same as a (large) \mathcal{M} -category in enriched category theory [53]. It is rather a triple consisting of a large category \mathcal{C} , a large \mathcal{M} -category \mathcal{D} , and an identity-on-objects functor $J: \mathcal{C} \rightarrow \mathcal{D}_0$, where \mathcal{D}_0 is the underlying category of \mathcal{D} .

In detail, given an enrichment $(\langle -, - \rangle, j, M)$ of \mathcal{C} in \mathcal{M} , we may define the \mathcal{M} -category \mathcal{D} with $\text{ob}(\mathcal{D}) = \text{ob}(\mathcal{C})$ using the data $(\langle -, - \rangle, j, M)$ of the enrichment (that is, $\mathcal{D}(A, B) = \langle A, B \rangle$ and so on). The identity-on-objects functor $J: \mathcal{C} \rightarrow \mathcal{D}_0$ may be defined by mapping a morphism $f: A \rightarrow B$ in \mathcal{C} to the composite $\langle A, f \rangle \circ j_A$, or equivalently, $\langle f, B \rangle \circ j_B$:

$$\begin{array}{ccc} I & \xrightarrow{j_B} & \langle B, B \rangle \\ j_A \downarrow & & \downarrow \langle f, B \rangle \\ \langle A, A \rangle & \xrightarrow{\langle A, f \rangle} & \langle A, B \rangle. \end{array}$$

We say that an enrichment is **normal** if the corresponding identity-on-objects functor J is an isomorphism of categories. We shall return to the relationship to enriched category theory at the end of this section.

From an enrichment, we now derive a definition of model of an algebraic theory. First observe that, given an enrichment $\langle -, - \rangle = (\langle -, - \rangle, j, M)$ of a large category \mathcal{C} over a large monoidal category \mathcal{M} and an object $C \in \mathcal{C}$, we have a monoid object $\text{End}_{\langle -, - \rangle}(C) = (\langle C, C \rangle, j_C, M_{C,C,C})$ in \mathcal{M} ; that these data define a monoid object may be seen immediately from Definition 3.12. Because we identify algebraic theories with monoid objects, we give a definition of model of a monoid object \mathbb{T} in \mathcal{M} .

Definition 3.13. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category, $\mathbb{T} = (T, e, m)$ be a monoid object in \mathcal{M} , \mathcal{C} be a large category, and $\langle -, - \rangle = (\langle -, - \rangle, j, M)$ be an enrichment of \mathcal{C} over \mathcal{M} .

1. A **model of \mathbb{T} in \mathcal{C} with respect to $\langle -, - \rangle$** is a pair (C, χ) consisting of an object C of \mathcal{C} and a monoid homomorphism $\chi: \mathbb{T} \rightarrow \text{End}_{\langle -, - \rangle}(C)$; that is, a morphism $\chi: T \rightarrow \langle C, C \rangle$ in \mathcal{M} making the following diagrams commute:

$$\begin{array}{ccc} I & \xrightarrow{e} & T \\ j_C \searrow & & \downarrow \chi \\ & & \langle C, C \rangle \end{array} \quad \begin{array}{ccc} T \otimes T & \xrightarrow{m} & T \\ \chi \otimes \chi \downarrow & & \downarrow \chi \\ \langle C, C \rangle \otimes \langle C, C \rangle & \xrightarrow{M_{C,C,C}} & \langle C, C \rangle. \end{array}$$

2. Let (C, χ) and (C', χ') be models of \mathbb{T} in \mathcal{C} with respect to $\langle -, - \rangle$. A **homomorphism from (C, χ) to (C', χ')** is a morphism $f: C \rightarrow C'$ in \mathcal{C} making the following diagram commute:

$$\begin{array}{ccc} T & \xrightarrow{\chi'} & \langle C', C' \rangle \\ \chi \downarrow & & \downarrow \langle f, C' \rangle \\ \langle C, C \rangle & \xrightarrow{\langle C, f \rangle} & \langle C, C' \rangle. \end{array}$$

We denote the (large) category of models of \mathbb{T} in \mathcal{C} with respect to $\langle -, - \rangle$ by $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \langle -, - \rangle))$. ■

The above definitions of model and homomorphism are reminiscent of ones for clones (Definitions 2.14 and 2.16), symmetric operads and non-symmetric operads (Definition 2.25). Indeed, we can restore the standard notions of model for these notions of algebraic theory via suitable enrichments.

Example 3.14. Recall that clones may be identified with monoids in $[\mathbf{F}, \mathbf{Set}]$ with the substitution monoidal structure. Let \mathcal{C} be a locally small² category with all finite powers. We have an enrichment of \mathcal{C} over $[\mathbf{F}, \mathbf{Set}]$ defined as follows:

- The functor $\langle -, - \rangle: \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow [\mathbf{F}, \mathbf{Set}]$ maps $A, B \in \mathcal{C}$ and $[n] \in \mathbf{F}$ to the set

$$\langle A, B \rangle_n = \mathcal{C}(A^n, B).$$

- The natural transformation $(j_C: I \longrightarrow \langle C, C \rangle)_{C \in \mathcal{C}}$ corresponds by the Yoneda lemma (recall that $I = \mathbf{F}([1], -)$) to the family

$$(\bar{j}_C = \text{id}_C \in \langle C, C \rangle_1)_{C \in \mathcal{C}}.$$

- The natural transformation $(M_{A,B,C}: \langle B, C \rangle \otimes \langle A, B \rangle \longrightarrow \langle A, C \rangle)_{A,B,C \in \mathcal{C}}$ corresponds by the universality of coends (recall that $(Y \otimes X)_n = \int^{[k] \in \mathbf{F}} Y_k \times (X_n)^k$) to the family whose (A, B, C) -th component is given by

$$(\bar{M}_{A,B,C})_{n,k}: \langle B, C \rangle_k \times (\langle A, B \rangle_n)^k \longrightarrow \langle A, B \rangle_n$$

mapping $(g, f_1, \dots, f_k) \in \mathcal{C}(B^k, C) \times \mathcal{C}(A^n, B)^k$ to $g \circ \langle f_1, \dots, f_k \rangle \in \mathcal{C}(A^n, B)$.

Clearly the definition of the clone $\mathbf{End}(A)$ from a set A (Definition 2.12) is derived from the above enrichment, by setting $\mathcal{C} = \mathbf{Set}$. Consequently, we restore the classical definitions of model (Definition 2.14) and homomorphism between models (Definition 2.16) for clones as instances of Definition 3.13. ■

Example 3.15. Symmetric operads may be identified with monoids in $[\mathbf{P}, \mathbf{Set}]$ with the substitution monoidal structure. Let $\mathcal{C} = (\mathcal{C}, I', \otimes')$ be a locally small symmetric monoidal category. We have an enrichment of \mathcal{C} over $[\mathbf{P}, \mathbf{Set}]$ defined as follows:

- The functor $\langle -, - \rangle: \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow [\mathbf{P}, \mathbf{Set}]$ maps $A, B \in \mathcal{C}$ and $[n] \in \mathbf{P}$ to the set

$$\langle A, B \rangle_n = \mathcal{C}(A^{\otimes' n}, B),$$

where $A^{\otimes' n}$ is the monoidal product of n many copies of A .

- The natural transformation $(j_C: I \longrightarrow \langle C, C \rangle)_{C \in \mathcal{C}}$ corresponds by the Yoneda lemma (recall that $I = \mathbf{P}([1], -)$) to the family

$$(\bar{j}_C = \text{id}_C \in \langle C, C \rangle_1)_{C \in \mathcal{C}}.$$

- The natural transformation $(M_{A,B,C}: \langle B, C \rangle \otimes \langle A, B \rangle \longrightarrow \langle A, C \rangle)_{A,B,C \in \mathcal{C}}$ corresponds by the universality of coends (recall Definition 3.6) to the family whose (A, B, C) -th component is given by

$$\begin{aligned} (\bar{M}_{A,B,C})_{n,k,n_1,\dots,n_k}: \langle B, C \rangle_k \times \mathbf{P}([n_1 + \dots + n_k], [n]) \\ \times \langle A, B \rangle_{n_1} \times \dots \times \langle A, B \rangle_{n_k} \longrightarrow \langle A, B \rangle_n, \end{aligned}$$

²Recall that by Convention 1.3, “locally small” implies “large”.

which is the unique function from the empty set if $n \neq n_1 + \dots + n_k$ and, if $n = n_1 + \dots + n_k$, maps $(g, u, f_1, \dots, f_k) \in \mathcal{C}(B^{\otimes' k}, C) \times \mathbf{P}([n_1 + \dots + n_k], [n]) \times \mathcal{C}(A^{\otimes' n_1}, B) \times \dots \times \mathcal{C}(A^{\otimes' n_k}, B)$ to $g \circ (f_1 \otimes' \dots \otimes' f_k) \circ A^{\otimes' u}$.

Via the above enrichment, we restore the classical definitions of model and homomorphism between models for symmetric operads. ■

Example 3.16. Non-symmetric operads may be identified with monoids in $[\mathbf{N}, \mathbf{Set}]$ with the substitution monoidal structure. Let $\mathcal{C} = (\mathcal{C}, I', \otimes')$ be a locally small monoidal category. We have an enrichment of \mathcal{C} over $[\mathbf{N}, \mathbf{Set}]$ which is similar to, and simpler than, the one in the previous example.

This enrichment restores the classical definitions of model and homomorphism between models for non-symmetric operads, including Definition 2.25 (take $\mathcal{C} = (\mathbf{Set}, 1, \times)$ for set models and $\mathcal{C} = (\mathbf{Ab}, \mathbb{Z}, \otimes)$ for abelian group models). ■

Example 3.17. We may also consider infinitary variants of Example 3.14. Here we take an extreme. Let \mathcal{C} be a locally small category with all small powers. Then we obtain an enrichment of \mathcal{C} over $[\mathbf{Set}, \mathbf{Set}]$, the category of endofunctors on \mathbf{Set} with composition as the monoidal product.

- The functor $\langle -, - \rangle: \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$ maps $A, B \in \mathcal{C}$ and $X \in \mathbf{Set}$ to the set

$$\langle A, B \rangle(X) = \mathcal{C}(A^X, B),$$

where A^X is the X -th power of A .

- The natural transformation $(j_C: \text{id}_{\mathbf{Set}} \longrightarrow \langle C, C \rangle)_{C \in \mathcal{C}}$ corresponds by the Yoneda lemma (note that $\text{id}_{\mathbf{Set}} \cong \mathbf{Set}(1, -)$, where 1 is a singleton) to the family

$$(\bar{j}_C = \text{id}_C \in \langle C, C \rangle(1))_{C \in \mathcal{C}}.$$

- The natural transformation $(M_{A,B,C}: \langle B, C \rangle \circ \langle A, B \rangle \longrightarrow \langle A, C \rangle)_{A,B,C \in \mathcal{C}}$ has the X -th component ($X \in \mathbf{Set}$)

$$\langle B, C \rangle \circ \langle A, B \rangle(X) = \mathcal{C}(B^{\mathcal{C}(A^X, B)}, C) \longrightarrow \mathcal{C}(A^X, C) = \langle A, C \rangle(X)$$

the function induced from the canonical morphism $A^X \longrightarrow B^{\mathcal{C}(A^X, B)}$ in \mathcal{C} .

Since monoids in $[\mathbf{Set}, \mathbf{Set}]$ are precisely monads on \mathbf{Set} , this enrichment gives us a definition of model of a monad \mathbf{T} on \mathbf{Set} in \mathcal{C} . To spell this out, first note that for any object $C \in \mathcal{C}$, the functor $\langle C, C \rangle: \mathbf{Set} \longrightarrow \mathbf{Set}$ which maps $X \in \mathbf{Set}$ to $\mathcal{C}(C^X, C)$ acquires the monad structure, giving rise to the monad $\text{End}_{\langle -, - \rangle}(C)$ on \mathbf{Set} . A model of \mathbf{T} is then an object $C \in \mathcal{C}$ together with a monad morphism $\mathbf{T} \longrightarrow \text{End}_{\langle -, - \rangle}(C)$. This is the definition of *relative algebra* of a monad on \mathbf{Set} by Hino, Kobayashi, Hasuo and Jacobs [37]. As noted in [37], in the case where $\mathcal{C} = \mathbf{Set}$, relative algebras of a monad \mathbf{T} on \mathbf{Set} agree with Eilenberg–Moore algebras of \mathbf{T} ; we shall later show this fact in Example 3.30. ■

Example 3.18. Let \mathcal{S} be a large category and consider the monoidal category $[\mathcal{S}, \mathcal{S}]$ of endofunctors on \mathcal{S} , with composition as the monoidal product. Then an enrichment over $[\mathcal{S}, \mathcal{S}]$ is the same thing as an \mathcal{S} -parameterised monad (without strength) in the sense of Atkey [2, Definition 1], introduced in the study of computational effects. ■

Having reformulated semantics of notions of algebraic theory in terms of enrichments, let us investigate some of its immediate consequences.

$\mathbf{Mod}(-, -)$ as a 2-functor

It is well-known that given clones \mathbb{T} and \mathbb{T}' , a clone homomorphism $f: \mathbb{T} \rightarrow \mathbb{T}'$, and a locally small category \mathcal{C} with finite products, we have the induced functor

$$\mathbf{Mod}(f, \mathcal{C}): \mathbf{Mod}(\mathbb{T}', \mathcal{C}) \rightarrow \mathbf{Mod}(\mathbb{T}, \mathcal{C})$$

between the categories of models. For instance, we can take \mathbb{T} to be the clone for monoids and \mathbb{T}' to be the clone for groups, with $f: \mathbb{T} \rightarrow \mathbb{T}'$ the canonical clone map easily obtained from the standard presentations of monoids and of groups. Then $\mathbf{Mod}(f, \mathcal{C})$ is the natural embedding of the category of group objects in \mathcal{C} to the category of monoid objects in \mathcal{C} ; in particular, if we let $\mathcal{C} = \mathbf{Set}$, we get the embedding of the category of groups into the category of monoids (in words, “groups are a special case of monoids”).

On the other hand, given a clone \mathbb{T} , locally small categories \mathcal{C} and \mathcal{C}' with finite products, and a functor $G: \mathcal{C} \rightarrow \mathcal{C}'$ preserving finite products, we obtain a functor

$$\mathbf{Mod}(\mathbb{T}, G): \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \rightarrow \mathbf{Mod}(\mathbb{T}, \mathcal{C}').$$

As a concrete example, let \mathbb{T} be the clone for groups, $\mathcal{C} = \mathbf{Top}$ (the category of topological spaces), $\mathcal{C}' = \mathbf{Set}$ and $G: \mathbf{Top} \rightarrow \mathbf{Set}$ be the functor mapping a topological space to its underlying set. Then we obtain a functor from the category of topological groups to the category of groups, which simply forgets the topology.

In order to formulate such functoriality of $\mathbf{Mod}(-, -)$, we introduce a 2-category of enrichments.

Definition 3.19. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category. The (locally large) 2-category $\mathcal{Enrich}(\mathcal{M})$ of enrichments over \mathcal{M} is defined as follows:

- An object is an enrichment $(\mathcal{C}, \langle -, - \rangle, j, M)$ over \mathcal{M} .
- A 1-cell from $(\mathcal{C}, \langle -, - \rangle, j, M)$ to $(\mathcal{C}', \langle -, - \rangle', j', M')$ is a functor $G: \mathcal{C} \rightarrow \mathcal{C}'$ together with a natural transformation $(g_{A,B}: \langle A, B \rangle \rightarrow \langle GA, GB \rangle')_{A,B \in \mathcal{C}}$ making the following diagrams commute for all $A, B, C \in \mathcal{C}$:

$$\begin{array}{ccc}
 I & \xrightarrow{j_{\mathcal{C}}} & \langle \mathcal{C}, \mathcal{C} \rangle \\
 & \searrow^{j'_{GC}} & \downarrow^{g_{\mathcal{C}, \mathcal{C}}} \\
 & & \langle GC, GC \rangle'
 \end{array}$$

$$\begin{array}{ccc}
 \langle B, C \rangle \otimes \langle A, B \rangle & \xrightarrow{M_{A,B,C}} & \langle A, C \rangle \\
 \downarrow^{g_{B,C} \otimes g_{A,B}} & & \downarrow^{g_{A,C}} \\
 \langle GB, GC \rangle' \otimes \langle GA, GB \rangle' & \xrightarrow{M'_{GA,GB,GC}} & \langle GA, GC \rangle'.
 \end{array}$$

- A 2-cell from (G, g) to (G', g') , both from $(\mathcal{C}, \langle -, - \rangle, j, M)$ to $(\mathcal{C}', \langle -, - \rangle', j', M')$, is a natural transformation $\theta: G \Rightarrow G'$ making the following diagram

commute for all $A, B \in \mathcal{C}$:

$$\begin{array}{ccc}
 \langle A, B \rangle & \xrightarrow{g_{A,B}} & \langle GA, GB \rangle' \\
 g'_{A,B} \downarrow & & \downarrow \langle GA, \theta_B \rangle' \\
 \langle G'A, G'B \rangle' & \xrightarrow{\langle \theta_A, G'B \rangle'} & \langle GA, G'B \rangle'.
 \end{array}$$

■

Example 3.20. Let $\mathcal{F}Pow$ be the 2-category of locally small categories with chosen finite powers, functors preserving finite powers (in the usual sense³) and all natural transformations. We have a canonical 2-functor

$$\mathcal{F}Pow \longrightarrow \mathit{Enrich}([\mathbf{F}, \mathbf{Set}])$$

which is fully faithful (see Section 1.5 for the definition of full faithfulness for 2-functors).

Let $\mathcal{F}Prod$ be the 2-category of locally small categories with chosen finite products, functors preserving finite products (in the usual sense) and all natural transformations. We have a canonical 2-functor

$$\mathcal{F}Prod \longrightarrow \mathit{Enrich}([\mathbf{F}, \mathbf{Set}])$$

which is locally fully faithful.

Hence we may restore the classical functoriality of $\mathbf{Mod}(\mathbb{T}, -)$ for a clone \mathbb{T} , recalled above, if we could show that it is functorial with respect to morphisms in $\mathit{Enrich}([\mathbf{F}, \mathbf{Set}])$. ■

We also have canonical (locally faithful) 2-functors

$$\mathit{Sym}.\mathit{Mon}\mathcal{CAT}_{\text{lax}}^{\text{ls}} \longrightarrow \mathit{Enrich}([\mathbf{P}, \mathbf{Set}]),$$

where the domain is the 2-category of locally small symmetric monoidal categories, symmetric lax monoidal functors and monoidal natural transformations, and

$$\mathit{Mon}\mathcal{CAT}_{\text{lax}}^{\text{ls}} \longrightarrow \mathit{Enrich}([\mathbf{N}, \mathbf{Set}]),$$

where the domain is the 2-category of locally small monoidal categories, lax monoidal functors and monoidal natural transformations.

Now the functoriality of $\mathbf{Mod}(-, -)$ may be expressed by saying that it is a 2-functor

$$\mathbf{Mod}(-, -): \mathbf{Mon}(\mathcal{M})^{\text{op}} \times \mathit{Enrich}(\mathcal{M}) \longrightarrow \mathcal{CAT} \quad (3.2)$$

(when we say that (3.2) is a 2-functor, we are identifying the category $\mathbf{Mon}(\mathcal{M})$ with the corresponding locally discrete 2-category). Actually, the 2-functor (3.2) arises immediately from the structure of the locally large 2-category $\mathit{Enrich}(\mathcal{M})$. Observe that we may identify a monoid object in \mathcal{M} with an enrichment of the terminal category 1 over \mathcal{M} . The full sub-2-category of $\mathit{Enrich}(\mathcal{M})$ consisting of all enrichments over the (fixed) terminal category 1 is in fact locally discrete, and is isomorphic to $\mathbf{Mon}(\mathcal{M})$. This way we obtain a fully faithful inclusion 2-functor

³That is, we do not require these functors to preserve the chosen finite powers on the nose.

$\mathbf{Mon}(\mathcal{M}) \longrightarrow \mathcal{E}nrich(\mathcal{M})$. It is straightforward to see that the appropriate 2-functor (3.2) is given by the composite

$$\begin{array}{c} \mathbf{Mon}(\mathcal{M})^{\text{op}} \times \mathcal{E}nrich(\mathcal{M}) \\ \downarrow \text{inclusion} \\ \mathcal{E}nrich(\mathcal{M})^{\text{op}} \times \mathcal{E}nrich(\mathcal{M}) \\ \downarrow \mathcal{E}nrich(\mathcal{M})(-, -) \\ \mathcal{CAT}, \end{array}$$

where $\mathcal{E}nrich(\mathcal{M})(-, -)$ is the hom-2-functor for $\mathcal{E}nrich(\mathcal{M})$.

Comparing different notions of algebraic theory

So far we have been working within a fixed notion of algebraic theory. We now turn to the question of comparing different notions of algebraic theory.

By way of illustration, let us consider the relationship of clones, symmetric operads and non-symmetric operads. On the ‘‘syntactical’’ side, we have inclusions of algebraic theories

$$\{\text{non-sym. operads}\} \subseteq \{\text{sym. operads}\} \subseteq \{\text{clones}\}, \quad (3.3)$$

in the sense that every symmetric operad may be derived from a regular presentation of an equational theory, which is at the same time a presentation of an equational theory and therefore defines a clone, etc. On the ‘‘semantical’’ side, in contrast, we have inclusions of (standard) notions of models in the other direction, namely:

$$\{\text{mon. cat.}\} \supseteq \{\text{sym. mon. cat.}\} \supseteq \{\text{cat. with fin. prod.}\}. \quad (3.4)$$

Furthermore, suppose we take the algebraic theory \mathbb{T} of monoids (which is expressible as a non-symmetric operad) and the category \mathbf{Set} (which has finite products). Then we can consider the category of models $\mathbf{Mod}(\mathbb{T}, \mathbf{Set})$ in three different ways: either thinking of \mathbb{T} as a clone and \mathbf{Set} as a category with finite products, \mathbb{T} as a symmetric operad and \mathbf{Set} as a symmetric monoidal category, or \mathbb{T} as a non-symmetric operad and \mathbf{Set} as a monoidal category. It turns out that the resulting three categories of models are isomorphic to each other, indicating certain compatibility between the three notions of algebraic theory.

The key to understand these phenomena in our framework is the functoriality of the $\mathcal{E}nrich(-)$ construction. That is, we may extend (just like base change of enriched categories) $\mathcal{E}nrich(-)$ to a 2-functor

$$\mathcal{E}nrich(-): \mathbf{MonCAT}_{\text{lax}} \longrightarrow \underline{\mathbf{2-CAT}} \quad (3.5)$$

from the 2-category $\mathbf{MonCAT}_{\text{lax}}$ of large monoidal categories, lax monoidal functors and monoidal natural transformations to the 2-category $\underline{\mathbf{2-CAT}}$ of huge 2-categories, 2-functors and 2-natural transformations. We just describe the action of a lax monoidal functor on an enrichment, as the rest of the data for the 2-functor (3.5) follows from that rather routinely.

Definition 3.21. Let $\mathcal{M} = (\mathcal{M}, I_{\mathcal{M}}, \otimes_{\mathcal{M}})$ and $\mathcal{N} = (\mathcal{N}, I_{\mathcal{N}}, \otimes_{\mathcal{N}})$ be large monoidal categories, $F = (F, f., f): \mathcal{M} \longrightarrow \mathcal{N}$ be a lax monoidal functor,⁴ \mathcal{C}

⁴In more detail, such a lax monoidal functor (also called monoidal functors in e.g., [71]) $(F, f., f)$ consists of a functor $F: \mathcal{M} \longrightarrow \mathcal{N}$, a morphism $f: I_{\mathcal{N}} \longrightarrow FI_{\mathcal{M}}$ and a natural transformation $(f_{X,Y}: FY \otimes_{\mathcal{N}} FX \longrightarrow F(Y \otimes_{\mathcal{M}} X))_{X,Y \in \mathcal{M}}$ satisfying the suitable axioms.

be a large category and $\langle -, - \rangle = (\langle -, - \rangle, j, M)$ be an enrichment of \mathcal{C} over \mathcal{M} . We define the enrichment $F_*(\langle -, - \rangle) = (\langle -, - \rangle', j', M')$ of \mathcal{C} over \mathcal{N} as follows:

- The functor $\langle -, - \rangle': \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{N}$ maps $(A, B) \in \mathcal{C}^{\text{op}} \times \mathcal{C}$ to $F\langle A, B \rangle$.
- The natural transformation $(j'_C: I_{\mathcal{N}} \rightarrow \langle C, C \rangle')_{C \in \mathcal{C}}$ is defined by $j'_C = Fj_C \circ f$:

$$I_{\mathcal{N}} \xrightarrow{f} FI_{\mathcal{M}} \xrightarrow{Fj_C} F\langle C, C \rangle.$$

- The natural transformation $(M'_{A,B,C}: \langle B, C \rangle' \otimes_{\mathcal{N}} \langle A, B \rangle' \rightarrow \langle A, C \rangle')_{A,B,C \in \mathcal{C}}$ is defined by $M'_{A,B,C} = FM_{A,B,C} \circ f_{\langle A, B \rangle, \langle B, C \rangle}$:

$$F\langle B, C \rangle \otimes_{\mathcal{N}} F\langle A, B \rangle \xrightarrow{f_{\langle A, B \rangle, \langle B, C \rangle}} F(\langle B, C \rangle \otimes_{\mathcal{M}} \langle A, B \rangle) \xrightarrow{FM_{A,B,C}} F\langle A, C \rangle. \quad \blacksquare$$

As an immediate consequence of the 2-functoriality (3.5), it follows that whenever we have a monoidal adjunction (adjunction in $\text{Mon}\mathcal{CAT}_{\text{lax}}$)

$$\mathcal{M} \begin{array}{c} \xrightarrow{L} \\ \perp \\ \xleftarrow{R} \end{array} \mathcal{N},$$

we obtain a 2-adjunction

$$\mathcal{E}nrich(\mathcal{M}) \begin{array}{c} \xrightarrow{\mathcal{E}nrich(L)} \\ \perp \\ \xleftarrow{\mathcal{E}nrich(R)} \end{array} \mathcal{E}nrich(\mathcal{N}).$$

Therefore, if we take $\mathbb{T} \in \mathbf{Mon}(\mathcal{M}) \subseteq \mathcal{E}nrich(\mathcal{M})$ and $(\mathcal{C}, \langle -, - \rangle) \in \mathcal{E}nrich(\mathcal{N})$ in this situation, then

$$\begin{aligned} \mathcal{E}nrich(\mathcal{M})(\mathbb{T}, \mathcal{E}nrich(R)(\mathcal{C}, \langle -, - \rangle)) \\ \cong \mathcal{E}nrich(\mathcal{N})(\mathcal{E}nrich(L)(\mathbb{T}), (\mathcal{C}, \langle -, - \rangle)). \end{aligned} \quad (3.6)$$

Since the action of $\mathcal{E}nrich(-)$ preserves the underlying categories, we may assume $\mathcal{E}nrich(L)(\mathbb{T}) \in \mathbf{Mon}(\mathcal{N})$. Therefore (3.6) may be seen as an isomorphism between the category of models of \mathbb{T} in \mathcal{C} with respect to $R_*(\langle -, - \rangle)$ and the category of models of $\mathcal{E}nrich(L)(\mathbb{T})$ in \mathcal{C} with respect to $\langle -, - \rangle$.

The relationship between clones, symmetric operads and non-symmetric operads mentioned above can be explained in this way. First note that there is a chain of inclusions

$$\mathbf{N} \xrightarrow{J} \mathbf{P} \xrightarrow{J'} \mathbf{F}.$$

Therefore, precomposition and left Kan extensions induce a chain of adjunctions

$$[\mathbf{N}, \mathbf{Set}] \begin{array}{c} \xrightarrow{\text{Lan}_J} \\ \perp \\ \xleftarrow{[J, \mathbf{Set}]} \end{array} [\mathbf{P}, \mathbf{Set}] \begin{array}{c} \xrightarrow{\text{Lan}_{J'}} \\ \perp \\ \xleftarrow{[J', \mathbf{Set}]} \end{array} [\mathbf{F}, \mathbf{Set}].$$

It turns out that these adjunctions acquire natural structures of monoidal adjunctions. Hence in our framework, the inclusions (3.3) are expressed as the functors

$$\mathbf{Mon}([\mathbf{N}, \mathbf{Set}]) \xrightarrow{\mathbf{Mon}(\text{Lan}_J)} \mathbf{Mon}([\mathbf{P}, \mathbf{Set}]) \xrightarrow{\mathbf{Mon}(\text{Lan}_{J'})} \mathbf{Mon}([\mathbf{F}, \mathbf{Set}])$$

between the categories of monoids, whereas the inclusions (3.4) are restrictions of the 2-functors

$$\mathcal{E}nrich([\mathbf{N}, \mathbf{Set}]) \xleftarrow{\mathcal{E}nrich([J, \mathbf{Set}])} \mathcal{E}nrich([\mathbf{P}, \mathbf{Set}]) \xleftarrow{\mathcal{E}nrich([J', \mathbf{Set}])} \mathcal{E}nrich([\mathbf{F}, \mathbf{Set}])$$

between the 2-categories of enrichments.

Relation to enriched category theory

Before concluding this section, we shall remark on the relationship between our notion of enrichment and the standard notions in enriched category theory [53]. The reader may move on to the next section on oplax actions, since the results obtained in the following discussion will not be used in this thesis, though they explain how our approach relates to an enriched categorical approach to clones (= Lawvere theories = finitary monads on **Set**) by Garner [31].

We have mentioned that an enrichment of \mathcal{C} over \mathcal{M} can be equivalently given as an \mathcal{M} -category \mathcal{D} and an identity-on-objects functor $J: \mathcal{C} \rightarrow \mathcal{D}_0$. Let us first make the relation of these two formulations precise. In order to compare them, we introduce a natural 2-category having the latter as objects.

Definition 3.22. Let \mathcal{M} be a large monoidal category. The 2-category $\mathcal{E}nrich'(\mathcal{M})$ is defined as follows:

- An object is a triple $(\mathcal{C}, \mathcal{D}, J)$ consisting of a large category \mathcal{C} , a large \mathcal{M} -category \mathcal{D} and an identity-on-objects functor $J: \mathcal{C} \rightarrow \mathcal{D}_0$.
- A 1-cell from $(\mathcal{C}, \mathcal{D}, J)$ to $(\mathcal{C}', \mathcal{D}', J')$ is given by a functor $G: \mathcal{C} \rightarrow \mathcal{C}'$ and an \mathcal{M} -functor $H: \mathcal{D} \rightarrow \mathcal{D}'$ such that $H_0 \circ J = J' \circ G$.
- A 2-cell from (G, H) to (G', H') , both from $(\mathcal{C}, \mathcal{D}, J)$ to $(\mathcal{C}', \mathcal{D}', J')$, is given by a natural transformation $\theta: G \Rightarrow G'$ and an \mathcal{M} -natural transformation $\phi: H \Rightarrow H'$ such that $\phi_0 \circ J = J' \circ \theta$. ■

Hence $\mathcal{E}nrich'(\mathcal{M})$ is a full sub-2-category of the comma 2-category defined by the diagram

$$\mathcal{C}\mathcal{A}\mathcal{T} \xrightarrow{\text{id}_{\mathcal{C}\mathcal{A}\mathcal{T}}} \mathcal{C}\mathcal{A}\mathcal{T} \xleftarrow{(-)_0} \mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T},$$

where $\mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}$ is the 2-category of large \mathcal{M} -categories, \mathcal{M} -functors and \mathcal{M} -natural transformations, and $(-)_0$ is the forgetful 2-functor described in [53, Section 1.3].

It is routine to check that the obvious construction (sketched just after Definition 3.12) from $(\mathcal{C}, \langle -, - \rangle, j, M) \in \mathcal{E}nrich(\mathcal{M})$ to $(\mathcal{C}, \mathcal{D}, J) \in \mathcal{E}nrich'(\mathcal{M})$ extends to an isomorphism of the 2-categories $\mathcal{E}nrich(\mathcal{M})$ and $\mathcal{E}nrich'(\mathcal{M})$. Therefore we may identify $\mathcal{E}nrich(\mathcal{M})$ with $\mathcal{E}nrich'(\mathcal{M})$ via this isomorphism; $\mathcal{E}nrich'(\mathcal{M})$ is better suited to establish connections to enriched category theory.

We may embed (fully faithfully) both $\mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}$ and the underlying category $\mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}_0$ of $\mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}$ into $\mathcal{E}nrich'(\mathcal{M})$. The embedding

$$K_N: \mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T} \rightarrow \mathcal{E}nrich'(\mathcal{M})$$

maps an \mathcal{M} -category \mathcal{A} to the normal enrichment $(\mathcal{A}_0, \mathcal{A}, \text{id}_{\mathcal{A}_0})$ over \mathcal{M} (recall that an enrichment $(\mathcal{C}, \mathcal{D}, J)$ is called *normal* iff J is an isomorphism). Clearly, an enrichment is normal precisely when it is isomorphic to an enrichment of the form $K_N \mathcal{A}$ for some $\mathcal{A} \in \mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}$ (i.e., when it is in the essential image of K_N). The embedding

$$K_D: \mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}_0 \rightarrow \mathcal{E}nrich'(\mathcal{M})$$

maps an \mathcal{M} -category \mathcal{A} to the enrichment $(\text{ob}(\mathcal{A}), \mathcal{A}, J)$ over \mathcal{M} of the set $\text{ob}(\mathcal{D})$ seen as a discrete category (J is the unique identity-on-objects functor $\text{ob}(\mathcal{A}) \rightarrow \mathcal{A}_0$). It is clear from the definition of $\mathcal{E}nrich'(\mathcal{M})$ that both K_N and K_D are

fully faithful 2-functors. The fully faithful embedding $\mathbf{Mon}(\mathcal{M}) \longrightarrow \mathcal{E}nrich(\mathcal{M})$ mentioned above is a restriction of $K_{\mathcal{D}}$.

The 2-functor $K_{\mathcal{N}}$ admits a left adjoint 2-functor

$$L: \mathcal{E}nrich'(\mathcal{M}) \longrightarrow \mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}$$

mapping $(\mathcal{C}, \mathcal{D}, J) \in \mathcal{E}nrich'(\mathcal{M})$ to $\mathcal{D} \in \mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}$ and so on. Therefore for a monoid \mathbb{T} in \mathcal{M} and a *normal* enrichment $(\mathcal{C}, \mathcal{D}, J)$ over \mathcal{M} , the category of models $\mathbf{Mod}(\mathbb{T}, \mathcal{C}) = \mathcal{E}nrich(\mathcal{M})(\mathbb{T}, (\mathcal{C}, \mathcal{D}, J)) \cong \mathcal{E}nrich(\mathcal{M})(\mathbb{T}, K_{\mathcal{N}}\mathcal{D})$ is isomorphic to $\mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}(L\mathbb{T}, \mathcal{D})$, where $L\mathbb{T}$ is just a monoid \mathbb{T} seen as a one-object \mathcal{M} -category.

The enrichments corresponding to the standard notions of model for clones, symmetric operads and non-symmetric operads are all normal, hence in order to capture the categories of models relative to these notions of model, we may work entirely within the 2-category $\mathcal{M}\text{-}\mathcal{C}\mathcal{A}\mathcal{T}$, as already observed (in the case of clones) in [31].

3.1.3 Notions of model as oplax actions

In order to capture models of monads and generalised operads, enrichments do not suffice in general. A suitable structure is *oplax action*, defined as follows.

Definition 3.23. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category. An **oplax action of \mathcal{M}** consists of:

- a large category \mathcal{C} ;
- a functor $*$: $\mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C}$;
- a natural transformation $(\varepsilon_C: I * C \longrightarrow C)_{C \in \mathcal{C}}$;
- a natural transformation $(\delta_{X,Y,C}: (Y \otimes X) * C \longrightarrow Y * (X * C))_{X,Y \in \mathcal{M}, C \in \mathcal{C}}$,⁵

making the following diagrams commute for all $X, Y, Z \in \mathcal{M}$ and $C \in \mathcal{C}$:

$$\begin{array}{ccc} (I \otimes X) * C & \xrightarrow{\delta_{X,I,C}} & I * (X * C) \\ & \searrow \cong & \downarrow \varepsilon_{X * C} \\ & & X * C \end{array} \quad \begin{array}{ccc} (X \otimes I) * C & \xrightarrow{\delta_{I,X,C}} & X * (I * C) \\ & \searrow \cong & \downarrow X * \varepsilon_C \\ & & X * C \end{array}$$

$$\begin{array}{ccc} ((Z \otimes Y) \otimes X) * C & \xrightarrow{\delta_{X,Z \otimes Y,C}} & (Z \otimes Y) * (X * C) \\ \cong \downarrow & & \downarrow \delta_{Y,Z,X * C} \\ (Z \otimes (Y \otimes X)) * C & \xrightarrow{\delta_{Y \otimes X,Z,C}} & Z * ((Y \otimes X) * C) \xrightarrow{Z * \delta_{X,Y,C}} Z * (Y * (X * C)) \end{array}$$

We say that $(\mathcal{C}, *, \varepsilon, \delta)$ is an oplax action of \mathcal{M} , or that $(*, \varepsilon, \delta)$ is an oplax action of \mathcal{M} on \mathcal{C} . ■

⁵We have chosen to set $\delta_{X,Y,C}: (Y \otimes X) * C \longrightarrow Y * (X * C)$ and not $\delta_{X,Y,C}: (X \otimes Y) * C \longrightarrow X * (Y * C)$, because the former agrees with the convention to write composition of morphisms in the anti-diagrammatic order, which we adopt throughout this thesis.

An oplax action $(*, \varepsilon, \delta)$ of \mathcal{M} on \mathcal{C} is called a **pseudo action** (resp. **strict action**) if both ε and δ are natural isomorphisms (resp. identities).

The definition of model we derive from an oplax action is the following.

Definition 3.24. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category, $\mathbb{T} = (T, e, m)$ be a monoid object in \mathcal{M} , \mathcal{C} be a large category, and $*$ $= (*, \varepsilon, \delta)$ be an oplax action of \mathcal{M} on \mathcal{C} .

1. A **model of \mathbb{T} in \mathcal{C} with respect to $*$** is a pair (C, γ) consisting of an object $C \in \mathcal{C}$ and a morphism $\gamma: T * C \rightarrow C$ in \mathcal{C} making the following diagrams commute:

$$\begin{array}{ccc} I * C & \xrightarrow{e * C} & T * C \\ & \searrow \varepsilon_C & \downarrow \gamma \\ & & C \end{array} \quad \begin{array}{ccc} (T \otimes T) * C & \xrightarrow{m * C} & T * C \\ \delta_{T, T, C} \downarrow & & \downarrow \gamma \\ T * (T * C) & \xrightarrow{T * \gamma} & T * C \xrightarrow{\gamma} C. \end{array}$$

2. Let (C, γ) and (C', γ') be models of \mathbb{T} in \mathcal{C} with respect to $*$. A **homomorphism from (C, γ) to (C', γ')** is a morphism $f: C \rightarrow C'$ in \mathcal{C} making the following diagram commute:

$$\begin{array}{ccc} T * C & \xrightarrow{T * f} & T * C' \\ \gamma \downarrow & & \downarrow \gamma' \\ C & \xrightarrow{f} & C'. \end{array}$$

We denote the (large) category of models of \mathbb{T} in \mathcal{C} with respect to $*$ by $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, *))$. ■

The above definition is standard; see e.g., [4, Section 2.2].

Example 3.25. Let \mathcal{C} be a large category. Recall that monads on \mathcal{C} are precisely monoids in the monoidal category $[\mathcal{C}, \mathcal{C}]$ whose monoidal product is given by composition. We have a strict action

$$*: [\mathcal{C}, \mathcal{C}] \times \mathcal{C} \rightarrow \mathcal{C}$$

given by evaluation: $(X, C) \mapsto XC$.

This clearly generates the definitions of Eilenberg–Moore algebra and homomorphism (Definition 2.30). ■

Example 3.26. Let \mathcal{C} be a large category with finite limits and $\mathbb{S} = (S, \eta, \mu)$ be a cartesian monad on \mathcal{C} . Recall that under these assumptions the slice category $\mathcal{C}/S1$ acquires a structure (I, \otimes) of monoidal category, and an \mathbb{S} -operad is a monoid in $(\mathcal{C}/S1, I, \otimes)$. Models of an \mathbb{S} -operad and their homomorphisms (Definition 2.35) were introduced by using the pseudo action

$$*: (\mathcal{C}/S1) \times \mathcal{C} \rightarrow \mathcal{C}$$

in the first place, and therefore are immediately an instance of the above general definitions. ■

The 2-category of oplax actions of \mathcal{M}

For a monoidal category \mathcal{M} , we can define the 2-category of oplax actions of \mathcal{M} (cf. Definition 3.19).

Definition 3.27. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category. The (locally large) 2-category $\mathcal{A}ct_{\text{oplax}}(\mathcal{M})$ of oplax actions of \mathcal{M} is defined as follows:

- An object is an oplax action $(\mathcal{C}, *, \varepsilon, \delta)$ of \mathcal{M} .
- A 1-cell from $(\mathcal{C}, *, \varepsilon, \delta)$ to $(\mathcal{C}', *, \varepsilon', \delta')$ is a functor $G: \mathcal{C} \rightarrow \mathcal{C}'$ together with a natural transformation $(g_{X,C}: X *' GC \rightarrow G(X * C))_{X \in \mathcal{M}, C \in \mathcal{C}}$ making the following diagrams commute for all $X, Y \in \mathcal{M}$ and $C \in \mathcal{C}$:

$$\begin{array}{ccc}
 I *' GC & \xrightarrow{g_{I,C}} & G(I * C) \\
 & \searrow \varepsilon'_{GC} & \downarrow G\varepsilon_C \\
 & & GC
 \end{array}$$

$$\begin{array}{ccc}
 (Y \otimes X) *' GC & \xrightarrow{g_{Y \otimes X, C}} & G((Y \otimes X) * C) \\
 \delta'_{X,Y,GC} \downarrow & & \downarrow G\delta_{X,Y,C} \\
 Y *' (X *' GC) & \xrightarrow{Y *' g_{X,C}} Y *' G(X * C) \xrightarrow{g_{Y, X * C}} & G(Y * (X * C)).
 \end{array}$$

- A 2-cell from (G, g) to (G', g') , both from $(\mathcal{C}, *, \varepsilon, \delta)$ to $(\mathcal{C}', *, \varepsilon', \delta')$, is a natural transformation $\theta: G \Rightarrow G'$ making the following diagram commute for all $X \in \mathcal{M}$ and $C \in \mathcal{C}$:

$$\begin{array}{ccc}
 X *' GC & \xrightarrow{g_{X,C}} & G(X * C) \\
 X *' \theta_C \downarrow & & \downarrow \theta_{X * C} \\
 X *' G' C & \xrightarrow{g'_{X,C}} & G'(X * C).
 \end{array}$$

Similarly as the case of enrichments, we may extend the $\mathbf{Mod}(-, -)$ construction into a 2-functor

$$\mathbf{Mod}(-, -): \mathbf{Mon}(\mathcal{M})^{\text{op}} \times \mathcal{A}ct_{\text{oplax}}(\mathcal{M}) \rightarrow \mathcal{C}\mathcal{A}\mathcal{T}.$$

On the other hand, $\mathcal{A}ct_{\text{oplax}}(-)$ extends to a 2-functor in an apparently different manner than $\mathcal{E}nrich(-)$. Namely, it is a 2-functor of type

$$\mathcal{A}ct_{\text{oplax}}(-): (\mathbf{Mon}\mathcal{C}\mathcal{A}\mathcal{T}_{\text{oplax}})^{\text{coop}} \rightarrow \underline{2}\text{-}\mathcal{C}\mathcal{A}\mathcal{T},$$

where $\mathbf{Mon}\mathcal{C}\mathcal{A}\mathcal{T}_{\text{oplax}}$ is the 2-category of large monoidal categories, oplax monoidal functors and monoidal natural transformations. The apparent discrepancy between functoriality of $\mathcal{A}ct_{\text{oplax}}(-)$ and $\mathcal{E}nrich(-)$ will be solved in Section 3.2.3.

We sketch the action of an oplax monoidal functor on an oplax action.

Definition 3.28. Let $\mathcal{M} = (\mathcal{M}, I_{\mathcal{M}}, \otimes_{\mathcal{M}})$ and $\mathcal{N} = (\mathcal{N}, I_{\mathcal{N}}, \otimes_{\mathcal{N}})$ be large monoidal categories, $G = (G, g, g): \mathcal{N} \rightarrow \mathcal{M}$ be an oplax monoidal functor,⁶ \mathcal{C} be a large category and $*$ = $(*, \varepsilon, \delta)$ be an oplax action of \mathcal{M} on \mathcal{C} . We define the oplax action $G^*(*) = (*', \varepsilon', \delta')$ of \mathcal{N} on \mathcal{C} as follows:

- The functor $*': \mathcal{N} \times \mathcal{C} \rightarrow \mathcal{C}$ maps $(X, C) \in \mathcal{N} \times \mathcal{C}$ to $(GX) * C$.
- The natural transformation $(\varepsilon'_C: I_{\mathcal{N}} *' C \rightarrow C)_{C \in \mathcal{C}}$ is defined by $\varepsilon'_C = \varepsilon_C \circ (g * C)$:

$$GI_{\mathcal{N}} * C \xrightarrow{g * C} I_{\mathcal{M}} * C \xrightarrow{\varepsilon_C} C.$$

- The natural transformation $(\delta'_{X,Y,C}: (Y \otimes_{\mathcal{N}} X) *' C \rightarrow Y *' (X *' C))_{X,Y \in \mathcal{N}, C \in \mathcal{C}}$ is defined by $\delta'_{X,Y,C} = \delta_{GX,GY,C} \circ (g_{X,Y} * C)$:

$$G(Y \otimes_{\mathcal{N}} X) * C \xrightarrow{g_{X,Y} * C} (GY \otimes_{\mathcal{M}} GX) * C \xrightarrow{\delta_{GX,GY,C}} GY * (GX * C). \quad \blacksquare$$

3.1.4 The relation between enrichments and oplax actions

We have introduced two types of structures—enrichment and oplax action—to formalise notions of model. The former captures the standard notions of model for clones, symmetric operads and non-symmetric operads, whereas the latter captures those for monads and generalised operads. We will unify enrichment and oplax action by the notion of *metamodel* in Section 3.2.2, but before doing so we remark on the relationship between them. Though the results in this section will be subsumed by the theory of metamodelling, we believe that the following direct comparison of enrichments and oplax actions would be more accessible to some readers. We also explain why in some good cases we can give definition of model both in terms of enrichment and oplax actions; for instances of this phenomenon in the literature, see e.g., [50, Section 3] and [64, Section 6.4].

Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category and \mathcal{C} be a large category. The relationship between enrichment and oplax action is summarised in the adjunction

$$\mathcal{M} \begin{array}{c} \xrightarrow{- * C} \\ \perp \\ \xleftarrow{\langle C, - \rangle} \end{array} \mathcal{C}. \quad (3.7)$$

In more detail, what we mean is the following. Suppose that we have an enrichment $(\langle -, - \rangle, j, M)$ of \mathcal{C} over \mathcal{M} . If, in addition, for each $C \in \mathcal{C}$ the functor $\langle C, - \rangle$ has a left adjoint as in (3.7), then—by the parameter theorem for adjunctions; see [71, Section IV.7]—the left adjoints canonically extend to a bifunctor $*$: $\mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C}$, and j and M define appropriate natural transformations ε and δ , giving rise to an oplax action $(*, \varepsilon, \delta)$ of \mathcal{M} on \mathcal{C} . And vice versa, if we start from an oplax action.

To make this idea into a precise mathematical statement, let us introduce the following 2-categories.

Definition 3.29. Let \mathcal{M} be a large monoidal category.

1. Let $\mathcal{E}nrich^r(\mathcal{M})$ be the full sub-2-category of $\mathcal{E}nrich(\mathcal{M})$ consisting of all enrichments $(\mathcal{C}, \langle -, - \rangle, j, M)$ such that for each $C \in \mathcal{C}$, $\langle C, - \rangle$ is a right adjoint.

⁶Such an oplax functor consists of a functor $G: \mathcal{N} \rightarrow \mathcal{M}$, a morphism $g: GI_{\mathcal{N}} \rightarrow I_{\mathcal{M}}$ and a natural transformation $(g_{X,Y}: G(Y \otimes_{\mathcal{N}} X) \rightarrow GY \otimes_{\mathcal{M}} GX)_{X,Y \in \mathcal{N}}$ satisfying the suitable axioms.

2. Let $\mathcal{A}ct_{\text{oplax}}^1(\mathcal{M})$ be the full sub-2-category of $\mathcal{A}ct_{\text{oplax}}(\mathcal{M})$ consisting of all oplax actions $(\mathcal{C}, *, \varepsilon, \delta)$ such that for each $C \in \mathcal{C}$, $- * C$ is a left adjoint. ■

The above discussion can be summarised into the statement that the two 2-categories $\mathcal{E}nrich^r(\mathcal{M})$ and $\mathcal{A}ct_{\text{oplax}}^1(\mathcal{M})$ are equivalent. A direct proof of this equivalence would be essentially routine, but seems to involve rather lengthy calculation. We shall defer a proof to Corollary 3.43.

This observation is a variant of well-known categorical folklore. In the literature, it is usually stated in a slightly more restricted form than the above, for example as a correspondence between tensored \mathcal{M} -categories and closed pseudo actions of \mathcal{M} [50, 32, 65, 43].

Furthermore, the above correspondence is compatible with the definitions of model (Definitions 3.13 and 3.24). Suppose that $(\mathcal{C}, \langle -, - \rangle, j, M)$ and $(\mathcal{C}, *, \varepsilon, \delta)$ form a pair of an enrichment over \mathcal{M} and an oplax action of \mathcal{M} connected by the adjunctions (3.7) (in a way compatible with the natural transformations j, M, ε and δ). Then for any monoid object $\mathbb{T} = (T, e, m)$ in \mathcal{M} and any object $C \in \mathcal{C}$, a morphism

$$\chi: T \longrightarrow \langle C, C \rangle$$

is a model of \mathbb{T} in \mathcal{C} with respect to $\langle -, - \rangle$ (Definition 3.13) if and only if its transpose under the adjunction $- * C \dashv \langle C, - \rangle$

$$\gamma: T * C \longrightarrow C$$

is a model of \mathbb{T} in \mathcal{C} with respect to $*$ (Definition 3.24), and similarly for homomorphism between models of \mathbb{T} . Hence we obtain an isomorphism of categories

$$\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \langle -, - \rangle)) \cong \mathbf{Mod}(\mathbb{T}, (\mathcal{C}, *))$$

commuting with the forgetful functors into \mathcal{C} .

Some of the enrichments and oplax actions we have introduced so far are good enough to obtain the corresponding oplax actions or enrichments, giving rise to alternative definitions of model.

Example 3.30. Let \mathcal{C} be a locally small category with all small powers. Recall the strict action

$$*: [\mathcal{C}, \mathcal{C}] \times \mathcal{C} \longrightarrow \mathcal{C}$$

of the monoidal category $[\mathcal{C}, \mathcal{C}]$ of endofunctors on \mathcal{C} on \mathcal{C} , used to capture Eilenberg–Moore algebras of monads on \mathcal{C} . For any object $C \in \mathcal{C}$, write by $[C]: 1 \longrightarrow \mathcal{C}$ the functor from the terminal category 1 which maps the unique object of 1 to $C \in \mathcal{C}$ ($[C]$ is sometimes called the *name of C*).

By the assumptions on \mathcal{C} , for any object $A \in \mathcal{C}$ the functor $- * A$ (which may be seen as the precomposition by $[A]: 1 \longrightarrow \mathcal{C}$) admits a right adjoint $\langle A, - \rangle$, which maps any $B \in \mathcal{C}$ (equivalently, $[B]: 1 \longrightarrow \mathcal{C}$) to the right Kan extension $\langle A, B \rangle = \text{Ran}_{[A]} [B]$ of $[B]$ along $[A]$. The functor $\text{Ran}_{[A]} [B]: \mathcal{C} \longrightarrow \mathcal{C}$ maps $C \in \mathcal{C}$ to $\text{Ran}_{[A]} [B](C) = B^{\mathcal{C}(C, A)}$.

For any object $C \in \mathcal{C}$, $\text{Ran}_{[C]} [C]$ exists and becomes a monad on \mathcal{C} in a canonical way (the *codensity monad of [C]*). For any monad \mathbb{T} on \mathcal{C} , to give a structure of an Eilenberg–Moore algebra on $C \in \mathcal{C}$ is equivalent to give a monad morphism from \mathbb{T} to $\text{Ran}_{[C]} [C]$. This observation is in e.g., [50, Section 3].

In particular, if we take $\mathcal{C} = \mathbf{Set}$, we see that the above enrichment agrees with the one given in Example 3.17. Hence the notion of relative algebra [37] agrees with that of Eilenberg–Moore algebra in this case. ■

3.2 Basic concepts

In the previous section, we have seen that for each notion of algebraic theory there exists a suitable monoidal category \mathcal{M} , and algebraic theories in that notion of algebraic theory corresponds to monoid objects in \mathcal{M} . We have also observed that suitable categorical structures to give definitions of model of algebraic theories (notions of model) may be formulated in terms of \mathcal{M} , either as enrichment over \mathcal{M} or as oplax action of \mathcal{M} .

Motivated by these observations, in this section we shall define basic concepts of our unified framework for notions of algebraic theory.

3.2.1 Metatheories and theories

Definition 3.31. A **metatheory** is a large monoidal category $\mathcal{M} = (\mathcal{M}, I, \otimes)$. ■

Metatheories are intended to formalise notions of algebraic theory. We remark that, in this thesis, we leave the term *notion of algebraic theory* informal and will not give any mathematical definitions to it.

Definition 3.32. Let \mathcal{M} be a metatheory. A **theory in \mathcal{M}** is a monoid object $T = (T, e, m)$ in \mathcal{M} .

We denote the category of theories in \mathcal{M} by $\mathbf{Th}(\mathcal{M})$, which we define to be the same as $\mathbf{Mon}(\mathcal{M})$, the category of monoid objects in \mathcal{M} . ■

Theories formalise what we have been calling *algebraic theories*.

The above definitions simply renames well-known concepts. Our hope is that, by using the terms which reflect our intention, statements and discussions become easier to follow; think of the terms such as *generalised element* (which is synonymous to morphism in a category) or *map* (used by some authors to mean left adjoint in a bicategory) which have been used with great benefit in the literature.

3.2.2 Metamodels and models

In Sections 3.1.2 and 3.1.3, we have seen that the standard notions of model for various notions of algebraic theory can be formalised either as enrichments or as oplax actions. With *two* definitions, however, we cannot claim to have formalised notions of model in a satisfactory way. We now unify enrichments and oplax actions by introducing a more general structure of *metamodel* (of a metatheory). We also derive a definition of models of theories and their homomorphisms from a metamodel, and show that they generalise the corresponding definitions for enrichments and oplax actions.

We may approach the concept of metamodel of a metatheory \mathcal{M} in two different ways, one by generalising enrichments over \mathcal{M} and the other by generalising oplax actions of \mathcal{M} . Before giving a formal (and neutral) definition of metamodel, we describe these two perspectives.

Metamodels as generalised enrichments

Let us first discuss how a generalisation of enrichments over \mathcal{M} leads to the notion of metamodel. For this, we use a construction known as the *Day convolution* [19]. Given any large monoidal category $\mathcal{M} = (\mathcal{M}, I, \otimes)$, this construction endows the

presheaf category $\widehat{\mathcal{M}} = [\mathcal{M}^{\text{op}}, \mathbf{SET}]$ with a (biclosed) monoidal structure $(\widehat{I}, \widehat{\otimes})$, in such a way that the Yoneda embedding $\mathcal{M} \rightarrow \widehat{\mathcal{M}}$ canonically becomes strong monoidal.

Definition 3.33 ([19]). Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a large monoidal category. The **convolution monoidal structure** $(\widehat{I}, \widehat{\otimes})$ on the presheaf category $\widehat{\mathcal{M}} = [\mathcal{M}^{\text{op}}, \mathbf{SET}]$ is defined as follows.

- The unit object \widehat{I} is the representable functor $\mathcal{M}(-, I): \mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$.
- Given $P, Q \in \widehat{\mathcal{M}}$, their monoidal product $Q \widehat{\otimes} P: \mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$ maps $Z \in \mathcal{M}$ to

$$(Q \widehat{\otimes} P)(Z) = \int^{X, Y \in \mathcal{M}} \mathcal{M}(Z, Y \otimes X) \times Q(Y) \times P(X). \quad (3.8)$$

■

For a metatheory \mathcal{M} , a *metamodel* of \mathcal{M} is simply an enrichment over $\widehat{\mathcal{M}} = (\widehat{\mathcal{M}}, \widehat{I}, \widehat{\otimes})$ ⁷. Thanks to the Yoneda embedding, it is immediate that every enrichment over \mathcal{M} induces a metamodel of \mathcal{M} .

We can find several uses of $\widehat{\mathcal{M}}$ -categories (in the sense of enriched category theory) in the literature. In particular, [57, Section 6] and [74] contain discussions on relationship between $\widehat{\mathcal{M}}$ -categories and various actions of \mathcal{M} .

Metamodels as generalised oplax actions

Let us move on to the second perspective on metamodels, namely as generalised oplax actions. First note that an oplax action $(\mathcal{C}, *, \varepsilon, \delta)$ of a large monoidal category \mathcal{M} can be equivalently given as an oplax monoidal functor

$$\mathcal{M} \rightarrow [\mathcal{C}, \mathcal{C}]$$

defined by $X \mapsto X * -$, or as a colax functor

$$\Sigma\mathcal{M} \rightarrow \mathcal{CAT}, \quad (3.9)$$

where $\Sigma\mathcal{M}$ denotes \mathcal{M} seen as a one-object bicategory [6].

To generalise this, we use the bicategory \mathcal{PROF} of *profunctors* (also called *distributors* or *bimodules*) [7, 60]. The notion of profunctor will recur in this thesis.

Definition 3.34 ([7]). We define the bicategory \mathcal{PROF} as follows.

- An object is a large category.
- A 1-cell from \mathcal{A} to \mathcal{B} is a **profunctor from \mathcal{A} to \mathcal{B}** , which we define to be a functor

$$H: \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \mathbf{SET}.$$

We write $H: \mathcal{A} \rightarrow \mathcal{B}$ if H is a profunctor from \mathcal{A} to \mathcal{B} . The identity 1-cell on a large category \mathcal{C} is the hom-functor $\mathcal{C}(-, -)$. Given profunctors $H: \mathcal{A} \rightarrow \mathcal{B}$ and $K: \mathcal{B} \rightarrow \mathcal{C}$, their composite $K \odot H: \mathcal{A} \rightarrow \mathcal{C}$ maps $(C, A) \in \mathcal{C}^{\text{op}} \times \mathcal{A}$ to

$$(K \odot H)(C, A) = \int^{B \in \mathcal{B}} K(C, B) \times H(B, A). \quad (3.10)$$

⁷Although we have defined enrichment (Definition 3.12) only for large monoidal categories, the definition does not depend on any size condition and it is clear what we mean by enrichments over non-large monoidal categories, such as $\widehat{\mathcal{M}}$.

- A 2-cell from H to H' , both from \mathcal{A} to \mathcal{B} , is a natural transformation $\alpha: H \Rightarrow H': \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \mathbf{SET}$. ■

It is well-known that both \mathcal{CAT} and $\mathcal{CAT}^{\text{coop}}$ canonically embed into \mathcal{PROF} . Both embeddings are identity-on-objects and locally fully faithful pseudofunctors. The embedding

$$(-)_*: \mathcal{CAT} \rightarrow \mathcal{PROF}$$

maps a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ to the profunctor $F_*: \mathcal{A} \dashv\vdash \mathcal{B}$ defined by $F_*(B, A) = \mathcal{B}(B, FA)$. Note that, given functors $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$,

$$\begin{aligned} (G_* \odot F_*)(C, A) &= \int^{B \in \mathcal{B}} \mathcal{C}(C, GB) \times \mathcal{B}(B, FA) \\ &\cong \mathcal{C}(C, GFA) \\ &= (G \circ F)_*(C, A) \end{aligned}$$

by the Yoneda lemma. The embedding

$$(-)^*: \mathcal{CAT}^{\text{coop}} \rightarrow \mathcal{PROF}$$

maps a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ to the profunctor $F^*: \mathcal{B} \dashv\vdash \mathcal{A}$ with $F^*(A, B) = \mathcal{B}(FA, B)$. For any functor $F: \mathcal{A} \rightarrow \mathcal{B}$, we have an adjunction $F_* \dashv F^*$ in \mathcal{PROF} .

A *metamodel* of \mathcal{M} is a colax functor

$$\Sigma\mathcal{M} \rightarrow \mathcal{PROF}^{\text{coop}},$$

or equivalently a lax functor

$$(\Sigma\mathcal{M})^{\text{co}} = \Sigma(\mathcal{M}^{\text{op}}) \rightarrow \mathcal{PROF}^{\text{op}}. \quad (3.11)$$

Clearly, oplax actions of \mathcal{M} , in the form (3.9), give rise to metamodels of \mathcal{M} by postcomposing the pseudofunctor $(-)^*$.

Let us restate what a lax functor of type (3.11) amounts to, in monoidal categorical terms.

Definition 3.35. Let \mathcal{C} be a large category. Define the monoidal category $[\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathbf{SET}] = ([\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathbf{SET}], \mathcal{C}(-, -), \odot^{\text{rev}})$ of **endo-profunctors on \mathcal{C}** to be the endo-hom-category $\mathcal{PROF}^{\text{op}}(\mathcal{C}, \mathcal{C})$. More precisely:

- The unit object is the hom-functor $\mathcal{C}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{SET}$.
- Given $H, K: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{SET}$, define their monoidal product $H \odot^{\text{rev}} K$ to be the functor which maps $(A, C) \in \mathcal{C}^{\text{op}} \times \mathcal{C}$ to

$$(H \odot^{\text{rev}} K)(A, C) = \int^{B \in \mathcal{C}} H(B, C) \times K(A, B). \quad \blacksquare$$

Note that $H \odot^{\text{rev}} K \cong K \odot H$ (i.e., \odot^{rev} is “ \odot reversed”).

Using this monoidal structure on $[\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathbf{SET}]$, a metamodel of \mathcal{M} in a large category \mathcal{C} may be written as a lax monoidal functor

$$\mathcal{M}^{\text{op}} \rightarrow [\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathbf{SET}].$$

The definition of metamodel

Definition 3.36. Let $\mathcal{M} = (\mathcal{M}, \otimes, I)$ be a metatheory. A **metamodel of \mathcal{M}** consists of:

- a large category \mathcal{C} ;
- a functor $\Phi: \mathcal{M}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{SET}$ (whose action we write as $(X, A, B) \longmapsto \Phi_X(A, B)$);
- a natural transformation $((\bar{\phi}.)_C: 1 \longrightarrow \Phi_I(C, C))_{C \in \mathcal{C}}$;
- a natural transformation

$$((\bar{\phi}_{X,Y})_{A,B,C}: \Phi_Y(B, C) \times \Phi_X(A, B) \longrightarrow \Phi_{Y \otimes X}(A, C))_{X,Y \in \mathcal{M}, A,B,C \in \mathcal{C}},$$

making the following diagrams commute for all $X, Y, Z \in \mathcal{M}$ and $A, B, C, D \in \mathcal{C}$:

$$\begin{array}{ccc}
1 \times \Phi_X(A, B) & \xrightarrow{(\bar{\phi}.)_B \times \Phi_X(A, B)} & \Phi_I(B, B) \times \Phi_X(A, B) \\
\cong \downarrow & & \downarrow (\bar{\phi}_{X,I})_{A,B,B} \\
\Phi_X(A, B) & \xrightarrow{\cong} & \Phi_{I \otimes X}(A, B) \\
\\
\Phi_X(A, B) \times 1 & \xrightarrow{\Phi_X(A, B) \times (\bar{\phi}.)_A} & \Phi_X(A, B) \times \Phi_I(A, A) \\
\cong \downarrow & & \downarrow (\bar{\phi}_{I,X})_{A,A,B} \\
\Phi_X(A, B) & \xrightarrow{\cong} & \Phi_{X \otimes I}(A, B) \\
\\
(\Phi_Z(C, D) \times \Phi_Y(B, C)) \times \Phi_X(A, B) & \xrightarrow{(\bar{\phi}_{Y,Z})_{B,C,D} \times \Phi_X(A, B)} & \Phi_{Z \otimes Y}(B, D) \times \Phi_X(A, B) \\
\cong \downarrow & & \downarrow (\bar{\phi}_{X,Z \otimes Y})_{A,B,D} \\
\Phi_Z(C, D) \times (\Phi_Y(B, C) \times \Phi_X(A, B)) & & \Phi_{(Z \otimes Y) \otimes X}(A, D) \\
\downarrow \Phi_Z(C, D) \times (\bar{\phi}_{X,Y})_{A,B,C} & & \downarrow \cong \\
\Phi_Z(C, D) \times \Phi_{Y \otimes X}(A, C) & \xrightarrow{(\bar{\phi}_{Y \otimes X,Z})_{A,C,D}} & \Phi_{Z \otimes (Y \otimes X)}(A, D).
\end{array}$$

We say that $(\mathcal{C}, \Phi, \bar{\phi}., \bar{\phi})$ is a metamodel of \mathcal{M} , or that $(\Phi, \bar{\phi}., \bar{\phi})$ is a metamodel of \mathcal{M} in \mathcal{C} . ■

The above definition perfectly makes sense even if we replace the category **SET** of large sets by the category **Set** of small sets. Indeed, most of the naturally occurring notions of model can be captured by these “small” metamodels. However, for later developments it turns out to be more convenient to define metamodels as above.

Note that we may replace $((\bar{\phi}.)_C)_{C \in \mathcal{C}}$ by

$$((j_C)_Z: \hat{I}(Z) \longrightarrow \Phi_Z(C, C))_{C \in \mathcal{C}, Z \in \mathcal{M}}$$

and $((\bar{\phi}_{X,Y})_{A,B,C})_{X,Y \in \mathcal{M}, A,B,C \in \mathcal{C}}$ by

$$((M_{A,B,C})_Z: (\Phi_{(-)}(B, C) \widehat{\otimes} \Phi_{(-)}(A, B))(Z) \longrightarrow \Phi_Z(A, C))_{A,B,C \in \mathcal{M}, Z \in \mathcal{M}}.$$

The axioms for metamodel then translate to the ones for enrichments (over $\widehat{\mathcal{M}}$).

On the other hand, we may also replace $((\bar{\phi}_{\cdot})_C)_{C \in \mathcal{C}}$ by

$$((\phi_{\cdot})_{A,B}: \mathcal{C}(A, B) \longrightarrow \Phi_I(A, B))_{A,B \in \mathcal{C}}$$

and $((\bar{\phi}_{X,Y})_{A,B,C})_{X,Y \in \mathcal{M}, A,B,C \in \mathcal{C}}$ by

$$((\phi_{X,Y})_{A,C}: (\Phi_Y \odot^{\text{rev}} \Phi_X)(A, C) \longrightarrow \Phi_{Y \otimes X}(A, C))_{A,C \in \mathcal{C}, X, Y \in \mathcal{M}}.$$

The axioms for metamodel then state that

$$(\Phi, \phi_{\cdot}, \phi): (\mathcal{M}^{\text{op}}, I, \otimes) \longrightarrow ([\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathbf{SET}], \mathcal{C}(-, -), \odot^{\text{rev}})$$

is an oplax monoidal functor.

Hence the attempts to generalise enrichments and oplax actions mentioned above coincide and both give rise to Definition 3.36.

The definitions of model and homomorphism we derive from a metamodel are the following.

Definition 3.37. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a metatheory, $\mathbb{T} = (T, e, m)$ be a theory in \mathcal{M} , \mathcal{C} be a large category and $\Phi = (\Phi, \bar{\phi}_{\cdot}, \bar{\phi})$ be a metamodel of \mathcal{M} in \mathcal{C} .

1. A **model of \mathbb{T} in \mathcal{C} with respect to Φ** is a pair (C, ξ) consisting of an object C of \mathcal{C} and an element $\xi \in \Phi_T(C, C)$ such that $(\Phi_e)_{C,C}(\xi) = (\bar{\phi}_{\cdot})_C(*)$ (where $*$ is the unique element of 1) and $(\Phi_m)_{C,C}(\xi) = (\bar{\phi}_{T,T})_{C,C,C}(\xi, \xi)$:

$$\begin{array}{ccc} \Phi_T(C, C) & & 1 \\ \searrow (\Phi_e)_{C,C} & & \swarrow (\bar{\phi}_{\cdot})_C \\ & \Phi_I(C, C) & \end{array} \quad \begin{array}{ccc} \Phi_T(C, C) & & \Phi_T(C, C) \times \Phi_T(C, C) \\ \searrow (\Phi_m)_{C,C} & & \swarrow (\bar{\phi}_{T,T})_{C,C,C} \\ & \Phi_{T \otimes T}(C, C) & \end{array}$$

2. Let (C, ξ) and (C', ξ') be models of \mathbb{T} in \mathcal{C} with respect to Φ . A **homomorphism from (C, ξ) to (C', ξ')** is a morphism $f: C \longrightarrow C'$ in \mathcal{C} such that $\Phi_T(C, f)(\xi) = \Phi_T(f, C')(\xi')$:

$$\begin{array}{ccc} \Phi_T(C, C) & & \Phi_T(C', C') \\ \searrow \Phi_T(C, f) & & \swarrow \Phi_T(f, C') \\ & \Phi_T(C, C') & \end{array}$$

We denote the (large) category of models of \mathbb{T} in \mathcal{C} with respect to Φ by $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi))$. ■

Example 3.38. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a metatheory, \mathcal{C} be a large category and $(\langle -, - \rangle, j, M)$ be an enrichment of \mathcal{C} over \mathcal{M} . This induces a metamodel $(\Phi, \bar{\phi}_{\cdot}, \bar{\phi})$ of \mathcal{M} in \mathcal{C} as follows.

- The functor $\Phi: \mathcal{M}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{SET}$ maps $(X, A, B) \in \mathcal{M}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C}$ to

$$\Phi_X(A, B) = \mathcal{M}(X, \langle A, B \rangle).$$

- For each $C \in \mathcal{C}$, $(\bar{\phi}.)_C: 1 \longrightarrow \Phi_I(C, C)$ is the name of j_C (i.e., $(\bar{\phi}.)_C$ maps the unique element of the singleton 1 to j_C).
- For each $A, B, C \in \mathcal{C}$ and $X, Y \in \mathcal{M}$, the function $(\bar{\phi}_{X,Y})_{A,B,C}: \Phi_Y(B, C) \times \Phi_X(A, B) \longrightarrow \Phi_{Y \otimes X}(A, C)$ maps $g: Y \longrightarrow \langle B, C \rangle$ and $f: X \longrightarrow \langle A, B \rangle$ to

$$Y \otimes X \xrightarrow{g \otimes f} \langle B, C \rangle \otimes \langle A, B \rangle \xrightarrow{M_{A,B,C}} \langle A, C \rangle.$$

The definition of model and homomorphism (Definition 3.13) we derive from an enrichment may be seen as a special case of the corresponding definition (Definition 3.37) for metamodel. ■

Example 3.39. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a metatheory, \mathcal{C} be a large category and $(*, \varepsilon, \delta)$ be an oplax action of \mathcal{M} on \mathcal{C} . This induces a metamodel $(\Phi, \bar{\phi}., \bar{\phi})$ of \mathcal{M} in \mathcal{C} as follows.

- The functor $\Phi: \mathcal{M}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{SET}$ maps $(X, A, B) \in \mathcal{M}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C}$ to

$$\Phi_X(A, B) = \mathcal{C}(X * A, B).$$

- For each $C \in \mathcal{C}$, $(\bar{\phi}.)_C: 1 \longrightarrow \Phi_I(C, C)$ is the name of ε_C .
- For each $A, B, C \in \mathcal{C}$ and $X, Y \in \mathcal{M}$, the function $(\bar{\phi}_{X,Y})_{A,B,C}: \Phi_Y(B, C) \times \Phi_X(A, B) \longrightarrow \Phi_{Y \otimes X}(A, C)$ maps $g: Y * B \longrightarrow C$ and $f: X * A \longrightarrow B$ to

$$(Y \otimes X) * A \xrightarrow{\delta_{X,Y,A}} Y * (X * A) \xrightarrow{Y * f} Y * B \xrightarrow{g} C.$$

The definition of model and homomorphism (Definition 3.24) we derive from an oplax action may be seen as a special case of the corresponding definition (Definition 3.37) for metamodel. ■

The 2-category of metamodels

Metamodels of a metatheory naturally form a 2-category, just like enrichments and oplax actions do.

Definition 3.40. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a metatheory. We define the (locally large) 2-category $\mathcal{M}\text{Mod}(\mathcal{M})$ of metamodels of \mathcal{M} as follows.

- An object is a metamodel $(\mathcal{C}, \Phi, \bar{\phi}., \bar{\phi})$ of \mathcal{M} .
- A 1-cell from $(\mathcal{C}, \Phi, \bar{\phi}., \bar{\phi})$ to $(\mathcal{C}', \Phi', \bar{\phi}'.., \bar{\phi}')$ is a functor $G: \mathcal{C} \longrightarrow \mathcal{C}'$ together with a natural transformation $(g_{X,A,B}: \Phi_X(A, B) \longrightarrow \Phi'_X(GA, GB))_{X \in \mathcal{M}, A, B \in \mathcal{C}}$ making the following diagrams commute for all $X, Y \in \mathcal{M}$ and $A, B, C \in \mathcal{C}$:

$$\begin{array}{ccc} 1 & \xrightarrow{(\bar{\phi}.)_C} & \Phi_I(C, C) \\ & \searrow^{(\bar{\phi}.)_{GC}} & \downarrow g_{I,C,C} \\ & & \Phi'_I(GC, GC) \end{array}$$

$$\begin{array}{ccc}
\Phi_Y(B, C) \times \Phi_X(A, B) & \xrightarrow{(\bar{\phi}_{X,Y})_{A,B,C}} & \Phi_{Y \otimes X}(A, C) \\
\downarrow g_{Y,B,C} \times g_{X,A,B} & & \downarrow g_{Y \otimes X, A, C} \\
\Phi'_Y(GB, GC) \times \Phi'_X(GA, GB) & \xrightarrow{(\bar{\phi}'_{X,Y})_{GA, GB, GC}} & \Phi'_{Y \otimes X}(GA, GC).
\end{array}$$

- A 2-cell from (G, g) to (G', g') , both from $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$ to $(\mathcal{C}', \Phi', \bar{\phi}', \bar{\phi}')$, is a natural transformation $\theta: G \Rightarrow G'$ making the following diagram commute for all $X \in \mathcal{M}$ and $A, B \in \mathcal{C}$:

$$\begin{array}{ccc}
\Phi_X(A, B) & \xrightarrow{g_{X,A,B}} & \Phi'_X(GA, GB) \\
\downarrow g'_{X,A,B} & & \downarrow \Phi'_X(GA, \theta_B) \\
\Phi'_X(G'A, G'B) & \xrightarrow{\Phi'_X(\theta_A, G'B)} & \Phi'_X(GA, G'B). \quad \blacksquare
\end{array}$$

Recall that for a functor (resp. a 2-functor) $F: \mathcal{A} \rightarrow \mathcal{B}$, the **essential image** of F is the full subcategory (resp. full sub-2-category) of \mathcal{B} consisting of all objects $B \in \mathcal{B}$ such that there exists an object $A \in \mathcal{A}$ and an isomorphism $FA \cong B$. If \mathcal{A} is a large category, a contravariant presheaf $\mathcal{A}^{\text{op}} \rightarrow \mathbf{SET}$ (resp. a covariant presheaf $\mathcal{A} \rightarrow \mathbf{SET}$) over \mathcal{A} is called **representable** if and only if it is in the essential image of the Yoneda embedding $\mathcal{A} \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{SET}]$ (resp. $\mathcal{A} \rightarrow [\mathcal{A}, \mathbf{SET}]^{\text{op}}$).

Proposition 3.41. *Let \mathcal{M} be a metatheory. The construction given in Example 3.38 canonically extends to a fully faithful 2-functor*

$$\mathcal{E}nrich(\mathcal{M}) \rightarrow \mathcal{M}.Mod(\mathcal{M}).$$

A metamodel $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$ of \mathcal{M} is in the essential image of this 2-functor if and only if for each $A, B \in \mathcal{C}$, the functor

$$\Phi_{(-)}(A, B): \mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$$

is representable.

Proof. The construction of the 2-functor $\mathcal{E}nrich(\mathcal{M}) \rightarrow \mathcal{M}.Mod(\mathcal{M})$ is straightforward. The rest can also be proved by a standard argument using the Yoneda lemma. We sketch the argument below.

Let us focus on the characterisation of the essential image. Suppose that $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$ is a metamodel of \mathcal{M} such that for each $A, B \in \mathcal{C}$, the functor $\Phi_{(-)}(A, B)$ is representable. From such a metamodel we obtain an enrichment $(\langle -, - \rangle, j, \mathcal{M})$ of \mathcal{C} over \mathcal{M} as follows. For each $A, B \in \mathcal{C}$, choose an object $\langle A, B \rangle \in \mathcal{M}$ and an isomorphism $\alpha_{A,B}: \mathcal{M}(-, \langle A, B \rangle) \rightarrow \Phi_{(-)}(A, B)$. By functoriality of Φ , $\langle -, - \rangle$ uniquely extends to a functor of type $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{M}$ while making $(\alpha_{A,B})_{A,B \in \mathcal{C}}$ natural. For each $C \in \mathcal{C}$, $(\bar{\phi}.)_C: 1 \rightarrow \Phi_I(C, C) \cong \mathcal{M}(I, \langle C, C \rangle)$ gives rise to a morphism $j_C: I \rightarrow \langle C, C \rangle$ in \mathcal{M} . For each $A, B, C \in \mathcal{C}$

\mathcal{M} , consider the function

$$\begin{array}{c}
\mathcal{M}(\langle B, C \rangle, \langle B, C \rangle) \times \mathcal{M}(\langle A, B \rangle, \langle A, B \rangle) \\
\downarrow (\alpha_{B,C})_{\langle B, C \rangle} \times (\alpha_{A,B})_{\langle A, B \rangle} \\
\Phi_{\langle B, C \rangle}(B, C) \times \Phi_{\langle A, B \rangle}(A, B) \\
\downarrow (\bar{\phi}_{\langle A, B \rangle, \langle B, C \rangle})_{A, B, C} \\
\Phi_{\langle B, C \rangle \otimes \langle A, B \rangle}(A, C) \\
\downarrow (\alpha_{A,C})_{\langle B, C \rangle \otimes \langle A, B \rangle}^{-1} \\
\mathcal{M}(\langle B, C \rangle \otimes \langle A, B \rangle, \langle A, C \rangle).
\end{array}$$

Let the image of $(\text{id}_{\langle B, C \rangle}, \text{id}_{\langle A, B \rangle})$ under this function be $M_{A, B, C}: \langle B, C \rangle \otimes \langle A, B \rangle \rightarrow \langle A, C \rangle$. The axioms of metamodel then shows that $(\langle -, - \rangle, j, M)$ is an enrichment.

Moreover, if we consider the metamodel induced from this enrichment (see Example 3.38), then it is isomorphic to our original $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$. In particular, for each $X, Y \in \mathcal{M}$ and $A, B, C \in \mathcal{C}$, the function $(\bar{\phi}_{X, Y})_{A, B, C}$ is completely determined by $M_{A, B, C}$, as in Example 3.38. To see this, note that for each $f \in \mathcal{M}(X, \langle A, B \rangle)$ and $g \in \mathcal{M}(Y, \langle B, C \rangle)$, the diagram

$$\begin{array}{ccc}
\mathcal{M}(\langle B, C \rangle, \langle B, C \rangle) \times \mathcal{M}(\langle A, B \rangle, \langle A, B \rangle) & \xrightarrow{\mathcal{M}(g, \text{id}) \times \mathcal{M}(f, \text{id})} & \mathcal{M}(Y, \langle B, C \rangle) \times \mathcal{M}(X, \langle A, B \rangle) \\
\alpha \times \alpha \downarrow & & \downarrow \alpha \times \alpha \\
\Phi_{\langle B, C \rangle}(B, C) \times \Phi_{\langle A, B \rangle}(A, B) & \xrightarrow{\Phi_g(B, C) \times \Phi_f(A, B)} & \Phi_Y(B, C) \times \Phi_X(A, B) \\
(\bar{\phi}_{\langle A, B \rangle, \langle B, C \rangle})_{A, B, C} \downarrow & & \downarrow (\bar{\phi}_{X, Y})_{A, B, C} \\
\Phi_{\langle B, C \rangle \otimes \langle A, B \rangle}(A, C) & \xrightarrow{\Phi_{g \otimes f}(A, C)} & \Phi_{Y \otimes X}(A, C) \\
\alpha^{-1} \downarrow & & \downarrow \alpha^{-1} \\
\mathcal{M}(\langle B, C \rangle \otimes \langle A, B \rangle, \langle A, C \rangle) & \xrightarrow{\mathcal{M}(g \otimes f, \text{id})} & \mathcal{M}(\langle B, C \rangle \otimes \langle A, B \rangle, \langle A, C \rangle)
\end{array}$$

commutes. Hence by chasing the element $(\text{id}_{\langle B, C \rangle}, \text{id}_{\langle A, B \rangle})$ in the top left set, we observe that (modulo the isomorphisms α) (g, f) is mapped by $(\bar{\phi}_{X, Y})_{A, B, C}$ to $M_{A, B, C} \circ (g \otimes f)$. \square

Proposition 3.42. *Let \mathcal{M} be a metatheory. The construction given in Example 3.39 canonically extends to a fully faithful 2-functor*

$$\mathcal{A}ct_{\text{oplax}}(\mathcal{M}) \rightarrow \mathcal{M}\text{Mod}(\mathcal{M}).$$

A metamodel $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$ of \mathcal{M} is in the essential image of this 2-functor if and only if for each $X \in \mathcal{M}$ and $A \in \mathcal{C}$, the functor

$$\Phi_X(A, -): \mathcal{C} \rightarrow \mathbf{SET}$$

is representable.

Proof. Similar to the proof of Proposition 3.41.

In particular, given a metamodel $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$ of \mathcal{M} such that for each $X \in \mathcal{M}$ and $A \in \mathcal{C}$, the functor $\Phi_X(A, -)$ is representable, we may construct an oplax

action $(*, \varepsilon, \delta)$ of \mathcal{C} as follows. For each $X \in \mathcal{M}$ and $A \in \mathcal{C}$, choose an object $X * A \in \mathcal{C}$ and an isomorphism $\beta_{X,A}: \mathcal{C}(X * A, -) \rightarrow \Phi_X(A, -)$. We easily obtain a functor $*$: $\mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $(\varepsilon_C)_{C \in \mathcal{C}}$. To get δ , for each $X, Y \in \mathcal{M}$ and $A \in \mathcal{C}$ consider the function

$$\begin{array}{c}
\mathcal{C}(Y * (X * A), Y * (X * A)) \times \mathcal{C}(X * A, X * A) \\
\downarrow (\beta_{Y, X * A})_{Y * (X * A)} \times (\beta_{X, A})_{X * A} \\
\Phi_Y(X * A, Y * (X * A)) \times \Phi_X(A, X * A) \\
\downarrow (\bar{\phi}_{X, Y})_{A, X * A, Y * (X * A)} \\
\Phi_{Y \otimes X}(A, Y * (X * A)) \\
\downarrow (\beta_{Y \otimes X, A})_{Y * (X * A)}^{-1} \\
\mathcal{C}((Y \otimes X) * A, Y * (X * A)).
\end{array}$$

We define $\delta_{X, Y, A}: (Y \otimes X) * A \rightarrow Y * (X * A)$ to be the image of $(\text{id}_{Y * (X * A)}, \text{id}_{X * A})$ under this function.

To verify that the metamodel induced from this oplax action (see Example 3.39) is isomorphic to $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$, essentially we only need to check that $(\bar{\phi}_{X, Y})_{A, B, C}$ for each $X, Y \in \mathcal{M}$ and $A, B, C \in \mathcal{C}$ is determined by $\delta_{X, Y, A}$ as in Example 3.39. Suppressing the isomorphisms β from now on, for each $f: X * A \rightarrow B$ and $g: Y * B \rightarrow C$ consider the following digram:

$$\begin{array}{ccc}
\mathcal{C}(Y * (X * A), Y * (X * A)) \times \mathcal{C}(X * A, X * A) & \xrightarrow{(\bar{\phi}_{X, Y})_{A, X * A, Y * (X * A)}} & \mathcal{C}((Y \otimes X) * A, Y * (X * A)) \\
\downarrow \mathcal{C}(\text{id}, g \circ (Y * f)) \times \mathcal{C}(\text{id}, \text{id}) & & \downarrow \mathcal{C}(\text{id}, g \circ (Y * f)) \\
\mathcal{C}(Y * (X * A), C) \times \mathcal{C}(X * A, X * A) & \xrightarrow{(\bar{\phi}_{X, Y})_{A, X * A, C}} & \mathcal{C}((Y \otimes X) * A, C) \\
\uparrow \mathcal{C}(Y * f, \text{id}) \times \mathcal{C}(\text{id}, \text{id}) & & \uparrow (\bar{\phi}_{X, Y})_{A, B, C} \\
\mathcal{C}(Y * B, C) \times \mathcal{C}(X * A, X * A) & \xrightarrow{\mathcal{C}(\text{id}, \text{id}) \times \mathcal{C}(\text{id}, f)} & \mathcal{C}(Y * B, C) \times \mathcal{C}(X * A, B).
\end{array}$$

The top square commutes by naturality in \mathcal{C} of $(\bar{\phi}_{X, Y})_{A, B, C}$ and the bottom square commutes by (extra) naturality in B of it. By chasing the appropriate elements as follows

$$\begin{array}{ccc}
(\text{id}_{Y * (X * A)}, \text{id}_{X * A}) & \xrightarrow{(\bar{\phi}_{X, Y})_{A, X * A, Y * (X * A)}} & \delta_{X, Y, A} \\
\downarrow \mathcal{C}(\text{id}, g \circ (Y * f)) \times \mathcal{C}(\text{id}, \text{id}) & & \downarrow \mathcal{C}(\text{id}, g \circ (Y * f)) \\
(g \circ (Y * f), \text{id}_{X * A}) & \xrightarrow{(\bar{\phi}_{X, Y})_{A, X * A, C}} & g \circ (Y * f) \circ \delta_{X, Y, A} \\
\uparrow \mathcal{C}(Y * f, \text{id}) \times \mathcal{C}(\text{id}, \text{id}) & & \uparrow (\bar{\phi}_{X, Y})_{A, B, C} \\
(g, \text{id}_{X * A}) & \xrightarrow{\mathcal{C}(\text{id}, \text{id}) \times \mathcal{C}(\text{id}, f)} & (g, f),
\end{array}$$

we conclude that $(\bar{\phi}_{X, Y})_{A, B, C}(g, f) = g \circ (Y * f) \circ \delta_{X, Y, A}$, as desired. \square

Recall the 2-categories $\mathcal{E}nrich^r(\mathcal{M})$ and $\mathcal{A}ct_{\text{oplax}}^1(\mathcal{M})$ defined in Definition 3.29.

Corollary 3.43. *Let \mathcal{M} be a metatheory.*

1. The 2-functors in Proposition 3.41 and Proposition 3.42 restrict to fully faithful 2-functors

$$\mathcal{E}nrich^{\mathbf{f}}(\mathcal{M}) \longrightarrow \mathcal{M}Mod(\mathcal{M}) \quad \mathcal{A}ct_{\text{oplax}}^1(\mathcal{M}) \longrightarrow \mathcal{M}Mod(\mathcal{M})$$

with the same essential image characterised as follows: a metamodel $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$ of \mathcal{M} is in the essential image if and only if for each $X \in \mathcal{M}$ and $A, B \in \mathcal{C}$, the functors

$$\Phi_{(-)}(A, B): \mathcal{M}^{\text{op}} \longrightarrow \mathbf{SET} \quad \Phi_X(A, -): \mathcal{C} \longrightarrow \mathbf{SET}$$

are representable.

2. The 2-categories $\mathcal{E}nrich^{\mathbf{f}}(\mathcal{M})$ and $\mathcal{A}ct_{\text{oplax}}^1(\mathcal{M})$ are equivalent.

Proof. The first clause is immediate from the definition of adjunction. For instance, an enrichment $(\mathcal{C}, \langle -, - \rangle, j, M)$ over \mathcal{M} is in $\mathcal{E}nrich^{\mathbf{f}}(\mathcal{M})$ if and only if for each $A \in \mathcal{C}$, $\langle A, - \rangle$ is a right adjoint, which in turn is the case if and only if for each $X \in \mathcal{M}$ and $A \in \mathcal{C}$, the functor

$$\mathcal{M}(X, \langle A, - \rangle): \mathcal{C} \longrightarrow \mathbf{SET}$$

is representable.

The second clause is a direct consequence of the first. \square

The reader might have noticed that there is another representability condition not covered by Propositions 3.41 and 3.42, namely metamodels $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$ such that for each $X \in \mathcal{M}$ and $B \in \mathcal{C}$, the functor

$$\Phi_X(-, B): \mathcal{C}^{\text{op}} \longrightarrow \mathbf{SET}$$

is representable. They correspond to right lax actions of \mathcal{M}^{op} on \mathcal{C} , or equivalently, to right oplax actions of \mathcal{M} on \mathcal{C}^{op} .

Extending the definition of enrichment (Definition 3.12) and the 2-category of enrichments (Definition 3.19) to huge monoidal categories, we obtain the following.

Proposition 3.44. *Let \mathcal{M} be a metatheory and $\widehat{\mathcal{M}} = ([\mathcal{M}^{\text{op}}, \mathbf{SET}], \widehat{I}, \widehat{\otimes})$ (see Definition 3.33). The 2-categories $\mathcal{M}Mod(\mathcal{M})$ and $\mathcal{E}nrich(\widehat{\mathcal{M}})$ are canonically isomorphic.*

Mod(−, −) as a 2-functor

Let \mathcal{M} be a metatheory. Similarly to the cases of enrichments and oplax actions, we can view the **Mod**(−, −) construction as a 2-functor using the 2-category $\mathcal{M}Mod(\mathcal{M})$. In fact, via the inclusion

$$\mathbf{Th}(\mathcal{M}) = \mathbf{Mon}(\mathcal{M}) \longrightarrow \mathcal{E}nrich(\mathcal{M}) \longrightarrow \mathcal{M}Mod(\mathcal{M}), \quad (3.12)$$

the 2-functor **Mod**(−, −) is simply given by the following composite:

$$\begin{array}{c} \mathbf{Th}(\mathcal{M})^{\text{op}} \times \mathcal{M}Mod(\mathcal{M}) \\ \downarrow \text{inclusion} \\ \mathcal{M}Mod(\mathcal{M})^{\text{op}} \times \mathcal{M}Mod(\mathcal{M}) \\ \downarrow \mathcal{M}Mod(\mathcal{M})(-, -) \\ \mathcal{SET}, \end{array}$$

where $\mathcal{M}\text{Mod}(\mathcal{M})(-, -)$ is the hom-2-functor for the locally large $\mathcal{M}\text{Mod}(\mathcal{M})$. The inclusion (3.12) identifies a theory $\mathbb{T} = (T, m, e)$ in \mathcal{M} with the metamodel $(\Phi^{(\mathbb{T})}, \overline{\phi^{(\mathbb{T})}}, \overline{\phi^{(\mathbb{T})}})$ of \mathcal{M} in the terminal category 1 (whose unique object we denote by $*$), defined as follows:

- the functor $\Phi^{(\mathbb{T})}: \mathcal{M}^{\text{op}} \times 1^{\text{op}} \times 1 \rightarrow \mathbf{SET}$ maps $(X, *, *)$ to $\mathcal{M}(X, T)$;
- the function $(\overline{\phi^{(\mathbb{T})}})_*: 1 \rightarrow \mathcal{M}(I, T)$ maps the unique element of 1 to e ;
- for each $X, Y \in \mathcal{M}$, the function $(\overline{\phi^{(\mathbb{T})}})_{X, Y}^{*, *, *}: \mathcal{M}(Y, T) \times \mathcal{M}(X, T) \rightarrow \mathcal{M}(Y \otimes X, T)$ maps (g, f) to $m \circ (g \otimes f)$.

3.2.3 Morphisms of metatheories

In this section, we introduce a notion of morphism between metatheories. The main purpose of morphisms of metatheories is to provide a uniform method to compare different notions of algebraic theory. A paradigmatic case of such a comparison is given in Section 3.1.2, where we compare clones, symmetric operads and non-symmetric operads. Recall that the crucial observation used there was the fact that the $\text{Enrich}(-)$ construction extends to a 2-functor

$$\text{Enrich}(-): \text{Mon}\mathcal{CAT}_{\text{lax}} \rightarrow \underline{2}\text{-CAT}. \quad (3.13)$$

Therefore, we want to define morphisms of metatheories with respect to which $\mathcal{M}\text{Mod}(-)$ behaves (2-)functorially.

On the other hand, recall from Section 3.1.3 that $\text{Act}_{\text{oplax}}(-)$ is a 2-functor of type

$$\text{Act}_{\text{oplax}}(-): (\text{Mon}\mathcal{CAT}_{\text{oplax}})^{\text{coop}} \rightarrow \underline{2}\text{-CAT}. \quad (3.14)$$

Since metamodels unify both enrichments and oplax actions, we would like to explain both (3.13) and (3.14) by introducing a sufficiently general notion of morphism of metatheories.

The requirement to unify both $\text{Mon}\mathcal{CAT}_{\text{lax}}$ and $(\text{Mon}\mathcal{CAT}_{\text{oplax}})^{\text{coop}}$ suggests the possibility of using a suitable variant of profunctors (Definition 3.34), leading to the following definition.

Definition 3.45. Let $\mathcal{M} = (\mathcal{M}, I_{\mathcal{M}}, \otimes_{\mathcal{M}})$ and $\mathcal{N} = (\mathcal{N}, I_{\mathcal{N}}, \otimes_{\mathcal{N}})$ be metatheories. A **morphism of metatheories from \mathcal{M} to \mathcal{N}** is a lax monoidal functor

$$H = (H, h., h): \mathcal{N}^{\text{op}} \times \mathcal{M} \rightarrow \mathbf{SET}.$$

More precisely, such a morphism consists of:

- a functor $H: \mathcal{N}^{\text{op}} \times \mathcal{M} \rightarrow \mathbf{SET}$;
- a function $h.: 1 \rightarrow H(I_{\mathcal{N}}, I_{\mathcal{M}})$;
- a natural transformation $(h_{N, N', M, M'}: H(N', M') \times H(N, M) \rightarrow H(N' \otimes_{\mathcal{N}} N, M' \otimes_{\mathcal{M}} M))_{N, N' \in \mathcal{N}, M, M' \in \mathcal{M}}$

making the following diagrams commute for each $N, N', N'' \in \mathcal{N}$ and $M, M', M'' \in \mathcal{M}$ (we omit subscripts on \otimes):

$$\begin{array}{ccc} 1 \times H(N, M) & \xrightarrow{h. \times H(N, M)} & H(I_{\mathcal{N}}, I_{\mathcal{M}}) \times H(N, M) \\ \cong \downarrow & & \downarrow h_{N, I_{\mathcal{N}}, M, I_{\mathcal{M}}} \\ H(N, M) & \xrightarrow{\cong} & H(I_{\mathcal{N}} \otimes N, I_{\mathcal{M}} \otimes M) \end{array}$$

$$\begin{array}{ccc}
H(N, M) \times 1 & \xrightarrow{H(N, M) \times h} & H(N, M) \times H(I_{\mathcal{N}}, I_{\mathcal{M}}) \\
\downarrow \cong & & \downarrow h_{I_{\mathcal{N}}, N, I_{\mathcal{M}}, M} \\
H(N, M) & \xrightarrow{\cong} & H(N \otimes I_{\mathcal{N}}, M \otimes, I_{\mathcal{M}}) \\
\\
(H(N'', M'') \times H(N', M')) \times H(N, M) & \xrightarrow{h_{N', N'', M', M''} \times H(N, M)} & H(N'' \otimes N', M'' \otimes M') \times H(N, M) \\
\downarrow \cong & & \downarrow h_{N, N'' \otimes N', M, M'' \otimes M'} \\
H(N'', M'') \times (H(N', M') \times H(N, M)) & & H((N'' \otimes N') \otimes N, (M'' \otimes M') \otimes M) \\
\downarrow H(N'', M'') \times h_{N, N', M, M'} & & \downarrow \cong \\
H(N'', M'') \times H(N' \otimes N, M' \otimes M) & \xrightarrow{h_{N' \otimes N, N'', M' \otimes M, M''}} & H(N'' \otimes (N' \otimes N), M'' \otimes (M' \otimes M)).
\end{array}$$

We write $H: \mathcal{M} \rightarrow \mathcal{N}$ if H is a morphism of metatheories from \mathcal{M} to \mathcal{N} . ■

Morphisms of metatheories are a monoidal version of profunctors, and indeed they are called *monoidal profunctors* in [41]. We may identify a morphism $H: \mathcal{M} \rightarrow \mathcal{N}$ with a lax monoidal functor

$$(M \mapsto H(-, M)): \mathcal{M} \rightarrow [\mathcal{N}^{\text{op}}, \mathbf{SET}],$$

or equivalently with an oplax monoidal functor

$$(N \mapsto H(N, -)): \mathcal{N} \rightarrow [\mathcal{M}, \mathbf{SET}]^{\text{op}},$$

where in both cases the codomain is equipped with the convolution monoidal structure.

Definition 3.46. We define the bicategory $\mathcal{MT}\mathcal{H}$ of metatheories as follows.

- An object is a metatheory.
- A 1-cell from \mathcal{M} to \mathcal{N} is a morphism of metatheories $\mathcal{M} \rightarrow \mathcal{N}$. The identity 1-cell on a metatheory \mathcal{M} is the hom-functor $\mathcal{M}(-, -)$, equipped with the evident structure for a morphism of metatheories. Given morphisms of metatheories $(H, h., h): \mathcal{M} \rightarrow \mathcal{N}$ and $(K, k., k): \mathcal{N} \rightarrow \mathcal{L}$, their composite is $(K \odot H, k \odot h., k \odot h): \mathcal{M} \rightarrow \mathcal{L}$ where $K \odot H$ is the composition of the profunctors H and K (Definition 3.34), and $k \odot h.$ and $k \odot h$ are the evident natural transformations.
- A 2-cell from H to H' , both from \mathcal{M} to \mathcal{N} , is a monoidal natural transformation $\alpha: H \Rightarrow H': \mathcal{N}^{\text{op}} \times \mathcal{M} \rightarrow \mathbf{SET}$. ■

Similarly to the case of profunctors, we have identity-on-objects fully faithful pseudofunctors

$$(-)_*: \text{Mon}\mathcal{CAT}_{\text{lax}} \rightarrow \mathcal{MT}\mathcal{H}$$

and

$$(-)^*: (\text{Mon}\mathcal{CAT}_{\text{oplax}})^{\text{coop}} \rightarrow \mathcal{MT}\mathcal{H}.$$

In detail, a lax monoidal functor

$$F = (F, f., f): \mathcal{M} \rightarrow \mathcal{N}$$

gives rise to a morphism of metatheories

$$F_* = (F_*, (f_*)_., f_*): \mathcal{M} \dashrightarrow \mathcal{N}$$

with $F_*(N, M) = \mathcal{N}(N, FM)$, $(f_*)_.: 1 \rightarrow \mathcal{N}(I_{\mathcal{N}}, FI_{\mathcal{M}})$ mapping the unique element of 1 to $f.: I_{\mathcal{N}} \rightarrow FI_{\mathcal{M}}$, and $(f_*)_{N, N', M, M'}: \mathcal{N}(N', FM') \times \mathcal{N}(N, FM) \rightarrow \mathcal{N}(N' \otimes_{\mathcal{N}} N, F(M' \otimes_{\mathcal{M}} M))$ mapping $g': N' \rightarrow FM'$ and $g: N \rightarrow FM$ to $f_{M, M'} \circ (g' \otimes g): N' \otimes_{\mathcal{N}} N \rightarrow F(M' \otimes_{\mathcal{M}} M)$. Given an oplax monoidal functor

$$F = (F, f_., f): \mathcal{M} \rightarrow \mathcal{N},$$

we obtain a morphism of metatheories

$$F^* = (F^*, (f^*)_., f^*): \mathcal{N} \dashrightarrow \mathcal{M}$$

analogously.

In particular, a *strong* monoidal functor

$$F: \mathcal{M} \rightarrow \mathcal{N}$$

gives rise to both $F_*: \mathcal{M} \dashrightarrow \mathcal{N}$ and $F^*: \mathcal{N} \dashrightarrow \mathcal{M}$, and it is straightforward to see that these form an adjunction $F_* \dashv F^*$ in \mathcal{MTH} .

A morphism of metatheories $H: \mathcal{M} \dashrightarrow \mathcal{N}$ induces a 2-functor

$$\mathcal{M}Mod(H): \mathcal{M}Mod(\mathcal{M}) \rightarrow \mathcal{M}Mod(\mathcal{N}).$$

Its action on objects is as follows.

Definition 3.47. Let $\mathcal{M} = (\mathcal{M}, I_{\mathcal{M}}, \otimes_{\mathcal{M}})$ and $\mathcal{N} = (\mathcal{N}, I_{\mathcal{N}}, \otimes_{\mathcal{N}})$ be metatheories, $H = (H, h_., h): \mathcal{M} \dashrightarrow \mathcal{N}$ a morphism of metatheories, \mathcal{C} a large category and $\Phi = (\Phi, \bar{\phi}_., \bar{\phi})$ a metamodel of \mathcal{M} in \mathcal{C} . We define the metamodel $H(\Phi) = (\Phi', \bar{\phi}'_., \bar{\phi}')$ of \mathcal{N} on \mathcal{C} as follows:

- The functor $\Phi': \mathcal{N}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{SET}$ maps $(N, A, B) \in \mathcal{N}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C}$ to the set

$$\Phi'_N(A, B) = \int^{M \in \mathcal{M}} H(N, M) \times \Phi_M(A, B). \quad (3.15)$$

- The natural transformation $((\bar{\phi}')_C): 1 \rightarrow \Phi'_{I_{\mathcal{N}}}(C, C)_{C \in \mathcal{C}}$ is defined by mapping the unique element $*$ of 1 to

$$\begin{aligned} [I_{\mathcal{M}} \in \mathcal{M}, h_.*) \in H(I_{\mathcal{N}}, I_{\mathcal{M}}), (\bar{\phi}_.)_C(*) \in \Phi_{I_{\mathcal{M}}}(C, C)] \\ \in \int^{M \in \mathcal{M}} H(I_{\mathcal{N}}, M) \times \Phi_M(C, C) \end{aligned}$$

for each $C \in \mathcal{C}$.

- The natural transformation

$$((\bar{\phi}'_{N, N'})_{A, B, C}: \Phi'_{N'}(B, C) \times \Phi'_N(A, B) \rightarrow \Phi'_{N' \otimes_{\mathcal{N}} N}(A, C))_{N, N' \in \mathcal{N}, A, B, C \in \mathcal{C}}$$

is defined by mapping a pair consisting of $[M', x', y'] \in \Phi'_{N'}(B, C)$ and $[M, x, y] \in \Phi'_N(A, B)$ to

$$[M' \otimes_{\mathcal{M}} M, h_{N, N', M, M'}(x', x), (\bar{\phi}_{M, M'})_{A, B, C}(y', y)]$$

for each $N, N' \in \mathcal{N}$ and $A, B, C \in \mathcal{C}$. ■

The above construction extends routinely, giving rise to a pseudofunctor

$$\mathcal{M}Mod(-): \mathcal{MTH} \rightarrow \underline{\mathbf{2-CAT}}.$$

3.3 Comparing different notions of algebraic theory

In this section, we shall demonstrate how we can compare different notions of algebraic theory via morphisms of metatheories.

We start with a few remarks on simplification of the action (Definition 3.47) of a morphism of metatheories on metamodels, in certain special cases. Let \mathcal{M} and \mathcal{N} be metatheories,

$$H: \mathcal{M} \dashrightarrow \mathcal{N}$$

be a morphism of metatheories, \mathcal{C} be a large category and $\Phi = (\Phi, \bar{\phi}, \bar{\phi})$ be a metamodel of \mathcal{M} in \mathcal{C} .

First consider the case where for each $A, B \in \mathcal{C}$, the functor $\Phi_{(-)}(A, B): \mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$ is representable. This means that Φ is in fact (up to an isomorphism) an enrichment $\langle -, - \rangle$; see Proposition 3.41. In this case, $\Phi_M(A, B)$ may be written as $\mathcal{M}(M, \langle A, B \rangle)$ and hence the formula (3.15) simplifies:

$$\Phi'_N(A, B) = \int^{M \in \mathcal{M}} H(N, M) \times \mathcal{M}(M, \langle A, B \rangle) \cong H(N, \langle A, B \rangle).$$

In particular, if moreover H is of the form

$$F_*: \mathcal{M} \dashrightarrow \mathcal{N}$$

for some lax monoidal functor $F: \mathcal{M} \rightarrow \mathcal{N}$, then we have

$$\Phi'_N(A, B) \cong \mathcal{N}(N, F\langle A, B \rangle),$$

implying that $H(\Phi) = F_*(\Phi)$ is again isomorphic to an enrichment; indeed, this case reduces to $F_*(\langle -, - \rangle)$ defined in Definition 3.21. Note that, as a special case, for any theory \mathbb{T} in \mathcal{M} (recall that such a theory is identified with a metamodel of \mathcal{M} in the terminal category $\mathbf{1}$), $F_*(\mathbb{T})$ is again isomorphic to a theory in \mathcal{N} . The 2-functor

$$\mathcal{M}\text{Mod}(F_*): \mathcal{M}\text{Mod}(\mathcal{M}) \rightarrow \mathcal{M}\text{Mod}(\mathcal{N})$$

extends the functor

$$\mathbf{Th}(F): \mathbf{Th}(\mathcal{M}) \rightarrow \mathbf{Th}(\mathcal{N})$$

between the categories of theories induced by F , using the well-known fact that a lax monoidal functor preserves theories (= monoid objects).

Next consider the case where H is of the form

$$G^*: \mathcal{M} \dashrightarrow \mathcal{N}$$

for some oplax monoidal functor $G: \mathcal{N} \rightarrow \mathcal{M}$. In this case $H(N, M) = \mathcal{M}(GN, M)$ and the formula (3.15) simplifies as follows:

$$\Phi'_N(A, B) = \int^{M \in \mathcal{M}} \mathcal{M}(GN, M) \times \Phi_M(A, B) \cong \Phi_{GN}(A, B).$$

Of course this construction reduces to $G^*(*)$ defined in Definition 3.28 for a metamodel induced from an oplax action.

Combining the above observations, suppose now that we have a strong monoidal functor

$$F: \mathcal{M} \rightarrow \mathcal{N}$$

between metatheories \mathcal{M} and \mathcal{N} . On the one hand, F induces a functor

$$\mathbf{Th}(F): \mathbf{Th}(\mathcal{M}) \rightarrow \mathbf{Th}(\mathcal{N})$$

between the categories of theories, which is a restriction of the 2-functor $\mathcal{M}\text{Mod}(F_*)$. On the other hand, F induces a 2-functor

$$\mathcal{M}\text{Mod}(F^*): \mathcal{M}\text{Mod}(\mathcal{N}) \longrightarrow \mathcal{M}\text{Mod}(\mathcal{M})$$

between the 2-categories of metamodels. The 2-adjointness $\mathcal{M}\text{Mod}(F_*) \dashv \mathcal{M}\text{Mod}(F^*)$ yields, for each theory \mathbb{T} in \mathcal{M} and each metamodel (\mathcal{C}, Φ) of \mathcal{N} , an isomorphism of categories

$$\mathbf{Mod}(F_*(\mathbb{T}), (\mathcal{C}, \Phi)) \cong \mathbf{Mod}(\mathbb{T}, (\mathcal{C}, F^*(\Phi))).$$

Observe that $F_*(\mathbb{T}) = \mathbf{Th}(F)(\mathbb{T})$ is the standard action of a strong monoidal functor on a theory, and $F^*(\Phi)$ is, in essence, simply precomposition by F .

Now we apply the above argument to some concrete cases.

Example 3.48. Recall from Section 3.1.2, where we have compared clones, symmetric operads and non-symmetric operads, that there is a chain of lax monoidal functors

$$[\mathbf{N}, \mathbf{Set}] \xrightarrow{\text{Lan}_J} [\mathbf{P}, \mathbf{Set}] \xrightarrow{\text{Lan}_{J'}} [\mathbf{F}, \mathbf{Set}].$$

These functors, being left adjoints in $\mathcal{M}\text{on}\mathcal{C}\mathcal{A}\mathcal{T}_{\text{lax}}$, are in fact strong monoidal [51]. Theories are mapped as follows, as noted in Section 3.1.2:

$$\mathbf{Th}([\mathbf{N}, \mathbf{Set}]) \xrightarrow{\mathbf{Th}(\text{Lan}_J)} \mathbf{Th}([\mathbf{P}, \mathbf{Set}]) \xrightarrow{\mathbf{Th}(\text{Lan}_{J'})} \mathbf{Th}([\mathbf{F}, \mathbf{Set}]).$$

In this case, the suitable 2-functors between 2-categories of metamodels can be given either as $\mathcal{M}\text{Mod}((\text{Lan}_J)^*)$ or $\mathcal{M}\text{Mod}([\mathbf{J}, \mathbf{Set}]_*)$ (and similarly for J'), because $(\text{Lan}_J)^* \cong ([\mathbf{J}, \mathbf{Set}]_*)$ in $\mathcal{M}\mathcal{T}\mathcal{H}$. \blacksquare

Example 3.49. Let us consider the relationship between clones and monads on \mathbf{Set} . The inclusion functor

$$J'': \mathbf{F} \longrightarrow \mathbf{Set}$$

induces a functor

$$\text{Lan}_{J''}: [\mathbf{F}, \mathbf{Set}] \longrightarrow [\mathbf{Set}, \mathbf{Set}],$$

which naturally acquires the structure of a strong monoidal functor. The essential image of this functor is precisely the *finitary* endofunctors on \mathbf{Set} , i.e., those endofunctors preserving filtered colimits. The functor $\mathbf{Th}(\text{Lan}_{J''})$ maps a clone to a finitary monad on \mathbf{Set} , in accordance with the well-known correspondence between clones (= Lawvere theories) and finitary monads on \mathbf{Set} [66]. Between the 2-categories of metamodels, we have a 2-functor

$$\mathcal{M}\text{Mod}((\text{Lan}_{J''})^*): \mathcal{M}\text{Mod}([\mathbf{Set}, \mathbf{Set}]) \longrightarrow \mathcal{M}\text{Mod}([\mathbf{F}, \mathbf{Set}]).$$

The standard metamodel of $[\mathbf{Set}, \mathbf{Set}]$ in \mathbf{Set} (corresponding to the definition of Eilenberg–Moore algebras) is given by the strict action described in Example 3.25; in particular, its functor part $\Phi: [\mathbf{Set}, \mathbf{Set}]^{\text{op}} \times \mathbf{Set}^{\text{op}} \times \mathbf{Set} \longrightarrow \mathbf{SET}$ maps (F, A, B) to $\mathbf{Set}(FA, B)$. The metamodel $(\text{Lan}_{J''})^*(\Phi)$ of $[\mathbf{F}, \mathbf{Set}]$ in \mathbf{Set} has the functor part $(\text{Lan}_{J''})^*(\Phi): [\mathbf{F}, \mathbf{Set}]^{\text{op}} \times \mathbf{Set}^{\text{op}} \times \mathbf{Set} \longrightarrow \mathbf{SET}$ mapping (X, A, B) to

$$\begin{aligned} \mathbf{Set}((\text{Lan}_{J''}X)A, B) &= \mathbf{Set}\left(\int^{[n] \in \mathbf{F}} A^n \times X_n, B\right) \\ &\cong \int_{[n] \in \mathbf{F}} \mathbf{Set}(X_n, \mathbf{Set}(A^n, B)) \\ &\cong [\mathbf{F}, \mathbf{Set}](X, \langle A, B \rangle), \end{aligned}$$

where $\langle A, B \rangle \in [\mathbf{F}, \mathbf{Set}]$ in the final line is the one in Example 3.14. Hence $\mathcal{M}\text{Mod}((\text{Lan}_{J''})^*)$ preserves the standard metamodells and this way we restore the well-known observation that the classical correspondence of clones and finitary monads on \mathbf{Set} preserves semantics.

Note that by combining the previous example we obtain the chain

$$[\mathbf{N}, \mathbf{Set}] \xrightarrow{\text{Lan}_J} [\mathbf{P}, \mathbf{Set}] \xrightarrow{\text{Lan}_{J'}} [\mathbf{F}, \mathbf{Set}] \xrightarrow{\text{Lan}_{J''}} [\mathbf{Set}, \mathbf{Set}]$$

of strong monoidal functors, connecting non-symmetric and symmetric operads with monads on \mathbf{Set} . ■

Example 3.50. Let \mathcal{M} be a metatheory, \mathcal{C} a large category, and $*$ a *pseudo* action of \mathcal{M} on \mathcal{C} . We obtain a strong monoidal functor

$$F: \mathcal{M} \longrightarrow [\mathcal{C}, \mathcal{C}]$$

(where $[\mathcal{C}, \mathcal{C}]$ is equipped with the composition monoidal structure) as the transpose of $*$: $\mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C}$. The functor $\mathbf{Th}(F)$ maps any theory $\mathbb{T} = (T, e, m)$ in \mathcal{M} to the monad $F(\mathbb{T}) = (T * (-), e * (-), m * (-))$ on \mathcal{C} . The 2-functor $\mathcal{M}\text{Mod}(F^*): \mathcal{M}\text{Mod}([\mathcal{C}, \mathcal{C}]) \longrightarrow \mathcal{M}\text{Mod}(\mathcal{M})$ maps the standard metamodell Φ of $[\mathcal{C}, \mathcal{C}]$ in \mathcal{C} (Example 3.25) to the metamodell $F^*(\Phi): \mathcal{M}^{\text{op}} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathbf{SET}$ mapping (X, A, B) to

$$\mathcal{C}((FX)A, B) = \mathcal{C}(X * A, B).$$

Therefore it maps the standard metamodell Φ to the metamodell induced from $*$.

As a special case, for a large category \mathcal{C} with finite limits and a cartesian monad \mathbf{S} on \mathcal{C} , the standard metamodell for \mathbf{S} -operads (Example 3.26) may be related to the standard metamodell of monads on \mathcal{C} , and models of an \mathbf{S} -operad \mathbb{T} may alternatively be defined as Eilenberg–Moore algebras of the monad on \mathcal{C} induced from \mathbb{T} (as noted in [64]). ■

We have introduced a notion of morphism between metatheories, which is more general than both lax monoidal functors and oplax monoidal functors (in the opposite direction). As we pointed out, an adjunction of morphisms between metatheories are rich enough to generate isomorphisms of categories of models. Moreover, such adjunctions abound, as every strong monoidal functor generates one.

3.4 Related work

There are a few recent papers [18, 39, 3] which develop unified account of various notions of algebraic theory.

The papers by Curien [18] and Hyland [39] concentrate on clones, symmetric operads and non-symmetric operads, and concern primarily the conceptual understanding of the substitution monoidal structures. Via the theory of pseudo-distributive laws [83], they reduce substitution monoidal structures to certain 2-monads on \mathcal{Cat} , for example the free cartesian category 2-monad in the case of clones. Their work illuminates the relationship between the notions of algebraic theory they treat and their standard metamodells, because the standard metamodells arise as Eilenberg–Moore algebras of the 2-monad from which the corresponding substitution monoidal structure is induced. On the other hand, monads and generalised operads do not seem to be captured by their framework.

The framework by Avery [3] is relative to a well-behaved 2-category (which he calls a *setting*). In the basic setting of \mathcal{CAT} he identifies algebraic theories with identity-on-objects functor from a certain category \mathcal{A} of arities, calling them *proto-theories*. In this case, the relationship to our work may be established by the fact that (putting size issues aside) identity-on-objects functors from \mathcal{A} correspond to monoid objects in $[\mathcal{A}^{\text{op}} \times \mathcal{A}, \mathbf{SET}]$ (with the profunctor composition as the monoidal structure). This way we may understand Avery's framework (with respect to the setting \mathcal{CAT}) within ours, although for general setting probably we cannot do so. However we remark that for specific examples of settings treated in [3], it seems that proto-theories therein can be identified with monoid objects in the category of a suitable variant of profunctors.

Avery's framework has an attractive feature that it can treat Lawvere theories, PROPs, PROs, symmetric and non-symmetric operads by choosing a suitable setting, without requiring any complicated calculation (cf. the definition of substitution monoidal product and the relevant enrichments in Section 3.1). Generalised operads do not seem to be captured in Avery's framework.

Avery does not consider the questions of functoriality that arise at various levels. Note that, in contrast, we have defined morphisms of metamodells, of metatheories, and so on, which suitably act on the relevant constructions.

Chapter 4

Structure-semantics adjunctions

Structure-semantics adjunctions are a classical topic in categorical algebra. They are a family of adjunctions parametrised by a metatheory \mathcal{M} and its metamodel $(\mathcal{C}, \Phi, \bar{\phi}, \bar{\phi})$; if we fix these parameters, the structure-semantics adjunction is *ideally* of type

$$\mathbf{Th}(\mathcal{M})^{\text{op}} \begin{array}{c} \xleftarrow{\text{Str}} \\ \perp \\ \xrightarrow{\text{Sem}} \end{array} \mathbf{CAT}/\mathcal{C}, \quad (4.1)$$

and the functor Sem is essentially $\mathbf{Mod}(-, (\mathcal{C}, \Phi))$. Various authors have constructed such adjunctions for a variety of notions of algebraic theory, most notably for clones [58, 66, 42] and monads [20, 81]. There were also some attempts to unify these results [67, 3]. See Section 4.1 for the ideas behind such adjunctions.

If we try to work this idea out, however, there turn out to be size-issues or other problems, and usually we cannot obtain an adjunction of type (4.1); we cannot find a suitable functor Str of that type. To get an adjunction, various conditions on objects in \mathbf{CAT}/\mathcal{C} were introduced in the literature in order to single out well-behaved (usually called *tractable*) objects, yielding a restricted version of (4.1):

$$\mathbf{Th}(\mathcal{M})^{\text{op}} \begin{array}{c} \xleftarrow{\text{Str}} \\ \perp \\ \xrightarrow{\text{Sem}} \end{array} (\mathbf{CAT}/\mathcal{C})_{\text{tr}}. \quad (4.2)$$

Here, $(\mathbf{CAT}/\mathcal{C})_{\text{tr}}$ is the full-subcategory of \mathbf{CAT}/\mathcal{C} consisting of all tractable objects.

In this chapter, we construct a structure-semantics adjunction for an arbitrary metatheory and an arbitrary metamodel of it. Of course, we cannot obtain an adjunction of type (4.1), for the same reasons that have prevented other authors from doing so. However, we shall obtain a modified adjunction by a strategy different from theirs (and similar to [67, 3]): instead of restricting \mathbf{CAT}/\mathcal{C} , we extend $\mathbf{Th}(\mathcal{M})$ to $\mathbf{Th}(\widehat{\mathcal{M}})^1$ (where $\widehat{\mathcal{M}} = [\mathcal{M}^{\text{op}}, \mathbf{SET}]$ is equipped with the convolution monoidal structure), and obtain an extended version of (4.1):

$$\mathbf{Th}(\widehat{\mathcal{M}})^{\text{op}} \begin{array}{c} \xleftarrow{\text{Str}} \\ \perp \\ \xrightarrow{\text{Sem}} \end{array} \mathbf{CAT}/\mathcal{C}. \quad (4.3)$$

We may then obtain known adjunctions of the form (4.2), at least for clones and monads, by suitably restricting (4.3).

4.1 The idea of structure-semantics adjunctions

This section is an introduction to the idea of structure-semantics adjunctions. We start with an informal explanation of a duality between *sentences* and *struc-*

¹The monoidal category $\widehat{\mathcal{M}}$ is not a metatheory because it is not large. Extending Definition 3.32, by $\mathbf{Th}(\widehat{\mathcal{M}})$ we mean the category of monoids in $\widehat{\mathcal{M}}$.

tures [59], which may be seen as a degenerate version of structure-semantics adjunctions. Given any sentence ϕ of a suitable type and any structure A of a suitable type, suppose we know whether ϕ holds in A (written as $A \models \phi$) or not. Then, from a set of sentences Φ we may define a set $\text{Mod}(\Phi)$ of structures, whose elements are called *models of Φ* :

$$\text{Mod}(\Phi) = \{ A \mid A \models \phi \text{ for all } \phi \in \Phi \}.$$

Conversely, from a set of structures \mathcal{A} we get a set $\text{Thm}(\mathcal{A})$ of sentences, whose elements we call *theorems of \mathcal{A}* :

$$\text{Thm}(\mathcal{A}) = \{ \phi \mid A \models \phi \text{ for all } A \in \mathcal{A} \}.$$

It is straightforward to see that Mod and Thm form a Galois connection: for any set Φ of sentences and any set \mathcal{A} of structures,

$$\Phi \subseteq \text{Thm}(\mathcal{A}) \iff \mathcal{A} \subseteq \text{Mod}(\Phi)$$

holds. The setting of universal algebra (see Section 2.1) provides a concrete example. For a fixed graded set (signature) Σ , the notions of Σ -equation (Definition 2.5) and Σ -algebra (Definition 2.2) play the roles of sentence and structure respectively, with the relation \models defined as in Definition 2.9.

In various fields in mathematics, it has been observed that behind classical Galois connections there often hide more profound adjunctions [59]; the structure-semantics adjunctions are what we may find behind the above duality between sentences and structures. For example, the structure-semantics adjunctions for clones refine and unify the dualities for universal algebra for arbitrary graded sets Σ . Given a *small* category \mathcal{C} with finite powers, the structure-semantics adjunction for clones with respect to \mathcal{C} may be formulated as an adjunction

$$\mathbf{Clo}^{\text{op}} \begin{array}{c} \xleftarrow{\text{Str}} \\ \perp \\ \xrightarrow{\text{Sem}} \end{array} \mathbf{Cat}/\mathcal{C}, \quad (4.4)$$

where $\mathbf{Clo} = \mathbf{Th}([\mathbf{F}, \mathbf{Set}])$ is the category of clones and \mathbf{Cat}/\mathcal{C} is a slice category. We already know what the functor Sem does: it maps a clone \mathbb{T} to the category $\mathbf{Mod}(\mathbb{T}, \mathcal{C})$ of models of \mathbb{T} in \mathcal{C} (with respect to the standard metamodel as in Example 3.14) equipped with the forgetful functor $U: \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \rightarrow \mathcal{C}$. The functor Str , in this case, maps any functor $V: \mathcal{A} \rightarrow \mathcal{C}$ with small domain \mathcal{A} to the clone whose underlying graded set is given by $([\mathcal{A}, \mathcal{C}]((-)^n \circ V, V))_{n \in \mathbb{N}}$, where $(-)^n: \mathcal{C} \rightarrow \mathcal{C}$ is the functor taking n -th powers, and whose clone operations canonically induced from powers in \mathcal{C} .

An object of \mathbf{Cat}/\mathcal{C} , say $V: \mathcal{A} \rightarrow \mathcal{C}$, may be seen as specifying an additional structure (of a very general type) on objects in \mathcal{C} , by viewing \mathcal{A} as the category of \mathcal{C} -objects with the additional structure and V as the associated forgetful functor. The functor Str extracts a clone from V , giving the best approximation of this additional structure by structures expressible by clones.

We remark that if we take a *locally small* category \mathcal{C} , as is often the case of interest (e.g., $\mathcal{C} = \mathbf{Set}$), then in general we cannot have an adjunction

$$\mathbf{Clo}^{\text{op}} \begin{array}{c} \xleftarrow{\text{Str}} \\ \perp \\ \xrightarrow{\text{Sem}} \end{array} \mathbf{CAT}/\mathcal{C}.$$

The above construction fails because for an object $V: \mathcal{A} \rightarrow \mathcal{C}$ in \mathbf{CAT}/\mathcal{C} and a natural number n , the set $[\mathcal{A}, \mathcal{C}]((-)^n \circ V, V)$ may not be small. Indeed, a functor $V: \mathcal{A} \rightarrow \mathcal{C}$ is called *tractable* in [66] precisely when the sets of the form $[\mathcal{A}, \mathcal{C}]((-)^n \circ V, V)$ are small. We obtain an adjunction if we restrict \mathbf{CAT}/\mathcal{C} to its full subcategory consisting of all tractable functors.

4.2 The structure and semantics functors

Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a metatheory, \mathcal{C} be a large category, and $\Phi = (\Phi, \bar{\phi}, \phi)$ be a metamodel of \mathcal{M} in \mathcal{C} . The metamodel Φ enables us to define, for each theory $\mathbb{T} \in \mathbf{Th}(\mathcal{M})$, the category of models $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi))$ together with the forgetful functor $U: \mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi)) \rightarrow \mathcal{C}$. This construction is functorial, and gives rise to a functor

$$\mathbf{Th}(\mathcal{M})^{\text{op}} \rightarrow \mathbf{CAT}/\mathcal{C}.$$

However, as we have remarked in Proposition 3.44, a metamodel of \mathcal{M} in \mathcal{C} corresponds to an enrichment of \mathcal{C} over $\widehat{\mathcal{M}}$; hence using Φ we can actually give the definition of models for any theory (i.e., monoid object) in $\widehat{\mathcal{M}}$. Therefore the previous functor can be extended to

$$\text{Sem}: \mathbf{Th}(\widehat{\mathcal{M}})^{\text{op}} \rightarrow \mathbf{CAT}/\mathcal{C}. \quad (4.5)$$

The category $\mathbf{Th}(\widehat{\mathcal{M}})$ is isomorphic to the category of lax monoidal functors of type $\mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$ and monoidal natural transformations between them. Indeed, an object (P, e, m) of $\mathbf{Th}(\widehat{\mathcal{M}})$ consists of:

- a functor $P: \mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$;
- a natural transformation $(e_X: \widehat{I}(X) \rightarrow P(X))_{X \in \mathcal{M}}$;
- a natural transformation $(m_X: (P \widehat{\otimes} P)(X) \rightarrow P(X))_{X \in \mathcal{M}}$

satisfying the monoid axioms, and such a data is equivalent to

- a functor $P: \mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$;
- a function $\bar{e}: 1 \rightarrow P(I)$;
- a natural transformation $(\bar{m}_{X,Y}: P(Y) \times P(X) \rightarrow P(Y \otimes X))_{X,Y \in \mathcal{M}}$

satisfying the axioms for (P, \bar{e}, \bar{m}) to be a lax monoidal functor $\mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$. We shall use these two descriptions of objects of the category $\mathbf{Th}(\widehat{\mathcal{M}})$ interchangeably.

Let us describe the action of the functor Sem concretely. For any $\mathbb{P} = (P, e, m) \in \mathbf{Th}(\widehat{\mathcal{M}})$, we define the category $\mathbf{Mod}(\mathbb{P}, (\mathcal{C}, \Phi))$ as follows:

- An object is a pair consisting of an object $C \in \mathcal{C}$ and a natural transformation

$$(\xi_X: P(X) \rightarrow \Phi_X(C, C))_{C \in \mathcal{M}}$$

making the following diagrams commute for each $X, Y \in \mathcal{M}$:

$$\begin{array}{ccc} 1 & \xrightarrow{\bar{e}} & P(I) & & P(Y) \times P(X) & \xrightarrow{\bar{m}_{X,Y}} & P(Y \otimes X) \\ & \searrow & \downarrow \xi_I & & \downarrow \xi_Y \times \xi_X & & \downarrow \xi_{Y \otimes X} \\ & & \Phi_I(C, C) & & \Phi_Y(C, C) \times \Phi_X(C, C) & \xrightarrow{(\bar{\phi}_{X,Y})_{C,C,C}} & \Phi_{Y \otimes X}(C, C). \end{array} \quad (4.6)$$

- A morphism from (C, ξ) to (C', ξ') is a morphism $f: C \rightarrow C'$ in \mathcal{C} making the following diagram commute for each $X \in \mathcal{M}$:

$$\begin{array}{ccc}
P(X) & \xrightarrow{\xi_X} & \Phi_X(C, C) \\
\xi'_X \downarrow & & \downarrow \Phi_X(C, f) \\
\Phi_X(C', C') & \xrightarrow{\Phi_X(f, C')} & \Phi_X(C, C').
\end{array} \tag{4.7}$$

There exists an evident forgetful functor $U: \mathbf{Mod}(\mathbf{P}, (\mathcal{C}, \Phi)) \rightarrow \mathcal{C}$ mapping (C, ξ) to C and f to f ; the functor \mathbf{Sem} maps \mathbf{P} to U .

We have a canonical fully faithful functor

$$J: \mathbf{Th}(\mathcal{M}) \rightarrow \mathbf{Th}(\widehat{\mathcal{M}})$$

mapping $(T, e, m) \in \mathbf{Th}(\mathcal{M})$ to the functor $\mathcal{M}(-, T)$ with the evident monoid structure induced from e and m . An object $(P, e, m) \in \mathbf{Th}(\widehat{\mathcal{M}})$ is in the essential image of J if and only if $P: \mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$ is representable.

Let us describe the left adjoint \mathbf{Str} to (4.5). Given an object $V: \mathcal{A} \rightarrow \mathcal{C}$ of \mathbf{CAT}/\mathcal{C} , we define $\mathbf{Str}(V) = (P^{(V)}, e^{(V)}, m^{(V)}) \in \mathbf{Th}(\widehat{\mathcal{M}})$ as follows:

- The functor $P^{(V)}: \mathcal{M}^{\text{op}} \rightarrow \mathbf{SET}$ maps $X \in \mathcal{M}$ to

$$P^{(V)}(X) = \int_{A \in \mathcal{A}} \Phi_X(VA, VA). \tag{4.8}$$

- The function $\overline{e^{(V)}}: 1 \rightarrow P^{(V)}(I)$ maps the unique element of 1 to $((\overline{\phi})_{VA}(*))_{A \in \mathcal{A}} \in P^{(V)}(I)$.
- The (X, Y) -th component of the natural transformation

$$\overline{m^{(V)}}_{X, Y}: P^{(V)}(Y) \times P^{(V)}(X) \rightarrow P^{(V)}(Y \otimes X)_{X, Y \in \mathcal{M}}$$

maps $((y_A)_{A \in \mathcal{A}}, (x_A)_{A \in \mathcal{A}})$ to $((\overline{\phi}_{X, Y})_{VA, VA, VA}(y_A, x_A))_{A \in \mathcal{A}}$.

The monoid axioms for $(P^{(V)}, e^{(V)}, m^{(V)})$ follow easily from the axioms for meta-models, and \mathbf{Str} routinely extends to a functor of type $\mathbf{CAT}/\mathcal{C} \rightarrow \mathbf{Th}(\widehat{\mathcal{M}})^{\text{op}}$.

Theorem 4.1. *Let \mathcal{M} be a metatheory, \mathcal{C} be a large category and $\Phi = (\Phi, \overline{\phi}, \overline{\phi})$ be a metamodel of \mathcal{M} in \mathcal{C} . The functors \mathbf{Sem} and \mathbf{Str} defined above form an adjunction:*

$$\mathbf{Th}(\widehat{\mathcal{M}})^{\text{op}} \begin{array}{c} \xleftarrow{\mathbf{Str}} \\ \perp \\ \xrightarrow{\mathbf{Sem}} \end{array} \mathbf{CAT}/\mathcal{C}.$$

Proof. We show that there are bijections

$$\mathbf{Th}(\widehat{\mathcal{M}})(\mathbf{P}, \mathbf{Str}(V)) \cong (\mathbf{CAT}/\mathcal{C})(V, \mathbf{Sem}(\mathbf{P}))$$

natural in $\mathbf{P} = (P, e, m) \in \mathbf{Th}(\widehat{\mathcal{M}})$ and $(V: \mathcal{A} \rightarrow \mathcal{C}) \in \mathbf{CAT}/\mathcal{C}$.

In fact, we show that the following three types of data naturally correspond to each other.

1. A morphism $\alpha: \mathbf{P} \rightarrow \mathbf{Str}(V)$ in $\mathbf{Th}(\widehat{\mathcal{M}})$; that is, a natural transformation

$$(\alpha_X: P(X) \rightarrow P^{(V)}(X))_{X \in \mathcal{M}}$$

making the suitable diagrams commute.

2. A natural transformation

$$(\xi_{A,X}: P(X) \longrightarrow \Phi_X(VA, VA))_{X \in \mathcal{M}, A \in \mathcal{A}}$$

making the following diagrams commute for each $A \in \mathcal{A}$ and $X, Y \in \mathcal{M}$:

$$\begin{array}{ccccc} 1 & \xrightarrow{\bar{e}} & P(I) & & P(Y) \times P(X) \xrightarrow{\bar{m}_{X,Y}} P(Y \otimes X) \\ & \searrow^{(\bar{\phi})_{VA}} & \downarrow \xi_{A,I} & & \downarrow \xi_{A,Y \otimes X} \\ & & \Phi_I(VA, VA) & \xrightarrow{\xi_{A,Y} \times \xi_{A,X}} & \Phi_Y(VA, VA) \times \Phi_X(VA, VA) \xrightarrow{(\bar{\phi}_{X,Y})_{VA, VA, VA}} \Phi_{Y \otimes X}(VA, VA). \end{array}$$

3. A morphism $F: V \longrightarrow \mathbf{Sem}(\mathbf{P})$ in \mathbf{CAT}/\mathcal{C} ; that is, a functor $F: \mathcal{A} \longrightarrow \mathbf{Mod}(\mathbf{P}, (\mathcal{C}, \Phi))$ such that $U \circ F = V$ ($U: \mathbf{Mod}(\mathbf{P}, (\mathcal{C}, \Phi)) \longrightarrow \mathcal{C}$ is the forgetful functor).

The correspondence between 1 and 2 is by the universality of ends (see (4.8)). To give ξ as in 2 *without* requiring naturality in $A \in \mathcal{A}$, is equivalent to give a function $\text{ob}(F): \text{ob}(\mathcal{A}) \longrightarrow \text{ob}(\mathbf{Mod}(\mathbf{P}, (\mathcal{C}, \Phi)))$ such that $\text{ob}(U) \circ \text{ob}(F) = \text{ob}(V)$ (see (4.6)). To say that ξ is natural also in $A \in \mathcal{A}$ is equivalent to saying that $\text{ob}(F)$ extends to a functor $F: \mathcal{A} \longrightarrow \mathbf{Mod}(\mathbf{P}, (\mathcal{C}, \Phi))$ by mapping each morphism f in \mathcal{A} to Vf . \square

4.3 The classical cases

We conclude this chapter by showing that we can restore the known structure-semantics adjunctions for clones and monads, by restricting our version of structure-semantics adjunctions (Theorem 4.1).

In both cases of clones and monads, we shall consider the diagram

$$\begin{array}{ccc} \mathbf{Th}(\widehat{\mathcal{M}})^{\text{op}} & \begin{array}{c} \xleftarrow{\text{Str}} \\ \perp \\ \xrightarrow{\text{Sem}} \end{array} & \mathbf{CAT}/\mathcal{C} \\ J \uparrow & & \uparrow K \\ \mathbf{Th}(\mathcal{M})^{\text{op}} & \begin{array}{c} \xleftarrow{\text{Str}'} \\ \perp \\ \xrightarrow{\text{Sem}'} \end{array} & (\mathbf{CAT}/\mathcal{C})_{\text{tr}} \end{array}$$

in which the top adjunction is the one we have constructed in the previous section, the bottom adjunction is a classical structure-semantics adjunction, and J and K are the canonical fully faithful functors (the precise definition of $(\mathbf{CAT}/\mathcal{C})_{\text{tr}}$ will be given below). We shall prove that the two squares, one involving Str and Str' , the other involving Sem and Sem' , commute, showing that Str' (resp. Sem') arises as a restriction of Str (resp. Sem).

First, that $K \circ \text{Sem}' \cong \text{Sem} \circ J$ holds is straightforward, and this is true as soon as Sem' maps any $\mathbf{T} \in \mathbf{Th}(\mathcal{M})$ to the forgetful functor $U: \mathbf{Mod}(\mathbf{T}, (\mathcal{C}, \Phi)) \longrightarrow \mathcal{C}$. Indeed, for any theory $\mathbf{T} = (T, e, m)$ in \mathcal{M} , $J(\mathbf{T}) \in \mathbf{Th}(\widehat{\mathcal{M}})$ has the underlying object $\mathcal{M}(-, T) \in \widehat{\mathcal{M}}$, and the description of $\mathbf{Mod}(J(\mathbf{T}), (\mathcal{C}, \Phi))$ in the previous section coincides with $\mathbf{Mod}(\mathbf{T}, (\mathcal{C}, \Phi))$ by the Yoneda lemma.

Let us check that $J \circ \text{Str}' \cong \text{Str} \circ K$ holds.² For this, we have to review the classical structure functors and the tractability conditions.

²This does not seem to follow formally from $K \circ \text{Sem}' \cong \text{Sem} \circ J$, even if we take into consideration the fact that J and K are fully faithful.

We begin with the case of clones as treated in [66], which we have already sketched in Section 4.1. Let \mathcal{C} be a locally small category with finite powers and consider the standard metamodel Φ of $[\mathbf{F}, \mathbf{Set}]$ in \mathcal{C} (derived from the enrichment $\langle -, - \rangle$ in Example 3.14). An object $V: \mathcal{A} \rightarrow \mathcal{C} \in \mathbf{CAT}/\mathcal{C}$ is called **tractable** if and only if for any natural number n , the set $[\mathcal{A}, \mathcal{C}]((-)^n \circ V, V)$ is small. Given a tractable V , $\text{Str}'(V) \in \mathbf{Th}([\mathbf{F}, \mathbf{Set}])$ has the underlying functor $|\text{Str}'(V)|$ mapping $[n] \in \mathbf{F}$ to $[\mathcal{A}, \mathcal{C}]((-)^n \circ V, V)$. On the other hand, our formula (4.8) reduces as follows:

$$\begin{aligned}
P^{(V)}(X) &= \int_{A \in \mathcal{A}} \Phi_X(VA, VA) \\
&= \int_{A \in \mathcal{A}} [\mathbf{F}, \mathbf{Set}](X, \langle VA, VA \rangle) \\
&\cong \int_{A \in \mathcal{A}, [n] \in \mathbf{F}} \mathbf{Set}(X_n, \mathcal{C}((VA)^n, VA)) \\
&\cong \int_{[n] \in \mathbf{F}} \mathbf{Set}\left(X_n, \int_{A \in \mathcal{A}} \mathcal{C}((VA)^n, VA)\right) \\
&\cong \int_{[n] \in \mathbf{F}} \mathbf{Set}(X_n, [\mathcal{A}, \mathcal{C}]((-)^n \circ V, V)) \\
&\cong [\mathbf{F}, \mathbf{Set}](X, |\text{Str}'(V)|).
\end{aligned}$$

It is routine from this to see that $J \circ \text{Str}' \cong \text{Str} \circ K$ holds.

Finally, for monads, we take as a classical structure-semantics adjunction the one in [20, Section II. 1]. Let \mathcal{C} be a large category and consider the standard metamodel Φ of $[\mathcal{C}, \mathcal{C}]$ in \mathcal{C} (derived from the standard strict action $*$ in Example 3.25). An object $V: \mathcal{A} \rightarrow \mathcal{C} \in \mathbf{CAT}/\mathcal{C}$ is called **tractable** if and only if the right Kan extension $\text{Ran}_V V$ of V along itself exists.³ It is known that a functor of the form $\text{Ran}_V V$ acquires a canonical monad structure, and the resulting monad is called the codensity monad of V . For a tractable V , $\text{Str}'(V)$ is defined to be the codensity monad of V . Now let us return to our formula (4.8):

$$\begin{aligned}
P^{(V)}(X) &= \int_{A \in \mathcal{A}} \Phi_X(VA, VA) \\
&= \int_{A \in \mathcal{A}} \mathcal{C}(XVA, VA) \\
&\cong [\mathcal{A}, \mathcal{C}](X \circ V, V) \\
&\cong [\mathcal{C}, \mathcal{C}](X, \text{Ran}_V V).
\end{aligned}$$

Again we see that $J \circ \text{Str}' \cong \text{Str} \circ K$ holds.

³In fact, in [20, p. 68] Dubuc defines tractability as a slightly stronger condition. However, the condition we have introduced above is the one which is used for the construction of structure-semantics adjunctions in [20].

Chapter 5

Categories of models as double limits

Let \mathcal{M} be a metatheory, \mathbb{T} be a theory in \mathcal{M} , \mathcal{C} be a large category and Φ be a metamodel of \mathcal{M} in \mathcal{C} . Given these data, in Definition 3.37 we have defined—in a concrete manner—the category $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi))$ of models of \mathbb{T} in \mathcal{C} with respect to Φ , equipped with the evident forgetful functor $U: \mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi)) \rightarrow \mathcal{C}$.

In this chapter, we give an abstract characterisation of the categories of models. A similar result is known for the Eilenberg–Moore category of a monad; Street [81] has proved that it can be abstractly characterised as the lax limit in \mathcal{CAT} of a certain diagram canonically constructed from the original monad. We prove that the categories of models in our framework can also be characterised by a certain universal property. A suitable language to express this universal property is that of *pseudo double categories* [34], reviewed in Section 5.2. We show that the category $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi))$, together with the forgetful functor U and some other natural data, form a double limit in the pseudo double category \mathbf{PROF} of large categories, profunctors, functors and natural transformations.

5.1 The universality of Eilenberg–Moore categories

In this section we review the 2-categorical characterisation in [81] of the Eilenberg–Moore category of a monad on a large category, in elementary terms.¹ Let \mathcal{C} be a large category and $\mathbb{T} = (T, \eta, \mu)$ be a monad on \mathcal{C} . The Eilenberg–Moore category $\mathcal{C}^{\mathbb{T}}$ of \mathbb{T} is equipped with a canonical forgetful functor $U: \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$ mapping an Eilenberg–Moore algebra (C, γ) of \mathbb{T} to its underlying object C . Moreover, there exists a canonical natural transformation $u: T \circ U \Rightarrow U$, i.e., of type

$$\begin{array}{ccc} \mathcal{C}^{\mathbb{T}} & \xrightarrow{\text{id}_{\mathcal{C}^{\mathbb{T}}}} & \mathcal{C}^{\mathbb{T}} \\ U \downarrow & \uparrow u & \downarrow U \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C}. \end{array}$$

We are depicting u in a square rather than in a triangle for later comparison with similar diagrams in a pseudo double category. For each $(C, \gamma) \in \mathcal{C}^{\mathbb{T}}$, the (C, γ) -th component of u is simply $\gamma: TC \rightarrow C$. We claim that the data $(\mathcal{C}^{\mathbb{T}}, U, u)$ is characterised by a certain universal property.

¹The main point of the paper [81] is the introduction of the notion of Eilenberg–Moore object in an *arbitrary 2-category* \mathcal{B} via a universal property and show that, if exists, it satisfies certain formal properties of Eilenberg–Moore categories. However, for our purpose, it suffices to consider the simple case $\mathcal{B} = \mathcal{CAT}$ only. It is left as future work to investigate whether we can develop a similar “formal theory” from the double-categorical universal property of categories of models.

To state this universal property, let us define a **left T-module** to be a triple (\mathcal{A}, V, v) consisting of a large category \mathcal{A} , a functor $V: \mathcal{A} \rightarrow \mathcal{C}$ and a natural transformation $v: T \circ V \Rightarrow V$, such that the following equations hold:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ V \downarrow & \uparrow v & \downarrow V \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ & \uparrow \eta & \\ & \text{id}_{\mathcal{C}} & \end{array} = \begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ V \downarrow & \uparrow \text{id}_V & \downarrow V \\ \mathcal{C} & \xrightarrow{\text{id}_{\mathcal{C}}} & \mathcal{C} \end{array}$$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ V \downarrow & \uparrow v & \downarrow V \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ & \uparrow \mu & \\ & T \circ T & \end{array} = \begin{array}{ccccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ V \downarrow & \uparrow v & \downarrow V & \uparrow v & \downarrow V \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array}$$

The triple $(\mathcal{C}^{\mathbb{T}}, U, u)$ is then a **universal left T-module**, meaning that it satisfies the following:

1. it is a left T-module;
2. for any left T-module (\mathcal{A}, V, v) , there exists a unique functor $K: \mathcal{A} \rightarrow \mathcal{C}^{\mathbb{T}}$ such that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ V \downarrow & \uparrow v & \downarrow V \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array} = \begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ K \downarrow & \uparrow \text{id}_K & \downarrow K \\ \mathcal{C}^{\mathbb{T}} & \xrightarrow{\text{id}_{\mathcal{C}^{\mathbb{T}}}} & \mathcal{C}^{\mathbb{T}} \\ U \downarrow & \uparrow u & \downarrow U \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array}$$

holds;

3. for any pair of left T-modules (\mathcal{A}, V, v) and (\mathcal{A}, V', v') on a common large category \mathcal{A} and any natural transformation $\theta: V \Rightarrow V'$ such that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ V \downarrow & v \uparrow & V \left(\begin{array}{c} \theta \\ \Rightarrow \end{array} \right) \downarrow \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array} = \begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{id}_{\mathcal{A}}} & \mathcal{A} \\ V \left(\begin{array}{c} \theta \\ \Rightarrow \end{array} \right) \downarrow & \uparrow v' & \downarrow V' \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array}$$

holds, there exists a unique natural transformation $\sigma: K \Rightarrow K'$ such that $\theta = U \circ \sigma$, where $K: \mathcal{A} \rightarrow \mathcal{C}^{\mathbb{T}}$ and $K': \mathcal{A} \rightarrow \mathcal{C}^{\mathbb{T}}$ are the functors corresponding to (\mathcal{A}, V, v) and (\mathcal{A}, V', v') respectively.

In more conceptual terms, this means that we have a family of isomorphisms of categories

$$\mathcal{CAT}(\mathcal{A}, \mathcal{C}^{\mathbb{T}}) \cong \mathcal{CAT}(\mathcal{A}, \mathcal{C})^{\mathcal{CAT}(\mathcal{A}, \mathbb{T})}$$

natural in $\mathcal{A} \in \mathcal{CAT}$, where the right hand side denotes the Eilenberg–Moore category of the monad $\mathcal{CAT}(\mathcal{A}, \mathbb{T})$; note that $\mathcal{CAT}(\mathcal{A}, -)$ is a 2-functor and therefore preserves monads.

It is straightforward to verify the above three statements on (\mathcal{C}^\top, U, u) . That (\mathcal{C}^\top, U, u) is a left \mathbb{T} -module follows from the definition of Eilenberg–Moore algebras. Given a left \mathbb{T} -module (\mathcal{A}, V, v) , for any object $A \in \mathcal{A}$ the pair (VA, v_A) is an Eilenberg–Moore algebra of \mathbb{T} . Hence the required functor $K: \mathcal{A} \rightarrow \mathcal{C}^\top$ can be defined by mapping an object $A \in \mathcal{A}$ to (VA, v_A) and a morphism f in \mathcal{A} to Vf . The final clause can be proved similarly. In fact, this automatically follows from the second clause since \mathcal{CAT} admits tensor products (= cartesian products) with the arrow category; see [54].

As with any universal characterisation, the above property characterises the triple (\mathcal{C}^\top, U, u) uniquely up to unique isomorphisms. One can also express this universal property in terms of the standard 2-categorical limit notions, such as lax limit or weighted 2-limit [82].

5.2 Pseudo double categories

We shall see that our category of models admit a similar characterisation, in a different setting: instead of the 2-category \mathcal{CAT} , we will work within the *pseudo double category* \mathbb{PROF} . The notion of pseudo double category is due to Grandis and Paré [34], and it generalises the classical notion of double category [21] in a way similar to the generalisation of 2-categories to bicategories, or to the generalisation of strict monoidal categories to monoidal categories. In this section we briefly review pseudo double categories, and introduce the pseudo double category \mathbb{PROF} .

Let us begin with an informal explanation of double categories. A double category consists of **objects** A , **vertical morphisms** $f: A \rightarrow A'$, **horizontal morphisms** $X: A \rightarrow B$ and **squares**

$$\begin{array}{ccc} A & \xrightarrow{X} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ A' & \xrightarrow{X'} & B' \end{array}$$

together with several identity and composition operations, namely:

- for each object A we have the **vertical identity morphism** $\text{id}_A: A \rightarrow A$;
- for each composable pair of vertical morphisms $f: A \rightarrow A'$ and $f': A' \rightarrow A''$ we have the **vertical composition** $f' \circ f: A \rightarrow A''$;
- for each horizontal morphism $X: A \rightarrow B$ we have the **vertical identity square**

$$\begin{array}{ccc} A & \xrightarrow{X} & B \\ \text{id}_A \downarrow & \Downarrow \text{id}_X & \downarrow \text{id}_B \\ A & \xrightarrow{X} & B \end{array}$$

- for each vertically composable pair of squares

$$\begin{array}{ccc} A & \xrightarrow{X} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ A' & \xrightarrow{X'} & B' \end{array} \quad \text{and} \quad \begin{array}{ccc} A' & \xrightarrow{X'} & B' \\ f' \downarrow & \Downarrow \alpha' & \downarrow g' \\ A'' & \xrightarrow{X''} & B'' \end{array}$$

we have the **vertical composition**

$$\begin{array}{ccc} A & \xrightarrow{X} & B \\ f' \circ f \downarrow & & \downarrow \alpha' \circ \alpha \\ A'' & \xrightarrow{X''} & B''; \end{array} \quad \begin{array}{c} \downarrow g' \circ g \\ \downarrow \end{array}$$

and symmetrically:

- for each object A we have the **horizontal identity morphism** $I_A: A \rightarrow A$;
- for each composable pair of horizontal morphisms $X: A \rightarrow B$ and $Y: B \rightarrow C$ we have the **horizontal composition** $Y \otimes X: A \rightarrow C$;
- for each vertical morphism $f: A \rightarrow A'$ we have the **horizontal identity square**

$$\begin{array}{ccc} A & \xrightarrow{I_A} & A \\ f \downarrow & & \downarrow f \\ A' & \xrightarrow{I_{A'}} & A'; \end{array} \quad \begin{array}{c} \downarrow I_f \\ \downarrow \end{array}$$

- for each horizontally composable pair of squares

$$\begin{array}{ccc} A & \xrightarrow{X} & B \\ f \downarrow & & \downarrow g \\ A' & \xrightarrow{X'} & B' \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{Y} & C \\ g \downarrow & & \downarrow h \\ B' & \xrightarrow{Y'} & C' \end{array} \quad \begin{array}{c} \downarrow \alpha \\ \downarrow \beta \end{array}$$

we have the **horizontal composition**

$$\begin{array}{ccc} A & \xrightarrow{Y \otimes X} & C \\ f \downarrow & & \downarrow h \\ A' & \xrightarrow{Y' \otimes X'} & C'. \end{array} \quad \begin{array}{c} \downarrow \beta \otimes \alpha \\ \downarrow \end{array}$$

These identity and composition operations are required to satisfy several axioms, such as the unit and associativity axioms for vertical (resp. horizontal) identity and composition, as well as the axiom $\text{id}_{I_A} = I_{\text{id}_A}$ for each object A and the *interchange law*, saying that whenever we have a configuration of squares as in

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array} \quad \begin{array}{c} \downarrow \alpha \\ \downarrow \beta \\ \downarrow \alpha' \\ \downarrow \beta' \end{array}$$

$(\beta' \otimes \alpha') \circ (\beta \otimes \alpha) = (\beta' \circ \beta) \otimes (\alpha' \circ \alpha)$ holds.

Some naturally arising double-category-like structure, including $\mathbb{P}\text{ROF}$, are such that whose vertical morphisms are homomorphism-like (e.g., functors) and whose horizontal morphisms are bimodule-like (e.g., profunctors); see [80, Section 1] for a discussion on these two kinds of morphisms. However, a problem crops up from the bimodule-like horizontal morphisms: in general, their composition is not unital nor associative on the nose. Therefore such structures fail to form (strict) double categories, but instead form *pseudo* (or *weak*) *double categories* [34, 64, 28, 80], in which horizontal composition is allowed to be unital and associative up to suitable isomorphism.²

Definition 5.1 ([34]). A **pseudo double category** \mathbb{D} consists of the following data.

- (DD1) A category \mathbb{D}_0 , whose objects are called **objects** of \mathbb{D} and whose morphisms **vertical morphisms** of \mathbb{D} .
- (DD2) A category \mathbb{D}_1 , whose objects are called **horizontal morphisms** of \mathbb{D} and whose morphisms **squares** of \mathbb{D} .
- (DD3) Functors

$$\begin{aligned} s, t: \mathbb{D}_1 &\longrightarrow \mathbb{D}_0, \\ I: \mathbb{D}_0 &\longrightarrow \mathbb{D}_1, \\ \otimes: \mathbb{D}_2 &\longrightarrow \mathbb{D}_1, \end{aligned}$$

where \mathbb{D}_2 is the pullback

$$\begin{array}{ccc} \mathbb{D}_2 & \xrightarrow{\pi_2} & \mathbb{D}_1 \\ \pi_1 \downarrow & \lrcorner & \downarrow t \\ \mathbb{D}_1 & \xrightarrow{s} & \mathbb{D}_0 \end{array}$$

of categories.

- (DD4) Natural isomorphisms with components

$$\begin{aligned} \mathbf{a}_{X,Y,Z}: (Z \otimes Y) \otimes X &\longrightarrow Z \otimes (Y \otimes X), \\ \mathbf{l}_X: I_B \otimes X &\longrightarrow X, \\ \mathbf{r}_X: X \otimes I_A &\longrightarrow X \end{aligned}$$

in \mathbb{D}_1 , where $(Z, Y, X) \in \mathbb{D}_3$ which is the pullback

$$\begin{array}{ccc} \mathbb{D}_3 & \longrightarrow & \mathbb{D}_1 \\ \downarrow & \lrcorner & \downarrow t \\ \mathbb{D}_2 & \xrightarrow{s \circ \pi_2} & \mathbb{D}_0 \end{array}$$

of categories and $X \in \mathbb{D}_1$ with $s(X) = A$ and $t(X) = B$.

These data are subject to the following axioms.

²In the literature, definitions of pseudo double category differ as to whether to weaken horizontal compositions or vertical compositions. We follow [28, 80] and weaken horizontal compositions, but note that the original paper [34] weakens vertical compositions.

(DA1) The diagrams

$$\begin{array}{ccc}
 & \mathbb{D}_0 & \\
 \text{id} \swarrow & \downarrow I & \searrow \text{id} \\
 \mathbb{D}_0 & \xleftarrow{s} \mathbb{D}_1 \xrightarrow{t} & \mathbb{D}_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathbb{D}_2 & \\
 s \circ \pi_2 \swarrow & \downarrow \otimes & \searrow t \circ \pi_1 \\
 \mathbb{D}_0 & \xleftarrow{s} \mathbb{D}_1 \xrightarrow{t} & \mathbb{D}_0
 \end{array}$$

commute (on the nose).

(DA2) The morphisms $s(\mathbf{a}_{X,Y,Z})$, $s(\mathbf{l}_X)$ and $s(\mathbf{r}_X)$ are equal to id_A for all $(Z, Y, X) \in \mathbb{D}_3$ and $X \in \mathbb{D}_1$, where $A = s(X)$. Similarly for t .

(DA3) The coherence axioms (triangle and pentagon) for \mathbf{a} , \mathbf{l} and \mathbf{r} . ■

See [28, Section 2.1] for the full details of the definition. Although Definition 5.1 might look quite different from the aforementioned informal description of double categories at the first sight, in fact it is not, and the only difference is the existence of isomorphisms \mathbf{a} , \mathbf{l} and \mathbf{r} instead of equalities. Perhaps it is worth remarking that the functors s and t are meant to assign the (horizontal) *sources* and *targets*, so given the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{X} & B \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 A' & \xrightarrow{X'} & B'
 \end{array}$$

in \mathbb{D} , we read as: A is the domain of f in \mathbb{D}_0 , X' is the codomain of α in \mathbb{D}_1 , $A = s(X)$, $g = t(\alpha)$, and so on.

We write the isomorphisms \mathbf{a} , \mathbf{l} and \mathbf{r} as

$$\begin{array}{ccc}
 A & \xrightarrow{X} & B & \xrightarrow{Z \otimes Y} & D \\
 & \searrow & \downarrow \cong & \searrow & \\
 Y \otimes X & \xrightarrow{\quad} & C & \xrightarrow{Z} & D
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{X} & B & \xrightarrow{I_B} & B \\
 & \searrow & \downarrow \cong & \searrow & \\
 & \xrightarrow{\quad} & X & \xrightarrow{\quad} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{I_A} & A & \xrightarrow{X} & B \\
 & \searrow & \downarrow \cong & \searrow & \\
 & \xrightarrow{\quad} & X & \xrightarrow{\quad} & B
 \end{array}$$

The suppression of the vertical morphisms in the above diagrams is justified by (DA2). Similarly we also denote inverses and composites of \mathbf{a} , \mathbf{l} and \mathbf{r} by unnamed double arrows labelled with \cong .

Example 5.2 ([34]). Let \mathcal{B} be a bicategory. This induces a pseudo double category $\mathbb{H}\mathcal{B}$, given as follows:

- an object of $\mathbb{H}\mathcal{B}$ is an object of \mathcal{B} ;
- all vertical morphisms of $\mathbb{H}\mathcal{B}$ are vertical identity morphisms;
- a horizontal morphism of $\mathbb{H}\mathcal{B}$ is a 1-cell of \mathcal{B} ;
- a square of $\mathbb{H}\mathcal{B}$ is a 2-cell of \mathcal{B} .

The isomorphisms \mathbf{a} , \mathbf{l} and \mathbf{r} of $\mathbb{H}\mathcal{B}$ is given by the corresponding iso-2-cells of \mathcal{B} .

Conversely, for any pseudo double category \mathbb{D} , we obtain a bicategory $\mathcal{H}\mathbb{D}$ given as follows:

- an object of $\mathcal{H}\mathbb{D}$ is an object of \mathbb{D} ;

- a 1-cell of $\mathcal{H}\mathbb{D}$ is a horizontal morphism of \mathbb{D} ;
- a 2-cell of $\mathcal{H}\mathbb{D}$ is a square in \mathbb{D} whose horizontal source and target are both vertical identity morphisms. \blacksquare

Let us introduce the pseudo double category \mathbf{PROF} .

Definition 5.3 ([34, Section 3.1]). We define the pseudo double category \mathbf{PROF} as follows.

- An object is a large category.
- A vertical morphism from \mathcal{A} to \mathcal{A}' is a functor $F: \mathcal{A} \rightarrow \mathcal{A}'$.
- A horizontal morphism from \mathcal{A} to \mathcal{B} is a profunctor $H: \mathcal{A} \multimap \mathcal{B}$, i.e., a functor $H: \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \mathbf{SET}$. Horizontal identities and horizontal compositions are the same as in Definition 3.34.
- A square as in

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{H} & \mathcal{B} \\ F \downarrow & \Downarrow \alpha & \downarrow G \\ \mathcal{A}' & \xrightarrow{H'} & \mathcal{B}' \end{array}$$

is a natural transformation

$$\alpha = (\alpha_{B,A}: H(B, A) \rightarrow H'(GB, FA))_{B \in \mathcal{B}, A \in \mathcal{A}},$$

that is, of type

$$\begin{array}{ccc} & H & \\ \mathcal{B}^{\text{op}} \times \mathcal{A} & \xrightarrow{\quad} & \mathbf{SET} \\ G^{\text{op}} \times F \searrow & \Downarrow \alpha & \nearrow H' \\ & \mathcal{B}'^{\text{op}} \times \mathcal{A}' & \end{array}$$

It is straightforward to define various compositions of these morphisms and squares. The isomorphisms \mathbf{a} , \mathbf{l} and \mathbf{r} are the same as those in the bicategory $\mathcal{P}\mathcal{R}\mathcal{O}\mathcal{F}$. (Indeed, using the construction introduced in Example 5.2, $\mathcal{P}\mathcal{R}\mathcal{O}\mathcal{F} = \mathcal{H}\mathbf{PROF}$.) \blacksquare

Given a pseudo double category \mathbb{D} , denote by \mathbb{D}^{op} , \mathbb{D}^{co} , and \mathbb{D}^{coop} the pseudo double categories obtained from \mathbb{D} by reversing the horizontal direction (swapping s and t), reversing the vertical direction (taking the opposites of \mathbb{D}_0 and \mathbb{D}_1) and reversing both the horizontal and vertical directions, respectively.³ In the following we shall mainly work within $\mathbf{PROF}^{\text{op}}$, though most of the diagrams are symmetric in the horizontal direction and this makes little difference. (In fact, the pseudo double category defined in [34, Section 3.1] amounts to our $\mathbf{PROF}^{\text{op}}$, because our convention on the direction of profunctors differs from theirs.)

³For a double category \mathbb{D} we also have the *transpose* \mathbb{D}^{tr} , obtained from \mathbb{D} by swapping the horizontal and vertical directions. However, in pseudo double categories the horizontal direction and the vertical direction are not symmetric and we no longer have this duality for them.

5.3 The universality of categories of models

Let \mathcal{M} be a metatheory, $\mathbb{T} = (T, e, m)$ be a theory in \mathcal{M} , \mathcal{C} be a large category, and $\Phi = (\Phi, \bar{\phi}, \bar{\phi})$ be a metamodel of \mathcal{M} in \mathcal{C} . Recall from Section 3.2.2 that in the data of the metamodel $(\Phi, \bar{\phi}, \bar{\phi})$, the natural transformations

$$((\bar{\phi})_C: 1 \longrightarrow \Phi_I(C, C))_{C \in \mathcal{C}}$$

and

$$((\bar{\phi}_{X,Y})_{A,B,C}: \Phi_Y(B, C) \times \Phi_X(A, B) \longrightarrow \Phi_{Y \otimes X}(A, C))_{X,Y \in \mathcal{M}, A,B,C \in \mathcal{C}},$$

may be replaced by the natural transformations

$$((\phi)_{A,B}: \mathcal{C}(A, B) \longrightarrow \Phi_I(A, B))_{A,B \in \mathcal{C}}$$

and

$$((\phi_{X,Y})_{A,B}: (\Phi_Y \odot^{\text{rev}} \Phi_X)(A, B) \longrightarrow \Phi_{Y \otimes X}(A, B))_{X,Y \in \mathcal{M}, A,B \in \mathcal{C}},$$

respectively. In this chapter we shall mainly use the expression of metamodel via the data (Φ, ϕ, ϕ) . The category of models $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi))$, henceforth abbreviated as $\mathbf{Mod}(\mathbb{T}, \mathcal{C})$, defined in Definition 3.37 admits a canonical forgetful functor $U: \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \longrightarrow \mathcal{C}$ and a natural transformation (a square in $\mathbf{PROF}^{\text{op}}$) u as in

$$\begin{array}{ccc} \mathbf{Mod}(\mathbb{T}, \mathcal{C}) & \xrightarrow{\mathbf{Mod}(\mathbb{T}, \mathcal{C})(-, -)} & \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \\ U \downarrow & \Downarrow u & \downarrow U \\ \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} \end{array}$$

Concretely, u is a natural transformation

$$(u_{(C, \xi), (C', \xi')}: \mathbf{Mod}(\mathbb{T}, \mathcal{C})((C, \xi), (C', \xi')) \longrightarrow \Phi_T(C, C'))_{(C, \xi), (C', \xi') \in \mathbf{Mod}(\mathbb{T}, \mathcal{C})}$$

whose $((C, \xi), (C', \xi'))$ -th component maps each morphism $f: (C, \xi) \longrightarrow (C', \xi')$ in $\mathbf{Mod}(\mathbb{T}, \mathcal{C})$ to the element $\Phi_T(C, f)(\xi) = \Phi_T(f, C')(\xi') \in \Phi_T(C, C')$. Alternatively, by the Yoneda lemma, u may be equivalently given as a natural transformation

$$(\bar{u}_{(C, \xi)}: 1 \longrightarrow \Phi_T(C, C))_{(C, \xi) \in \mathbf{Mod}(\mathbb{T}, \mathcal{C})}$$

whose (C, ξ) -th component maps the unique element of 1 to $\xi \in \Phi_T(C, C)$.

We claim that the triple $(\mathbf{Mod}(\mathbb{T}, \mathcal{C}), U, u)$ has a certain universal property.

Definition 5.4. Define a **vertical double cone over $\Phi(\mathbb{T})$** to be a triple (\mathcal{A}, V, v) consisting of a large category \mathcal{A} , a functor $V: \mathcal{A} \longrightarrow \mathcal{C}$, and a square v in $\mathbf{PROF}^{\text{op}}$ of type

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-, -)} & \mathcal{A} \\ V \downarrow & \Downarrow v & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} \end{array}$$

satisfying the following equations:

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-, -)} & \mathcal{A} \\ V \downarrow & \Downarrow v & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} \\ \downarrow \Phi_e & & \\ \mathcal{C} & \xrightarrow{\Phi_I} & \mathcal{C} \end{array} & = & \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-, -)} & \mathcal{A} \\ V \downarrow & \Downarrow I_V & \downarrow V \\ \mathcal{C} & \xrightarrow{\mathcal{C}(-, -)} & \mathcal{C} \\ \downarrow \phi & & \\ \mathcal{C} & \xrightarrow{\Phi_I} & \mathcal{C} \end{array} \end{array} \quad (5.1)$$

$$\begin{array}{ccc}
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\
\downarrow V & \Downarrow \Phi_T & \downarrow V \\
\mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} \\
\downarrow \Phi_{T \otimes T} & & \\
\mathcal{C} & \xrightarrow{\Phi_{T \otimes T}} & \mathcal{C}
\end{array} & = &
\begin{array}{ccccc}
& & \mathcal{A}(-,-) & & \\
& & \downarrow \cong & & \\
\mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\
\downarrow V & \Downarrow \Phi_T & \downarrow V & \Downarrow \Phi_T & \downarrow V \\
\mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} \\
\downarrow \Phi_{T \otimes T} & & \downarrow \Phi_{T \otimes T} & & \\
\mathcal{C} & \xrightarrow{\Phi_{T \otimes T}} & \mathcal{C} & \xrightarrow{\Phi_{T \otimes T}} & \mathcal{C}
\end{array}
\end{array} \quad (5.2)$$

■

Using this notion, we can state the universal property of the triple $(\mathbf{Mod}(\mathbb{T}, \mathcal{C}), U, u)$, just as in the case of Eilenberg–Moore categories.

Theorem 5.5. *Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a metatheory, $\mathbb{T} = (T, e, m)$ be a theory in \mathcal{M} , \mathcal{C} be a large category, and $\Phi = (\Phi, \phi, \phi)$ be a metamodel of \mathcal{M} in \mathcal{C} . The triple $(\mathbf{Mod}(\mathbb{T}, \mathcal{C}), U, u)$ defined above is a universal vertical double cone over $\Phi(\mathbb{T})$, namely:*

1. *it is a vertical double cone over $\Phi(\mathbb{T})$;*
2. *for any vertical double cone (\mathcal{A}, V, v) over $\Phi(\mathbb{T})$, there exists a unique functor $K: \mathcal{A} \rightarrow \mathbf{Mod}(\mathbb{T}, \mathcal{C})$ such that*

$$\begin{array}{ccc}
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\
\downarrow V & \Downarrow v & \downarrow V \\
\mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C}
\end{array} & = &
\begin{array}{ccccc}
& & \mathcal{A}(-,-) & & \\
& & \downarrow I_K & & \\
\mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\
\downarrow K & \Downarrow \text{Mod}(\mathbb{T}, \mathcal{C})(-,-) & \downarrow K & \Downarrow \text{Mod}(\mathbb{T}, \mathcal{C})(-,-) & \downarrow K \\
\mathbf{Mod}(\mathbb{T}, \mathcal{C}) & \xrightarrow{\text{Mod}(\mathbb{T}, \mathcal{C})(-,-)} & \mathbf{Mod}(\mathbb{T}, \mathcal{C}) & \xrightarrow{\text{Mod}(\mathbb{T}, \mathcal{C})(-,-)} & \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \\
\downarrow U & \Downarrow u & \downarrow U & \Downarrow u & \downarrow U \\
\mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C}
\end{array}
\end{array}$$

holds;

3. *for any pair of vertical cones (\mathcal{A}, V, v) and (\mathcal{A}', V', v') over $\Phi(\mathbb{T})$, any horizontal morphism $H: \mathcal{A} \rightarrow \mathcal{A}'$ in $\mathbf{PROF}^{\text{op}}$ and any square*

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{H} & \mathcal{A}' \\
\downarrow V & \Downarrow \theta & \downarrow V' \\
\mathcal{C} & \xrightarrow{\mathcal{C}(-,-)} & \mathcal{C}
\end{array}$$

in $\mathbf{PROF}^{\text{op}}$ such that

$$\begin{array}{ccc}
\begin{array}{ccccc}
& & H & & \\
& & \downarrow \cong & & \\
\mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} & \xrightarrow{H} & \mathcal{A}' \\
\downarrow V & \Downarrow v & \downarrow V & \Downarrow \theta & \downarrow V' \\
\mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} & \xrightarrow{\mathcal{C}(-,-)} & \mathcal{C} \\
\downarrow \Phi_T & & \downarrow \cong & & \\
\mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C}
\end{array} & = &
\begin{array}{ccccc}
& & H & & \\
& & \downarrow \cong & & \\
\mathcal{A} & \xrightarrow{H} & \mathcal{A}' & \xrightarrow{\mathcal{A}'(-,-)} & \mathcal{A}' \\
\downarrow V & \Downarrow \theta & \downarrow V' & \Downarrow v' & \downarrow V' \\
\mathcal{C} & \xrightarrow{\mathcal{C}(-,-)} & \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} \\
\downarrow \Phi_T & & \downarrow \cong & & \\
\mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C}
\end{array}
\end{array} \quad (5.3)$$

holds, there exists a unique square

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{H} & \mathcal{A}' \\
K \downarrow & & \downarrow \sigma \\
\mathbf{Mod}(\mathbb{T}, \mathcal{C}) & \xrightarrow{\mathbf{Mod}(\mathbb{T}, \mathcal{C})(-, -)} & \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \\
& & \downarrow K'
\end{array}$$

in $\mathbf{PROF}^{\text{op}}$ such that

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{H} & \mathcal{A}' \\
V \downarrow & \downarrow \theta & \downarrow V' \\
\mathcal{C} & \xrightarrow{\mathcal{C}(-, -)} & \mathcal{C}
\end{array}
=
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{H} & \mathcal{A}' \\
K \downarrow & & \downarrow \sigma \\
\mathbf{Mod}(\mathbb{T}, \mathcal{C}) & \xrightarrow{\mathbf{Mod}(\mathbb{T}, \mathcal{C})(-, -)} & \mathbf{Mod}(\mathbb{T}, \mathcal{C}) \\
U \downarrow & & \downarrow I_U \\
\mathcal{C} & \xrightarrow{\mathcal{C}(-, -)} & \mathcal{C}
\end{array}$$

holds, where K and K' are the functors corresponding to (\mathcal{A}, V, v) and (\mathcal{A}', V', v') respectively.

The above statements are taken from the definition of double limit [34, Section 4.2].

Proof of Theorem 5.5. First, that $(\mathbf{Mod}(\mathbb{T}, \mathcal{C}), U, u)$ is a vertical double cone over $\Phi(\mathbb{T})$ follows directly from the definition of model of \mathbb{T} in \mathcal{C} with respect to Φ (Definition 3.37).

Given a vertical double cone (\mathcal{A}, V, v) over $\Phi(\mathbb{T})$, for each object $A \in \mathcal{A}$, the pair $(VA, v_{A,A}(\text{id}_A))$ is a \mathbb{T} -model in \mathcal{C} with respect to Φ , and for each morphism $f: A \rightarrow A'$ in \mathcal{A} , the morphism Vf is a \mathbb{T} -model homomorphism from $(VA, v_{A,A}(\text{id}_A))$ to $(VA', v_{A',A'}(\text{id}_{A'}))$. The functor $K: \mathcal{A} \rightarrow \mathbf{Mod}(\mathbb{T}, \mathcal{C})$ can therefore be given as $KA = (VA, v_{A,A}(\text{id}_A))$ and $Kf = Vf$. The uniqueness is clear.

Finally, given H and θ as in the third clause, the equation (5.3) says that for each $A \in \mathcal{A}$, $A' \in \mathcal{A}'$ and $x \in H(A, A')$, the morphism $\theta_{A,A'}(x): VA \rightarrow VA'$ in \mathcal{C} satisfies $\Phi_T(VA, \theta_{A,A'}(x))(v_{A,A}(\text{id}_A)) = \Phi_T(\theta_{A,A'}(x), VA')(v'_{A',A'}(\text{id}_{A'}))$; in other words, that $\theta_{A,A'}(x)$ is a \mathbb{T} -model homomorphism from KA to $K'A'$. The square σ can then be given as the natural transformation with $\sigma_{A,A'}(x) = \theta_{A,A'}(x)$. \square

5.4 Relation to double limits

In this final section of this chapter, we sketch how the double categorical universal property (Theorem 5.5) of categories of models in our framework can be expressed via the notion of double limit [34], connecting our characterisation to a well-established notion. A short outline of this reduction is as follows.

1. A theory \mathbb{T} in a metatheory \mathcal{M} may be equivalently given as a strong monoidal functor $\mathbb{T}: \Delta_a \rightarrow \mathcal{M}$, where Δ_a is the augmented simplex category with monoidal structure given by ordinal sum; see Definition 5.6.
2. A metamodel Φ of a metatheory \mathcal{M} may be identified with a lax double functor $\Phi: \mathbb{H}\Sigma(\mathcal{M}^{\text{op}}) \rightarrow \mathbf{PROF}^{\text{op}}$, where Σ turns a monoidal category to the corresponding one-object bicategory and \mathbb{H} turns a bicategory to the corresponding vertically discrete pseudo double category (see Example 5.2).

3. Therefore given a theory \mathbb{T} and a metamodel Φ (in \mathcal{C}) of a metatheory \mathcal{M} , we obtain a lax double functor $\Phi(\mathbb{T}): \mathbb{H}\Sigma(\Delta_a^{\text{op}}) \rightarrow \text{PROF}^{\text{op}}$ as the following composition:

$$\mathbb{H}\Sigma(\Delta_a^{\text{op}}) \xrightarrow{\mathbb{H}\Sigma(\mathbb{T}^{\text{op}})} \mathbb{H}\Sigma(\mathcal{M}^{\text{op}}) \xrightarrow{\Phi} \text{PROF}^{\text{op}}.$$

Theorem 5.5 may then be interpreted as establishing that $\mathbf{Mod}(\mathbb{T}, \mathcal{C})$ is (the apex of) the double limit of $\Phi(\mathbb{T})$ in the sense of [34].

We remark that the 2-categorical universal property of Eilenberg–Moore categories (Section 5.1) can also be interpreted as establishing $\mathcal{C}^{\mathbb{T}}$ as (the apex of) the lax limit of the 2-functor of type $\Sigma\Delta_a \rightarrow \mathcal{CAT}$ corresponding to a monad \mathbb{T} on a large category \mathcal{C} ; see [82]. The following reduction is essentially routine and rather peripheral, so those readers contented with the above outline may safely skip the rest of this section.

We start from the first step, namely a well-known observation (see e.g., [71, Section VII. 5]) that monoid objects (= theories) may be identified with strong monoidal functors out of Δ_a .

Definition 5.6. We define the **augmented simplex category** (also known as the **algebraists’ simplex category**) Δ_a as follows.

- Objects are all finite ordinals $\mathbf{n} = \{0 < 1 < \dots < n - 1\}$, including the empty ordinal $\mathbf{0} = \{\}$.
- Morphisms are all monotone functions.

Note that a morphism in Δ_a is mono (resp. epi) iff it is an injective (resp. surjective) monotone function.

This category has a natural monoidal structure, given as follows.

- The unit object is $\mathbf{0}$.
- The monoidal product $+: \Delta_a \times \Delta_a \rightarrow \Delta_a$ maps a pair of objects \mathbf{n} and \mathbf{m} in Δ_a to $\mathbf{n} + \mathbf{m}$, and maps a pair of morphisms $u: \mathbf{n} \rightarrow \mathbf{n}'$ and $v: \mathbf{m} \rightarrow \mathbf{m}'$ in Δ_a to $u + v: \mathbf{n} + \mathbf{m} \rightarrow \mathbf{n}' + \mathbf{m}'$ defined as

$$(u + v)(i) = \begin{cases} u(i) & \text{if } i \leq n - 1 \\ n' + v(i) & \text{if } i \geq n. \end{cases}$$

In the following, whenever we talk about a monoidal structure on Δ_a , we always mean this (strict) monoidal structure $(\mathbf{0}, +)$. ■

The morphisms in the category Δ_a are generated by certain simple morphisms. For each $\mathbf{n} \in \Delta_a$ and $i \in \{0, \dots, n\}$, let $\delta_i^{(n)}: \mathbf{n} \rightarrow \mathbf{n} + \mathbf{1}$ be the unique injective monotone function whose image does not contain $i \in \mathbf{n} + \mathbf{1}$, and for each $\mathbf{n} \in \Delta_a$ and $i \in \{0, \dots, n - 1\}$, let $\sigma_i^{(n)}: \mathbf{n} \rightarrow \mathbf{n} - \mathbf{1}$ be the unique surjective monotone function such that $\sigma_i^{(n)}(i) = \sigma_i^{(n)}(i + 1) = i$. Morphisms of the form $\delta_i^{(n)}$ are called **face maps** and those of the form $\sigma_i^{(n)}$ **degeneracy maps**. It is easy to see that every monomorphism in Δ_a can be expressed as a composition of face maps⁴, and every epimorphism in Δ_a as a composition of degeneracy maps. Furthermore, an arbitrary morphism in Δ_a can be written uniquely as the composition of

⁴An identity morphism in Δ_a is interpreted as the result of 0-ary composition of morphisms.

an epimorphism followed by a monomorphism (the image factorisation). Hence all morphisms in Δ_a can be written as a composition of face and degeneracy maps. This means that an arbitrary *functor* from Δ_a to a category is completely determined by its images of all objects in Δ_a and face and degeneracy maps. Conversely, such an assignment of the images of objects and face and degeneracy maps extends to a functor if and only if it satisfies the well-known *simplicial identities*; see [71, Section VII. 5].

Moreover, if we take into account the monoidal structure of Δ_a , we can further cut down the generating data. Clearly, every object in Δ_a is written as the monoidal product of finitely many copies of $\mathbf{1}$. Consider the unique morphism $!_0: \mathbf{0} \rightarrow \mathbf{1}$ in Δ_a . Every face map $\delta_i^{(n)}: \mathbf{n} \rightarrow \mathbf{n} + \mathbf{1}$ can be written as $\text{id}_i + !_0 + \text{id}_{\mathbf{n}-i}$ using this morphism and the monoidal product. Similarly, using the unique morphism $!_2: \mathbf{2} \rightarrow \mathbf{1}$, every degeneracy map $\sigma_i^{(n)}: \mathbf{n} \rightarrow \mathbf{n} - \mathbf{1}$ can be written as $\text{id}_i + !_2 + \text{id}_{\mathbf{n}-i-2}$. Hence every *strict monoidal functor* of type $F: \Delta_a \rightarrow \mathcal{M}$ to a strict monoidal category $(\mathcal{M}, I, \otimes)$ is completely determined by the object $T = F(\mathbf{1}) \in \mathcal{M}$ and the morphisms $e = F(!_0): I \rightarrow T$ and $m = F(!_2): T \otimes T \rightarrow T$ in \mathcal{M} . It turns out that, conversely, such a data (T, e, m) defines a strict monoidal functor if and only if (T, e, m) is a monoid object in \mathcal{M} .

The following proposition is a mild variant of this.

Proposition 5.7. *Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a metatheory. There is an equivalence of categories between the category $\text{MonCAT}_{\text{strong}}(\Delta_a, \mathcal{M})$ of all strong monoidal functors $\Delta_a \rightarrow \mathcal{M}$ and monoidal natural transformations, and the category $\mathbf{Th}(\mathcal{M})$.*

Proof. Recall that a strong monoidal functor $(F, f., f): \Delta_a \rightarrow \mathcal{M}$ consists of a functor $F: \Delta_a \rightarrow \mathcal{M}$, an isomorphism $f.: I \rightarrow F(\mathbf{0})$ and a natural isomorphism

$$f = (f_{\mathbf{m}, \mathbf{n}}: F(\mathbf{n}) \otimes F(\mathbf{m}) \rightarrow F(\mathbf{n} + \mathbf{m}))_{\mathbf{m}, \mathbf{n} \in \Delta_a}$$

satisfying the suitable axioms. The functor

$$\text{MonCAT}_{\text{strong}}(\Delta_a, \mathcal{M}) \rightarrow \mathbf{Th}(\mathcal{M})$$

mapping an object $(F, f., f) \in \text{MonCAT}_{\text{strong}}(\Delta_a, \mathcal{M})$ to $(F(\mathbf{1}), F(!_0) \circ f., F(!_2) \circ f_{\mathbf{1}, \mathbf{1}})$ and a morphism $\phi: (F, f., f) \rightarrow (G, g., g)$ in $\text{MonCAT}_{\text{strong}}(\Delta_a, \mathcal{M})$ to $\phi_{\mathbf{1}}$ is well-defined and is an equivalence of categories. \square

The second step, that a metamodel of a metatheory \mathcal{M} corresponds to a lax double functor of type $\mathbb{H}\Sigma(\mathcal{M}^{\text{op}}) \rightarrow \text{PROF}^{\text{op}}$, is straightforward. Rather than introducing a general definition of lax double functor (for this, see [34, Section 7.2]), we shall use the following fact: for any bicategory \mathcal{B} and any pseudo double category \mathbb{D} , lax double functors of type $\mathbb{H}\mathcal{B} \rightarrow \mathbb{D}$ bijectively correspond to lax functors of type $\mathcal{B} \rightarrow \mathcal{H}\mathbb{D}$ in a canonical way. Hence it suffices to see that a metamodel of \mathcal{M} corresponds to a lax functor of type $\Sigma(\mathcal{M}^{\text{op}}) \rightarrow \mathcal{H}(\text{PROF}^{\text{op}}) = \text{PROF}^{\text{op}}$, which we have already remarked in Section 3.2.2.

As a sketch for the final step, we show that a vertical double cone over the lax double functor $\Phi(\mathbb{T})$ in the sense of [34, Section 4.1, 7.3] is indeed equivalent to a triple (\mathcal{A}, V, v) defined in Definition 5.4. Specialising the original definition, a vertical double cone over $\Phi(\mathbb{T}): \mathbb{H}\Sigma(\mathcal{M}^{\text{op}}) \rightarrow \text{PROF}^{\text{op}}$ consists of the following data:

(DCD1) A category \mathcal{A} .

(DCD2) A functor $V: \mathcal{A} \rightarrow \mathcal{C}$.

(DCD3') For each $\mathbf{n} \in \Delta_a$, a square in PROF^{op}

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow v_{\mathbf{n}} & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_{T^{\otimes n}} (= \Phi(\mathbf{T})_{\mathbf{n}})} & \mathcal{C} \end{array}$$

satisfying the following axioms:

(DCA1')

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow v_{\mathbf{0}} & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_I} & \mathcal{C} \end{array} = \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow \text{id}_V & \downarrow V \\ \mathcal{C} & \xrightarrow{\mathcal{C}(-,-)} & \mathcal{C} \\ & \Downarrow \phi & \\ & \Phi_I & \end{array}$$

(DCA2') For each pair of objects $\mathbf{n}, \mathbf{m} \in \Delta_a$,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow v_{\mathbf{m}+\mathbf{n}} & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_{T^{\otimes(m+n)}}} & \mathcal{C} \end{array} = \begin{array}{ccccc} & & \mathcal{A}(-,-) & & \\ & & \downarrow \cong & & \\ \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow v_{\mathbf{m}} & \downarrow V & \Downarrow v_{\mathbf{n}} & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_{T^{\otimes m}}} & \mathcal{C} & \xrightarrow{\Phi_{T^{\otimes n}}} & \mathcal{C} \\ & & \Downarrow \phi_{T^{\otimes n}, T^{\otimes m}} & & \\ & & \Phi_{T^{\otimes(m+n)}} & & \end{array}$$

(DCA3') For each morphism $u: \mathbf{n} \rightarrow \mathbf{n}'$ in Δ_a ,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow v_{\mathbf{n}} & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_{T^{\otimes n}}} & \mathcal{C} \end{array} = \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow v_{\mathbf{n}'} & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_{T^{\otimes n'}}} & \mathcal{C} \\ & \Downarrow \Phi(\mathbf{T})_u & \\ & \Phi_{T^{\otimes n}} & \end{array}$$

By (DCA1') and (DCA2'), $v_{\mathbf{1}}$ determines all $v_{\mathbf{n}}$. Also, it suffices to check the condition (DCA3') with respect to all face and degeneracy maps. In fact, it suffices to check (DCA3') only with respect to two maps, namely $!_{\mathbf{0}}: \mathbf{0} \rightarrow \mathbf{1}$ and $!_{\mathbf{2}}: \mathbf{2} \rightarrow \mathbf{1}$. This is because, as noted above, any face map $\delta_i^{(n)}: \mathbf{n} \rightarrow \mathbf{n} + \mathbf{1}$ can be written as $\text{id}_i + !_{\mathbf{0}} + \text{id}_{\mathbf{n}-i}$ and any degeneracy map $\sigma_i^{(n)}: \mathbf{n} \rightarrow \mathbf{n} - \mathbf{1}$ as $\text{id}_i + !_{\mathbf{2}} + \text{id}_{\mathbf{n}-i-2}$. Therefore if

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow v_{\mathbf{0}} & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_I} & \mathcal{C} \end{array} = \begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathcal{A}(-,-)} & \mathcal{A} \\ V \downarrow & \Downarrow v_{\mathbf{1}} & \downarrow V \\ \mathcal{C} & \xrightarrow{\Phi_T} & \mathcal{C} \\ & \Downarrow \Phi_e & \\ & \Phi_I & \end{array}$$

holds (we have used $\Phi(\mathbb{T})_{!_0} = \Phi_e$, where $e: I \rightarrow T$ is given by the theory $\mathbb{T} = (T, e, m)$), then for $\delta_i^{(n)} = \text{id}_i + !_0 + \text{id}_{n-i}: \mathbf{n} \rightarrow \mathbf{n} + \mathbf{1}$,

$$\begin{array}{c}
\mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \\
V \downarrow \quad \Downarrow v_n \quad \downarrow V \\
\mathcal{C} \xrightarrow{\Phi_{T^{\otimes n}}} \mathcal{C}
\end{array}
=
\begin{array}{c}
\mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \\
V \downarrow \quad \Downarrow v_{n-i} \quad \downarrow V \quad \Downarrow v_0 \quad \downarrow V \quad \Downarrow v_i \quad \downarrow V \\
\mathcal{C} \xrightarrow{\Phi_{T^{\otimes(n-i)}}} \mathcal{C} \xrightarrow{\Phi_I} \mathcal{C} \xrightarrow{\Phi_{T^{\otimes i}}} \mathcal{C} \\
\Downarrow \phi_{T^{\otimes i}, I, T^{\otimes(n-i)}} \\
\Phi_{T^{\otimes n}}
\end{array}
=
\begin{array}{c}
\mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \\
V \downarrow \quad \Downarrow v_{n-i} \quad \downarrow V \quad \Downarrow v_1 \quad \downarrow V \quad \Downarrow v_i \quad \downarrow V \\
\mathcal{C} \xrightarrow{\Phi_{T^{\otimes(n-i)}}} \mathcal{C} \xrightarrow{\Phi_T} \mathcal{C} \xrightarrow{\Phi_{T^{\otimes i}}} \mathcal{C} \\
\Downarrow \Phi_e \quad \Downarrow \phi_{T^{\otimes i}, I, T^{\otimes(n-i)}} \\
\Phi_I \quad \Phi_{T^{\otimes n}}
\end{array}
=
\begin{array}{c}
\mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \\
V \downarrow \quad \Downarrow v_{n-i} \quad \downarrow V \quad \Downarrow v_1 \quad \downarrow V \quad \Downarrow v_i \quad \downarrow V \\
\mathcal{C} \xrightarrow{\Phi_{T^{\otimes(n-i)}}} \mathcal{C} \xrightarrow{\Phi_T} \mathcal{C} \xrightarrow{\Phi_{T^{\otimes i}}} \mathcal{C} \\
\Downarrow \phi_{T^{\otimes i}, T, T^{\otimes(n-i)}} \\
\Phi_{T^{\otimes(n+1)}} \quad \Downarrow \Phi_{\delta_i^{(n)}} \\
\Phi_{T^{\otimes n}}
\end{array}
=
\begin{array}{c}
\mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \\
V \downarrow \quad \Downarrow v_{n+1} \quad \downarrow V \\
\mathcal{C} \xrightarrow{\Phi_{T^{\otimes(n+1)}}} \mathcal{C} \\
\Downarrow \Phi_{\delta_i^{(n)}} \\
\Psi_{T^{\otimes n}}
\end{array}$$

(where ϕ with three subscripts denote suitable composites of $\phi_{X,Y}$), and similarly for the degeneracy maps.

Therefore, a vertical double cone for $\Phi(\mathbb{T})$ is given equivalently as the data (DCD1), (DCD2) together with:

(DCD3) a square in PROF^{OP}

$$\begin{array}{c}
\mathcal{A} \xrightarrow{\mathcal{A}(-, -)} \mathcal{A} \\
V \downarrow \quad \Downarrow v \quad \downarrow V \\
\mathcal{C} \xrightarrow{\Phi_T} \mathcal{C},
\end{array}$$

satisfying the equations (5.1) and (5.2). This coincides with Definition 5.4.

Arguing similarly, we obtain the following corollary of Theorem 5.5.

Corollary 5.8. *Let \mathcal{M} be a metatheory, \mathbb{T} be a theory in \mathcal{M} , \mathcal{C} be a large category and Φ be a metamodel of \mathcal{M} in \mathcal{C} . The category $\mathbf{Mod}(\mathbb{T}, (\mathcal{C}, \Phi))$ of models of \mathbb{T} in \mathcal{C} with respect to Φ is the apex of the double limit of the lax double functor $\Phi(\mathbb{T})$.*

Part II

Weak n -dimensional \mathcal{V} -categories

Chapter 6

Extensive categories

From this chapter on we shall turn to the study of weak n -categories. In this chapter, we introduce *extensive categories*, a central notion in our study of weak n -categories, and prove useful lemmas for them.

The results in this section have been published in [15, 16].

6.1 The definition and examples

Extensive categories were first introduced by Lawvere [61, 62] and their basic properties established by Carboni, Lack and Walters [12] and by Cockett [14]. Roughly speaking, an extensive category is a category with well-behaved coproducts.¹

Let \mathcal{V} be a large category with all small coproducts, I be a small set and $(X_i)_{i \in I}$ be an I -indexed family of objects of \mathcal{V} . We have the functor

$$\coprod: \prod_{i \in I} (\mathcal{V}/X_i) \longrightarrow \mathcal{V}/\coprod_{i \in I} X_i \quad (6.1)$$

which maps $(f_i: A_i \longrightarrow X_i)_{i \in I}$ to $(\coprod_{i \in I} f_i: \coprod_{i \in I} A_i \longrightarrow \coprod_{i \in I} X_i)$.

Definition 6.1 ([13], cf. [12, 14]). A large category \mathcal{V} is **extensive** if and only if it admits all small coproducts and for any small set I and I -indexed family $(X_i)_{i \in I}$ of objects of \mathcal{V} , the functor \coprod in (6.1) is an equivalence of categories. ■

Our leading examples of extensive categories are **Set** and the category ω -**Cpo** of (small) posets with sups of ω -chains and monotone functions preserving sups of ω -chains, together with, for any extensive category \mathcal{V} with finite limits, the categories \mathcal{V} -**Gph**⁽ⁿ⁾ and \mathcal{V} -**Cat**⁽ⁿ⁾, which are defined recursively. In order to define the former, we first need to define the category of \mathcal{V} -graphs.

Definition 6.2 ([88]). Let \mathcal{V} be a large category.

1. A **small \mathcal{V} -graph** G consists of a small set $\text{ob}(G)$ together with, for each $x, y \in \text{ob}(G)$, an object $G(x, y) \in \mathcal{V}$.
2. A **morphism of \mathcal{V} -graphs** from G to G' is a function $f: \text{ob}(G) \longrightarrow \text{ob}(G')$ together with, for each $x, y \in \text{ob}(G)$, a morphism $f_{x,y}: G(x, y) \longrightarrow G'(fx, fy)$ in \mathcal{V} . ■

¹The original notion of extensive category requires well-behaved *finite* coproducts, but what we shall use below is an infinitary variant of this, requiring well-behaved *small* coproducts; such a notion is previously used in e.g., [13, Section 4]. In this thesis, the term “extensive category” always refer to this infinitary variant as defined in Definition 6.1.

Clearly, a **Set**-graph is nothing but a directed multigraph.

We denote the category of all small \mathcal{V} -graphs and morphisms by $\mathcal{V}\text{-Gph}$. The construction $(-)\text{-Gph}$ routinely extends to an endo-2-functor on the 2-category \mathcal{CAT} of large categories.

Definition 6.3. For any natural number n and any large category \mathcal{V} , the category $\mathcal{V}\text{-Gph}^{(n)}$ is defined as follows:

$$\mathcal{V}\text{-Gph}^{(0)} = \mathcal{V}; \quad \mathcal{V}\text{-Gph}^{(n+1)} = (\mathcal{V}\text{-Gph}^{(n)})\text{-Gph}.$$

An object of $\mathcal{V}\text{-Gph}^{(n)}$ is called an n -dimensional \mathcal{V} -graph. ■

Definition 6.4. For each natural number n and any large category \mathcal{V} with finite products, the category $\mathcal{V}\text{-Cat}^{(n)}$ is defined as follows (using the cartesian structure for enrichment):

$$\mathcal{V}\text{-Cat}^{(0)} = \mathcal{V}; \quad \mathcal{V}\text{-Cat}^{(n+1)} = (\mathcal{V}\text{-Cat}^{(n)})\text{-Cat}.$$

An object of $\mathcal{V}\text{-Cat}^{(n)}$ is called a **strict** n -dimensional \mathcal{V} -category (to avoid confusion with *weak* n -dimensional \mathcal{V} -category which we are trying to define). ■

From now on, whenever we mention enriched categories, we always mean enrichment with respect to the cartesian structure. When $\mathcal{V} = \mathbf{Set}$, we abbreviate $\mathcal{V}\text{-Gph}^{(n)}$ by $n\text{-Gph}$ (whose object we call an n -graph), and we abbreviate $\mathcal{V}\text{-Cat}^{(n)}$ by $n\text{-Cat}$ (whose object we call a **strict** n -category).

We now show that if \mathcal{V} is an extensive category with finite limits, then so are $\mathcal{V}\text{-Gph}$ and $\mathcal{V}\text{-Cat}$. Actually, to ensure that $\mathcal{V}\text{-Gph}$ and $\mathcal{V}\text{-Cat}$ are extensive, the much weaker requirement of \mathcal{V} having a strict initial object suffices. Recall that an initial object 0 in a category is called **strict** if every morphism going into 0 is an isomorphism. Every extensive category has a strict initial object; consider the case $I = \emptyset$ in (6.1).

Proposition 6.5. *If \mathcal{V} is a large category with a strict initial object 0 , then $\mathcal{V}\text{-Gph}$ is extensive.*

Proof. The coproduct of a family $(G_i)_{i \in I}$ of \mathcal{V} -graphs is given by $\text{ob}(\coprod_{i \in I} G_i) = \coprod_{i \in I} \text{ob}(G_i)$ and

$$\left(\coprod_{i \in I} G_i \right) ((i, x), (i', x')) = \begin{cases} G_i(x, x') & \text{if } i = i', \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the functor $\coprod: \prod_{i \in I} (\mathcal{V}\text{-Gph}/G_i) \rightarrow \mathcal{V}\text{-Gph}/(\coprod_{i \in I} G_i)$ (as in (6.1)) is full and faithful. For any object $(f: H \rightarrow \coprod_{i \in I} G_i)$ in $\mathcal{V}\text{-Gph}/(\coprod_{i \in I} G_i)$, define an object $(f_i: H_i \rightarrow G_i)_{i \in I} \in \prod_{i \in I} (\mathcal{V}/G_i)$ by the pullbacks of f along the coprojections $\sigma_i: G_i \rightarrow \coprod_{i \in I} G_i$; note that these pullbacks always exist, and H_i are just the suitable “full sub” \mathcal{V} -graphs of H . Since 0 is strict, $(\coprod_{i \in I} f_i: \coprod_{i \in I} H_i \rightarrow \coprod_{i \in I} G_i)$ is isomorphic to f . Hence \coprod is also essentially surjective. □

Proposition 6.6. *If \mathcal{V} is a large category with a strict initial object and finite products, then $\mathcal{V}\text{-Cat}$ is extensive.*

Proof. Coproducts in $\mathcal{V}\text{-Cat}$ are formed just as in $\mathcal{V}\text{-Gph}$; namely, given a family $(\mathcal{C}_i)_{i \in I}$ of \mathcal{V} -categories, we have $\text{ob}(\coprod_{i \in I} \mathcal{C}_i) = \coprod_{i \in I} \text{ob}(\mathcal{C}_i)$ and

$$\left(\coprod_{i \in I} \mathcal{C}_i\right)((i, x), (i', x')) = \begin{cases} \mathcal{C}_i(x, x') & \text{if } i = i', \\ 0 & \text{otherwise.} \end{cases}$$

Note that to define a composition law for $\coprod_{i \in I} \mathcal{C}_i$, we use the fact that for a category \mathcal{V} with a strict initial object 0 , $0 \times B \cong 0$ for every object $B \in \mathcal{V}$.² The rest of the proof is identical to that of Proposition 6.5. \square

When \mathcal{V} has finite limits, then so do $\mathcal{V}\text{-Gph}$ and $\mathcal{V}\text{-Cat}$. Given a finite category \mathcal{I} and a functor $F: \mathcal{I} \rightarrow \mathcal{V}\text{-Gph}$, the limit $\lim F$ of F can be constructed as follows.³ The set of objects is $\text{ob}(\lim F) = \lim(\text{ob} \circ F)$, where $\text{ob}: \mathcal{V}\text{-Gph} \rightarrow \mathbf{Set}$ is the functor mapping a \mathcal{V} -graph to its set of objects. Explicitly, an object of $\lim F$ is an $\text{ob}(\mathcal{I})$ -indexed family $(a_i)_{i \in \mathcal{I}}$ where a_i is an object of the \mathcal{V} -graph F_i and such that for any morphism $u: i \rightarrow j$ in \mathcal{I} , $(Fu)(a_i) = a_j$ holds. Given any pair of objects $a = (a_i)_{i \in \mathcal{I}}, b = (b_i)_{i \in \mathcal{I}} \in \text{ob}(\lim F)$, we obtain a functor $F_{a,b}: \mathcal{I} \rightarrow \mathcal{V}$ by mapping an object $i \in \mathcal{I}$ to $(F_i)(a_i, b_i)$ and a morphism $u: i \rightarrow j$ in \mathcal{I} to $(Fu)_{a_i, b_i}$; observe that $(Fu)(a_i) = a_j$ and $(Fu)(b_i) = b_j$ hold and we indeed have a morphism $(Fu)_{a_i, b_i}: (F_i)(a_i, b_i) \rightarrow (F_j)(a_j, b_j)$ in \mathcal{V} . The object $(\lim F)(a, b)$ is given by $\lim F_{a,b}$.

Finite limits in $\mathcal{V}\text{-Cat}$ may be constructed similarly, noting that limits commute with products; we remind the reader that $\mathcal{V}\text{-Cat}$ is defined using the cartesian structure of \mathcal{V} .

Below we record the case of pullbacks, as they will play an important role later.

Proposition 6.7. *Let \mathcal{V} have finite limits. A commutative square*

$$\begin{array}{ccc} P & \xrightarrow{k} & B \\ h \downarrow & & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$

in $\mathcal{V}\text{-Gph}$ or in $\mathcal{V}\text{-Cat}$ is a pullback if and only if the square

$$\begin{array}{ccc} \text{ob}(P) & \xrightarrow{k} & \text{ob}(B) \\ h \downarrow & & \downarrow g \\ \text{ob}(A) & \xrightarrow{f} & \text{ob}(X) \end{array}$$

is a pullback in \mathbf{Set} , and for any pair $p_1, p_2 \in \text{ob}(P)$, writing $a_i = h(p_i), b_i = k(p_i)$

²In fact, for a category \mathcal{V} with an initial object 0 and finite products, 0 is strict if and only if $0 \times B \cong 0$ for every $B \in \mathcal{V}$.

³This construction is valid for limits indexed by an arbitrary small category \mathcal{I} , provided that \mathcal{V} has all \mathcal{I} -indexed limits.

and $x_i = f(a_i) = g(b_i)$ for $i = 1, 2$, the square

$$\begin{array}{ccc} P(p_1, p_2) & \xrightarrow{k_{p_1, p_2}} & B(b_1, b_2) \\ \downarrow h_{p_1, p_2} & & \downarrow g_{b_1, b_2} \\ A(a_1, a_2) & \xrightarrow{f_{a_1, a_2}} & X(x_1, x_2) \end{array}$$

is a pullback in \mathcal{V} .

Combining the above observation with Propositions 6.5 and 6.6, we immediately obtain the following result.

Corollary 6.8. *If \mathcal{V} is an extensive category with finite limits, then so are $\mathcal{V}\text{-Gph}^{(n)}$ and $\mathcal{V}\text{-Cat}^{(n)}$, for each natural number n .*

6.2 Properties of coproducts in an extensive category

We need several results about behaviour of coproducts in extensive categories later, so in this section we collect such results.

The first proposition gives a characterisation of extensive categories.

Proposition 6.9 ([13, Section 4.2, Exercise 1], cf. [12, Proposition 2.2]). *A category \mathcal{V} with small coproducts is extensive if and only if it has all pullbacks along coprojections associated with small coproducts, and for any small set I , I -indexed family $(X_i)_{i \in I}$ of objects of \mathcal{V} , morphism $f: A \rightarrow \coprod_{i \in I} X_i$ in \mathcal{V} , and I -indexed family of commutative squares*

$$\begin{array}{ccc} A_i & \xrightarrow{\tau_i} & A \\ \downarrow f_i & & \downarrow f \\ X_i & \xrightarrow{\sigma_i} & \coprod_{i \in I} X_i \end{array} \quad (6.2)$$

in \mathcal{V} (in which σ_i is the i -th coprojection), each square (6.2) is a pullback square if and only if $(\tau_i)_{i \in I}$ defines a coproduct (that is, $A = \coprod_{i \in I} A_i$ with τ_i the i -th coprojection).

Proof. If \mathcal{V} has small coproducts, then the functor (6.1) has a right adjoint if and only if all pullbacks along σ_i exists in \mathcal{V} , and in that case the right adjoint

$$\langle \sigma_i^* \rangle_{i \in I}: \mathcal{V} / \coprod_{i \in I} X_i \rightarrow \prod_{i \in I} (\mathcal{V} / X_i)$$

has the i -th component $\sigma_i^*: \mathcal{V} / \coprod_{i \in I} X_i \rightarrow \mathcal{V} / X_i$ mapping $(f: A \rightarrow \coprod_{i \in I} X_i) \in \mathcal{V} / \coprod_{i \in I} X_i$ to $(\sigma_i^* f: \sigma_i^* A \rightarrow X_i) \in \mathcal{V} / X_i$, defined by the pullback

$$\begin{array}{ccc} \sigma_i^* A & \longrightarrow & A \\ \sigma_i^* f \downarrow & \lrcorner & \downarrow f \\ X_i & \xrightarrow{\sigma_i} & \coprod_{i \in I} X_i \end{array}$$

in \mathcal{V} .

In general, a functor is an equivalence of categories if and only if it has a right adjoint and the associated unit and counit are natural isomorphisms. Applying this fact to the functors of the form (6.1), we obtain the desired result. \square

Proposition 6.10. *Let \mathcal{V} be an extensive category. For any small set I and I -indexed family of pullback squares in \mathcal{V} as on the left of the following diagram, the square as on the right is a pullback.*

$$\begin{array}{ccc}
P_i & \xrightarrow{q_i} & B_i \\
\downarrow p_i & \lrcorner & \downarrow g_i \\
A_i & \xrightarrow{f_i} & X_i
\end{array}
\qquad
\begin{array}{ccc}
\coprod_{i \in I} P_i & \xrightarrow{\coprod_{i \in I} q_i} & \coprod_{i \in I} B_i \\
\downarrow \coprod_{i \in I} p_i & \lrcorner & \downarrow \coprod_{i \in I} g_i \\
\coprod_{i \in I} A_i & \xrightarrow{\coprod_{i \in I} f_i} & \coprod_{i \in I} X_i
\end{array}$$

Proof. By the definition of extensivity, the functor $\coprod: \prod_{i \in I} (\mathcal{V}/X_i) \rightarrow \mathcal{V}/(\prod_{i \in I} X_i)$ is an equivalence of categories and, in particular, it preserves binary products. \square

Proposition 6.11. *Let \mathcal{V} be an extensive category with finite products. For any $B \in \mathcal{V}$, the functor $(-) \times B: \mathcal{V} \rightarrow \mathcal{V}$ preserves small coproducts.*

Proof. In any category, a square as on the left of the following diagram is always a pullback. Hence for any object $B \in \mathcal{V}$, small set I , and I -indexed family $(X_i)_{i \in I}$ of objects of \mathcal{V} , for each $i \in I$ the square as on the right is a pullback.

$$\begin{array}{ccc}
A \times B & \xrightarrow{h \times B} & C \times B \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
A & \xrightarrow{h} & C
\end{array}
\qquad
\begin{array}{ccc}
X_i \times B & \xrightarrow{\sigma_i \times B} & (\prod_{i \in I} X_i) \times B \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
X_i & \xrightarrow{\sigma_i} & \prod_{i \in I} X_i
\end{array}$$

Therefore by Proposition 6.9, $(\prod_{i \in I} X_i) \times B \cong \prod_{i \in I} (X_i \times B)$. \square

Proposition 6.12. *Let \mathcal{V} be an extensive category. For any object $Y \in \mathcal{V}$, the slice category \mathcal{V}/Y is again extensive.*

Proof. Clearly \mathcal{V}/Y has small coproducts given by $\prod_{i \in I} (f_i: X_i \rightarrow Y) = ([f_i]_{i \in I}: \prod_{i \in I} X_i \rightarrow Y)$. Also note that for any object $(f: X \rightarrow Y)$ of \mathcal{V}/Y , the canonical functor $(\mathcal{V}/Y)/f \rightarrow \mathcal{V}/X$ which maps $(h: (g: A \rightarrow Y) \rightarrow f) \in (\mathcal{V}/Y)/f$ to $(h: A \rightarrow X) \in \mathcal{V}/X$ is an isomorphism of categories. For any small set I and I -indexed family $(f_i: X_i \rightarrow Y)_{i \in I}$ of objects of \mathcal{V}/Y , the diagram

$$\begin{array}{ccc}
\prod_{i \in I} ((\mathcal{V}/Y)/f_i) & \xrightarrow{\coprod} & (\mathcal{V}/Y)/[f_i]_{i \in I} \\
\cong \downarrow & & \downarrow \cong \\
\prod_{i \in I} (\mathcal{V}/X_i) & \xrightarrow{\coprod} & \mathcal{V}/(\prod_{i \in I} X_i)
\end{array}$$

(in which the vertical arrows are the canonical isomorphisms mentioned above) commutes. Since the lower \coprod is an equivalence by the assumption, so is the upper one. \square

Corollary 6.13. *Let \mathcal{V} be an extensive category with pullbacks.*

1. For any morphism $g: B \rightarrow X$ in \mathcal{V} , small set I , and I -indexed family of pullback squares in \mathcal{V} as on the left of the following diagram, the square as on the right is a pullback.

$$\begin{array}{ccc}
 P_i & \xrightarrow{q_i} & B \\
 \downarrow p_i & \lrcorner & \downarrow g \\
 A_i & \xrightarrow{f_i} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \coprod_{i \in I} P_i & \xrightarrow{[q_i]_{i \in I}} & B \\
 \downarrow \coprod_{i \in I} p_i & \lrcorner & \downarrow g \\
 \coprod_{i \in I} A_i & \xrightarrow{[f_i]_{i \in I}} & X
 \end{array}$$

2. For any object $X \in \mathcal{V}$, small set I , I -indexed family of morphisms $(f_i: A_i \rightarrow X)_{i \in I}$ in \mathcal{V} , small set J , J -indexed family of morphisms $(g_j: B_j \rightarrow X)_{j \in J}$ in \mathcal{V} , and $(I \times J)$ -indexed family of pullback squares in \mathcal{V} as on the left of the following diagram, the square as on the right is a pullback.

$$\begin{array}{ccc}
 P_{i,j} & \xrightarrow{q_{i,j}} & B_j \\
 \downarrow p_{i,j} & \lrcorner & \downarrow g_j \\
 A_i & \xrightarrow{f_i} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \coprod_{i \in I, j \in J} P_{i,j} & \xrightarrow{\coprod_{j \in J} ([q_{i,j}]_{i \in I})} & \coprod_{j \in J} B_j \\
 \downarrow \coprod_{i \in I} ([p_{i,j}]_{j \in J}) & \lrcorner & \downarrow [g_j]_{j \in J} \\
 \coprod_{i \in I} A_i & \xrightarrow{[f_i]_{i \in I}} & X
 \end{array}$$

- Proof.* 1. By the assumption, the slice category \mathcal{V}/X has finite products \times_X (given by pullbacks in \mathcal{V}), and is extensive (Proposition 6.12). Hence by Proposition 6.11, binary product by $(g: B \rightarrow X) \in \mathcal{V}/X$ preserves small coproducts, that is, $(\coprod_{i \in I} f_i) \times_X g \cong \coprod_{i \in I} (f_i \times_X g)$.
2. Using the first clause iteratively, we obtain $(\coprod_{i \in I} f_i) \times_X (\coprod_{j \in J} g_j) \cong \coprod_{i \in I, j \in J} (f_i \times_X g_j)$. \square

Chapter 7

The free strict n -dimensional \mathcal{V} -category monad on $\mathcal{V}\text{-Gph}^{(n)}$

The construction of the free category FG generated by a (**Set**-)graph G is well-known: the set of objects of FG is the same as that of G , and a morphism in FG is a (directed) *path* in G (see Section 7.1). This construction is the left adjoint to the forgetful functor $U: \mathbf{Cat} \rightarrow \mathbf{Gph} = \mathbf{Set}\text{-Gph}$, and gives rise to a monad \mathbb{T} on \mathbf{Gph} , the *free category monad*. This monad and its higher dimensional analogues, the free strict n -category monad $\mathbb{T}^{(n)}$ on $n\text{-Gph}$ for each natural number n , play a crucial role in the Batanin–Leinster approach to weak n -categories, because they turn out to be *cartesian* monads and therefore we may consider $\mathbb{T}^{(n)}$ -operads. The structure of weak n -category is expressed via a certain $\mathbb{T}^{(n)}$ -operad.

In this chapter, we show a generalisation of these facts; rather than starting from the category **Set**, we start from an arbitrary extensive category \mathcal{V} with finite limits, and show that we have the *free strict n -dimensional \mathcal{V} -category monad* $\mathbb{T}^{(n)}$ on $\mathcal{V}\text{-Gph}^{(n)}$, and that it is cartesian.

The results in this section have been published in [16].

7.1 The free \mathcal{V} -category monad

In this section we deal with the one-dimensional case; that is, we define the free \mathcal{V} -category monad on $\mathcal{V}\text{-Gph}$ and show it is cartesian.

Let us start with reviewing the construction of free categories over graphs. Suppose that $G = (\text{ob}(G), (G(x, y))_{x, y \in \text{ob}(G)})$ is an object of **Gph**, i.e., a directed multigraph. For $x, y \in \text{ob}(G)$, a **path** in G from x to y is a sequence

$$(w_0, f_1, w_1, f_2, \dots, f_n, w_n)$$

where n is a natural number called the **length** of the path, $w_i \in \text{ob}(G)$ and $f_i \in G(w_{i-1}, w_i)$ such that $w_0 = x$ and $w_n = y$:

$$x = w_0 \xrightarrow{f_1} w_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} w_n = y.$$

The set of all paths in G from x to y is therefore given by

$$\coprod_{n \in \mathbb{N}} \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y}} G(w_{n-1}, w_n) \times \dots \times G(w_0, w_1) \quad (7.1)$$

(we have written $G(w_{n-1}, w_n) \times \dots \times G(w_0, w_1)$ instead of $G(w_0, w_1) \times \dots \times G(w_{n-1}, w_n)$ because the former agrees with our convention to write compositions in a category in the anti-diagrammatic order).

The free category FG over G has the same objects as G , and its hom-set $(FG)(x, y)$ is given by (7.1). Note that the set of all paths in G from x to y of length n is given as

$$(FG)(x, y)_n = \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y}} G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1), \quad (7.2)$$

and using this, we may rewrite (7.1) as

$$(FG)(x, y) = \coprod_{n \in \mathbb{N}} (FG)(x, y)_n.$$

The identities in FG are given by the paths of length 0 (note that $(FG)(x, y)_0$ is a singleton if $x = y$ and is empty otherwise), and compositions in FG are given by the evident compositions of paths.

The following construction is a straightforward generalisation of the above ‘‘path’’ construction for free categories over graphs.

Proposition 7.1. *If \mathcal{V} has finite products and small coproducts, and if for any $B \in \mathcal{V}$ the functor $(-)\times B: \mathcal{V} \rightarrow \mathcal{V}$ preserves small coproducts, then the forgetful functor $U: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Gph}$ admits a left adjoint F .*

Proof. Given a \mathcal{V} -graph $G = (\text{ob}(G), (G(x, y))_{x, y \in \text{ob}(G)})$, the free \mathcal{V} -category FG on G has the same objects as G and the hom-object given by

$$(FG)(x, y) = \coprod_{n \in \mathbb{N}} \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y}} G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1)$$

for all $x, y \in \text{ob}(FG) = \text{ob}(G)$. To spell out the identity elements and composition laws in FG , let us write

$$(FG)(x, y)_n = \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y}} G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1)$$

for all $x, y \in \text{ob}(FG)$ and $n \in \mathbb{N}$.

Note that $(FG)(x, y)_0$ is the terminal object 1 of \mathcal{V} if $x = y$ (otherwise, it is the initial object 0 of \mathcal{V}). Hence the identity element on $x \in \text{ob}(FG)$ can be given as

$$1 \xrightarrow{\cong} (FG)(x, x)_0 \xrightarrow{\sigma_0} \coprod_{n \in \mathbb{N}} (FG)(x, x)_n = (FG)(x, y),$$

where σ_0 denotes the 0-th coprojection. Given any triple $x, y, z \in \text{ob}(FG)$ of objects, by the assumption we have

$$(FG)(y, z) \times (FG)(x, y) \cong \coprod_{k, l \in \mathbb{N}} (FG)(y, z)_l \times (FG)(x, y)_k.$$

Using the assumption once again, we see that $(FG)(y, z)_l \times (FG)(x, y)_k$ is isomorphic to

$$\coprod_{\substack{u_0, \dots, u_k, v_0, \dots, v_l \in \text{ob}(G) \\ u_0 = x, u_k = v_0 = y, v_l = z}} G(v_{l-1}, v_l) \times \cdots \times G(v_0, v_1) \times G(u_{k-1}, u_k) \times \cdots \times G(u_0, u_1),$$

and therefore naturally embeds into $(FG)(x, z)_{k+l}$. The universality of coproducts induce the composition laws for FG from these embeddings. \square

Examples of categories \mathcal{V} satisfying the assumptions of Proposition 7.1 include cartesian closed categories with small coproducts (in this case, Proposition 7.1 appears in [88, Proposition 2.2]) and extensive categories with finite products (by Proposition 6.11).

For any extensive category \mathcal{V} with finite limits, the **free \mathcal{V} -category monad** $\mathbb{T} = (T, \eta, \mu)$ is the monad on $\mathcal{V}\text{-Gph}$ generated by the adjunction $F \dashv U$ in Proposition 7.1. The rest of this section is devoted to a proof of the fact that \mathbb{T} is cartesian. We show this by inspecting the adjunction $F \dashv U$ rather than the monad \mathbb{T} itself, because we will use certain properties of $F \dashv U$ in an inductive argument in the next section.

As a preliminary for the proof of the next proposition, let us examine the action of the functor $F: \mathcal{V}\text{-Gph} \rightarrow \mathcal{V}\text{-Cat}$ on morphisms. Suppose that $f: G \rightarrow H$ is a morphism in $\mathcal{V}\text{-Gph}$. The \mathcal{V} -functor $Ff: FG \rightarrow FH$ is given as follows. Its action on objects is the same as f . Given $x, y \in \text{ob}(G)$, the morphism $(Ff)_{x,y}: (FG)(x, y) \rightarrow (FH)(fx, fy)$ is induced by the universality of coproducts, as the unique morphism making the following diagram commute for all $n \in \mathbb{N}$ and $w_0, \dots, w_n \in \text{ob}(G)$ such that $w_0 = x$ and $w_n = y$:

$$\begin{array}{ccc}
G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1) & \xrightarrow{\sigma_{(n, w_0, \dots, w_n)}} & \coprod_{n \in \mathbb{N}} \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y}} G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1) \\
\downarrow f_{w_{n-1}, w_n} \times \cdots \times f_{w_0, w_1} & & \downarrow (Ff)_{x,y} \\
H(fw_{n-1}, fw_n) \times \cdots \times H(fw_0, fw_1) & & \downarrow \\
\downarrow \sigma_{(n, fw_0, \dots, fw_n)} & \rightarrow & \coprod_{n \in \mathbb{N}} \coprod_{\substack{v_0, \dots, v_n \in \text{ob}(H) \\ v_0 = fx, v_n = fy}} H(v_{n-1}, v_n) \times \cdots \times H(v_0, v_1),
\end{array}$$

where σ denotes the appropriate coprojections. Note that the morphism $(Ff)_{x,y}$ may be written as

$$(Ff)_{x,y} = \coprod_{n \in \mathbb{N}} (Ff)_{x,y,n}, \quad (7.3)$$

where $(Ff)_{x,y,n}: (FG)(x, y)_n \rightarrow (FH)(fx, fy)_n$ is characterised by the condition that the diagram

$$\begin{array}{ccc}
G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1) & \xrightarrow{\sigma_{(w_0, \dots, w_n)}} & \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y}} G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1) \\
\downarrow f_{w_{n-1}, w_n} \times \cdots \times f_{w_0, w_1} & & \downarrow (Ff)_{x,y,n} \\
H(fw_{n-1}, fw_n) \times \cdots \times H(fw_0, fw_1) & & \downarrow \\
\downarrow \sigma_{(fw_0, \dots, fw_n)} & \rightarrow & \coprod_{\substack{v_0, \dots, v_n \in \text{ob}(H) \\ v_0 = fx, v_n = fy}} H(v_{n-1}, v_n) \times \cdots \times H(v_0, v_1)
\end{array}$$

commutes, and this morphism $(Ff)_{x,y,n}$ may in turn be rewritten, using

$$\begin{aligned} \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y}} G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1) \\ \cong \coprod_{\substack{v_0, \dots, v_n \in \text{ob}(H) \\ v_0 = fx, v_n = fy}} \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y \\ f(w_i) = v_i}} G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1), \end{aligned}$$

as

$$(Ff)_{x,y,n} = \coprod_{\substack{v_0, \dots, v_n \in \text{ob}(H) \\ v_0 = fx, v_n = fy}} (Ff)_{x,y,n,v_0, \dots, v_n}, \quad (7.4)$$

where

$$(Ff)_{x,y,n,v_0, \dots, v_n} = [f_{w_{n-1}, w_n} \times \cdots \times f_{w_0, w_1}]_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y \\ f(w_i) = v_i}}. \quad (7.5)$$

Proposition 7.2. *If \mathcal{V} is an extensive category with finite limits, then the functor $F: \mathcal{V}\text{-Gph} \rightarrow \mathcal{V}\text{-Cat}$ given in Proposition 7.1 preserves pullbacks.*

Proof. Suppose we have a pullback

$$\begin{array}{ccc} P & \xrightarrow{k} & B \\ \downarrow h & \lrcorner & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$

in $\mathcal{V}\text{-Gph}$. Since F does nothing on the set of objects, by Proposition 6.7 it suffices to show that for any pair $p = (a, b), p' = (a', b') \in \text{ob}(P)$ with $f(a) = g(b) = x$ and $f(a') = g(b') = x'$, the square

$$\begin{array}{ccc} (FP)(p, p') & \xrightarrow{(Fk)_{p,p'}} & (FB)(b, b') \\ (Fh)_{p,p'} \downarrow & & \downarrow (Fg)_{b,b'} \\ (FA)(a, a') & \xrightarrow{(Ff)_{a,a'}} & (FX)(x, x') \end{array}$$

is a pullback in \mathcal{V} . Recall that

$$(FP)(p, p') = \coprod_{n \in \mathbb{N}} (FP)(x, y)_n = \coprod_{n \in \mathbb{N}} \coprod_{\substack{p_0, \dots, p_n \in \text{ob}(P) \\ p_0 = p, p_n = p'}} P(p_{n-1}, p_n) \times \cdots \times P(p_0, p_1),$$

and similarly for other objects in the above diagram. Decomposing the morphisms by (7.3), we may apply Proposition 6.10 and now it suffices to show that for each $n \in \mathbb{N}$, the square

$$\begin{array}{ccc} (FP)(p, p')_n & \xrightarrow{(Fk)_{p,p',n}} & (FB)(b, b')_n \\ (Fh)_{p,p',n} \downarrow & & \downarrow (Fg)_{b,b',n} \\ (FA)(a, a')_n & \xrightarrow{(Ff)_{a,a',n}} & (FX)(x, x')_n \end{array}$$

is a pullback. Decomposing the morphisms by (7.4) and applying Proposition 6.10 once again, we see that it suffices to show that for each $n \in \mathbb{N}$ and $x_0, \dots, x_n \in \text{ob}(X)$ with $x_0 = x$ and $x_n = x'$, the square

$$\begin{array}{ccc}
\coprod_{\substack{p_0, \dots, p_n \in \text{ob}(P) \\ p_0 = p, p_n = p' \\ f \circ h(p_i) = x_i}} P(p_{n-1}, p_n) \times \cdots \times P(p_0, p_1) & \xrightarrow{\coprod_{\substack{b_0, \dots, b_n \in \text{ob}(B) \\ b_0 = b, b_n = b' \\ g(b_i) = x_i}} (Fk)_{p, p', n, b_0, \dots, b_n}} & \coprod_{\substack{b_0, \dots, b_n \in \text{ob}(B) \\ b_0 = b, b_n = b' \\ g(b_i) = x_i}} B(b_{n-1}, b_n) \times \cdots \times B(b_0, b_1) \\
\downarrow \coprod_{\substack{a_0, \dots, a_n \in \text{ob}(A) \\ a_0 = a, a_n = a' \\ f(a_i) = x_i}} (Fh)_{p, p', n, a_0, \dots, a_n} & & \downarrow (Fg)_{b, b', n, x_0, \dots, x_n} \\
\coprod_{\substack{a_0, \dots, a_n \in \text{ob}(A) \\ a_0 = a, a_n = a' \\ f(a_i) = x_i}} A(a_{n-1}, a_n) \times \cdots \times A(a_0, a_1) & & \\
\downarrow (Ff)_{a, a', n, x_0, \dots, x_n} & \longrightarrow & X(x_{n-1}, x_n) \times \cdots \times X(x_0, x_1)
\end{array}$$

is a pullback. Writing the indexing sets of the coproducts appearing in the above diagram as

$$\begin{aligned}
I &= \{ (a_0, \dots, a_n) \mid a_i \in \text{ob}(A), a_0 = a, a_n = a', f(a_i) = x_i \}, \\
J &= \{ (b_0, \dots, b_n) \mid b_i \in \text{ob}(B), b_0 = b, b_n = b', g(b_i) = x_i \}, \\
K &= \{ (p_0, \dots, p_n) \mid p_i \in \text{ob}(P), p_0 = p, p_n = p', f \circ h(p_i) = x_i \},
\end{aligned}$$

we have $I \times J \cong K$ by the description of $\text{ob}(P)$ as a pullback. Using (7.5) and the second clause of Corollary 6.13, it suffices to show that for any n, x_i, a_i, b_i, p_i with $p_i = (a_i, b_i)$, $f(a_i) = g(b_i) = x_i$, the square

$$\begin{array}{ccc}
P(p_{n-1}, p_n) \times \cdots \times P(p_0, p_1) & \xrightarrow{k_{p_{n-1}, p_n} \times \cdots \times k_{p_0, p_1}} & B(b_{n-1}, b_n) \times \cdots \times B(b_0, b_1) \\
\downarrow h_{p_{n-1}, p_n} \times \cdots \times h_{p_0, p_1} & & \downarrow g_{b_{n-1}, b_n} \times \cdots \times g_{b_0, b_1} \\
A(a_{n-1}, a_n) \times \cdots \times A(a_0, a_1) & \xrightarrow{f_{a_{n-1}, a_n} \times \cdots \times f_{a_0, a_1}} & X(x_{n-1}, x_n) \times \cdots \times X(x_0, x_1)
\end{array}$$

is a pullback. This follows from the fact that each $P(p_i, p_{i+1})$ is the pullback of $A(a_i, a_{i+1})$ and $B(b_i, b_{i+1})$ over $X(x_i, x_{i+1})$, as pullbacks commute with products. \square

Proposition 7.3. *If \mathcal{V} has a strict initial object 0 and finite products, then the categories $\mathcal{V}\text{-Gph}$ and $\mathcal{V}\text{-Cat}$ admit small coproducts and the forgetful functor $U: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Gph}$ preserves small coproducts.*

Proof. In both $\mathcal{V}\text{-Gph}$ and $\mathcal{V}\text{-Cat}$, small coproducts are given by taking disjoint union of objects and setting the hom-objects between objects from different components to be 0 (see the proofs of Propositions 6.5 and 6.6). \square

Proposition 7.4. *If \mathcal{V} is an extensive category with finite limits, then the unit $\eta: \text{id}_{\mathcal{V}\text{-Gph}} \Rightarrow UF$ of the adjunction $F \dashv U$ in Proposition 7.1 is cartesian.*

Proof. By Proposition 6.7, it suffices to show that for any morphism $f: G \rightarrow H$ of \mathcal{V} -graphs, and $x, y \in \text{ob}(G)$, the square

$$\begin{array}{ccc} G(x, y) & \xrightarrow{\sigma_{(1,x,y)}} & (FG)(x, y) \\ \downarrow f_{x,y} & & \downarrow (Ff)_{x,y} \\ H(fx, fy) & \xrightarrow{\sigma_{(1,fx,fy)}} & (FH)(fx, fy) \end{array}$$

is a pullback in \mathcal{V} . This follows from Proposition 6.9 (note that if we rewrite $(Ff)_{x,y}$ via (7.3) and (7.4), we have $(Ff)_{x,y,1,fx,fy} = f_{x,y}$). \square

We only need the case $m = 1$ of the following proposition in order to show that \mathbb{T} is cartesian; the full generality of this stronger version will be needed in the next section.

Proposition 7.5. *If \mathcal{V} is an extensive category with finite limits, then for each natural number m , the natural transformation*

$$(\mathcal{V}\text{-Gph})^m \xrightarrow{F^m} (\mathcal{V}\text{-Cat})^m \xrightarrow{\prod} \mathcal{V}\text{-Cat} \xrightarrow{\text{id}_{\mathcal{V}\text{-Cat}}} \mathcal{V}\text{-Cat},$$

$\begin{array}{ccc} & U \nearrow & \mathcal{V}\text{-Gph} & \searrow F \\ & & \downarrow \varepsilon & \\ & & \mathcal{V}\text{-Cat} & \end{array}$

where ε is the counit of the adjunction $F \dashv U$ in Proposition 7.1 and \prod is the m -ary product functor, is cartesian.

Proof. Let $f = (f^{(1)}, \dots, f^{(m)}): (G^{(1)}, \dots, G^{(m)}) \rightarrow (H^{(1)}, \dots, H^{(m)})$ be a morphism in $(\mathcal{V}\text{-Gph})^m$. Our aim is to show that the square

$$\begin{array}{ccc} FU(FG^{(1)} \times \dots \times FG^{(m)}) & \xrightarrow{\varepsilon_{FG^{(1)} \times \dots \times FG^{(m)}}} & FG^{(1)} \times \dots \times FG^{(m)} \\ \downarrow FU(Ff^{(1)} \times \dots \times Ff^{(m)}) & & \downarrow Ff^{(1)} \times \dots \times Ff^{(m)} \\ FU(FH^{(1)} \times \dots \times FH^{(m)}) & \xrightarrow{\varepsilon_{FH^{(1)} \times \dots \times FH^{(m)}}} & FH^{(1)} \times \dots \times FH^{(m)} \end{array}$$

in $\mathcal{V}\text{-Cat}$ is a pullback. By Proposition 6.7 it suffices to show that for every pair of objects $x = (x^{(1)}, \dots, x^{(m)}), y = (y^{(1)}, \dots, y^{(m)}) \in \text{ob}(FU(FG^{(1)} \times \dots \times FG^{(m)})) = \text{ob}(G^{(1)}) \times \dots \times \text{ob}(G^{(m)})$, the square

$$\begin{array}{ccc} (FU \prod_{i=1}^m (FG^{(i)}))(x, y) & \longrightarrow & \prod_{i=1}^m (FG^{(i)})(x^{(i)}, y^{(i)}) \\ \downarrow & & \downarrow \\ (FU \prod_{i=1}^m (FH^{(i)}))(fx, fy) & \longrightarrow & \prod_{i=1}^m (FH^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)}) \end{array} \quad (7.6)$$

in \mathcal{V} is a pullback. Using Proposition 6.11, we may rewrite the bottom right object $\prod_{i=1}^m (FH^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)})$ as a coproduct of products. Precisely, we define an (indexing) set I to be

$$I = \{ (n_1, v_0^{(1)}, \dots, v_{n_1}^{(1)}, \dots, n_m, v_0^{(m)}, \dots, v_{n_m}^{(m)}) \mid n_i \in \mathbb{N}, v_j^{(i)} \in \text{ob}(H^{(i)}), v_0^{(i)} = f^{(i)}x^{(i)}, v_{n_i}^{(i)} = f^{(i)}y^{(i)} \}.$$

Then $\prod_{i=1}^m (FH^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)})$ is isomorphic to

$$\prod_I H^{(1)}(v_{n_1-1}^{(1)}, v_{n_1}^{(1)}) \times \cdots \times H^{(1)}(v_0^{(1)}, v_1^{(1)}) \\ \times \cdots \times H^{(m)}(v_{n_m-1}^{(m)}, v_{n_m}^{(m)}) \times \cdots \times H^{(m)}(v_0^{(m)}, v_1^{(m)}). \quad (7.7)$$

We may now decompose the diagram (7.6) into coproducts over I and apply Proposition 6.10. Fix an element $(n_1, v_0^{(1)}, \dots, v_{n_1}^{(1)}, \dots, n_m, v_0^{(m)}, \dots, v_{n_m}^{(m)}) \in I$, and introduce new indexing sets

$$J = \{ (k, j_1^{(1)}, \dots, j_{k-1}^{(1)}, \dots, j_1^{(m)}, \dots, j_{k-1}^{(m)}) \\ | k, j_l^{(i)} \in \mathbb{N}, 0 \leq j_1^{(i)} \leq \cdots \leq j_{k-1}^{(i)} \leq n_i \}, \\ K = \{ (w_0^{(1)}, \dots, w_{n_1}^{(1)}, \dots, w_0^{(m)}, \dots, w_{n_m}^{(m)}) \\ | w_j^{(i)} \in \text{ob}(G^{(i)}), w_0^{(i)} = x^{(i)}, w_{n_i}^{(i)} = y^{(i)}, f^{(i)}w_j^{(i)} = v_j^{(i)} \}.$$

It suffices to show that the square

$$\begin{array}{ccc} \prod_J \prod_K G^{(1)}(w_{n_1-1}^{(1)}, w_{n_1}^{(1)}) \times \cdots \times G^{(m)}(w_0^{(m)}, w_1^{(m)}) & \xrightarrow{\prod_K [\text{id}]_J} & \prod_K G^{(1)}(w_{n_1-1}^{(1)}, w_{n_1}^{(1)}) \times \cdots \times G^{(m)}(w_0^{(m)}, w_1^{(m)}) \\ \downarrow & & \downarrow \\ \prod_J [f_{w_{n_1-1}^{(1)}, w_{n_1}^{(1)}}^{(1)} \times \cdots \times f_{w_0^{(m)}, w_1^{(m)}}^{(m)}]_K & & \prod_K G^{(1)}(w_{n_1-1}^{(1)}, w_{n_1}^{(1)}) \times \cdots \times G^{(m)}(w_0^{(m)}, w_1^{(m)}) \\ \downarrow & & \downarrow \\ \prod_J H^{(1)}(v_{n_1-1}^{(1)}, v_{n_1}^{(1)}) \times \cdots \times H^{(m)}(v_0^{(m)}, v_1^{(m)}) & & [f_{w_{n_1-1}^{(1)}, w_{n_1}^{(1)}}^{(1)} \times \cdots \times f_{w_0^{(m)}, w_1^{(m)}}^{(m)}]_K \\ \downarrow & & \downarrow \\ \prod_J H^{(1)}(v_{n_1-1}^{(1)}, v_{n_1}^{(1)}) \times \cdots \times H^{(m)}(v_0^{(m)}, v_1^{(m)}) & \xrightarrow{[\text{id}]_J} & \prod_J H^{(1)}(v_{n_1-1}^{(1)}, v_{n_1}^{(1)}) \times \cdots \times H^{(m)}(v_0^{(m)}, v_1^{(m)}) \end{array}$$

is a pullback, which follows from the second clause of Corollary 6.13. \square

Theorem 7.6. *If \mathcal{V} is an extensive category with finite limits, then the free \mathcal{V} -category monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathcal{V}\text{-Gph}$ is cartesian.*

Proof. The functor $T = UF$ preserves pullbacks since both U (because it is a right adjoint) and F (by Proposition 7.2) do. The unit η is cartesian by Proposition 7.4. The multiplication $\mu = U\varepsilon F$ is cartesian because εF is so by Proposition 7.5 (take $m = 1$), and U preserves pullbacks. \square

7.2 The free strict n -dimensional \mathcal{V} -category monad

In this section we show that the forgetful functor from the category of strict n -dimensional \mathcal{V} -categories to that of n -dimensional \mathcal{V} -graphs has a left adjoint. We assume throughout that \mathcal{V} is extensive and has finite limits. It follows that $\mathcal{V}\text{-Gph}$ and $\mathcal{V}\text{-Cat}$ are likewise (by Propositions 6.5, 6.6 and 6.7), and so, by induction, for each natural number n , the categories $\mathcal{V}\text{-Gph}^{(n)}$ and $\mathcal{V}\text{-Cat}^{(n)}$ are also extensive with finite limits.

Recall that, by Propositions 6.11 and 7.1, the forgetful functor

$$U: (\mathcal{V}\text{-Cat}^{(n)})\text{-Cat} \longrightarrow (\mathcal{V}\text{-Cat}^{(n)})\text{-Gph}$$

admits a left adjoint F .

Definition 7.7. For each natural number n , we define an adjunction $F^{(n)} \dashv U^{(n)}: \mathcal{V}\text{-Cat}^{(n)} \longrightarrow \mathcal{V}\text{-Gph}^{(n)}$ recursively as follows:

1. $F^{(0)} = U^{(0)} = \text{id}_{\mathcal{V}}$;
2. $F^{(n+1)} \dashv U^{(n+1)}$ is the composite:

$$(\mathcal{V}\text{-Gph}^{(n)})\text{-Gph} \begin{array}{c} \xrightarrow{F^{(n)\text{-Gph}}} \\ \perp \\ \xleftarrow{U^{(n)\text{-Gph}}} \end{array} (\mathcal{V}\text{-Cat}^{(n)})\text{-Gph} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} (\mathcal{V}\text{-Cat}^{(n)})\text{-Cat}.$$

■

The adjunction $F^{(n)} \dashv U^{(n)}$ induces a monad $\mathbb{T}^{(n)} = (T^{(n)}, \eta^{(n)}, \mu^{(n)})$ on $\mathcal{V}\text{-Gph}^{(n)}$. We call $\mathbb{T}^{(n)}$ the **free strict n -dimensional \mathcal{V} -category monad**, and now show that it is cartesian.

Proposition 7.8. For each natural number n , $F^{(n)}: \mathcal{V}\text{-Gph}^{(n)} \longrightarrow \mathcal{V}\text{-Cat}^{(n)}$ preserves pullbacks.

Proof. For $n = 0$, the assertion is trivial. Proceeding inductively, if $F^{(n)}$ preserves pullbacks, so does $F^{(n)\text{-Gph}}$ by Proposition 6.7. The functor $F: (\mathcal{V}\text{-Cat}^{(n)})\text{-Gph} \longrightarrow (\mathcal{V}\text{-Cat}^{(n)})\text{-Cat}$ preserves pullbacks by Proposition 7.2. \square

Proposition 7.9. For each natural number n , $U^{(n)}: \mathcal{V}\text{-Cat}^{(n)} \longrightarrow \mathcal{V}\text{-Gph}^{(n)}$ preserves small coproducts.

Proof. For $n = 0$, the assertion is trivial. Proceeding inductively, if $U^{(n)}$ preserves small coproducts, it preserves initial objects, and so the functor $U^{(n)\text{-Gph}}$ preserves small coproducts. The functor $U: (\mathcal{V}\text{-Cat}^{(n)})\text{-Cat} \longrightarrow (\mathcal{V}\text{-Cat}^{(n)})\text{-Gph}$ also preserves small coproducts by Proposition 7.3. \square

Proposition 7.10. For each natural number n , the unit $\eta^{(n)}: \text{id}_{\mathcal{V}\text{-Gph}^{(n)}} \Longrightarrow U^{(n)}F^{(n)}$ of the adjunction $F^{(n)} \dashv U^{(n)}$ is cartesian.

Proof. Observe that adjunctions whose units are cartesian are closed under composition. Proceeding inductively, if $\eta^{(n)}$ is cartesian, so is $\eta^{(n)\text{-Gph}}$ by Proposition 6.7. The unit of the adjunction $F \dashv U: (\mathcal{V}\text{-Cat}^{(n)})\text{-Cat} \longrightarrow (\mathcal{V}\text{-Cat}^{(n)})\text{-Gph}$ is cartesian by Proposition 7.4. \square

Proposition 7.11. For each pair of natural numbers n and m , the natural transformation

$$(\mathcal{V}\text{-Gph}^{(n)})^m \xrightarrow{(F^{(n)})^m} (\mathcal{V}\text{-Cat}^{(n)})^m \xrightarrow{\prod} \mathcal{V}\text{-Cat}^{(n)} \xrightarrow{\text{id}_{\mathcal{V}\text{-Cat}^{(n)}}} \mathcal{V}\text{-Cat}^{(n)},$$

$\begin{array}{ccc} & U^{(n)} \nearrow & \mathcal{V}\text{-Gph}^{(n)} \\ & & \downarrow \varepsilon^{(n)} \\ & & \mathcal{V}\text{-Cat}^{(n)} \\ & & \searrow F^{(n)} \end{array}$

where $\varepsilon^{(n)}$ is the counit of the adjunction $F^{(n)} \dashv U^{(n)}$ and \prod is the m -ary product functor, is cartesian.

Proof. By induction on n . Suppose the claim is true for $n = k$ and for all m . For brevity, we will write the adjunction $F^{(k)\text{-Gph}} \dashv U^{(k)\text{-Gph}}$ as $F' \dashv U'$, and whose counit $\varepsilon^{(k)\text{-Gph}}$ as ε' . We aim to show that for every morphism

$(f^{(1)}, \dots, f^{(m)}): (G^{(1)}, \dots, G^{(m)}) \longrightarrow (H^{(1)}, \dots, H^{(m)})$ in $(\mathcal{V}\text{-Gph}^{(k+1)})^m$, the outer rectangle in the diagram

$$\begin{array}{ccccc}
FF'U'U(\prod_{i=1}^m FF'G^{(i)}) & \xrightarrow{F\varepsilon'_{U(\prod_{i=1}^m FF'G^{(i)})}} & FU(\prod_{i=1}^m FF'G^{(i)}) & \xrightarrow{\varepsilon_{\prod_{i=1}^m FF'G^{(i)}}} & \prod_{i=1}^m FF'G^{(i)} \\
\downarrow & & \downarrow & & \downarrow \\
FF'U'U(\prod_{i=1}^m FF'f^{(i)}) & & FU(\prod_{i=1}^m FF'f^{(i)}) & & \prod_{i=1}^m FF'f^{(i)} \\
\downarrow & & \downarrow & & \downarrow \\
FF'U'U(\prod_{i=1}^m FF'H^{(i)}) & \xrightarrow{F\varepsilon'_{U(\prod_{i=1}^m FF'H^{(i)})}} & FU(\prod_{i=1}^m FF'H^{(i)}) & \xrightarrow{\varepsilon_{\prod_{i=1}^m FF'H^{(i)}}} & \prod_{i=1}^m FF'H^{(i)}
\end{array}$$

in $\mathcal{V}\text{-Cat}^{(k+1)}$ is a pullback. The right square is a pullback by Proposition 7.5, so we shall show that the left square is also a pullback. Since F preserves pullbacks by Proposition 7.2, it suffices to show that the square

$$\begin{array}{ccc}
F'U'U(\prod_{i=1}^m FF'G^{(i)}) & \xrightarrow{\varepsilon'_{U(\prod_{i=1}^m FF'G^{(i)})}} & U(\prod_{i=1}^m FF'G^{(i)}) \\
\downarrow & & \downarrow \\
F'U'U(\prod_{i=1}^m FF'f^{(i)}) & & U(\prod_{i=1}^m FF'f^{(i)}) \\
\downarrow & & \downarrow \\
F'U'U(\prod_{i=1}^m FF'H^{(i)}) & \xrightarrow{\varepsilon'_{U(\prod_{i=1}^m FF'H^{(i)})}} & U(\prod_{i=1}^m FF'H^{(i)})
\end{array}$$

in $(\mathcal{V}\text{-Cat}^{(k)})\text{-Gph}$ is a pullback. By Proposition 6.7, it suffices to show that for every pair of objects $(x^{(1)}, \dots, x^{(m)}), (y^{(1)}, \dots, y^{(m)}) \in \text{ob}(F'U'U(\prod_{i=1}^m FF'G^{(i)})) = \text{ob}(G^{(1)}) \times \dots \times \text{ob}(G^{(m)})$, the square

$$\begin{array}{ccc}
F^{(k)}U^{(k)}(\prod_{i=1}^m (FF'G^{(i)})(x^{(i)}, y^{(i)})) & \longrightarrow & \prod_{i=1}^m (FF'G^{(i)})(x^{(i)}, y^{(i)}) \\
\downarrow & & \downarrow \\
F^{(k)}U^{(k)}(\prod_{i=1}^m (FF'H^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)})) & \longrightarrow & \prod_{i=1}^m (FF'H^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)})
\end{array} \tag{7.8}$$

in $\mathcal{V}\text{-Cat}^{(k)}$ is a pullback. The bottom right object may be rewritten, using the set

$$\begin{aligned}
I = \{ & (n_1, v_0^{(1)}, \dots, v_{n_1}^{(1)}, \dots, n_m, v_0^{(m)}, \dots, v_{n_m}^{(m)}) \\
& \mid n_i \in \mathbb{N}, v_j^{(i)} \in \text{ob}(H^{(i)}), v_0^{(i)} = f^{(i)}x^{(i)}, v_{n_i}^{(i)} = f^{(i)}y^{(i)} \},
\end{aligned}$$

as

$$\begin{aligned}
& \prod_I F^{(k)}(H^{(1)}(v_{n_1-1}^{(1)}, v_{n_1}^{(1)})) \times \dots \times F^{(k)}(H^{(1)}(v_0^{(1)}, v_1^{(1)})) \\
& \times \dots \times F^{(k)}(H^{(m)}(v_{n_m-1}^{(m)}, v_{n_m}^{(m)})) \times \dots \times F^{(k)}(H^{(m)}(v_0^{(m)}, v_1^{(m)}));
\end{aligned}$$

cf. (7.7). Because both $F^{(k)}$ and $U^{(k)}$ (by Proposition 7.9) preserve small coproducts, we may decompose (7.8) as the coproduct over the set I and apply Proposition 6.10. Fix an element $(n_1, v_0^{(1)}, \dots, v_{n_1}^{(1)}, \dots, n_m, v_0^{(m)}, \dots, v_{n_m}^{(m)}) \in I$ and introduce the set

$$\begin{aligned}
K = \{ & (w_0^{(1)}, \dots, w_{n_1}^{(1)}, \dots, w_0^{(m)}, \dots, w_{n_m}^{(m)}) \\
& \mid w_j^{(i)} \in \text{ob}(G^{(i)}), w_0^{(i)} = x^{(i)}, w_{n_i}^{(i)} = y^{(i)}, f^{(i)}w_j^{(i)} = w_j^{(i)} \}.
\end{aligned}$$

It suffices to show that the square

$$\begin{array}{ccc}
 \prod_K F^{(k)} U^{(k)} F^{(k)} G^{(1)}(w_{n_1-1}^{(1)}, w_{n_1}^{(1)}) \times \cdots & & \\
 \cdots \times F^{(k)} U^{(k)} F^{(k)} G^{(m)}(w_0^{(m)}, w_1^{(m)}) & \xrightarrow{\quad} & \prod_K F^{(k)} G^{(1)}(w_{n_1-1}^{(1)}, w_{n_1}^{(1)}) \times \cdots \\
 & & \cdots \times F^{(k)} G^{(m)}(w_0^{(m)}, w_1^{(m)}) \\
 \downarrow & & \downarrow \\
 F^{(k)} U^{(k)} F^{(k)} H^{(1)}(v_{n_1-1}^{(1)}, v_{n_1}^{(1)}) \times \cdots & & \\
 \cdots \times F^{(k)} U^{(k)} F^{(k)} H^{(m)}(v_0^{(m)}, v_1^{(m)}) & & \\
 \downarrow & & \downarrow \\
 & \xrightarrow{\quad} & F^{(k)} H^{(1)}(v_{n_1-1}^{(1)}, v_{n_1}^{(1)}) \times \cdots \\
 & & \cdots \times F^{(k)} H^{(m)}(v_0^{(m)}, v_1^{(m)})
 \end{array}$$

is a pullback. This follows from the first clause of Corollary 6.13, and the induction hypothesis. \square

Theorem 7.12. *For each natural number n , the free strict n -dimensional \mathcal{V} -category monad $\mathbb{T}^{(n)}$ is cartesian.*

Proof. This follows from Proposition 7.8, Proposition 7.10, and Proposition 7.11 (take $m = 1$); cf. the proof of Theorem 7.6. \square

Chapter 8

The definition of weak n -dimensional \mathcal{V} -category

Building upon the results of the previous chapters, in this chapter we define weak n -dimensional \mathcal{V} -category for each natural number n and *locally presentable* extensive category \mathcal{V} .¹ Local presentability is a certain size condition on a category, and we need to assume this in the final step of the definition. Our definition follows and enriches that of Leinster [64], which in turn was inspired by Batanin's work [5].

Leinster's definition of weak n -category may be summarised as follows. Consider the free strict n -category monad $\mathbb{T}^{(n)}$ on the category $n\text{-Gph}$ of n -graphs; this is the case $\mathcal{V} = \mathbf{Set}$ of the monad $\mathbb{T}^{(n)}$ studied in the previous chapter. As we have already seen in Theorem 7.12 in the enriched setting, this monad is cartesian, hence we may consider $\mathbb{T}^{(n)}$ -operads. Now, Leinster has introduced the notion of *contraction* on morphisms in $n\text{-Gph}$. Recall that a $\mathbb{T}^{(n)}$ -operad is a monoid object in the slice category $n\text{-Gph}/\mathbb{T}^{(n)}1$. By defining a contraction on a $\mathbb{T}^{(n)}$ -operad to be a contraction on its underlying object in $n\text{-Gph}/\mathbb{T}^{(n)}1$, we may also talk about $\mathbb{T}^{(n)}$ -operads with contractions. Let $\mathbb{L}^{(n)}$ be the *initial* $\mathbb{T}^{(n)}$ -operad with a contraction. Leinster defines weak n -categories to be the models of $\mathbb{L}^{(n)}$.

In this chapter, we will carry out the enriched version of the above development. Leinster's original formulation of contraction depends heavily on set-theoretic manipulations, so we shall use Garner's reformulation [29] of contractions in more categorical terms. We define contractions on morphisms in $\mathcal{V}\text{-Gph}^{(n)}$, and then on $\mathbb{T}^{(n)}$ -operads. We show the existence of the initial $\mathbb{T}^{(n)}$ -operad with a contraction $\mathbb{L}^{(n)}$ using our new assumption that \mathcal{V} is locally presentable, and finally define weak n -dimensional \mathcal{V} -categories to be the models of $\mathbb{L}^{(n)}$.

The results in this section have been published in [16].

8.1 Contractions

In this section we describe the notion of *contraction*, introduced by Leinster [64], and generalise it to the enriched setting. We follow Garner [29] and define contraction as a choice of certain diagonal fillers. The following definition is an example of the construction described in [30, Proposition 3.8].

Definition 8.1. Let \mathcal{C} be a category, J a set, and $\mathcal{F} = (f_j: A_j \rightarrow B_j)_{j \in J}$ a J -indexed family of morphisms in \mathcal{C} .

¹Locally presentable categories are both complete and cocomplete, so we do not have to write the condition that \mathcal{V} admits all finite limits separately.

1. A **contraction** (with respect to \mathcal{F}) on a morphism $g: C \rightarrow D$ in \mathcal{C} is a J -indexed family of functions $(\kappa_j)_{j \in J}$ such that for each $j \in J$, κ_j assigns to every pair of morphisms (h, k) in \mathcal{V} which makes the perimeter of (8.1) commute, a morphism $\kappa_j(h, k)$ making the whole diagram (8.1) commute.

$$\begin{array}{ccc}
 A_j & \xrightarrow{h} & C \\
 \downarrow f_j & \nearrow \kappa_j(h, k) & \downarrow g \\
 B_j & \xrightarrow{k} & D
 \end{array} \tag{8.1}$$

2. A **map of morphisms with contractions** from $(g: C \rightarrow D, (\kappa_j)_{j \in J})$ to $(g': C' \rightarrow D', (\kappa'_j)_{j \in J})$ is a map of morphisms $(u: C \rightarrow C', v: D \rightarrow D')$ from g to g' which commutes with contractions: for each $j \in J$ and (h, k) in the domain of κ_j , $u \circ \kappa_j(h, k) = \kappa'_j(uh, vk)$.

$$\begin{array}{ccccccc}
 A_j & \xrightarrow{h} & C & \xrightarrow{u} & C' & & \\
 \downarrow f_j & \nearrow \kappa_j(h, k) & \downarrow g & \nearrow \kappa'_j(uh, vk) & \downarrow g' & & \\
 B_j & \xrightarrow{k} & D & \xrightarrow{v} & D' & &
 \end{array}$$

We write the category of morphisms in \mathcal{C} with contractions (with respect to \mathcal{F}) as $\mathbf{Contr}(\mathcal{F})$. Note that we have the evident forgetful functor $V: \mathbf{Contr}(\mathcal{F}) \rightarrow \mathcal{C}^{\mathbf{2}}$ where $\mathbf{2}$ denotes the arrow category (i.e., the ordinal $\mathbf{2}$ seen as a category) and $\mathcal{C}^{\mathbf{2}} = [\mathbf{2}, \mathcal{C}]$ is the functor category. \blacksquare

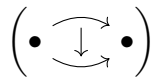
In other words, for each $j \in J$, κ_j is a section of the function ρ_j below, induced by the universality of pullback.

$$\begin{array}{ccc}
 \mathcal{C}(B_j, C) & \xrightarrow{\mathcal{C}(B_j, g)} & \mathcal{C}(B_j, D) \\
 \downarrow \mathcal{C}(f_j, C) & \dashrightarrow \rho_j & \downarrow \mathcal{C}(f_j, D) \\
 \mathcal{C}(A_j, C) & \xrightarrow{\mathcal{C}(A_j, g)} & \mathcal{C}(A_j, D)
 \end{array}$$

As observed in [29], Leinster's notion of contraction, for each natural number n , is a special case of Definition 8.1 where $\mathcal{C} = n\text{-Gph}$ and \mathcal{F} is a certain family $\mathcal{F}^{(n)} = (f_0^{(n)}, \dots, f_{n+1}^{(n)})$ consisting of $n+2$ morphisms in $n\text{-Gph}$. Before giving a precise definition, we try to give an intuitive idea of them by drawing a suggestive picture. For example, when $n = 2$ the family can be drawn as

$$\mathcal{F}^{(2)} = \left(\begin{array}{cccc}
 \left(\begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \right) & \left(\begin{array}{cc} \bullet & \bullet \end{array} \right) & \left(\begin{array}{ccc} \bullet & \rightleftarrows & \bullet \end{array} \right) & \left(\begin{array}{ccc} \bullet & \rightleftarrows & \bullet \\ \downarrow & & \downarrow \end{array} \right) \\
 \downarrow f_0^{(2)} & \downarrow f_1^{(2)} & \downarrow f_2^{(2)} & \downarrow f_3^{(2)} \\
 \left(\bullet \right) & \left(\bullet \longrightarrow \bullet \right) & \left(\begin{array}{ccc} \bullet & \rightleftarrows & \bullet \\ \downarrow & & \downarrow \end{array} \right) & \left(\begin{array}{ccc} \bullet & \rightleftarrows & \bullet \\ \downarrow & & \downarrow \end{array} \right)
 \end{array} \right).$$

Just in case it is not clear how to read the above picture, let us explain one object. The picture



denotes the 2-graph G with two objects ($\text{ob}(G) = \{s, t\}$, represented by the black dots), such that the 1-graphs $G(s, s)$, $G(t, s)$ and $G(t, t)$ have no objects, and $G(s, t)$ is the 1-graph with two objects ($\text{ob}(G(s, t)) = \{x, y\}$, represented by the two horizontal arrows between the black dots) such that $(G(s, t))(x, x) = (G(s, t))(y, x) = (G(s, t))(y, y) = \emptyset$ and $(G(s, t))(x, y) = \{z\}$ (the vertical arrow).

The morphisms $f_0^{(2)}$, $f_1^{(2)}$ and $f_2^{(2)}$ are monomorphisms, and $f_3^{(2)}$ is an epimorphism in $n\text{-Gph}$. The idea is that an element of $\mathcal{F}^{(n)}$ is “the inclusion of the boundary of a ball”, although $f_{n+1}^{(n)}$ is no longer a monomorphism due to lack of cells of dimension greater than n .

To give a recursive definition of $\mathcal{F}^{(n)}$ in the enriched setting, we start with auxiliary definitions. For any category \mathcal{V}' with an initial object 0 , define the **suspension** functor $\Sigma: \mathcal{V}' \rightarrow \mathcal{V}'\text{-Gph}$ which maps $X \in \mathcal{V}'$ to

$$\Sigma X = (\{s, t\}, (\Sigma X(i, j))_{i, j \in \{s, t\}})$$

given by $\Sigma X(s, t) = X$, $\Sigma X(i, j) = 0$ if $(i, j) \neq (s, t)$; cf. [64, Section 9.3]. Also define the **discrete \mathcal{V}' -graph** functor $D: \mathbf{Set} \rightarrow \mathcal{V}'\text{-Gph}$ which maps a set I to $DI = (I, (0)_{i, j \in I})$. The functor D is the left adjoint of $\text{ob}(-): \mathcal{V}'\text{-Gph} \rightarrow \mathbf{Set}$.

Definition 8.2. Let \mathcal{V} be a category with a terminal object and finite coproducts. For each natural number n , define a family $\mathcal{F}^{(n)} = (f_0^{(n)}, \dots, f_{n+1}^{(n)})$ of morphisms in $\mathcal{V}\text{-Gph}^{(n)}$ recursively as follows.

1. $f_0^{(0)}: 0 \rightarrow 1$ and $f_1^{(0)}: 1 + 1 \rightarrow 1$ are the unique morphisms in \mathcal{V} into the terminal object 1 .
2. $f_0^{(n)}: D\emptyset \rightarrow D\{*\}$, where \emptyset and $\{*\}$ are the empty set and a singleton respectively, is the unique morphism in $\mathcal{V}\text{-Gph}^{(n)}$ out of the initial object $D\emptyset$, and for each $i \in \{1, \dots, n+1\}$, $f_i^{(n)} = \Sigma f_{i-1}^{(n-1)}$. ■

For each object $X \in \mathcal{V}\text{-Gph}^{(n)}$, define the category $\mathbf{Contr}(\mathcal{F}^{(n)})_X$ of morphisms into X with contractions (with respect to $\mathcal{F}^{(n)}$) as the following pullback of categories:

$$\begin{array}{ccc} \mathbf{Contr}(\mathcal{F}^{(n)})_X & \longrightarrow & \mathbf{Contr}(\mathcal{F}^{(n)}) \\ \downarrow V_X & \lrcorner & \downarrow V \\ \mathcal{V}\text{-Gph}^{(n)}/X & \longrightarrow & (\mathcal{V}\text{-Gph}^{(n)})^2, \end{array} \quad (8.2)$$

where $\mathcal{V}\text{-Gph}^{(n)}/X \rightarrow (\mathcal{V}\text{-Gph}^{(n)})^2$ is the inclusion functor. Explicitly, the category $\mathbf{Contr}(\mathcal{F}^{(n)})_X$ is given as follows.

- An object is a morphism g in $\mathcal{V}\text{-Gph}^{(n)}$ with a contraction as in Definition 8.1 such that the codomain of g is X .
- A morphism is a map of morphisms with contractions (u, v) as in Definition 8.1 such that $v = \text{id}_X$.

We will in particular be concerned with the case where $X = T^{(n)}\mathbf{1}$.

Now we can describe our definition of weak n -dimensional \mathcal{V} -category in more detail. We have already mentioned at the beginning of this chapter that we define a weak n -dimensional \mathcal{V} -category to be a model of a certain $\mathbb{T}^{(n)}$ -operad $\mathbf{L}^{(n)}$, characterised as the *initial $\mathbb{T}^{(n)}$ -operad with a contraction*. Let us define what this means in more precise terms. We define the category $\mathbb{T}^{(n)}\text{-OC}$ of $\mathbb{T}^{(n)}$ -operads with contractions to be the following pullback of categories:

$$\begin{array}{ccc}
 \mathbb{T}^{(n)}\text{-OC} & \longrightarrow & \mathbf{Contr}(\mathcal{F}^{(n)})_{T^{(n)}\mathbf{1}} \\
 \downarrow & \lrcorner & \downarrow V_{T^{(n)}\mathbf{1}} \\
 \mathbb{T}^{(n)}\text{-Opd} & \xrightarrow{W} & \mathcal{V}\text{-Gph}^{(n)}/T^{(n)}\mathbf{1},
 \end{array} \tag{8.3}$$

where the functor $V_{T^{(n)}\mathbf{1}}$ is the appropriate instance of (8.2) and W forgets the $\mathbb{T}^{(n)}$ -operad structure (recall that $\mathbb{T}^{(n)}\text{-Opd} = \mathbf{Mon}(\mathcal{V}\text{-Gph}^{(n)}/T^{(n)}\mathbf{1})$). Provided that the category $\mathbb{T}^{(n)}\text{-OC}$ has an initial object ($(\text{ar}_{L^{(n)}} : L^{(n)} \rightarrow T^{(n)}\mathbf{1}), m, e, \kappa$), by the initial $\mathbb{T}^{(n)}$ -operad with contraction we mean its underlying $\mathbb{T}^{(n)}$ -operad $\mathbf{L}^{(n)} = ((\text{ar}_{L^{(n)}} : L^{(n)} \rightarrow T^{(n)}\mathbf{1}), m, e)$ (forgetting the contraction κ).

Thus the remaining step in our definition of weak n -dimensional \mathcal{V} -category is to show that the category $\mathbb{T}^{(n)}\text{-OC}$ indeed has an initial object. This can be shown, under the additional assumption that \mathcal{V} is locally presentable.

8.2 Local presentability and algebraic weak factorisation systems

We first provide a minimal introduction to locally presentable categories; see [1, Chapter 1] for more details.

A cardinal α is called **regular** if for any set I and I -indexed family of sets $(x_i)_{i \in I}$, $|I| < \alpha$ and $|x_i| < \alpha$ for all $i \in I$ imply $|\coprod_{i \in I} x_i| < \alpha$. We shall only talk about small regular cardinals.

From now on, let α be a (small) regular cardinal. A small poset \mathcal{I} is said to be **α -directed** if any subset of \mathcal{I} whose cardinality is less than α admits an upper bound in \mathcal{I} . For any category \mathcal{C} , an **α -directed diagram** is a functor $\mathcal{I} \rightarrow \mathcal{C}$ from an α -directed poset \mathcal{I} (seen as a category). By an **α -directed colimit** we mean the colimit of an α -directed diagram.

Suppose that \mathcal{C} and \mathcal{D} are locally small categories admitting all α -directed colimits (i.e., admitting all colimits indexed by small α -directed posets). A functor $\mathcal{C} \rightarrow \mathcal{D}$ is said to be **α -accessible** if it preserves all α -directed colimits. An object $C \in \mathcal{C}$ is called **α -presentable** if the functor $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$ is α -accessible.

A locally small category \mathcal{C} is called **locally α -presentable** if it is cocomplete and there exists a small full subcategory $\mathcal{C}_\alpha \subseteq \mathcal{C}$ such that (i) all objects in \mathcal{C}_α are α -presentable, and (ii) any object in \mathcal{C} can be expressed as an α -directed colimit of objects in \mathcal{C}_α .

Finally, a locally small category \mathcal{C} is called **locally presentable** if there exists a (small) regular cardinal α such that \mathcal{C} is locally α -presentable. A functor F between cocomplete categories is called **accessible** if there exists a (small) regular cardinal α such that F is α -accessible.

It is known that \mathbf{Set} is locally \aleph_0 -presentable (also called *locally finitely presentable*), and $\omega\text{-Cpo}$ is locally \aleph_1 -presentable (see [1, Example 1.18]). It is also known that whenever \mathcal{V} is locally presentable, so is $\mathcal{V}\text{-Gph}$ ([48, Proposition 4.4]).

Among others, local presentability is used as a standard condition on categories in order to ensure that certain transfinite constructions converge [52]. An example of such constructions relevant to our purpose is Garner’s version [30] of the *small object argument* originally developed by Quillen [79]. We have the following result, easily deducible from [9, Proposition 16].

Proposition 8.3. *Let \mathcal{V} be a locally presentable category. Then for each $n \in \mathbb{N}$ and $X \in \mathcal{V}\text{-Gph}^{(n)}$, the functor $V_X: \mathbf{Contr}(\mathcal{F}^{(n)})_X \rightarrow \mathcal{V}\text{-Gph}^{(n)}/X$ is monadic and accessible.*

8.3 Weak n -dimensional \mathcal{V} -categories

Finally we prove that $\mathbb{T}^{(n)\text{-OC}}$ actually has an initial object, for any category \mathcal{V} which is locally presentable and extensive.

Theorem 8.4. *If \mathcal{V} is a locally presentable and extensive category, then for any natural number n the category $\mathbb{T}^{(n)\text{-OC}}$ has an initial object.*

Proof. We shall follow the argument in [64, Appendix G] (where $\mathcal{V} = \mathbf{Set}$ and $n = \omega$) and show that $\mathcal{V}\text{-Gph}^{(n)}/T^{(n)}1$ is locally presentable (hence is both complete and cocomplete), and that W and $V_{T^{(n)}1}$ are monadic and accessible. Then by [52, Theorem 27.1] it follows that the forgetful functor from $\mathbb{T}^{(n)\text{-OC}}$ to $\mathcal{V}\text{-Gph}^{(n)}/T^{(n)}1$ (the composite of functors in (8.3)) is also monadic, thus in particular $\mathbb{T}^{(n)\text{-OC}}$ has an initial object, given by the free algebra over the initial object in $\mathcal{V}\text{-Gph}^{(n)}/T^{(n)}1$.

Because $\mathcal{V}\text{-Gph}^{(n)}$ is locally presentable, so is $\mathcal{V}\text{-Gph}^{(n)}/T^{(n)}1$, being its slice. The functor W is monadic because it is the forgetful functor from a category of monoids and admits a left adjoint G (which, incidentally, is of a particularly simple form $GP = \coprod_{n \in \mathbb{N}} P^{\otimes n}$ thanks to Proposition 7.9). It is routine to show that W is accessible. The functor $V_{T^{(n)}1}$ is monadic and accessible by Proposition 8.3. \square

The condition of \mathcal{V} being locally presentable and extensive is an axiomatic reason why Batanin and Leinster’s approach works. Of course the category \mathbf{Set} satisfies this condition, but in their work this fact is used only implicitly, often in the form of concrete set-theoretic manipulation.

Definition 8.5. Let \mathcal{V} be a locally presentable extensive category and n be a natural number. A **weak n -dimensional \mathcal{V} -category** is a model of the initial $\mathbb{T}^{(n)}$ -operad with contraction, where $\mathbb{T}^{(n)}$ is the free strict n -dimensional \mathcal{V} -category monad on $\mathcal{V}\text{-Gph}^{(n)}$. \blacksquare

We remark that when $\mathcal{V}(1, -)$ is not conservative, it might be more appropriate to replace 1 of Definition 8.2 by the family of morphisms $0 \rightarrow X$ and $X + X \rightarrow X$ (codiagonal) where X ranges over a small set of strong generators of \mathcal{V} (exists if \mathcal{V} is locally presentable). We thank an anonymous reviewer of [16] for pointing this out. Even if we alter Definition 8.2 this way, all arguments so far hold unchanged.

Example 8.6. If we let $\mathcal{V} = \mathbf{Set}$ and $n = 2$, then weak 2-categories (weak 2-dimensional \mathbf{Set} -categories) are equivalent to *unbiased bicategories*, which are a variant of bicategories equipped with for each natural number m , an m -ary horizontal composition operation. See [64, Section 9.4] for details. \blacksquare

Example 8.7. If we let $\mathcal{V} = \omega\text{-Cpo}$ and $n = 2$, then weak 2-dimensional $\omega\text{-Cpo}$ -categories are the unbiased version of $\omega\text{-Cpo}$ -enriched bicategories as in [78]. \blacksquare

Chapter 9

Conclusion

9.1 Summary

In this thesis, we have investigated aspects of algebraic structure. In the first part, we have developed a unified framework for various notions of algebraic theory. In the second part, we focused on a particular algebraic structure, weak n -categories à la Batanin and Leinster, and generalised the known definition by allowing enrichment over any extensive and locally presentable category.

Our unified framework for notions of algebraic theory is based on a number of more or less independent observations made by many researchers over years, which we have summarised in Section 3.1. The concepts of metatheory and theory, being identical to (large) monoidal category and monoid object, are of course well-known. As for these, the novelty is not in the concepts themselves but in our attitude to identify them with notion of algebraic theory and algebraic theory respectively. To the best of our knowledge, no one seems to have proposed such identification.

We have supported this rather bold proposal by modelling the semantical aspect of notions of algebraic theory as well in our framework. Here, in order to unify enrichments and oplax actions, which have been observed to underlie notions of model, we have introduced a new concept of metamodel. Although one can reduce metamodels (of \mathcal{M} in \mathcal{C}) to combinations of known concepts, such as enrichment¹ of \mathcal{C} over $\widehat{\mathcal{M}} = [\mathcal{M}^{\text{op}}, \mathbf{SET}]$ or as a lax monoidal functor $\mathcal{M}^{\text{op}} \rightarrow [\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathbf{SET}]$, they do not seem to have been studied extensively so far, let alone in connection to notions of algebraic theory. The fact that we can give a definition of model relative to a metamodel in a way compatible with those relative to an enrichment or an oplax action, though not particularly difficult to show, seems to testify to the inherent coherence underlying various notions of algebraic theory.

We have also introduced morphism between metatheories. An appropriate notion of morphism turned out to be more general than the ones usually considered, namely lax, oplax or strong monoidal functors; it is a monoidal version of profunctors. If the morphisms come in an adjoint pair, then (by the pseudo-functoriality of $\mathcal{M}\text{Mod}(-)$) we obtain a 2-adjunction between the 2-categories of metamodels. Because our morphisms between metatheories are quite general, it is not difficult to obtain an adjoint pair of them; any strong monoidal functor generates an adjoint pair. In this case, we immediately obtain isomorphisms of categories of models in different notions of algebraic theory, by a purely formal categorical argument (see Section 3.3).

¹To be precise, the concept of enrichment (Definition 3.12) also seems to have been newly introduced in this thesis, though it is fairly similar to the well-known concept of \mathcal{M} -category [53].

Within our framework, we have also obtained a general structure-semantics adjointness result (Chapter 4) and a double categorical universal characterisation of categories of models (Chapter 5). The former result supports our claim that the framework is appropriate for notions of algebraic theory, by incorporating the topic which has been studied extensively in the categorical algebra community. The latter result may be taken as an evidence of the naturality or canonicity of our framework, as it gives an abstract characterisation of categories of models arising in our framework, generalising the characterisation of Eilenberg–Moore categories by Street [81] in a natural direction. In addition, we believe that it provides a non-trivial example of double limits, which are a newly introduced notion [34] and seem to be in need of examples.

Our generalisation of Batanin and Leinster’s definition of weak n -category clarifies the structure of their original definition, by pointing out the fact that the categorical properties of extensivity and local presentability play a key role in the definition.

We have established in Chapter 6 a number of properties on (infinite) extensive categories. Since these properties are not very hard to show, we expect that they are either known to or immediately recognisable by the experts, but we have not been able to find a suitable reference. The papers [12, 14] are excellent sources of information, but they only treat finitary extensive categories.

In Chapter 7 we have shown by induction on n that the free strict n -dimensional \mathcal{V} -category monad $\mathbb{T}^{(n)}$ on $\mathcal{V}\text{-Gph}^{(n)}$ is cartesian. Our inductive argument is more delicate than one might first imagine, and we had to choose properties more general than is strictly necessary for our goal (see e.g., Proposition 7.11). The proofs fully exploit the properties of extensive categories established in Chapter 6.

Our definition of weak n -dimensional \mathcal{V} -category for any extensive and locally presentable category \mathcal{V} is given in Chapter 8. Here, in order to generalise Leinster’s notion of contraction, we have applied Garner’s theory of algebraic weak factorisation systems [30].

9.2 Future work

As future work, we would like to further investigate various aspects of our unified framework for notions of algebraic theory. One natural open problem is to characterise the categories of models arising in our framework—or rather, the associated forgetful functors—by their intrinsic properties. For the case of monad, the corresponding result is various *monadicity theorems* (such as Beck’s theorem [71, Section VI. 7]), characterising the *monadic functors*, i.e., those functors isomorphic to the forgetful functors from Eilenberg–Moore categories. The forgetful functors arising in our framework are more general than the monadic functors; for example, they need not admit left adjoints, as is the case for the forgetful functor $\mathbf{FinGrp} \rightarrow \mathbf{FinSet}$ from the category of finite groups to the category of finite sets (this functor arises if we consider the metatheory $[\mathbf{F}, \mathbf{Set}]$ for clones, the clone of groups and the standard metamodel of $[\mathbf{F}, \mathbf{Set}]$ in the category \mathbf{FinSet} with finite powers). However, they are far from being arbitrary. For example, it is immediate from the definition of categories of models (Definition 3.37) that such functors are faithful and conservative. We would like to identify what additional condition on a functor is enough to ensure that it arises (up to an isomorphism) as the forgetful functor associated with a category of models in our framework. Such a result would help us to better understand the generality of our framework.

We would also like to incorporate more examples of notions of algebraic theory into our framework. We have already listed some possible examples in Section 2.7. As for PROs and PROPs, we expect that monoidal and symmetric monoidal versions of profunctors (cf. morphisms of metatheories in Definition 3.45) would be useful. For example, a PRO is defined as a strict monoidal category together with an identity-on-objects strict monoidal functor from \mathbf{N} , the free strict monoidal category generated by one object. By considering the monoidal category of monoidal endo-profunctors on \mathbf{N} , we would obtain PROs as monoids therein. As for multi-sorted algebraic theories, we think that the best way to model them is to identify them with pseudo double categories, in such a way that objects, vertical morphisms, horizontal morphisms and squares correspond to sorts, translations between sorts, functional signatures (with designated input/output sorts) and translations of functional signatures, respectively. This view is compatible with our current framework, because pseudo double categories with one object and one vertical morphism correspond to monoidal categories. In fact, the pseudo double categories suitable for multi-sorted clones, symmetric operads, non-symmetric operads and generalised operads are already studied in [17]; this paper would lay foundations for the syntactic aspect of the multi-sorted version of our framework.

Our framework shows that whenever we have a monoidal category, we can regard it as a notion of algebraic theory. This observation provides a novel, particularly simple way to define new notions of algebraic theory. Need for new notions of algebraic theory would arise, for example, in study of computational effects. The monad and Lawvere theory approaches to computational effects (see Section 1.1) have captured different aspects of computational effects, and the characteristic features of these notions of algebraic theory are reflected in their major applications: the simplicity of monad makes it into a popular design pattern in functional programming [87], and the modularity of Lawvere theory neatly explains how to model combinations of effects [40]. One naturally expects that suitable notions of algebraic theory would be useful in capturing other aspects of computational effects. Here we mention one such possibility: the quantitative aspect as measured by *effect systems* [68]. A categorical semantics of effect systems is given via the notion of *graded monad* [47], which is a monad in a suitable 2-category [26] and hence a monoid in a monoidal category, but a suitable notion of *graded Lawvere theory* is yet to be defined.

Another future work is to apply our framework to the study of higher dimensional categories. As we have mentioned in the introduction, currently there are many definitions of weak n -category and a conceptual understanding of the relationship between these definitions is in need. An obstruction to the direct comparison is the fact that different (algebraic) definitions of weak n -category are given in terms of algebraic theories belonging to different notion of algebraic theory, such as generalised operads, symmetric operads and monads; cf. [63]. We expect that our unified framework may overcome this difficulty thanks to its generality, incorporating a wide range of notions of algebraic theory.

Finally we mention that there are also a lot to be done around Batanin and Leinster's weak n -categories. In Leinster's definition, weak n -categories are defined as models of a certain $\mathbf{T}^{(n)}$ -operad $\mathbf{L}^{(n)}$. However, if we consider homomorphisms in the usual sense between models of $\mathbf{L}^{(n)}$, then these correspond to *strict n -functors* and the more natural *weak n -functors* are not treated in [64]. Batanin gives a definition of weak n -functor in [5, Definition 8.8], and it would be interesting to adapt that definition to Leinster's version of weak n -categories, and to enrich it over an extensive and locally presentable category \mathcal{V} in order to clarify

the structure of the definition. We believe that a substantial theory of weak n -categories would have applications in computer science as well, for instance by suggesting new semantically motivated axioms to homotopy type theory.

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