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Imprecise Information and Second-Order Beliefs*

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Abstract

A decision problem under uncertainty is often given with a piece of objective but imprecise information about the states of the world such as in the Ellsberg urn. By incorporating such information into the smooth ambiguity model of Seo (2009), we characterize a class of smooth ambiguity representations whose second-order beliefs are consistent with the objective information. As a corollary, we provide an axiomatization for the second-order expected utility, which has been studied by Nau (2001), Neilson (2009), Grant, Polak, and Strzalecki (2009), Strzalecki (2011), and Ghirardato and Pennesi (2019). In our model, attitude toward uncertainty can be disentangled from a perception about uncertainty and connected with attitude toward reduction of compound lotteries.

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1 Introduction

1.1 Objective

Choice under uncertainty is an important aspect of decision making. Since Ellsberg's seminal work, it is admitted that a decision maker's behavior may be inconsistent with a probabilistic belief (or a subjective probability measure) about the states of the world. Such a situation is called ambiguity and has been studied by several models of decision making, such as the Choquet expected utility (Schmeidler [31]), the maxmin expected utility (Gilboa and Schmeidler [12]), and the smooth ambiguity model (Klibanoff, Marinacci, and Mukerji [22] and Seo [33]).

In applications of decision making under uncertainty, a decision problem is often given with a piece of objective but imprecise information about the states of the world. This feature is most salient in dynamic environments. A decision maker obtains a piece of information through observations of signals over time, which is given as a set of events or a set of conditional distributions over states. Then, the decision maker's belief should be updated according to the objective information.

Even in static environments, objective information is relevant for decision making. For instance, in the Ellsberg's three-color urn experiment, there is an urn including red, blue and green balls with unknown proportion. Prior to betting on the color of a ball drawn from the urn, a decision maker is informed that the proportion of red balls is one-third. In a portfolio choice problem, a decision maker may be partially informed about marginal distributions of random variables, but ignorant about their joint distribution or correlations (Ellis and Piccione [6]). Under these circumstances, the decision maker should have a belief constrained by the objective information.

In the model of maxmin expected utility, Gajdos, Hayashi, Tallon, and Vergnaud [10] and Hayashi [18] explicitly consider a decision theoretic model with objective but imprecise information. They take preference over the Cartesian product of the set of acts and some set of probabilities over states as a primitive, and characterize a representation where an act is evaluated according to the maxmin expected utility of which subjective multiple priors are consistent with the objective information. Giraud [13] adopts the same primitive and considers consistency between the preference and the objective information within the smooth ambiguity model and its variants.

In the present paper, we take a more parsimonious approach where preference over the set of acts is given together with a fixed piece of objective information, rather than preference over the Cartesian product. More precisely, we introduce a piece of objective but imprecise information into the smooth ambiguity model of Seo [33]. As assumed in Seo [33], let S be a finite set of states of the world and Z be a separable metric space of outcomes. Let $\Delta(Z)$ be the set of all Borel probability measures over Z. An (Anscombe-Aumann, or AA) act is a function $f : S \to \Delta(Z)$ and the set of all acts is denoted by $\mathcal{F}$. We consider preference $\succsim$ over the set $\Delta(\mathcal{F})$ of all Borel probability measures over $\mathcal{F}$. Each element $P \in \Delta(\mathcal{F})$ is called a random act. On top of these primitives, we assume that a measurable subset $\mathcal{T} \subset \Delta(S)$ is given as a piece of objective but imprecise information.

Say that $\succsim$ admits a $\mathcal{T}$-consistent smooth ambiguity representation if there exist a continuous expected utility function $u : \Delta(Z) \to \mathbb{R}$, a second-order belief $m \in \Delta(\Delta(S))$ with $m(\mathcal{T}) = 1$, and a continuous increasing function $\varphi : u(\Delta(Z)) \to \mathbb{R}$ such that

$$W(P) = \int_{\mathcal{F}} V(f)\, \mathrm{d}P(f), \text{ where } V(f) = \int_{\mathcal{T}} \varphi\left(\int_S u(f(s))\, \mathrm{d}\mu(s)\right) \mathrm{d}m(\mu), \tag{1}$$

represents $\succsim$. Compared with Seo [33], $\mathcal{T}$-consistency additionally requires $m(\mathcal{T}) = 1$. Alternatively, Seo [33] is interpreted as the case where no piece of objective information is given, that is, $\mathcal{T} = \Delta(S)$. By specifying $\mathcal{T}$ in various ways, we can accommodate several examples (such as correlation misperception, unambiguous events within the smooth ambiguity model, statistical inference, and so on) in terms of $\mathcal{T}$-consistent smooth ambiguity representations.

A key axiom of Seo [33] is called Dominance. For each act $f \in \mathcal{F}$ and each probability measure $\mu \in \Delta(S)$, let the reduced lottery induced from f and μ be denoted by $\Psi(f, \mu) \in \Delta(Z)$. For each random act P, we define $\Psi(P, \mu) \in \Delta(\Delta(Z))$ by the distribution induced from $\Psi(f, \mu)$ and P. The Dominance axiom requires that one random act P is preferred to another Q if $\Psi(P, \mu)$ is preferred to $\Psi(Q, \mu)$ for all $\mu \in \Delta(S)$. If there is a piece of objective information, the Dominance axiom should be adjusted to the information: say that $\succsim$ satisfies $\mathcal{T}$-Dominance if $P \succsim Q$ whenever $\Psi(P, \mu) \succsim \Psi(Q, \mu)$ for all $\mu \in \mathcal{T}$. Our main finding is that $\mathcal{T}$-Dominance and the other axioms of Seo [33] characterize a $\mathcal{T}$-consistent smooth ambiguity representation.

We establish this result by applying the subjective expected utility representation theorem for continuous acts shown by Zhou [36], which considerably simplifies Seo [33]'s original proof. See an outline of the proof provided in Section 2.4.1 for more details. Moreover, our alternative proof allows any measurable subset $\mathcal{T} \subset \Delta(S)$, which is not necessarily a compact subset.[1]

[1]Seo [33]'s original proof relies on compactness of $\mathcal{T} = \Delta(S)$ (see Step 1 of Lemma B.7 in Seo [33]).

As a main application, we provide an axiomatization for second-order expected utility (SOEU) (see Nau [27], Neilson [29], Grant, Polak, and Strzalecki [15], Ghirardato and Pennesi [11]): There exist an expected utility function $u : \Delta(Z) \to \mathbb{R}$ and a continuous increasing function φ such that $\succsim$ on $\mathcal{F}$ is represented by

$$V(f) = \int_S \varphi(u(f(s)))\, \mathrm{d}\mu(s). \tag{2}$$

As pointed out by, for instance, Nau [27], a second-order expected utility allows for sensitivity to different sources of uncertainty. In particular, if φ is concave, the model is consistent with ambiguity aversion of Schmeidler [31] and Gilboa and Schmeidler [12] even though it is probabilistically sophisticated.

An SOEU representation (2) can be regarded as a special case of (1) if the support of the second-order belief m consists only of Dirac measures on S. As an immediate corollary of our main result, an SOEU representation can be axiomatized by specifying $\mathcal{T}$ to the set of all Dirac measures on S, that is, $\mathcal{T}^S := \{\delta_s \in \Delta(S) \mid s \in S\}$.

Our model allows for general φ and hence, perception and attitude toward uncertainty can be disentangled. Moreover, it is possible to connect the curvature of φ to attitude toward reduction of compound lotteries in $\Delta(\Delta(Z))$. Our behavioral comparisons show that concavity (or convexity) of φ corresponds to preference for (or aversion to) reduction of compound lotteries. In particular, if φ is linear, the decision maker does not care about reduction, which corresponds to a subjective expected utility representation of Anscombe and Aumann [2].

1.2 Related literature

1.2.1 Dominance and subjective probabilities

Following Seo [33], we consider preference over random acts and a dominance-type axiom in order to derive a second-order belief. Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci [4] provide a general framework for preferences over random (Savage) acts. This choice object has a natural interpretation in a setting of game theory. The set S of states is the set of strategy profiles of the opponents, the set Z is regarded as the set of social outcomes, and then an act $f : S \to Z$ is interpreted as a strategy of a particular player. Note that P can be viewed as his mixed strategy. Compared with the choice objects in Anscombe-Aumann [2], the timing of realization of states and lotteries is reversed. An AA act $f : S \to \Delta(Z)$ is interpreted as a compound lottery of horse races and roulette wheels, that is, a state is resolved first and subsequently an outcome is realized according to the lottery corresponding to the state. This is a model of ex post randomization. On the other hand, a random act is interpreted as a compound lottery of roulette wheels and horse races, that is, this is a model of ex ante randomization. Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci [4] show that the two models are equivalent if a dominance-type axiom (State-Wise Dominance, as stated in Section 4) is taken appropriately.

Ellis and Piccione [6] consider a decision maker who may choose a bundle of several actions without knowing their correlation across payoffs. Each action $a \in A$ generates a payoff $a(\theta)$ depending on a state $\theta \in \Theta$. The decision maker may choose a bundle of several actions. An action profile $\langle a_i \rangle_{i=1}^n$ generates a monetary payoff $\sum_i a_i(\theta)$. Though Θ describes the objective or correct structure of joint returns, the decision maker may not be able to perceive each action a as a mapping $a : \Theta \to \mathbb{R}$, and hence, may misperceive correlation among a bundle of actions. For example, even if $b(\theta) + c(\theta) = a(\theta)$ for all θ, the decision maker may misperceive that the background uncertainties among these actions are mutually independent, in which case, $\langle b, c \rangle$ may be strictly preferred to $\langle a \rangle$ because of hedging motives. Ellis and Piccione [6] consider preference over lotteries over action profiles $\langle a_i \rangle_{i=1}^n$ and by using the VNM axioms and a dominance-type axiom, characterize a representation where the decision maker has a subjective probability measure over an enlarged state space $S := \Theta^A$, where Θ^A is the Cartesian product where one copy of Θ is assigned to each action in A. A subjective belief over S expresses a perceived correlation across actions. Section 3.3 illustrates how correlation misperception is accommodated in our model.

Aoyama [3] considers unambiguous events within smooth ambiguity models. Though his main objective is an endogenous derivation of unambiguous events from preference, as an intermediate result, he takes a set of events, denoted by $\mathcal{U}$, as a primitive and shows that $\mathcal{U}$ can be interpreted as the set of unambiguous events if preference satisfies a form of dominance axiom. As illustrated in Section 3.4, his observation is obtained by an application of our $\mathcal{T}$-Dominance.

1.2.2 Second-order expected utilities

The second-order expected utility model with concave φ exhibits uncertainty aversion even though it is consistent with probabilistic sophistication. That is, uncertainty aversion is not necessarily attributed to ambiguous beliefs over states. This observation has been pointed out by Epstein [8] and Halevy and Feltkamp [16].

The model also allows for sensitivity to the source of uncertainty. Suppose that φ is strictly concave. For example, take the lottery p over outcomes that gives either x or y with the equal probability. By the representation, the evaluation of p is given by $U(p) = \varphi(\frac{1}{2}u(x) + \frac{1}{2}u(y))$. On the other hand, consider an act $f : S \to Z$ that gives x on event E of probability $\frac{1}{2}$ and y otherwise. The representation implies $U(f) = \frac{1}{2}\varphi(u(x)) + \frac{1}{2}\varphi(u(y))$. Even though both p and f induce the same lottery over outcomes, $U(p) > U(f)$, that is, the decision maker exhibits more risk aversion for subjective risks (probabilities over states) rather than objective risks (randomization over outcomes). The same intuition can be applied to explain the Ellsberg-type behavior (for example, Ellsberg's two-color experiments). Ergin and Gul [7] and Chew and Sagi [5] study sensitivity to the source of uncertainty in the Savage setting.

As we characterize in Section 4.2, the curvature of φ captures attitude toward reduction of compound lotteries. Even though this attitude seems to be conceptually different from risk aversion, as the above illustration suggests, it contributes also to risk aversion on the domain of $\mathcal{F}$. The theory of compound lotteries or preference for reduction and its interaction with risk aversion have been considered both in the setting of two-stage lotteries; Segal [32], and Grant, Kajii, and Polak [14], and in multi-stage settings; Kreps and Porteus [25], Hayashi [17], and Klibanoff and Ozdenoren [24].

Alternative axiomatizations of the second-order expected utility are provided by Neilson [29] and Grant, Polak, and Strzalecki [15]. Neilson [29] considers the Savage model augmented by lotteries over outcomes such as Anscombe and Aumann [2]. In particular, subjective probabilities are identified directly by Savage's theorem. Grant, Polak, and Strzalecki [15] consider preference over $\mathcal{F}$. Since uncertainty aversion axioms are required for their axiomatization, φ is forced to be concave. Compared with Grant, Polak, and Strzalecki [15], our result accommodates a general φ but requires preference over $\Delta(\mathcal{F})$, which is larger than the set of AA acts.

Ghirardato and Pennesi [11] identify a utility midpoint from preference over a product space by exploiting the property of its additive separable utility representation. They extend this framework to preference over Savage acts into this product space. Without relying on any objective lotteries, they axiomatize the second-order expected utility representation for this preference. Their result also allows for a general attitude toward uncertainty.

1.2.3 Updating of second-order beliefs

In a dynamic setting under uncertainty, a decision maker obtains a piece of information through observations of signals over time. Klibanoff, Marinacci, and Mukerji [23], which is a dynamic extension of Klibanoff, Marinacci, and Mukerji [22], consider an updating of second-order beliefs. Their model is built on preference over second-order acts (acts defined over probability measures over S), which are typically unobservable.

Hayashi and Miao [19] also consider a recursive version of the smooth ambiguity representation and show that the auxiliary domain of second-order acts is dispensable if Seo's static model is extended into a dynamic setting. Take S^{∞} as a set of sequences of observable signals. For a history s^t of signal observations up to period t, the decision maker is given a set $\mathcal{P}_{s^t}$ of one-step-ahead probability measures over the next signal S. The decision maker forms an updated subjective probability measure m_{s^t} over $\mathcal{P}_{s^t}$, which can be interpreted as a second-order belief.

This framework is flexible enough to accommodate statistical inference models. For example, let Θ be a set of unknown parameters and consider a statistical model $(\pi_\theta)_{\theta\in\Theta}$, where π_θ is a probability distribution over S^{∞}. The decision maker has an initial prior m_0 over Θ. If $\mathcal{P}_{s^t}$ consists of one-step-ahead conditional distributions $\pi_\theta(\cdot|s^t) \in \Delta(S)$ indexed by an unknown parameter $\theta \in \Theta$, a second-order belief is a probability measure over Θ and is updated according to signals s^t over time.

Hayashi and Miao [19] impose $\mathcal{P}_{s^t}$-Dominance on preference on a recursive domain and derive a recursive version of $\mathcal{P}_{s^t}$-consistent smooth ambiguity representation. They establish this theorem by scrutinizing Seo's proof and verifying that the same proof works even if $\Delta(S)$ is replaced with $\mathcal{P}_{s^t} \subset \Delta(S)$. As stated in the introduction, however, Seo's proof relies on compactness of $\Delta(S)$ and, hence, Hayashi and Miao [19] must assume $\mathcal{P}_{s^t}$ to be compact. Our result allows for any measurable $\mathcal{P}_{s^t} \subset \Delta(S)$.

1.2.4 Randomization and ambiguity aversion

Our domain of choice objects involves two types of randomization. One is an ex ante randomization, which is a lottery over acts, while the other is an ex post randomization, which is resolved after state realization. A relationship between preference for randomization and ambiguity aversion has been studied by Saito [30] and Ke and Zhang [21]. Saito [30] considers a decision maker who endogenously randomizes over a feasible set (or menu) of acts in order to eliminate the effects of uncertainty. Ke and Zhang [21] consider preference over $\Delta(\mathcal{F})$ as in our model and axiomatize a representation, called a double maxmin expected utility. They require an axiom, called indifference to timing of constant act randomization, which reduces φ to be linear in our model. Thus, as they point out, the only intersection between their representation and the smooth ambiguity model (and hence second-order expected utility) is subjective expected utility of Anscombe and Aumann [2].

2 Model

2.1 Primitives

The primitives of our basic model are as follows:

- S: a finite set of states of the world
- Z: a set of outcomes (assumed to be a separable metric space)
- $\Delta(Z)$: the set of all Borel probability measures over Z
- $\mathcal{F}$: the set of AA-acts, that is, $\mathcal{F} = \{f \mid f : S \to \Delta(Z)\}$
- $\Delta(\mathcal{F})$: the set of all Borel probability measures over $\mathcal{F}$

Preference $\succsim$ is defined on $\Delta(\mathcal{F})$. A general element of $\Delta(\mathcal{F})$ is denoted by P, Q, and R, and is called a random act. A degenerate random act δ_f is identified with an AA-act f. A random act P whose support consists only of constant acts is identified with a two-stage compound lottery in $\Delta(\Delta(Z))$. Thus, $\mathcal{F}$ and $\Delta(\Delta(Z))$ are subsets of $\Delta(\mathcal{F})$.

Now we introduce another primitive into the model to incorporate a piece of objective but imprecise information. Let $\Delta(S)$ be the set of all probability measures over S. Since S is finite, $\Delta(S)$ is identified with a unit simplex in an Euclidean space. Let $\Delta(S)$ be endowed with the Borel measurable sets. Take any measurable subset $\mathcal{T} \subset \Delta(S)$ as a piece of objective information. The decision maker is sure that this set contains the "true" probability distribution unknown to him.

Example 1 (Ellsberg's urn)**.** Ellsberg's three-color urn experiment is a typical example of choice under uncertainty with a piece of objective but imprecise information. There are three states

$S = \{R, B, G\}$. The decision maker is only informed that R happens with a probability of $1/3$. This piece of information is modeled by

$$\mathcal{T}^{Urn3} := \left\{ \mu \in \Delta(S) \,\middle|\, \mu(\{R\}) = \frac{1}{3},\ \mu(\{B, G\}) = \frac{2}{3} \right\}.$$

In Example 1, $\mathcal{T}^{Urn3}$ is closed (and hence compact in this setup). As illustrated below, the set $\mathcal{T}$ is not necessarily given as a closed subset in $\Delta(S)$.

Example 2 (Full support probabilities)**.** Suppose that the decision maker is informed that any state in S may happen with positive probabilities. This objective information is modeled as

$$\mathcal{T}^{FS} := \{\mu \in \Delta(S) \mid \mu(s) > 0 \text{ for all } s \in S\}.$$

Note that $\mathcal{T}^{FS}$ coincides with the relative interior of $\Delta(S)$, and hence, not a closed subset in $\Delta(S)$.

Example 3 (Full-Bayesian updating)**.** Suppose that the decision maker has objective but imprecise information $M \subset \Delta(S)$. Assume that M is a compact subset. In addition, the decision maker is informed that a true state is contained in an event $E \subset S$. Then, he will update each $\mu \in M$ and obtain a full-Bayesian updated set given by

$$\mathcal{T}^{(M,E)} := \{\mu(\cdot|E) \in \Delta(S) \mid \mu \in M\}.$$

Note that $\mu(\cdot|E)$ is well-defined only when $\mu(E) > 0$. This set is not necessarily a closed subset in $\Delta(S)$.

For example, take $S = \{s, s', s''\}$. For all $n \geq 1$, let $\mu^n(s) = 1 - \frac{1}{n}$, $\mu^n(s') = (1 - \frac{1}{n})\frac{1}{n}$, and $\mu^n(s'') = \frac{1}{n^2}$. Also let $\mu^0(s) = 1$. Finally, let $M = \{\mu^n \in \Delta(S) \mid n \geq 0\}$. Since $\mu^n \to \mu^0$, M is closed and hence compact. If $E = S \setminus \{s\}$,

$$\mathcal{T}^{(M,E)} = \{\mu^n(\cdot|E) \in \Delta(S) \mid n \geq 1\}.$$

Note that $\mu^n(\cdot|E)$ satisfies $\mu^n(s'|E) = 1 - \frac{1}{n}$ and $\mu^n(s''|E) = \frac{1}{n}$, but its limit probability measure μ^* (that is, $\mu^*(s') = 1$) does not belong to $\mathcal{T}^{(M,E)}$. Thus, $\mathcal{T}^{(M,E)}$ is not closed.

2.2 Functional form

Definition 1. $\succsim$ admits a $\mathcal{T}$-consistent smooth ambiguity representation (u, m, φ) if there exist a continuous expected utility function $u : \Delta(Z) \to \mathbb{R}$, a second-order belief $m \in \Delta(\Delta(S))$ with

$m(\mathcal{T}) = 1$, and an increasing continuous function $\varphi : u(\Delta(Z)) \to \mathbb{R}$ such that

$$W(P) = \int_{\mathcal{F}} V(f)\,\mathrm{d}P(f)\,, \text{where } V(f) = \int_{\mathcal{T}} \varphi\left(\int_S u(f(s))\,\mathrm{d}\mu(s)\right)\mathrm{d}m(\mu)\,,$$

represents $\succsim$.[2]

Preference is represented by an expected utility form W of which the VNM function is denoted by V, which, in turn, takes a form of a subjective expected utility with a second-order belief. $\mathcal{T}$-consistency requires that the second-order belief assigns positive probabilities only over (first-order) probability measures in $\mathcal{T}$.

2.3 Axioms

Seo [33] imposes the following axioms on $\succsim$. The first three axioms are the same as the VNM axioms.

Axiom 1 (Order)**.** $\succsim$ *is complete and transitive.*

The next continuity axiom is stronger than the mixture continuity axiom.

Axiom 2 (Continuity)**.** *For all* $P \in \Delta(\mathcal{F})$*,* $\{Q \mid P \succsim Q\}$ *and* $\{Q \mid Q \succsim P\}$ *are closed.*

The mixture on $\Delta(\mathcal{F})$ is defined as usual. For all $P, Q \in \Delta(\mathcal{F})$ and $\lambda \in [0,1]$, $\lambda P + (1-\lambda)Q \in \Delta(\mathcal{F})$ is defined as

$$(\lambda P + (1-\lambda)Q)(B) = \lambda P(B) + (1-\lambda)Q(B)$$

for all measurable sets $B \subset \mathcal{F}$.

Axiom 3 (First-Stage Independence)**.** *For all* $P, Q, R \in \Delta(\mathcal{F})$ *and* $\lambda \in [0,1]$*,*

$$P \succsim Q \iff \lambda P + (1-\lambda)R \succsim \lambda Q + (1-\lambda)R.$$

Take any lottery $p \in \Delta(Z)$. This lottery is identified with a degenerate random act whose support consists of a constant act p, that is, δ_p. We call p a second-stage lottery. The second-stage mixture on $\Delta(Z)$ is defined as follows. For all p, $q \in \Delta(Z)$ and $\lambda \in [0,1]$, $\lambda p \oplus (1-\lambda)q \in \Delta(Z)$ is defined as

$$(\lambda p \oplus (1-\lambda)q)(B) = \lambda p(B) + (1-\lambda)q(B)$$

for all measurable sets $B \subset Z$.

[2]Let $u(\Delta(Z))$ denote the range of u, that is, $u(\Delta(Z)) = \{u(p) \in \mathbb{R} \mid p \in \Delta(Z)\}$.

Axiom 4 (Second-Stage Independence)**.** *For all* $p, q, r \in \Delta(Z)$ *and* $\lambda \in [0, 1]$*,*

$$p \succsim q \iff \lambda p \oplus (1-\lambda) r \succsim \lambda q \oplus (1-\lambda) r.$$

To see the last axiom of Seo [33], take any $\mu \in \Delta(S)$ and $f \in \mathcal{F}$. These two objects generate a two-stage compound lottery. Define $\Psi(f, \mu) \in \Delta(Z)$ by its reduced one-stage lottery. For example, if $S = \{s_1, s_2\}$ and if f gives p if $s = s_1$ and q if $s = s_2$, then $\Psi(f, \mu) = \mu(s_1)p \oplus (1 - \mu(s_1))q$. Next, for any $P \in \Delta(\mathcal{F})$, define $\Psi(P, \mu) \in \Delta(\Delta(Z))$ by the distribution induced from $\Psi(f, \mu)$ and P, that is, for all $B \subset \Delta(Z)$,

$$\Psi(P, \mu)(B) = P(\{f \in \mathcal{F} \mid \Psi(f, \mu) \in B\}).$$

If P gives f with probability λ and g with probability $1 - \lambda$, $\Psi(P, \mu)$ is a two-stage compound lottery that gives $\Psi(f, \mu)$ with probability λ and $\Psi(g, \mu)$ with probability $1 - \lambda$.

Seo [33]'s dominance axiom is defined as follows:

Axiom 5 (Dominance)**.** *For all* $P, Q \in \Delta(\mathcal{F})$*,*

$$\Psi(P, \mu) \succsim \Psi(Q, \mu) \text{ for all } \mu \in \Delta(S) \implies P \succsim Q.$$

The decision maker may be uncertain about probabilities over S. No matter what belief about states he has, however, P induces a better compound lottery than does Q, in which case he should prefer P to Q.

It is unlikely that the decision maker cares about probability measures which are inconsistent with the objective information given by $\mathcal{T}$. Now the Dominance axiom is adjusted properly as follows:

Axiom 6 ($\mathcal{T}$-Dominance)**.** *For all* $P, Q \in \Delta(\mathcal{F})$*,*

$$\Psi(P, \mu) \succsim \Psi(Q, \mu) \text{ for all } \mu \in \mathcal{T} \implies P \succsim Q.$$

Since $\mathcal{T} \subset \Delta(S)$, the presumption of $\mathcal{T}$-Dominance is weaker than that of Dominance, which is $\Delta(S)$-Dominance. Since $\mathcal{T}$-Dominance requires $P \succsim Q$ under the weaker presumption, it is a stronger axiom than Dominance.

2.4 Result

We are ready to state our main theorem.

Theorem 1. *A nondegenerate preference $\succsim$ on $\Delta(\mathcal{F})$ satisfies Order, Continuity, First-Stage Independence, Second-Stage Independence, and $\mathcal{T}$-Dominance if and only if it admits a $\mathcal{T}$-consistent smooth ambiguity representation (u, m, φ).*

Note that Seo [33] corresponds to the case where $\mathcal{T} = \Delta(S)$. The other extreme is the case of precise information, that is, $\mathcal{T} = \{\mu^*\}$ for some $\mu^* \in \Delta(S)$. Then, $m(\{\mu^*\}) = 1$, that is, the second-order belief m is degenerate at μ^*, and the smooth ambiguity representation on $\mathcal{F}$ is reduced to a subjective expected utility representation,

$$V(f) = \int_S u(f(s))\,\mathrm{d}\mu^*(s).$$

It is easy to see that u and $\varphi(u)$ are unique up to positive affine transformations. As pointed out by Seo [33], a second-order belief m in the smooth ambiguity representation is not unique in general. Lemma C.1 of Seo [33] characterizes the uniqueness class of m for any given u and φ. A $\mathcal{T}$-consistent smooth ambiguity representation has exactly the same uniqueness property.

2.4.1 Outline of the proof

We establish Theorem 1 by applying the subjective expected utility representation theorem for continuous acts shown by Zhou [36], which considerably simplifies Seo [33]'s original proof. A proof sketch is as follows. First of all, the first three axioms, that is, the VNM axioms, deliver the expected utility form of

$$W(P) = \int_{\mathcal{F}} V(f)\,\mathrm{d}P(f)$$

with a continuous VNM function $V : \mathcal{F} \to \mathbb{R}$. Let v be its restriction on $\Delta(Z)$.

Next, we show that $\Delta(\mathcal{F})$ can be embedded into the set $C(\mathcal{T})$ of real-valued bounded continuous functions on $\mathcal{T}$. Take any $P \in \Delta(\mathcal{F})$. Regarding $\mathcal{T}$ as a set of fictitious states, P is translated into an AA act $\Psi(P, \cdot) : \mathcal{T} \to \Delta(\Delta(Z))$. By using the representation, $\Psi(P, \cdot)$ is in turn translated into a utility act $\omega(P) : \mathcal{T} \to \mathbb{R}$ as defined by $\omega(P, \mu) := W(\Psi(P, \mu))$. We show that $\omega(P)$ is bounded and continuous. This establishes a mapping ω from $\Delta(\mathcal{F})$ into $C(\mathcal{T})$. $\mathcal{T}$-Dominance implies that for all P, Q,

$$\Psi(P, \mu) \sim \Psi(Q, \mu) \text{ for all } \mu \in \mathcal{T} \implies P \sim Q.$$

That is, if $\omega(P) = \omega(Q)$ then $P \sim Q$. This means that ω is one-to-one up to indifference. Therefore, ω can be regarded as an embedding into $C(\mathcal{T})$. Let $C^* \subset C(\mathcal{T})$ be the image of the function ω. We can regard C^* as a mixture space since ω is linear.

By using this embedding, we define preference $\succsim^*$ on C^* induced from $\succsim$ on $\Delta(\mathcal{F})$. We can verify that $\succsim^*$ satisfies all the Anscombe-Aumann axioms. The theorem of Zhou [36] ensures that $\succsim^*$ admits a subjective expected utility representation. In particular, there exists a subjective probability measure m on $\mathcal{T}$ such that $\succsim^*$ is represented by

$$\int_{\mathcal{T}} \omega(P, \mu)\, \mathrm{d}m(\mu).$$

Equivalently, $\succsim$ is represented by

$$\int_{\mathcal{T}} W(\Psi(P, \mu))\, \mathrm{d}m(\mu) = \int_{\mathcal{T}} \int_{\mathcal{F}} v(\Psi(f, \mu))\, \mathrm{d}P(f)\, \mathrm{d}m(\mu). \tag{3}$$

Finally, by Second-Stage Independence, $\succsim$ on $\Delta(Z)$ satisfies all the VNM axioms. Thus, there exists a continuous expected utility representation $u : \Delta(Z) \to \mathbb{R}$. Since v and u represent the same preference on $\Delta(Z)$, there exists a continuous increasing function φ such that $v = \varphi(u)$. Therefore, (3) is rearranged to

$$\begin{aligned}\int_{\mathcal{T}} \int_{\mathcal{F}} \varphi\left(u(\Psi(f, \mu)\right) \mathrm{d}P(f)\, \mathrm{d}m(\mu) &= \int_{\mathcal{T}} \int_{\mathcal{F}} \varphi\left(\int_S u(f(s))\, \mathrm{d}\mu(s)\right) \mathrm{d}P(f)\, \mathrm{d}m(\mu) \\ &= \int_{\mathcal{F}} \int_{\mathcal{T}} \varphi\left(\int_S u(f(s))\, \mathrm{d}\mu(s)\right) \mathrm{d}m(\mu)\, \mathrm{d}P(f),\end{aligned}$$

as desired.

3 Examples

In this section, we provide several examples of $\mathcal{T}$, that is, a piece of objective information, and discuss implications of $\mathcal{T}$-Dominance.

3.1 The Ellsberg urn

Recall Example 1 as given in Section 2.1. If $\succsim$ on $\Delta(\mathcal{F})$ satisfies $\mathcal{T}^{Urn3}$-Dominance together with the other axioms, then it admits a $\mathcal{T}^{Urn3}$-consistent smooth ambiguity representation (u, m, φ). Since $m(\mathcal{T}^{Urn3}) = 1$, $\mu(\{R\}) = 1/3$ and $\mu(\{B, G\}) = 2/3$ for all $\mu \in \Delta(S)$, m-a.e. Therefore, events $\{R\}$ and $\{B, G\}$ are interpreted as unambiguous events under this smooth ambiguity representation. See Section 3.4 for a more general discussion about unambiguous events.

Ellsberg's two-color urn example is also an experiment with objective information. There are two urns, urn I and urn II, each of which contains red and black balls. The decision maker is informed that urn I contains the same number of red and black balls, while he is completely ignorant about the composition of red and black balls in urn II. The decision maker bets on colors of balls drawn from both urns. Then, the decision maker faces with the state space $S = \{RR, RB, BR, BB\}$, where, for instance, RB means that a red ball is drawn from urn I and a black ball is drawn from urn II. Note that betting on the red color in urn I corresponds to the event $\{RR, RB\} \subset S$. The piece of objective information in this experiment is modeled by

$$\mathcal{T}^{Urn2} := \left\{ \mu \in \Delta(S) \,\middle|\, \mu(\{RR, RB\}) = \mu(\{BR, BB\}) = \frac{1}{2} \right\}.$$

Similarly, a $\mathcal{T}^{Urn2}$-consistent representation is obtained by $\mathcal{T}^{Urn2}$-Dominance.

3.2 Full support probabilities

Recall Example 2 as given in Section 2.1. If $\succsim$ on $\Delta(\mathcal{F})$ satisfies $\mathcal{T}^{FS}$-Dominance together with the other axioms, then it admits a $\mathcal{T}^{FS}$-consistent smooth ambiguity representation (u, m, φ). Since $m(\mathcal{T}^{FS}) = 1$, $\mu(s) > 0$ for all $s \in S$, m-a.e.

Because Dominance is $\Delta(S)$-Dominance and $\mathcal{T}^{FS}$ is the relative interior of $\Delta(S)$, the difference between Dominance and $\mathcal{T}^{FS}$-dominance arises only on the boundary of $\Delta(S)$, which may look minor difference. However, a standard smooth ambiguity representation obtained by Dominance allows m with $m(\mathcal{T}^{FS}) = 0$. $\mathcal{T}^{FS}$-Dominance is useful to exclude such a representation. Note that $\mathcal{T}^{FS}$ is not a compact set, so we cannot directly use the proof technique of Seo [33] to obtain the $\mathcal{T}^{FS}$-consistent representation.

3.3 Correlation misperception

Ellis and Piccione [6] consider preference over action profiles and derive a subjective probability measure which allows for correlation misperception across actions. For example, suppose that there are two actions b_C and b_F that pay monetary payoffs depending on tomorrow's high temperature. Action b_C pays according to the temperature in Celsius, while action b_F pays in Fahrenheit. Take the product set of the temperatures measured in Celsius and in Fahrenheit as the whole state space. Since there is a linear transformation translating one metric into the other, only a strict subset of the state space is objectively possible. However, if the decision maker is not aware of such a formula between the two metrics, he may misunderstand a joint distribution or correlation across the two coordinates.

We can accommodate such a correlation misperception in our setup. For simplicity, suppose there are two state spaces, denoted by S_1 and S_2. Let $S = S_1 \times S_2$ be the whole state space. A decision maker is well-informed about a distribution over each (coordinate) state space. Let $\mu_i^* \in \Delta(S_i)$, $i = 1, 2$, denote such an objective distribution. However, the decision maker may be unaware of joint distributions or correlations between the two coordinates. In such a situation, a conceivably possible probability measure $\mu \in \Delta(S)$ should have the marginal distribution, denoted by μ_i, which coincides with the objective distribution μ_i^* for all coordinates.

This kind of objective information is modeled by

$$\mathcal{T}^M := \left\{\mu \in \Delta(S) \mid \mu_i = \mu_i^* \text{ for all } i\right\}.$$

If $\succsim$ on $\Delta(\mathcal{F})$ satisfies $\mathcal{T}^M$-Dominance together with the other axioms, then it admits a $\mathcal{T}^M$-consistent smooth ambiguity representation.

When it is also known that the two coordinates are not independent, the corresponding objective information is modeled by

$$\mathcal{T}^{M_{cor}} := \left\{\mu \in \Delta(S) \mid \mu_i = \mu_i^* \text{ for all } i, \text{ and } \mu \neq \mu_1^* \otimes \mu_2^*\right\},$$

where $\mu_1^* \otimes \mu_2^*$ is the product measure of μ_1^* and μ_2^*. Although $\mathcal{T}^{M_{cor}}$ is not a compact set, we can obtain a $\mathcal{T}^{M_{cor}}$-consistent representation.

Alternatively, it is possible to replace objectively given marginal distributions with the assumption that only events in each coordinate state space are unambiguous for the decision maker. See Section 3.4 for more details.

3.4 Unambiguous events

Aoyama [3] considers unambiguous events within the framework of smooth ambiguity model.[3] Though his main objective is an endogenous derivation of unambiguous events from preference, as an intermediate result, he takes a set of events, denoted by $\mathcal{U}$, as a primitive and shows that $\mathcal{U}$ can be interpreted as the set of unambiguous events if preference satisfies a dominance axiom which is appropriately adjusted to $\mathcal{U}$. As explained below, this observation is obtained by an application of $\mathcal{T}$-Dominance.

[3]Assuming unambiguous events exogenously, Epstein [8] investigates a behavioral definition of ambiguous aversion and its implications including the maxmin expected utility. Epstein and Zhang [9] derive subjectively unambiguous events from preference over Savage acts.

To see the idea of Aoyama [3], let $\mathcal{U}$ be a set of events in S. Say that an act $f \in \mathcal{F}$ is $\mathcal{U}$-measurable if $f^{-1}(B) \in \mathcal{U}$ for all measurable subsets $B \subset \Delta(Z)$. Suppose that there exists $\mu \in \Delta(S)$ such that $f \sim \Psi(f, \mu)$ for all $\mathcal{U}$-measurable acts f. Since f is indifferent to the lottery induced from f and μ, the decision maker should believe that probabilities on events in $\mathcal{U}$ are consistent with a probability measure μ. In other words, $\mathcal{U}$ is perceived as the set of unambiguous events. Consequently, $\mathcal{U}$-measurable acts are supposed to be unambiguous acts.

Now define $\mathcal{T}^{\mathcal{U}} \subset \Delta(S)$ as the set of such consistent probability measures:

$$\mathcal{T}^{\mathcal{U}} := \{\mu \in \Delta(S) \mid f \sim \Psi(f, \mu),\ \forall\, \mathcal{U}\text{-measurable } f\}.$$

Since $\Psi(f, \cdot)$ is continuous and linear, $\mathcal{T}^{\mathcal{U}}$ is a closed and convex subset of $\Delta(S)$.

As a corollary of Theorem 1, if $\succsim$ satisfies $\mathcal{T}^{\mathcal{U}}$-Dominance together with the other axioms, a smooth ambiguity representation (u, m, φ) is $\mathcal{T}^{\mathcal{U}}$-consistent, that is, $m(\mathcal{T}^{\mathcal{U}}) = 1$. We can verify that events in $\mathcal{U}$ are unambiguous events with respect to the representation.

Corollary 1. *For all $E \in \mathcal{U}$, $\mu(E) = \mu'(E)$ for all $\mu, \mu' \in \Delta(S)$, m-a.e.*

Proof. If $m(\mathcal{T}^{\mathcal{U}}) = 1$, then $f \sim \Psi(f, \mu) \sim \Psi(f, \mu')$ for all μ, μ', m-a.e., and for all $\mathcal{U}$-measurable acts f. By the smooth ambiguity representation, $V(\Psi(f, \mu)) = V(\Psi(f, \mu'))$, that is,

$$\int u(f(s))\, \mathrm{d}\mu(s) = \int u(f(s))\, \mathrm{d}\mu'(s). \tag{4}$$

In particular, for any $E \in \mathcal{U}$, if f is taken as a bet which gives x^* on E and $x_* \prec x^*$ otherwise, then (4) implies

$$\mu(E)u(x^*) + (1 - \mu(E))u(x_*) = \mu'(E)u(x^*) + (1 - \mu'(E))u(x_*),$$

or equivalently, $\mu(E) = \mu'(E)$, as desired. □

3.5 Stochastic orders

Suppose that S is a totally ordered set by some order $\geq_S$.[4] For example, S is assumed to be a finite set of real numbers, interpreted as tomorrow's high temperature. Alternatively, S may consist of tomorrow's weather conditions such as $S = \{sunny, cloudy, rainy\}$ with the order of less likeliness of rain, that is, $sunny \geq_S cloudy \geq_S rainy$. A decision maker has a base prior $\mu^* \in \Delta(S)$. In the case of weather conditions, suppose that the decision maker has in mind a

[4]We thank Koji Abe for suggesting this example.

scenario of $\mu^* = (1/3, 1/3, 1/3)$. Then, it is less likely that he also has in mind another scenario of $(1/2, 0, 1/2)$. Indeed, if he strongly believes that it is sunny tomorrow, the decision maker would also believe that it is more likely to be cloudy rather than rainy.

We can specify a set of plausible beliefs by considering upward or downward shifts of probabilities from the base prior μ^*. Let $\triangleright$ be a stochastic order over $\Delta(S)$ defined according to $\geq_S$ such as the first-order stochastic dominance, the hazard-rate order, the reverse hazard-rate order, or the monotone likelihood ratio order.[5] Then, a piece of objective information can be modeled by

$$\mathcal{T}^{\triangleright} := \{\mu \in \Delta(S) \mid \mu \triangleright \mu^*\}, \tag{5}$$

or

$$\mathcal{T}^{\bowtie} := \{\mu \in \Delta(S) \mid \mu \triangleright \mu^* \text{ or } \mu^* \triangleright \mu\}. \tag{6}$$

Formulation (5) states that plausible beliefs are restricted to the class which stochastically dominates the base prior. Similarly, formulation (6) states that plausible beliefs are restricted to the class which either stochastically dominates or dominated over the base prior.

3.6 Statistical inference

Finally, we take an example in intertemporal decision making. Consider a learning process about some unknown parameter $\theta \in \Theta$, where Θ is finite. The decision maker has an initial prior m_0 over Θ. Take S^∞ as a set of sequences of signal observations. Each $\theta \in \Theta$ corresponds to a probability distribution π_θ over S^∞, which is interpreted as a statistical model.

Each history of signals $s^t = (s_1, \cdots, s_{t-1})$ induces a piece of objective information in period t. Let

$$\mathcal{T}_{s^t} := \{\pi_\theta(\cdot|s^t) \in \Delta(S) \mid \theta \in \Theta\} \tag{7}$$

be a set of one-step-ahead conditional probability distributions over the next signals, derived from $\{\pi_\theta\}_{\theta\in\Theta}$. Moreover, by Bayes' rule, a posterior $m_{s^t} \in \Delta(\Theta)$ is derived as

$$m_{s^t}(\theta) = \frac{\pi_\theta(s^t) m_0(\theta)}{\sum_{\theta'} \pi_{\theta'}(s^t) m_0(\theta')}.$$

Together with (7), m_{s^t} is interpreted as a distribution over $\mathcal{T}_{s^t}$ or a second-order belief over the next period signals S.

[5] See Shaked and Shanthikumar [34] for definitions and properties of these stochastic orders.

Hayashi and Miao [19] adopt a recursive domain of AA acts and axiomatize an intertemporal utility function consistent with the above learning process by applying Seo [33]'s idea. For each preference conditional on history s^t of observable signals and some subset $\mathcal{T}_{s^t} \subset \Delta(S)$, Hayashi and Miao [19] consider $\mathcal{T}_{s^t}$-Dominance. Since Seo [33]'s original proof relies on the compactness of $\Delta(S)$, their argument also relies on the compactness of $\mathcal{T}_{s^t}$. However, as illustrated in Example 3 of Section 2.1, the set of conditional probabilities induced by the full Bayesian updating is not necessarily compact. In contract, our result allows for any measurable subsets $\mathcal{T}_{s^t} \subset \Delta(S)$.

4 Second-order expected utilities

4.1 Axiomatization

This section provides another application of our main theorem and characterizes the second-order expected utility representation described as below:

Definition 2. $\succsim$ on $\Delta(\mathcal{F})$ admits a second-order expected utility (SOEU) representation (u, μ, φ) if there exist a continuous expected utility function $u : \Delta(Z) \to \mathbb{R}$, a subjective probability measure $\mu \in \Delta(S)$, and an increasing continuous function $\varphi : u(\Delta(Z)) \to \mathbb{R}$ such that

$$W(P) = \int_{\mathcal{F}} V(f)\, \mathrm{d}P(f), \text{ where } V(f) = \int_S \varphi(u(f(s)))\, \mathrm{d}\mu(s),$$

represents $\succsim$.

If φ is linear, the above model is called a subjective expected utility representation or an Anscombe-Aumann representation.

As explained in the introduction, the SOEU representation can allow for sensitivity to the source of uncertainty. A decision maker behaves as if he distinguishes objective and subjective risks and uses different risk attitudes toward them. In particular, if φ is concave, he is more risk averse toward subjective risks. This behavior is consistent with the ambiguity aversion axiom by Schmeidler [31]. Therefore, the model can be applied to explain the Ellsberg-type behavior.

The SOEU representation can be interpreted as a $\mathcal{T}$-consistent smooth ambiguity representation when a piece of imprecise information is given as

$$\mathcal{T}^S := \{\delta_s \in \Delta(S) \mid s \in S\},$$

where δ_s is the Dirac measure at s, which is degenerate at $s \in S$. If $m(\mathcal{T}^S) = 1$, m can be identified with a probability measure on S (or, a first-order belief) because δ_s is identified with the state s.

Let $\Psi(P, s)$ denote $\Psi(P, \delta_s)$. The $\mathcal{T}^S$-Dominance axiom is equivalently written as follows:

Axiom 7 (State-Wise Dominance)**.** *For all $P, Q \in \Delta(\mathcal{F})$,*

$$\Psi(P, s) \succsim \Psi(Q, s) \text{ for all } s \in S \implies P \succsim Q.$$

If P and Q are degenerate, State-Wise Dominance implies State-Wise Monotonicity of Anscombe and Aumann [2], that is,

$$f(s) \succsim g(s) \text{ for all } s \in S \implies f \succsim g.$$

Note that $\Psi(P, \cdot) : S \to \Delta(\Delta(Z))$ can be regarded as an AA act by considering $\Delta(Z)$ as an outcome space. See Figure 1 for the connection between P and $\Psi(P, \cdot)$. An implication of State-Wise Dominance is that two random acts P and Q are translated into the induced AA acts, $\Psi(P, \cdot)$ and $\Psi(Q, \cdot)$, and as long as they give indifferent compound lotteries for each state, P and Q are indifferent. In this sense, State-Wise Dominance is some form of reversal of order.

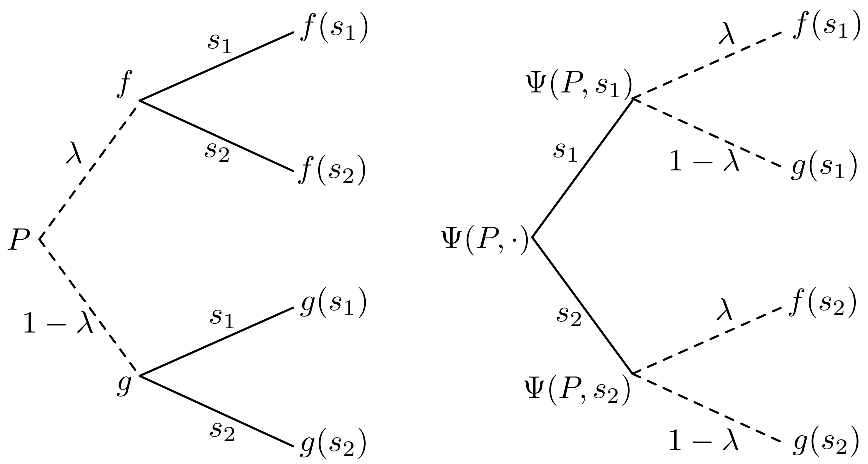

Figure 1: Random act P and the induced AA act $\Psi(P, \cdot)$

Corollary 2. *A nondegenerate preference $\succsim$ on $\Delta(\mathcal{F})$ satisfies Order, Continuity, First-Stage Independence, Second-Stage Independence, and $\mathcal{T}^S$-Dominance (or State-Wise Dominance) if and only if it admits a second-order expected utility representation (u, μ, φ). Moreover, if both (u, μ, φ) and (u', μ', φ') represent the same preference, we have $\mu = \mu'$, and there exists $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$, $i = 1, 2$, such that $u' = \alpha_2 u + \beta_2$ and $\varphi'(\alpha_2 w + \beta_2) = \alpha_1 \varphi(w) + \beta_1$ for all $w \in u(\Delta(Z))$.*

4.2 Attitudes toward reduction of compound lotteries

In the SOEU model, φ contributes to risk attitude to evaluate AA-acts. The curvature of φ also captures attitude toward reduction of two-stage compound lotteries.

Take second-stage lotteries $p, q \in \Delta(Z)$ and $\lambda \in [0,1]$. Recall that $\lambda p \oplus (1-\lambda)q \in \Delta(Z)$ is also a second-stage lottery. On the other hand, $\lambda p + (1-\lambda)q$ is the first-stage lottery that gives p with probability λ and q with probability $1-\lambda$. Thus, $\lambda p + (1-\lambda)q$ is a two-stage compound lottery, that is, $\lambda p + (1-\lambda)q \in \Delta(\Delta(Z))$.[6]

Different attitudes toward reduction of compound lotteries are classified as follows:

Definition 3. For all $p, q \in \Delta(Z)$ and $\lambda \in [0,1]$,

(1) $\succsim$ has preference for reduction if $\lambda p + (1-\lambda)q \precsim \lambda p \oplus (1-\lambda)q$.

(2) $\succsim$ has aversion to reduction if $\lambda p + (1-\lambda)q \succsim \lambda p \oplus (1-\lambda)q$.

(3) $\succsim$ is indifferent about reduction if $\lambda p + (1-\lambda)q \sim \lambda p \oplus (1-\lambda)q$.

Proposition 3. *Assume that $\succsim$ admits an SOEU representation (u, μ, φ).*

(1) If $\succsim$ has preference for reduction, then φ is concave.

(2) If $\succsim$ has aversion to reduction, then φ is convex.

(3) If $\succsim$ is indifferent about reduction, then φ is linear.

Proof. We only show (1). The symmetric argument can be applied for the other cases. Take any $p, q \in \Delta(Z)$ and $\lambda \in [0,1]$. The representation implies that

$$\lambda\varphi(u(p)) + (1-\lambda)\varphi(u(q)) \leq \varphi(u(\lambda p \oplus (1-\lambda)q)).$$

Since u is linear,

$$\lambda\varphi(u(p)) + (1-\lambda)\varphi(u(q)) \leq \varphi(\lambda u(p) + (1-\lambda)u(q)).$$

Thus, for all values $u_1, u_2 \in u(\Delta(Z))$,

$$\lambda\varphi(u_1) + (1-\lambda)\varphi(u_2) \leq \varphi(\lambda u_1 + (1-\lambda)u_2),$$

that is, φ is concave. □

[6]More precisely, $\lambda p + (1-\lambda)q$ should be written as $\lambda\delta_p + (1-\lambda)\delta_q$.

We have the following corollary:

Corollary 4. *Assume that $\succsim$ admits an SOEU representation. Then, $\succsim$ is indifferent about reduction if and only if the representation is reduced to an Anscombe-Aumann representation.*

It is instructive to compare this corollary with the original work of Anscombe and Aumann [2]. They consider preference $\succsim$ on $\Delta(\mathcal{F})$ satisfying Order, Continuity, First-Stage Independence, and State-Wise Monotonicity, and explicitly assume the reduction axiom, called Reversal of Order (ROO), such as

$$\lambda f + (1-\lambda)g \sim \lambda f \oplus (1-\lambda)g,$$

for all $f, g \in \mathcal{F}$ and $\lambda \in [0,1]$, where $(\lambda f \oplus (1-\lambda)g)(s) = \lambda f(s) \oplus (1-\lambda)g(s)$ for all s. Note that First-Stage Independence and ROO imply the independence on $\mathcal{F}$, that is, for all $f, g, h \in \mathcal{F}$ and $\lambda \in [0,1]$,

$$f \succsim g \iff \lambda f \oplus (1-\lambda)h \succsim \lambda g \oplus (1-\lambda)h. \tag{8}$$

Therefore, a subjective expected utility representation is obtained.

5 Comparisons with alternative models

In this section, we summarize relations between a $\mathcal{T}$-consistent smooth ambiguity representation and other preferences under uncertainty. As explained in Section 4.2, the SOEU representation includes the subjective expected utility as a special case, but is nested in $\mathcal{T}$-consistent smooth ambiguity models when $\mathcal{T}$ is specified to $\mathcal{T}^S$. This class is in turn nested into the smooth ambiguity representation.[7] Figure 2 shows these nested relationships.

Strzalecki [35] shows that the intersection between the SOEU and the variational preference of Maccheroni, Marinacci, and Rustichini [26] is the class of multiplier preferences. Since the multiplier preference exhibits uncertainty aversion, it corresponds to the SOEU representation with concave φ.

Ke and Zhang [21] also consider preference $\succsim$ on $\Delta(\mathcal{F})$ in order to distinguish an ex ante randomization from an ex post randomization. Their main representation, called a double maxmin

[7] Nascimento and Riella [28] provide a generalization of the smooth ambiguity representation, where incomplete preference and ambiguity about second-order beliefs are allowed. Izhakian [20] proposes an alternative of the smooth ambiguity model yet relying on second-order beliefs in order to make a complete distinction between beliefs and attitudes, and between risk and ambiguity.

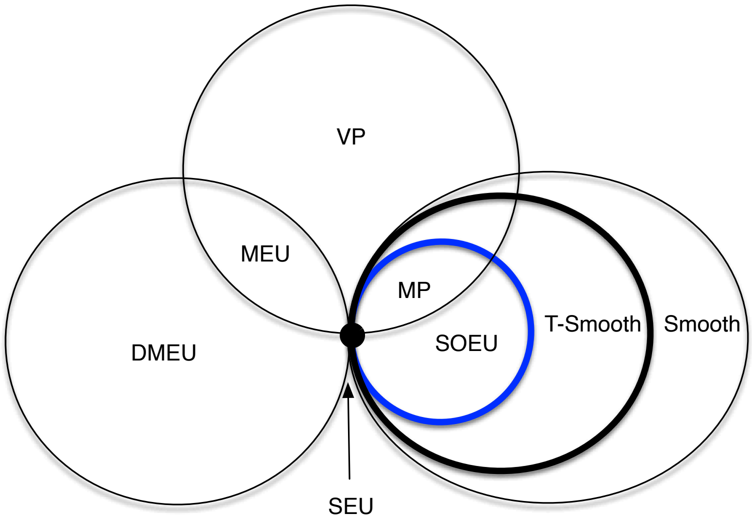

Figure 2: Relations between preferences: SOEU=second-order expected utility, $\mathcal{T}$-Smooth=$\mathcal{T}$-consistent smooth ambiguity preference, Smooth=smooth ambiguity preference, SEU=subjective expected utility, MP=multiplier preference, VP=variational preference, MEU=maxmin expected utility, DMEU=double maxmin expected utility

expected utility (DMEU), generalizes a maxmin expected utility of Gilboa and Schmeidler [12] as follows: There exist an expected utility $u : \Delta(Z) \to \mathbb{R}$, a compact set $\mathcal{M}$ of multiple priors over S such that $\succsim$ on $\Delta(\mathcal{F})$ is represented by

$$W(P) = \min_{M \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_S u(f(s))\, \mathrm{d}\mu(s) \right) \mathrm{d}P(f).$$

One of their axioms requires that a decision maker is indifferent to reduction of compound lotteries, that is, condition (3) of Definition 3. From Corollary 4, the only intersection between DMEU and SOEU is the subjective expected utility.

Appendices

A Proof of Theorem 1

Necessity for the first four axioms is standard and is omitted. We prove the necessity of $\mathcal{T}$-Dominance. First of all, note that for all P, the representation implies

$$\begin{aligned}
W(P) &= \int_{\mathcal{F}} \int_{\mathcal{T}} \varphi\left(\int_S u(f(s))\,\mathrm{d}\mu(s)\right) \mathrm{d}m(\mu)\,\mathrm{d}P(f) \\
&= \int_{\mathcal{T}} \int_{\mathcal{F}} \varphi\left(\int_S u(f(s))\,\mathrm{d}\mu(s)\right) \mathrm{d}P(f)\,\mathrm{d}m(\mu) \\
&= \int_{\mathcal{T}} \int_{\mathcal{F}} \varphi\left(u(\Psi(f,\mu)\right) \mathrm{d}P(f)\,\mathrm{d}m(\mu) = \int_{\mathcal{T}} \int_{\mathcal{F}} W(\Psi(f,\mu))\,\mathrm{d}P(f)\,\mathrm{d}m(\mu) \\
&= \int_{\mathcal{T}} W(\Psi(P,\mu))\,\mathrm{d}m(\mu). \qquad (9)
\end{aligned}$$

Now suppose $\Psi(P,\mu) \gtrsim \Psi(Q,\mu)$ for all $\mu \in \mathcal{T}$. Since $W(\Psi(P,\mu)) \geq W(\Psi(Q,\mu))$ for all $\mu \in \mathcal{T}$, $W(P) \geq W(Q)$ follows from (9), as desired.

We show sufficiency. The first three axioms are the VNM axioms, thus delivering the expected utility representation

$$W(P) = \int_{\mathcal{F}} V(f)\,\mathrm{d}P(f)$$

with a continuous VNM function $V : \mathcal{F} \to \mathbb{R}$ by Grandmont (1974). Define $v : \Delta(Z) \to \mathbb{R}$ as the restriction of V on $\Delta(Z) \subset \mathcal{F}$.

Regarding $\mathcal{T}$ as a set of fictitious states, for any $P \in \Delta(\mathcal{F})$, we can define an AA act $\Psi(P,\cdot) : \mathcal{T} \to \Delta(\Delta(Z))$ induced by P. For $P \in \Delta(\mathcal{F})$, define

$$\omega(P) := (W(\Psi(P,\mu)))_{\mu\in\mathcal{T}} \subseteq \mathbb{R}^{\mathcal{T}}.$$

We call $\omega(P)$ a utility act induced from P. We write $\omega(P,\mu) = W(\Psi(P,\mu))$ for the μ-coordinate of $\omega(P)$. Note that ω is linear, that is, $\omega(\lambda P + (1-\lambda)Q) = \lambda\omega(P) + (1-\lambda)\omega(Q)$. We write $\omega(P) \geq \omega(Q)$ if $\omega(P,\mu) \geq \omega(Q,\mu)$ for all $\mu \in \mathcal{T}$.

$\mathcal{T}$-Dominance implies that for all P, Q,

$$\Psi(P,\mu) \sim \Psi(Q,\mu) \text{ for all } \mu \in \mathcal{T} \implies P \sim Q.$$

That is, if $\omega(P) = \omega(Q)$ then $P \sim Q$.

As preliminaries, we prepare the following two lemmas.

Lemma 1. *$\gtrsim$ satisfies State-Wise Monotonicity: for all $f, g \in \mathcal{F}$ such that $f(s) \gtrsim g(s)$ for all $s \in S$, we have $f \gtrsim g$.*

Proof. We follow Lemma 4.1 (ii) of Seo [33]. First, consider any $f, g \in \mathcal{F}$ such that $f(s) \gtrsim g(s)$ for some s and $f(s') = g(s') = h(s')$ for all $s' \neq s$. By Second-Stage Independence, for all $\mu \in \mathcal{T}$,

$$\Psi(f,\mu) = \mu(s)f(s) + \sum_{s' \neq s} \mu(s')h(s') \gtrsim \mu(s)g(s) + \sum_{s' \neq s} \mu(s')h(s') = \Psi(g,\mu).$$

By $\mathcal{T}$-Dominance, $f \gtrsim g$. Now take any f, g satisfying $f(s) \gtrsim g(s)$ for all $s \in S$. By applying the above argument repeatedly, we have $f \gtrsim g$, as desired. □

Let $C(\mathcal{T})$ denote the set of all real-valued bounded continuous functions defined on $\mathcal{T}$.

Lemma 2. *$\omega(P) \in C(\mathcal{T})$ for each $P \in \Delta(\mathcal{F})$.*

Proof. We show continuity, that is, $\omega(P,\mu^n) \to \omega(P,\mu)$ as $\mu^n \to \mu$. First we show the claim when P is degenerate at f. Note that

$$\omega(f,\mu) = W(\Psi(f,\mu)) = W(\sum_s \mu(s)f(s)).$$

Since W is continuous,

$$\omega(f,\mu^n) = W(\sum_s \mu^n(s)f(s)) \to W(\sum_s \mu(s)f(s)) = \omega(f,\mu),$$

as desired.

Next, take a general P. Notice that for all P and $\mu \in \mathcal{T}$,

$$\omega(P,\mu) = W(\Psi(P,\mu)) = \int_{\mathcal{F}} W(\Psi(f,\mu))\,\mathrm{d}P(f) = \int_{\mathcal{F}} \omega(f,\mu)\,\mathrm{d}P(f).$$

We will claim that there exists an integrable function $\xi : \mathcal{F} \to v(\Delta(Z))$ with respect to P such that $|\omega(\cdot,\mu^n)| \leq |\xi|$ for all n. Note that $v(\Delta(Z))$ is an interval because it is the image of a continuous function with a convex domain. Consider the following two cases: (i) $v(\Delta(Z))$ is either unbounded above or bounded below, or (ii) $v(\Delta(Z))$ is bounded above and unbounded below. We start with case (i). If $v(\Delta(Z))$ is bounded below, by linearity of v, we can assume $\inf v(\Delta(Z)) = 0$ without loss of generality. For each $f \in \mathcal{F}$, define

$$\xi(f) = \max_{s \in S} |v(f(s))| \geq 0.$$

There exists $p^f \in \Delta(Z)$ such that $v(p^f) = \xi(f)$. Since $\gtrsim$ on $\Delta(Z)$ satisfies Second-Stage Independence, for each n,

$$|\omega(f,\mu^n)| = |W(\sum_s \mu^n(s)f(s))| \leq \max_{s \in S} |W(f(s))| = \max_{s \in S} |v(f(s))| = \xi(f). \tag{10}$$

In the case of (ii), we can assume $\sup v(\Delta(Z)) = 0$ without loss of generality. Define $\xi(f) = \min_{s\in S} v(f(s)) \le 0$. There exists $p^f \in \Delta(Z)$ such that $v(p^f) = \xi(f)$. By the same argument as in (10), $|\omega(f, \mu^n)| \le |\xi(f)|$.

Since ξ defined above is measurable, it induces a distribution $P \circ \xi^{-1}$ over $v(\Delta(Z))$ by $P(\xi^{-1}(E))$ for all Borel measurable subsets E in $v(\Delta(Z))$. We will claim that there exists some $Q \in \Delta(\Delta(Z))$ of which utility in the representation coincides with the expected value of $P \circ \xi^{-1}$.

Take p_0 with $v(p_0) = 0$ without loss of generality. Let $k_0 = 0$. If $\overline{k} = \sup v(\Delta(Z))$ is attained in $v(\Delta(Z))$, let $p_1 \in \Delta(Z)$ with $\overline{k} = v(p_1)$. Otherwise, take a monotone sequence $k_n \to \overline{k}$ (possibly ∞) for all positive integers n. Similarly, if $\underline{k} = \inf v(\Delta(Z))$ is attained in $v(\Delta(Z))$, let $p_{-1} \in \Delta(Z)$ with $\underline{k} = v(p_{-1})$. Otherwise, take a monotone sequence $k_n \to \underline{k}$ (possibly $-\infty$) for all negative integers n. For any integer n, there exists $p_n \in \Delta(Z)$ with $k_n = v(p_n)$.

Now take any $k \in v(\Delta(Z))$. If $\overline{k} = \sup v(\Delta(Z))$ is attained in $v(\Delta(Z))$ and if $k = \overline{k}$, then let $q_k = p_1$. If $\underline{k} = \inf v(\Delta(Z))$ is attained in $v(\Delta(Z))$ and if $k = \underline{k}$, then let $q_k = p_{-1}$. Otherwise, there exists a unique integer n such that $k \in [k_{n-1}, k_n)$. Let

$$q_k = \frac{k - k_{n-1}}{k_n - k_{n-1}} p_n + \frac{k_n - k}{k_n - k_{n-1}} p_{n-1}.$$

Note that

$$\begin{aligned} v(q_k) &= \frac{k - k_{n-1}}{k_n - k_{n-1}} v(p_n) + \frac{k_n - k}{k_n - k_{n-1}} v(p_{n-1}) \\ &= \frac{k - k_{n-1}}{k_n - k_{n-1}} k_n + \frac{k_n - k}{k_n - k_{n-1}} k_{n-1} = k. \end{aligned}$$

For any $k \in v(\Delta(Z))$, define $\zeta(k) = q_k$. By construction, $v(\zeta(k)) = k$. Moreover, $\zeta : v(\Delta(Z)) \to \Delta(Z)$ is continuous and thus measurable.

Let $Q := (P \circ \xi^{-1}) \circ \zeta^{-1} \in \Delta(\Delta(Z))$ be the distribution induced by $\zeta : v(\Delta(Z)) \to \Delta(Z)$ and $P \circ \xi^{-1} \in \Delta(v(\Delta(Z)))$. By the representation,

$$\begin{aligned} W(Q) &= \int_{\Delta(Z)} v(p)\, \mathrm{d}Q(p) = \int_{\Delta(Z)} v(p)\, \mathrm{d}(P \circ \xi^{-1}) \circ \zeta^{-1} \\ &= \int_{v(\Delta(Z))} v(\zeta(k))\, \mathrm{d}(P \circ \xi^{-1})(k) = \int_{v(\Delta(Z))} k\, \mathrm{d}(P \circ \xi^{-1})(k) \\ &= \int_{\mathcal{F}} \xi(f)\, \mathrm{d}P(f). \end{aligned}$$

Since $-\infty < W(Q) < \infty$, in both of (i) and (ii),

$$\int_{\mathcal{F}} |\xi(f)|\, \mathrm{d}P(f) < \infty, \tag{11}$$

that is, ξ is integrable, as desired.

Since $\omega(f, \mu^n) \to \omega(f, \mu)$ and the above claim, the dominated convergence theorem (Aliprantis and Border [1, Theorem 11.21, p.415]) implies that

$$\omega(P, \mu^n) = \int_{\mathcal{F}} \omega(f, \mu^n) \,\mathrm{d}P(f) \to \int_{\mathcal{F}} \omega(f, \mu) \,\mathrm{d}P(f) = \omega(P, \mu),$$

as desired.

Finally, by (10) and (11), for all $\mu \in \mathcal{T}$,

$$|\omega(P, \mu)| \leq \int_{\mathcal{F}} |\omega(f, \mu)| \,\mathrm{d}P(f) \leq \int_{\mathcal{F}} |\xi(f)| \,\mathrm{d}P(f) < \infty.$$

Hence, $\omega(P)$ is bounded. □

Let

$$C^* := \{\omega(P) \mid P \in \Delta(\mathcal{F})\}$$

be the set of all induced utility acts. By Lemma 2, $C^* \subset C(\mathcal{T})$.

Take any P_1, $P_2 \in \Delta(\mathcal{F})$. For all $\lambda \in [0, 1]$, it holds that $\lambda\omega(P_1) + (1 - \lambda)\omega(P_2) = \omega(\lambda P_1 + (1 - \lambda)P_2)$. Thus, C^* is a convex subset in $C(\mathcal{T})$. In particular, C^* is a mixture space with respect to this mixture operation.

For all $p \in \Delta(Z)$, $\omega(p, \mu) = W(\Psi(p, \mu)) = v(p)$ for all $\mu \in \mathcal{T}$. That is, $\omega(p) \in C^*$ is a constant function. Thus, C^* includes all constant functions.

Define $\succsim^*$ on C^* by

$$\omega(P) \succsim^* \omega(Q) \iff W(P) \geq W(Q).$$

By $\mathcal{T}$-Dominance, $\succsim^*$ is well-defined. Thus, W can be regarded as a continuous linear utility representation for $\succsim^*$. $\mathcal{T}$-Dominance also implies that if $\omega(P) \geq \omega(Q)$, then $\omega(P) \succsim^* \omega(Q)$. Since $\succsim^*$ on C^* satisfies the axioms of Anscombe and Aumann [2], that is, Order, Continuity, Independence, Nondegeneracy, and Monotonicity, Theorem 2 of Zhou [36] ensures that there exists a subjective expected utility representation for $\succsim^*$, that is, there exists an affine function $\tilde{v}$ and a subjective probability measure m on $\mathcal{T}$ such that

$$\widetilde{W}(\omega(P)) = \int_{\mathcal{T}} \tilde{v}(\omega(P, \mu)) \,\mathrm{d}m(\mu)$$

represents $\succsim^*$. Since $\tilde{v}$ is affine, $\tilde{v}$ can be assumed to be an identity mapping $\tilde{v}(x) = x$.

Both $\widetilde{W}$ and W represent the same preference $\succsim^*$, and hence they are ordinally equivalent. There exists an increasing function ψ such that $W(P) = \psi(\widetilde{W}(\omega(P))$ for all P. Note that for all $p \in \Delta(Z)$, $v(p) = W(p) = \psi(\widetilde{W}(\omega(p))) = \psi(v(p))$. Thus, ψ must be an identify mapping $\psi(x) = x$. Consequently, for all P, we have

$$W(P) = \int_{\mathcal{T}} \omega(P, \mu)\, \mathrm{d}m(\mu).$$

By Second-Stage Independence, $\succsim$ on $\Delta(Z)$ satisfies all the VNM axioms. Thus, there exists a continuous expected utility representation $u : \Delta(Z) \to \mathbb{R}$. Since v and u represent the same preference on $\Delta(Z)$, there exists a continuous increasing function φ such that $v = \varphi(u)$.

Therefore,

$$\begin{aligned}
W(P) &= \int_{\mathcal{T}} \omega(P, \mu)\, \mathrm{d}m(\mu) = \int_{\mathcal{T}} W(\Psi(P, \mu))\, \mathrm{d}m(\mu) \\
&= \int_{\mathcal{T}} \int_{\mathcal{F}} v(\Psi(f, \mu))\, \mathrm{d}P(f)\, \mathrm{d}m(\mu) = \int_{\mathcal{T}} \int_{\mathcal{F}} \varphi\left(u(\Psi(f, \mu)\right)\, \mathrm{d}P(f)\, \mathrm{d}m(\mu) \\
&= \int_{\mathcal{T}} \int_{\mathcal{F}} \varphi\left(\int_S u(f(s))\, \mathrm{d}\mu(s)\right) \mathrm{d}P(f)\, \mathrm{d}m(\mu) \\
&= \int_{\mathcal{F}} \int_{\mathcal{T}} \varphi\left(\int_S u(f(s))\, \mathrm{d}\mu(s)\right) \mathrm{d}m(\mu)\, \mathrm{d}P(f),
\end{aligned}$$

as desired.

B Proof of Corollary 2

The axiomatization is a direct consequence of Theorem 1 when $\mathcal{T}$ is specified to $\mathcal{T}^S$. We show uniqueness. Suppose that (u, μ, φ) and (u', μ', φ') represent the same preference $\succsim$. Let V and V' be the corresponding representations over $\mathcal{F}$. By the uniqueness property of VNM functions, V is unique up to positive affine transformation, that is, there exist $\alpha_1 > 0$ and $\beta_1 \in \mathbb{R}$ such that $V' = \alpha_1 V + \beta_1$. In particular, $v' = \alpha_1 v + \beta_1$, where v and v' are the restrictions on $\Delta(Z)$, respectively. By the above uniqueness of v,

$$\widetilde{V}(f) = \int_S v(f(s))\, \mathrm{d}\mu'(s)$$

also represents the same preference over $\mathcal{F}$. Note that the range of v, denoted by $v(\Delta(Z))$, is a non-degenerate interval of $\mathbb{R}$. If $\mu \neq \mu'$, then we can always find some $f, g \in \mathcal{F}$ such that

$$\int_S v(f(s))\, \mathrm{d}\mu(s) > \int_S v(g(s))\, \mathrm{d}\mu(s), \text{ and } \int_S v(f(s))\, \mathrm{d}\mu'(s) < \int_S v(g(s))\, \mathrm{d}\mu'(s),$$

which contradicts to the assumption that (u, μ, φ) and (u', μ', φ') represent the same preference. Hence, we have $\mu = \mu'$.

The property $v' = \alpha_1 v + \beta_1$ implies that $\varphi'(u') = \alpha_1 \varphi(u) + \beta_1$. Moreover, since u and u' must be cardinally equivalent, there exist $\alpha_2 > 0$ and $\beta_2 \in \mathbb{R}$ such that $u' = \alpha_2 u + \beta_2$. For all $p \in \Delta(Z)$, $\varphi'(\alpha_2 u(p) + \beta_2) = \alpha_1 \varphi(u(p)) + \beta_1$. This completes the proof.

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