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Isomonodromic deformation and Painlevé equations, and the Garnier system

Dedicated to Professor Seizô Itô on his sixtieth birthday

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Introduction

The present article concerns the studies on the isomonodromic deformation of the linear differential equation of the second order:

$$\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_2(x)y=0, \tag{0.1}$$

$p_1(x)$ and $p_2(x)$ being rational functions. In the first part of this paper (§§ 1.2.3), we shall mainly investigate the linear differential equation (0.1) of the Fuchsian type, and prove that the isomonodromic deformation of (0.1) is governed by the completely integrable Hamiltonian system of partial differential equations. By the use of this result, Painlevé equations will be studied and the Hamiltonian structure will be given to Painlevé systems in the second part.

In 1907, R. Fuchs [1] considered the linear differential equation of the Fuchsian type:

$$\frac{d^2z}{dx^2}=p(x)z, \tag{0.2}$$

with the four regular singular points, $x=0,\ 1,\ \infty,\ t$ and the non logarithmic singularity, $x=\lambda$. Viewing t as a variable singularity, he proved that, if and only if (0.2) has a fundamental system of solutions whose monodromy is independent of t, λ satisfies as a function of t the sixth Painlevé equation, P_{VI}. This non linear differential equation is obtained from the complete integrability condition of the extended system of (0.2):

$$\begin{aligned}&\frac{\partial^2 z}{\partial x^2}=p(x)z\\ &\frac{\partial z}{\partial t}=\mathsf{B}(x)z+\mathsf{A}(x)\frac{\partial z}{\partial x},\end{aligned} \tag{0.3}$$

such that $\mathsf{A}(x)$ and $\mathsf{B}(x)$ are rational functions of x.

The result of R. Fuchs was generalized by R. Garnier [2] in two different ways. The first generalization is related to the linear differential equation (0.2) of the Fuchsian type having the N variable singularities, $x=t_j$ $(j=1, \cdots, N)$ and the N non logarithmic singular points, $x=\lambda_k$ $(k=1, \cdots, N)$. He obtained from the complete integrability condition of (0.3) the completely integrable system of non linear partial differential equations. We write it below:

$$
\begin{aligned}
&\frac{T'(t_i)(t_i-\lambda_k)}{\Lambda(t_i)}\frac{\partial\lambda_k}{\partial t_i}-\frac{T'(t_j)(t_j-\lambda_k)}{\Lambda(t_j)}\frac{\partial\lambda_k}{\partial t_j}\\
&=\frac{(t_i-t_j)T(\lambda_k)}{(\lambda_k-t_i)(\lambda_k-t_j)\Lambda'(\lambda_k)}, \qquad (i, j, k=1, \cdots, N),
\end{aligned}
\tag{0.4}
$$

$$
\begin{aligned}
\frac{\partial^2\lambda_k}{\partial t_i^2}=&\frac{1}{2}\left[\frac{T'(\lambda_k)}{T(\lambda_k)}-\frac{1}{2}\frac{\Lambda''(\lambda_k)}{\Lambda'(\lambda_k)}\right]\left(\frac{\partial\lambda_k}{\partial t_i}\right)^2-\left[\frac{1}{2}\frac{T''(t_i)}{T'(t_i)}-\frac{\Lambda'(t_i)}{\Lambda(t_i)}\right]\frac{\partial\lambda_k}{\partial t_i}\\
&+\frac{1}{2}\sum_{(l)}^{(k)}\frac{T(\lambda_k)\Lambda'(\lambda_l)(\lambda_l-t_i)^2}{T(\lambda_l)\Lambda'(\lambda_k)(\lambda_k-t_i)^2(\lambda_k-\lambda_l)}\left(\frac{\partial\lambda_l}{\partial t_i}\right)^2\\
&-\sum_{(l)}^{(k)}\frac{\lambda_k-t_i}{(\lambda_l-t_i)(\lambda_l-\lambda_k)}\frac{\partial\lambda_k}{\partial t_i}\frac{\partial\lambda_l}{\partial t_i}\\
&+\frac{\Lambda(t_i)^2T(\lambda_k)}{2T'(t_i)^2(\lambda_k-t_i)^2\Lambda'(\lambda_k)}\left[\kappa_\infty^2+\frac{T'(0)}{\Lambda(0)}\frac{\kappa_0^2}{\lambda_k}+\frac{T'(1)}{\Lambda(1)}\frac{\kappa_1^2}{\lambda_k-1}\right.\\
&\left.+\sum_{(j)}^{(i)}\frac{T'(t_j)}{\Lambda(t_j)}\frac{\theta_j^2}{\lambda_k-t_j}+\frac{T'(t_i)}{\Lambda(t_i)}\frac{\theta_i^2-1}{\lambda_k-t_i}\right], \qquad (i, j, k, l=1, \cdots, N),
\end{aligned}
\tag{0.5}
$$

where

$$
T(x)=x(x-1)\prod_{i=1}^{N}(x-t_i),
$$
$$
\Lambda(x)=\prod_{k=1}^{N}(x-\lambda_k), \quad \left({}'=\frac{d}{dx}\right),
$$

κ_Δ $(\Delta=0, 1, \infty)$, θ_j $(j=1, \cdots, N)$ being constants. Here we denote by $\sum_{(m)}^{(n)}$ the sum for $m=1, \cdots, N$ except for $m=n$. This system will be called in the following of this paper as the Garnier system.

The other result obtained by R. Garnier is connected to the isomonodromic deformation of (0.2) with irregular singularities. He showed that the other five Painlevé equations, $\mathrm{P_I}$, $\mathrm{P_{II}}$, $\mathrm{P_{III}}$, $\mathrm{P_{IV}}$, $\mathrm{P_V}$, are obtained from completely integrability conditions of extendes systems; however he did not mention about any monodromy property. We can deduce from the complete integrable system (0.3) that, if the linear differential equation (0.2) is not of the Fuchsian type, then Stokes multipliers given in a neighbourhood of an irregular singularity, are independent of t. Therefore, (0.3) defines the isomonodromic deformation of (0.2) in this meaning (cf. [6], [15], [22]).

On the other hand, L. Schlesinger [21] considered the isomonodromic deformation of the linear system of differential equations:

$$\frac{dY}{dx}=\Big(\sum_{n=1}^{r}\frac{R_n}{x-t_n}\Big)Y, \tag{0.6}$$

and obtained the completely integrable system of non linear differential equations:

$$\frac{\partial R_m}{\partial t_n}=\frac{[R_m, R_n]}{t_m-t_n}, \qquad (m\neq n)$$
$$\frac{\partial}{\partial t_n}\Big(\sum_{m=1}^{r}R_m\Big)=0,$$

where

$$[R_m, R_n]=R_mR_n-R_nR_m.$$

Moreover, R. Garnier [3] showed that, in the case when R_n $(n=1, \cdots, r)$ are 2×2 matrices, the isomonodromic deformation of (0.6) can be reduced to that of (0.2).

The purpose of this paper is to rewrite the results mentioned above by the use of the Hamiltonian structure, which is induced in a natural way from the isomonodromic deformation. We shall show in the main theorem of the present article that the Garnier system (0.4), (0.5) is written as the Hamiltonian system

$$\begin{cases}\dfrac{\partial\lambda_k}{\partial t_i}=\dfrac{\partial H_i}{\partial\mu_k}\\[2mm] \dfrac{\partial\mu_k}{\partial t_i}=-\dfrac{\partial H_i}{\partial\lambda_k} \qquad (i, k=1, \cdots, N),\end{cases} \tag{0.7}$$

where μ_k denotes the canonical variable conjugated to λ_k, and H_i is a rational function of t_j $(j=1, \cdots, N)$, λ_k, μ_k $(k=1, \cdots, N)$. In the case $N=1$, the Garnier system is reduced to the sixth Painlevé equation P_{VI}, hence, it is equivalent to the Hamiltonian system.

The linear differential equation (0.1) is transformed to (0.2) by a transformation of the form

$$y=\Phi(x)z. \tag{0.8}$$

We shall prove that (0.8) induces the canonical transformation for the Hamiltonian system (0.7) (Lemma 2.1). By considering canonical transformations of the Hamiltonian structure associated with P_{VI}, we shall have the Hamiltonian, H_{VI}, of P_{VI}, which is a polynomial of the two canonical variables, λ and μ.

As is well-known, P_{VI} yields the other five equations P_J $(J=\mathrm{I}, \cdots, \mathrm{V})$ by a process of coalescence according to the following scheme:

$$\begin{tikzcd}[row sep=small] & \mathrm{P}_{\mathrm{III}} \arrow[dr] & & \\ \mathrm{P}_{\mathrm{VI}} \arrow[r] & \mathrm{P}_{\mathrm{V}} \arrow[u] \arrow[d] & \mathrm{P}_{\mathrm{II}} \arrow[r] & \mathrm{P}_{\mathrm{I}}. \\ & \mathrm{P}_{\mathrm{IV}} \arrow[ur] & & \end{tikzcd}$$

This fact stands also for the Hamiltonian H_{VI}. By the use of the process of step-by-step degeneration, we shall derive from H_{VI}, the Hamiltonians H_J associated with P_J. This is defined by successive canonical transformation with a parameter and carried out according to a similar scheme to that of Painlevé equations. The Hamiltonians H_J $(J=\mathrm{I}, \cdots, \mathrm{V})$ thus obtained are polynomials of the two canonical variables. We shall show that the degeneration of the Hamiltonians causes simultaneously the confluence of singularities of linear differential equations.

It was J. Malmquist who pointed out for the first time that each Painlevé equation can be written as a Hamiltonian system. In the studies [13] on a system of differential equations of the form

$$\frac{dq}{dt}=F(t;\, q,\, p)$$
$$\frac{dp}{dt}=G(t;\, q,\, p)$$

without any movable branch point, F and G being rational functions in q and p, an explicit form of a Hamiltonian function $H(t;\, q,\, p)$, polynomial in q and p, is obtained for each of the Painlevé equations except for the third one. He gave there, however, no reference to the isomonodromic deformation of a linear ordinary differential equation. See also [4], [12].

The Hamiltonian structure of the Painlevé equations can be derived in a natural way from the structure of analytic foliation associated with the equations: cf. [16].

In this paper, we assume that the difference, c, of the two exponents at each apparent singularity of (0.1) is two. By considering the general cases with $c=3, 4, \cdots$, we will obtain also a Hamiltonian system whose Hamiltonian function is rational or algebraic in the two canonical variables. It is known [11] that, when $c=3$, such system is reduced to the Painlevé equation through a canonical transformation. As for exponents at an apparent singular point, see [9].

In §1, we shall state the main theorem concerning the Garnier system. Brief review of the theory of the isomonodromic deformation of the linear ordinary differential equation (0.1) will be given in §1.3. The proof of the main theorem will be completed in §2. We shall study in §3 the correspondence between the Garnier system and the 2×2 Schlesinger system. The rest of the paper, §4, will be devoted to the studies on the Hamiltonian structures of the Painlevé equations.

The results about the Painlevé equations have been announced in [17]. The present article is the reformation of [19].

§1. Isomonodromic deformation of a linear equation of the second order.

1.1. Main Theorem.

Consider the linear differential equation of the Fuchsian type:

$$\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_2(x)y=0, \tag{1.1}$$

with the Riemannian scheme

$$\left\{\begin{matrix} x=0, & x=1, & x=t_j, & x=\lambda_k, & x=\infty \\ \alpha_0 & \alpha_1 & \beta_j & \gamma_k & \alpha_\infty \\ \alpha_0+\kappa_0 & \alpha_1+\kappa_1 & \beta_j+\theta_j & \gamma_k+2 & \alpha_\infty+\kappa_\infty \end{matrix}\right\} \qquad (j, k=1, \cdots, N). \tag{1.2}$$

We assume that none of $\kappa_\varDelta$ ($\varDelta=0, 1, \infty$), θ_j ($j=1, \cdots, N$) is an integer and make the following assumption:

(A) *None of the singular points* $x=\lambda_k$ ($k=1, \cdots, N$) *is a logarithmic singularity.*

Under this assumption, viewing

$$t=(t_1, \cdots, t_N)$$

as a variable singularities, we shall study the following problem: (*Isomonodromic deformation of the linear equation* (1.1)) determine the coefficients $p_1(x)$, $p_2(x)$ of (1.1) as functions of t such that (1.1) has a fundamental system of solutions whose monodromy is independent of t.

Now we state the main theorem:

THEOREM. *Under the assumption* (A), *the isomonodromic deformation of the linear equation* (1.1) *is governed by the completely integrable Hamiltonian system of partial differential equations*:

$$(\mathrm{H})_N \qquad \begin{cases} \dfrac{\partial \lambda_k}{\partial t_j} = \dfrac{\partial \mathsf{H}_j}{\partial \mu_k} \\ \dfrac{\partial \mu_k}{\partial t_j} = -\dfrac{\partial \mathsf{H}_j}{\partial \lambda_k} \end{cases} \qquad (j, k=1, \cdots, N),$$

where

$$(1.3) \qquad \mathsf{H}_j = -\operatorname*{Res}_{x=t_j} p_2(x) \qquad (j=1, \cdots, N),$$

$$(1.4) \qquad \mu_k = \operatorname*{Res}_{x=\lambda_k} p_2(x) \qquad (k=1, \cdots, N).$$

This system is equivalent to the Garnier system.

In the case $N=1$, the Garnier system is reduced to the sixth Painlevé equation. Then we obtain:

COROLLARY. *The sixth Painlevé equation is equivalent to the Hamiltonian system*

$$(\mathrm{H})_1 \qquad \begin{cases} \dfrac{d\lambda}{dt} = \dfrac{\partial \mathsf{H}}{\partial \mu} \\ \dfrac{d\mu}{dt} = -\dfrac{\partial \mathsf{H}}{\partial \lambda} \end{cases}$$

which gives the isomonodromic deformation of the linear equation of the form (1.1).

The proof of the theorem will be given in the following section.

1.2. Notations and Remarks.

The coefficients $p_1(x)$, $p_2(x)$ of (1.1) can be written as follows:[1)]

1) In the following of this paper, we denote by $\sum_{(n)}$ ($\prod_{(n)}$, resp.) the summation (the product, resp.) for $n=1, \cdots, N$, and by $\sum_{(m)}^{(n)}$ ($\prod_{(m)}^{(n)}$, resp.) that for $m=1, \cdots, N$ except for $m=n$.

$$p_1(x)=\frac{c_0}{x}+\frac{c_1}{x-1}+\sum_{(j)}\frac{e_j}{x-t_j}+\sum_{(k)}\frac{f_k}{x-\lambda_k}, \tag{1.5}$$

$$\begin{aligned} p_2(x)=&\frac{\alpha_0(\alpha_0+\kappa_0)}{x^2}+\frac{\alpha_1(\alpha_1+\kappa_1)}{(x-1)^2}+\frac{\kappa}{x(x-1)}\\ &+\sum_{(j)}\left[\frac{\beta_j(\beta_j+\theta_j)}{(x-t_j)^2}-\frac{t_j(t_j-1)\mathrm{H}_j}{x(x-1)(x-t_j)}\right]\\ &+\sum_{(k)}\left[\frac{\gamma_k(\gamma_k+2)}{(x-\lambda_k)^2}+\frac{\lambda_k(\lambda_k-1)\mu_k}{x(x-1)(x-\lambda_k)}\right], \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} &c_\Delta=1-2\alpha_\Delta-\kappa_\Delta \qquad (\Delta=0,\ 1)\\ &e_j=1-2\beta_j-\theta_j \qquad (j=1,\ \cdots,\ N)\\ &f_k=-1-2\gamma_k \qquad (k=1,\ \cdots,\ N)\\ &\kappa=\alpha_\infty(\alpha_\infty+\kappa_\infty)-\alpha_0(\alpha_0+\kappa_0)-\alpha_1(\alpha_1+\kappa_1)\\ &\qquad -\sum_{(j)}\beta_j(\beta_j+\theta_j)-\sum_{(k)}\gamma_k(\gamma_k+2). \end{aligned}$$

It is easy to see by the use of the method of Frobenius that the assumption (A) is equivalent to the following N equalities:

$$\gamma_k[(\gamma_k+1)\mathscr{V}_k^2+\mathscr{V}_k'+\mu_k\mathscr{V}_k]+\mu_k^2+\mu_k\mathscr{V}_k+\mathscr{U}_k=0 \qquad (k=1,\ \cdots,\ N) \tag{1.7}$$

where

$$\begin{aligned} \mathscr{U}_k=&\frac{\alpha_0(\alpha_0+\kappa_0)}{\lambda_k^2}+\frac{\alpha_1(\alpha_1+\kappa_1)}{(\lambda_k-1)^2}+\frac{\kappa}{\lambda_k(\lambda_k-1)}-\frac{2\lambda_k-1}{\lambda_k(\lambda_k-1)}\cdot\mu_k\\ &+\sum_{(j)}\left[\frac{\beta_j(\beta_j+\theta_j)}{(\lambda_k-t_j)^2}-\frac{t_j(t_j-1)\mathrm{H}_j}{\lambda_k(\lambda_k-1)(\lambda_k-t_j)}\right]\\ &+\sum_{(l)}^{(k)}\left[\frac{\gamma_l(\gamma_l+2)}{(\lambda_k-\lambda_l)^2}+\frac{\lambda_l(\lambda_l-1)\mu_l}{\lambda_k(\lambda_k-1)(\lambda_k-\lambda_l)}\right],\\ \mathscr{V}_k=&\frac{c_0}{\lambda_k}+\frac{c_1}{\lambda_k-1}+\sum_{(j)}\frac{e_j}{\lambda_k-t_j}+\sum_{(l)}^{(k)}\frac{f_l}{\lambda_k-\lambda_l},\\ \mathscr{V}_k'=&\frac{\partial}{\partial\lambda_k}\mathscr{V}_k. \end{aligned}$$

Consider now a transformation of (1.1) of the form

$$y=\Phi(x)z. \tag{1.8}$$

Putting in (1.8)

$$\Phi(x)=x^{\alpha_0}(x-1)^{\alpha_1}\prod_{(j)}(x-t_j)^{\beta_j}\prod_{(k)}(x-\lambda_k)^{\gamma_k}.$$

We obtain the linear equation whose Riemannian scheme (1.2) is normalized as

$$\alpha_0=\alpha_1=\beta_j=\gamma_k=0 \qquad (j,\ k=1,\ \cdots,\ N).$$

A linear equation of this type will be called of *canonical* type. Furthermore, if we put

$$\Phi(x)=\exp\left(-\frac{1}{2}\int^x p_1(x)dx\right)$$

(1.1) is transformed to an equation of the form

$$\frac{d^2z}{dx^2}=p(x)z, \tag{1.9}$$

where

$$p(x)=-p_2(x)+\frac{1}{4}p_1(x)^2+\frac{1}{2}\frac{d}{dx}p_1(x). \tag{1.10}$$

The equation (1.9) is said to be a *SL-type equation*: remark that a monodromy of (1.9) is a subgroup of $SL(2, \boldsymbol{C})$. The coefficient $p(x)$ can be written in the form:

$$\begin{aligned}p(x)=&\frac{a_0}{x^2}+\frac{a_1}{(x-1)^2}+\frac{a_\infty}{x(x-1)}\\&+\sum_{(j)}\left[\frac{b_j}{(x-t_j)^2}+\frac{t_j(t_j-1)\mathsf{K}_j}{x(x-1)(x-t_j)}\right]\\&+\sum_{(k)}\left[\frac{3}{4(x-\lambda_k)^2}-\frac{\lambda_k(\lambda_k-1)\nu_k}{x(x-1)(x-\lambda_k)}\right],\end{aligned} \tag{1.11}$$

where

$$a_0=-\frac{1}{4}(1-\kappa_0^2),\quad a_1=-\frac{1}{4}(1-\kappa_1^2),\quad b_j=-\frac{1}{4}(1-\theta_j^2)$$

$$a_\infty=-\frac{1}{4}\left(\kappa_0^2+\kappa_1^2+\sum_{(j)}\theta_j^2-\kappa_\infty^2-1\right)-\frac{1}{2}N,$$

$$\mathsf{K}_j=\mathsf{H}_j+\frac{1}{2}e_j\left[\frac{c_0}{t_j}+\frac{c_1}{t_j-1}+\sum_{(i)}^{(j)}\frac{e_i}{t_j-t_i}+\sum_{(k)}\frac{f_k}{t_j-\lambda_k}\right], \tag{1.12}$$

$$\nu_k = \mu_k - \frac{1}{2} f_k \left[\frac{c_0}{\lambda_k} + \frac{c_1}{\lambda_k - 1} + \sum_{(j)} \frac{e_j}{\lambda_k - t_j} + \sum_{(l)}^{(k)} \frac{f_l}{\lambda_k - \lambda_l} \right]. \tag{1.13}$$

The relations (1.12), (1.13) will be used in the proof of the theorem.

The Fuchsian relation for the linear differential equation (1.1) reads:

$$2\alpha_0 + \kappa_0 + 2\alpha_1 + \kappa_1 + 2\alpha_\infty + \kappa_\infty + \sum_{(j)} (2\beta_j + \theta_j) + 2 \sum_{(k)} \gamma_k = 1. \tag{1.14}$$

Remark that the characteristic exponents, α_Δ, κ_Δ ($\Delta = 0, 1, \infty$), β_j, θ_j ($j = 1, \cdots, N$) and γ_k ($k = 1, \cdots, N$) remain invariant under the isomonodromic deformation of (1.1).

1.3. Isomonodromic deformation.

We recall some known results on the isomonodromic deformation of the linear equation (1.1): cf. [6], [9], [15].

PROPOSITION 1.1. (1.1) *has a fundamental system of solutions whose monodromy is independent of t, if and only if there exist rational functions of x, $\mathsf{A}_j(x)$, $\mathsf{B}_j(x)$ ($j = 1, \cdots, N$) such that the extended system of the differential equations*

$$\begin{cases} \dfrac{\partial^2 y}{\partial x^2} + p_1(x) \dfrac{\partial y}{\partial x} + p_2(x) y = 0, \\ \dfrac{\partial y}{\partial t_j} = \mathsf{B}_j(x) y + \mathsf{A}_j(x) \dfrac{\partial y}{\partial x} \end{cases} \tag{1.15}$$

is completely integrable.

REMARK 1.1. Under the transformation (1.8) of the linear equation (1.1), the function $\mathsf{A}_j(x)$ is unchanged.

REMARK 1.2. The statement of Proposition 1.1 is also valid for a linear equation of the second order with irregular singularities, provided that Stokes multipliers at an irregular singular point are independent of the deformation parameters, t_j (cf. [15], [22]). The general theory of isomonodromic deformation of a linear differential system with rational coefficients is established in [5].

The complete integrability condition of (1.15) is written as follows:

$$2\frac{\partial B_j}{\partial x}+\frac{\partial^2 A_j}{\partial x^2}=\frac{\partial}{\partial x}(p_1 A_j)-\frac{\partial p_1}{\partial t_j},$$

$$2\frac{\partial^2 B_j}{\partial x^2}-4p_2\frac{\partial A_j}{\partial x}-2\frac{\partial p_2}{\partial x}A_j+2\frac{\partial p_2}{\partial t_j}=p_1\Big(\frac{\partial^2 A_j}{\partial x^2}-\frac{\partial}{\partial x}(p_1 A_j)+\frac{\partial p_1}{\partial t_j}\Big), \tag{1.16}$$

$$\frac{\partial B_j}{\partial t_i}+A_j\frac{\partial B_i}{\partial x}=\frac{\partial B_i}{\partial t_j}+A_i\frac{\partial B_j}{\partial x}, \tag{1.17}$$

$$\frac{\partial A_j}{\partial t_i}+A_j\frac{\partial A_i}{\partial x}=\frac{\partial A_i}{\partial t_j}+A_i\frac{\partial A_j}{\partial x}, \qquad (i,\ j=1,\ \cdots,\ N). \tag{1.18}$$

For each j, a function, $C_j(x)$, defined by

$$\frac{\partial C_j}{\partial x}=\frac{\partial p_1}{\partial t_j}$$

is rational in x. Consequently, if $A_j(x)$ is rational, so is the function $B_j(x)$, which is given by

$$B_j=\frac{1}{2}\Big(-\frac{\partial A_j}{\partial x}+p_1 A_j-C_j\Big). \tag{1.19}$$

We obtained from (1.16), (1.19) the linear differential equation

$$\frac{\partial^3 A_j}{\partial x^3}-4p\frac{\partial A_j}{\partial x}-2\frac{\partial p}{\partial x}A_j+2\frac{\partial p}{\partial t_j}=0 \tag{1.20}$$

where

$$p=-p_2+\frac{1}{4}p_1^2+\frac{1}{2}\frac{\partial p_1}{\partial x}.$$

Note that this relation is nothing but (1.10).

On the other hand, (1.17) is written as

$$\Big(\frac{\partial}{\partial x}-p_1\Big)\Big(\frac{\partial A_j}{\partial t_i}+A_j\frac{\partial A_i}{\partial x}\Big)=\Big(\frac{\partial}{\partial x}-p_1\Big)\Big(\frac{\partial A_i}{\partial t_j}+A_i\frac{\partial A_j}{\partial x}\Big),$$

by virtue of (1.19). Then we arrive at the

PROPOSITION 1.2 (cf. [9], [14]). *The isomonodromic deformation of the linear differential equation* (1.1) *is reduced to that of the SL-type equation defined by* (1.10).

PROPOSITION 1.3. *The isomonodromic deformation of* (1.1) *is equivalent to the existence of the rational functions* $\mathsf{A}_j(x)$ $(j=1, \cdots, N)$, *satisfying the system of differential equations* (1.18), (1.20).

Concerning $\mathsf{A}_j(x)$, the following result was shown by R. Garnier [2]:

PROPOSITION 1.4. $\mathsf{A}_j(x)$ *can be written in the form*

$$\mathsf{A}_j(x)=\mathsf{M}_j\frac{T(x)}{\Lambda(x)(x-t_j)}, \tag{1.21}$$

where

$$T(x)=x(x-1)\prod_{(j)}(x-t_j), \tag{1.22}$$

$$\Lambda(x)=\prod_{(k)}(x-\lambda_k) \tag{1.23}$$

$$\mathsf{M}_j=-\frac{\Lambda(t_j)}{T'(t_j)} \qquad \left(T'=\frac{d}{dx}T\right). \tag{1.24}$$

The three propositions given above will play an important role in the proof of the main theorem.

§ 2. Proof of the main theorem.

2.1. Canonical transformation.

In this section we shall give the complete proof of the theorem by using notation of Section 1.2. At first, we show that the theorem can be reduced to the case of the SL-type equation (1.9) with the coefficient (1.10).

LEMMA 2.1. *The change of the variables given by* (1.12) *and* (1.13),

$$(\lambda_k, \mu_k, \mathsf{H}_j)\longrightarrow(\lambda_k, \nu_k, \mathsf{K}_j) \qquad (j, k=1, \cdots, N)$$

is a canonical transformation of the Hamiltonian system $(\mathrm{H})_N$.

PROOF. By the use of (1.12), (1.13), it is not difficult to see that

$$\sum_{(k)}\nu_k d\lambda_k=\sum_{(k)}\mu_k d\lambda_k-\frac{1}{2}\sum_{(k)}d\lambda_k\sum_{(j)}\frac{f_k e_j}{\lambda_k-t_j}-d\Omega_1,$$

$$\sum_{(j)} \mathsf{K}_j dt_j = \sum_{(j)} \mathsf{H}_j dt_j + \frac{1}{2} \sum_{(j)} dt_j \sum_{(k)} \frac{e_j f_k}{t_j - \lambda_k} + d\Omega_2,$$

where

$$\Omega_1 = \frac{1}{2} \sum_{(k)} f_k \Big[c_0 \log \lambda_k + c_1 \log(\lambda_k - 1) + \frac{1}{2} \sum_{(l)}^{(k)} f_l \log(\lambda_k - \lambda_l) \Big],$$

$$\Omega_2 = \frac{1}{2} \sum_{(j)} e_j \Big[c_0 \log t_j + c_1 \log(t_j - 1) + \frac{1}{2} \sum_{(i)}^{(j)} c_i \log(t_j - t_i) \Big],$$

d$\cdot$ denoting the differential with respect to λ_k, t_j.

Noting

$$\frac{1}{2} \sum_{(k)} f_k \sum_{(j)} \frac{e_j}{\lambda_k - t_j} d\lambda_k + \frac{1}{2} \sum_{(j)} e_j \sum_{(k)} \frac{f_k}{t_j - \lambda_k} dt_j = \frac{1}{2} \sum_{(j)} \sum_{(k)} f_k e_j d \log(\lambda_k - t_j)$$
$$\equiv d\Omega_3,$$

we obtain

$$\sum_{(k)} \nu_k d\lambda_k - \sum_{(j)} \mathsf{K}_j dt_j = \sum_{(k)} \mu_k d\lambda_k - \sum_{(j)} \mathsf{H}_j dt_j - d\Omega,$$
$$\Omega = \Omega_1 + \Omega_2 + \Omega_3,$$

so that

$$\sum_{(k)} d\nu_k \wedge d\lambda_k - \sum_{(j)} d\mathsf{K}_j \wedge dt_j = \sum_{(k)} d\mu_k \wedge d\lambda_k - \sum_{(j)} d\mathsf{H}_j \wedge dt_j.$$

This proves the lemma.

2.2. Hamiltonians K_j, H_j.

In what follows we consider the isomonodromic deformation of the SL-type equation

$$\frac{d^2 z}{dx^2} = p(x) z, \tag{1.9}$$

with the coefficient $p(x)$ given by (1.11). It is just the case studied by R. Garnier. However, our goal is different from his one, hence we have to make again computations to obtain the desired results.

We determine first the Hamiltonians:

PROPOSITION 2.1. *The Hamiltonians* K_j $(j=1, \cdots, N)$ *are given by*

$$\mathsf{K}_j = \mathsf{M}_j \sum_{(k)} [M^{k,j} \nu_k^2 - M^{k,j,0} \nu_k - M^{k,j} U_k] \tag{2.1}$$

where

$$M^{k,j}=\frac{T(\lambda_k)}{\Lambda'(\lambda_k)(\lambda_k-t_j)}\qquad\left(\Lambda'=\frac{d\Lambda}{dx}\right)$$

(2.2)
$$=\frac{\lambda_k(\lambda_k-1)\prod_{(i)}^{(j)}(\lambda_k-t_i)}{\prod_{(l)}^{(k)}(\lambda_k-\lambda_l)},$$

(2.3)
$$M^{k,j,0}=\sum_{(l)}^{(k)}\frac{\lambda_k(\lambda_k-1)}{\lambda_l(\lambda_l-1)(\lambda_k-\lambda_l)}M^{l,j}+\frac{2\lambda_k-1}{\lambda_k(\lambda_k-1)}M^{k,j},$$

$$U_k=\frac{a_0}{\lambda_k^2}+\frac{a_1}{(\lambda_k-1)^2}+\frac{a_\infty}{\lambda_k(\lambda_k-1)}+\sum_{(j)}\frac{b_j}{(\lambda_k-t_j)^2}+\sum_{(l)}^{(k)}\frac{3}{4(\lambda_k-\lambda_l)^2},$$

$T(x)$, $\Lambda(x)$ *and* M_j *being given by* (1.22), (1.23) *and* (1.24) *respectively.*

PROOF. In this case, the relation (1.7) read:

(2.4)
$$\mathscr{U}_k=\nu_k^2$$

where

(2.5)
$$\mathscr{U}_k=U_k+\sum_{(j)}\frac{t_j(t_j-1)\mathsf{K}_j}{\lambda_k(\lambda_k-1)(\lambda_k-t_j)}-\sum_{(l)}^{(k)}\frac{\lambda_l(\lambda_l-1)\nu_l}{\lambda_k(\lambda_k-1)(\lambda_k-\lambda_l)}+\frac{2\lambda_k-1}{\lambda_k(\lambda_k-1)}\nu_k.$$

For the sake of obtaining K_j from the linear system (2.4), we shall compute the inverse matrix

$$\boldsymbol{F}=((F_{jk}))_{j,k=1,\cdots,N}$$

of $\boldsymbol{E}=((E_{kj}))$ where

(2.6)
$$E_{kj}=\frac{1}{\lambda_k(\lambda_k-1)(\lambda_k-t_j)}.$$

Consider the auxiliary rational functions

$$Z_j(x)=\frac{T(x)}{x(x-1)(x-t_j)\Lambda(x)}$$

(2.7)
$$=\frac{\prod_{(i)}^{(j)}(x-t_i)}{\prod_{(k)}(x-\lambda_k)}$$

$$=\sum_{(k)}\frac{M^{k,j}}{\lambda_k(\lambda_k-1)}\frac{1}{x-\lambda_k}\qquad (j=1,\cdots,N);$$

see (2.2). Since

$$Z_j(t_j)=\frac{T'(t_j)}{t_j(t_j-1)\Lambda(t_j)}=\frac{1}{t_j(t_j-1)\mathsf{M}_j},$$
$$Z_j(t_i)=0\qquad (i\neq j),$$

we have

$$1=\sum_{(k)}E_{kj}t_j(t_j-1)\mathsf{M}_jM^{k,j}$$
$$0=\sum_{(k)}E_{ki}t_j(t_j-1)\mathsf{M}_jM^{k,j}\qquad (i\neq j),$$

which show

(2.8) $$F_{jk}=t_j(t_j-1)\mathsf{M}_jM^{k,j}.$$

We rewrite (2.4) as

(2.4)′ $$\sum_{(j)}E_{kj}t_j(t_j-1)\mathsf{K}_j+\sum_{(l)}G_{kl}\nu_l+U_k=\nu_k^2,$$

where

$$G_{kl}=\frac{\lambda_l(\lambda_l-1)}{\lambda_k(\lambda_k-1)(\lambda_l-\lambda_k)}\qquad (l\neq k),$$
$$G_{kk}=\frac{2\lambda_k-1}{\lambda_k(\lambda_k-1)}.$$

It follows that

$$t_j(t_j-1)\mathsf{K}_j=\sum_{(k)}F_{jk}\Big[\nu_k^2-\sum_{(l)}G_{kl}\nu_l-U_k\Big],$$

from which we obtain (2.1) by using (2.3), (2.8).

REMARK 2.1. We can verify

$$\det \boldsymbol{E}=\prod_{i<j}(t_i-t_j)\prod_{l<k}(\lambda_k-\lambda_l)\Big(\prod_{(k)}T(\lambda_k)\Big)^{-1}.$$

Concerning the differential equation of the canonical type, that is, the case when in (1.5), (1.6)

$$\alpha_0=\alpha_1=\beta_j=\nu_k=0,$$

we prove the following:

PROPOSITION 2.2. *The Hamiltonians* H_j *are written as follows*:

$$\mathsf{H}_j=\mathsf{M}_j\Big[\sum_{(k)} M^{k,j}\Big\{\mu_k^2-\Big(\frac{\kappa_0}{\lambda_k}+\frac{\kappa_1}{\lambda_k-1}+\sum_{(i)}\frac{\theta_i-\delta_{ij}}{\lambda_k-t_i}\Big)\mu_k\Big\}+\kappa\Big], \tag{2.9}$$

δ_{ij} *being Kronecker's* δ.

PROOF. In a way similar to the proof of the preceding proposition, we obtain from (1.7)

$$\mathsf{H}_j=\mathsf{M}_j\sum_{(k)}\Big[M^{k,j}\mu_k^2+(M^{k,j}\mathscr{V}_k-M^{k,j,0})\mu_k+\frac{M^{k,j}}{\lambda_k(\lambda_k-1)}\kappa\Big],$$

where

$$\mathscr{V}_k=\frac{1-\kappa_0}{\lambda_k}+\frac{1-\kappa_1}{\lambda_k-1}+\sum_{(j)}\frac{1-\theta_j}{\lambda_k-t_j}-\sum_{(l)}^{(k)}\frac{1}{\lambda_k-\lambda_l}.$$

We claim:

$$M^{k,j,0}=M^{k,j}\Big(\frac{1}{\lambda_k}+\frac{1}{\lambda_k-1}+\sum_{(i)}^{(j)}\frac{1}{\lambda_k-t_i}-\sum_{(l)}^{(k)}\frac{1}{\lambda_k-\lambda_l}\Big); \tag{2.10}$$

in fact, consider the rational function

$$\begin{aligned}x(x-1)Z_j(x)&=\frac{T(x)}{(x-t_j)\Lambda(x)}\\&=x(x-1)\sum_{(k)}\frac{M^{k,j}}{\lambda_k(\lambda_k-1)}\cdot\frac{1}{x-\lambda_k}.\end{aligned}$$

We obtain firstly by virtue of (2.3)

$$\Big(x(x-1)Z_j(x)-\frac{M^{k,j}}{x-\lambda_k}\Big)\Big|_{x=\lambda_k}=M^{k,j,0},$$

on the other hand

$$\begin{aligned}&\frac{d}{dx}(x(x-1)(x-\lambda_k)Z_j(x))\Big|_{x=\lambda_k}\\&=\Big(\frac{(x-\lambda_k)T(x)}{(x-t_j)\Lambda(x)}\Big\{\frac{1}{x}+\frac{1}{x-1}+\sum_{(i)}^{(j)}\frac{1}{x-t_i}-\sum_{(l)}^{(k)}\frac{1}{x-\lambda_l}\Big\}\Big)\Big|_{x=\lambda_k}\\&=M^{k,j,0},\end{aligned}$$

which establishes (2.10). Moreover, it is easily seen by a residue calculus for the rational function (2.7) that

$$\sum_{(k)} \frac{M^{k,j}}{\lambda_k(\lambda_k-1)}=1; \tag{2.11}$$

note that

$$\operatorname*{Res}_{x=\infty} Z_j(x)dx=-1.$$

2.3. System of differential equations.

Now we shall derive, from the differential equations (1.18) and (1.20), the system of differential equations for λ_k, ν_k ($k=1, \cdots, N$), by using (1.11) and (1.21). First we rewrite (1.18) in the following form:

$$\frac{1}{\mathsf{A}_i\mathsf{A}_j}\left(\frac{\partial \mathsf{A}_j}{\partial t_i}-\frac{\partial \mathsf{A}_i}{\partial t_j}\right)+\frac{\partial}{\partial x}\log \mathsf{A}_i-\frac{\partial}{\partial x}\log \mathsf{A}_j=0 \qquad (i\neq j). \tag{2.12}$$

Noting

$$\begin{aligned}
&\frac{\partial}{\partial x}\log \mathsf{A}_j=\sum_{(h)}^{(j)}\frac{1}{x-t_h}+\frac{1}{x}+\frac{1}{x-1}-\sum_{(k)}\frac{1}{x-\lambda_k},\\
&\frac{\partial}{\partial t_i}\log \mathsf{A}_j=\frac{\partial}{\partial t_i}\log \mathsf{M}_j-\frac{1}{x-t_i}+\sum_{(k)}\frac{1}{x-\lambda_k}\frac{\partial \lambda_k}{\partial t_i},\\
&\mathsf{A}_j(t_j)=-1,
\end{aligned}$$

we see that the left hand side of (2.12), $\mathsf{A}_{ij}(x)$, is a rational function having N poles ($x=0, 1, t_h$ $(h\neq i, j)$) and a simple zero at $x=\infty$. Therefore (2.12) is established if

$$\mathsf{A}_{ij}(\lambda_k)=0, \qquad \text{for} \quad k=1, \cdots, N,$$

which read as:

$$\frac{\lambda_k-t_i}{M_i}\frac{\partial\lambda_k}{\partial t_i}-\frac{\lambda_k-t_j}{M_j}\frac{\partial\lambda_k}{\partial t_j}+\frac{(t_j-t_i)T(\lambda_k)}{(\lambda_k-t_i)(\lambda_k-t_j)\Lambda'(\lambda_k)}=0 \tag{2.13}$$
$$(i, j, k=1, \cdots, N).$$

The differential equation (1.20) can be written as

$$\frac{1}{2}\mathsf{A}_j\frac{\partial^3 \mathsf{A}_j}{\partial x^3}-\frac{\partial}{\partial x}(p\mathsf{A}_j^2)+\mathsf{A}_j\frac{\partial p}{\partial t_j}=0 \qquad (j=1, \cdots, N); \tag{2.14}$$

let $\mathscr{A}_j(x)$ be the left hand side of (2.14).

PROPOSITION 2.3. *$\mathscr{A}_j(x)$ can be written in the form*

$$\mathscr{A}_j(x)=\frac{w^j}{x-t_j}+\sum_{(k)}\sum_{m=1}^{4}\frac{w_m^{j,k}}{(x-\lambda_k)^m},$$

and the equation (2.14) *is established if*

$$w_m^{j,k}=0, \qquad (j,\ k=1,\ \cdots,\ N,\ m=1,\ \cdots,\ 4).$$

PROOF. It is easy to verify that $\mathscr{A}_j(x)$ has the following properties:

(i) $\mathscr{A}_j(x)$ is holomorphic at $x=0,1$ and t_i $(i\neq j)$,

(ii) $x=t_j$ is a simple pole,

(iii) $x=\lambda_k$ is a pole of the fourth order for each k,

(iv) $\mathscr{A}_j(x)$ has a zero of the second order at $x=\infty$.

The first assertion of the proposition is a consequence of (i), (ii), (iii), and the second one follows from (iv) at once.

We compute $w_m^{j,k}$ explicitly by using (2.14). Let

$$\mathsf{A}_j(x)=\mathsf{M}_j\left[\frac{M^{k,j}}{x-\lambda_k}+\sum_{n=0}^{\infty}M^{k,j,n}(x-\lambda_k)^n\right],$$

$$p(x)=\frac{3}{4(x-\lambda_k)^2}-\frac{\nu_k}{x-\lambda_k}+\sum_{n=0}^{\infty}\mathscr{U}_{k,n}(x-\lambda_k)^n,$$

be local expansions of $\mathsf{A}_j(x)$, $p(x)$ around $x=\lambda_k$, where $\mathscr{U}_{k,0}=\mathscr{U}_k$ (see (2.5)). We obtain the following system of differential equations:

$$\frac{\partial\lambda_k}{\partial t_j}-\mathsf{M}_j[2M^{k,j}\nu_k-M^{k,j,0}]=0, \quad \text{from } w_4^{j,k}=0, \tag{2.15}$$

$$\frac{\partial\nu_k}{\partial t_j}-\mathsf{M}_j\left[M^{k,j}\mathscr{U}_{k,1}+M^{k,j,1}\nu_k-\frac{3}{2}M^{k,j,2}\right]=0, \quad \text{from } w_2^{j,k}=0, \tag{2.16}$$

and finally from $w_1^{j,k}=0$,

$$M^{k,j}\frac{\partial\mathscr{U}_{k,0}}{\partial t_j}-M^{k,j,0}\frac{\partial\nu_k}{\partial t_j}=\left(M^{k,j}\mathscr{U}_{k,1}+M^{k,j,1}\nu_k-\frac{3}{2}M^{k,j,2}\right)\frac{\partial\lambda_k}{\partial t_j}. \tag{2.17}$$

The coefficient $w_3^{j,k}$ vanishes identically because of the constraint (2.4):

$$\mathscr{U}_{k,0}=\nu_k^2.$$

The equation (2.17) is a repetition of (2.15), (2.16); in fact we have from (2.4)

$$\frac{\partial \mathscr{U}_{k,0}}{\partial t_j}=2\nu_k\frac{\partial \nu_k}{\partial t_j}.$$

We have obtained the differential equations (2.13), (2.15), (2.16) for λ_k, ν_k $(k=1, \cdots, N)$.

We prove the following:

PROPOSITION 2.4. *The system of differential equations* (2.13), (2.15), (2.16) *is equivalent to the Hamiltonian system*:

$$(\mathrm{H})'_{\mathrm{N}} \qquad \begin{cases} \dfrac{\partial \lambda_k}{\partial t_j}=\dfrac{\partial \mathsf{K}_j}{\partial \nu_k} \\ \dfrac{\partial \nu_k}{\partial t_j}=-\dfrac{\partial \mathsf{K}_j}{\partial \lambda_k} \qquad (j,\ k=1,\ \cdots,\ N). \end{cases}$$

PROPOSITION 2.5. $(\mathrm{H})'_{\mathrm{N}}$ *is completely integrable.*

The theorem is established by virtue of these propositions.

2.4. Verification of Propositions.

PROOF OF PROPOSITION 2.4. The first equation of $(\mathrm{H})'_{\mathrm{N}}$ follows from (2.1) and (2.15) immediately. To verify the second one, differentiate the constraint (2.4)′ with respect to λ_l regarding $\nu_1, \cdots, \nu_N, t_1, \cdots, t_N$ as constants. For $l\neq k$, we obtain

$$\sum_{(j)} E_{kj}t_j(t_j-1)\frac{\partial \mathsf{K}_j}{\partial \lambda_l}+\frac{3}{2(\lambda_k-\lambda_l)^3}-\nu_l\frac{\partial G_{kl}}{\partial \lambda_l}=0,$$

and for $l=k$,

$$\sum_{(j)} E_{kj}t_j(t_j-1)\frac{\partial \mathsf{K}_j}{\partial \lambda_k}+\mathscr{U}_{k,1}+\frac{\nu_k}{\lambda_k(\lambda_k-1)}=0,$$

since

$$\begin{aligned}\mathscr{U}_{k,1}&=\left(\frac{\partial}{\partial x}\left(p(x)-\frac{3}{4(x-\lambda_k)^2}+\frac{\nu_k}{x-\lambda_k}\right)\right)\Big|_{x=\lambda_k}\\ &=\left(\frac{\partial}{\partial \lambda_k}\right)\mathscr{U}_{k,0}-\frac{\nu_k}{\lambda_k(\lambda_k-1)}.\end{aligned}$$

Here we denote by $(\partial/\partial\lambda_k)$ the differentiation with respect to λ_k regarding $t_j, K_j, \nu_1, \cdots, \nu_N$ as constants. It follows from (2.6), (2.8) that

$$\frac{\partial \mathsf{K}_j}{\partial \lambda_k} = -\mathsf{M}_j\left[M^{k,j}\mathscr{U}_{k,1} + \frac{3}{2}\sum_{(l)}^{(k)} \frac{M^{l,j}}{(\lambda_l-\lambda_k)^3} + \nu_k\left\{ \frac{M^{k,j}}{\lambda_k(\lambda_k-1)} - \sum_{(l)}^{(k)} M^{l,j}\frac{\partial G_{lk}}{\partial \lambda_k}\right\}\right].$$

We claim:

$$M^{k,j,2} = -\sum_{(l)}^{(k)} \frac{M^{l,j}}{(\lambda_l-\lambda_k)^3} \tag{2.18}$$

$$M^{k,j,1} = \frac{M^{k,j}}{\lambda_k(\lambda_k-1)} - \sum_{(l)}^{(k)} M^{l,j}\frac{\partial G_{lk}}{\partial \lambda_k}. \tag{2.19}$$

In fact, we have from the definition of $M^{k,j,n}$

$$\begin{aligned} M^{k,j,1} &= \left(\frac{\partial}{\partial x}\left(x(x-1)Z_j(x) - \frac{M^{k,j}}{x-\lambda_k}\right)\right)\Big|_{x=\lambda_k} \\ &= \sum_{(l)}^{(k)} M^{l,j}\left(\frac{\partial}{\partial x}\left(\frac{x(x-1)}{\lambda_l(\lambda_l-1)(x-\lambda_k)}\right)\right)\Big|_{x=\lambda_k} + \frac{M^{k,j}}{\lambda_k(\lambda_k-1)} \end{aligned}$$

which show (2.19). Moreover, since

$$\begin{aligned} x(x-1)Z_j(x) &= \frac{T(x)}{(x-t_j)\Lambda(x)} \\ &= \sum_{(l)} \frac{M^{l,j}}{x-\lambda_l} + \sum_{(l)} \frac{x+\lambda_l-1}{\lambda_l(\lambda_l-1)}M^{l,j}, \end{aligned}$$

we obtain (2.18).

The constraint equation (2.13) is deduced a priori from (2.15), which will be easily seen by means of (2.10).

PROOF OF PROPOSITION 2.5. Put

$$\Gamma = \sum_{(k)} d\nu_k \wedge d\lambda_k - \sum_{(j)} d\mathsf{K}_j \wedge dt_j.$$

If we regard λ_k, ν_k and K_j as functions of $t_1, \cdots, t_N$, it can be written in the form

$$\begin{aligned} \Gamma &= \sum_{i<j} \Gamma_{i,j} dt_i \wedge dt_j, \\ \Gamma_{i,j} &= \sum_{(k)} \left[\frac{\partial \mathsf{K}_j}{\partial \lambda_k}\frac{\partial \mathsf{K}_i}{\partial \mu_k} - \frac{\partial \mathsf{K}_i}{\partial \lambda_k}\frac{\partial \mathsf{K}_j}{\partial \mu_k}\right] + \left(\frac{\partial}{\partial t_i}\right)\mathsf{K}_j - \left(\frac{\partial}{\partial t_j}\right)\mathsf{K}_i, \end{aligned}$$

where $(\partial/\partial t_j)$ denotes the differentiation with respect to t_j such that λ_k, ν_k are viewed to be independent of t_j.

Note:

$$2\Gamma_{i,j}=\frac{\partial \mathsf{K}_j}{\partial t_i}-\frac{\partial \mathsf{K}_i}{\partial t_j}+\left(\frac{\partial}{\partial t_i}\right)\mathsf{K}_j-\left(\frac{\partial}{\partial t_j}\right)\mathsf{K}_i. \tag{2.20}$$

To verify the proposition, it suffices to show that the 2-form Γ is vanished along a solution leaf of the foliation defined by the system $(\mathrm{H})'_N$.

Let

$$\mathsf{A}_j(x)=\mathsf{M}_j(x-t_i)(m_0^{j,i}+m_1^{j,i}(x-t_i)+\cdots),$$
$$p(x)=\frac{b_i}{(x-t_i)^2}+\frac{\mathsf{K}_i}{x-t_i}+\cdots$$

be local expansions around $x=t_i$ $(i\neq j)$, where

$$m_0^{j,i}=\frac{1}{\mathsf{M}_i}\frac{1}{t_j-t_i},$$
$$m_1^{j,i}=-\sum_{(k)}\frac{M^{k,j}}{(\lambda_k-t_i)^3}.$$

The left hand side of (2.14), $\mathscr{A}_j(x)$, vanishing identically, we obtain from the identities

$$\mathscr{A}_j(t_i)\equiv 0,$$

the following differential equations:

$$\frac{\partial \mathsf{K}_i}{\partial t_j}=\mathsf{M}_j[m_0^{j,i}\mathsf{K}_i+2m_1^{j,i}b_i].$$

By using this equation, we can see that the right hand side of (2.20) vanishes; this proves Proposition 2.5. We do not enter into details of computation.

§ 3. Hamiltonian system and Schlesinger system.

3.1. Notation and results.

In this section, we shall study the correspondence between the 2×2 Schlesinger system and the Garnier system $(\mathrm{H})_N$.

Consider the linear system of differential equations:

$$
(3.1) \qquad \begin{cases} \dfrac{dy_1}{dx} = R^{11}(x)y_1 + R^{12}(x)y_2, \\[2mm] \dfrac{dy_2}{dx} = R^{21}(x)y_1 + R^{22}(x)y_2, \end{cases}
$$

where

$$
R^{\alpha\beta}(x) = \frac{q_0^{\alpha\beta}}{x} + \frac{q_1^{\alpha\beta}}{x-1} + \sum_{(j)} \frac{r_j^{\alpha\beta}}{x-t_j} \qquad (\alpha, \beta = 1, 2, \ j = 1, \cdots, N).
$$

Put

$$
\begin{aligned} Q_\Delta &= ((q_\Delta^{\alpha\beta}))_{\alpha,\beta=1,2}, \qquad && (\Delta = 0, 1) \\ R_j &= ((r_j^{\alpha\beta}))_{\alpha,\beta=1,2}, \qquad && (j = 1, \cdots, N) \end{aligned}
$$

and make the following assumptions:

$$
(3.2) \qquad \begin{cases} \operatorname{trace} Q_\Delta = \kappa_\Delta \\ \operatorname{trace} R_j = \theta_j, \end{cases}
$$

$$
(3.3) \qquad \det Q_\Delta = \det R_j = 0,
$$

$$
(3.4) \qquad Q_0 + Q_1 + \sum_{(j)} R_j = -\begin{pmatrix} \alpha_\infty, & 0 \\ 0, & \alpha_\infty + \kappa_\infty - 1 \end{pmatrix}
$$

κ_Δ, θ_j, α_∞, κ_∞ being constants.

PROPOSITION 3.1 (cf. [3]). *y_1 verifies the linear differential equation of the canonical type,*

$$
(3.5) \qquad \frac{d^2y_1}{dx^2} + p_1(x)\frac{dy_1}{dx} + p_2(x)y_1 = 0,
$$

having N non logarithmic singular points, $x = \lambda_k$ $(k = 1, \cdots, N)$, in addition to the $N+3$ singularities, $x = 0, 1, \infty, t_j$ $(j = 1, \cdots, N)$.

This proposition assures us the equivalence between the isomonodromic deformation of (3.1) and that of (3.5), therefore the 2×2 Schlesinger system is reduced to the Hamiltonian system

$$
(\mathrm{H})_N \qquad \begin{cases} \dfrac{\partial \lambda_k}{\partial t_j} = \dfrac{\partial \mathsf{H}_j}{\partial \mu_k} \\[2mm] \dfrac{\partial \mu_k}{\partial t_j} = -\dfrac{\partial \mathsf{H}_j}{\partial \lambda_k}. \end{cases}
$$

We shall determine the explicit forms of the Hamiltonians H_j and of the canonical variables μ_k by the use of the elements of Q_Δ, R_j.

Inversely, we shall prove the following results:

PROPOSITION 3.2. *The matrices* Q_Δ, R_j $(\Delta=0, 1,\ j=1, \cdots, N)$ *are determined by* (λ_k, μ_k) $(k=1, \cdots, N)$ *up to the following multiplicative quantity*:

$$X=q_1^{12}+\sum_{(j)} t_j r_j^{12}. \tag{3.6}$$

PROPOSITION 3.3. X *satisfies the completely integrable Pfaffian equation*

$$d\log X=-\kappa_\infty \sum_{(j)} \mathsf{M}_j dt_j, \tag{3.7}$$

where M_j $(j=1, \cdots, N)$ *are given by* (1.24).

3.2. Correspondence between (3.1) and (3.5).

In this paragraph, we prove Propositions 3.1 and 3.2. The proof of the third one will be given in the next paragraph.

PROOF OF PROPOSITION 3.1. It is easy to see that the coefficients $p_1(x)$, $p_2(x)$ of the differential equation (3.5) is written as follows:

$$\begin{aligned} p_1(x)&=-R^{11}(x)-R^{22}(x)-\frac{d}{dx}\log R^{12}(x)\\ p_2(x)&=R^{11}(x)R^{22}(x)-R^{12}(x)R^{21}(x)\\ &\quad+R^{11}(x)\frac{d}{dx}\log R^{12}(x)-\frac{d}{dx}R^{11}(x). \end{aligned} \tag{3.8}$$

If we put

$$R^{12}(x)=T(x)^{-1}\Lambda^{12}(x),$$

$$T(x)=x(x-1)\prod_{(j)}(x-t_j)$$

then $\Lambda^{12}(x)$ is a polynomial of degree N; let λ_k $(k=1, \cdots, N)$ be zeros of $\Lambda^{12}(x)$. For each k, $x=\lambda_k$ is a singular point of the linear differential equation (3.5) such that the characteristic exponents are 0 and 2. Clearly it is not a logarithmic singularity. Therefore, we can put

$$\Lambda^{12}(x)=X\Lambda(x),$$

$$\Lambda(x)=\prod_{(k)}(x-\lambda_k),$$

where

$$X=q_1^{12}+\sum_{(j)}t_jr_j^{12}.$$

Moreover, (3.8) can be written in the following form:

$$(3.8)' \qquad \begin{aligned} p_1(x)&=\frac{1-\kappa_0}{x}+\frac{1-\kappa_1}{x-1}+\sum_{(j)}\frac{1-\theta_j}{x-t_j}-\sum_{(k)}\frac{1}{x-\lambda_k},\\ p_2(x)&=\frac{\kappa}{x(x-1)}-\sum_{(j)}\frac{t_j(t_j-1)\mathsf{H}_j}{x(x-1)(x-t_j)}+\sum_{(k)}\frac{\lambda_k(\lambda_k-1)\mu_k}{x(x-1)(x-\lambda_k)},\end{aligned}$$

where

$$\kappa=\alpha_\infty(\alpha_\infty+\kappa_\infty)$$

$$(3.9) \qquad \mu_k=\frac{q_0^{11}}{\lambda_k}+\frac{q_1^{11}}{\lambda_k-1}+\sum_{(j)}\frac{r_j^{11}}{\lambda_k-t_j},$$

$$(3.10) \qquad \begin{aligned}\mathsf{H}_j=\sum_{(k)}\frac{r_j^{11}}{\lambda_k-t_j}&+\frac{1}{t_j}(q_0^{11}+r_j^{11}-\kappa_0\theta_j)\\ &+\frac{1}{t_j-1}(q_1^{11}+r_j^{11}-\kappa_1\theta_j)\\ &+\sum_{(i)}^{(j)}\frac{1}{t_j-t_i}(r_i^{11}+r_j^{11}-\theta_i\theta_j)\\ &+\mathrm{trace}\left[R_j\left(\frac{Q_0}{t_j}+\frac{Q_1}{t_j-1}+\sum_{(i)}^{(j)}\frac{R_i}{t_j-t_i}\right)\right].\end{aligned}$$

Here we have used (3.2). The proof of Proposition 3.1 is thus completed; the explicit forms of μ_k, H_j are given by (3.9) and (3.10).

PROOF OF PROPOSITION 3.2. Since $R^{12}(x)$ vanishes at $x=\lambda_k$, we have:

$$\frac{q_0^{12}}{\lambda_k}+\frac{q_1^{12}}{\lambda_k-1}+\sum_{(j)}\frac{r_j^{12}}{\lambda_k-t_j}=0.$$

By the use of (3.4) it can be written in the form:

$$\sum_{(j)}E_{kj}t_j(t_j-1)r_j^{12}=-\frac{X}{\lambda_k(\lambda_k-1)},$$

E_{kj} being given by (2.6). We obtain from this

$$r_j^{12} = -\mathsf{M}_j X \qquad (j=1, \cdots, N), \tag{3.11}$$

in the same way as the proof of Proposition 2.1. Moreover, if we define $\mathsf{M}^{(\varDelta)}$ $(\varDelta=0, 1)$ by

$$\frac{\varLambda(x)}{T(x)} = -\frac{\mathsf{M}^{(0)}}{x} - \frac{\mathsf{M}^{(1)}}{x-1} - \sum_{(j)} \frac{\mathsf{M}_j}{x-t_j},$$

then $q_\varDelta^{12}$ is given by

$$q_\varDelta^{12} = -\mathsf{M}^{(\varDelta)} X, \tag{3.12}$$

which is easily verified by the use of the relations:

$$\mathsf{M}^{(0)} + \mathsf{M}^{(1)} + \sum_{(j)} \mathsf{M}_j = 0,$$

$$\mathsf{M}^{(1)} + \sum_{(j)} t_j \mathsf{M}_j + 1 = 0.$$

To determine $q_\varDelta^{11}$, r_j^{11}, we put

$$W = q_1^{11} + \sum_{(j)} t_j r_j^{11}, \tag{3.13}$$

and rewrite (3.9) as follows:

$$\sum_{(j)} E_{kj} t_j (t_j - 1) r_j^{11} = -\frac{W}{\lambda_k(\lambda_k - 1)} + \mu_k + \frac{\alpha_\infty}{\lambda_k}.$$

Therefore, using (2.11), we obtain

$$\begin{aligned} r_j^{11} &= \sum_{(k)} M^j M^{k,j} \Big(\mu_k + \frac{\alpha_\infty}{\lambda_k} - \frac{W}{\lambda_k(\lambda_k - 1)} \Big) \\ &= \mathsf{M}_j (W_j - W), \end{aligned} \tag{3.14}$$

where

$$W_j = \sum_{(k)} M^{k,j} \Big(\mu_k + \frac{\alpha_\infty}{\lambda_k} \Big). \tag{3.15}$$

On the other hand, defining $W^{(\varDelta)}$ $(\varDelta=0, 1)$ by

$$\mathsf{M}^{(0)} W^{(0)} = \sum_{(j)} (t_j - 1) \mathsf{M}_j W_j - \alpha_\infty,$$

$$\mathsf{M}^{(1)} W^{(1)} = -\sum_{(j)} t_j \mathsf{M}_j W_j,$$

we have:

$$q_{\Delta}^{11}=\mathsf{M}^{(\Delta)}(W^{(\Delta)}-W). \tag{3.16}$$

The other elements of Q_{Δ}, R_j are given by (3.2) and (3.3); in fact we obtain from (3.2)

$$\begin{aligned} q_{\Delta}^{22}&=\mathsf{M}^{(\Delta)}(W-W^{(\Delta)})+\kappa_{\Delta}, \\ r_j^{22}&=\mathsf{M}_j(W-W_j)+\theta_j, \end{aligned} \tag{3.17}$$

and moreover, from (3.3),

$$\begin{aligned} q_{\Delta}^{21}&=X^{-1}(W-W^{(\Delta)})[\mathsf{M}^{(\Delta)}(W-W^{(\Delta)})+\kappa_{\Delta}], \\ r_j^{21}&=X^{-1}(W-W_j)[\mathsf{M}_j(W-W_j)+\theta_j], \qquad (\Delta=0,1,\ j=1,\cdots,N). \end{aligned} \tag{3.18}$$

Note:

$$\mathsf{M}^{(0)}W^{(0)}+\mathsf{M}^{(1)}W^{(1)}+\sum_{(j)}\mathsf{M}_jW_j+\alpha_{\infty}=0.$$

Now we can determine W by (3.18); in fact, we obtain from (3.18) and the Fuchsian relation (1.14),

$$W=(\kappa_{\infty})^{-1}[W^{(0)}(\mathsf{M}^{(0)}W^{(0)}-\kappa_0)+W^{(1)}(\mathsf{M}^{(1)}W^{(1)}-\kappa_1)+\sum_{(j)}W_j(\mathsf{M}_jW_j-\theta_j)]. \tag{3.19}$$

The elements of the matrices Q_{Δ}, R_j have been thus determined by means of λ_k, μ_k, up to X. They are given by (3.11), (3.12), (3.14), (3.16), (3.17), (3.18) and (3.19).

REMARK 3.1. In the case $N=1$, (3.19) reads:

$$\begin{aligned} W=-(\kappa_{\infty})^{-1}[&\lambda(\lambda-1)(\lambda-t)\mu^2-\{\kappa_0(\lambda-1)(\lambda-t) \\ &+\kappa_1\lambda(\lambda-t)+\theta_1\lambda(\lambda-1)\}\mu-\alpha_{\infty}^2(\kappa-t-1) \\ &-\alpha_{\infty}\{\kappa_0(\lambda-t-1)+\kappa_1(\lambda-t)+\theta_1(\lambda-1)\}], \end{aligned}$$

where we omit indices.

3.3. Proof of Proposition 3.3.

For the determination of X, we consider the isomonodromic deformation of the linear differential equations, (3.1) and (3.5). So, assume that (λ_k, μ_k) $(k=1,\cdots,N)$ satisfy the Hamiltonian system $(\mathrm{H})_N$ and Q_{Δ}, R_j $(\Delta=0,1,\ j=1,\cdots,N)$ verify the Schlesinger system:

$$(3.20)\qquad \begin{aligned}&\frac{\partial Q_0}{\partial t_i}=-\frac{[Q_0, R_i]}{t_i}, \quad \frac{\partial Q_1}{\partial t_i}=\frac{[Q_1, R_i]}{1-t_i},\\ &\frac{\partial R_j}{\partial t_i}=\frac{[R_j, R_i]}{t_j-t_i} \qquad (i\neq j).\end{aligned}$$

We prove the following two lemmas.

LEMMA 3.1. *$\sigma=\sum_{(k)}\lambda_k$ verifies the differential equation,*

$$(3.21)\qquad \frac{\partial\sigma}{\partial t_i}=\mathsf{M}_i[2W_i-L_i],$$

where

$$(3.22)\qquad \begin{aligned}&L_i=\sum_{(h)}(\theta_h-\delta_{ih})\Big(t_h-1+\frac{\delta_{ih}}{\mathsf{M}_i}\Big)-\kappa_0-2\alpha_\infty-\kappa_\infty m^i,\\ &m^i=\sigma-\sum_{(h)}^{(i)}t_h,\end{aligned}$$

W_i being given by (3.15).

PROOF. We obtain from (2.9) and $(\mathrm{H})_N$:

$$\frac{\partial\lambda_k}{\partial t_i}=\mathsf{M}_i\Big[2M^{k,i}\mu_k-M^{k,i}\Big(\frac{\kappa_0}{\lambda_k}+\frac{\kappa_1}{\lambda_k-1}+\sum_{(h)}\frac{\theta_h-\delta_{ih}}{\lambda_k-t_h}\Big)\Big].$$

Moreover we can verify the following formula:

$$(3.23)\qquad \begin{aligned}&\sum_{(k)}\frac{M^{k,i}}{\lambda_k}=-1+m^i, \qquad \sum_{(k)}\frac{M^{k,i}}{\lambda_k-1}=m^i,\\ &\sum_{(k)}\frac{M^{k,i}}{\lambda_k-t_h}=t_h-1+m^i+\frac{\delta_{ih}}{M_i}.\end{aligned}$$

For example, we prove (3.23) for $h=i$ with the help of the auxiliary rational function,

$$\bar{Z}_i(x)=\frac{T(x)}{(x-t_i)^2\Lambda(x)}.$$

In fact, it is easy to see that:

$$\begin{aligned}&\operatorname*{Res}_{x=\lambda_k}\bar{Z}_i(x)dx=\frac{T(\lambda_k)}{(\lambda_k-t_i)^2\Lambda'(\lambda_k)}=\frac{M^{k,i}}{\lambda_k-t_i}\\ &\operatorname*{Res}_{x=t_j}\bar{Z}_i(x)dx=\frac{T'(t_i)}{\Lambda(t_i)}=-\frac{1}{\mathsf{M}_i}\end{aligned}$$

and finally

$$\operatorname*{Res}_{x=\infty} \bar{Z}_i(x)dx = t_i - 1 + \sum_{(k)} \lambda_k - \sum_{(h)}^{(i)} t_h = t_i - 1 + m^i.$$

Hence we have (3.23) for $h=i$. Note that W_i can be written as:

$$W_i = \sum_{(k)} M^{k,i}\mu_k + \alpha_\infty(-1+m^i).$$

Now Lemma 3.1 is immediately established by virtue of the Fuchsian relation (1.14).

LEMMA 3.2. $\partial \mathsf{M}_i/\partial t_j = \partial \mathsf{M}_j/\partial t_i$.

PROOF. Note firstly that, if $i \neq j$,

$$\frac{\partial \mathsf{M}_j}{\partial t_i} = \mathsf{M}_j\left[-\sum_{(k)} \frac{1}{t_j-\lambda_k}\frac{\partial \lambda_k}{\partial t_i} + \frac{1}{t_j-t_i}\right]. \tag{3.24}$$

Then we have

$$\begin{aligned} \frac{\partial \mathsf{M}_i}{\partial t_j} - \frac{\partial \mathsf{M}_j}{\partial t_i} = (t_i - t_j)\mathsf{M}_i\mathsf{M}_j \sum_{(k)} \frac{T(\lambda_k)}{(\lambda_k - t_i)^2(\lambda_k - t_j)^2 \Lambda'(\lambda_k)} \\ - \frac{1}{t_i - t_j}(\mathsf{M}_i + \mathsf{M}_j), \end{aligned} \tag{3.25}$$

by means of the constraint (2.13) which is written in the form:

$$\frac{\mathsf{M}_j}{\lambda_k - t_j}\frac{\partial \lambda_k}{\partial t_i} - \frac{\mathsf{M}_i}{\lambda_k - t_i}\frac{\partial \lambda_k}{\partial t_j} + \frac{(t_j - t_i)\mathsf{M}_i\mathsf{M}_j T(\lambda_k)}{(\lambda_k - t_i)^2(\lambda_k - t_j)^2 \Lambda'(\lambda_k)} = 0.$$

The right hand side of (3.25) vanishes actually as is easily verified by the use of the auxiliary function:

$$\frac{T(x)}{(x-t_i)^2(x-t_j)^2\Lambda(x)}.$$

This proves the lemma.

Now we verify Proposition 3.3. The system of differential equations (3.20) implies that, if $i \neq j$,

$$\frac{\partial r_j^{12}}{\partial t_i} = \frac{1}{t_j - t_i}[r_j^{11} r_i^{12} + r_j^{12} r_i^{22} - r_i^{11} r_j^{12} - r_i^{12} r_j^{22}].$$

Inserting (3.11), (3.14), (3.17) in this equation, we have the differential equations for X:

$$\frac{\partial}{\partial t_i}\log X=\frac{1}{t_i-t_j}\left[2\mathsf{M}_i(W_i-W_j)+\theta_j\frac{\mathsf{M}_i}{\mathsf{M}_j}-\theta_i\right]-\frac{\partial}{\partial t_i}\log \mathsf{M}_j. \tag{3.26}$$

We obtain firstly from (3.21)

$$2\mathsf{M}_i(W_i-W_j)=\left(\frac{\partial}{\partial t_i}-\frac{\mathsf{M}_i}{\mathsf{M}_j}\frac{\partial}{\partial t_j}\right)\sigma+\mathsf{M}_i(L_i-L_j).$$

On the other hand, since (2.13) is written as:

$$\frac{\mathsf{M}_i}{\mathsf{M}_j}\frac{\partial\lambda_k}{\partial t_j}=\frac{\lambda_k-t_i}{\lambda_k-t_j}\frac{\partial\lambda_k}{\partial t_i}-\mathsf{M}_i(t_i-t_j)\frac{M^{k,i}}{(\lambda_k-t_j)^2}\qquad (i\neq j),$$

it follows that

$$\begin{aligned}\left(\frac{\partial}{\partial t_i}-\frac{\mathsf{M}_i}{\mathsf{M}_j}\frac{\partial}{\partial t_j}\right)\sigma&=(t_i-t_j)\left[\sum_{(k)}\frac{1}{\lambda_k-t_j}\frac{\partial\lambda_k}{\partial t_i}+\mathsf{M}_i\sum_{(k)}\frac{M^{k,i}}{(\lambda_k-t_j)^2}\right]\\&=(t_i-t_j)\left[\sum_{(k)}\frac{1}{\lambda_k-t_j}\frac{\partial\lambda_k}{\partial t_i}+\mathsf{M}_i\right]-\frac{\mathsf{M}_i}{\mathsf{M}_j}.\end{aligned}$$

Here the second equality is verified with the help of a residue calculus of the rational function,

$$\frac{T(x)}{(x-t_i)(x-t_j)^2\Lambda(x)}.$$

Finally we can show by computations that (3.26) reads:

$$\frac{\partial}{\partial t_i}\log X=-\kappa_\infty\mathsf{M}_i.$$

The complete integrability condition for (3.7) has been assured in Lemma 3.2.

§ 4. Painlevé equations.

4.1. Isomonodromic deformation giving Painlevé equations.

In this section we shall determine the Hamiltonian structure associated with Painlevé equations by considering the isomonodromic deformation of linear equations of the form:

$$\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_2(x)y=0. \tag{4.1}$$

We shall refer to Painlevé equations as P_J ($J=\mathrm{I}, \cdots, \mathrm{VI}$).

Consider firstly the sixth Painlevé equation P_{VI}:

$$\begin{aligned}\frac{d^2\lambda}{dt^2}=&\frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\\&+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{\lambda^2}+\gamma\frac{t-1}{(\lambda-1)^2}+\delta\frac{t(t-1)}{(\lambda-t)^2}\right],\end{aligned}$$

where α, β, γ and δ denote constants. Since the Garnier system is reduced to P_{VI} for $N=1$, we obtain the Hamiltonian K_{VI} from (2.1):

$$\begin{aligned}\mathrm{K}_{\mathrm{VI}}=\frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\Big[&\nu^2-\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)\nu-\frac{a_0}{\lambda^2}-\frac{a_1}{(\lambda-1)^2}\\&-\frac{a_\infty}{\lambda(\lambda-1)}-\frac{b_1}{(\lambda-t)^2}\Big].\end{aligned}$$

For the sake of simplification of notation, we omit the indices for the variables λ, ν, μ, t.

The isomonodromic deformation of the linear differential equation of the SL-type,

$$\frac{d^2z}{dx^2}=p(x)z \tag{4.2}$$

such that

$$\begin{aligned}p(x)=&\frac{a_0}{x^2}+\frac{a_1}{(x-1)^2}+\frac{a_\infty}{x(x-1)}+\frac{b_1}{(x-t)^2}+\frac{3}{4(x-\lambda)^2}\\&+\frac{t(t-1)\mathrm{K}_{\mathrm{VI}}}{x(x-1)(x-t)}-\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)},\end{aligned} \tag{4.3}$$

was studied by R. Fuchs [1]. We denote by $\mathrm{SL}_{\mathrm{VI}}$ the linear equation (4.2) with (4.3). The following proposition is deduced from the main theorem:

PROPOSITION 4.1. *The isomonodromic deformation of* $\mathrm{SL}_{\mathrm{VI}}$ *is governed by the Hamiltonian system*

$$\left\{\begin{aligned}\frac{d\lambda}{dt}&=\frac{\partial \mathrm{K}}{\partial \nu}\\ \frac{d\nu}{dt}&=-\frac{\partial \mathrm{K}}{\partial \lambda},\end{aligned}\right. \tag{H$'$}$$

with $\mathsf{K}=\mathsf{K}_{\mathrm{VI}}$. *This system of differential equations is equivalent to* P_{VI}.

REMARK 4.1. The constants a_Δ ($\Delta=0, 1, \infty$) and b_1 are related to $\alpha, \beta, \gamma, \delta$ of P_{VI} as follows:

$$a_0=-\frac{1}{2}\beta-\frac{1}{4}, \quad a_1=\frac{1}{2}\gamma-\frac{1}{4}, \quad b_1=-\frac{1}{2}\delta,$$
$$a_\infty=\frac{1}{2}(\alpha+\beta-\gamma+\delta-1).$$

Let L_{VI} be the canonical type equation for $\mathsf{SL}_{\mathrm{VI}}$. We obtain from (1.5), (1.6):

$$p_1(x)=\frac{1-\kappa_0}{x}+\frac{1-\kappa_1}{x-1}+\frac{1-\theta_1}{x-t}-\frac{1}{x-\lambda}, \tag{4.4}$$

$$p_2(x)=\frac{\kappa}{x(x-1)}-\frac{t(t-1)\mathsf{H}_{\mathrm{VI}}}{x(x-1)(x-t)}+\frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}, \tag{4.5}$$
$$\kappa=\frac{1}{4}[(\kappa_0+\kappa_1+\theta_1-1)^2-\kappa_\infty^2].$$

The Hamiltonian H_{VI} is written as:

$$\begin{aligned}\mathsf{H}_{\mathrm{VI}}=\frac{1}{t(t-1)}[&\lambda(\lambda-1)(\lambda-t)\mu^2-\{\kappa_0(\lambda-1)(\lambda-t)\\ &+\kappa_1\lambda(\lambda-t)+(\theta_1-1)\lambda(\lambda-1)\}\mu+\kappa(\lambda-t)].\end{aligned}$$

Note that H_{VI} is a polynomial in λ, μ. We shall call H_{VI} *the canonical Hamiltonian associated with* P_{VI}.

REMARK 4.2. The extended system for the linear equation (4.1) is of the form:

$$\begin{cases}\dfrac{\partial^2 y}{\partial x^2}+p_1(x)\dfrac{\partial y}{\partial x}+p_2(x)y=0\\ \dfrac{\partial y}{\partial t}=\mathsf{B}(x)y+\mathsf{A}(x)\dfrac{\partial y}{\partial x}.\end{cases} \tag{4.6}$$

For the canonical type equation L_{VI}, we have

$$\mathsf{A}(x)=\frac{\lambda-t}{t(t-1)}\cdot\frac{x(x-1)}{x-\lambda},$$

$$\mathsf{B}(x)=\frac{1}{2}\left(-\frac{\partial \mathsf{A}(x)}{\partial x}+p_1(x)\mathsf{A}(x)+\frac{1-\theta_1}{x-t}-\frac{1}{x-\lambda}\frac{d\lambda}{dt}\right).$$

As we have mentioned in Section 1, the complete integrability condition of the extended system (4.6) with the rational function $\mathsf{A}(x)$, $\mathsf{B}(x)$, gives the isomonodromic deformation of the linear differential equation (4.1), even if it is not of the Fuchsian type. The isomonodromic deformation of (4.1) is reduced to that of (4.2). By computing the complete integrability condition of (4.6) for the SL-type equation (4.2), R. Garnier [2] obtained the Painlevé equations without mentioning about the Hamiltonian structure.

In the following of this article we shall derive the Hamiltonian structure by using a process of coalescence, and show that the canonical Hamiltonian H_{VI} yields the Hamiltonian H_J associated with P_J. The process of step-by-step degeneration is carried out according to the following scheme:

$$\begin{array}{ccccc} & & \mathsf{H}_{\mathrm{III}} & & \\ \mathsf{H}_{\mathrm{VI}} \longrightarrow \mathsf{H}_{\mathrm{V}} & \nearrow\!\!\!\searrow & & \searrow\!\!\!\nearrow & \mathsf{H}_{\mathrm{II}} \longrightarrow \mathsf{H}_{\mathrm{I}} \\ & & \mathsf{H}_{\mathrm{IV}} & & \end{array}$$

The main result of this section is the following:

PROPOSITION 4.2. (i) *The Painlevé equation* P_J *is equivalent to the Hamiltonian system*

$$\text{(H)}\qquad \begin{cases} \dfrac{d\lambda}{dt}=\dfrac{\partial \mathsf{H}}{\partial \mu} \\[2mm] \dfrac{d\mu}{dt}=-\dfrac{\partial \mathsf{H}}{\partial \lambda} \end{cases}$$

with the Hamiltonian H_J, *which is a polynomial of the two canonical variables* λ, μ.

(ii) *The system* (H) *with* H_J *governs the isomonodromic deformation of the linear differential equation* L_J, *obtained from the canonical type equation* L_{VI} *by the process of step-by-step confluence of singularities.*

We shall call L_J as the canonical type equation and H_J as the canonical Hamiltonian. A result similar to Proposition 4.2 is obtained concerning the SL-type equation SL_J and the Hamiltonian K_J associated with P_J. Each equation of SL_J, L_J ($J=\mathrm{I}, \cdots, \mathrm{VI}$) has a regular singularity at $x=\lambda$, which is not of the logarithmic type with the exponents 0, 2.

4.2. Degeneration of Painlevé equations.

We shall recall in the following the results obtained by P. Painlevé [20] concerning the process of step-by-step degeneration.

$\mathrm{P_{VI}} \to \mathrm{P_V}$ Replace in $\mathrm{P_{VI}}$, t by $1+\varepsilon t_1$, γ by $-\delta_1\varepsilon^{-2}+\gamma_1\varepsilon^{-1}$ and δ by $\delta_1\varepsilon^{-2}$ and let ε tend to zero. Then we obtain the fifth Painlevé equation:

$$\frac{d^2\lambda}{dt^2}=\left(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt}+\frac{(\lambda-1)^2}{t^2}\left(\alpha\lambda+\frac{\beta}{\lambda}\right)+\frac{\gamma}{t}\lambda+\delta\frac{\lambda(\lambda+1)}{\lambda-1},$$

where we replace again t_1 by t and γ_1, δ_1 by γ, δ, respectively. For the sake of simplification of notation, the replacement and the succeeding limiting process will be written as follows:

$$\begin{aligned}&t\longrightarrow 1+\varepsilon t;\\ &\gamma\longrightarrow -\delta\varepsilon^{-2}+\gamma\varepsilon^{-1},\quad \delta\longrightarrow\delta\varepsilon^{-2},\qquad (\varepsilon\to 0).\end{aligned}$$

$\mathrm{P_V} \to \mathrm{P_{IV}}$ By the use of notation as above, the degeneration from $\mathrm{P_V}$ to $\mathrm{P_{IV}}$ is given by the following scheme:

$$\begin{aligned}&t\longrightarrow 1+\sqrt{2}\,\varepsilon t,\quad &&\lambda\longrightarrow\frac{\varepsilon}{\sqrt{2}}\lambda;\\ &\alpha\longrightarrow\frac{1}{2}\varepsilon^{-4}, &&\beta\longrightarrow\frac{1}{4}\beta,\\ &\gamma\longrightarrow -\varepsilon^{-4}, &&\delta\longrightarrow -\frac{1}{2}\varepsilon^{-4}+\alpha\varepsilon^{-2},\qquad (\varepsilon\to 0).\end{aligned}$$

The fourth Painlevé equation is:

$$\frac{d^2\lambda}{dt^2}=\frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2+\frac{3}{2}\lambda^3+4t\lambda^2+2(t^2-\alpha)\lambda+\frac{\beta}{\lambda}.$$

$\mathrm{P_V} \to \mathrm{P_{III}}$ In $\mathrm{P_V}$, make the substitution:

$$\begin{aligned}&\lambda\longrightarrow 1+\varepsilon\lambda,\\ &\alpha\longrightarrow\frac{1}{8}\varepsilon^{-2}\gamma+\frac{1}{4}\varepsilon^{-1}\alpha,\quad \beta\longrightarrow -\frac{1}{8}\varepsilon^{-2}\gamma,\\ &\gamma\longrightarrow\frac{1}{4}\varepsilon\beta,\quad \delta\longrightarrow\frac{1}{8}\varepsilon^2\delta,\qquad (\varepsilon\to 0).\end{aligned}$$

Then the limit form of the equation $\mathrm{P_V}$ is written as:

$$\frac{d^2\lambda}{dt^2}=\frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt}+\frac{\lambda^2}{4t^2}(\gamma\lambda+\alpha)+\frac{\beta}{4t}+\frac{\delta}{4\lambda},$$

which we denote by $\mathrm{P}'_{\mathrm{III}}$. Moreover the change of the variables

$$t\longrightarrow t^2,\quad \lambda\longrightarrow t\lambda,$$

yields from $\mathrm{P}'_{\mathrm{III}}$ the third Painlevé equation:

$$\frac{d^2\lambda}{dt^2}=\frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt}+\frac{1}{t}(\alpha\lambda^2+\beta)+\gamma\lambda^3+\frac{\delta}{\lambda}.$$

It is easy to see that the properties of $\mathrm{P}_{\mathrm{III}}$ are derived from these of $\mathrm{P}'_{\mathrm{III}}$ and vice versa.

$\mathrm{P}_{\mathrm{III}}\to\mathrm{P}_{\mathrm{II}}$ The second Painlevé equation

$$\frac{d^2\lambda}{dt^2}=2\lambda^3+t\lambda+\alpha$$

is the limit from both of P_{IV} and of $\mathrm{P}_{\mathrm{III}}$. We obtain P_{II} from $\mathrm{P}_{\mathrm{III}}$ by:

$$\begin{aligned}&t\longrightarrow 1+\varepsilon^2 t, && \lambda\longrightarrow 1+2\varepsilon\lambda;\\ &\alpha\longrightarrow -\frac{1}{2}\varepsilon^{-6}, && \beta\longrightarrow \frac{1}{2}\varepsilon^{-6}(1+4\alpha\varepsilon^3),\\ &\gamma\longrightarrow \frac{1}{4}\varepsilon^{-6}, && \delta\longrightarrow -\frac{1}{4}\varepsilon^{-6}, \qquad (\varepsilon\to 0).\end{aligned}$$

$\mathrm{P}_{\mathrm{IV}}\to\mathrm{P}_{\mathrm{II}}$ P_{IV} is reduced to P_{II} as follows:

$$\begin{aligned}&t\longrightarrow -\varepsilon^{-3}(1-2^{-2/3}\varepsilon^4 t),\\ &\lambda\longrightarrow \varepsilon^{-3}(1+2^{2/3}\varepsilon^2\lambda);\\ &\alpha\longrightarrow -\frac{1}{2}\varepsilon^{-6}\alpha,\quad \beta\longrightarrow -\frac{1}{2}\varepsilon^{-12}, \qquad (\varepsilon\to 0).\end{aligned}$$

$\mathrm{P}_{\mathrm{II}}\to\mathrm{P}_{\mathrm{I}}$ We obtain from P_{II} the first Painlevé equation

$$\frac{d^2\lambda}{dt^2}=6\lambda^2+t,$$

by the following replacement and the passage to the limit

$$\begin{aligned}&t\longrightarrow -6\varepsilon^{-10}\left(1-\frac{1}{6}\varepsilon^{12}t\right),\\ &\lambda\longrightarrow \varepsilon^{-5}(1+\varepsilon^6\lambda);\end{aligned}$$

$$\alpha \longrightarrow 4\varepsilon^{-15}, \qquad (\varepsilon \to 0).$$

4.3. The canonical Hamiltonians and the linear equations.

We shall determine in this paragraph the canonical Hamiltonians H_J ($J=\mathrm{I}, \cdots, \mathrm{V}$) by the process of coalescence. First we give below the list of H_J, and then examine the degeneration of H_J as well as the step-by-step confluence of the linear differential equations L_J of the canonical type.

Table of H_J

$$\mathsf{H}_{\mathrm{V}}=\frac{1}{t}[\lambda(\lambda-1)^2\mu^2-\{\kappa_0(\lambda-1)^2+\theta_1\lambda(\lambda-1)-\eta_1 t\lambda\}\mu+\kappa(\lambda-1)],$$

$$\left(\kappa=\frac{1}{4}[(\kappa_0+\theta_1)^2-\kappa_\infty^2]\right),$$

$$\mathsf{H}_{\mathrm{IV}}=2\lambda\mu^2-\{\lambda^2+2t\lambda+2\kappa_0\}\mu+\theta_\infty\lambda,$$

$$\mathsf{H}'_{\mathrm{III}}=\frac{1}{t}\left[\lambda^2\mu^2-\{\eta_\infty\lambda^2+\theta_0\lambda-\eta_0 t\}\mu+\frac{1}{2}\eta_\infty(\theta_0+\theta_\infty)\lambda\right],$$

$$\mathsf{H}_{\mathrm{III}}=\frac{1}{t}[2\lambda^2\mu^2-\{2\eta_\infty t\lambda^2+(2\theta_0+1)\lambda-2\eta_0 t\}\mu+\eta_\infty(\theta_0+\theta_\infty)t\lambda],$$

$$\mathsf{H}_{\mathrm{II}}=\frac{1}{2}\mu^2-\left(\lambda^2+\frac{t}{2}\right)\mu-\frac{1}{2}(2\alpha+1)\lambda,$$

$$\mathsf{H}_{\mathrm{I}}=\frac{1}{2}\mu^2-2\lambda^3-t\lambda.$$

Here we denote by $\mathsf{H}'_{\mathrm{III}}$ the canonical Hamiltonian associated with the equation $\mathsf{P}'_{\mathrm{III}}$.

$\mathsf{H}_{\mathrm{VI}}\to\mathsf{H}_{\mathrm{V}}$ In the Hamiltonian H_{VI}, replace t by $1+\varepsilon t$ H_{VI} by $\varepsilon^{-1}\mathsf{H}(\varepsilon)$. This defines a canonical transformation with the parameter ε. Moreover if we substitute $\eta_1\varepsilon^{-1}+\theta_1+1$ for κ_1 and $-\eta_1\varepsilon^{-1}$ for θ_1, then $\mathsf{H}(\varepsilon)$ is holomorphic in ε and $\mathsf{H}(0)$ gives the Hamiltonian H_{V}.
We express this procedure by the following scheme:

$$(\lambda, \mu, t, \mathsf{H}_{\mathrm{VI}})\longrightarrow(\lambda, \mu, 1+\varepsilon t, \varepsilon^{-1}\mathsf{H}_{\mathrm{V}});$$
$$\kappa_1\longrightarrow\eta_1\varepsilon^{-1}+\theta_1+1, \quad \theta_1\longrightarrow-\eta_1\varepsilon^{-1}, \qquad (\varepsilon\to 0).$$

This replacement and the passage to the limit cause simultaneously in the linear differential equation L_{VI} the confluence of singular point $x=t$ to $x=1$. In fact we obtain from (4.4), (4.5),

$$(4.7)\qquad \begin{cases} p_1(x)=\dfrac{1-\kappa_0}{x}+\dfrac{\eta_1 t}{(x-1)^2}+\dfrac{1-\theta_1}{x-1}-\dfrac{1}{x-\lambda}, \\ p_2(x)=\dfrac{\kappa}{x(x-1)}-\dfrac{t\mathsf{H}_{\mathrm{V}}}{x(x-1)^2}+\dfrac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}. \end{cases}$$

This gives the linear differential equation L_{V} associated with H_{V}. The Riemannian scheme for L_{V} reads:

$$\begin{cases} x=0 & \overbrace{\quad x=1 \quad} & x=\lambda & x=\infty \\ 0 & 0 \quad 0 & 0 & -\dfrac{1}{2}(\kappa_0+\theta_1+\kappa_\infty) \\ \kappa_0 & \eta_1 t \quad \theta_1+1 & 2 & -\dfrac{1}{2}(\kappa_0+\theta_1-\kappa_\infty). \end{cases}$$

Here, the symbol

$$\begin{array}{c} \overbrace{\qquad x=\varDelta \qquad} \\ \zeta_r^1,\ \cdots,\ \zeta_1^1,\ \zeta_0^1 \\ \zeta_r^2,\ \cdots,\ \zeta_1^2,\ \zeta_0^2, \end{array}$$

means the existence of formal solutions at an irregular singular point $x=\varDelta$ of the form:

$$\exp\Big(\sum_{m=1}^{r}\zeta_m^{(i)}\omega^{-m}+\zeta_0^{(i)}\log\omega\Big)\sum_{n=0}^{\infty}g_n^{(i)}\omega^n \qquad (i=1,\,2)$$

with $\omega=x-\varDelta$ if $\varDelta\neq\infty$ or $\omega=x^{-1}$ if $\varDelta=\infty$.

$\mathsf{H}_{\mathrm{V}}\to\mathsf{H}_{\mathrm{IV}}$ We have H_{IV} by the canonical transformation:

$$(\lambda,\,\mu,\,t,\,\mathsf{H}_{\mathrm{V}}+\kappa)\longrightarrow\Big(\frac{\varepsilon}{\sqrt{2}}\lambda,\ \sqrt{2}\,\varepsilon^{-1}\mu,\ 1+\sqrt{2}\,\varepsilon t,\ \frac{1}{\sqrt{2}}\varepsilon^{-1}\mathsf{H}_{\mathrm{IV}}\Big);$$
$$\eta_1\longrightarrow-\varepsilon^{-2},\quad \theta_1\longrightarrow\varepsilon^{-2}+2\theta_\infty-\kappa_0,\quad \kappa_\infty\longrightarrow\varepsilon^{-2},\qquad (\varepsilon\to 0).$$

Moreover, if we substitute $\varepsilon x/\sqrt{2}$ for x:

$$x\longrightarrow\frac{\varepsilon}{\sqrt{2}}x,$$

besides the above replacement, L_{V} is reduced to the linear differential equation L_{IV} in the limit, $\varepsilon\to 0$. The Riemannian scheme of L_{IV} is of the form:

$$\begin{cases} x=0 & x=\lambda & \overbrace{\qquad\qquad x=\infty \qquad\qquad} \\ 0 & 0 & 0, \quad 0, \qquad \theta_\infty \\ \kappa_0 & 2 & \frac{1}{4}, \quad t, \quad -\theta_\infty-\kappa_0+1, \end{cases}$$

and we obtain from (4.7)

$$\begin{cases} p_1(x)=\dfrac{1-\kappa_0}{x}-\dfrac{x+2t}{2}-\dfrac{1}{x-\lambda}, \\ p_2(x)=\dfrac{1}{2}\theta_\infty-\dfrac{\mathsf{H}_{\mathrm{IV}}}{2x}+\dfrac{\lambda\mu}{x(x-\lambda)}. \end{cases}$$

$\mathsf{H}_{\mathrm{V}} \to \mathsf{H}'_{\mathrm{III}}$ H_{V} degenerates to $\mathsf{H}'_{\mathrm{III}}$ by:

$$(\lambda, \mu, t, \mathsf{H}_{\mathrm{V}}) \longrightarrow (1+\varepsilon\lambda, \varepsilon^{-1}\mu, t, \mathsf{H}'_{\mathrm{III}});$$
$$\kappa_0 \longrightarrow \varepsilon^{-1}\eta_\infty, \quad \eta_1 \longrightarrow \varepsilon\eta_0, \quad \theta_1 \longrightarrow \theta_0, \quad \kappa_\infty \longrightarrow \eta_\infty\varepsilon^{-1}-\theta_\infty, \qquad (\varepsilon\to 0).$$

Moreover, by the change of the variable x in L_{V},

$$x \longrightarrow 1+\varepsilon x, \qquad (\varepsilon\to 0),$$

we have the linear differential equation, $\mathsf{L}'_{\mathrm{III}}$, associated with $\mathsf{H}'_{\mathrm{III}}$:

$$\begin{cases} p_1(x)=\dfrac{\eta_0 t}{x^2}+\dfrac{1-\theta_0}{x}-\eta_\infty-\dfrac{1}{x-\lambda}, \\ p_2(x)=\dfrac{\eta_\infty(\theta_0+\theta_\infty)}{2x}-\dfrac{t\mathsf{H}'_{\mathrm{III}}}{x^2}+\dfrac{\lambda\mu}{x(x-\lambda)}. \end{cases}$$

$\mathsf{H}'_{\mathrm{III}} \to \mathsf{H}_{\mathrm{III}}$ The Hamiltonian system with the Hamiltonian $\mathsf{H}'_{\mathrm{III}}$ is equivalent to the other associated with $\mathsf{H}_{\mathrm{III}}$ by means of the canonical transformation

$$(\lambda, \mu, t, \mathsf{H}'_{\mathrm{III}}) \longrightarrow \left(t\lambda, t^{-1}\mu, t^2, \frac{1}{2t}\left(\mathsf{H}_{\mathrm{III}}+\frac{\lambda\mu}{t}\right)\right),$$

and by the change of the variable:

$$x \longrightarrow tx,$$

$\mathsf{L}'_{\mathrm{III}}$ is reduced to $\mathsf{L}_{\mathrm{III}}$, which is of the form:

$$\begin{cases} p_1(x)=\dfrac{\eta_0 t}{x^2}+\dfrac{1-\theta_0}{x}-\eta_\infty t-\dfrac{1}{x-\lambda}, \\ p_2(x)=\dfrac{\eta_\infty(\theta_0+\theta_\infty)t}{2x}-\dfrac{t\mathsf{H}_{\mathrm{III}}+\lambda\mu}{2x^2}+\dfrac{\lambda\mu}{x(x-\lambda)}. \end{cases}$$

The Riemannian scheme of $\mathrm{L_{III}}$ is:

$$\left\{\begin{array}{cccccc} \overbrace{x=0}^{} & & x=\lambda & \overbrace{\qquad x=\infty \qquad}^{} & \\ 0 & 0 & 0 & 0 & -\frac{1}{2}(\theta_0+\theta_\infty) \\ \eta_0 t & \theta_0+1 & 2 & \eta_\infty t & -\frac{1}{2}(\theta_0-\theta_\infty). \end{array}\right.$$

$\mathrm{H_{III}} \to \mathrm{H_{II}}$ Consider the following canonical transformation:

$$(\lambda,\ \mu,\ t,\ \mathrm{H_{III}}) \longrightarrow \left(1+2\varepsilon\lambda,\ \frac{1}{2}\varepsilon^{-1}\mu,\ 1+\varepsilon^2 t,\ \varepsilon^{-2}\mathrm{H_{II}}\right);$$

$$\eta_0 \longrightarrow -\frac{1}{4}\varepsilon^{-3}, \quad \eta_\infty \longrightarrow \frac{1}{4}\varepsilon^{-3},$$

$$\theta_0 \longrightarrow -\frac{1}{2}\varepsilon^{-3}-2\alpha-1, \quad \theta_\infty \longrightarrow -\frac{1}{2}\varepsilon^{-3},$$

and the change of the variable

$$x \longrightarrow 1+2\varepsilon x, \qquad (\varepsilon \to 0).$$

This gives $\mathrm{H_{II}}$ from $\mathrm{H_{III}}$ and $\mathrm{L_{II}}$ from $\mathrm{L_{III}}$.
The linear differential equation $\mathrm{L_{II}}$ is written in the form:

$$\begin{cases} p_1(x) = -2x^2 - t - \dfrac{1}{x-\lambda}, \\ p_2(x) = -(2\alpha+1)x - 2\mathrm{H_{II}} + \dfrac{\mu}{x-\lambda}. \end{cases}$$

The Riemannian scheme reads:

$$\left\{\begin{array}{ccccc} x=\lambda & \overbrace{\qquad\quad x=\infty \qquad\quad}^{} & & & \\ 0 & 0 & 0 & 0 & \alpha+\frac{1}{2} \\ 2 & \frac{2}{3} & 0 & t & -\alpha+\frac{1}{2}. \end{array}\right.$$

$\mathrm{H_{IV}} \to \mathrm{H_{II}}$ The degeneration from $\mathrm{H_{IV}}$ to $\mathrm{H_{II}}$ and the confluence from $\mathrm{L_{IV}}$ to $\mathrm{L_{II}}$ are carried out according to the following scheme:

$$(\lambda,\ \mu,\ t,\ \mathsf{H}_{\mathrm{IV}})$$
$$\longrightarrow\Big(\varepsilon^{-3}(1+2^{2/3}\varepsilon^{2}\lambda),\ \ 2^{-2/3}\varepsilon\mu,\ \ -\varepsilon^{-3}(1-2^{-2/3}\varepsilon^{4}t),\ \ 2^{2/3}\varepsilon^{-1}\mathsf{H}_{\mathrm{II}}-\frac{1}{2}(2\alpha+1)\varepsilon^{-3}\Big);$$

$$\kappa_0\longrightarrow\frac{1}{2}\varepsilon^{-6},\quad \theta_\infty\longrightarrow-\frac{1}{2}(2\alpha+1),$$

and

$$x\longrightarrow\varepsilon^{-3}(1+2^{2/3}\varepsilon^{2}x),\qquad (\varepsilon\to 0).$$

$\mathsf{H}_{\mathrm{II}}\to\mathsf{H}_{\mathrm{I}}$ The procedure of the degeneration from H_{II} to H_{I} is slightly different from the others. First H_{II} degenerates to H_{I} as follows:

$$\Big(\lambda,\ \mu-\lambda^2-\frac{t}{2},\ t,\ \mathsf{H}_{\mathrm{II}}+\frac{\lambda}{2}+\frac{1}{8}t^2\Big)$$
$$\longrightarrow\Big(\varepsilon^{-5}(1+\varepsilon^{6}\lambda),\ \varepsilon^{-1}\mu,\ -6\varepsilon^{-10}\Big(1-\frac{1}{6}\varepsilon^{12}t\Big),\ \ \varepsilon^{-2}\Big(\mathsf{H}_{\mathrm{I}}-\frac{t}{2}\varepsilon^{-6}-\frac{3}{2}\varepsilon^{-18}\Big)\Big);$$

$$\alpha\longrightarrow 4\varepsilon^{-15},\qquad (\varepsilon\to 0).$$

To obtain the linear equation L_{I} from L_{II}, change the dependent variable y of L_{II} as:

$$y\longrightarrow\exp\Big(\frac{1}{3}x^3+\frac{1}{2}tx\Big)y,$$

and then replace x by $\varepsilon x+\varepsilon^{-5}$:

$$x\longrightarrow\varepsilon x+\varepsilon^{-5}.$$

In the limit $(\varepsilon\to 0)$ we have L_{I}, given by:

$$\begin{cases} p_1(x)=-\dfrac{1}{x-\lambda},\\[2mm] p_2(x)=-4x^3-2tx-2\mathsf{H}_{\mathrm{I}}+\dfrac{\mu}{x-\lambda}.\end{cases}$$

The linear differential equation L_{I} admit at $x=\infty$ formal solution of the form:

$$\exp\Big(\frac{4}{5}\xi^5+t\xi\Big)\xi^{-1/2}\sum_{n=0}^{\infty}g_n\xi^{-n},$$
$$\xi^2=x.$$

Therefore, we write the Riemannian scheme of L_{I} as follows:

$$\left\{\begin{array}{ccccccc} x=\lambda & \multicolumn{6}{c}{x=\infty\left(\frac{1}{2}\right)} \\ & \multicolumn{6}{c}{\overbrace{\hphantom{-\frac{4}{5}\quad 0\quad 0\quad 0\quad -t\quad \frac{1}{2}}}} \\ 0 & -\frac{4}{5} & 0 & 0 & 0 & -t & \frac{1}{2} \\ 2 & \frac{4}{5} & 0 & 0 & 0 & t & \frac{1}{2} \end{array}\right.$$

REMARK 4.3. The process of step-by-step degeneration of the Hamiltonians H_J is carried out by the successive canonical transformations with parameters.

A regular singular point is said to be apparent, if it is of the non-logarithmic type singularities with only integer characteristic exponents. The singular point $x=\lambda$ of the linear differential equation L_J is an apparent singularity. Consider a linear differential equation of the form (4.1) with N singular points, $x=t_j$ and N' apparent singularities, $x=\lambda_k$. We associate with each of $x=t_j$ $(j=1, \cdots, N)$ a rational number r_j such that the Poincaré rank of $x=t_j$ is given by r_j-1. Then we can represent such a linear differential equation by the following symbol:

$$(r_1+r_2+\cdots+r_N)_{N'}.$$

The linear differential equation considered in the studies on the Garnier system is of the type:

$$\underbrace{(1+1+\cdots+1)_N}_{N+3},$$

where a regular singular point is regarded as a singularity with the Poincaré rank zero. By the use of this notation, the confluence scheme of the linear differential equations L_J $(J=\mathrm{I}, \cdots, \mathrm{VI})$ associated with the canonical Hamiltonians is written in the following form:

$$\begin{tikzcd}[column sep=small, row sep=tiny] & & (2+2)_1 \arrow[dr] & & \\ (1+1+1+1)_1 \arrow[r] & (1+1+2)_1 \arrow[ur] \arrow[dr] & & (4)_1 \arrow[r] & \left(\frac{7}{2}\right)_1. \\ & & (1+3)_1 \arrow[ur] & & \end{tikzcd}$$

The degeneration scheme of the two-dimensional Garnier system is studied in [8] and [10].

REMARK 4.4. We define the τ-function related to the canonical Hamiltonian H_J by:

$$\mathsf{H}_J(t;\lambda(t),\mu(t))=\frac{d}{dt}\log\tau_J(t),$$

$(\lambda(t),\mu(t))$ being a solution of the Hamiltonian system (H). The τ-function $\tau_J(t)$ has not so much as a movable pole ([18]).

Finally we prove the following result:

PROPOSITION 4.3 (cf. [5]). *The third Painlevé equation* $\mathsf{P}_{\mathrm{III}}$ *with* $\gamma\delta\neq 0$ *is equivalent to the fifth equation* P_{V} *with* $\delta=0$.

PROOF. Since $\mathsf{P}_{\mathrm{III}}$ is equivalent to $\mathsf{P}'_{\mathrm{III}}$, we verify this proposition concerning $\mathsf{P}'_{\mathrm{III}}$. Let $\mathsf{H}'_{\mathrm{III}}$ be the canonical Hamiltonian associated with $\mathsf{P}'_{\mathrm{III}}$. By the assumption $\gamma\delta\neq 0$, we can put $\eta_\Delta=1$ $(\Delta=0,\infty)$ by changing the scales of t and λ, for we obtain by computations

$$\gamma=4\eta_\infty^2,\quad \delta=-4\eta_0^2.$$

It follows from the Hamiltonian system $(\mathrm{H})_1$ that

$$\mu=(2\lambda^2)^{-1}\left(t\frac{d\lambda}{dt}+\lambda^2+\theta_0\lambda+t\right),$$

$$\lambda=(2\mu-2\mu^2)^{-1}\left(t\frac{d\mu}{dt}-\theta_0\mu+\frac{1}{2}(\theta_0+\theta_\infty)\right).$$

Moreover, it is not difficult to verify, if we put

$$\lambda_1=\frac{\mu}{\mu-1},$$

that it satisfies the fifth Painlevé equation P_{V} such that

$$\alpha=\frac{1}{8}(\theta_0-\theta_\infty)^2,\quad \beta=-\frac{1}{8}(\theta_0+\theta_\infty)^2,\quad \gamma=2,\quad \delta=0.$$

REMARK 4.5. If $\gamma=\delta=0$ in $\mathsf{P}_{\mathrm{III}}$, substitute λ^2 for λ and t^2 for t. The resulting equation is:

$$\frac{d^2\lambda}{dt^2}=\frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt}+2\alpha\lambda^3+\frac{2\beta}{\lambda}.$$

Therefore, this substitution causes in $\mathsf{P}_{\mathrm{III}}$ the change of constants:

$$\alpha \longrightarrow 0, \quad \beta \longrightarrow 0, \quad 0 \longrightarrow 2\alpha, \quad 0 \longrightarrow 2\beta.$$

4.4. Case of the SL-type equations.

We shall give below the table of the SL-type linear differential equation SL_J and the Hamiltonians K_J. The process of step-by-step confluence of SL_J and degeneration of K_J will be carried out in a similar way to that of L_J and that of H_J respectively. The extended system of SL_J is as follows:

$$(4.8) \qquad \begin{aligned} &\frac{\partial^2 z}{\partial x^2} = p(x) z \\ &\frac{\partial z}{\partial t} = -\frac{1}{2}\frac{\partial \mathsf{A}(x)}{\partial x} z + \mathsf{A}(x)\frac{\partial z}{\partial x}. \end{aligned}$$

Given the rational function $\mathsf{A}(x)$ in the explicit form, we can verify that the complete integrability condition of (4.8) implies actually the Hamiltonian system (H) with the Hamiltonian K_J. We do not enter into details of computations. The list of the coefficients $\mathsf{A}(x)$ of (4.8) is included in the following table; for $\mathsf{SL}_{\mathrm{VI}}$, see (4.3) and (4.6).

Table of SL_J *and* K_J

SL_{V}: $$p(x) = \frac{a_0}{x^2} + \frac{a_1 t^2}{(x-1)^4} + \frac{a_2 t}{(x-1)^3} + \frac{a_\infty}{(x-1)^2} + \frac{3}{4(x-\lambda)^2} + \frac{t\mathsf{K}_{\mathrm{V}}}{x(x-1)^2} - \frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)},$$

$$\mathsf{A}(x) = \frac{\lambda-1}{t}\cdot\frac{x(x-1)}{x-\lambda},$$

$$\mathsf{K}_{\mathrm{V}} = \frac{\lambda(\lambda-1)^2}{t}\left[\nu^2 - \left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)\nu - \frac{a_0}{\lambda^2} - \frac{a_1 t^2}{(\lambda-1)^4} - \frac{a_2 t}{(\lambda-1)^3} - \frac{a_\infty}{(\lambda-1)^2}\right],$$

$$a_0 = -\frac{1}{2}\beta - \frac{1}{4}, \quad a_1 = -\frac{1}{2}\delta, \quad a_2 = -\frac{1}{2}\gamma, \quad a_\infty = \frac{1}{2}(\alpha+\beta) - \frac{3}{4}.$$

$\mathsf{SL}_{\mathrm{IV}}$: $$p(x) = \frac{a_0}{x^2} + \frac{\mathsf{K}_{\mathrm{IV}}}{2x} + a_1 + \left(\frac{x+2t}{4}\right)^2 + \frac{3}{4(x-\lambda)^2} - \frac{\lambda\nu}{x(x-\lambda)},$$

$$\mathsf{A}(x) = \frac{2x}{x-\lambda},$$

$$\mathsf{K}_{\mathrm{IV}} = 2\lambda\nu^2 - 2\nu - \frac{2a_0}{\lambda} - 2a_1\lambda - 2\lambda\left(\frac{\lambda^2+2t}{4}\right)^2,$$

$$a_0=-\frac{1}{8}\beta-\frac{1}{4}, \quad a_1=-\frac{1}{4}\alpha.$$

$\mathsf{SL}'_{\mathrm{III}}$: $$p(x)=\frac{a_0t^2}{x^4}+\frac{a'_0t}{x^3}+\frac{t\mathsf{K}'_{\mathrm{III}}}{x^2}+\frac{a'_\infty}{x}+a_\infty+\frac{3}{4(x-\lambda)^2}-\frac{\lambda\nu}{x(x-\lambda)},$$

$$\mathsf{A}(x)=\frac{\lambda x}{t(x-\lambda)},$$

$$\mathsf{K}'_{\mathrm{III}}=\frac{1}{t}\left[\lambda^2\nu^2-\lambda\nu-\frac{a_0t^2}{\lambda^2}-\frac{a'_0t}{\lambda}-a'_\infty\lambda-a_\infty\lambda^2\right],$$

$$a_0=-\frac{1}{16}\delta, \quad a'_0=-\frac{1}{8}\beta, \quad a_\infty=\frac{1}{16}\gamma, \quad a'_\infty=\frac{1}{8}\alpha.$$

$\mathsf{SL}_{\mathrm{III}}$: $$p(x)=\frac{a_0t^2}{x^4}+\frac{a'_0t}{x^3}+\frac{t\mathsf{K}_{\mathrm{III}}+\lambda\nu}{2x^2}+\frac{a'_\infty t}{x}+a_\infty t^2+\frac{3}{4(x-\lambda)^2}-\frac{\lambda\nu}{x(x-\lambda)},$$

$$\mathsf{A}(x)=\frac{2\lambda x}{t(x-\lambda)}+\frac{x}{t},$$

$$\mathsf{K}_{\mathrm{III}}=\frac{1}{t}\left[2\lambda^2\nu^2-3\lambda\nu-\frac{2a_0t^2}{\lambda^2}-\frac{2a'_0t}{\lambda}-2a'_\infty t\lambda-2a_\infty t^2\lambda^2\right],$$

$$a_0=-\frac{1}{16}\delta, \quad a'_0=-\frac{1}{8}\beta, \quad a_\infty=\frac{1}{16}\gamma, \quad a'_\infty=\frac{1}{8}\alpha.$$

$\mathsf{SL}_{\mathrm{II}}$: $$p(x)=x^4+tx^2+2\alpha x+2\mathsf{K}_{\mathrm{II}}+\frac{3}{4(x-\lambda)^2}-\frac{\nu}{x-\lambda},$$

$$\mathsf{A}(x)=\frac{1/2}{x-\lambda},$$

$$\mathsf{K}_{\mathrm{II}}=\frac{1}{2}\nu^2-\frac{1}{2}\lambda^4-\frac{1}{2}t\lambda^2-\alpha\lambda.$$

SL_{I}: $$p(x)=4x^3+2tx+2\mathsf{K}_{\mathrm{I}}+\frac{3}{4(x-\lambda)^2}-\frac{\nu}{x-\lambda},$$

$$\mathsf{A}(x)=\frac{1/2}{x-\lambda},$$

$$\mathsf{K}_{\mathrm{I}}=\frac{1}{2}\nu^2-2\lambda^3-t\lambda$$
$$=\mathsf{H}_{\mathrm{I}}.$$

REMARK 4.6 ([7]). We can obtain the Painlevé equations by considering the isomonodromic deformation of systems of differential equations of the form:

$$\frac{dY}{dx} = A(x)Y,$$

where $A(x)$ is a 2×2 matrix, rational in x.

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