博士論文

論文題目　Categories of operators for multicategories with various symmetries

(多彩な対称性を持つマルチ圏のオペレーターの圏)

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Contents

Introduction

In this thesis, we aim at new models of operads and multicategories which admit various symmetries including those of symmetric groups, of braid groups, and of ribbon braid groups. They are based on the idea of categories of operators proposed by May and Thomason [58]. We construct a 2-category $\mathbb{B}_{\mathcal{G}}$ for each group operad $\mathcal{G}$ in the sense of Zhang [76] and show that $\mathcal{G}$-symmetric multicategories give rise to 2-categories fibered over $\mathbb{B}_{\mathcal{G}}$. This point of view clarifies how symmetries on monoidal structures are used in defining the Hochschild homology and the homologies associated to crossed simplicial groups, which was introduced by Fiedorowicz and Loday [24], of associative algebras. In other words, we can parametrize the Hochschild homology of algebras over *noncommutative* bases. We expect that these techniques admit homotopy algebraic analogues and enables geometry of algebras over more general bases.

The central motivation lies in the geometry of higher algebras, which has been developed based on the duality of "spaces" and "algebras." One of the most important examples of dualities was pointed out by Gelfand and Naimark [26]; compact Hausdorff spaces corresponds in one-to-one to unital commutative C^*-algebras contravariantly. More precisely, for a compact Hausdorff space X, it is dual to the C^*-algebra $C(X) = C(X, \mathbb{C})$ of complex-valued continuous functions. The original space X is recovered as the space of maximal ideals of C^*. The point is that the duality enables us to do "geometry" not only on spaces but in the algebra side. For instance, it is an important problem in geometry to classify vector bundles. According to Serre-Swan Theorem [73], the category of finite-rank complex vector bundles over X is equivalent to that of finitely generated projective modules over $C(X)$ for every compact Hausdorff space X. This relates the complex topological K-theory KU of compact Hausdorff spaces to the (0-th) K-theory K_0 of commutative unital C^*-algebras. We also note that there are similar duality between affine varieties and commutative rings and correspondence of vector bundles to projective modules; the latter is due to Serre [69].

Connes' *noncommutative geometry* [14] is one of attempts to extending the meaning of "geometry;" namely he got rid of the assumption of the *commutativity* on algebras. Hochschild homology and the cyclic homology play important roles. We here recall the description of them. For a symmetric monoidal abelian category $\mathcal{C}$, one can think of an algebra object in $\mathcal{C}$ as a monoidal functor

$$A : \widetilde{\Delta} \to \mathcal{C} \tag{0.0.1}$$

out of the category $\widetilde{\Delta}$ of finite cardinals and arbitrary maps equipped with the disjoint unions as the monoidal structure. Thanks to the symmetric structure on $\mathcal{C}$, the functor (0.0.1) extends to a symmetric monoidal functor

$$A_{\mathfrak{S}} : \widetilde{\Delta}_{\mathfrak{S}} \to \mathcal{C} \ , \tag{0.0.2}$$

where $\widetilde{\Delta}_{\mathfrak{S}}$ is the *free symmetrization* of $\widetilde{\Delta}$ by the sequence of symmetric groups $\mathfrak{S} = \{\mathfrak{S}_n\}_n$. It turns out that the category $\widetilde{\Delta}_{\mathfrak{S}}$ has *Connes' cyclic category* Λ as a subcategory. Using the self-duality of Λ pointed out in [12], one obtains the composition

$$A^\natural : \Lambda^{\mathrm{op}} \cong \Lambda \hookrightarrow \widetilde{\Delta}_{\mathfrak{S}} \xrightarrow{A_{\mathfrak{S}}} \mathcal{C} \ . \tag{0.0.3}$$

As Λ is an extension of the simplex category Δ, in view of the Dold-Kan correspondence [16] [17] [44], the restriction $A^\natural|_{\Delta^{\mathrm{op}}} : \Delta^{\mathrm{op}} \to \mathcal{C}$ corresponds to the *cyclic bar resolution* of the algebra A. As a result, we obtain the Hochschild homology and the cyclic homology of A as

$$HH_*(A) = H_*(A^\natural|_{\Delta^{\mathrm{op}}}) \ , \quad HC_*(A) = H_*(\operatorname{hocolim} A^\natural) \ , \tag{0.0.4}$$

here the homotopy colimit is taken in the derived category $D(\mathcal{C})$. We are mainly interested in the case $\mathcal{C}$ is the category of modules over a commutative ring. In particular, if $\mathcal{C}$ is the category of vector spaces over a field k, and if A is smooth and finitely generated over k, we have Hochschild-Kostant-Rosenberg isomorphism [35]

$$\Omega^*_{A/k} \cong HH_*(A) \ ,$$

here $\Omega^*_{A/k}$ is the exterior algebra of Kähler differentials. Furthermore, in the same situation, the comparison of the differential on $\Omega_{A/k}$ and the "circle-action" on $HH_*(A)$ coming from the cyclic structure on $A^\natural$, one obtains an isomorphism

$$HC_n(A) \cong (\Omega^n_{A/k}/d\Omega^{n-1}_{A/k}) \oplus \bigoplus_{k \geq 1} H^{n-2k}_{\mathrm{dR}}(A) \ .$$

We here also mention that there are similar isomorphisms due to [13] for the de Rham cohomology on a smooth manifold X and the cyclic homology of the ring of complex-valued smooth functions. In this case, the circle-action on Hochschild homology can be seen in the isomorphisms given by Jones [38]: for a simply connected manifold X, we have

$$HH_*(\Omega^*_{\mathrm{dR}}(X)) \cong H^*(LX; \mathbb{R}) \ , \quad HC_*(\Omega^*_{\mathrm{dR}}(X)) \cong H^*(LX /\!/ S^1; \mathbb{R}) \ ,$$

here LX is the free loop space of X, and $LX /\!/ S^1$ is its homotopy quotient by the canonical circle action. Therefore, in noncommutative situations, we can make use of the Hochschild homology and the cyclic homology instead of the de Rham cohomology.

The perspective above is also convenient when the geometry of higher algebras is developed; yet what are higher algebras? In order to formulate it, it is convenient to describe algebraic structures in terms of *operads*. The notion first appeared implicitly in Stasheff's work [71] on the homotopy associativity, and the abstract definition was given by May in his book [57]. For a symmetric monoidal category $\mathcal{V}$, a *symmetric operad* $\mathcal{O}$ in $\mathcal{V}$ consists of a family $\mathcal{O} = \{\mathcal{O}(n)\}_n$ of objects of $\mathcal{V}$ together with an $\mathfrak{S}_n$-action on each $\mathcal{O}(n)$ and a composition operation

$$\gamma_{\mathcal{O}} : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$$

which is associative, unital, and compatible with $\mathfrak{S}_n$-actions. More generally, there is a "colored" version of operads: a $\mathcal{V}$-enriched *symmetric multicategory* is similarly defined while it is indexed with finite sequences of "objects" instead of natural numbers. When $\mathcal{V}$ is omitted, the enrichment in sets is understood. For example, symmetric operads can be seen as symmetric multicategories with single objects, and every symmetric monoidal category $\mathcal{C}$ gives rise to a symmetric multicategory $\mathcal{C}^{\otimes}$. The latter assignment in particular defines a 2-functor $\mathbf{MonCat}_{\mathfrak{S}} \to \mathbf{MultCat}_{\mathfrak{S}}$ from the 2-category of symmetric monoidal categories, symmetric monoidal functors, and monoidal natural transformations to that of symmetric multicategories, symmetric multifunctors, and multinatural transformations. Then, for a symmetric operad $\mathcal{O}$ and a symmetric monoidal category $\mathcal{C}$, $\mathcal{O}$-algebras in $\mathcal{C}$ may be defined as multifunctors $\mathcal{O} \to \mathcal{C}^{\otimes}$, which form a category $\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) := \mathbf{MultCat}_{\mathfrak{S}}(\mathcal{O}, \mathcal{C}^{\otimes})$. If we write $\mathbf{Assoc}_{\mathfrak{S}}$ the symmetric operad with each $\mathbf{Assoc}(n)$ being an $\mathfrak{S}_n$-torsor, then $\mathbf{Assoc}_{\mathfrak{S}}$-algebras are nothing but associative algebra objects.

There are mainly two approaches to higher algebras using the theory of operads. One uses *cofibrant operads*: if the enriching category $\mathcal{V}$ admits a model structure in the sense of Quillen [65] which is good enough, then the category of operads in $\mathcal{V}$ also admits a model structure [5]. In the case $\mathcal{V}$ is the category of topological spaces or chain complexes, cofibrant replacements have been discussed by many authors after Stasheff's invention of the associahedra. We would however use the other approach, namely the *category of operators*. The concept was first discussed by May and Thomason [58]. We put Γ the opposite category of a skeleton of the category of finite pointed sets. Then, categories of operators are categories fibered over Γ^{op} with certain *universal lifting properties*. It was pointed out by May and Thomason, and later by Lurie [54], that each symmetric multicategory gives rise to a category of operators satisfying a version of *Segal condition*. It was shown in [58] that, if $\mathcal{K} \to \Gamma^{\mathrm{op}}$ and $\mathcal{C}^{\triangledown\mathfrak{S}} \to \Gamma^{\mathrm{op}}$ are categories of operators associated to a symmetric operad $\mathcal{O}$ and a symmetric monoidal category $\mathcal{C}$ respectively, then $\mathcal{O}$-algebras in $\mathcal{C}$ correspond in one-to-one to functors $\mathcal{K} \to \mathcal{C}^{\triangledown\mathfrak{S}}$ over Γ^{op} preserving universal liftings. According to Lurie [54], the argument actually goes well not only in the ordinary category theory but also in the theory of quasi-categories, which are models of ∞-categories, developed by Boardman-Vogt [6], Joyal [40] [41], and Lurie [53]. In the latter situation, one obtains the notion of "∞-operads" and their "∞-algebras."

In terms of categories of operators, Hochschild homology of algebras is given as follows: First, in view of the *representability* of multicategories considered by Hermida [33], symmetric monoidal categories correspond to Grothendieck opfibrations over Γ^{op}. Lurie gave a construction of a Grothendieck opfibration $\mathrm{Env}_{\mathfrak{S}}(\mathcal{K}) \to \Gamma^{\mathrm{op}}$ for every category of operators $\mathcal{K} \to \Gamma^{\mathrm{op}}$ and showed that it is the *free symmetric monoidal category* generated by the operad associated with $\mathcal{K}$; in other words, if $\mathcal{C}^{\triangledown\mathfrak{S}} \to \Gamma^{\mathrm{op}}$ is a Grothendieck opfibration associated with a symmetric monoidal category $\mathcal{C}$, maps $\mathcal{K} \to \mathcal{C}^{\triangledown\mathfrak{S}}$ of categories of operators correspond in one-to-one to maps $\mathrm{Env}_{\mathfrak{S}}(\mathcal{K}) \to \mathcal{C}^{\triangledown\mathfrak{S}}$ of Grothendieck opfibrations. In particular, taking $\mathcal{K}$ to be the category of operators associated to $\mathbf{Assoc}_{\mathfrak{S}}$, one can regard an algebra object A in $\mathcal{C}$ as a functor $A^{\triangledown} : \mathrm{Env}_{\mathfrak{S}}(\mathcal{K}) \to \mathcal{C}^{\triangledown\mathfrak{S}}$ over Γ^{op}. We denote by $\langle 1 \rangle_+ \in \Gamma^{\mathrm{op}}$ the object consisting of exactly one element except the base. Then, it is verified that the fiber $\mathcal{C}^{\triangledown\mathfrak{S}}_{\langle 1 \rangle_+}$ is identified with $\mathcal{C}$ itself. Moreover, the cyclic category Λ admits a canonical functor $\Lambda \to \mathrm{Env}_{\mathfrak{S}}(\mathcal{K})_{\langle 1\rangle_+}$. In the case $\mathcal{C}$ is an ordinary category and A is an ordinary algebra, the composition

$$\Lambda^{\mathrm{op}} \cong \Lambda \to \mathrm{Env}_{\mathfrak{S}}(\mathcal{K})_{\langle 1\rangle_+} \xrightarrow{A^{\triangledown}} \mathcal{C}^{\triangledown\mathfrak{S}}_{\langle 1\rangle_+} \cong \mathcal{C}$$

coincides with the functor $A^{\natural}$ given in (0.0.3) so that we can define the Hochschild homology and the cyclic homology provided $\mathcal{C}$ is abelian. Hence, if $\mathcal{C}$ is a symmetric monoidal stable ∞-category in the sense in [54], we obtain the Hochschild homology for higher algebras, and geometry of higher algebras, or the *derived algebraic geometry*, is enabled. In particular, if $\mathcal{C} = \mathcal{S}pc$ is the ∞-category of spectra, the Hochschild homology is called the *topological Hochschild homology*, which is first considered by Bökstedt [7].

In the argument above, we always assume the base category $\mathcal{C}$ is symmetric. From the geometric point of view, it reflects the assumption that the base ring of algebras is commutative. On the other hand, there are some examples of algebraic structures on noncommutative bases studied especially in higher algebras and in quantum mathematics. In the first area, May [57] considered levels of homotopy commutativity of products on spaces; namely, if $\mathbb{E}_k$ denotes the little k-disks operad, then an $\mathbb{E}_k$-algebra in the category of topological spaces, or an $\mathbb{E}_k$*-space* briefly, is a homotopy associative topological monoid which is commutative in homotopy k-types. For a few k, the commutativity of $\mathbb{E}_k$-algebras is classically known as below:

$$\begin{array}{ccccccc} \mathbb{E}_1 & \hookrightarrow & \mathbb{E}_2 & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{E}_\infty \\ \text{associative} & & \text{braided} & & \cdots & & \text{commutative} \end{array}.$$

According to the *stabilization hypothesis* [2], which was proved in [27], for an $\mathbb{E}_k$-algebra A, modules over A form an (∞-)category which is not fully symmetric but $\mathbb{E}_l$-monoidal for some $l \leq k-1$. Hence, in order to consider geometry with base A, we will need the Hochschild homology of algebras in $\mathbb{E}_k$-monoidal categories. Also, in quantum mathematics, the symmetry of braid groups appears frequently. For instance, for a Hopf algebra H, the category H-**Mod** of left H-modules admits a canonical monoidal structure, where algebra objects are called *H-module algebras* according to [61]. It is known that symmetries on the monoidal structure on H-**Mod** may be encoded as special elements of tensor products of H. In particular, according to [11], braidings on H-**Mod** precisely correspond to elements of $H \otimes H$ called *R-matrices.* A Hopf algebra H is said to be *quasitriangular* if it is equipped with a fixed R-matrix. The quantum groups $U_{\mathfrak{q}}(\mathfrak{g})$ are important examples [18]. Thus, for quasitriangular Hopf algebra H, geometry of H-module algebras will requires Hochschild homology with braided base category.

Fortunately, there is a framework to define some sort of symmetries on multicategories in the realm of ordinary category theory. The key observation is that the family $\mathfrak{S} = \{\mathfrak{S}_n\}_n$ of symmetric groups admits a canonical structure of (planar) operads. Zhang introduced in his paper [76] the notion of *group operads* as generalizations of the operad $\mathfrak{S}$, though the axioms were already stated by Wahl [75]. An operad $\mathcal{G}$ is called a group operad if each set $\mathcal{G}(n)$ is equipped with a structure of a group and a group homomorphism $\mathcal{G}(n) \to \mathfrak{S}_n$ satisfying a compatibility condition with the operad structure. The examples include the operad $\mathcal{B}$ of braid groups and $\mathcal{RB}$ of ribbon braid groups as well as

$\mathfrak{S}$ itself. The important feature is, as pointed out in [15] and in [31], that each group operad $\mathcal{G}$ gives rise to a notion of $\mathcal{G}$-symmetries on (non-symmetric) multicategories. Namely, a $\mathcal{G}$-symmetric structure on a multicategory $\mathcal{M}$ is given as a right action

$$\left(\coprod_{\sigma\in\mathfrak{S}_n}\mathcal{M}(a_{\sigma(1)}\dots a_{\sigma(n)};a)\right)\times\mathcal{G}(n)\to\left(\coprod_{\sigma\in\mathfrak{S}_n}\mathcal{M}(a_{\sigma(1)}\dots a_{\sigma(n)};a)\right)$$

for objects $a,a_1,\dots,a_n\in\mathcal{M}$ with appropriate coherence conditions. A multifunctor is said to be *$\mathcal{G}$-symmetric* provided it preserves $\mathcal{G}$-symmetric structures. We obtain a 2-category $\mathbf{MultCat}_{\mathcal{G}}$ of $\mathcal{G}$-symmetric multicategories, $\mathcal{G}$-symmetric multifunctors, and multinatural transformations. Since every monoidal category $\mathcal{C}$ gives rise to a multicategory $\mathcal{C}^{\otimes}$ in the same way as the symmetric case, we can also consider $\mathcal{G}$-symmetric structures on monoidal categories. Thus, the notion of $\mathcal{G}$-symmetric monoidal categories arises so that they form a 2-category $\mathbf{MonCat}_{\mathcal{G}}$ with a locally fully faithful and conservative functor $(-)^{\otimes}:\mathbf{MonCat}_{\mathcal{G}}\to\mathbf{MultCat}_{\mathcal{G}}$.

Looking ahead to the higher algebraic analogue, we are interested in categories of operators for $\mathcal{G}$-symmetric multicategories for arbitrary group operads $\mathcal{G}$, which is exactly the main theme of this thesis. We begin with the non-symmetric case. For non-symmetric operads and multicategories, we use the category ∇ instead of Γ^{op}, where ∇ is defined such that

- objects are linearly ordered sets of the form below for $n\in\mathbb{N}$:

$$\langle\!\langle n\rangle\!\rangle:=\{-\infty,1,\dots,n,\infty\}\ ;$$

- morphisms are order-preserving maps which sends $\pm\infty$ to $\pm\infty$ respectively.

Since there is a canonical functor $\nabla\to\Gamma^{\mathrm{op}}$ with $\langle\!\langle n\rangle\!\rangle\mapsto\langle\!\langle n\rangle\!\rangle/\{\pm\infty\}\cong\langle n\rangle_+$, it derives the *universal lifting problems* for fibrations from Γ^{op}. In addition, since ∇ is isomorphic to the opposite of the simplex category Δ by [39], we can also consider *Segal condition* on categories over ∇. We can then define a category of non-symmetric operators, as an analogue of one in [58] and [54], as a functor $p:\mathcal{K}\to\nabla$ such that

(i) it admits coCartesian lifts for *inert morphisms* in ∇, that is, morphisms $\varphi:\langle\!\langle m\rangle\!\rangle\to\langle\!\langle n\rangle\!\rangle$ with $\varphi^{-1}\{1,\dots,n\}\to\{1,\dots,n\}$ bijective;

(ii) for the inert morphisms $\rho_i:\langle\!\langle n\rangle\!\rangle\to\langle\!\langle 1\rangle\!\rangle$ with $\rho_i(j)=1$ precisely when $j=i$, the square below induced by coCartesian lifts $\widehat{\rho}_i:X\to X_i\in\mathcal{K}$ is a pullback:

$$\begin{array}{ccc}\mathcal{K}(W,X) & \longrightarrow & \prod_{i=1}^n\mathcal{K}(W,X_i)\\ \downarrow & \lrcorner & \downarrow\\ \nabla(p(W),\langle\!\langle n\rangle\!\rangle) & \longrightarrow & \nabla(p(W),\langle\!\langle 1\rangle\!\rangle)^{\times n}\end{array}\ .$$

(iii) Segal condition: the functor $\mathcal{K}_{\langle\!\langle n\rangle\!\rangle}\to\mathcal{K}_{\langle\!\langle 1\rangle\!\rangle}^{\times n}$ induced by the coCartesian lifts of the inert morphisms $\rho_1,\dots,\rho_n:\langle\!\langle n\rangle\!\rangle\to\langle\!\langle 1\rangle\!\rangle$ is an equivalence of categories.

As a result, we can exhibit (non-symmetric) operads and multicategories as categories fibered over ∇ and see they form a subcategory of the slice 2-category $\mathbf{Cat}^{/\nabla}$.

For categories of operators of $\mathcal{G}$-symmetric multicategories, we need to find a category $\mathbb{B}_{\mathcal{G}}$ which places the position of Γ^{op} or ∇ in the ($\mathfrak{S}$-)symmetric or in the non-symmetric case. At this stage, things get involved with the higher category theory. Indeed, it turns out that, if $\mathcal{G}$ is neither trivial nor $\mathfrak{S}$, $\mathbb{B}_{\mathcal{G}}$ becomes not an ordinary category but a 2-category actually. We denote by $\mathbf{Cat}_2$ the 3-category of 2-categories, normalized pseudofunctors, pseudonatural transformations, and modifications. The main results are then stated as follows.

Theorem A. *Let $\mathcal{G}$ be a group operad. Then, there are 3-subcategories*

$$\mathbf{RepOper}^{\mathsf{geom}}_{\mathcal{G}} \subset \mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}} \subset \mathbf{Cat}_2^{/\mathbb{B}_{\mathcal{G}}}$$

of the strict slice over $\mathbb{B}_{\mathcal{G}}$ with a commutative square

$$\begin{array}{ccc} \mathbf{MonCat}_{\mathcal{G}} & \longrightarrow & \mathbf{RepOper}^{\mathsf{geom}}_{\mathcal{G}} \\ {\scriptstyle (-)^{\otimes}}\downarrow & & \downarrow \\ \mathbf{MultCat}_{\mathcal{G}} & \xrightarrow{(-)^{\nabla\mathcal{G}}} & \mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}} \end{array}$$

with horizontal arrows triequivalences.

Theorem B. *For every group operad $\mathcal{G}$, the inclusion*

$$\mathbf{RepOper}^{\mathsf{geom}}_{\mathcal{G}} \hookrightarrow \mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}}$$

admits a left adjoint

$$\mathrm{Env}_{\mathcal{G}}(-) : \mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}} \to \mathbf{RepOper}^{\mathsf{geom}}_{\mathcal{G}} \ .$$

As in the symmetric or non-symmetric case, our models above of categories of operators admit straightforward higher analogue. Indeed, one can relax the "truncatedness" in the definition of the 3-category $\mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}}$; cf. covering spaces v.s. fiber bundles over BG for a group G. Furthermore, it will turn out that it provides a coherent way to define Hochschild homology for algebras in $\mathcal{G}$-symmetric monoidal categories. Indeed, for a $\mathcal{G}$-symmetric monoidal abelian category $\mathcal{C}$, the theorems above give the following sequence of biequivalences:

$$\begin{aligned} \mathbf{Alg}(\mathcal{C}) &\simeq \mathbf{MultCat}_{\mathcal{G}}(\mathcal{G}, \mathcal{C}^{\otimes}) \\ &\simeq \mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}}(\mathcal{G}^{\nabla\mathcal{G}}, (\mathcal{C}^{\otimes})^{\nabla\mathcal{G}}) \\ &\simeq \mathbf{RepOper}^{\mathsf{geom}}_{\mathcal{G}}(\mathrm{Env}_{\mathcal{G}}(\mathcal{G}^{\nabla\mathcal{G}}), (\mathcal{C}^{\otimes})^{\nabla\mathcal{G}}) \ . \end{aligned} \tag{0.0.5}$$

We have a canonical identification $(\mathcal{C}^{\otimes})^{\nabla\mathcal{G}}_{\langle\langle 1\rangle\rangle} \cong \mathcal{C}$, and the direct computation shows that there is an ordinary category $\widetilde{\Delta}_{\mathfrak{J}^\natural\mathcal{G}}$ with $\mathrm{Env}_{\mathcal{G}}(\mathcal{G}^{\nabla\mathcal{G}})_{\langle\langle 1\rangle\rangle} \simeq \widetilde{\Delta}_{\mathfrak{J}^\natural\mathcal{G}}$. Thus, taking the fibers over $\langle\langle 1\rangle\rangle \in \mathbb{B}_{\mathcal{G}}$ in the last term of (0.0.5), one obtains a pseudofunctor

$$(-)^{\natural} : \mathbf{Alg}(\mathcal{C}) \to \mathbf{Cat}_2(\widetilde{\Delta}_{\mathfrak{J}^\natural\mathcal{G}}, \mathcal{C}) \ .$$

Therefore, putting Λ_∞ the *paracyclic category* [23], we obtain

$$\begin{aligned}\mathbf{Alg}(\mathcal{C}) \times \mathbf{Cat}^{\Delta/}(\Lambda_\infty, \widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}}) &\xrightarrow{(-)^\natural \times \mathrm{Id}} \mathbf{Cat}(\widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}}, \mathcal{C}) \times \mathbf{Cat}(\Lambda_\infty, \widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}}) \\ &\xrightarrow{\text{comp.}} \mathbf{Cat}(\Lambda_\infty, \mathcal{C}) \simeq \mathbf{Cat}(\Lambda_\infty^{\mathrm{op}}, \mathcal{C}) \to \mathbf{Cat}(\Delta^{\mathrm{op}}, \mathcal{C}) \ ,\end{aligned} \tag{0.0.6}$$

where $\mathbf{Cat}^{\Delta/}$ is the 2-category of functors out of Δ. We note that the cyclic category Λ is a quotient category of Λ_∞ so that the self-duality $\Lambda \cong \Lambda^{\mathrm{op}}$ comes from the isomorphism $\Lambda_\infty \cong \Lambda_\infty^{\mathrm{op}}$ constructed by Elmendorf [23], so (0.0.6) is regarded as the *parametrized cyclic resolution.* As a result, we obtain a family of Hochschild homologies of each algebra A in $\mathcal{C}$. As for the parameter category $\mathbf{Cat}^{\Delta/}(\Lambda_\infty, \widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}})$, since Λ_∞ is the total category of a crossed simplicial group as pointed out by Fiedorowicz and Loday [24], the structure of $\mathbf{Cat}^{\Delta/}(\Lambda_\infty, \widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}})$ is deeply related to the adjunctions in the following theorem.

Theorem C. *There are adjunctions*

$$\mathbf{CrsGrp}_{\Delta}^{/\boldsymbol{\mathcal{S}}} \rightleftarrows \mathbf{CrsGrp}_{\widetilde{\Delta}}^{/\mathfrak{S}} \underset{\mathfrak{J}^\natural}{\rightleftarrows} \mathbf{GrpOp}$$

between the slice categories of crossed simplicial groups and of augmented crossed simplicial groups over the ones consisting of the symmetric groups and the category of group operads.

The notion of crossed groups was introduced by Fiedorowicz and Loday [24] and Krasauskas [47] independently in the simplicial case, and it is easily generalized over arbitrary base category. It was pointed out that the paracyclic category Λ_∞ is the total category of a crossed simplicial group $\boldsymbol{\mathcal{Z}}$, and we have

$$\mathbf{Cat}^{\Delta/}(\Lambda_\infty, \widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}}) \cong \mathbf{CrsGrp}_{\Delta}(\boldsymbol{\mathcal{Z}}, \mathfrak{J}^\natural \mathcal{G}) \ . \tag{0.0.7}$$

In the case $\mathcal{G} = \mathfrak{S}$, the augmented crossed simplicial group $\mathfrak{J}^\natural \mathfrak{S}$ is an augmented crossed simplicial subgroup of the terminal object in $\mathbf{CrsGrp}_{\widetilde{\Delta}}$ so the both sides in (0.0.7) consists of a unique element. The composition (0.0.6) is hence identified with a functor $\mathbf{Alg}(\mathcal{C}) \to \mathbf{Cat}(\Delta^{\mathrm{op}}, \mathcal{C})$ which is exactly the usual *cyclic resolution* (0.0.3). Furthermore, it is also an interesting problem to consider maps $f : G \to \mathfrak{J}^\natural \mathcal{G}$ of crossed simplicial groups for a general crossed simplicial group G. Using the formula

$$\mathbf{Cat}^{\Delta/}(\Delta_G, \widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}}) \cong \mathbf{CrsGrp}_{\Delta}(G, \mathfrak{J}^\natural \mathcal{G})$$

analogous to (0.0.7), we obtain the *associated homology* defined in [24] as the functor

$$HG^f(-) : \mathbf{Alg}(\mathcal{C}) \xrightarrow{f^*(-)^\natural} \mathbf{Cat}(\Delta_G, \mathcal{C}) \xrightarrow{\mathrm{hocolim}} D(\mathcal{C})$$

into the derived category $D(\mathcal{C})$, where the last functor takes the homotopy colimits of diagrams. It would be an exciting problem to consider their geometric interpretation; such as the relation to the free loop spaces described in [38] or [52] in the commutative case. In particular the case $\mathcal{C}$ is the category of modules over quasitriangular Hopf algebra, explicit computations may be possible, which would be a future task.

We will establish the theorems above in the following way: after reviewing in Chapter 1 the basic notions and results we need in the later sections, in Chapter 2, we establish the basic theory of crossed groups. For a small category $\mathcal{A}$, a crossed $\mathcal{A}$-group is an $\mathcal{A}$-set G with a group structure on each $G(a)$ for $a \in \mathcal{A}$ which is compatible in a *crossed* sense with the $\mathcal{A}$-set structure. Note that the notion contains $\mathcal{A}$-groups in the usual sense as a subclass. In the cases $\mathcal{A} = \Delta, \widetilde{\Delta}, \nabla$, we obtain the notions of crossed simplicial groups, of augmented crossed simplicial groups, and of crossed interval groups, where the third terminology is due to [3]. Although crossed simplicial groups have been investigated by many authors after [24] and [47], the general theory of crossed groups over arbitrary base categories seems not to be well-studied. Above all, we will verify that the category $\mathbf{CrsGrp}_{\mathcal{A}}$ of crossed $\mathcal{A}$-groups is locally presentable and admits all small limits and colimits for every base category $\mathcal{A}$. This result has some important consequences. First, it shows that there is the terminal crossed $\mathcal{A}$-group $\mathfrak{T}_{\mathcal{A}}$. Since the category $\mathbf{CrsGrp}_{\mathcal{A}}$ admits pointwise kernels and cokernels, for every crossed $\mathcal{A}$-group G, the unique map $f : G \to \mathfrak{T}_{\mathcal{A}}$ induces a pointwise exact sequence

$$* \to G^{\mathsf{nc}} \hookrightarrow G \twoheadrightarrow G^{\mathsf{red}} \to *$$

with

- G^{nc} is a (non-crossed) $\mathcal{A}$-group;
- G^{red} is a crossed $\mathcal{A}$-subgroup of the terminal crossed $\mathcal{A}$-group $\mathfrak{T}_{\mathcal{A}}$.

Thus, we obtain a classification of crossed $\mathcal{A}$-groups up to non-crossed ones by computing all the crossed $\mathcal{A}$-subgroups of $\mathfrak{T}_{\mathcal{A}}$. In fact, in the simplicial case, they were all computed in [24], and it will see that we also have the straightforward analogue in augmented case. In addition, we will achieve the classification in the interval case using the crossed analogue of *Goursat Lemma*. Second, the local presentability enables us to use *General Adjoint Functor Theorem*. Hence, we obtain the basechange adjunctions

$$\Phi^{\widetilde{G}}_{\flat}, \Phi^{\widetilde{G}}_{\sharp} : \mathbf{CrsGrp}^{/\Phi^*\widetilde{G}}_{\mathcal{A}} \underset{\perp}{\overset{\perp}{\rightleftarrows}} \mathbf{CrsGrp}^{/\widetilde{G}}_{\widetilde{\mathcal{A}}} : \Phi^{\natural}_{\widetilde{G}} .$$

on the slice categories over $\widetilde{G} \in \mathbf{CrsGrp}^{/\widetilde{G}}_{\widetilde{\mathcal{A}}}$ along a faithful functor $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ which is stable under the action of $\widetilde{G}$. Actually, we obtain not only the existence of adjunctions but also explicit formulas. In particular, the canonical functors $\Delta \to \widetilde{\Delta} \to \nabla$ induce the adjunctions in more or less computable forms.

In Chapter 3, we will prove that the category $\mathbf{GrpOp}$ of group operads can be fully faithfully embedded into the category $\mathbf{CrsGrp}_{\nabla}$ of crossed interval groups. Moreover, The explicit computation shows that it exhibits $\mathbf{GrpOp}$ as a reflective subcategory of $\mathbf{CrsGrp}_{\nabla}$; i.e. there is a functor

$$\mathbf{CrsGrp}_{\nabla} \to \mathbf{GrpOp} \ ; \quad G \mapsto \mathcal{O}_G$$

which is left adjoint to the embedding. Combining with the basechange adjunctions, we obtain Theorem C.

As for the other theorems, we will prove them in the last two chapters. In view of the results obtained in the previous chapters, we will regard a group operad $\mathcal{G}$ as a crossed interval group throughout these chapters. Then, two models of categories of operators for $\mathcal{G}$-symmetric multicategories will be proposed. In Chapter 4, we will construct a category $\widetilde{\mathbb{E}}_{\mathcal{G}}$ which is a quotient of the total category $\nabla_{\mathcal{G}}$. It will be also seen that it is the horizontal part of the double groupoid, or the "object" of an internal category

$$\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}} \ . \tag{0.0.8}$$

Then, we will define a *category of algebraic $\mathcal{G}$-operators* as a category $\mathcal{C}$ over $\widetilde{\mathbb{E}}_{\mathcal{G}}$ such that

(i) it satisfies the three conditions similar to the ones on categories of non-symmetric operators above;

(ii) it admits a structure of an *internal presheaf* over the internal category $\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}}$; i.e. a functor
$$\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathcal{C}$$
which is unital and associative.

We write $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}$ the 2-category of categories of algebraic $\mathcal{G}$-operators. We will also see that every multicategory $\mathcal{M}$ gives rise to a category $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ over $\widetilde{\mathbb{E}}_{\mathcal{G}}$ and that $\mathcal{G}$-symmetric structures on $\mathcal{M}$ can be presented as internal presheaf structures. This observation gives rise to a 2-functor $\mathbf{MultCat}_{\mathcal{G}} \to \mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}$, and we will prove it is actually a biequivalence.

On the other hand, in Chapter 5, we will construct the other model using so-called *internal Grothendieck construction.* Namely, for a category of algebraic $\mathcal{G}$-operators $\mathcal{C}$, the action and the projection defines a double category

$$\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \mathcal{C} \ . \tag{0.0.9}$$

Notice that since the double category (0.0.8) has the discrete vertical category, so does (0.0.9). Hence, we can see (0.0.9) as a 2-category, which we write $\mathcal{C} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$. In particular, we put

$$\mathbb{B}_{\mathcal{G}} := \widetilde{\mathbb{E}}_{\mathcal{G}} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \ ,$$

and we call it the *classifying category of $\mathcal{G}$*. Using the bicategorical analogue of the theory of coCartesian morphisms developed in [10], one can see the three conditions on categories of operators still make sense for 2-categories over $\mathbb{B}_{\mathcal{G}}$, and we obtain the notion of *categories of geometric $\mathcal{G}$-operators*, which form a 2-*truncated* 3-category $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$. The 2-functor given by the internal Grothendieck fibration causes a triequivalence

$$(-) /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} : \mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}} \to \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}} \ .$$

Hence, we obtain a biequivalence $\mathbf{MultCat}_{\mathcal{G}} \simeq \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$. Moreover, we will see the notion of *representability* on multicategories introduced by Hermida [33] corresponds to the coCartesian lifting property of arbitrary morphisms, and biequivalence above restricts to that between the 2-category $\mathbf{RepMultCat}_{\mathcal{G}}$ of representable $\mathcal{G}$-symmetric multicategories and the 3-subcategory $\mathbf{RepOper}_{\mathcal{G}}^{\mathsf{geom}}$ of $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$ consisting of Grothendieck opfibrations over $\mathbb{B}_{\mathcal{G}}$. This result is exactly Theorem A since $\mathbf{MonCat}_{\mathcal{G}} \simeq \mathbf{RepMultCat}_{\mathcal{G}}$. This observation also provides a recipe to construct the left adjoint $\mathrm{Env}_{\mathcal{G}} : \mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}} \to \mathbf{RepOper}^{\mathsf{geom}}_{\mathcal{G}}$ in Theorem B.

It is worth noting that, regarding models of categories of operators, we use the terminologies *algebraic* and *geometric* in the same meaning as in the theory of models for homotopy theories and higher categories [51] [70]. In this theory, structures in a model are realized either as additional data or as a sort of properties. We say the medel is algebraic in the first case while geometric otherwise. This is why we say objects in $\mathbf{Oper}^{\mathsf{alg}}_{\mathcal{G}}$ are algebraic; their symmetries associated with $\mathcal{G}$ are presented in an algebraic way; i.e. by the functor $\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathcal{C}$. On the other hand, the $\mathcal{G}$-symmetries on objects of $\mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}}$ are presented by the lifting properties on the action groupoid $\mathfrak{S}_n /\!\!/ \mathcal{G}(n)$ for $n \in \mathbb{N}$. We also mention that the composition structures on both models of categories of operators can be said to be geometric in the sense above. In this point of view, our models are compared with the one Gurski proposed in [31] as in the following table.

	composition	**symmetry**
$\mathcal{G}$-symmetric multicategory	algebraic	algebraic
$\mathbf{Mnd}_d(\mathbf{Kl}_{\widetilde{E\mathcal{G}}})$ [31]	algebraic	geometric
$\mathbf{Oper}^{\mathsf{alg}}_{\mathcal{G}}$	geometric	algebraic
$\mathbf{Oper}^{\mathsf{geom}}_{\mathcal{G}}$	geometric	geometric

It also shows the advantage of our models in view of higher algebras. In particular, since categories of geometric $\mathcal{G}$-operators are fully geometric models, it is expected that there are ∞-categorical analogues.

Acknowledgment

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Chapter 1

Preliminaries

1.1 2-categories

In this paper, we use the notions and terminologies coming from the 2-category theory, so we quickly review them. We essentially follow the literature [50]

Definition. A 2-*category* is a category enriched in the category **Cat** of small categories and functors equipped with the Cartesian monoidal structure. More precisely, a 2-category $\mathcal{K}$ consists of the following data

- a collection $\operatorname{Ob}\mathcal{K}$, whose members are called *objects of* $\mathcal{K}$;
- for each pair U and V of objects of $\mathcal{K}$, a small category $\mathcal{K}(U,V)$, whose objects are called 1-*morphisms* and whose morphisms 2-*morphisms*;
- for each triple U, V, and W of objects of $\mathcal{K}$, a functor

$$\gamma : \mathcal{K}(V,W) \times \mathcal{K}(U,V) \to \mathcal{K}(U,W) \ ;$$

- for each object $U \in \mathcal{K}$, and object $\mathrm{id}_U \in \mathcal{K}(U,U)$;

such that the diagrams

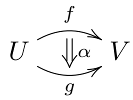
$$\begin{array}{ccc}
\mathcal{K}(V,W) \times \mathcal{K}(U,V) \times \mathcal{K}(T,U) & \xrightarrow{\mathrm{Id}\times\gamma} & \mathcal{K}(V,W) \times \mathcal{K}(T,V) \\
{\scriptstyle \gamma\times\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \gamma} \\
\mathcal{K}(U,W) \times \mathcal{K}(T,U) & \longrightarrow & \mathcal{K}(T,W)
\end{array} \tag{1.1.1}$$

$$\begin{array}{ccc}
\mathcal{K}(U,V) \times \{\mathrm{id}_U\} & \hookrightarrow & \mathcal{K}(U,V) \times \mathcal{K}(U,U) \\
& \searrow \quad \swarrow{\scriptstyle \gamma} & \\
& \mathcal{K}(U,V) &
\end{array} \tag{1.1.2}$$

$$\begin{array}{ccc}
\{\mathrm{id}_V\} \times \mathcal{K}(U,V) & \hookrightarrow & \mathcal{K}(V,V) \times \mathcal{K}(U,V) \\
& \searrow \quad \swarrow{\scriptstyle \gamma} & \\
& \mathcal{K}(U,V) &
\end{array} \tag{1.1.3}$$

Remark 1.1.1. There is a weaker notion of 2-categories; namely *bicategories*. In bicategories, the associativity of 1-morphisms holds only up to specific invertible 2-morphisms which is coherent in an appropriate sense. In fact, from the ∞-categorical point of view, many authors think of bicategories as *right* models for two dimensional categories. Nevertheless, we stick to the strict notion since it is enough for our purpose.

If U and V are objects in a 2-category $\mathcal{K}$, then we represent a morphism $\alpha : f \to g \in \mathcal{K}(U, V)$ by a "2-cell"



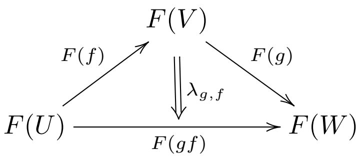
in a diagram or by $\alpha : f \to g : U \to V$ in one-liner. The former notation is sometimes called the *pasting diagram of the* 2*-morphism* α.

Example 1.1.2. Let $\mathcal{C}$ be an ordinary category. Then, we can see $\mathcal{C}$ as a 2-category with only trivial 2-morphisms.

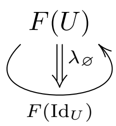
Example 1.1.3. We have the 2-category **Cat** of (small) categories, functors, and natural transformations.

Definition. Let $\mathcal{K}$ and $\mathcal{L}$ be 2-categories. Then, a *pseudofunctor* F from $\mathcal{K}$ to $\mathcal{L}$, written as $F : \mathcal{K} \to \mathcal{L}$, consists of data

- a map $F : \operatorname{Ob} \mathcal{K} \to \operatorname{Ob} \mathcal{L}$;
- a functor
$$F : \mathcal{K}(U, V) \to \mathcal{L}(F(U), F(V))$$
for each $U, V \in \operatorname{Ob} \mathcal{K}$;
- an invertible 2-morphism

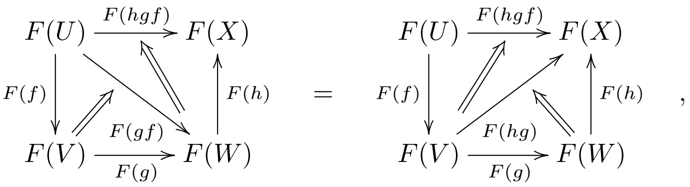


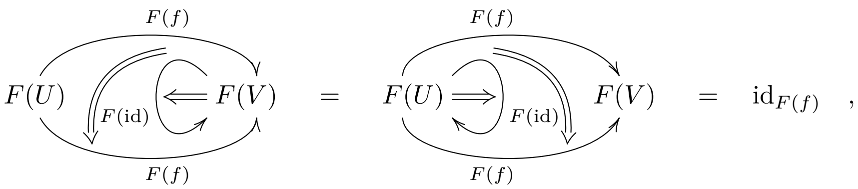
for each 1-morphisms $f : U \to V$ and $g : V \to W$;
- an invertible 2-morphism



for each object $U \in \operatorname{Ob} \mathcal{K}$;

such that the following equations of 2-morphisms hold:

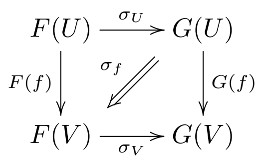




for every 1-morphisms $f : U \to V$, $g : V \to W$, and $h : W \to X$ in $\mathcal{K}$. In particular, a pseudofunctor $F : \mathcal{K} \to \mathcal{L}$ is said to be *normalized* if the 2-morphisms $\lambda_\varnothing$ are trivial. We say moreover F is a 2-*functor* if it is normalized with even $\lambda_{g,f}$ trivial.

Definition. Let $F, G : \mathcal{K} \to \mathcal{L}$ be two pseudofunctors. Then a *pseudonatural transformation σ from F to G*, written $\sigma : F \to G$, consists of

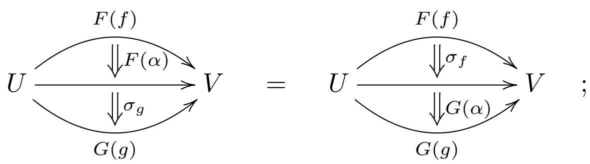
- a 1-morphism $\sigma_U : F(U) \to G(U) \in \mathcal{L}$ for each object $U \in \mathcal{K}$;
- a 2-morphism in $\mathcal{L}$ depicted as



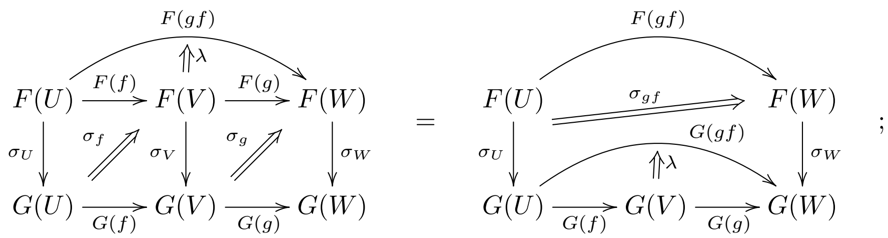
  for each 1-morphism $f : U \to V \in \mathcal{K}$;

satisfying the following coherence conditions:

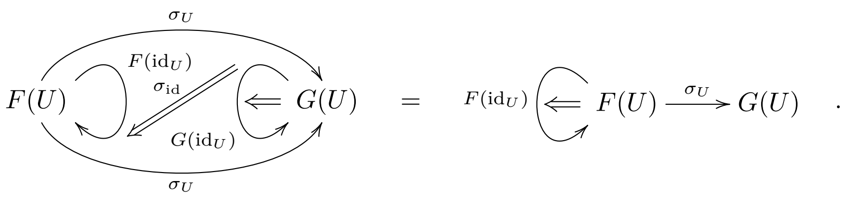
(i) for each $\alpha : f \to g \in \mathcal{K}(U, V)$,



(ii) For every sequence of 1-morphisms $U \xrightarrow{f} V \xrightarrow{g} W \in \mathcal{K}$,



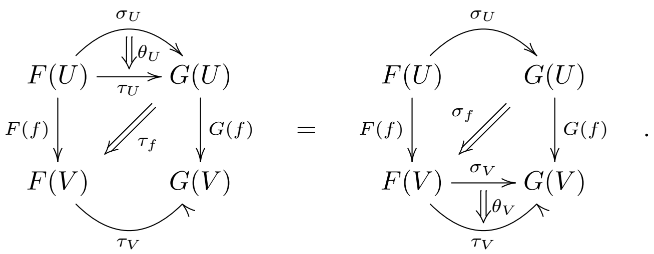
(iii) for each object $U \in \mathcal{K}$,



We call σ a 2-*natural transformation* if the 2-morphism σ_f in $\mathcal{L}$ is trivial for every 1-morphism f in $\mathcal{K}$.

In contrast to the ordinary category theory, as for 2-categories, there is yet another "transformation;" namely between pseudonatural transformations.

Definition. Let $\sigma, \tau : F \to G : \mathcal{K} \to \mathcal{L}$ be two pseudonatural transformations between pseudofunctors. Then, a *modification* θ *from* σ *to* τ consists of a 2-morphism $\theta_U : \sigma_U \to \tau_U$ in $\mathcal{L}$ for each object $U \in \mathcal{K}$ satisfying the following coherence condition: for each 1-morphism $f : U \to V \in \mathcal{K}$,



Note that 2-categories, normalized pseudofunctors, pseudonatural transformations, and modifications form a 3-category $\mathbf{Cat}_2$; i.e. it is a category enriched over 2-categories with respect to the Cartesian products. Hence, we have the notion of *equivalences* of 2-categories. We call an equivalence in $\mathbf{Cat}_2$ a *biequivalence*. There is a convenient criterion.

Proposition 1.1.4 (e.g. see [49])**.** *Let $F : \mathcal{K} \to \mathcal{L}$ be a normalized pseudofunctor between 2-categories. Then, it is a biequivalence if and only if it satisfies the following conditions:*

(i) *it is* essentially surjective*; i.e. for every object $X \in \mathcal{L}$, there is an object $U \in \mathcal{K}$ together with an equivalence $F(U) \simeq X$ in $\mathcal{L}$;*

(ii) *it is* essentially fully faithful*; i.e. for every objects $U, V \in \mathcal{K}$, the functor*

$$F : \mathcal{K}(U, V) \to \mathcal{L}(F(U), F(V))$$

is an equivalence of categories.

Remark 1.1.5. Actually, one can relax in Proposition 1.1.4 the assumption that F is normalized.

We finally discuss *monads* on 2-categories. Though there are several conventions on the *weakness*, the most strict one suffices in our theory.

Definition. Let $\mathcal{K}$ be a 2-category. Then, a 2-*monad* on $\mathcal{K}$ is a 2-functor $T : \mathcal{K} \to \mathcal{K}$ together with 2-natural transformations

$$\mu : T \circ T \to T \ , \quad \eta : \mathrm{Id}_{\mathcal{K}} \to T$$

such that the diagrams below are strictly commutative:

$$\begin{array}{ccc} T \circ T \circ T & \xrightarrow{T\mu} & T \circ T \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T \circ T & \xrightarrow{\mu} & T \end{array} \ , \qquad \begin{array}{ccccc} \mathrm{Id}_{\mathcal{K}} \circ T & \xrightarrow{\eta T} & T \circ T & \xleftarrow{T\eta} & T \circ \mathrm{Id}_{\mathcal{K}} \\ & \searrow\!\!\!\!\!= & \downarrow{\scriptstyle \mu} & =\!\!\!\!\!\swarrow & \\ & & T & & \end{array} \ .$$

Definition. Let $\mathcal{K}$ be a 2-category and (T, μ, η) a 2-monad on it. Then, a T-*algebra* is an object $A \in \mathcal{K}$ together with a 1-morphism $\mathcal{A} : TA \to A$ such that the diagrams below are strictly commutative:

$$\begin{array}{ccc} TTA & \xrightarrow{\mu} & TA \\ {\scriptstyle \mathcal{A}}\downarrow & & \downarrow{\scriptstyle \mathcal{A}} \\ TA & \xrightarrow{\mathcal{A}} & A \end{array} \ , \qquad \begin{array}{ccc} A & \xrightarrow{\eta} & TA \\ & \searrow\!\!\!\!\!= & \downarrow{\scriptstyle \mathcal{A}} \\ & & A \end{array} \ .$$

Remark 1.1.6. Even if T is a 2-monad on a 2-category $\mathcal{K}$ in the strict sense, we can still consider the weak analogues of T-algebras; which are sometimes called *pseudo-T-algebras*. As for these structures, the commutativity above are replaced by specific coherent invertible 2-morphisms. Though we do not write down it here, we mention it is important in the higher algebraic theory.

1.2 Unbiased monoidal categories

We next review the basic definitions of monoidal categories. Though it is common to define them as bicategories with a single object, we rather use the unbiased convention, which is convenient to discuss higher coherence.

Definition (Definition 3.1.1 in [51])**.** Let $\mathcal{C}$ be a category. An *unbiased monoidal structure* on a category $\mathcal{C}$ consists of a family of functors

$$\otimes_n : \mathcal{C}^{\times n} \to \mathcal{C}$$

for $n \in \mathbb{N}$ together with natural isomorphisms

$$\theta_{k_1,\dots,k_n} : \otimes_{\sum_i k_i} \cong \otimes_n \circ (\otimes_{k_1} \times \cdots \times \otimes_{k_n})$$

such that the following diagram of natural transformations made from θ commutes:

$$\begin{array}{ccc} \otimes_{\sum_j \sum_i k_j^{(i)}} & \longrightarrow & \otimes_{\sum_j m_j} \circ \prod_{j=1}^{n} \prod_{i=1}^{m_j} \otimes_{k_i^{(j)}} \\ \downarrow & & \downarrow \\ \otimes_n \circ \prod_{j=1}^{n} \otimes_{\sum_i k_i^{(j)}} & \longrightarrow & \otimes_n \circ \prod_{j=1}^{n} \otimes_{m_j} \circ \prod_{j=1}^{n} \prod_{i=1}^{m_j} \otimes_{k_i^{(j)}} \end{array} \ .$$

An *unbiased monoidal category* is a category equipped with an (unbiased) monoidal structure. For objects $a_1, \dots, a_n$ in an unbiased monoidal category, we often write

$$a_1 \otimes \cdots \otimes a_n := \otimes_n(a_1, \dots, a_n) \ .$$

Note that, for an unbiased monoidal category $\mathcal{C}$, the category $\mathcal{C}^{\times 0}$ is the trivial category, we can think of the functor $\otimes_0 : * \to \mathcal{C}$ as an object of $\mathcal{C}$, which we call the *unit object* in the monoidal structure.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two unbiased monoidal categories. A *monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor on the underlying categories together with natural isomorphisms

$$\lambda_n : F(a_1) \otimes \cdots \otimes F(a_n) \cong F(a_1 \otimes \cdots \otimes a_n)$$

for $n \in \mathbb{N}$ which make the following diagram commute:

$$\begin{array}{ccc} \otimes_{k_1+\cdots+k_n} \circ F^{\times k_1+\cdots+k_n} & \xrightarrow{\theta} & \otimes_n \circ \prod_{i=1}^{n} \otimes_{k_i} \circ F^{\times k_1+\cdots+k_n} \\ \downarrow{\scriptstyle \lambda_{k_1+\cdots+k_n}} & & \downarrow{\scriptstyle \lambda_n \circ \prod_i \lambda_{k_i}} \\ F \circ \otimes_{k_1+\cdots+k_n} & \xrightarrow{F(\theta)} & F \circ \otimes_n \circ \prod_{i=1}^{n} \otimes_{k_i} \end{array} \quad . \tag{1.2.1}$$

Definition. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two monoidal functors between unbiased monoidal categories. A natural transformation $\alpha : F \to G$ is said to be *monoidal* if the square

$$\begin{array}{ccc} F(a_1) \otimes \cdots \otimes F(a_n) & \xrightarrow{\lambda_n} & F(a_1 \otimes \cdots \otimes a_n) \\ \downarrow{\scriptstyle \alpha_{a_1} \otimes \cdots \otimes \alpha_{a_n}} & & \downarrow{\scriptstyle \alpha_{a_1 \otimes \cdots \otimes a_n}} \\ G(a_1) \otimes \cdots \otimes G(a_n) & \xrightarrow{\lambda_n} & G(a_1 \otimes \cdots \otimes a_n) \end{array}$$

is commutative for each objects $a_1, \dots, a_n \in \mathcal{C}$.

We denote by **MonCat** the 2-category of unbiased monoidal categories, monoidal functors, and monoidal natural transformations.

Remark 1.2.1. Thanks to the *Coherence Theorem* (see Appendix B in [51]), the notion of unbiased monoidal categories is equivalent to that of ordinary monoidal categories. In fact, it is shown that the 2-category **MonCat** is equivalent to that of ordinary monoidal categories, monoidal functors, and monoidal natural transformations. It enables us to identify these two notions, and we just say a category $\mathcal{M}$ is monoidal no matter whether it is unbiased or not.

There are alternative descriptions for monoidal categories based on the idea of Segal [68]. To explain this, we define a category ∇ so that

- objects are the totally ordered set

  $$\langle\langle n \rangle\rangle := \{-\infty, 1, \dots, n, \infty\}$$

  for natural numbers n (possibly 0);

- morphisms are order-preserving maps which sends $\pm\infty$ to $\pm\infty$ respectively;
- the composition is the obvious one.

We call ∇ the *category of intervals*. According to Joyal's unpublished note [39], ∇ is isomorphic to the opposite of the simplex category Δ. Segal pointed out that the category ∇ classifies a certain algebraic structure in categories with pullbacks. Fortunately, we can apply Segal's construction to monoidal categories.

Let $\mathcal{C}$ be an (unbiased) monoidal category. Note that the morphisms $\varphi : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$ correspond in one-to-one to $(n+2)$-tuples $\vec{k} = (k_{-\infty}, k_1, \dots, k_n, k_\infty)$ of non-negative integers with

$$k_{-\infty} + k_1 + \cdots + k_n + k_\infty = m$$

via the assignment $\varphi \mapsto \vec{k}^{(\varphi)}$ such that

$$k_j^{(\varphi)} := \begin{cases} \#\varphi^{-1}\{j\} & 1 \le j \le n \ , \\ \#\varphi^{-1}\{j\} - 1 & j = \pm\infty \ . \end{cases} \tag{1.2.2}$$

Using the description, we define a pseudofunctor

$$\mathcal{C}^{\circledS} : \nabla \to \mathbf{Cat}$$

so that

- for each $n \in \mathbb{N}$, we set $\mathcal{C}^{\circledS}(\langle\!\langle n\rangle\!\rangle) := \mathcal{C}^{\times n}$;
- for a morphism $\varphi : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$, define $\mathcal{C}^{\circledS}(\varphi) : \mathcal{C}^{\circledS}(\langle\!\langle m\rangle\!\rangle) \to \mathcal{C}^{\circledS}(\langle\!\langle n\rangle\!\rangle)$ to be the composition
$$\mathcal{C}^{\times m} \cong \mathcal{C}^{\times k_{-\infty}^{(\varphi)}} \times \mathcal{C}^{\times k_1^{(\varphi)}} \times \cdots \times \mathcal{C}^{\times k_n^{(\varphi)}} \times \mathcal{C}^{\times k_\infty^{(\varphi)}} \xrightarrow{* \times \otimes_{k_1^{(\varphi)}} \times \cdots \times \otimes_{k_n^{(\varphi)}} \times *} \mathcal{C}^{\times n} \ ;$$
- for each composition $\psi\varphi$ in ∇, the functoriality natural isomorphism $\mathcal{C}^{\circledS}(\psi) \circ \mathcal{C}^{\circledS}(\varphi) \cong \mathcal{C}^{\circledS}(\psi\varphi)$ defined in terms of the associativity isomorphisms $\theta : \otimes_{k_1+\cdots+k_n} \cong \otimes_n \times \prod_i \otimes_{k_i}$ in the obvious way.

It is clear that $\mathcal{C}^{\circledS}$ is a normalized pseudofunctor. The important observation is that we can recover all the monoidal structure on $\mathcal{C}$ from the pseudofunctor $\mathcal{C}^{\circledS}$. Namely, we have $\mathcal{C} = \mathcal{C}^{\circledS}(\langle\!\langle 1\rangle\!\rangle)$ by definition. Moreover, writing $\rho_i : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle 1\rangle\!\rangle$ the morphism in ∇ such that $\rho_i(j) = 1$ precisely when $i = j$, one can see the functor

$$(\mathcal{C}^{\circledS}(\rho_1), \dots, \mathcal{C}^{\circledS}(\rho_n)) : \mathcal{C}^{\circledS}(\langle\!\langle n\rangle\!\rangle) \to \mathcal{C}^{\circledS}(\langle\!\langle 1\rangle\!\rangle)^{\times n}$$

is an equivalence of categories (even an isomorphism in fact). This condition is sometimes called the *Segal condition*. The n-fold monoidal product $\otimes_n : \mathcal{C}^{\times n} \to \mathcal{C}$ is now identified with the composition

$$\mathcal{C}^{\circledS}(\langle\!\langle 1\rangle\!\rangle)^{\times n} \xleftarrow{\cong} \mathcal{C}^{\circledS}(\langle\!\langle n\rangle\!\rangle) \xrightarrow{\mathcal{C}^{\circledS}(\mu_n)} \mathcal{C}^{\circledS}(\langle\!\langle 1\rangle\!\rangle) \ ,$$

where $\mu_n : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle 1\rangle\!\rangle$ is the morphism of ∇ such that $\mu_n(j) = 1$ precisely when $1 \le j \le n$.

1.3 Multicategories

To review basics on multicategories, we mainly follow [33] for definitions and results.

Definition. A *multicategory* $\mathcal{M}$ consists of

- a set $\operatorname{Ob}\mathcal{M}$, whose elements are called *objects* of $\mathcal{M}$;
- a set $\mathcal{M}(a_1 \dots a_n; a)$ for each $a_i, a \in \operatorname{Ob}\mathcal{M}$, whose elements are called *multimorphisms*;
- an element $\mathrm{id} = \mathrm{id}_a \in \mathcal{M}(a; a)$ for each $a \in \operatorname{Ob}\mathcal{M}$ called the identity on a;
- a map

  $$\gamma : \mathcal{M}(a_1 \dots a_n; a) \times \prod_{i=1}^{n} \mathcal{M}(\vec{a}^{(i)}; a_i) \to \mathcal{M}(\vec{a}^{(1)} \dots \vec{a}^{(n)}; a)$$

  called the composition operation for each $a, a_i \in \operatorname{Ob}\mathcal{M}$ and finite sequences $\vec{a}^{(i)} = a_1^{(i)} \dots a_{k_i}^{(i)}$ of members of $\operatorname{Ob}\mathcal{M}$, where $\vec{a}^{(1)} \dots \vec{a}^{(n)}$ is the concatenation of the sequences;

satisfying the following conditions

(i) **associativity**: for $f \in \mathcal{M}(a_1 \dots a_n; a)$, $f_i \in \mathcal{M}(a_1^{(i)} \dots a_{k_i}^{(i)}; a_i)$, and $f_j^{(i)} \in \mathcal{M}(\vec{a}^{(i,j)}; a_j^{(i)})$,

$$\begin{aligned}&\gamma(f; \gamma(f_1; f_1^{(1)}, \dots, f_{k_1}^{(1)}), \dots, \gamma(f_n; f_1^{(n)}, \dots, f_{k_n}^{(n)})) \\ &\qquad = \gamma(\gamma(f; f_1, \dots, f_n); f_1^{(1)}, \dots, f_{k_1}^{(1)}, \dots, f_{k_n}^{(n)}) \ .\end{aligned}$$

(ii) **unitality**: for $f \in \mathcal{M}(w; a)$,

$$\gamma(f; \mathrm{id}, \dots, \mathrm{id}) = \gamma(\mathrm{id}; f) = f$$

Remark 1.3.1. In the definition above, we require the set $\operatorname{Ob}\mathcal{M}$ of objects to be actually a *set*, so our multicategories are sometimes said to be *small*. On the other hand, one can easily relax this assumption to obtain the notion of *large* multicategories. Actually, it turn out that some important examples of multicategories are large; e.g. Example 1.3.4. Note that, if we assume so-called the *Axiom of Universe*, we can rethink of them as small ones by taking a larger universe. In the following discussion, we hence do not really distinguish small ones from large ones as long as it does not cause a problem.

Remark 1.3.2. In spite of the definition above, some authors assume an additional structure on multicategories, namely actions of the symmetric groups $\mathfrak{S}_n$. In this convention, our multicategories and operads are called *planar* ones. Actually, the assumption is equivalent to saying that a multicategory is equipped with a $\mathfrak{S}$-symmetric structure in the sense of Section 3.2.

Example 1.3.3. Let $\mathcal{C}$ be a category. Then, we can think of it as a multicategory with the same objects, the multi-hom set

$$\mathcal{C}(a_1 \dots a_n; a) = \begin{cases} \mathcal{C}(a_1, a) & n = 1 \\ \varnothing & n \neq 1 \end{cases},$$

and the same composition operation.

Example 1.3.4. For a monoidal category $\mathcal{C}$, we define a multicategory $\mathcal{C}^{\otimes}$ as follows: the objects of $\mathcal{C}^{\otimes}$ are those of $\mathcal{C}$. For $X, X_1, \dots, X_n \in \mathcal{C}$, we set

$$\mathcal{C}^{\otimes}(X_1 \dots X_n; X) := \mathcal{C}(X_1 \otimes \dots \otimes X_n, X) \ .$$

Then $\mathcal{C}^{\otimes}$ has an obvious composition operation and the unit which make $\mathcal{C}^{\otimes}$ into a multicategory. We call $\mathcal{C}^{\otimes}$ the *multicategory associated to* $\mathcal{C}$.

As a special case, an *operad* is a multicategory which has exactly one object. If $\mathcal{O}$ is an operad, then there is a canonical bijection $\operatorname{Ob}\mathcal{O} \cong \mathbb{N}$. Hence, it makes sense to denote by $\mathcal{O}(n)$ the set of multimorphisms of *arity* n. Unwinding the definition, the composition operation of an operad $\mathcal{O}$ is a map

$$\gamma : \mathcal{O}(n) \times \prod_{i=1}^{n} \mathcal{O}(k_i) \to \mathcal{O}(k_1 + \dots + k_n) \ .$$

Example 1.3.5. We define an operad $*$ with $*(n)$ singleton and the obvious composition operation. We call it the *terminal operad*.

Example 1.3.6. Let $\mathcal{M}$ be a multicategory. For each object $a \in \mathcal{M}$, we set

$$\operatorname{End}_a(n) := \mathcal{M}(\overbrace{a \dots a}^{n}; a) \ .$$

Then, the multicategory structure restricts to $\operatorname{End}_a := \{\operatorname{End}_a(n)\}_n$ so that End_a is an operad. We call it the *endomorphism operad on* a *in* $\mathcal{M}$.

We can define *multi*-analogues of functors and natural transformations in the ordinary category theory.

Definition. Let $\mathcal{M}$ and $\mathcal{N}$ be two multicategories. Then, a *multifunctor* F from $\mathcal{M}$ to $\mathcal{N}$, written $F : \mathcal{M} \to \mathcal{N}$, consists of

- a map $F : \operatorname{Ob}\mathcal{M} \to \operatorname{Ob}\mathcal{N}$ and
- for each $a_i, b \in \operatorname{Ob}\mathcal{M}$, a map

$$F : \mathcal{M}(a_1 \dots a_n; a) \to \mathcal{N}(F(a_1) \dots F(a_n); F(a)) \ ;$$

which satisfy the equations

$$F(\gamma(f; f_1, \dots, f_n)) = \gamma(F(f); F(f_1), \dots, F(f_n)) \tag{1.3.1}$$

$$F(\mathrm{id}_a) = \mathrm{id}_{F(a)} \tag{1.3.2}$$

for objects $a \in \operatorname{Ob}\mathcal{M}$ and multimorphisms f and f_i whenever they make sense.

Definition. Let $F, G : \mathcal{M} \to \mathcal{N}$ be two multifunctors between multicategories. A *multinatural transformation* α from F to G, written $\alpha : F \to G$, is a family $\{\alpha_a\}_{a \in \mathrm{Ob}\,\mathcal{M}}$ of elements $\alpha_a \in \mathcal{N}(F(a); G(a))$ indexed by the objects of $\mathcal{M}$ such that, for every multimorphism $f \in \mathcal{M}(a_1 \ldots a_n; a)$, we have

$$\gamma(f; \alpha_{a_1}, \ldots, \alpha_{a_n}) = \gamma(\alpha_a; f) \ .$$

Multicategories, multifunctors, and multinatural transformations form a 2-category **MultCat**; i.e. a category enriched over the category **Cat** of (small) categories (with respect to the cartesian monoidal structure). Indeed, for each pair $(\mathcal{M}, \mathcal{N})$ of multicategories, we have the category $\mathbf{MultCat}(\mathcal{M}, \mathcal{N})$ of multifunctors from $\mathcal{M}$ to $\mathcal{N}$ and multinatural transformations between them. The composition operations are defined in a canonical way.

Example 1.3.7. In the case $\mathcal{M} = \mathcal{C}^{\otimes}$ is a multicategory obtained from a monoidal category $\mathcal{C}$ in the way of Example 1.3.4, for another multicategory $\mathcal{N}$, we write

$$\mathbf{Alg}_{\mathcal{N}}(\mathcal{C}) := \mathbf{MultCat}(\mathcal{N}, \mathcal{C}^{\otimes})$$

and call the objects *$\mathcal{N}$-algebras in $\mathcal{C}$*.

Example 1.3.8. It is easily verified that the construction Example 1.3.3 gives rise to a 2-functor $\mathbf{Cat} \to \mathbf{MultCat}$, here **Cat** is the 2-category of (small) categories, functors, and natural transformations. Moreover, it is strictly fully faithful; i.e. for every pair of categories $\mathcal{C}$ and $\mathcal{D}$, the functor

$$\mathbf{Cat}(\mathcal{C}, \mathcal{D}) \to \mathbf{MultCat}(\mathcal{C}, \mathcal{D})$$

is an isomorphism of categories. Note that there is a construction in the inverse direction; for a multicategory $\mathcal{M}$, define a category $\underline{\mathcal{M}}$ with the same objects, the hom-sets

$$\underline{\mathcal{M}}(a, b) := \mathcal{M}(a; b) \ ,$$

and the restriction of the composition operation. We call $\underline{\mathcal{M}}$ the *underlying category of* $\mathcal{M}$. It easily extends to a functor $\mathbf{MultCat} \to \mathbf{Cat}$ which is right adjoint to the 2-functor above.

Example 1.3.9. Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. We define a multifunctor $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ as follows:

- it is identical to F on objects;
- for a multimorphism $f \in \mathcal{C}^{\otimes}(X_1 \ldots X_n; X)$, we define a multimorphism

  $$F^{\otimes}(f) \in \mathcal{D}^{\otimes}(F(X_1) \ldots F(X_n); F(X)) = \mathcal{D}(F(X_1) \otimes \cdots \otimes F(X_n); F(X))$$

  to be the composition

  $$F(X_1) \otimes \cdots \otimes F(X_n) \xrightarrow{\cong} F(X_1 \otimes \cdots \otimes X_n) \xrightarrow{F(f)} F(X) \ .$$

The *multifunctoriality* is easily verified. Similarly, one can assign a multinatural transformation to each monoidal natural transformation. These construction defines a 2-functor

$$(-)^{\otimes} : \mathbf{MonCat} \to \mathbf{MultCat} \ .$$

In the rest of this section, we discuss a necessary and sufficient condition for a multicategory to come from a monoidal category in the way of Example 1.3.4. It will be enable us to discuss monoidal categories inside the theory of multicategories.

Definition (Definition 8.3 in [33])**.** Let $\mathcal{M}$ be a multicategory.

(1) A multimorphism $u \in \mathcal{M}(\vec{a}; u_!\vec{a})$ is said to be *strongly universal* if the map

$$\mathcal{M}(\vec{b}\,(u_!\vec{a})\,\vec{c}; d) \to \mathcal{M}(\vec{b}\vec{a}\vec{c}; d)$$

induced by the precomposition with u is a bijection for every finite sequences $\vec{b}$ and $\vec{c}$ of objects of $\mathcal{M}$.

(2) $\mathcal{M}$ is said to be *representable* if every finite sequence $\vec{a}$ of objects admits a strongly universal multimorphism $u \in \mathcal{M}(\vec{a}; u_!\vec{a})$.

We note that strongly universal multimorphisms are determined by its domain up to unique isomorphisms; indeed, if $u\mathcal{M}(\vec{a}; u_!\vec{a})$ and $u' \in \mathcal{M}(\vec{a}; u'_!\vec{a}$ are strongly universal, then we have isomorphisms

$$\begin{aligned} \mathcal{M}(u'_!\vec{a}; u_!\vec{a}) &\cong \mathcal{M}(\vec{a}; u_!\vec{a}) \cong \mathcal{M}(u_!\vec{a}; u_!\vec{a}) \ , \\ \mathcal{M}(u_!\vec{a}; u'_!\vec{a}) &\cong \mathcal{M}(\vec{a}; u'_!\vec{a}) \cong \mathcal{M}(u'_!\vec{a}; u'_!\vec{a}) \ . \end{aligned}$$

Chasing the elements, one can find a unique pair of morphisms $\tau : u'_!\vec{a} \to u_!\vec{a}$ and $\tau' : u_!\vec{a} \to u'_!\vec{a}$ so that they are inverse to each other and that $u = \gamma(\tau; u')$ and $u' = \gamma(\tau'; u)$.

Using the uniqueness of strongly universal morphisms, for each representable multicategory $\mathcal{M}$, we can endow the underlying category $\mathcal{M}$ with an (unbiased) monoidal structure. For each sequence $\vec{a} = a_1 \dots a_n$ of objects of $\mathcal{M}$, choose a strongly universal multimorphism $u^{\vec{a}} \in \mathcal{M}(\vec{a}; u^{\vec{a}}_!\vec{a})$, and put $\otimes_n(\vec{a}) := u^{\vec{a}}_!\vec{a}$. We extend $\otimes_n$ to a functor $\underline{\mathcal{M}}^{\times n} \to \underline{\mathcal{M}}$ as follows: for morphisms $f_i : a_i \to b_i$ for $1 \le i \le n$, define $\otimes_n(f_1, \dots, f_n)$ to be the image of $(f_1, \dots, f_n)$ under the composition

$$\otimes_n : \prod_{i=1}^{n} \underline{\mathcal{M}}(a_i, b_i) \xrightarrow{u^{\vec{b}}_*} \mathcal{M}(a_1 \dots a_n; \otimes_n(\vec{b})) \underset{\cong}{\xleftarrow{(u^{\vec{a}})^*}} \underline{\mathcal{M}}(\otimes_n(\vec{a}), \otimes_n(\vec{b})) \ .$$

The functoriality is easily verified. Moreover, the uniqueness of strongly universal morphisms implies that, for sequences $\vec{a}^{(1)}, \dots, \vec{a}^{(n)}$ of length $k_1, \dots, k_n$ respectively, there is a unique isomorphism

$$\tau : \otimes_{\sum_i k_i}(\vec{a}^{(1)} \dots \vec{a}^{(n)}) \cong \otimes_n(\otimes_{k_1}(\vec{a}^{(1)}) \dots \otimes_{k_n}(\vec{a}^{(n)}))$$

such that

$$\gamma(u^{\otimes_{k_1}(\vec{a}^{(1)}) \dots \otimes_{k_n}(\vec{a}^{(n)})}; u^{\vec{a}^{(1)}}, \dots, u^{\vec{a}^{(n)}}) = \gamma(\tau, u^{\vec{a}^{(1)} \dots \vec{a}^{(n)}}) \ .$$

It turns out that τ defines a natural transformation so $(\{\otimes_n\}, \{\tau\})$ forms a monoidal structure on $\underline{\mathcal{M}}$. This construction actually supplies the inverse to the following equivalence.

Theorem 1.3.10 (Corollary 8.13 in [33])**.** *The functor given in Example 1.3.9 restricts to a biequivalence of* **MonCat** *and the* 2*-category of representable multicategories, multifunctors preserving strongly universal multimorphisms, and multinatural transformations.*

1.4 CoCartesian morphisms

In this section, we review coCartesian morphisms, which are a kind of universal morphisms. The notion is a key of the correspondence of fibered categories and pseudofunctors into **Cat**, which is pointed out in Section VI.8 in [30].

Definition. Let $p : \mathcal{C} \to \mathcal{B}$ be a functor. A morphism $f : X \to Y \in \mathcal{C}$ is said to be *p-coCartesian* if it satisfies the following condition: given commutative diagrams in $\mathcal{C}$ and in $\mathcal{B}$ depicted as the solid part below:

$$\left(\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & \searrow^{g} & \\ Y & \overset{\exists! g'}{\dashrightarrow} & Z \end{array}\right) \quad \overset{p}{\to} \quad \left(\begin{array}{ccc} p(X) & & \\ {\scriptstyle p(f)}\downarrow & \searrow^{p(g)} & \\ p(Y) & \xrightarrow{\varphi} & p(Z) \end{array}\right)$$

there is a unique morphism $g' : W \to Y \in \mathcal{C}$ with $g = g' \circ f$ and $p(g') = \varphi$.

We often omit the indication of the functor p and just use the terminology *coCartesian morphisms* when there is no danger of confusion.

The condition above can be rephrased as follows: for a functor $p : \mathcal{C} \to \mathcal{B}$, a morphism $f : X \to Y \in \mathcal{C}$ is p-coCartesian if and only if, for every $Z \in \mathcal{C}$, the square below is a pullback:

$$\begin{array}{ccc} \mathcal{C}(Y,Z) & \xrightarrow{f^*} & \mathcal{C}(X,Z) \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathcal{B}(p(Y),p(Z)) & \xrightarrow{p(f)^*} & \mathcal{B}(p(X),p(Z)) \end{array} \quad . \tag{1.4.1}$$

This observation leads to the invariance of coCartesian morphisms under isomorphisms; indeed, a morphism $f : X \to Y \in \mathcal{C}$ is p-coCartesian if and only if it is isomorphic to a p-coCartesian morphism.

We are often interested in the lifting problem of a morphism in the base category to a coCartesian morphism. Let $p : \mathcal{C} \to \mathcal{B}$ be a functor. Given a morphism $\varphi : a \to b \in \mathcal{B}$ and an object $X \in \mathcal{C}$ with $p(X) = a$, a *p-coCartesian lift of* φ *along* X is a p-coCartesian morphism $f : X \to Y$ with $p(f) = \varphi$. The universal property of coCartesian morphisms implies that if we have another coCartesian lift $f' : X \to Y'$ o φ along X, there is a unique isomorphism $h : Y \cong Y'$ with $f'h = f$. We say p *admits coCartesian lifts of a class* M *of morphisms* if every morphism in M admits p-coCartesian lifts along every object in the fiber over the domain.

Lemma 1.4.1. *Let* $p : \mathcal{C} \to \mathcal{B}$ *be a functor, and suppose* p *admits coCartesian lifts of a morphism* $\varphi : a \to b \in \mathcal{B}$*. We write*

$$\mathcal{C}_a := p^{-1}\{a\} = \mathcal{C} \times_{\mathcal{B}} \{a\} \ , \quad \mathcal{C}_b := p^{-1}\{b\} \ .$$

Then, there is a functor $\varphi_! : \mathcal{C}_a \to \mathcal{C}_b$ *together with a natural transformation*

$$\begin{array}{ccc} & \mathcal{C}_b & \\ {\scriptstyle \varphi_!}\nearrow & \Downarrow{\scriptstyle \widehat{\varphi}} & \searrow \\ \mathcal{C}_a & \hookrightarrow & \mathcal{C} \end{array}$$

such that each component $\widehat{\varphi}_X : X \to \varphi_!(X)$ is a p-coCartesian morphism covering φ. Moreover, the pair $(\varphi_!, \widehat{\varphi})$ is unique up to a unique coherent natural isomorphism.

Proof. For each $X \in \mathcal{C}_a$, choose a p-coCartesian lift of φ along X, and put $\varphi_!(X)$ its codomain. In other words, we have a p-coCartesian lift $\widehat{\varphi} : X \to \varphi_!(X) \in \mathcal{C}$. To extend $\varphi_!$ to a functor, for each $W, Z \in \mathcal{C}$, and for each $\psi : p(W) \to p(Z) \in \mathcal{B}$, we write

$$\mathcal{C}(W,Z)_\psi := \mathcal{C}(W,Z) \times_{\mathcal{B}(p(W),p(Z))} \{\psi\} \ .$$

Then, the universal property (1.4.1) enables us to consider the composition

$$\begin{aligned}\varphi_! : \mathcal{C}_a(X,Y) = \mathcal{C}(X,Y)_{\mathrm{id}_a} &\xrightarrow{(\widehat{\varphi}_Y)_*} \mathcal{C}(X, \varphi_!(Y))_\varphi \\ &\xleftarrow[\cong]{\widehat{\varphi}_X^*} \mathcal{C}(\varphi_!(X), \varphi_!(Y))_{\mathrm{id}_b} = \mathcal{C}_b(\varphi_!(X), \varphi_!(Y)) \ .\end{aligned} \tag{1.4.2}$$

It is straightforward that (1.4.2) makes $\varphi_!$ into a functor so that $\widehat{\varphi} := \{\widehat{\varphi}_X\}_X$ is a natural transformation. On the other hand, the uniqueness of the pair $(\varphi_!, \widehat{\varphi})$ is a direct consequence of that of coCartesian lifts. ☐

Although the functor $\varphi_! : \mathcal{C}_a \to \mathcal{C}_b$ is not unique in the strict sense, we call it a *induced functor*.

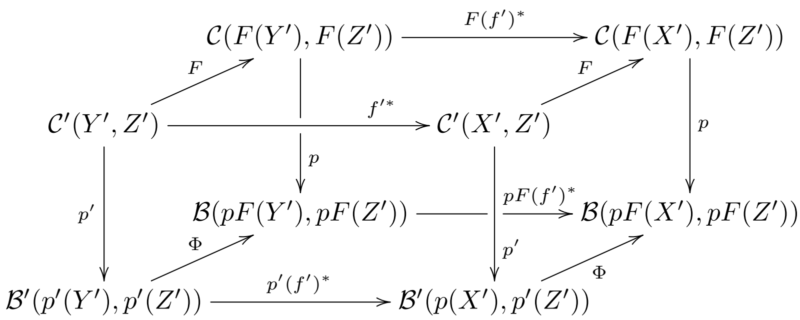
Definition. A functor $p : \mathcal{C} \to \mathcal{B}$ is called a *Grothendieck opfibration* if it admits coCartesian lifts of all the morphisms in $\mathcal{B}$.

Lemma 1.4.2. *Suppose we have a pullback square below of functors*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathcal{B}' & \xrightarrow{\Phi} & \mathcal{B} \end{array}$$

such that p is a Grothendieck opfibration. Then, a morphism $f' : X' \to Y' \in \mathcal{C}'$ is p'-coCartesian if and only if $F(f')$ is p-coCartesian. Consequently, p' is also a Grothendieck opfibration.

Proof. We first show that $f' : X' \to Y'$ is p'-coCartesian provided its image in $\mathcal{C}$ is p-coCartesian. For each $Z' \in \mathcal{C}'$, consider the following commutative cube:



(1.4.3)

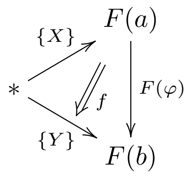
Since the left and the right faces are pullbacks by the assumption, the front face is a pullback as soon as so is the back face. The required statement then follows.

Conversely, consider the class S of morphisms f' in $\mathcal{C}'$ with $F(f')$ p-coCartesian. Since p is a Grothendieck opfibration, it is closed under composition and isomorphisms. Moreover, every morphism in $\mathcal{B}'$ admits lifts in S along every object in the fiber over the domain. It follows that S contains all the p'-coCartesian morphisms, which completes the proof. □

Typical examples of Grothendieck opfibrations are given by the *Grothendieck construction* for pseudofunctors into **Cat**. Let $F : \mathcal{B} \to \mathbf{Cat}$ be a pseudofunctor with $\mathcal{B}$ seen as a 2-category with only trivial 2-morphisms. Then, the Grothendieck construction for F, usually denoted by $\int_{\mathcal{B}} F$, is the category such that

- objects are pairs (a, X) with $a \in \mathcal{B}$ and $X \in F(a)$;
- morphisms $(a, X) \to (b, Y)$ are pairs (φ, f) with $\varphi : a \to b \in \mathcal{B}$ and $f : F(\varphi)(X) \to Y$;
- composition is the obvious one.

More diagrammatically, objects of $\int_{\mathcal{B}} F$ are functors of the form $* \to F(a)$ while morphisms are 2-cells



.

Hence, if we write $\mathbf{Cat}^{*/\!/}$ the weak coslice 2-category of the 2-category **Cat** over the trivial category $*$, we obtain a pullback square

$$\begin{array}{ccc} \int_{\mathcal{B}} F & \longrightarrow & \mathbf{Cat}^{*/\!/} \\ \downarrow & \lrcorner & \downarrow \text{codomain} \\ \mathcal{B} & \xrightarrow{F} & \mathbf{Cat} \end{array}$$

of pseudofunctors. Since the right vertical arrow is a Grothendieck opfibration as a functor between the underlying (1-)categories, $\int_{\mathcal{B}} F \to \mathcal{B}$ is a Grothendieck opfibration. Actually, every Grothendieck opfibration can be written in this form up to equivalences.

Proposition 1.4.3 (see Exposé VI in [30])**.** *Let $\mathcal{B}$ be a small category. Then, the Grothendieck construction gives rise to a biequivalence between the 2-category of*

- *pseudofunctors $\mathcal{B} \to$ **Cat**,*
- *pseudonatural transformations, and*
- *modifications;*

and that of

- *Grothendieck opfibrations over $\mathcal{B}$,*

- *functors over $\mathcal{B}$ preserving coCartesian morphisms, and*
- *natural transformations over $\mathcal{B}$.*

For a Grothendieck opfibration $p : \mathcal{C} \to \mathcal{B}$, the corresponding pseudofunctor $\widehat{\mathcal{C}} : \mathcal{B} \to \mathbf{Cat}$ is constructed roughly as follows: for each morphism $\varphi : a \to b \in \mathcal{B}$, set $\widehat{\mathcal{C}}(\varphi)$ to be an induced functor $\varphi_! : \mathcal{C}_a \to \mathcal{C}_b$. Then, the uniqueness guarantees that there is a unique natural isomorphism

$$\widehat{\mathcal{C}}(\psi\varphi) \cong \widehat{\mathcal{C}}(\psi) \circ \widehat{\mathcal{C}}(\varphi) \ .$$

This determines the assignments of 1- and 2-morphisms.

Remark 1.4.4. We also have the *dual* notions; namely, *Cartesian morphisms* and *Grothendieck fibrations*. The dual argument shows there is a biequivalence between 2-categories of Grothendieck fibrations over $\mathcal{B}$ and of pseudofunctors $\mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$, which is the original statement in [30].

Remark 1.4.5. We will discuss the bicategorical analogue of the results above in Section 5.3.

Chapter 2

Theory of crossed groups

The notion of crossed groups was originally introduced in the simplicial case by Fiedorowicz and Loday [24] and by Krasauskas [47] individually to obtain a categorical description of the cyclic homology and its variants. It however still makes sense in the other categories and would be an important research subject. The goal of this chapter is to make a comprehensive understanding of crossed groups for arbitrary base categories. In particular, we focus on the local presentability and monadicity over presheaf topoi. We will obtain explicit formulas for limits, colimits, and even the terminal crossed group when the base category is enough good. Furthermore, it is shown that there is a closed monoidal structure on a presheaf topoi for which crossed groups are monoid objects. This leads to the notion of crossed monoids and the monadicity. Combining these results, one obtains Kan extensions along certain sorts of functors with explicit formulas. We also prove a crossed analogue of Goursat's lemma for the sake of the classification of crossed interval groups, which Batanin and Markl were concerned about in their paper.

2.1 Definition and examples

In this first section, we recall the definition and examples of crossed groups.

Notation. If $\mathcal{A}$ is a small category, we denote by $\mathbf{Set}_{\mathcal{A}}$ the category of $\mathcal{A}$-sets; i.e. presheaves over $\mathcal{A}$.

Definition. Let $\mathcal{A}$ be a small category. Then, a *crossed $\mathcal{A}$-group* is an $\mathcal{A}$-set $G \in \mathbf{Set}_{\mathcal{A}}$ together with data

- a group structure on $G(a)$ for each $a \in \mathcal{A}$, and
- a left action $G(b) \times \mathcal{A}(a,b) \to \mathcal{A}(a,b)$; $(x,\varphi) \mapsto \varphi^x$ of the group $G(b)$ on $\mathcal{A}(a,b)$ for each $a,b \in \mathcal{A}$;

which satisfy the following conditions:

(i) for $\varphi : a \to b$ and $\psi : b \to c$ in $\mathcal{A}$, and for $x \in G(c)$, we have

$$(\psi\varphi)^x = \psi^x \varphi^{\psi^*(x)} \ ;$$

(ii) for $\varphi : a \to b$ and $x, y \in G(b)$, we have

$$\varphi^*(xy) = (\varphi^y)^*(x)\varphi^*(y) \ .$$

We can describe two conditions (i) and (ii) more categorically: consider a map

$$\mathrm{crs} : G(b) \times \mathcal{A}(a,b) \to \mathcal{A}(a,b) \times G(a) \ ; \quad (x,\varphi) \mapsto (\varphi^x, \varphi^*(x)) \ . \tag{2.1.1}$$

Then, the two conditions are respectively equivalent to the commutativities of the diagrams

$$\begin{array}{ccc} G(c)\times\mathcal{A}(b,c)\times\mathcal{A}(a,b) & \xrightarrow{(\mathrm{id}\times\mathrm{crs})(\mathrm{crs}\times\mathrm{id})} & \mathcal{A}(b,c)\times\mathcal{A}(a,b)\times G(a) \\ {\scriptstyle \mathrm{id}\times\mathrm{comp}}\downarrow & & \downarrow{\scriptstyle \mathrm{comp}\times\mathrm{id}} \\ G(c)\times\mathcal{A}(a,c) & \xrightarrow{\mathrm{crs}} & \mathcal{A}(a,c)\times G(a) \end{array} \tag{2.1.2}$$

and

$$\begin{array}{ccc} G(c)\times G(c)\times\mathcal{A}(b,c) & \xrightarrow{(\mathrm{crs}\times\mathrm{id})(\mathrm{id}\times\mathrm{crs})} & \mathcal{A}(b,c)\times G(b)\times G(b) \\ {\scriptstyle \mathrm{mul}\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times\mathrm{mul}} \\ G(c)\times\mathcal{A}(b,c) & \xrightarrow{\mathrm{crs}} & \mathcal{A}(b,c)\times G(b) \end{array} \ . \tag{2.1.3}$$

Lemma 2.1.1. *Let $\mathcal{A}$ be a small category, and let G be a crossed $\mathcal{A}$-group.*

(1) *For each $a \in \mathcal{A}$, the action of $G(a)$ on $\mathcal{A}(a,a)$ preserves the identity morphism; i.e.* $\mathrm{id}^x = \mathrm{id}$ *for any $x \in G(a)$.*

(2) *For each $a, b \in \mathcal{A}$, the action of $G(b)$ on $\mathcal{A}(a,b)$ preserves monomorphisms and split epimorphisms.*

(3) *For every morphism $\varphi : a \to b \in \mathcal{A}$, the map $\varphi^* : G(b) \to G(a)$ preserves the units of the groups. Moreover, if φ is $G(b)$-invariant, then φ^* is a group homomorphism.*

Proof. The assertions (1) and (3) easily follow from the conditions (i) and (ii) respectively of crossed groups. It remains to show (2). It immediately follows from (i) and the part (1) that the action of $G(a)$ on $\mathcal{A}(a,b)$ preserves split epimorphisms. To see it also preserves monomorphisms, take an arbitrary monomorphism $\delta : a \to b$ in $\mathcal{A}$, and let $x \in G(b)$. Given two morphisms $\varphi_1, \varphi_2 : c \to a$, suppose $\delta^x\varphi_1 = \delta^x\varphi_2$. By the condition (i), we have

$$\delta^x\varphi_i = \delta^x(\varphi_i^{\delta^*(x)^{-1}})^{\delta^*(x)} = (\delta\varphi_i^{\delta^*(x)^{-1}})^x$$

for $i = 1, 2$. Hence, $\delta^x\varphi_1 = \delta^x\varphi_2$ if and only if $\delta\varphi_1^{\delta^*(x)^{-1}} = \delta\varphi_2^{\delta^*(x)^{-1}}$. Since δ is a monomorphism, this happens precisely if $\varphi_1^{\delta^*(x)^{-1}} = \varphi_2^{\delta^*(x)^{-1}}$, or equivalently $\varphi_1 = \varphi_2$. This implies δ^x is a monomorphism, and we obtain (2). $\square$

Corollary 2.1.2. *Let $\mathcal{A}$ be a small category. If $t \in \mathcal{A}$ is a terminal object, for every crossed $\mathcal{A}$-group G, and for each object $a \in \mathcal{A}$, the unique map $a \to t$ induces a group homomorphism $G(t) \to G(a)$. Dually, if $s \in \mathcal{A}$ is an initial object, the unique map $s \to a$ induces a group homomorphism $G(a) \to G(s)$.*

Corollary 2.1.3. *Let $\mathcal{A}$ be a small category, and let G be a crossed $\mathcal{A}$-group. Then, if the action $G(a)$ on $\mathcal{A}(b,a)$ is trivial for each $a, b \in \mathcal{A}$, then G is $\mathcal{A}$-group; i.e. a group object in the category $\mathbf{Set}_{\mathcal{A}}$.*

Remark 2.1.4. The converse of Corollary 2.1.3 holds: every $\mathcal{A}$-group can be seen as a crossed $\mathcal{A}$-group with the trivial actions on each $\mathcal{A}(a,b)$. To distinguish such crossed groups from the others, we often say they are *non-crossed*.

Definition. Let $\mathcal{A}$ be a small category, and let G and H be crossed $\mathcal{A}$-groups. Then, a map $G \to H$ of crossed $\mathcal{A}$-groups is a map of $\mathcal{A}$-sets which is a degreewise group homomorphism respecting the actions on $\mathcal{A}(a,b)$ for each $a, b \in \mathcal{A}$.

Clearly, crossed $\mathcal{A}$-groups and maps of them form a category, which we will denote by $\mathbf{CrsGrp}_{\mathcal{A}}$. Then, the following result is obvious.

Proposition 2.1.5. *For every small category $\mathcal{A}$, the category $\mathbf{CrsGrp}_{\mathcal{A}}$ admits an initial object; namely, the terminal $\mathcal{A}$-set $*$ with the unique crossed group structure.*

Because of the compatibility condition in the definition of maps of crossed groups, the terminal $\mathcal{A}$-set $*$ is no longer terminal in the category $\mathbf{CrsGrp}_{\mathcal{A}}$ in general. Nevertheless, for each crossed $\mathcal{A}$-group G, there still exists the unique $\mathcal{A}$-set map $G \to *$, and it makes sense to ask whether it is a map of crossed groups or not. In this point of view, we can rephrase Corollary 2.1.3 as follows.

Corollary 2.1.6. *Let $\mathcal{A}$ be a small category. Then, a crossed $\mathcal{A}$-group G is a non-crossed if and only if the unique $\mathcal{A}$-map $G \to *$ is a map of crossed $\mathcal{A}$-groups.*

We denote by $\mathbf{Grp}_{\mathcal{A}}$ the category of $\mathcal{A}$-groups. In view of Remark 2.1.4, there is an embedding $\mathbf{Grp}_{\mathcal{A}} \hookrightarrow \mathbf{CrsGrp}_{\mathcal{A}}$. Corollary 2.1.6 says that it factors through the slice category $\mathbf{CrsGrp}_{\mathcal{A}}^{/*}$ and induces an equivalence

$$\mathbf{Grp}_{\mathcal{A}} \simeq \mathbf{CrsGrp}_{\mathcal{A}}^{/*}$$

of categories.

In Section 2.3, we will see the category $\mathbf{CrsGrp}_{\mathcal{A}}$ also admits a terminal object, which is hard to compute in general. In particular, it does not necessarily conincides with the initial object $* \in \mathbf{CrsGrp}_{\mathcal{A}}$, so the category $\mathbf{CrsGrp}_{\mathcal{A}}$ may not be pointed. Nevertheless, we can consider the notion of *images and kernels* of maps of crossed groups in an intuitive way. Indeed, if $\varphi : G \to H$ is a map of crossed $\mathcal{A}$-groups, then it turns out that its degreewise image and kernel respectively form crossed $\mathcal{A}$-groups. An important consequence of Corollary 2.1.6 is that the kernel of a map of crossed $\mathcal{A}$-groups is always non-crosssed.

Before seeing examples, we review on three categories, which would be taken as the base category $\mathcal{A}$. The most famous one is the simplex category Δ, whose objects are the totally ordered sets

$$[n] := \{0, \dots, n\}$$

for $n \in \mathbb{N}$ and whose morphisms are order-preserving maps. Crossed Δ-groups are usually called *crossed simplicial groups*, and they were of central interest in previous works such as [47], [24], and [21]. Next, we define a category $\widetilde{\Delta}$ to consists of the totally ordered sets

$$\langle n\rangle := \{1,\dots,n\}$$

for $n \in \mathbb{N}$ as objects and order-preserving maps as morphisms. A $\widetilde{\Delta}$-set is sometimes called an *augmented simplicial set*, so we call a crossed $\widetilde{\Delta}$-group an *augmented crossed simplicial group*. The third category ∇ is the*category of intervals*, which is already given in Section 1.2. We call crossed ∇-groups *crossed interval groups*, which is due to [3]. We have canonical functors

$$\begin{array}{ccc} \Delta \xrightarrow{\quad j \quad} \widetilde{\Delta} \xrightarrow{\quad \mathfrak{J} \quad} \nabla \\ [k] \longmapsto \langle k+1\rangle ,\ \langle l\rangle \longmapsto \langle\langle l\rangle\rangle \end{array} . \tag{2.1.4}$$

The first functor is fully faithful, and the second is faithful and bijective on objects.

Remark 2.1.7. In the cases the base category $\mathcal{A}$ is Δ, $\widetilde{\Delta}$, or ∇, for a crossed $\mathcal{A}$-group G, we write G_n instead of $G([n])$, $G(\langle n\rangle)$, or $G(\langle\langle n\rangle\rangle)$. This abuse of notation sometimes causes a problem because of the difference of the convention on the degree in (2.1.4). Hence, the reader should be careful in the following discussion.

Now, we are giving several examples below. For this, it is convenient to use *joins of ordered sets*: for two partially ordered sets P and Q, we denote by $P\star Q$ their join. For example, there is a unique isomorphism $\langle m\rangle \star \langle n\rangle \cong \langle m+n\rangle$ of ordered sets.

Example 2.1.8. We define a crossed interval group $\mathfrak{S}$ as follows:

- for each $n \in \mathbb{N}$, $\mathfrak{S}_n$ is the n-th permutation group, or the permutation group on the set $\langle\langle n\rangle\rangle$ fixing $\pm\infty$ respectively;
- for $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, we define $\varphi^* : \mathfrak{S}_n \to \mathfrak{S}_m$ as follows: for $\sigma \in \mathfrak{S}_n$, $\varphi^*(\sigma)$ is the permutation on $\langle\langle m\rangle\rangle$ given as the composition

$$\begin{aligned} \langle m\rangle &\cong \varphi^{-1}\{-\infty\} \star \varphi^{-1}\{1\} \star \dots \star \varphi^{-1}\{n\} \star \varphi^{-1}\{\infty\} \\ &\to \varphi^{-1}\{-\infty\} \star \varphi^{-1}\{\sigma^{-1}(1)\} \star \dots \star \varphi^{-1}\{\sigma^{-1}(n)\} \star \varphi^{-1}\{\infty\} \cong \langle m\rangle , \end{aligned}$$

  where the first and the last maps are the unique order-preserving bijections;
- for $\sigma \in \mathfrak{S}_n$, the action on $\nabla(\langle\langle m\rangle\rangle, \langle\langle n\rangle\rangle)$ is given by

$$\begin{aligned} \varphi^\sigma : \langle\langle m\rangle\rangle &\cong \varphi^{-1}\{-\infty\} \star \varphi^{-1}\{\sigma^{-1}(1)\} \star \dots \star \varphi^{-1}\{\sigma^{-1}(n)\} \star \varphi^{-1}\{\infty\} \\ &\to \{-\infty\} \star \{1\} \dots \{n\} \star \{\infty\} \cong \langle\langle n\rangle\rangle . \end{aligned}$$

The conditions on crossed groups are easily verified. We also have similar constructions for braid groups, pure braid groups, and so on.

Example 2.1.9. For each natural number n, we denote by C_n the cyclic group of order n, which is canonically embedded in the symmetric group $\mathfrak{S}_n$. Although the subsets $C_n \subset \mathfrak{S}_n$ do not form an interval subset of $\mathfrak{S}$ defined in Example 2.1.8, one can see they actually form an augmented simplicial subset, which we denote by $\boldsymbol{\mathcal{C}}$, so $\boldsymbol{\mathcal{C}}_n = C_n$. The augmented simplicial structure is explicitly described as follows: for $\mu : \langle m \rangle \to \langle n \rangle \in \widetilde{\Delta}$, $\mu^* : \Lambda_n \to \Lambda_m$ is given by

$$\mu^*(\sigma)(i) \equiv i + \sigma(\mu(i)) - \mu(i) \pmod{n} .$$

Clearly $\boldsymbol{\mathcal{C}}$ is a crossed augmented simplicial group, so it restricted to a crossed simplicial group. Note that the category $\Delta_{\boldsymbol{\mathcal{C}}}$ defined in Proposition 2.1.12 is called the *Connes' cycle category* after Connes' work [12] on the cyclic homology and sometimes denoted by Λ (see Remark 2.1.13).

Example 2.1.10. Recall that the wreath product of a group G by $\mathfrak{S}_n$ is the group

$$G \wr \mathfrak{S}_n := \mathfrak{S}_n \ltimes G^{\times n}$$

whose underlying set is $\mathfrak{S}_n \times G^{\times n}$ with multiplication given by

$$(\sigma; \vec{x})(\tau; \vec{y}) = (\sigma\tau; \tau^*(\vec{x}) \cdot \vec{y}) ,$$

where, if $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$,

$$\tau^*(\vec{x}) \cdot \vec{y} := (x_{\tau(1)} y_1, \dots, x_{\tau(n)} y_n) .$$

In particular the case $G = C_2$ is the cyclic group of order 2, $H_n := C_2 \wr \mathfrak{S}_n$ is the Weyl group of the root system B_n, which is called the *n-th hyperoctahedral group*.

We claim that the family $\{H_n\}_n$ forms a crossed interval group $\mathfrak{H}$. Set $\mathfrak{H}(\langle\!\langle n \rangle\!\rangle) := H_n$, and for $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$, we define $\varphi^* : H_n \to H_m$ by

$$\varphi^*(\sigma; \vec{\varepsilon}) := (\varphi^*(\sigma)\beta_\varphi(\varepsilon); \varphi^*(\vec{\varepsilon})) ,$$

where

- $\varphi^*(\sigma)$ is the permutation on $\langle\!\langle m \rangle\!\rangle$ defined in Example 2.1.8;
- $\beta_\varphi(\vec{\varepsilon})$ is the permutation given by

  $$\begin{aligned} &\langle\!\langle m \rangle\!\rangle \cong \varphi^{-1}\{-\infty\} \star \varphi^{-1}\{1\} \star \cdots \star \varphi^{-1}\{n\} \star \varphi^{-1}\{\infty\} \\ &\xrightarrow{\mathrm{id} \amalg \beta^{\varepsilon_1} \amalg \cdots \amalg \beta^{\varepsilon_n} \amalg \mathrm{id}} \varphi^{-1}\{-\infty\} \star \varphi^{-1}\{1\} \star \cdots \star \varphi^{-1}\{n\} \star \varphi^{-1}\{\infty\} \cong \langle\!\langle m \rangle\!\rangle , \end{aligned}$$

  where each $\beta : \varphi^{-1}\{j\} \to \varphi^{-1}\{j\}$ is the order-reversing map;
- if $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$, then

  $$\varphi^*(\vec{\varepsilon}) = (\varepsilon_{\varphi(1)}, \dots, \varepsilon_{\varphi(m)})$$

  with the assumption $\varepsilon_{\pm\infty} = 1$.

This actually defines an interval set

$$\mathfrak{H} : \nabla^{\mathrm{op}} \to \mathbf{Set} .$$

It is moreover verified that $\mathfrak{H}$ together with the action of H_n on $\nabla(\langle\!\langle m \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle)$ through $H_n \to \mathfrak{S}_n$ is a crossed interval group, which is called the *Hyperoctahedral crossed interval group*.

Example 2.1.11. Let G be a crossed $\mathcal{A}$-group. If $\varphi : \mathcal{A}' \to \mathcal{A}$ is a faithful functor such that the image of the map

$$\varphi : \mathcal{A}'(a,b) \to \mathcal{A}(F(a), F(b))$$

is stable under $G(\varphi(b))$-action, then the restricted $\mathcal{A}$-set $\varphi^*(G)$ inherits a structure of crossed $\mathcal{A}'$-groups. For example, it is verified that the canonical functors $\Delta \hookrightarrow \widetilde{\Delta} \hookrightarrow \nabla$ given in (2.1.4) pull back the Hyperoctahedral crossed interval group $\mathfrak{H}$ to crossed groups on Δ and $\widetilde{\Delta}$ respectively. We call both of them *Hyperoctahedral crossed simplicial groups.* In particular, to avoid the confusion on the degree convention, we denote by $\boldsymbol{\mathcal{H}}$ the resulting crossed simplicial group while we use the same notation $\mathfrak{H}$ for the augmented one, so we have $\boldsymbol{\mathcal{H}}_n = \mathfrak{H}_{n+1}$. This construction is discussed in more detail in Section 2.5.

We finally mention that we can associate each crossed $\mathcal{A}$-group with a category that is an extension of $\mathcal{A}$. More precisely, we have the following result, which is a direct consequence of the commutative squares (2.1.2) and (2.1.3) together with Lemma 2.1.1.

Proposition 2.1.12. *Let $\mathcal{A}$ be a small category. Suppose we are given an $\mathcal{A}$-set G together with a group structure on $G(a)$ and a left action on $\mathcal{A}(b,a)$ for each $a, b \in \mathcal{A}$. Then, G is a crossed $\mathcal{A}$-group if and only if the following data defines a category $\mathcal{A}_G$:*

- ***object**: the same as $\mathcal{A}$;*
- ***morphism**: for $a, b \in \mathcal{A}$, $\mathcal{A}_G(a,b) = \mathcal{A}(a,b) \times G(a)$;*
- ***composition**: given by*

$$\begin{aligned}\mathcal{A}(b,c) \times G(b) \times \mathcal{A}(a,b) \times G(a) &\xrightarrow{\mathrm{id}\times\mathrm{crs}\times\mathrm{id}} \mathcal{A}(b,c) \times \mathcal{A}(a,b) \times G(a) \times G(a) \\ &\xrightarrow{\mathrm{comp}\times\mathrm{mul}} \mathcal{A}(a,c) \times G(a)\ ,\end{aligned}$$

where the map crs *is one defined by* (2.1.1)*. In this case, the canonical map $\mathcal{A}(a,b) \to \mathcal{A}_G(a,b)$ defines a functor which is faithful and bijective on objects.*

The category $\mathcal{A}_G$ is sometimes called the *total category* of G.

Remark 2.1.13. If $\mathcal{A}$ has no non-trivial isomorphism, then for each crossed $\mathcal{A}$-group G, and for each $a \in \mathcal{A}$, we have

$$G(a) = \mathrm{Aut}_{\mathcal{A}_G}(a)\ .$$

Moreover, the whole crossed $\mathcal{A}$-group structure on G is recovered as follows: for $x \in G(a) = \mathrm{Aut}_{\mathcal{A}_G}(a)$ and for $f : b \to a \in \mathcal{A}$, the composition $xf : b \to a \in \mathcal{A}_G$ is uniquely represented by a pair (g, y) with $g : b \to a \in \mathcal{A}$ and $y \in G(b)$. Clearly, $g = f^x$ and $y = f^*(x)$. In other words, the crossed $\mathcal{A}$-group structure on G involves the unique factorization property of the category $\mathcal{A}_G$. In the case $\mathcal{A}$ is the simplex category Δ, formal statements and proofs will be found in [24] (Proposition 1.7). Note that, in these papers, crossed simplicial groups are defined as extensions of the category Δ.

Some crossed groups have more natural descriptions in terms of total categories.

Example 2.1.14. Fiedorowicz and Loday introduced in [24] a crossed simplicial group $\mathcal{Z}$, whose total category $\Delta_{\mathcal{Z}}$ has the following description due to [28] and [23] (see also Section 3 of [63]):

- the objects are natural numbers $n \in \mathbb{N}$;
- the hom-set $\Lambda_\infty(m, n)$ consists of order-preserving maps $f : \mathbb{Z} \to \mathbb{Z}$ (with respect to the standard linear order on the integers) such that for each $i \in \mathbb{Z}$, we have
$$f(i + m + 1) = f(i) + n + 1 \ ;$$
- the composition is the obvious one.

The category $\Delta_{\mathcal{Z}}$ is often denoted by Λ_∞ and called the *paracyclic category.* Combining the two conditions, for every map $f : \mathbb{Z} \to \mathbb{Z}$ lying in $\Lambda_\infty(m, n)$, we have
$$f(0) \le f(1) \le \cdots \le f(m) \le f(m + 1) = f(0)n + 1 \ .$$
On the other hand, we have group homomorphisms

$$\begin{array}{rcl} \operatorname{Aut}_{\Lambda_\infty}(n) & \rightleftarrows & \mathbb{Z} \\ f & \longmapsto & f(0) \\ (\bullet + k) & \longleftarrow\!\shortmid & k \end{array}$$

which are inverses to each other. It hence turns out that every morphism $f \in \Lambda_\infty(m, n)$ uniquely factors as $f = \varphi\tau$ with $\tau \in \operatorname{Aut}_{\Lambda_\infty}(m)$ and $\varphi : \mathbb{Z} \to \mathbb{Z}$ such that
$$0 \le \varphi(0) \le \varphi(1) \le \cdots \le \varphi(m) \le n + 1 \ . \tag{2.1.5}$$
Note that $\varphi \in \Lambda_\infty(m, n)$ satisfying (2.1.5) can be seen as a morphism $\varphi : [m] \to [n]$ in the simplex category Δ. Thus, in view of Remark 2.1.13, the unique factorization property exhibits $\{\mathcal{Z}_n := \operatorname{Aut}_{\Lambda_\infty}(n)\}_n$ as a crossed simplicial group, which is called the *duplicial crossed simplicial group* in [24] after the notion introduced in [20].

2.2 Cocompleteness and completeness

We investigate elementary properties of the category $\mathbf{CrsGrp}_{\mathcal{A}}$ with regard to colimits and limits. Throughout the section, we fix a small category $\mathcal{A}$. Note that the problem is not as easy as in the case of usual algebraic categories over $\mathbf{Set}_{\mathcal{A}}$. For example, as pointed out in the previous section, terminal objects in $\mathbf{CrsGrp}_{\mathcal{A}}$ are already highly non-trivial, and it shows that the forgetful functor $\mathbf{CrsGrp}_{\mathcal{A}} \to \mathbf{Set}_{\mathcal{A}}$ does not preserve limits.

We begin with colimits since the situation is somehow easier than limits. For a small category $\mathcal{A}$, we denote by $\mathcal{A}_0$ the maximal discrete subcategory of $\mathcal{A}$; that is, the subcategory with the same objects and with only the identities. Since crossed $\mathcal{A}$-groups are by definition degreewise groups, we have the forgetful functor $\mathbf{CrsGrp}_{\mathcal{A}} \to \mathbf{Grp}_{\mathcal{A}_0}$.

Proposition 2.2.1. *The forgetful functor* $\mathbf{CrsGrp}_{\mathcal{A}} \to \mathbf{Grp}_{\mathcal{A}_0}$ *creates arbitrary small colimits. Consequently, the category* $\mathbf{CrsGrp}_{\mathcal{A}}$ *is cocomplete.*

Proof. Thanks to Proposition 2.1.5, $\mathbf{CrsGrp}_{\mathcal{A}}$ has an initial object which is still initial in $\mathbf{Grp}_{\mathcal{A}_0}$. Thus, it suffices to show the functor $\mathbf{CrsGrp}_{\mathcal{A}} \to \mathbf{Grp}_{\mathcal{A}_0}$ creates both pushouts and filtered colimits. The latter is obvious since only finitely many variables appear simultaneously in the axioms of crossed $\mathcal{A}$-groups.

As for pushouts, suppose we are given a span

$$G_1 \xleftarrow{f_1} H \xrightarrow{f_2} G_2$$

in the category $\mathbf{CrsGrp}_{\mathcal{A}}$. It is well-known that the pushout of the span in $\mathbf{Grp}_{\mathcal{A}_0}$, which we write $G_1 *_H G_2$ to distinguish it with the pushout of the underlying $\mathcal{A}$-sets, is given as follows: for each $a \in \mathcal{A}_0$, the group $(G_1 *_H G_2)(a)$ is the quotient of the free monoid over the set $G_1(a) \amalg_{\{e\}} G_2(a)$ by the relation $\sim$ generated by

- insertions and deletions of the unit, i.e. for $x_i \in G_1(a) \amalg_{\{e\}} G_2(a)$,

$$(x_1, \dots, x_n) \sim (x_1, \dots, x_k, e, x_{k+1}, \dots, x_n) \ ;$$

- multiplicativities of $G_1(a)$ and $G_2(a)$, i.e. for $x_i \in G_1(a) \amalg_{\{e\}} G_2(a)$ with x_k and x_{k+1} lying in the common group,

$$(x_1, \dots, x_n) \sim (x_1, \dots, x_{k-1}, x_k x_{k+1}, x_{k+2}, \dots, x_n) \ ;$$

- H-invariance, i.e. for $x_i \in G_1(a) \amalg_{\{e\}} G_2(a)$ with $x_k \in G_j(a)$ and $x_{k+1} \in G_{j'}(a)$, and for $h \in H(a)$,

$$(x_1, \dots, x_n) \sim (x_1, \dots, x_{k-1}, x_k f_j(h)^{-1}, f_{j'}(h) x_{k+1}, x_{k+2}, \dots, x_n) \ .$$

For each morphism $\varphi : a \to b \in \mathcal{A}$, we define a map $\varphi^* : (G_1 *_H G_2)(b) \to (G_1 *_H G_2)(a)$ inductively by

$$\begin{aligned} \varphi^*(e) &= e \\ \varphi^*(x_1, \dots, x_n) &= (\varphi^{x_n})^*(x_1, \dots, x_{n-1}) \varphi^*(x_n) \ . \end{aligned}$$

It is easily verified the definition is invariant under the relation above, so φ^* is well-defined. Now, it is tedious but not difficult to see it is a unique crossed $\mathcal{A}$-group structure on $G_1 *_H G_2$ so that both injections $G_1, G_2 \to G_1 *_H G_2$ are maps of crossed $\mathcal{A}$-groups, which completes the proof. ☐

Corollary 2.2.2. *The forgetful functor* $\mathbf{CrsGrp}_{\mathcal{A}} \to \mathbf{Set}_{\mathcal{A}}$ *creates filtered colimits.*

Proof. We have the following commutative square of forgetful functors:

$$\begin{array}{ccc} \mathbf{CrsGrp}_{\mathcal{A}} & \longrightarrow & \mathbf{Set}_{\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathbf{Grp}_{\mathcal{A}_0} & \longrightarrow & \mathbf{Set}_{\mathcal{A}_0} \end{array}$$

By Proposition 2.2.1, the left functor creates filtered colimits, and it is well-known that so do the right and the bottom. Hence, the result follows. ☐

We now get into a difficult part: the limits. Fortunately, it turns out that the essential difficulty comes only from terminal objects and not from pullbacks.

Proposition 2.2.3. *Let C be a category with a terminal object. Then, the forgetful functor $\mathbf{CrsGrp}_{\mathcal{A}} \to \mathbf{Set}_{\mathcal{A}}$ creates C-limits. In particular, the category $\mathbf{CrsGrp}_{\mathcal{A}}$ has pullbacks which are computed degreewisely.*

Proof. Put $t \in C$ a terminal object, and suppose we are given a functor $G_\bullet : C \to \mathbf{CrsGrp}_{\mathcal{A}}$. Taking the limit in the category $\mathbf{Set}_{\mathcal{A}}$, we put $G_\infty = \lim_C G_\bullet \in \mathbf{Set}_{\mathcal{A}}$. Note that G_∞ admits a unique degreewise group structure so that the canonical $\mathcal{A}$-map $G_\infty \to G_c$ is a degreewise group homomorphism for each $c \in C$. In particular, for each $a \in \mathcal{A}$, the group $G_\infty(a)$ inherits an action on $\mathcal{A}(b,a)$ for each $b \in \mathcal{A}$ through the homomorphism $G_\infty(a) \to G_t(a)$. One can then see these structures exhibit G_∞ as a crossed $\mathcal{A}$-group. □

In order to show the existence of a terminal object, we instead show the category $\mathbf{CrsGrp}_{\mathcal{A}}$ is locally presentable. Indeed, it is known that, in locally presentable categories, limits can be realized as *colimits*. Now, $\mathbf{CrsGrp}_{\mathcal{A}}$ is cocomplete thanks to Proposition 2.2.1. Hence, in view of Corollary 2.47 in [1], we only have to show the accessibility. For a small category C, we denote by $|C|$ the cardinality of the set of morphisms of C. In the rest of the section, we are to prove the following theorem.

Theorem 2.2.4. *Let $\mathcal{A}$ be a small category, and let κ be an (infinite) regular cardinal greater than $\omega \times |\mathcal{A}|$, here ω is the smallest infinite cardinal. Then, the category $\mathbf{CrsGrp}_{\mathcal{A}}$ is locally κ-presentable; i.e. it is cocomplete, and every object can be written as a κ-filtered colimit of κ-small objects.*

Corollary 2.2.5. *For every small category $\mathcal{A}$, the category $\mathbf{CrsGrp}_{\mathcal{A}}$ is both complete and cocomplete. In particular, it admits a terminal object.*

We first give a criterion for the smallness of object in $\mathbf{CrsGrp}_{\mathcal{A}}$. Recall that, for a set S, it is κ-small in $\mathbf{Set}$ precisely if $|S| < \kappa$. Thus, a key idea is considering the functor

$$\underline{(\,\cdot\,)} : \mathbf{CrsGrp}_{\mathcal{A}} \to \mathbf{Set} \;; \quad G \mapsto \underline{G} := \coprod_{a \in \mathcal{A}} G(a)$$

to compare smallnesses in these two categories.

Lemma 2.2.6. *Let κ be a regular cardinal greater than $|\mathcal{A}|$. Then, a crossed $\mathcal{A}$-group G is κ-small provided the cardinality of the set $\underline{G}$ is less than κ.*

Proof. Let $G \in \mathbf{CrsGrp}_{\mathcal{A}}$ with $|\underline{G}| < \kappa$. We have to show that the functor

$$\mathbf{CrsGrp}_{\mathcal{A}}(G, \,\cdot\,) : \mathbf{CrsGrp}_{\mathcal{A}} \to \mathbf{Set}$$

preserves κ-filtered colimits. In view of Corollary 1.7 in [1], it suffices to verify the preservation only for sequential ones. Suppose λ is an ordinal of cofinality at least κ; i.e. every subset $S \subset \lambda$ with $|S| < \kappa$ has a supremum $\sup S \in \lambda$, and take a diagram $H_\bullet : \lambda \to \mathbf{CrsGrp}_{\mathcal{A}}$. Writing $H_\infty := \operatorname{colim} H_\bullet$ for simplicity, we show the canonical map

$$\operatorname*{colim}_{\alpha < \lambda} \mathbf{CrsGrp}_{\mathcal{A}}(G, H_\alpha) \to \mathbf{CrsGrp}_{\mathcal{A}}(G, H_\infty) \tag{2.2.1}$$

is bijective.

Note that, by virtue of Corollary 2.2.2, there is a canonical identification of $\underline{H_\infty}$ with $\operatorname{colim}_{\alpha<\lambda} \underline{H_\alpha}$, so we obtain a commutative square

$$\begin{array}{ccc} \operatorname{colim}_{\alpha<\lambda} \mathbf{CrsGrp}_{\mathcal{A}}(G, H_\alpha) & \longrightarrow & \mathbf{CrsGrp}_{\mathcal{A}}(G, H_\infty) \\ \downarrow \underline{(\,\cdot\,)} & & \downarrow \underline{(\,\cdot\,)} \\ \operatorname{colim}_{\alpha<\lambda} \mathbf{Set}(\underline{G}, \underline{H_\alpha}) & \xrightarrow{\simeq} & \mathbf{Set}(\underline{G}, \operatorname{colim}_{\alpha<\lambda} \underline{H_\alpha}) \end{array} \tag{2.2.2}$$

In view of the criterion of the smallness of sets, the bottom map is bijective. On the other hand, since the functor $\underline{(\,\cdot\,)}$ is faithful, and since filtered colimits in **Set** preserves injections, the vertical maps are injective. Hence, it immediately follows the map (2.2.1) is injective. Moreover, for each map $f : G \to H_\infty$ of crossed $\mathcal{A}$-group, the underlying map $\underline{f} : \underline{G} \to \underline{H_\infty}$ factors through a map $\bar{f} : \underline{G} \to \underline{H_{\alpha'_0}}$ followed by the structure map $\underline{H_{\alpha'_0}} \to \underline{H_\infty}$ for some ordinal $\alpha'_0 < \lambda$. This does not imply $\bar{f}$ is an underlying map of a map of crossed $\mathcal{A}$-group. Nevertheless, there are functions

$$\beta : \coprod_{a,b\in\mathcal{A}} \mathcal{A}(b,a) \times G(a) \to \lambda \ , \quad \gamma : \coprod_{a\in\mathcal{A}} G(a) \times G(a) \to \lambda \ ,$$

such that

(i) $\beta(\varphi, x), \gamma(x, y) > \alpha'_0$ for each $\varphi \in \mathcal{A}(a, b)$ and $x, y \in G(b)$;

(ii) the map $\underline{H_{\alpha'_0}} \to \underline{H_{\beta(\varphi,x)}}$ identifies the elements $\bar{f}(\varphi^*(x))$ and $\varphi^*(\bar{f}(x))$;

(iii) the map $\underline{H_{\alpha'_0}} \to \underline{H_{\gamma(x,y)}}$ identifies the elements $\bar{f}(xy^{-1})$ with $\bar{f}(x) \cdot \bar{f}(y)^{-1}$.

The set $\{\beta(\varphi, x) \mid \varphi, x\} \cup \{\gamma(x, y) \mid x, y\}$ is of cardinality $|\mathcal{A}| \times (|\mathcal{A}| + |\underline{G}|) < \kappa$, so the cofinality of λ implies there is an ordinal $\alpha_0 < \lambda$ with $\alpha_0 > \beta(\varphi, x), \gamma(x, y)$ for every φ, x, and y. Now, it is easily verified that the composition

$$\underline{G} \xrightarrow{\bar{f}} \underline{H_{\alpha'_0}} \to \underline{H_{\alpha_0}}$$

underlies a map $f_0 : G \to H_{\alpha_0}$ of crossed $\mathcal{A}$-group, and the map $f : G \to H_\infty$ factors through f_0. In other words, f is the image of $f_0 \in \mathbf{CrsGrp}_{\mathcal{A}}(G, H_{\alpha_0})$. This implies that the map (2.2.1) is also surjective. □

Fix a regular cardinal κ as in Theorem 2.2.4, and define $\mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$ to be the full subcategory of $\mathbf{CrsGrp}_{\mathcal{A}}$ spanned by crossed $\mathcal{A}$-groups G with $|\underline{G}| < \kappa$. According to Lemma 2.2.6, all objects in $\mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$ is κ-small in $\mathbf{CrsGrp}_{\mathcal{A}}$.

Lemma 2.2.7. *The category* $\mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$ *is essentially small.*

Proof. Since the category $\mathbf{CrsGrp}_{\mathcal{A}}$ is locally small, it suffices to show it has only finitely many isomorphism classes. Notice that, for an indexed family $\{X(a)\}_{a\in\mathcal{A}}$ of sets with $|X(a)| < \kappa$, a structure of crossed $\mathcal{A}$-groups can be seen as an element of the set

$$\prod_{\varphi:a\to b\in\mathcal{A}} \mathbf{Set}(X(b), X(a)) \times \prod_{a\in\mathcal{A}} \mathbf{Set}(X(a) \times X(a), X(a)) \\ \times \prod_{a,b\in\mathcal{A}} \mathbf{Set}(X(a), \mathrm{Aut}_{\mathbf{Set}}(\mathcal{A}(b, a)))$$

whose cardinality is bounded above by

$$(\kappa^\kappa)^{|\mathcal{A}|} \times (\kappa^{\kappa\times\kappa})^{|\mathcal{A}|} \times ((|\mathcal{A}|^{|\mathcal{A}|})^\kappa)^{|\mathcal{A}|\times|\mathcal{A}|} = \kappa^\kappa$$

Then, together with the cardinality of choices of the family $\{X(a)\}_{a\in\mathcal{A}}$, one can see there are only at most $\kappa^{|\mathcal{A}|} \times \kappa^\kappa = \kappa^\kappa$ isomorphism classes. □

Lemma 2.2.8. *The subcategory* $\mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa} \subset \mathbf{CrsGrp}_{\mathcal{A}}$ *is closed under* κ*-small colimits.*

Proof. Let $G_\bullet : C \to \mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$ be a diagram with $|C| < \kappa$. Since the category $\mathbf{CrsGrp}_{\mathcal{A}}$ is cocomplete by Proposition 2.2.1, we put $G_\infty := \operatorname{colim}_{c\in C} G_c \in \mathbf{CrsGrp}_{\mathcal{A}}$. According to Proposition 2.2.1, for each $a \in \mathcal{A}$, there is a surjection

$$\left(\coprod_{c\in C} G_c(a)\right)^* \twoheadrightarrow G_\infty(a) \ ,$$

where $(\,\cdot\,)^* : \mathbf{Set} \to \mathbf{Set}$ is the *Kleene star*, or the monad for monoids, while we have

$$\left|\left(\coprod_{c\in C} G_c(a)\right)^*\right| \leq \omega \times \sum_{c\in C} |G_c(a)| \leq \omega \times |C| \times \sup_{c\in C} |G_c(a)|$$

for each $a \in \mathcal{A}$. Since κ is regular and $|\mathcal{A} \times C| < \kappa$, it follows

$$|\underline{G_\infty}| \leq |\mathcal{A}| \times \omega \times |C| \times \sup_{a\in\mathcal{A}, c\in C} |G_c(a)| < \kappa \ ,$$

which shows $G_\infty \in \mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$. □

Lemma 2.2.9. *Let* G *be a crossed* $\mathcal{A}$*-group. Suppose we are given a subset* $S \subset \underline{G}$ *of cardinality less than* κ*. Then, there is a crossed* $\mathcal{A}$*-subgroup* $G' \subset G$ *such that* $S \subset \underline{G'}$ *and* $G' \in \mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$*.*

Proof. If $S = \varnothing$, the statement is obvious. In the case S is a singleton, say $S = \{x\}$ with $x \in G(a_0)$, for each $a \in \mathcal{A}$, we set $G'(a) \subset G(a)$ to be the subgroup generated by the subset

$$\{\varphi^*(x^\varepsilon) \mid \varepsilon = \pm 1,\ \varphi : a \to a_0 \in \mathcal{A}\} \ . \tag{2.2.3}$$

Since $G'(a) \subset G(a)$ is clearly a subgroup, in order to see G' actually forms a crossed $\mathcal{A}$-subgroup of G, it is enough to check it is an $\mathcal{A}$-subset. Note that since the subset (2.2.3) is closed under inverses, every element in $G'(a)$ can be written as products of elements of (2.2.3). For $\psi : b \to a \in \mathcal{A}$, $\varepsilon_i = \pm 1$, and $\varphi_i : a \to a_0 \in \mathcal{A}$, we have

$$\psi^*(\varphi_1^*(x^{\varepsilon_1}) \dots \varphi_k^*(x^{\varepsilon_k})) = (\varphi_1 \psi^{\varphi_2^*(x_2^{\varepsilon_2}) \dots \varphi_k^*(x_k^{\varepsilon_k})})^*(x_1^{\varepsilon_1}) \dots (\varphi_k\psi)^*(x_k^{\varepsilon_k}) \ ,$$

which shows $\psi^* : G(a) \to G(b)$ carries $G'(a)$ into $G'(b)$, and hence $G' \subset G$ is a crossed $\mathcal{A}$-subgroup. On the other hand, the definition of G' directly implies

$$|\underline{G'}| = \left|\coprod_{a\in\mathcal{A}} G'(a)\right| \leq \omega \times 2 \times |\mathcal{A}| < \kappa$$

so $G' \in \mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$.

In general case, for each $x \in S$, the proof of the singleton case implies that there is a crossed $\mathcal{A}$-subgroup $G'_x \subset G$ so that $x \in \underline{G'_x}$ and $G'_x \in \mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$. We take G' to be the image of the map

$$G'_1 * \cdots * G'_n \to G$$

of crossed $\mathcal{A}$-groups induced by the inclusions $G'_i \hookrightarrow G$. Clearly $S \subset \underline{G'}$, and Lemma 2.2.8 implies that $G' \in \mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa}$. This completes the proof. □

of Theorem 2.2.4. In view of Corollary 2.47 in [1], it suffices to show the category $\mathbf{CrsGrp}_{\mathcal{A}}$ is accessible; i.e. there is a set $\mathcal{S}$ of κ-small objects such that every object in $\mathbf{CrsGrp}_{\mathcal{A}}$ can be written as a κ-filtered colimit of objects from $\mathcal{S}$. According to Lemma 2.2.6, Lemma 2.2.7, and Lemma 2.2.9, we can take $\mathcal{S}$ to be any skeleton of the subcategory $\mathbf{CrsGrp}_{\mathcal{A}}^{<\kappa} \subset \mathbf{CrsGrp}_{\mathcal{A}}$. □

2.3 Computation of the terminal object

In the previous section, we proved Corollary 2.2.5 that asserts the category $\mathbf{CrsGrp}_{\mathcal{A}}$ has all small limits and colimits. In fact, Proposition 2.2.1, Corollary 2.2.2, and Proposition 2.2.3 provide relatively practical ways to compute colimits and limits except for the terminal object. On the other hand, as for a few categories $\mathcal{A}$, the terminal crossed $\mathcal{A}$-group was constructed by hand. For example, if $\mathcal{A} = \Delta$ or $\widetilde{\Delta}$, it is precisely the Hyperoctahedral crossed group $\mathfrak{H}$; see [24]. The goal of this section is to generalize their results and compute the terminal crossed $\mathcal{A}$-group for more general $\mathcal{A}$ including ∇ as well as Δ and $\widetilde{\Delta}$. Note that, although the arguments are highly abstract, the reader should keep the concrete examples in the mind; such as $\mathcal{A} = \Delta$, $\widetilde{\Delta}$, or ∇.

Definition. Let $\mathcal{A}$ be a category and $s \in \mathcal{A}$ an object. Then, an *internal co-relation* on s is a tuple $(\bar{s}; \iota_0, \iota_1)$ of an object $\bar{s} \in \mathcal{A}$ and a jointly-epimorphic pair $\iota_0, \iota_1 : s \rightrightarrows \bar{s}$ in $\mathcal{A}$. By abuse of notation, we often denote it just by $\bar{s}$.

If $\bar{s}$ is an internal co-relation on $s \in \mathcal{A}$, for each $a \in \mathcal{A}$, we have a map

$$(\iota_0^*, \iota_1^*) : \mathcal{A}(\bar{s}, a) \to \mathcal{A}(s, a) \times \mathcal{A}(s, a)$$

which is by definition injective since (ι_0, ι_1) is jointly-epimorphic. In other words, it exhibits $\mathcal{A}(\bar{s}, a)$ as a (binary) *relation* on $\mathcal{A}(s, a)$. Explicitly, for two morphisms $\alpha_0, \alpha_1 : s \rightrightarrows a$, they are connected by the relation if and only if there is a morphism $\bar{\alpha} : \bar{s} \to a$ such that $\alpha_0 = \bar{\alpha}\iota_0$ and $\alpha_1 = \bar{\alpha}\iota_1$. Hence, every morphism $a \to b \in \mathcal{A}$ induces a map $\mathcal{A}(s, a) \to \mathcal{A}(s, b)$ which preserves the relation induced by $\bar{s}$.

Definition. Let $\mathcal{A}$ be a category. An *internal well-co-order* on an object $s \in \mathcal{A}$ is an internal co-relation $\bar{s}$ on s such that the set $\mathcal{A}(s, a)$ is well-ordered by the induced relation for every $a \in \mathcal{A}$.

Example 2.3.1. Let **Ord** be the category of well-ordered sets and order-preserving maps. In particular, **Ord** contains all the finite ordinals $\underline{n} = \{0, \dots, n-1\}$. We have an obvious internal well-co-order on $\underline{1}$ as

$$\iota_0, \iota_1 : \underline{1} \rightrightarrows \underline{2} \ . \ ; \quad 0 \mapsto 0, 1$$

Hence, every full subcategory of **Ord** containing a diagram isomorphic to the above one, such as Δ and $\widetilde{\Delta}$, $\underline{1}$ is an internally well-co-ordered object. Namely, the generators $[0] \in \Delta$ and $\langle 1 \rangle \in \widetilde{\Delta}$ are canonically well-co-ordered objects.

Example 2.3.2. The object $\langle\!\langle 1 \rangle\!\rangle \in \nabla$ admits a canonical internal co-order:

$$\iota_0, \iota_1 : \langle\!\langle 1 \rangle\!\rangle \to \langle\!\langle 2 \rangle\!\rangle \ ; \quad 1 \mapsto 1, 2 \ .$$

It is easily verified that, for each $n \in \mathbb{N}$, the set $\nabla(\langle\!\langle 1 \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle)$ together with the order induced by the internal co-order above is identified with the ordered set $\langle\!\langle n \rangle\!\rangle$ itself. Thus, $\langle\!\langle 1 \rangle\!\rangle$ is a well-co-ordered object.

If $s \in \mathcal{A}$ admits an internal well-co-order $\bar{s}$, the corepresentable functor $\mathcal{A}(s, \cdot\,)$ lifts to the functor

$$\mathcal{A}(s, \cdot\,) : \mathcal{A} \to \mathbf{Ord} \ .$$

Hence, we can make use of the following splendid property of the category **Ord** through this functor.

Lemma 2.3.3. *Let $\varphi : A \to B$ be an order-preserving map between well-ordered sets such that the inverse image $\varphi^{-1}\{b\}$ is finite for each $b \in B$. Suppose we are given a permutation σ on A with the composition $\varphi\sigma$ again order-preserving. Then, we have $\varphi\sigma = \varphi$.*

Proof. It suffices to prove the permutation σ is restricted to each $A_b := \varphi^{-1}\{b\}$. Suppose we have $b \in B$ with $\sigma(A_b) \not\subset A_b$; in particular, we may assume b is the minimum among such elements of B since B is well-ordered. Take $a \in A_b$ such that $\varphi\sigma(a) \neq b$. Note that we have $\varphi\sigma(a) > b$; otherwise, the minimality of b implies σ restricts to a permutation on $A_{\varphi\sigma(a)}$. We have $a \notin A_{\varphi\sigma(a)}$ while $\sigma(\{a\} \cup A_{\varphi\sigma(a)}) \subset A_{\varphi\sigma(a)}$, which contradicts to the injectivity of σ since $A_{\varphi\sigma(a)}$ is finite.

Since A_b and $\sigma^{-1}(A_b)$ are finite sets with the same cardinality, so are two subsets $A_b \setminus \sigma^{-1}(A_b)$ and $\sigma^{-1}(A_b) \setminus A_b$ of A. The first one is non-empty, e.g. containing a, so we can take an element $a' \in \sigma^{-1}(A_b) \setminus A_b$. The minimality of b again implies $\varphi(a') > b = \varphi(a)$ so $a' > a$. We however have

$$\varphi\sigma(a') = b < \varphi\sigma(a) \ ,$$

which contradicts to the assumption that $\varphi\sigma$ preserves the order. ☐

Corollary 2.3.4. *Let A and B be finite well-ordered set. Then, the permutation group $\mathfrak{S}(B)$ on B admits a unique left action*

$$\mathfrak{S}(B) \times \mathbf{Ord}(A, B) \to \mathbf{Ord}(A, B) \ ; \quad (\sigma, \varphi) \mapsto \varphi^\sigma$$

such that, for each $\sigma \in \mathfrak{S}(B)$ and $\varphi \in \mathbf{Ord}(A, B)$, φ^σ is a map through which the composition $\sigma\varphi : A \to B$ factors after a permutation on A.

Proof. We may assume $B = \underline{n} = \{0, \dots, n-1\}$. Since the map $\sigma\varphi$ factors as

$$A \cong \varphi^{-1}\{0\} \star \cdots \star \varphi^{-1}\{n-1\} \xrightarrow{\text{perm.}} \varphi^{-1}\{\sigma^{-1}(0)\} \star \cdots \star \varphi^{-1}\{\sigma^{-1}(n-1)\} \to \underline{n} \ ,$$

the existence of φ^σ follows. We show the uniqueness of φ^σ. Say $\sigma\varphi = \varphi^\sigma\sigma'$ with $\sigma' \in \mathfrak{S}(B)$, and suppose we have another factorization $\sigma\varphi = \psi\tau$. Then, we have $\psi = \varphi^\sigma\sigma'\tau^{-1}$. Since A is finite, and since both φ^σ and ψ are order-preserving, Lemma 2.3.3 implies $\psi = \varphi^\sigma$. This guarantees the uniqueness of φ^σ. Now, the associativity of the action easily follows from the property of φ^σ. ☐

In the rest of the section, we assume $\mathcal{A}$ to be a category such that

(i) $\mathcal{A}$ is locally finite; i.e. each hom-set $\mathcal{A}(a,b)$ is finite;

(ii) it is equipped with a generator $s \in \mathcal{A}$, so the corepresentable functor $\mathcal{A}(s,\,\cdot\,) : \mathcal{A} \to \mathbf{Set}$ is by definition faithful;

(iii) s is internally well-co-ordered.

As seen in Example 2.3.1 and Example 2.3.2, the examples of $\mathcal{A}$ include Δ, $\widetilde{\Delta}$, and ∇.

Remark 2.3.5. If $\mathcal{A}$ is a category satisfying the conditions above, the relation on each $\mathcal{A}(s,a)$ induced by the internal co-relation $\bar{s}$ is reflexive. This means that every morphism $\alpha : s \to a$ determines a morphism $\bar{\alpha} : \bar{s} \to a$ such that $\bar{\alpha}\iota_0 = \bar{\alpha}\iota_1 = \alpha$. In addition, since ι_0 and ι_1 are jointly epimorphic, such $\bar{\alpha}$ is unique. Hence, we obtain a unique map

$$\texttt{refl}_a : \mathcal{A}(s,a) \to \mathcal{A}(\bar{s},a) \tag{2.3.1}$$

which is a common section of the precomposition maps with ι_0 and ι_1. It is easily verified that $\texttt{refl}_a$ is natural with respect to $a \in \mathcal{A}$; for every morphism $\varphi : a \to b$, we have

$$\varphi_*(\texttt{refl}(\alpha)) = \texttt{refl}(\varphi_*(\alpha)) \ .$$

Yoneda Lemma thus implies we have a map $\bar{s} \to s$ corepresenting $\texttt{refl}$.

For $\mathcal{A}$ above, note that we can think of $\mathcal{A}(a,b)$ as a subset of $\mathbf{Ord}(\mathcal{A}(s,a),\mathcal{A}(s,b))$ on which the group $\mathfrak{S}(\mathcal{A}(s,a))$ acts from the left. We define a group $\mathfrak{S}^{\mathcal{A}}(a)$ by

$$\mathfrak{S}^{\mathcal{A}}(a) := \{\sigma \in \mathfrak{S}(\mathcal{A}(s,a)) \mid \forall b \in \mathcal{A} : \sigma(\mathcal{A}(b,a)) \subset \mathcal{A}(b,a)\} \ .$$

In particular, $\mathfrak{S}^{\mathcal{A}}(a)$ acts on $\mathcal{A}(b,a)$ from the left for each $b \in \mathcal{A}$.

Example 2.3.6. If $\mathcal{A}$ is a full subcategory of $\widetilde{\Delta}$ containing $\langle 1\rangle$ and $\langle 2\rangle$, and if we take $\langle 1\rangle$ as the generator with the canonical co-well-order, then we have

$$\mathfrak{S}^{\mathcal{A}}(\langle n\rangle) = \mathfrak{S}(\mathcal{A}(\langle 1\rangle,\langle n\rangle)) \cong \mathfrak{S}^n \ .$$

In particular,

$$\mathfrak{S}^{\widetilde{\Delta}}(\langle n\rangle) \cong \mathfrak{S}_n \ , \quad \mathfrak{S}^{\Delta}([n]) \cong \mathfrak{S}_{n+1} \ .$$

Example 2.3.7. In the case $\mathcal{A} = \nabla$, the evaluation map

$$\nabla(\langle\!\langle 1\rangle\!\rangle,\langle\!\langle n\rangle\!\rangle) \to \langle\!\langle n\rangle\!\rangle \ ; \quad \alpha \mapsto \alpha(1)$$

is bijective, so we identify the two sets by the map. We claim

$$\mathfrak{S}^{\nabla}(\langle\!\langle n\rangle\!\rangle) = \mathfrak{S}(\langle n\rangle) \times \mathfrak{S}(\{-\infty,\infty\}) \tag{2.3.2}$$

for each $n \in \mathbb{N}$ respecting the action on $\langle\!\langle n\rangle\!\rangle$. Note that the action of the right hand side group on $\nabla(\langle\!\langle m\rangle\!\rangle,\langle\!\langle n\rangle\!\rangle)$ is given as follows: for $\varphi : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$, $\varphi^{(\sigma,\varepsilon)}$ is the map

$$\begin{aligned}\langle\!\langle m\rangle\!\rangle &\cong \varphi^{-1}\{\varepsilon(-\infty)\} \star \varphi^{-1}\{\sigma^{-1}(1)\} \star \cdots \star \varphi^{-1}\{\sigma^{-1}(n)\} \star \varphi^{-1}\{\varepsilon(\infty)\} \\ &\to \{-\infty\} \star \{1\} \star \cdots \star \{n\} \star \{\infty\} \cong \langle\!\langle n\rangle\!\rangle \ .\end{aligned}$$

For the left-to-right inclusion of (2.3.2), it suffices to see the subset $\{-\infty,\infty\} \subset \langle\!\langle n\rangle\!\rangle$ is stable under the action of $\mathfrak{S}^{\nabla}(\langle\!\langle n\rangle\!\rangle)$. This follows from the observation that the set is the image of the unique map $\langle\!\langle 0\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$. The other direction is obvious, so we obtain (2.3.2).

For each crossed $\mathcal{A}$-group G, the action of $G(a)$ on $\mathcal{A}(s,a)$ determines a group homomorphism

$$R_a : G(a) \to \mathfrak{S}^{\mathcal{A}}(a) \tag{2.3.3}$$

Lemma 2.3.8. *Let G be a crossed $\mathcal{A}$-group. Then, the action of $G(a)$ on each $\mathcal{A}(b,a)$ in the structure of crossed $\mathcal{A}$-group agrees with the one induced from $\mathfrak{S}^{\mathcal{A}}(a)$ through the group homomorphism R_a given in* (2.3.3)*.*

Proof. For each $x \in G(a)$ and $\varphi : b \to a \in \mathcal{A}$, we have the following commutative square:

$$\begin{array}{ccc} \mathcal{A}(s,b) & \xrightarrow{\varphi_*} & \mathcal{A}(s,a) \\ {\scriptstyle f^*(x)}\downarrow & & \downarrow{\scriptstyle x=R_a(x)} \\ \mathcal{A}(s,b) & \xrightarrow{\varphi_*^x} & \mathcal{A}(s,a) \end{array} .$$

Hence, the uniqueness of the action of $\mathfrak{S}^{\mathcal{A}}(a)$ on $\mathcal{A}(b,a)$ given in Corollary 2.3.4 implies $\varphi^x = \varphi^{R_a(x)}$. ☐

Lemma 2.3.9. *Let G be a crossed $\mathcal{A}$-group. Suppose $x \in G(a)$ and $\alpha \in \mathcal{A}(s,a)$. Then, for every morphism $\varphi : b \to a \in \mathcal{A}$, the permutation*

$$\varphi^*(x) : \mathcal{A}(s,b) \to \mathcal{A}(s,b)$$

restricts to a map $(\varphi_*)^{-1}\{\alpha\} \to (\varphi_*^x)^{-1}\{\alpha^x\}$ which is either order-preserving or order-reversing, depending only on x and α but not on φ.

Proof. Let $\varphi : b \to a \in \mathcal{A}$ be an arbitrary morphism, so we have a map $\varphi_* : \mathcal{A}(s,b) \to \mathcal{A}(s,a)$. Take any two elements $\psi_0, \psi_1 \in \mathcal{A}(s,b)$ with $\varphi_*(\psi_0) = \varphi_*(\psi_1) = \alpha$. We may assume $\psi_0 \le \psi_1$ so there is a morphism $\bar{\psi} : \bar{s} \to b$ with $\bar{\psi}\iota_0 = \psi_0$ and $\bar{\psi}\iota_1 = \psi_1$. Since $\varphi\bar{\psi}\iota_0 = \varphi\bar{\psi}\iota_1 = \alpha$, we have $\varphi\bar{\psi} = \mathtt{refl}(\alpha)$, where `refl` is the map defined in Remark 2.3.5, and the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{A}(s,\bar{s}) & \xrightarrow{\bar{\psi}_*} & \mathcal{A}(s,b) & \xrightarrow{\varphi_*} & \mathcal{A}(s,a) \\ {\scriptstyle \mathtt{refl}(\alpha)^*(x)}\downarrow & & {\scriptstyle \varphi^*(x)}\downarrow & & \downarrow{\scriptstyle x} \\ \mathcal{A}(s,\bar{s}) & \xrightarrow{\bar{\psi}^{\varphi^*(x)}} & \mathcal{A}(s,b) & \xrightarrow{\varphi_*^x} & \mathcal{A}(s,a) \end{array}$$

Hence, the order of two elements $\varphi^*(x)(\psi_0), \varphi^*(x)(\psi_1) \in \mathcal{A}(s,b)$ is determined by that of $\mathtt{refl}(\alpha)^*(x)(\iota_0)$ and $\mathtt{refl}(\alpha)^*(x)(\iota_1)$. It clearly no longer depends on φ nor elements $\psi_0, \psi_1 \in (\varphi_*)^{-1}\{\alpha\}$, so we obtain the result. ☐

We now construct a candidate for the terminal crossed $\mathcal{A}$-group. For each morphism $\varphi : b \to a \in \mathcal{A}$, we define a map

$$\beta_\varphi : C_2^{\times \mathcal{A}(s,a)} \to \mathfrak{S}(\mathcal{A}(s,b))$$

as follows: for $\vec{\varepsilon} = (\varepsilon_i)_{i \in \mathcal{A}(s,a)}$, $\beta_\varphi(\vec{\varepsilon}) \in \mathfrak{S}(\mathcal{A}(s,b))$ is the unique permutation such that

- it preserves the fibers of the map $\varphi_* : \mathcal{A}(s,b) \to \mathcal{A}(s,a)$;
- for each $i \in \mathcal{A}(s,a)$, the restricted map $\beta_\varphi(\vec{\varepsilon}) : (\varphi_*)^{-1}\{i\} \to (\varphi_*)^{-1}\{i\}$ is either order-preserving or order-reversing depending on $\varepsilon_i \in C_2$.

The map β_φ is obviously a group homomorphism. Also, we define a map

$$\varphi^* : \mathfrak{S}^{\mathcal{A}}(a) \to \mathfrak{S}(\mathcal{A}(s,b))$$

so that $\varphi^*(\sigma)$ is the unique permutation satisfying

(i) the square below is commutative:

$$\begin{CD} \mathcal{A}(s,b) @>{\varphi_*}>> \mathcal{A}(s,a) \\ @V{\varphi^*(\sigma)}VV @VV{\sigma}V \\ \mathcal{A}(s,b) @>{\varphi_*^\sigma}>> \mathcal{A}(s,a) \end{CD}$$

(ii) $\varphi^*(\sigma) : \mathcal{A}(s,b) \to \mathcal{A}(s,b)$ restricts to an order-preserving map

$$\varphi^*(\sigma) : (\varphi_*)^{-1}\{i\} \to (\varphi_*^x)^{-1}\{i^x\}$$

for each $i \in \mathcal{A}(s,a)$.

The action of $\mathfrak{S}^{\mathcal{A}}(a)$ on $\mathcal{A}(s,b)$ enables us to consider the semidirect product $\mathfrak{S}^{\mathcal{A}}(a) \ltimes C_2^{\times \mathcal{A}(s,a)}$, which is just a cartesian product $\mathfrak{S}^{\mathcal{A}}(a) \times C_2^{\times \mathcal{A}(s,a)}$ as a set together with the multiplication

$$(\tau; \vec{\zeta})(\sigma; \vec{\varepsilon}) = (\tau\sigma; \vec{\zeta} \cdot \sigma^*(\vec{\varepsilon})) = \Big(\tau\sigma; \big(\zeta_i \varepsilon_{\sigma(i)}\big)_{i \in \mathcal{A}(s,a)}\Big) \ .$$

We define

$$\mathfrak{W}^{\mathcal{A}}(a) := \Big\{ (\sigma; \vec{\varepsilon}) \in \mathfrak{S}^{\mathcal{A}}(a) \ltimes C_2^{\times \mathcal{A}(s,a)} \ \Big| \ \forall \varphi \in \mathcal{A}(b,a) : \varphi^*(\sigma)\beta_\varphi(\vec{\varepsilon}) \in \mathfrak{S}^{\mathcal{A}}(b) \Big\} \ . \tag{2.3.4}$$

Remark 2.3.10. The permutation $\tau = \varphi^*(\sigma)\beta_\varphi(\vec{\varepsilon})$ on $\mathcal{A}(s,b)$ appearing in (2.3.4) is the permutation characterized by the following two properties:

(i) the square below is commutative:

$$\begin{CD} \mathcal{A}(s,b) @>{\varphi_*}>> \mathcal{A}(s,a) \\ @V{\tau}VV @VV{\sigma}V \\ \mathcal{A}(s,b) @>{\varphi_*^\sigma}>> \mathcal{A}(s,a) \end{CD}$$

(ii) the restriction $\tau : (\varphi_*)^{-1}\{i\} \to (\varphi_*^\sigma)^{-1}\{i^\sigma\}$ is either order-preserving or order-reversing depending on $\varepsilon_i \in C_2$.

Proposition 2.3.11. *Let $\mathcal{A}$ be as above. Then, the subset $\mathfrak{W}^{\mathcal{A}}(a) \subset \mathfrak{S}^{\mathcal{A}}(a) \ltimes C_2^{\times \mathcal{A}(s,a)}$ given in* (2.3.4) *has the following properties:*

(1) *$\mathfrak{W}^{\mathcal{A}}(a)$ is closed under multiplication and the inverses; hence it is a subgroup;*

(2) *for each morphism $\varphi : b \to a \in \mathcal{A}$, define*

$$\varphi^* : \mathfrak{W}^{\mathcal{A}}(a) \to \mathfrak{W}^{\mathcal{A}}(b) \ ; \quad (\sigma; \vec{\varepsilon}) \mapsto (\varphi^*(\sigma)\beta_\varphi(\vec{\varepsilon}); \varphi^*(\vec{\varepsilon})) \ .$$

Then, this defines a structure of $\mathcal{A}$-sets on $\mathfrak{W}^{\mathcal{A}}$.

Moreover, the family $\{\mathfrak{W}^{\mathcal{A}}(a)\}_{a\in\mathcal{A}}$ forms a crossed $\mathcal{A}$-group.

Proof. We first show (1). For each $(\sigma;\vec{\varepsilon}),(\tau;\vec{\zeta})\in\mathfrak{W}^{\mathcal{A}}(a)$, and for every morphism $\varphi:b\to a\in\mathcal{A}$, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{A}(s,b) & \xrightarrow{\varphi_*} & \mathcal{A}(s,a) \\
{\scriptstyle \varphi^*(\sigma)\beta_\varphi(\vec{\varepsilon})}\downarrow & & \downarrow{\scriptstyle \sigma} \\
\mathcal{A}(s,b) & \xrightarrow{\varphi_*^{\sigma}} & \mathcal{A}(s,a) \\
{\scriptstyle (\varphi^\sigma)^*(\tau)\beta_{\varphi^\sigma}(\vec{\zeta})}\downarrow & & \downarrow{\scriptstyle \tau} \\
\mathcal{A}(s,b) & \xrightarrow{\varphi_*^{\tau\sigma}} & \mathcal{A}(s,a)
\end{array}$$

Verifying the conditions in Remark 2.3.10, one will see the left vertical composition coincides with the permutation $\varphi^*(\tau\sigma)\beta_\varphi(\vec{\zeta}\cdot\sigma^*(\vec{\varepsilon}))$. It follows that $(\vec{\zeta};\tau)(\vec{\varepsilon};\sigma)\in\mathfrak{W}^{\mathcal{A}}(a)$. Closedness under the inverses is proved similarly.

As for the part (2), note that the map $\varphi^*:\mathfrak{W}^{\mathcal{A}}(a)\to\mathfrak{W}^{\mathcal{A}}(b)$ is actually well-defined. Indeed, for every morphism $\psi:c\to b\in\mathcal{A}$, we have a commutative diagram below.

$$\begin{array}{ccccc}
\mathcal{A}(s,c) & \xrightarrow{\psi_*} & \mathcal{A}(s,b) & \xrightarrow{\varphi_*} & \mathcal{A}(s,a) \\
{\scriptstyle \psi^*(\varphi^*(\sigma)\beta_\varphi(\vec{\varepsilon}))\beta_\psi(\varphi^*(\vec{\varepsilon}))}\downarrow & & \downarrow{\scriptstyle \varphi^*(\sigma)\beta_\varphi(\vec{\varepsilon})} & & \downarrow{\scriptstyle \sigma} \\
\mathcal{A}(s,c) & \xrightarrow{\psi^{\varphi^*(\sigma)\beta_\varphi(\vec{\varepsilon})}} & \mathcal{A}(s,b) & \xrightarrow{\varphi^{\sigma}} & \mathcal{A}(s,a)
\end{array}$$

Similarly to the part (1), one can check the conditions in Remark 2.3.10 to see the left vertical arrow equals to the permutation $(\varphi\psi)^*(\sigma)\beta_{\varphi\psi}(\vec{\varepsilon})$, which belongs to $\mathfrak{S}^{\mathcal{A}}(c)$ since $(\sigma;\vec{\varepsilon})\in\mathfrak{W}^{\mathcal{A}}(a)$. This argument also shows the functoriality, so $\mathfrak{W}^{\mathcal{A}}$ is an $\mathcal{A}$-set.

It remains to show $\mathfrak{W}^{\mathcal{A}}$ is a crossed $\mathcal{A}$-group. However, one can notice the proofs of (1) and (2) above also show $\mathfrak{W}^{\mathcal{A}}$ satisfies the two conditions on crossed groups respectively. ☐

Unfortunately, for a crossed $\mathcal{A}$-group G, the group homomorphism $R_a:G(a)\to\mathfrak{S}^{\mathcal{A}}(a)$ given in (2.3.3) does not define a map of $\mathcal{A}$-sets in general. Nevertheless, it turns out that the only obstruction is *parities*, so we can put that information on the codomain of R_a. More precisely, define a map $\vec{\varepsilon}_a:G(a)\to C_2^{\times\mathcal{A}(s,a)}$ as follows: recall that for each $x\in G(a)$ and each $\alpha\in\mathcal{A}(s,a)$, the square below is commutative:

$$\begin{array}{ccc}
\mathcal{A}(s,\bar{s}) & \xrightarrow{\mathtt{refl}(\alpha)_*} & \mathcal{A}(s,a) \\
{\scriptstyle \mathtt{refl}(\alpha)^*(x)}\downarrow & & \downarrow{\scriptstyle x} \\
\mathcal{A}(s,\bar{s}) & \xrightarrow{\mathtt{refl}(\alpha^x)_*} & \mathcal{A}(s,a)
\end{array}$$

We set $\vec{\varepsilon}_a(x)=(\varepsilon_a(x)_\alpha)_{\alpha\in\mathcal{A}(s,a)}$ by

$$\varepsilon_a(x)_\alpha := \begin{cases} 0 & \mathtt{refl}(\alpha)^*(x)(\iota_0) < \mathtt{refl}(\alpha)^*(x)(\iota_1) \\ 1 & \mathtt{refl}(\alpha)^*(x)(\iota_0) > \mathtt{refl}(\alpha)^*(x)(\iota_1)\ . \end{cases}$$

Lemma 2.3.12. *Let G be as above. Then, for each $a \in \mathcal{A}$ and $x \in G(a)$, the pair*

$$(R_a(x); \vec{\varepsilon}_a(x)) \in \mathfrak{S}(\mathcal{A}(s,a)) \ltimes C_2^{\times \mathcal{A}(s,a)}$$

belongs to the subgroup $\mathfrak{W}^{\mathcal{A}}(a)$ defined in (2.3.4)*. Moreover, the induced maps*

$$\tilde{R}_a := (R_a; \vec{\varepsilon}_a) : G(a) \to \mathfrak{W}^{\mathcal{A}}(a)$$

form a map $G \to \mathfrak{W}^{\mathcal{A}}$ of crossed $\mathcal{A}$-groups.

Proof. Verifying the conditions in Remark 2.3.10, one can observe that, for each $x \in G(a)$, and for each $\varphi : b \to a \in \mathcal{A}$, we have

$$R_a(\varphi^*(x)) = \varphi^*(R_a(x)) \cdot \beta_\varphi(\vec{\varepsilon}_a(x)) \ . \tag{2.3.5}$$

The left hand side clearly belongs to $\mathfrak{S}^{\mathcal{A}}(a)$, this implies $(R_a(x); \vec{\varepsilon}_a(x)) \in \mathfrak{W}^{\mathcal{A}}(a)$ by the definition (2.3.4) of $\mathfrak{W}^{\mathcal{A}}$. On the other hand, it is also verified that

$$\vec{\varepsilon}_a(\varphi^*(x)) = \varphi^*(\vec{\varepsilon}_a(x)) \ . \tag{2.3.6}$$

The equation (2.3.6) and (2.3.5) imply that $\tilde{R}_a = (R_a; \vec{\varepsilon}_a)$ is natural with respect to $a \in \mathcal{A}$, so it defines a map $\tilde{R} : G \to \mathfrak{W}^{\mathcal{A}}$ of $\mathcal{A}$-sets.

It is obvious that $\tilde{R} : G \to \mathfrak{W}^{\mathcal{A}}$ respects the action on each $\mathcal{A}(a,b)$, so it remains to prove it is a degreewise group homomorphism. Since the preservation of the unit is straightforward, it suffices to show, for each $a \in \mathcal{A}$, $\tilde{R}_a : G(a) \to \mathfrak{W}^{\mathcal{A}}(a)$ preserves multiplications. For each $x, y \in G(a)$, and for each $\alpha \in \mathcal{A}(s,a)$, we have a commutative diagram below:

$$\begin{array}{ccc}
\mathcal{A}(s,\bar{s}) & \xrightarrow{\mathtt{refl}(\alpha)_*} & \mathcal{A}(s,a) \\
{\scriptstyle \mathtt{refl}(\alpha)^*(x)}\downarrow & & \downarrow{\scriptstyle x} \\
\mathcal{A}(s,\bar{s}) & \xrightarrow{\mathtt{refl}(\alpha^x)_*} & \mathcal{A}(s,a) \\
{\scriptstyle \mathtt{refl}(\alpha^x)^*(y)}\downarrow & & \downarrow{\scriptstyle y} \\
\mathcal{A}(s,\bar{s}) & \xrightarrow{\mathtt{refl}(\alpha^{yx})_*} & \mathcal{A}(s,a)
\end{array} \tag{2.3.7}$$

One can deduce from (2.3.7) that

$$\varepsilon_a(yx)_\alpha = \varepsilon_a(y)_{\alpha^x} \varepsilon_a(x) \ ,$$

which implies $\vec{\varepsilon}_a(yx) = R_a(x)_*(\vec{\varepsilon}_a(y))\vec{\varepsilon}_a(x)$. Thus, we obtain

$$\begin{aligned}
\tilde{R}(yx) &= (R_a(yx); \vec{\varepsilon}_a(yx)) \\
&= (R_a(y)R_a(x); R_a(x)_*(\vec{\varepsilon}_a(y))\vec{\varepsilon}_a(x)) \\
&= (R_a(y); \vec{\varepsilon}_a(y)) \cdot (R_a(x); \vec{\varepsilon}_a(x))
\end{aligned}$$

so that $R_a : G(a) \to \mathfrak{W}^{\mathcal{A}}(a)$ is a group homomorphism. □

Theorem 2.3.13 (cf. Theorem 1.4 in [47])**.** *Let $\mathcal{A}$ be a small locally finite category equipped with an internally well-co-ordered generator $s \in \mathcal{A}$. Then, the Weyl crossed $\mathcal{A}$-group $\mathfrak{W}^{\mathcal{A}}$ is a terminal object in the category* $\mathbf{CrsGrp}_{\mathcal{A}}$.

Proof. By virtue of Lemma 2.3.12, it suffices to show that, for each crossed $\mathcal{A}$-group, the map $\tilde{R} : G \to \mathfrak{W}^{\mathcal{A}}$ is the unique map of crossed $\mathcal{A}$-group for each $G \in \mathbf{CrsGrp}_{\mathcal{A}}$. Since $\mathfrak{W}^{\mathcal{A}}(a)$ is, as a set, a subset of the direct product

$$\mathfrak{S}^{\mathcal{A}}(a) \times C_2^{\mathcal{A}(s,a)}$$

for each $a \in \mathcal{A}$, an $\mathcal{A}$-map $f : G \to \mathfrak{W}^{\mathcal{A}}$ is determined by maps

$$f_{\mathsf{perm}} : G(a) \to \mathfrak{S}^{\mathcal{A}}(a) \ , \quad f_{\mathsf{sign}} = (f_{\mathsf{sign}}^{(\alpha)})_{\alpha \in \mathcal{A}(s,a)} : G(a) \to C_2^{\times \mathcal{A}(s,a)}$$

for each $a \in \mathcal{A}$. If f is a map of crossed $\mathcal{A}$-group, the map f_{perm} clearly has to be the one associated to the action of $G(a)$ on $\mathcal{A}(s,a)$. On the other hand, each $f_{\mathsf{sign}}^{(\alpha)}$ is also determined automatically thanks to the naturality of f and Lemma 2.3.9. It follows that $\tilde{R}$ is the only map into $\mathfrak{W}^{\mathcal{A}}$. □

Corollary 2.3.14. *Let $\mathcal{A}$ be as above. Suppose G is an $\mathcal{A}$-set which is equipped with a degreewise group structure. Then, the following data are equivalent:*

(a) *left actions of $G(a)$ on $\mathcal{A}(b,a)$ for $a, b \in \mathcal{A}$ which exhibit G as a crossed $\mathcal{A}$-group;*

(b) *a map $G \to \mathfrak{W}^{\mathcal{A}}$ of $\mathcal{A}$-sets which is a degreewise group homomorphism.*

Corollary 2.3.15. *Let $\mathcal{A}$ be as above, and let G and H be crossed $\mathcal{A}$-groups. Then, a map $f : G \to H$ of $\mathcal{A}$-sets which is a degreewise group homomorphism is a map of crossed $\mathcal{A}$-groups if and only if the triangle below is commutative:*

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ & {\scriptstyle \tilde{R}} \searrow \quad \swarrow {\scriptstyle \tilde{R}} & \\ & \mathfrak{W}^{\mathcal{A}} & \end{array}$$

Example 2.3.16. In the case $\mathcal{A} = \widetilde{\Delta}$, since it is a full subcategory of $\mathbf{Ord}$, we have

$$\mathfrak{S}^{\widetilde{\Delta}}(\langle n \rangle) = \mathbf{Ord}(\widetilde{\Delta}(\langle 1 \rangle, \langle n \rangle)) \cong \mathfrak{S}_n$$
$$\mathfrak{W}^{\widetilde{\Delta}}(\langle n \rangle) = \mathfrak{S}_n \ltimes C_2^{\times n} \cong H_n \ .$$

It is easily verified that these induce an isomorphism from $\mathfrak{W}^{\widetilde{\Delta}}$ to the hyperoctahedral crossed simplicial group $\mathfrak{H}$ described in Example 2.1.11

Example 2.3.17. Recall that, in the case $\mathcal{A} = \nabla$, we saw in Example 2.3.7 that $\mathfrak{S}^{\nabla}(\langle\langle n \rangle\rangle) \cong \mathfrak{S}(\langle n \rangle) \times \mathfrak{S}(\{-\infty, \infty\})$. We claim that an element of $\mathfrak{S}^{\nabla}(\langle\langle n \rangle\rangle) \ltimes C_2^{\times \langle\langle n \rangle\rangle}$ of the form

$$x = ((\sigma, \theta); \varepsilon_{-\infty}, \varepsilon_1 \dots, \varepsilon_n, \varepsilon_\infty)$$

for $(\sigma, \theta) \in \mathfrak{S}(\langle n \rangle) \times \mathfrak{S}(\{-\infty, \infty\})$ and $\varepsilon_i \in C_2$ belongs to $\mathfrak{W}^{\nabla}(\langle\langle n \rangle\rangle)$ if and only if $\theta = \varepsilon_{-\infty} = \varepsilon_\infty$ under the canonical identification $\mathfrak{S}(\{-\infty, \infty\}) \cong C_2$. For a map $\varphi : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \nabla$, the permutation $\varphi^*(x)$ restricts to bijections

$$\varphi^{-1}\{-\infty\} \to \varphi^{-1}\{\theta(-\infty)\} \ , \quad \varphi^{-1}\{\infty\} \to \varphi^{-1}\{\theta(\infty)\}$$

which are either order-preserving or order-reversing according to $\varepsilon_{-\infty}$ and ε_∞ respectively. On the other hand, in view of the computation of Example 2.3.7,

$\varphi^*(x)$ belongs to $\mathfrak{S}^\nabla(\langle\!\langle m\rangle\!\rangle)$ if and only if it preserves the subset $\{-\infty,\infty\}$. It is easily seen that this happens for every $\varphi : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle$ precisely when the elements θ and $\varepsilon_{\pm\infty}$ all coincide. As a consequence, we obtain an isomorphism

$$\mathfrak{W}^\nabla(\langle\!\langle n\rangle\!\rangle) \cong H_n \times C_2 \ .$$

We finally note that the explicit computation of terminal crossed $\mathcal{A}$-group leads to a classification.

Proposition 2.3.18 (cf. Theorem 3.6 in [24])**.** *Let $\mathcal{A}$ be an arbitrary small category. Then, for every crossed $\mathcal{A}$-group G, there is a sequence*

$$G^{\mathsf{nc}} \hookrightarrow G \twoheadrightarrow G^{\mathsf{red}} \tag{2.3.8}$$

of maps of crossed $\mathcal{A}$-groups such that

(i) *the sequence* (2.3.8) *is degreewisely a short exact sequence of groups;*

(ii) G^{nc} *is a (non-crossed) $\mathcal{A}$-group;*

(iii) G^{red} *is a crossed $\mathcal{A}$-subgroup of the terminal crossed $\mathcal{A}$-group.*

Moreover, the sequence (2.3.8) *extends to a functor from* $\mathbf{CrsGrp}_{\mathcal{A}}$ *into the category of degreewise short exact sequences in* $\mathbf{CrsGrp}_{\mathcal{A}}$*.*

Proof. Put $\mathfrak{T}_{\mathcal{A}}$ the terminal crossed $\mathcal{A}$-group, and set G^{red} to be the image of the unique map $G \to \mathfrak{T}_{\mathcal{A}}$. Then, we can define G^{nc} by the following pullback square:

$$\begin{array}{ccc} G^{\mathsf{nc}} & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ G & \longrightarrow & G^{\mathsf{red}} \end{array}$$

The required properties are verified easily. □

Example 2.3.19. In Proposition 3.5 in [24], there is a complete list of crossed simplicial subgroups of $\mathfrak{W}^\Delta \cong \boldsymbol{\mathcal{H}}$ as in Table 2.1.

name	**symbol**	**n-th group**
Trivial	$*$	1
Reflexive	C_2	C_2
Cyclic	$\boldsymbol{\mathcal{C}}$	C_{n+1}
Dihedral	$\boldsymbol{\mathcal{D}}$	D_{n+1}
Symmetric	$\boldsymbol{\mathcal{S}}$	$\mathfrak{S}_{n+1}$
Reflexosymmetric	$\widetilde{\boldsymbol{\mathcal{S}}}$	$\mathfrak{S}_{n+1} \times C_2$
Weyl (Hyperoctahedral)	$\mathfrak{W}^\Delta \cong \boldsymbol{\mathcal{H}}$	H_{n+1}

Table 2.1: The crossed simplicial subgroups of $\mathfrak{W}^\Delta$

Example 2.3.20. We will see in Example 2.5.5 in Section 2.5 that the embedding $\Delta \hookrightarrow \widetilde{\Delta}$ induces a fully faithful embedding

$$\mathbf{CrsGrp}_\Delta \hookrightarrow \mathbf{CrsGrp}_{\widetilde{\Delta}}$$

which sends $\mathfrak{W}^\Delta$ to $\mathfrak{W}^{\widetilde{\Delta}}$. Hence, we obtain the same list of augmented crossed simplicial subgroups of $\mathfrak{W}^{\widetilde{\Delta}}$ as Table 2.1 while the indices are shifted by 1 because of the identification $[n] \cong \langle n+1\rangle$.

In the appendix, we will compute all the crossed interval subgroups of $\mathfrak{W}^{\nabla}$.

2.4 Crossed groups as monoid objects

We proved Theorem 2.2.4 that asserts the category $\mathbf{CrsGrp}_{\mathcal{A}}$ is locally presentable for each small category $\mathcal{A}$ by highly abstract argument, and it was for the sake of the existence of terminal crossed groups. On the other hand, we also proved the latter independently in a more explicit way in Theorem 2.3.13 for some special categories $\mathcal{A}$. Combining this with Proposition 2.2.1 and Proposition 2.2.3, which are proved in more or less constructive ways, one can recover the completeness and the cocompleteness of $\mathbf{CrsGrp}_{\mathcal{A}}$ mentioned in Corollary 2.2.5. In this section, we are going further: we see the category $\mathbf{CrsGrp}_{\mathcal{A}}$ is even *algebraic*, in some sense, over a presheaf category, which guarantees the locally presentability. More precisely, for a crossed $\mathcal{A}$-group G, we have the forgetful functor

$$\mathbf{CrsGrp}_{\mathcal{A}}^{/G} \to \mathbf{Set}_{\mathcal{A}}^{/G} \tag{2.4.1}$$

between slice categories. One has the category $\mathbf{CrsGrp}_{\mathcal{A}}$ on the left hand side when he takes G to be the terminal crossed $\mathcal{A}$-group, say $G = \mathfrak{T}_{\mathcal{A}}$. Note that though the functor (2.4.1) really forgets degreewise group structures, it remembers the actions on each hom-set $\mathcal{A}(a,b)$ through that of G, while the category $\mathbf{Set}_{\mathcal{A}}^{/G}$ is just a presheaf topos. This suggests that the *true* underlying category of $\mathbf{CrsGrp}_{\mathcal{A}}$ should be not $\mathbf{Set}_{\mathcal{A}}$ but the slice category $\mathbf{Set}_{\mathcal{A}}^{/\mathfrak{T}_{\mathcal{A}}}$. Throughout this section, we fix a small category $\mathcal{A}$ and a crossed $\mathcal{A}$-group G and aim to see (2.4.1) has a *good* left adjoint.

Notation. For each object $X \in \mathbf{Set}_{\mathcal{A}}^{/G}$, say $p : X \to G$ is the structure map, we consider its "action" on hom-sets as

$$X(b) \times \mathcal{A}(a,b) \to \mathcal{A}(a,b)\ ; \quad (x,\varphi) \mapsto \varphi^x := \varphi^{p(x)}$$

for $a, b \in \mathcal{A}$ even though $X(b)$ is no longer a group.

To begin with, we introduce the following construction.

Definition. For $K \in \mathbf{Set}_{\mathcal{A}}$ and $X \in \mathbf{Set}_{\mathcal{A}}^{/G}$, we define an $\mathcal{A}$-set $K \rtimes_G X$ as follows:

- for each $a \in \mathcal{A}$, we set $(K \rtimes_G X)(a) := K(a) \times X(a)$;
- for each $\varphi : a \to b \in \mathcal{A}$, we set

$$\varphi^* : (K \rtimes_G X)(b) \to (K \rtimes_G X)(a)\ ; \quad (k,x) \mapsto ((\varphi^x)^*(k), \varphi^*(x))\ .$$

Remark 2.4.1. The operation $\rtimes$ was originally introduced by Krasauskas in Definition 2.1 in [47] in the case $\mathcal{A} = \Delta$ and $G = \mathfrak{W}^{\Delta}$.

To see $K \rtimes_G X$ above actually defines an $\mathcal{A}$-set, it is convenient to consider the map

$$\mathrm{crs}_X : X(b) \times \mathcal{A}(a,b) \to \mathcal{A}(a,b) \times X(a)\ ; \quad (x,\varphi) \mapsto (\varphi^x, \varphi^*(x)) \tag{2.4.2}$$

for each $X \in \mathbf{Set}_{\mathcal{A}}^{/G}$. Similarly to the case of crossed groups, we have the following commutative diagram:

$$\begin{array}{ccc} X(c) \times \mathcal{A}(b,c) \times \mathcal{A}(a,b) & \xrightarrow{(\mathrm{id} \times \mathrm{crs}_X)(\mathrm{crs}_X \times \mathrm{id})} & \mathcal{A}(b,c) \times \mathcal{A}(a,b) \times X(a) \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\mathrm{comp}} & & \Big\downarrow{\scriptstyle \mathrm{comp}\times\mathrm{id}} \\ X(c) \times \mathcal{A}(a,c) & \xrightarrow{\mathrm{crs}_X} & \mathcal{A}(a,c) \times X(a) \end{array} \tag{2.4.3}$$

Note that the $\mathcal{A}$-set structure on $K \rtimes_G X$ is given by

$$\begin{aligned} (K \rtimes_G X)(b) \times \mathcal{A}(a,b) &= K(b) \times X(b) \times \mathcal{A}(a,b) \\ &\xrightarrow{\mathrm{id}\times\mathrm{crs}_X} K(b) \times \mathcal{A}(a,b) \times X(a) \\ &\xrightarrow{\mathrm{act}_K\times\mathrm{id}} K(a) \times X(a) \\ &= (K \rtimes_G X)(a) \ . \end{aligned} \tag{2.4.4}$$

Then, combining (2.4.3) and (2.4.4), one can verify $K \rtimes_G X$ is actually an $\mathcal{A}$-set.

Lemma 2.4.2. *If two objects $a, b \in \mathcal{A}$ are fixed, the map* crs *given in* (2.4.2) *is natural with respect to $X \in \mathbf{Set}_{\mathcal{A}}^{/G}$.*

Proof. Suppose $f : X \to Y \in \mathbf{Set}_{\mathcal{A}}^{/G}$. We have to show the square below commutes:

$$\begin{array}{ccc} X(b) \times \mathcal{A}(a,b) & \xrightarrow{\mathrm{crs}_X} & \mathcal{A}(a,b) \times X(a) \\ \Big\downarrow{\scriptstyle f\times\mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}\times f} \\ Y(b) \times \mathcal{A}(a,b) & \xrightarrow{\mathrm{crs}_Y} & \mathcal{A}(a,b) \times Y(a) \end{array}$$

For $(x, \varphi) \in X(b) \times \mathcal{A}(a,b)$, we have

$$\begin{aligned} (\mathrm{id} \times f) \circ \mathrm{crs}_X(x,\varphi) &= (\varphi^x, f\varphi^*(x)) \ , \\ \mathrm{crs}_Y \circ (f \times \mathrm{id})(x,\varphi) &= (\varphi^{f(x)}, \varphi^* f(x)) \ . \end{aligned}$$

Since f is a map of $\mathcal{A}$-set over G, the actions of x and $f(x)$ on $\mathcal{A}(a,b)$ agree with each other, so the required result follows. ☐

Corollary 2.4.3. *The assignment $(K, X) \to K \rtimes_G X$ defines a functor*

$$\rtimes_G : \mathbf{Set}_{\mathcal{A}} \times \mathbf{Set}_{\mathcal{A}}^{/G} \to \mathbf{Set}_{\mathcal{A}} \ .$$

The key observation is that we can lift the functor $\rtimes_G$ to a monoidal structure on $\mathbf{Set}_{\mathcal{A}}^{/G}$. For this, note that the degreewise multiplication gives rise to an $\mathcal{A}$-map

$$\mu : G \rtimes_G G \to G \ .$$

Indeed, for $\varphi : a \to b \in \mathcal{A}$, the condition (ii) on crossed groups implies the following square is commutative:

$$\begin{array}{ccc} (G \rtimes_G G)(b) & \xrightarrow{\mu} & G(b) \\ \Big\downarrow{\scriptstyle \varphi^*} & & \Big\downarrow{\scriptstyle \varphi^*} \\ (G \rtimes_G G)(a) & \xrightarrow{\mu} & G(a) \end{array}$$

Hence, we can define a functor

$$\rtimes_G : \mathbf{Set}_{\mathcal{A}}^{/G} \times \mathbf{Set}_{\mathcal{A}}^{/G} \to \mathbf{Set}_{\mathcal{A}}^{/G} \tag{2.4.5}$$

by

$$(X, Y) \mapsto (X \rtimes_G Y \to G \rtimes_G G \xrightarrow{\mu} G)\ .$$

Proposition 2.4.4. *The functor* (2.4.5) *gives a monoidal structure on* $\mathbf{Set}_{\mathcal{A}}^{/G}$ *where the unit is the terminal* $\mathcal{A}$*-set* $*$ *with the unique map* $* \to G$ *of crossed* $\mathcal{A}$*-groups. Moreover, the monoidal structure is biclosed; i.e. there are functors*

$$\mathrm{Hom}_G^{\mathsf{L}}, \mathrm{Hom}_G^{\mathsf{R}} : (\mathbf{Set}_{\mathcal{A}}^{/G})^{\mathrm{op}} \times \mathbf{Set}_{\mathcal{A}}^{/G} \to \mathbf{Set}_{\mathcal{A}}^{/G}$$

together with natural isomorphisms

$$\begin{aligned}\mathbf{Set}_{\mathcal{A}}^{/G}(X \rtimes_G Y, Z) &\cong \mathbf{Set}_{\mathcal{A}}^{/G}(X, \mathrm{Hom}_G^{\mathsf{R}}(Y, Z)) \\ &\cong \mathbf{Set}_{\mathcal{A}}^{/G}(Y, \mathrm{Hom}_G^{\mathsf{L}}(X, Z))\ .\end{aligned}$$

Proof. We first show that, for $X, Y, Z \in \mathbf{Set}_{\mathcal{A}}^{/G}$, the degreewise canonical identification actually gives an isomorphism

$$(X \rtimes_G Y) \rtimes_G Z \cong Z \rtimes_G (Y \rtimes_G Z)\ .$$

Since it is clearly a degreewise bijection, it suffices to show it is actually a map of $\mathcal{A}$-sets over G. Suppose $\varphi : a \to b \in \mathcal{A}$ is a morphism in $\mathcal{A}$. Then, an easy computation shows that both maps

$$\begin{aligned}\varphi^* &: ((X \rtimes_G Y) \rtimes_G Z)(b) \to ((X \rtimes_G Y) \rtimes_G Z)(a) \\ \varphi^* &: (X \rtimes_G (Y \rtimes_G Z))(b) \to (X \rtimes_G (Y \rtimes_G Z))(a)\end{aligned}$$

are identified with the map $\varphi^* : X(b) \times Y(b) \times Z(b) \to X(a) \times Y(a) \times Z(a)$ given by

$$\varphi^*(x, y, z) = (((\varphi^y)^z)^*(x), (\varphi^z)^*(y), z)\ .$$

Thus, we obtain a canonical identification $(X \rtimes_G Y) \rtimes_G Z \cong X \rtimes_G (Y \rtimes_G Z)$ as $\mathcal{A}$-sets. Actually it is an isomorphism over G; indeed, the structure maps into G are given by the common formula

$$X(a) \times Y(a) \times Z(a) \to G(a) \qquad (x, y, z) \mapsto p(x)q(y)r(z)\ ,$$

where $p : X \to G$, $q : Y \to G$, and $r : Z \to G$ are the structure maps. Therefore, we obtain an associativity isomorphism for the functor $\rtimes_G$. The unitality of $*$ is easily verify, so the first assertion follows.

To see the monoidal structure is biclosed, note that the category $\mathbf{Set}_{\mathcal{A}}^{/G}$ is a presheaf topos and so locally presentable. Hence, by General Adjoint Functor Theorem, it suffices to show the functor $\rtimes_G$ preserves arbitrary small colimits in each variable. Notice that colimits in $\mathbf{Set}_{\mathcal{A}}^{/G}$ are computed in the category $\mathbf{Set}_{\mathcal{A}}$ and so agree with the degreewise ones. Now, the functor $\rtimes_G$ is degreewisely just the cartesian product, so the problem is reduced to the case $\mathcal{A} = *$ where the result is obvious. □

Remark 2.4.5. We have an explicit description of $\mathcal{A}$-sets $\mathrm{Hom}^{\mathsf{R}}_G(Y,Z)$ and $\mathrm{Hom}^{\mathsf{L}}_G(X,Z)$ as follows: for each $a \in \mathcal{A}$, we denote by $\mathcal{A}[a]$ the $\mathcal{A}$-set represented by a. Then, we have

$$\mathrm{Hom}^{\mathsf{R}}_G(Y,Z)(a) = \coprod_{\mathcal{A}[a]\to G} \mathbf{Set}_{\mathcal{A}}(\mathcal{A}[a] \rtimes_G Y, Z)\ ,$$
$$\mathrm{Hom}^{\mathsf{L}}_G(X,Z)(a) = \coprod_{\mathcal{A}[a]\to G} \mathbf{Set}_{\mathcal{A}}(X \rtimes_G \mathcal{A}[a], Z)\ .$$

Remark 2.4.6. Similarly to Proposition 2.4.4, one can also prove that the functor $\rtimes_G : \mathbf{Set}_{\mathcal{A}} \times \mathbf{Set}^{/G}_{\mathcal{A}} \to \mathbf{Set}_{\mathcal{A}}$ given in Corollary 2.4.3 defines a right action of the monoidal category $(\mathbf{Set}^{/G}_{\mathcal{A}}, \rtimes_G)$ on the category $\mathbf{Set}_{\mathcal{A}}$. Note that, in the case $\mathcal{A} = \Delta$, this functor was discussed in Section 4 and 5 in [24], where they wrote $F_G(X) := X \rtimes_G G$ for $X \in \mathbf{Set}_{\Delta}$ and a crossed simplicial group G.

Lemma 2.4.7. *Let $G \to H$ be a map of crossed $\mathcal{A}$-groups. Then, the induced functor $\mathbf{Set}^{/G}_{\mathcal{A}} \to \mathbf{Set}^{/H}_{\mathcal{A}}$ is monoidal with respect to monoidal structures $\rtimes_G$ and $\rtimes_H$.*

Proof. For each $X, Y \in \mathbf{Set}^{/G}_{\mathcal{A}}$, $X \rtimes_G Y$ and $X \rtimes_H Y$ are clearly identical as $\mathcal{A}$-sets. In addition, since $G \to H$ is a map of crossed $\mathcal{A}$-groups, the square

$$\begin{array}{ccc} X \rtimes_G Y & = & X \rtimes_H Y \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

is commutative. Hence, the result is obvious. □

We are interested in *monoid objects* in the category $\mathbf{Set}^{/G}_{\mathcal{A}}$ with respect to the monoidal structure $\rtimes_G$. Recall that a monoid object in $\mathbf{Set}^{/G}_{\mathcal{A}}$ is an object M equipped with two morphisms

$$\eta : * \to M$$
$$\mu : M \rtimes_S M \to M$$

satisfying the ordinary conditions on monoids, namely the *associativity* and the *unitality*. The next lemma shows crossed $\mathcal{A}$-groups are examples of monoid objects.

Lemma 2.4.8. *Let H be a crossed $\mathcal{A}$-group over G; i.e. a crossed $\mathcal{A}$-group equipped with a map $H \to G$ of crossed $\mathcal{A}$-groups. Then, the maps $\eta_H : * \to H$ and $\mu_H : H \rtimes_G H \to H$ given by*

$$\eta_H : * \to H(a)\ ; \quad * \mapsto e_a$$
$$\mu_H : (H \rtimes_G H)(a) \to H(a)\ ; \quad (x,y) \mapsto xy$$

are maps of $\mathcal{A}$-sets over G. Moreover, they exhibit H as a monoid object in $\mathbf{Set}^{/G}_{\mathcal{A}}$ with respect to $\rtimes_G$.

Proof. The first statement follows from the assumption that the structure map $H \to G$ is a map of crossed $\mathcal{A}$-groups and the formula

$$\mu_H(\varphi^*(x,y)) = (\varphi^y)^*(x)\varphi^*(y) = \varphi^*(xy) = \varphi^*(\mu_H(x,y))\ .$$

The associativity and the unitality are obvious since $H(a)$ is a group for each $a \in \mathcal{A}$. □

We denote by $\mathbf{Mon}(\mathbf{Set}_{\mathcal{A}}^{/G}, \rtimes_G)$ the category of monoid object in $\mathbf{Set}_{\mathcal{A}}^{/G}$ with respect to the monoidal structure $\rtimes_G$. In particular, when G is the terminal crossed $\mathcal{A}$-group $\mathfrak{T}_{\mathcal{A}}$, we write

$$\mathbf{CrsMon}_{\mathcal{A}} := \mathbf{Mon}(\mathbf{Set}_{\mathcal{A}}^{/\mathfrak{T}_{\mathcal{A}}}, \rtimes_{\mathfrak{T}_{\mathcal{A}}}) \ .$$

Definition. A *crossed $\mathcal{A}$-monoid* is just an object of $\mathbf{CrsMon}_{\mathcal{A}}$; i.e. a monoid object in the category $\mathbf{Set}_{\mathcal{A}}^{/\mathfrak{T}}$ with respect to the monoidal structure $\rtimes_{\mathfrak{T}_{\mathcal{A}}}$. We call maps in $\mathbf{CrsMon}_{\mathcal{A}}$ *maps of crossed $\mathcal{A}$-monoids.*

Note that since every crossed $\mathcal{A}$-group can be seen as one over the terminal crossed A-group $\mathfrak{T}_{\mathcal{A}}$, we can think of it as a crossed $\mathcal{A}$-monoid by virtue of Lemma 2.4.8. Hence, it makes sense to consider the slice category $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$. On the other hand, in view of Lemma 2.4.7, the map $G \to \mathfrak{T}_{\mathcal{A}}$ also induces a functor

$$\mathbf{Mon}(\mathbf{Set}_{\mathcal{A}}^{/G}, \rtimes_G) \to \mathbf{CrsMon}_{\mathcal{A}} \ . \tag{2.4.6}$$

It immediately follows from the definition of $\rtimes_G$ that G is itself a terminal object in $\mathbf{Mon}(\mathbf{Set}_{\mathcal{A}}^{/G}, \rtimes_G)$. Thus, the functor (2.4.6) factors through

$$\mathbf{Mon}(\mathbf{Set}_{\mathcal{A}}^{/G}, \rtimes_G) \to \mathbf{CrsMon}_{\mathcal{A}}^{/G} \ . \tag{2.4.7}$$

Proposition 2.4.9. *The functor* (2.4.7) *is an equivalence of categories.*

Proof. To see (2.4.7) is essentially surjective, put $\mathfrak{T}_{\mathcal{A}}$ to be the terminal crossed $\mathcal{A}$-group, and observe that we have $X \rtimes_G X = X \rtimes_{\mathfrak{T}_{\mathcal{A}}} X$ as $\mathcal{A}$-sets for each $X \in \mathbf{Set}_{\mathcal{A}}^{/G}$. It turns out that a monoid structure on X with respect to $\rtimes_{\mathfrak{T}_{\mathcal{A}}}$ defines one with respect to $\rtimes_G$ if and only if the map $X \to G$ is a monoid homomorphism with respect to $\rtimes_{\mathfrak{T}_{\mathcal{A}}}$. This implies that (2.4.7) is essentially surjective. It also follows from the similar observation that (2.4.7) is fully faithful. ☐

Theorem 2.4.10. *Let $\mathcal{A}$ be a small category and G a crossed $\mathcal{A}$-group. Then, the forgetful functor $\mathbf{CrsMon}_{\mathcal{A}}^{/G} \to \mathbf{Set}_{\mathcal{A}}^{/G}$ admits a left adjoint so to form a monadic adjunction:*

$$F_{\mathcal{A}}^G : \ \mathbf{Set}_{\mathcal{A}}^{/G} \rightleftarrows \mathbf{CrsMon}_{\mathcal{A}}^{/G} \ : U_{\mathcal{A}}^G \ .$$

Moreover, the associated monad on $\mathbf{Set}_{\mathcal{A}}^{/G}$ is finitely; i.e. it commutes with filtered colimits. Consequently, the category $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$ is locally presentable.

Proof. According to Proposition 2.4.9, we can regard $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$ as the category of monoid objects in $\mathbf{Set}_{\mathcal{A}}^{/G}$ with respect to the monoidal structure $\rtimes_G$. Since it is biclosed by Proposition 2.4.4, we have an explicit description for the monad T_{mon}^G of free monoids; namely

$$T_{\mathsf{mon}}^G(X) := \coprod_{n=0}^{\infty} X^{\rtimes_G n} = \coprod_{n=0}^{\infty} \overbrace{X \rtimes_G \cdots \rtimes_G X}^{n} \ .$$

Thus, the first statement follows. To see T_{mon}^G is finitely, it suffices to show that the functor $X \mapsto X^{\rtimes_G n}$ commutes with filtered colimits for each $n \in \mathbb{N}$. Now, the colimits are computed degreewisely in $\mathbf{Set}_{\mathcal{A}}$, and $X^{\rtimes_G n}$ is degreewisely nothing but the n-fold cartesian product, it follows from the same result for the category $\mathbf{Set}$. The last statement now follows from 2.78 in [1]. ☐

Corollary 2.4.11. *For every small category $\mathcal{A}$, the category $\mathbf{CrsMon}_{\mathcal{A}}$ is locally presentable.*

Corollary 2.4.12. *The category $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$ is complete and cocomplete. Moreover, arbitrary limits and filtered colimits can be computed in the category $\mathbf{Set}_{\mathcal{A}}^{/G}$.*

We want to make use of Theorem 2.4.10 to establish a required adjunction between $\mathbf{CrsGrp}_{\mathcal{A}}^{/G}$ and $\mathbf{Set}_{\mathcal{A}}^{/G}$. In view of Lemma 2.4.8, we can think of each crossed $\mathcal{A}$-group over G as an object of $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$. This assignment actually defines a functor

$$\mathbf{CrsGrp}_{\mathcal{A}}^{/G} \to \mathbf{CrsMon}_{\mathcal{A}}^{/G} . \tag{2.4.8}$$

Indeed, every map $f : H \to K$ of crossed $\mathcal{A}$-groups over G clearly preserves the monoid structure described in Lemma 2.4.8.

Proposition 2.4.13. *In the situation above, the functor* (2.4.8) *is fully faithful. Moreover, a crossed $\mathcal{A}$-monoid M over G belongs to the essential image if and only if it is a degreewise group.*

Proof. Notice first that the following triangle is commutative:

$$\begin{array}{ccc} \mathbf{CrsGrp}_{\mathcal{A}}^{/G} & \longrightarrow & \mathbf{CrsMon}_{\mathcal{A}}^{/G} \\ & {\scriptstyle \text{forget}} \searrow \quad \swarrow {\scriptstyle \text{forget}} & \\ & \mathbf{Set}_{\mathcal{A}}^{/G} & \end{array}$$

Since both of the forgetful functors are faithful, the top one is also faithful. To see it is also full, take two crossed $\mathcal{A}$-groups H and K over G and an arbitrary homomorphism $f : H \to K$ of crossed $\mathcal{A}$-monoids over G. We show f is actually a map of crossed $\mathcal{A}$-groups. Since it is clearly a map of $\mathcal{A}$-sets that is a degreewise group homomorphism, it suffices to show f respects the actions of $H(a)$ and $K(a)$ on $\mathcal{A}(b,a)$ for each $a, b \in \mathcal{A}$. This follows from the observation that the actions of $H(a)$ and $K(a)$ factor through $G(a)$ and that we have a commutative triangle below:

$$\begin{array}{ccc} H(a) & \xrightarrow{\quad f \quad} & K(a) \\ & \searrow \quad \swarrow & \\ & G(a) & \end{array}$$

We finally prove the last assertion. Let M be a crossed $\mathcal{A}$-monoid which is a degreewise group. Then, it is easily verified that the structure map $M \to G$ is a degreewise group homomorphism. Hence, for each $a, b \in \mathcal{A}$, the group $M(a)$ inherits an action on $\mathcal{A}(b,a)$ from $G(a)$. One can see this action together with the group structure make M into a crossed $\mathcal{A}$-group. In addition, $M \to G$ is clearly a map of crossed $\mathcal{A}$-groups, so that we obtain $M \in \mathbf{CrsGrp}_{\mathcal{A}}^{/G}$ as required. □

By virtue of Proposition 2.4.13, we may regard $\mathbf{CrsGrp}_{\mathcal{A}}^{/G}$ as a full subcategory of $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$. In fact, it is more than just a subcategory but *special one*. Namely, it is both reflective and coreflective.

We just need one lemma.

Lemma 2.4.14. *Let M be a monoid object in $\mathbf{Set}_{\mathcal{A}}^{/G}$, and let $\varphi : a \to b \in \mathcal{A}$. Then, the map $\varphi^* : M(b) \to M(a)$ preserves invertible elements in the monoid structures.*

Proof. If $x \in M(b)$ is invertible in its monoid structure, we can describe the inverse of $\varphi^*(x) \in M(a)$ explicitly as follows:

$$\varphi^*(x)^{-1} = (\varphi^x)^*(x^{-1}) \ .$$

Indeed, we have

$$\varphi^*(x)(\varphi^x)^*(x^{-1}) = (\varphi^{x^{-1}x})^*(x)(\varphi^x)^*(x^{-1}) = (\varphi^x)^*(xx^{-1}) = 1$$
$$(\varphi^x)^*(x^{-1})\varphi^*(x) = \varphi^*(xx^{-1}) = 1$$

□

Theorem 2.4.15. *Let $\mathcal{A}$ be a small category, and let G be a crossed $\mathcal{A}$-group. Then, the subcategory $\mathbf{CrsGrp}_{\mathcal{A}}^{/G} \subset \mathbf{CrsMon}_{\mathcal{A}}^{/G}$ is closed under arbitrary (small) limits and colimits. Consequently, $\mathbf{CrsGrp}_{\mathcal{A}}^{/G} \subset \mathbf{CrsMon}_{\mathcal{A}}^{/G}$ is both reflective and coreflective as a subcategory; i.e. the inclusion admits both left and right adjoints.*

Proof. We first show $\mathbf{CrsGrp}_{\mathcal{A}}^{/G} \subset \mathbf{CrsMon}_{\mathcal{A}}^{/G}$ is coreflective. The right adjoint functor $J : \mathbf{CrsMon}_{\mathcal{A}}^{/G} \to \mathbf{CrsGrp}_{\mathcal{A}}^{/G}$ is described as follows: for a crossed $\mathcal{A}$-monoid M over G, the underlying $\mathcal{A}$-set of $J(M)$ is degreewisely the group of invertible elements of $M(a)$, which actually forms an $\mathcal{A}$-subset of M thanks to Lemma 2.4.14. It is easily verified that the composition $J(M) \hookrightarrow M \to G$ and the restricted operations exhibit $J(M)$ as a crossed $\mathcal{A}$-group over G. Note that, for a crossed $\mathcal{A}$-group H over G, each map $f : H \to M$ of crossed $\mathcal{A}$-monoids preserves invertible elements so it factors through $J(M) \hookrightarrow M$. Moreover, since $J(M) \to M$ is a monomorphism, this factorization is unique. This implies we have a natural bijection

$$\mathbf{CrsMon}_{\mathcal{A}}^{/G}(H, M) \cong \mathbf{CrsGrp}_{\mathcal{A}}^{/G}(H, J(M)) \ .$$

Hence, J is right adjoint to the inclusion.

Next, we prove the closedness properties. As for colimits, it follows from the coreflectivity proved above, Proposition 2.2.1, and Corollary 2.4.12. To see it is also the case for limits, let $H_\bullet : \mathcal{I} \to \mathbf{CrsGrp}_{\mathcal{A}}^{/G}$ be a functor from a small category $\mathcal{I}$. Since the category $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$ is complete by Corollary 2.4.12, we can take the limit in the category $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$ and write $H_\infty := \lim_{i \in \mathcal{I}} H_i$. We have to show H_∞ is a crossed $\mathcal{A}$-group. Note that H_∞ is also the limit in $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$ of the extended diagram $H_\bullet^\triangleright : \mathcal{I}^\triangleright = \mathcal{I} \star \{v\} \to \mathbf{CrsGrp}_{\mathcal{A}}^{/G}$ given by

$$H_x^\triangleright = \begin{cases} H_x & x \in \mathcal{I} \\ G & x = v \ . \end{cases}$$

In view of Corollary 2.4.12, limits of cocone diagrams in $\mathbf{CrsMon}_{\mathcal{A}}^{/G}$ are computed degreewisely, so for each $a \in \mathcal{A}$, we have

$$H_\infty(a) \cong \lim_{x \in \mathcal{I}^\triangleright} H_x^\triangleright(a) \ .$$

The right hand side is a limit of groups, so it is again a group. It follows that H_∞ is a degreewise group so it belongs to $\mathbf{CrsGrp}_{\mathcal{A}}^{/G}$ by virtue of Proposition 2.4.13.

To see the last statement, we have to show the embedding $\mathbf{CrsGrp}_{\mathcal{A}}^{/G} \to \mathbf{CrsMon}_{\mathcal{A}}^{/G}$ admits a left adjoint. This follows from the first assertion, Proposition 2.2.1, and the *General Adjoint Functor Theorem.* □

In summary, for every crossed $\mathcal{A}$-group G, we obtain the following adjunctions:

$$\mathbf{CrsGrp}_{\mathcal{A}}^{/G} \underset{\perp}{\overset{\perp}{\hookrightarrow}} \mathbf{CrsMon}_{\mathcal{A}}^{/G} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Set}_{\mathcal{A}}^{/G}$$

where the right one is monadic, and each rightward arrow creates arbitrary limits and filtered colimits.

Remark 2.4.16. If one knows a terminal crossed $\mathcal{A}$-group $\mathfrak{T}_{\mathcal{A}}$, then he can prove Theorem 2.2.4 in a more concrete way. Indeed, it is a consequence of Theorem 2.4.10, Theorem 2.4.15 above, and Corollary 2.4 in [56].

2.5 Basechange of crossed monoids

It is often the case that the category $\mathcal{A}$ is in nature related to another category, say $\widetilde{\mathcal{A}}$, by a functor $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$. Such a functor gives rise to adjunctions

$$\Phi_!, \Phi_* : \ \mathbf{Set}_{\mathcal{A}} \underset{\perp}{\overset{\perp}{\rightleftarrows}} \mathbf{Set}_{\widetilde{\mathcal{A}}} \ : \Phi^* \ ,$$

where $\Phi_!$ and Φ_* are the left and right Kan extensions of Φ respectively along the Yoneda embedding. More generally, for an $\widetilde{\mathcal{A}}$-set $\widetilde{S}$, we also have adjunctions

$$\Phi_!^{\widetilde{S}}, \Phi_*^{\widetilde{S}} : \ \mathbf{Set}_{\mathcal{A}}^{/\Phi^*\widetilde{S}} \underset{\perp}{\overset{\perp}{\rightleftarrows}} \mathbf{Set}_{\widetilde{\mathcal{A}}}^{/\widetilde{S}} \ : \Phi_{\widetilde{S}}^* \ . \tag{2.5.1}$$

Namely, $\Phi_{\widetilde{S}}^*$ is just the canonical lift of Φ^*, and two functors $\Phi_!^{\widetilde{S}}, \Phi_*^{\widetilde{S}}$ are defined as follows: for $X \in \mathbf{Set}_{\mathcal{A}}^{/\Phi^*\widetilde{S}}$, $\Phi_!^{\widetilde{S}} X := \Phi_! X$ with the adjoint morphism $\Phi_! X \to S$, and $\Phi_*^{\widetilde{S}} X$ is the object in the pullback square below:

$$\begin{array}{ccc} \Phi_*^{\widetilde{S}} X & \longrightarrow & \Phi_* X \\ \big\downarrow & \lrcorner & \big\downarrow \\ \widetilde{S} & \longrightarrow & \Phi_*\Phi^*\widetilde{S} \end{array}$$

A lifts of the adjunction (2.5.1) is the central interest in this section.

Now, fix a functor $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$, and let $\widetilde{G}$ be a crossed $\widetilde{\mathcal{A}}$-group. The notion of crossed monoids anyway arises from the monoidal structure on the category of presheaves as mentioned in Section 2.4, we investigate how the functor $\Phi_{\widetilde{G}}^*$ given above relates monoidal structures. Notice that, for this question to make sense, we have to give $\Phi^*\widetilde{G}$ a structure of $\mathcal{A}$-groups. Unfortunately, it immediately turns out that there is no canonical way to do this, so we give up the general cases and concentrate only on *faithful* functors.

Definition. Let $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ be a faithful functor between small categories. Then, for a crossed $\widetilde{\mathcal{A}}$-group $\widetilde{G}$, Φ is said to be $\widetilde{G}$-*stable* if for each $a, b \in \mathcal{A}$, the image of the map

$$\Phi : \mathcal{A}(a,b) \to \widetilde{\mathcal{A}}(\Phi(a), \Phi(b))$$

is $\widetilde{G}(b)$-stable.

For example, fully faithful functors are stable for every crossed group. The following result is straightforward.

Lemma 2.5.1. *Let $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ be a faithful functor between small categories. Suppose $\widetilde{G}$ is a crossed $\widetilde{\mathcal{A}}$-group such that Φ is $\widetilde{G}$-stable. Then, the $\mathcal{A}$-set $\Phi^*\widetilde{G}$ admits a unique structure of crossed $\mathcal{A}$-groups such that*

(i) *for each $a \in \mathcal{A}$, the group structure on $\Phi^*\widetilde{G}(a) = \widetilde{G}(\Phi(a))$ agrees with the original one;*

(ii) *for each $a, b \in \mathcal{A}$, the map*

$$\Phi : \mathcal{A}(a,b) \to \widetilde{\mathcal{A}}(\Phi(a), \Phi(b))$$

is $\Phi^*\widetilde{G}(b)$-equivariant.

In the following, we always regard $\Phi^*\widetilde{G}$ as a crossed $\mathcal{A}$-group with the structure in Lemma 2.5.1 whenever Φ is $\widetilde{G}$-stable faithful functor. Hence, the monoidal structure $\rtimes_{\Phi^*\widetilde{G}}$ on the category $\mathbf{Set}_{\mathcal{A}}^{/\Phi^*\widetilde{G}}$ makes sense (see Section 2.4).

Proposition 2.5.2. *Let $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ be a faithful functor between small categories which is $\widetilde{G}$-stable for a crossed $\widetilde{\mathcal{A}}$-group $\widetilde{G}$. Then, the induced functor*

$$\Phi^*_{\widetilde{G}} : \mathbf{Set}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}} \to \mathbf{Set}_{\mathcal{A}}^{/\Phi^*\widetilde{G}}$$

is monoidal with respect to the monoidal structures $\rtimes_{\widetilde{G}}$ and $\rtimes_{\Phi^*\widetilde{G}}$.

Proof. Note that, for each $\widetilde{X}, \widetilde{Y} \in \mathbf{Set}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}}$, and for each $a \in \mathcal{A}$, we have a canonical identification

$$\begin{aligned}\Phi^*(\widetilde{X} \rtimes_{\widetilde{G}} \widetilde{Y})(a) &= (\widetilde{X} \rtimes_{\widetilde{G}} \widetilde{Y})(\Phi(a)) = \widetilde{X}(\Phi(a)) \times \widetilde{Y}(\Phi(a)) \\ &= \Phi^*\widetilde{X}(a) \times \Phi^*\widetilde{Y}(a) = (\Phi^*\widetilde{\rtimes}_{\Phi^*\widetilde{G}}\Phi^*\widetilde{Y})(a) \ .\end{aligned}$$

On the other hand, for each morphism $\varphi : a \to b \in \mathcal{A}$, the induced map

$$\varphi^* : \widetilde{X}(\Phi(b)) \times \widetilde{Y}(\Phi(b)) \to \widetilde{X}(\Phi(a)) \times \widetilde{Y}(\Phi(a))$$

is, no matter whether it is considered in $\Phi^*(\widetilde{X} \rtimes_{\widetilde{G}} \widetilde{Y})$ or $\Phi^*\widetilde{X} \rtimes_{\Phi^*\widetilde{G}} \Phi^*\widetilde{Y}$, given by

$$\Phi^*(x,y) = \Phi(\Phi)^*(x,y) = ((\Phi(\varphi)^y)^*(x), \Phi(\varphi)^*(y)) = (\Phi(\varphi^y)^*(x), \Phi(\varphi)^*(y)) \ .$$

Thus, we obtain a canonical identification $\Phi^*(\widetilde{X} \rtimes_{\widetilde{G}} \widetilde{Y}) = \Phi^*\widetilde{X} \rtimes_{\Phi^*\widetilde{G}} \Phi^*\widetilde{Y}$, which makes Φ into a monoidal functor. □

Corollary 2.5.3. *Let $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ and $\widetilde{G} \in \mathbf{CrsGrp}_{\widetilde{\mathcal{A}}}$ be as in Proposition 2.5.2. Then, the adjunction $\Phi^*_{\widetilde{G}} \dashv \Phi^{\widetilde{G}}_*$ induces an adjunction*

$$\Phi^\natural_{\widetilde{G}} : \mathbf{CrsMon}^{/\widetilde{G}}_{\widetilde{\mathcal{A}}} \rightleftarrows \mathbf{CrsMon}^{/\Phi^*\widetilde{G}}_{\mathcal{A}} : \Phi^{\widetilde{G}}_\sharp .$$

Proof. The statement is a consequence of Proposition 2.5.2 and the fact that, for a monoidal functor $F : \mathcal{C} \to \mathcal{D}$, the induced functor $\mathbf{Mon}(\mathcal{C}) \to \mathbf{Mon}(\mathcal{D})$ admits a right adjoint as soon as so does F. The reader will find the full proof of this in the section 2.3 in [64]. □

Note that the fact we used in the proof of Corollary 2.5.3 not only shows the existence of the adjunction but also provides a way to compute it. Indeed, if $\widetilde{M}$ is a crossed $\widetilde{\mathcal{A}}$-monoid over $\widetilde{G}$, then $\Phi^*_{\widetilde{G}}\widetilde{M}$ has a canonical structure of crossed $\mathcal{A}$-monoids in view of Proposition 2.5.2. On the other hand, since the functor $\Phi^{\widetilde{G}}_*$ is right adjoint to the monoidal functor $\Phi^*_{\widetilde{G}}$, it admits a structure of lax monoidal functors; namely we have $\Phi^{\widetilde{G}}_*(*) \cong *$ and a natural transformation

$$\mu : \Phi^{\widetilde{G}} X \rtimes_{\widetilde{G}} \Phi^{\widetilde{G}} Y \to \Phi^{\widetilde{G}}_*(X \rtimes_{\Phi^*\widetilde{G}} Y)$$

subject to an appropriate coherence conditions. Then, for each crossed $\mathcal{A}$-monoid M over $\Phi^*\widetilde{G}$, $\Phi^{\widetilde{G}}_* M$ admits a canonical structure of crossed $\widetilde{\mathcal{A}}$-monoids over $\widetilde{G}$ as

$$\Phi^{\widetilde{G}}_* M \rtimes_{\widetilde{G}} \Phi^{\widetilde{G}}_* M \xrightarrow{\mu} \Phi^{\widetilde{G}}_*(M \rtimes_{\Phi^*\widetilde{G}} M) \to \Phi^{\widetilde{G}}_* M$$
$$* \cong \Phi^{\widetilde{G}}_*(*) \to \Phi^{\widetilde{G}}_* M .$$

This actually gives the right adjoint $\Phi^{\widetilde{G}}_\sharp$ in the adjunction in Corollary 2.5.3.

Proposition 2.5.4. *Let $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ be a fully faithful functor, and let $\widetilde{G}$ be a crossed $\widetilde{\mathcal{A}}$-group. Then, the functor $\Phi^{\widetilde{G}}_\sharp : \mathbf{CrsMon}^{/\Phi^*\widetilde{G}}_{\mathcal{A}} \to \mathbf{CrsMon}^{/\widetilde{G}}_{\widetilde{\mathcal{A}}}$ induced by the right Kan extension of Φ is fully faithful.*

Proof. Since Φ is fully faithful, the induced functor

$$\Phi_{/\widetilde{G}} : \mathcal{A}/(\Phi^*\widetilde{G}) \to \widetilde{\mathcal{A}}/\widetilde{G}$$

between the categories of elements is also fully faithful. Note that we have canonical identifications

$$\mathbf{Set}^{/\Phi^*\widetilde{G}}_{\mathcal{A}} \simeq \mathbf{Set}_{\mathcal{A}/\Phi^*\widetilde{G}} \quad \text{and} \quad \mathbf{Set}^{/\widetilde{G}}_{\widetilde{\mathcal{A}}} \simeq \mathbf{Set}_{\widetilde{\mathcal{A}}/\widetilde{G}}$$

so that the adjunction $\Phi^*_{\widetilde{G}} \dashv \Phi^{\widetilde{G}}_*$ is identified with the one obtained by the right Kan extension of $\Phi_{/\widetilde{G}}$. Since Kan extensions of fully faithful functors along the Yoneda embedding are again fully faithful, e.g see Proposition 4.23 in [45], $\Phi^{\widetilde{G}}_*$ is fully faithful. In other words, for each $X \in \mathbf{Set}^{/\Phi^*\widetilde{G}}_{\mathcal{A}}$, the counit

$$\Phi^*_{\widetilde{G}}\Phi^{\widetilde{G}}_* X \to X \tag{2.5.2}$$

is an isomorphism. Note that if X is a crossed $\mathcal{A}$-monoid over $\Phi^*\widetilde{G}$, then (2.5.2) underlies the counit map $\Phi^\natural_{\widetilde{G}}\Phi^{\widetilde{G}}_\sharp X \to X$. This implies that the right adjoint $\Phi^{\widetilde{G}}_\sharp$ is fully faithful. □

Example 2.5.5. Take $\mathcal{A} \to \widetilde{\mathcal{A}}$ to be the inclusion $j : \Delta \hookrightarrow \widetilde{\Delta}$. The right adjoint functor j_* to the restriction $j^* : \mathbf{Set}_{\widetilde{\Delta}} \to \mathbf{Set}_{\Delta}$ sends each simplicial set $X_\bullet$ to the augmented one j_*X given by

$$j_*X(\langle n\rangle) = \begin{cases} \{\mathrm{pt}\} & n = 0 \\ X_{n-1} & n \geq 1 \ . \end{cases} \tag{2.5.3}$$

In particular, we have a canonical isomorphisms

$$j^*\mathfrak{W}^{\widetilde{\Delta}} \cong \mathfrak{W}^{\Delta} \ , \quad j_*\mathfrak{W}^{\Delta} \cong \mathfrak{W}^{\widetilde{\Delta}} \ . \tag{2.5.4}$$

Now, since the functor j is fully faithful, it is $\mathfrak{W}^{\widetilde{\Delta}}$-stable, so we obtain an adjunction

$$j^*_{\mathfrak{W}^{\widetilde{\Delta}}} : \mathbf{Set}^{/\mathfrak{W}^{\widetilde{\Delta}}}_{\widetilde{\Delta}} \rightleftarrows \mathbf{Set}^{/\mathfrak{W}^{\Delta}}_{\Delta} : j^{\mathfrak{W}^{\widetilde{\Delta}}}_*$$

with $j^*_{\mathfrak{W}^{\widetilde{\Delta}}}$ monoidal by Proposition 2.5.2. We finally obtain an adjunction

$$j^\natural : \mathbf{CrsMon}_{\widetilde{\Delta}} \rightleftarrows \mathbf{CrsMon}_{\Delta} : j_\sharp \ .$$

Note that, thanks to the equation (2.5.4), the functor $j_\sharp$ is also given by (2.5.3) and fully faithful by virtue of Proposition 2.5.4.

Example 2.5.6. By its construction, the category ∇ is the Kleisli category of the monad

$$\mathfrak{J} : \widetilde{\Delta} \to \widetilde{\Delta} \ ; \quad \langle n\rangle \mapsto \langle n+2\rangle \cong \{-\infty\} \star \langle n\rangle \star \{\infty\} \cong \langle\!\langle n\rangle\!\rangle \ .$$

In view of this, one can find the right adjoint to the canonical embedding $\mathfrak{J} : \widetilde{\Delta} \to \nabla$; namely

$$U : \nabla \to \widetilde{\Delta} \ ; \quad \langle\!\langle n\rangle\!\rangle \to \langle n+2\rangle \ .$$

It turns out that the pullbacks along these functors gives rise to an adjunction $\mathfrak{J}^* \dashv U^* : \mathbf{Set}_{\widetilde{\Delta}} \to \mathbf{Set}_{\nabla}$, and the uniqueness of right adjoints implies $U^* \cong \mathfrak{J}_*$. More explicitly, the right adjoint $\mathfrak{J}_* : \mathbf{Set}_{\widetilde{\Delta}} \to \mathbf{Set}_{\nabla}$ is given by

$$\mathfrak{J}_*X(\langle\!\langle n\rangle\!\rangle) = X(\langle n+2\rangle) \ .$$

The counit $X \to \mathfrak{J}_*\mathfrak{J}^*X$ is the monomorphism described as follows: consider the map $\tau_n : \langle\!\langle n+2\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle$ defined by

$$\tau_n(i) = \begin{cases} -\infty & i = -\infty, 1 \\ i-1 & 2 \leq i \leq n+1 \\ \infty & i = n+2, \infty \ . \end{cases}$$

Then, each component $X \to \mathfrak{J}_*\mathfrak{J}^*X$ is the map induced by τ_n:

$$\tau_n^* : X(\langle\!\langle n\rangle\!\rangle) \to \mathfrak{J}_*\mathfrak{J}^*X(\langle\!\langle n\rangle\!\rangle) = X(\langle\!\langle n+2\rangle\!\rangle) \ .$$

In particular, as for the Weyl crossed interval group $\mathfrak{W}^\nabla$, the unit map $\mathfrak{W}^\nabla \to \mathfrak{J}_*\mathfrak{J}^*\mathfrak{W}^\nabla$ exhibits $\mathfrak{W}^\nabla(\langle\!\langle n\rangle\!\rangle)$ as a subset of $\mathfrak{W}^\nabla(\langle\!\langle n+2\rangle\!\rangle)$ given by

$$\left\{ (\sigma; \varepsilon_1, \dots, \varepsilon_{n+2}; \theta) \in \mathfrak{W}^\nabla(\langle\!\langle n+2\rangle\!\rangle) \;\middle|\; \begin{array}{c} \sigma(\{1, n+2\}) = \{1, n+2\} \ , \\ \sigma|_{\{1,n+2\}} = \varepsilon_1 = \varepsilon_{n+2} = \theta \end{array} \right\} \ .$$

On the other hand, we have

$$\mathfrak{J}^*\mathfrak{W}^{\nabla} \cong \mathfrak{H} \times C_2 \cong \mathfrak{W}^{\widetilde{\Delta}} \times C_2 \ .$$

Since $\mathfrak{W}^{\widetilde{\Delta}}$ is the terminal object in $\mathbf{CrsMon}_{\widetilde{\Delta}}$, giving a map $M \to \mathfrak{J}^*\mathfrak{W}^{\nabla}$ of augmented crossed simplicial monoid is equivalent to giving an augmented simplicial map $M \to C_2$ which is a degreewise monoid homomorphism. Hence, for $M \in \mathbf{CrsMon}_{\widetilde{\Delta}}^{/\mathfrak{W}^{\widetilde{\Delta}} \times C_2}$ with associated map $\theta : M \to C_2$, we have

$$\mathfrak{J}_{\sharp}^{\mathfrak{W}^{\nabla}} M(\langle\!\langle n \rangle\!\rangle) = \left\{ x \in M(\langle n+2 \rangle) \;\middle|\; \begin{array}{c} x(\{1, n+2\}) = \{1, n+2\}, \\ x|_{\{1,n+2\}} = \varepsilon_1(x) = \varepsilon_{n+2}(x) = \theta(x) \end{array} \right\} ,$$

where we write $(x; \varepsilon_1(x), \dots, \varepsilon_{n+2}(x))$ the image of x in $\mathfrak{W}^{\widetilde{\Delta}}$. It exactly gives the right adjoint in the adjunction

$$\mathfrak{J}^{\natural}_{\mathfrak{W}^{\nabla}} : \ \mathbf{CrsMon}_{\nabla} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{CrsMon}_{\widetilde{\Delta}}^{/\mathfrak{W}^{\widetilde{\Delta}} \times C_2} \ : \mathfrak{J}_{\sharp}^{\mathfrak{W}^{\nabla}} \ . \tag{2.5.5}$$

Note that this adjunction extends to the right: namely, the projection $\mathfrak{W}^{\widetilde{\Delta}} \times C_2 \to \mathfrak{W}^{\widetilde{\Delta}}$, which is the unique map of augmented crossed simplicial groups between them, gives rise to an adjunction

$$\begin{array}{ccc} \mathbf{CrsMon}_{\widetilde{\Delta}}^{/\mathfrak{W}^{\widetilde{\Delta}} \times C_2} & \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} & \mathbf{CrsMon}_{\widetilde{\Delta}} \\ \left(M \to \mathfrak{W}^{\widetilde{\Delta}} \times C_2 \right) & \longmapsto & M \\ \left(M \times C_2 \to \mathfrak{W}^{\widetilde{\Delta}} \times C_2 \right) & \longleftarrow\!\!\shortmid & M \end{array} \tag{2.5.6}$$

Combining (2.5.5) with (2.5.6), we obtain an adjunction

$$\mathfrak{J}^{\natural} : \ \mathbf{CrsMon}_{\nabla} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{CrsMon}_{\widetilde{\Delta}} \ : \mathfrak{J}_{\sharp} \ .$$

We next discuss the other adjunction $\Phi_!^{\widetilde{G}} \dashv \Phi_{\widetilde{G}}^*$ induced by the left Kan extension of a $\widetilde{G}$-stable faithful functor $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$. In contrast to the right Kan extension, the functor $\Phi_!^{\widetilde{G}}$ itself does not induce any functor on the category of monoids in general. We hence need to construct directly the left adjoint to the functor

$$\Phi_{\widetilde{G}}^{\natural} : \mathbf{CrsMon}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}} \to \mathbf{CrsMon}_{\mathcal{A}}^{/\Phi^*\widetilde{G}} \ .$$

Fortunately, we can make use of the following theorem.

Theorem 2.5.7 (Adjoint Lifting Theorem, Theorem 4.5.6 in [9])**.** *Suppose we have a square of functors*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{Q} & \mathcal{N} \\ {\scriptstyle U}\downarrow & \cong & \downarrow{\scriptstyle V} \\ \mathcal{C} & \xrightarrow{R} & \mathcal{D} \end{array}$$

which is commutative up to a natural isomorphism, and suppose $\mathcal{M}$ has all coequalizers. Then, Q has a left adjoint as soon as so does R. More precisely, *if $F \dashv U$, $G \dashv V$, and $L \dashv R$, so $FL \dashv RU \cong VQ$, then the left adjoint $K : \mathcal{N} \to \mathcal{M}$ to Q is defined by the coequalizer sequence*

$$FLVGV(N) \overset{FLV\varepsilon,\omega V}{\rightrightarrows} FLV(N) \longrightarrow K(N) \tag{2.5.7}$$

in $\mathcal{M}$, where $\varepsilon : GV \to \mathrm{Id}$ is the counit of the adjunction, and

$$\alpha : G \xrightarrow{G\eta^{FL\dashv VQ}} GVQFL \xrightarrow{\varepsilon QFL} QFL$$
$$\omega : FLVG \xrightarrow{FLV\alpha} FLVQFL \xrightarrow{\varepsilon^{FL\dashv VQ}FL} FL\ .$$

Remark 2.5.8. One can deduce Theorem 2.5.7 from, besides the direct proof, a more general theorem called Adjoint Triangle Theorem. We refer the reader to [19] and [72].

We can apply Theorem 2.5.7 in our situation, i.e. to the square

$$\begin{array}{ccc} \mathbf{CrsMon}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}} & \xrightarrow{\Phi^{\natural}_{\widetilde{G}}} & \mathbf{CrsMon}_{\mathcal{A}}^{/\Phi^*\widetilde{G}} \\ {\scriptstyle U^{\widetilde{G}}}\downarrow & & \downarrow{\scriptstyle U^{\Phi^*\widetilde{G}}} \\ \mathbf{Set}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}} & \xrightarrow{\Phi^{*}_{\widetilde{G}}} & \mathbf{Set}_{\mathcal{A}}^{/\Phi^*\widetilde{G}} \end{array} \quad . \tag{2.5.8}$$

To obtain a more explicit description, however, we need to know more about each involved functor. In particular, since all the right adjoints just forget structures, it is enough to care about the left adjoints. We first look at the free functor

$$F^G : \mathbf{Set}_{\mathcal{A}}^{/G} \to \mathbf{CrsMon}_{\mathcal{A}}^{/G}$$

for a crossed $\mathcal{A}$-group G. This functor is the one in Theorem 2.4.10 and computed as follows: for each $X \in \mathbf{Set}_{\mathcal{A}}^{/G}$ with the structure map $p : X \to G$, $F^G X$ is, as an $\mathcal{A}$-set, degreewisely the free monoid generated by X with the structure map

$$\begin{array}{rccl} \varphi^* : & F^G X(b) & \to & F^G X(a) \\ & x_1 x_2 \dots x_n & \mapsto & (\varphi^{p(x_2)\dots p(x_n)})^*(x_1)(\varphi^{p(x_3)\dots p(x_n)})^*(x_2)\dots\varphi^*(x_n)\ . \end{array}$$

for each $\varphi : a \to b \in \mathcal{A}$. The map $F^G X \to G$ is the induced one.

On the other hand, for a functor $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$, its left Kan extension $\Phi_! : \mathbf{Set}_{\mathcal{A}} \to \mathbf{Set}_{\widetilde{\mathcal{A}}}$ along the Yoneda embedding is realized as follows: for $X \in \mathbf{Set}_{\mathcal{A}}$ and for $\tilde{a} \in \widetilde{\mathcal{A}}$, $\Phi_! X(\tilde{a})$ is the quotient set

$$\left\{ (x,\widetilde{\varphi}) \;\middle|\; x \in X(b),\ \widetilde{\varphi} \in \widetilde{\mathcal{A}}(\tilde{a},\Phi(b)) \text{ for } b \in \mathcal{A} \right\} / \sim$$

by the equivalence relation $\sim$ generated by

$$(\theta^*(x),\widetilde{\varphi}) \sim (x,\Phi(\theta)\widetilde{\varphi})$$

for each triples $(x,\widetilde{\varphi},\theta)$ such that both sides make sense. We write $[x,\widetilde{\varphi}] \in \Phi_! X(\tilde{a})$ the equivalence class represented by the pair $(x,\widetilde{\varphi})$. If X is equipped with an $\mathcal{A}$-map $f : X \to \Phi^*\widetilde{S}$, then we have an $\widetilde{\mathcal{A}}$-map

$$\Phi_! X \to \widetilde{S}\ ; \quad [x,\widetilde{\varphi}] \mapsto \widetilde{\varphi}^*(f(x))\ ,$$

which exactly gives the functor $\Phi_!^{\widetilde{S}} : \mathbf{Set}_{\mathcal{A}}^{/\Phi^*\widetilde{S}} \to \mathbf{Set}_{\widetilde{\mathcal{A}}}^{/\widetilde{S}}$.

Combining the observations above and Theorem 2.5.7, we obtain the following result.

Theorem 2.5.9. *Let $\widetilde{G}$ be a crossed $\widetilde{\mathcal{A}}$-group, and let $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ be a $\widetilde{G}$-stable faithful functor. Then, the pullback $\Phi^\natural_{\widetilde{G}}$ admits a left adjoint functor $\Phi_\flat^{\widetilde{G}}$ so to form an adjunction*

$$\Phi_\flat^{\widetilde{G}} : \mathbf{CrsMon}_{\mathcal{A}}^{/\Phi^*\widetilde{G}} \rightleftarrows \mathbf{CrsMon}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}} : \Phi^\natural_{\widetilde{G}} .$$

More precisely, for each $M \in \mathbf{CrsMon}_{\mathcal{A}}^{/\Phi^*\widetilde{G}}$ with the structure map $p : M \to \Phi^*\widetilde{G}$, the crossed $\widetilde{\mathcal{A}}$-monoid $\Phi_\flat^{\widetilde{G}} M$ over $\widetilde{G}$ is given as follows: for each $\tilde{a} \in \widetilde{\mathcal{A}}$, the monoid $\Phi_\flat^{\widetilde{G}} M(\tilde{a})$ is obtained as the quotient of the free monoid with generating set

$$\left\{ (x, \widetilde{\varphi}) \;\middle|\; x \in M(b),\ \widetilde{\varphi} \in \widetilde{\mathcal{A}}(\tilde{a}, \Phi(b)) \text{ for } b \in \mathcal{A} \right\} \tag{2.5.9}$$

by the congruence relation $\sim$ generated by

$$(e_b, \widetilde{\varphi}) \sim e_a \ , \quad (xy, \widetilde{\varphi}) \sim (x, \widetilde{\varphi}^y)(y, \widetilde{\varphi}) \ , \quad (\theta^*(z), \widetilde{\varphi}) \sim (z, \Phi(\theta)\widetilde{\varphi})$$

for $x, y, z, \widetilde{\varphi}, \theta$ such that each term makes sense. For each $\widetilde{\psi} : \tilde{b} \to \tilde{a} \in \widetilde{\mathcal{A}}$, the map $\widetilde{\psi}^* : \Phi_\flat^{\widetilde{G}} M(\tilde{a}) \to \Phi_\flat^{\widetilde{G}} M(\tilde{b})$ is given by

$$\widetilde{\psi}([x_1, \widetilde{\varphi}_1] \dots [x_n, \widetilde{\varphi}_n]) = [x_1, \widetilde{\psi}^{\widetilde{\varphi}_2^*(p(x_2)) \dots \widetilde{\varphi}_n^*(p(x_n))} \widetilde{\varphi}_1] \dots [x_n, \widetilde{\psi}\widetilde{\varphi}_n] \ .$$

Finally, $\Phi_\flat^{\widetilde{G}} M \to \widetilde{G}$ is the one generated by $[x, \widetilde{\varphi}] \mapsto \widetilde{\varphi}^*(p(x))$.

Proof. We apply Theorem 2.5.7 to the diagram (2.5.8). To simplify the notation, we write $G := \Phi^*\widetilde{G}$ and omit all the forgetful functors from formulas. Then, the first thing we need is to know the two morphisms

$$F^{\widetilde{G}} \Phi_!^{\widetilde{G}} \varepsilon,\ \omega : F^{\widetilde{G}} \Phi_!^{\widetilde{G}} F^G M \rightrightarrows F^{\widetilde{G}} \Phi_!^{\widetilde{G}} M \tag{2.5.10}$$

of (2.5.7) for each $M \in \mathbf{CrsGrp}_{\mathcal{A}}^{/G}$. According to the discussion above, for each $\tilde{a} \in \widetilde{\mathcal{A}}$, the elements of $F^{\widetilde{G}} \Phi_!^{\widetilde{G}} F^G M(\tilde{a})$ are finite words in the quotient set

$$\left\{ (x_1, \dots, x_n; \widetilde{\varphi}) \;\middle|\; n \in \mathbb{N},\ x_i \in M(b),\ \widetilde{\varphi} \in \widetilde{\mathcal{A}}(\tilde{a}, \Phi(b)) \text{ for } b \in \mathcal{A} \right\} / \sim$$

by the equivalence relation $\sim$ generated by

$$(x_1, \dots, x_n; \Phi(\theta)\widetilde{\varphi}) \sim ((\theta^{x_2 \dots x_n})^*(x_1), \dots, \theta^*(x_n); \widetilde{\varphi}) \ .$$

We write $[x_1, \dots, x_n; \widetilde{\varphi}]$ the equivalence class represented by the tuple $(x_1, \dots, x_n; \widetilde{\varphi})$. On the other hand, $F^{\widetilde{G}} \Phi_!^{\widetilde{G}} M(\tilde{a})$ is the set of words in the set $\Phi_!^{\widetilde{G}} M(\tilde{a})$, which is obtained as the quotient of the set (2.5.9) as mentioned just before Theorem 2.5.9. Then, the direct computation shows the two maps in (2.5.10) are the monoid homomorphisms generated by the maps

$$\begin{aligned} &[x_1, \dots, x_n; \widetilde{\varphi}] \\ &\quad \mapsto [x_1 \dots x_n, \widetilde{\varphi}],\ [x_1, \widetilde{\varphi}^{x_2 \dots x_n}] \dots [x_n, \widetilde{\varphi}] \ , \end{aligned}$$

where one multiplies $x_1, \dots, x_n$ in M while the other distributes the brackets. Therefore one obtains the required presentation. ☐

Corollary 2.5.10. *Let $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ be a fully faithful functor, and let $\widetilde{G}$ be a crossed $\widetilde{\mathcal{A}}$-group. Then, the left adjoint functor $\Phi_\flat^{\widetilde{G}} : \mathbf{CrsMon}_{\mathcal{A}}^{/\Phi^*\widetilde{G}} \to \mathbf{CrsMon}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}}$ to the pullback $\Phi^\natural_{\widetilde{G}}$ is fully faithful.*

Proof. Suppose Φ is fully faithful, so we may regard $\mathcal{A}$ as a full subcategory of $\widetilde{\mathcal{A}}$. It suffices to show the unit $M \to \Phi^\natural_{\widetilde{G}}\Phi_\flat^{\widetilde{G}}M$ of the adjunction $\Phi_\flat^{\widetilde{G}} \dashv \Phi^\natural_{\widetilde{G}}$ is an isomorphism for each $M \in \mathbf{CrsMon}_{\mathcal{A}}^{/\Phi^*\widetilde{G}}$. Note that, for $a \in \mathcal{A}$, it is given by

$$M(a) \to \Phi^\natural_{\widetilde{G}}\Phi_\flat^{\widetilde{G}}M(a) = \Phi_\flat^{\widetilde{G}}M(a) \ ; \quad x \mapsto [x, \mathrm{id}] \ . \tag{2.5.11}$$

On the other hand, since $\mathcal{A}$ is a full subcategory of $\widetilde{\mathcal{A}}$, we have

$$[y, \widetilde{\varphi}] = [\widetilde{\varphi}^*(y), \mathrm{id}]$$

for each $y \in M(b)$ and $\widetilde{\varphi} \in \widetilde{\mathcal{A}}(a,b) = \mathcal{A}(a,b)$. This implies that the map (2.5.11) is surjective. Moreover, the faithfulness of the left Kan extension $\Phi_!^{\widetilde{G}} : \mathbf{Set}_{\mathcal{A}}^{/\Phi^*\widetilde{G}} \to \mathbf{Set}_{\widetilde{\mathcal{A}}}^{\widetilde{G}}$ implies that $[x, \mathrm{id}] = [x', \mathrm{id}]$ if and only if $x = x'$. Thus, (2.5.11) is also injective, so we obtain the result. ☐

Remark 2.5.11. Corollary 2.5.10 also follows from Proposition 2.5.4 and the fact that, for adjunctions $L \dashv F \dashv R$, L is fully faithful if and only if so is R. It seems to be a kind of *folklore* while the reader will find proofs in [22] and [46].

Example 2.5.12. Take Φ to be the embedding $j : \Delta \to \widetilde{\Delta}$ and $\widetilde{G} = \mathfrak{W}^{\widetilde{\Delta}}$, then we obtain an adjunction

$$j_\flat : \mathbf{CrsMon}_{\Delta} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{CrsMon}_{\widetilde{\Delta}} : j^\natural \tag{2.5.12}$$

by Theorem 2.5.9. Since j is fully faithful, by virtue of Corollary 2.5.10, for every every crossed simplicial monoid $M_\bullet$, we have a canonical identification

$$j_\flat M(\langle n \rangle) \cong M_{n-1}$$

for each $n \geq 1$. On the other hand, since the only object $\langle 0 \rangle \in \widetilde{\Delta}$ outside the image of j is initial, for the left Kan extension $j_!M$, we have

$$j_!M(\langle 0 \rangle) \cong \operatorname*{colim}_{\Delta} M_\bullet \cong \mathrm{coeq}\left(M_1 \overset{d_0, d_1}{\rightrightarrows} M_0 \right) \cong \pi_0(M) \ ,$$

where $\pi_0(M)$ is the *set of connected components* of the simplicial set M. We claim $\pi_0(M)$ inherits a structure of monoids through the quotient map $M_0 \to \pi_0(M)$. Note that $\pi_0(M)$ is obtained as the quotient of M_0 by the equivalence relation $\sim$ generated by

$$d_0 x \sim d_1 x$$

for each $x \in M_1$. Hence, to verify the claim, it suffices to find $z, z' \in M_1$ which support

$$u \cdot d_0 x \sim u \cdot d_1 x \ , \quad d_0 x \cdot u \sim d_1 x \cdot u$$

respectively for every $x \in M_1$ and $u \in M_0$. Actually, we can take $z := s_0 u \cdot x$ and $z' := x \cdot s_0 u$; indeed, we have

$$d_i z = d_{x(i)} s_0 u \cdot d_i x = u \cdot d_i x \ ,$$
$$d_i z' = d_{s_0 u(i)} x \cdot d_i s_0 u = d_{s_0 u(i)} x \cdot u \ .$$

It turns out that the monoid structure on $\pi_0 M$ above makes not only the map $M_0 \to \pi_0 M$ but also $M_n \to \pi_0 M$ into a monoid homomorphism for arbitrary $n \in \mathbb{N}$. Finally, the left adjoint functor $j_\flat$ in (2.5.12) is given by

$$j_\flat M(\langle n\rangle) \cong \begin{cases} \pi_0 M & n = 0 \\ M_{n-1} & n \geq 1 \ . \end{cases}$$

Example 2.5.13. Take Φ to be the functor $\mathfrak{J} : \widetilde{\Delta} \to \nabla$. Set $\mathcal{I}$ to be the set of morphisms $\varphi : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$ such that it restricts to a bijection

$$\varphi^{-1}\{1, \dots, n\} \to \{1, \dots, n\} \ .$$

It is known that every morphism in ∇ uniquely factors through a morphism in $\mathcal{I}$ followed by one in the image of $\mathfrak{J}$. This factorization gives us a nice description of the left Kan extension functor

$$\mathfrak{J}_! : \mathbf{Set}_{\widetilde{\Delta}} \to \mathbf{Set}_\nabla$$

as follows: for an augmented simplicial set X, $\mathfrak{J}_! X(\langle\!\langle n\rangle\!\rangle)$ is the set

$$\{(x, \rho) \mid k \in \mathbb{N},\ x \in X(k),\ \rho : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle k\rangle\!\rangle \in \mathcal{I}\} \tag{2.5.13}$$

For $\varphi : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$, the induced map $\varphi^* : \mathfrak{J}_! X(\langle\!\langle n\rangle\!\rangle) \to \mathfrak{J}_! X(\langle\!\langle m\rangle\!\rangle)$ is given by

$$\varphi^*(x, \rho) = (\mu^*(x), \rho_\varphi) \ ,$$

where (μ, ρ_φ) is the unique pair of morphisms with $\mu \in \mathfrak{J}(\widetilde{\Delta})$, $\rho_\varphi \in \mathcal{I}$, and $\mu\rho_\varphi = \rho\varphi$. Thus, the left adjoint $\mathfrak{J}_\flat^{\mathfrak{W}^\nabla}$ in the adjunction

$$\mathfrak{J}_\flat^{\mathfrak{W}^\nabla} \ : \ \mathbf{CrsMon}_{\widetilde{\Delta}}^{/\mathfrak{W}^{\widetilde{\Delta}} \times C_2} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{CrsMon}_\nabla \ : \mathfrak{J}^\natural_{\mathfrak{W}^\nabla} \tag{2.5.14}$$

is described as follows: for $M \in \mathbf{CrsMon}_{\widetilde{\Delta}}^{/\mathfrak{W}^{\widetilde{\Delta}} \times C_2}$ with the associated augmented simplicial map $\theta : M \to C_2$, $\mathfrak{J}_\flat^{\mathfrak{W}^\nabla} M(\langle\!\langle n\rangle\!\rangle)$ is the quotient of the free monoid over the set defined similarly to (2.5.13) by the congruence relation generated by

$$(xy, \rho) \sim (x, \rho^{\theta(y)})(y, \rho) \ , \quad (e_k, \rho) \sim e_n \ . \tag{2.5.15}$$

In particular, if $\theta : M \to C_2$ is trivial, the relation (2.5.15) gives rise to an isomorphism

$$\mathfrak{J}_\flat M(\langle\!\langle n\rangle\!\rangle) \cong \mathop{\ast}_{\rho : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle k\rangle\!\rangle \in \mathcal{I}} M(\langle k\rangle) \ ,$$

where the right hand side is the free product of monoids.

To end the section, we mention crossed groups. We saw above that the Kan extensions along stable faithful functors give rise to adjunctions between the category of crossed monoids. Notice that all the construction can be described as limits and colimits, at least degreewisely. On the other hand, in view of Theorem 2.4.15, the subcategory $\mathbf{CrsGrp}_{\mathcal{A}} \subset \mathbf{CrsMon}_{\mathcal{A}}$ is closed under both limits and colimits. This implies all the discussion above restricts to crossed groups, so we obtain the following result.

Theorem 2.5.14. *Let $\Phi : \mathcal{A} \to \widetilde{\mathcal{A}}$ be a faithful functor, and let $\widetilde{G}$ be a crossed $\widetilde{\mathcal{A}}$-group such that Φ is $\widetilde{G}$-stable. Then, the adjunctions*

$$\Phi_\flat^{\widetilde{G}}, \Phi_\sharp^{\widetilde{G}} : \mathbf{CrsMon}_{\mathcal{A}}^{/\Phi^*\widetilde{G}} \underset{\perp}{\overset{\perp}{\leftrightarrows}} \mathbf{CrsMon}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}} : \Phi^\natural_{\widetilde{G}}$$

given in Corollary 2.5.3 and Theorem 2.5.9 restrict to

$$\Phi_\flat^{\widetilde{G}}, \Phi_\sharp^{\widetilde{G}} : \mathbf{CrsGrp}_{\mathcal{A}}^{/\Phi^*\widetilde{G}} \underset{\perp}{\overset{\perp}{\leftrightarrows}} \mathbf{CrsGrp}_{\widetilde{\mathcal{A}}}^{/\widetilde{G}} : \Phi^\natural_{\widetilde{G}} .$$

Moreover, if Φ is fully faithful, so are $\Phi_\flat^{\widetilde{G}}$ and $\Phi_\sharp^{\widetilde{G}}$ even after restricted to crossed groups.

2.6 Classification of crossed interval groups

We give a complete list of crossed interval subgroups of the terminal crossed interval group $\mathfrak{W}^\nabla$. As mentioned at the end of Section 2.3, this gives us a classification of crossed interval groups up to *non-crossed* parts (see Proposition 2.3.18). Recall that, according to the computation in Example 2.3.17, we have

$$\mathfrak{W}^\nabla(\langle\langle n\rangle\rangle) \cong H_n \times C_2 , \tag{2.6.1}$$

where H_n is the n-th hyperoctahedral group, and C_2 is the group of order 2. At first glance, this looks pretty nice; we have an excellent theorem, namely, *Goursat's Lemma* [29] to seek subgroups of a product of groups. Unfortunately, the "product" in (2.6.1) is, however, not actually the product of interval sets; the second component C_2 is not really closed under the interval set structure so that the projections $H_n \times C_2 \to H_n$ fails to define a map of interval sets. This is because of the morphisms in ∇ outside the image of $\widetilde{\Delta}$, so we first consider its restriction to $\widetilde{\Delta}$. Indeed, let $\mathfrak{J} : \widetilde{\Delta} \to \nabla$ be the functor given by

$$\mathfrak{J}(\langle n\rangle) = \langle\langle n\rangle\rangle$$

(see Example 2.5.6). Since $\mathfrak{J}$ is $\mathfrak{W}^\nabla$-stable faithful functor, in view of Corollary 2.5.3, it induces a functor

$$\mathfrak{J}^\natural : \mathbf{CrsGrp}_\nabla \to \mathbf{CrsGrp}_{\widetilde{\Delta}}^{/\mathfrak{J}^*\mathfrak{W}^\nabla} .$$

A good news is that the isomorphism (2.6.1) now exhibits $\mathfrak{J}^\natural\mathfrak{W}^\nabla$ as a product $\mathfrak{W}^{\widetilde{\Delta}} \times C_2$ of augmented simplicial sets, where C_2 is the constant augmented simplicial set at C_2. Note that $\mathfrak{J}^\natural$ is faithful so it preserves monomorphisms. Hence every crossed interval subgroups of $\mathfrak{W}^\nabla$ is sent to an augmented crossed simplicial subgroup of $\mathfrak{J}^\natural\mathfrak{W}^\nabla \cong \mathfrak{W}^{\widetilde{\Delta}} \times C_2$.

To compute all the augmented crossed simplicial subgroups of $\mathfrak{W}^{\widetilde{\Delta}} \times C_2$, we establish a crossed analogue of Goursat's Lemma. This can actually be done for general base categories $\mathcal{A}$.

Lemma 2.6.1. *Let $\mathcal{A}$ be a small category. Suppose we are given an inclusion $N \hookrightarrow G$ of crossed $\mathcal{A}$-groups such that $N(a) \subset G(a)$ is a normal subgroup for each $a \in \mathcal{A}$. Then, the family $\{G(a)/N(a)\}_a$ admits a unique structure of $\mathcal{A}$-sets such that the canonical map $G(a) \twoheadrightarrow G(a)/N(a)$ is a map of $\mathcal{A}$-sets.*

Proof. The uniqueness follows from the surjectivity of each map $G(a) \to G(a)/N(a)$. We show that, for each $\varphi : b \to a \in \mathcal{A}$, the map

$$\varphi^* : G(a)/N(a) \to G(b)/N(b) \ ; \quad xN(a) \to \varphi^*(x)N(b) \tag{2.6.2}$$

is well-defined. Note that, since $N(a)$ and $N(b)$ are normal, it is equivalent to see the same statement for right cosets. For each $x \in G(a)$ and for every $u \in N(a)$, we have

$$\varphi^*(ux) = (\varphi^x)^*(u)\varphi^*(x) \ .$$

Since N is a crossed $\mathcal{A}$-subgroup of G, $(\varphi^x)^*(u) \in N(b)$ so we obtain $\varphi^*(ux) \in N(b)\varphi^*(x)$. This immediately implies that (2.6.2) in fact defines an $\mathcal{A}$-set structure on the family $\{G(a)/N(a)\}_a$. The required property is easily verified. □

In what follows, we write G/N the $\mathcal{A}$-set obtained in Lemma 2.6.1. Notice that it admits a canonical degreewise group structure induced from G.

Similarly to the ordinary Goursat's Lemma, we aim to present subgroups of a product of crossed $\mathcal{A}$-groups in terms of subgroups of each components. Here, the term "product" is ambiguous; indeed, *cartesian products* in the category $\mathbf{CrsGrp}_{\mathcal{A}}$ do not always agree with products of $\mathcal{A}$-sets, while the latter do not always produce crossed $\mathcal{A}$-groups even if they made from crossed $\mathcal{A}$-groups. For example, our target $\mathfrak{W}^{\widetilde{\Delta}} \times C_2$ is not a cartesian product in the category $\mathbf{CrsGrp}_{\widetilde{\Delta}}$. Hence, we need to find an appropriate notion to substitute for *products*. A key observation is that, for a group G, to establish an isomorphism $G \cong G^{(1)} \times G^{(2)}$, it suffices to find a pair $(G^{(1)}, G^{(2)})$ of subgroups of G such that G is generated by $G^{(1)} \cup G^{(2)}$ and

$$G^{(1)} \cap G^{(2)} = [G^{(1)}, G^{(2)}] = \{e\} \ ,$$

where the middle is the commutator subgroup.

Definition. Let $\mathcal{A}$ be a small category. A crossed $\mathcal{A}$-group G is said to be a *virtual product* of crossed $\mathcal{A}$-subgroups $G^{(1)}$ and $G^{(2)}$ if the following conditions hold:

- (i) the map $G^{(1)} * G^{(2)} \to G$ induced by the inclusions is an epimorphism in $\mathbf{CrsGrp}_{\mathcal{A}}$, where $G^{(1)} * G^{(2)}$ is the coproduct in $\mathbf{CrsGrp}_{\mathcal{A}}$ (see Proposition 2.2.1);
- (ii) the pullback $G^{(1)} \times_G G^{(2)}$ is trivial; roughly, we often write $G^{(1)} \cap G^{(2)} = *$;
- (iii) for each $a \in \mathcal{A}$, the commutator subgroup $[G^{(1)}(a), G^{(2)}(a)] \subset G(a)$ is trivial; in other words, elements of $G^{(1)}$ and $G^{(2)}$ commute with each other.

Lemma 2.6.2. *Let $\mathcal{A}$ be a small category, and let G be a crossed $\mathcal{A}$-group which is a virtual product of crossed $\mathcal{A}$-subgroups $G^{(1)}$ and $G^{(2)}$. Then, for every morphism $\varphi : a \to b$, and for each element $x_i \in G^{(i)}$ for $i = 1, 2$, we have*

$$(\varphi^{x_2})^*(x_1) = \varphi^*(x_1) \ , \quad (\varphi^{x_1})^*(x_2) = \varphi^*(x_2) \ .$$

Proof. The condition on crossed groups implies

$$\varphi^*(x_1x_2) = (\varphi^{x_2})^*(x_1)\varphi^*(x_2) \ , \quad \varphi^*(x_2x_1) = (\varphi^{x_1})^*(x_2)\varphi^*(x_1) \ .$$

Since the commutator subgroups $[G^{(1)}(a), G^{(2)}(a)] \subset G(a)$ and $[G^{(1)}(b), G^{(2)}(b)] \subset G(b)$ are trivial, the elements above equal, and we obtain

$$\varphi^*(x_1)^{-1}(\varphi^{x_2})^*(x_1) = (\varphi^{x_1})^*(x_2)\varphi^*(x_2)^{-1} .$$

The left hand side belongs to $G^{(1)}(b)$ while the right to $G^{(2)}(b)$, so both belong to $G^{(1)}(b) \cap G^{(2)}(b) = \{e\}$. Thus, the result follows. ☐

Theorem 2.6.3 (Goursat's Lemma for crossed groups)**.** *Let $\mathcal{A}$ be a small category. Suppose G is a crossed $\mathcal{A}$-group which is a virtual product of crossed $\mathcal{A}$-subgroups $G^{(1)}, G^{(2)} \subset G$. Then, there is a 1-1 correspondence between the following data:*

(a) *a crossed $\mathcal{A}$-subgroup H of G;*

(b) *a quintuple $(\widetilde{H}^{(1)}, H^{(1)}; \widetilde{H}^{(2)}, H^{(2)}; \chi)$ of*

  (i) *crossed $\mathcal{A}$-subgroups $H^{(i)} \subset \widetilde{H}^{(i)} \subset G^{(i)}$ for $i = 1, 2$ so that $H^{(i)}(a)$ is a normal subgroup in $\widetilde{H}^{(i)}(a)$ for each $a \in \mathcal{A}$;*

  (ii) *a map $\chi : \widetilde{H}^{(1)}/H^{(1)} \to \widetilde{H}^{(2)}/H^{(2)}$ of $\mathcal{A}$-sets which is a degreewise group isomorphism.*

Proof. We denote by $\mathrm{Sub}(G)$ the set of crossed $\mathcal{A}$-subgroups of G and by $\mathrm{Gou}(G^{(1)}, G^{(2)})$ the set of quintuples as in (b). For $Q = (\widetilde{H}^{(1)}, H^{(1)}; \widetilde{H}^{(2)}, H^{(2)}; \chi) \in \mathrm{Gou}(G^{(1)}, G^{(2)})$, consider the subset

$$H^Q(a) := \left\{ x_1 x_2 \;\middle|\; x_1 \in \widetilde{H}^{(1)}(a),\ x_2 \in \widetilde{H}^{(2)}(a),\ \chi\left(x_1 H^{(1)}(a)\right) = x_2 H^{(2)}(a) \right\}$$

of $G(a)$ for each $a \in \mathcal{A}$. We see the family $H^Q = \{H^Q(a)\}_a$ forms a crossed $\mathcal{A}$-subgroup of G. Since $[G^{(1)}(a), G^{(2)}(a)] = *$ and χ is a degreewise group isomorphism, H^Q is a degreewise subgroup of G. On the other hand, for $\varphi : b \to a \in \mathcal{A}$, Lemma 2.6.2 implies that, for each $x_1 x_2 \in H^Q(a)$ with $x_i \in \widetilde{H}^{(i)}(a)$ and $\chi\left(x_1 H^{(1)}(a)\right) = x_2 H^{(2)}(a)$,

$$\varphi^*(x_1 x_2) = (\varphi^{x_2})^*(x_1)\varphi^*(x_2) = \varphi^*(x_1)\varphi^*(x_2) .$$

Since χ is an $\mathcal{A}$-map, we have

$$\chi\left(\varphi^*(x_1) H^{(1)}(b)\right) = \varphi^* \chi\left(x_1 H^{(1)}(a)\right) = \varphi^*(x_2) H^{(2)}(b)$$

so $\varphi^*(x_1 x_2) \in H^Q(b)$. Hence, $H^Q \subset G$ is closed under both the degreewise group structure and the $\mathcal{A}$-set structure so to define a crossed $\mathcal{A}$-subgroup.

Now, consider the map

$$\mathrm{Gou}(G^{(1)}, G^{(2)}) \to \mathrm{Sub}(G) \ ; \quad Q \mapsto H^Q . \tag{2.6.3}$$

We show it admits an inverse. Suppose $H \subset G$ is a crossed $\mathcal{A}$-subgroup. For $\{i, j\} = \{1, 2\}$, and for $a \in \mathcal{A}$, we define

$$\widetilde{H}^{(i)}(a) := \left\{ x_i \in G^{(i)}(a) \;\middle|\; \exists x_j \in G^{(j)}(a) : x_i x_j \in H(a) \right\}$$
$$H^{(i)}(a) = G^{(i)}(a) \cap H(a) .$$

Note that the family $\widetilde{H}^{(i)} = \{\widetilde{H}^{(i)}(a)\}_a$ forms a crossed $\mathcal{A}$-subgroup of $G^{(i)}$ as well as $H^{(i)} = \{H^{(i)}(a)\}_a$. Indeed, if $(x_1, x_2), (x_1', x_2') \in G^{(1)}(a) \times G^{(2)}(a)$ are pairs with $x_1x_2, x_1'x_2' \in H(a)$, and if $\varphi : b \to a \in \mathcal{A}$, then by Lemma 2.6.2,

$$\begin{aligned} (x_1x_1')(x_2x_2') = (x_1x_2)(x_1'x_2') \in H(a) \ , \\ \varphi^*(x_1)\varphi^*(x_2) = \varphi^*(x_1x_2) \in H(b) \ . \end{aligned} \tag{2.6.4}$$

On the other hand, $H^{(i)}(a)$ is clearly a normal subgroup of $H(a)$. Moreover, for $x_i \in \widetilde{H}^{(i)}(a)$, take an element $x_j \in G^{(j)}(a)$ with $x_ix_j \in H(a)$, then

$$x_iH^{(i)}(a) = x_ix_jH^{(i)}(a)x_j^{-1} = (x_ix_jH^{(i)}(a)(x_ix_j)^{-1})x_i = H^{(i)}(a)x_i \ ,$$

which implies $H^{(i)}(a)$ is a normal subgroup of $\widetilde{H}^{(i)}(a)$. Hence, by Lemma 2.6.1, we obtain two $\mathcal{A}$-sets $\widetilde{H}^{(1)}/H^{(1)}$ and $\widetilde{H}^{(2)}/H^{(2)}$ that are degreewise groups. Define a map $\chi_a^H : (\widetilde{H}^{(1)}/H^{(1)})(a) \to (\widetilde{H}^{(2)}/H^{(2)})(a)$ so that

$$\chi_a^H\left(x_1H^{(1)}(a)\right) = x_2H^{(2)}(a)$$

if and only if $x_1x_2 \in H(a)$. It is easily verified that such a map χ_a is uniquely determined by the crossed $\mathcal{A}$-subgroup $H \subset G$. Furthermore, the formulas (2.6.4) implies that $\chi^H = \{\chi_a^H\}_a$ defines an $\mathcal{A}$-map $\widetilde{H}^{(1)}/H^{(1)} \to \widetilde{H}^{(2)}/H^{(2)}$ that is a degreewise group isomorphism. We write

$$Q^H := (\widetilde{H}^{(1)}, H^{(1)}; \widetilde{H}^{(2)}, H^{(2)}; \chi^H)$$

the resulting quintuple. Then, clearly $Q^H \in \mathrm{Gou}(G^{(1)}, G^{(2)})$, and the classical Goursat's Lemma for groups shows that the assignment $H \mapsto Q^H$ gives the inverse of the map (2.6.3). ☐

In the rest, we compute all the crossed interval subgroups of $\mathfrak{W}^{\nabla}$. To begin with, we focus on the crossed interval subgroup $\mathfrak{H} \subset \mathfrak{W}^{\nabla}$ of hyperoctahedral groups whose structure is given in Example 2.1.10. Since we have $\mathfrak{J}^*\mathfrak{H} \cong \mathfrak{W}^{\widetilde{\Delta}}$, crossed interval subgroups of $\mathfrak{H}$ are augmented crossed simplicial subgroups of $\mathcal{W}^{\widetilde{\Delta}}$ closed under the structure of interval sets. As a result of [24], we have a complete list of crossed simplicial subgroups of $\mathfrak{W}^{\Delta}$ as in Table 2.1 in Example 2.3.19. In view of Example 2.5.5, to obtain a complete list of augmented crossed simplicial subgroups of $\mathfrak{W}^{\widetilde{\Delta}}$, we only have to shift the indices in Table 2.1 by 1. The result is indicated in Table 2.2. It is seen that, among Table 2.2,

name	**symbol**	**group at $\langle n\rangle$**
Trivial	$*$	1
Reflexive	C_2	C_2
Cyclic	$\mathfrak{C}$	C_n
Dihedral	$\mathfrak{D}$	D_n
Symmetric	$\mathfrak{S}$	$\mathfrak{S}_n$
Reflexosymmetric	$\widetilde{\mathfrak{S}}$	$\mathfrak{S}_n \times C_2$
Weyl (Hyperoctahedral)	$\mathfrak{W}^{\widetilde{\Delta}} \cong \mathfrak{H}$	H_n

Table 2.2: The augmented crossed simplicial subgroups of $\mathfrak{W}^{\widetilde{\Delta}}$

crossed *interval* subgroups of $\mathfrak{H}$ are precisely the trivial one $*$, the symmetric one $\mathfrak{S}$, and the hyperoctahedral one $\mathfrak{H}$ itself.

Now suppose $H \subset \mathfrak{W}^{\nabla}$ is a crossed interval subgroup. Since the augmented crossed simplicial group $\mathfrak{J}^{*}\mathfrak{W}^{\nabla} \cong \mathfrak{H} \times C_2$ is, as suggested by the notation, a virtual product of $\mathfrak{H}$ and C_2, Theorem 2.6.3 implies there is an associated quintuple $(\widetilde{H}^{(1)}, H^{(1)}; \widetilde{H}^{(2)}, H^{(2)}; \chi^H)$ of augmented crossed simplicial subgroups $H^{(1)} \subset \widetilde{H}^{(1)} \subset \mathfrak{H}$, $H^{(2)} \subset \widetilde{H}^{(2)} \subset C_2$, and $\chi : \widetilde{H}^{(1)}/H^{(1)} \cong \widetilde{H}^{(2)}/H^{(2)}$. In particular, according to the proof of Theorem 2.6.3, $H^{(1)} = H \cap \mathfrak{H}$ is an intersection of crossed interval subgroups of $\mathfrak{W}^{\nabla}$, so $H^{(1)}$ is a crossed interval subgroup of $\mathfrak{H}$, which is either $*$, $\mathfrak{S}$, or $\mathfrak{H}$ by the argument above. On the other hand, since $\widetilde{H}^{(2)}/H^{(2)}$ is a subquotient of the group C_2, it is, degreewisely, of order at most 2. Thus, the isomorphism χ^H is, if exists, uniquely determined by the other data $(\widetilde{H}^{(1)}, H^{(1)}; \widetilde{H}^{(2)}, H^{(2)})$, so we can omit it in what follows. As a result, all the possibilities of the quadruples are listed below:

$$\begin{gathered}(*,*;*,*)\ ,(\mathfrak{S},\mathfrak{S};*,*)\ ,(\mathfrak{H},\mathfrak{H};*,*)\ ,(*,C_2;*,C_2)\ ,\\ (\mathfrak{S},\widetilde{\mathfrak{S}};*,C_2)\ ,(*,*;C_2,C_2)\ ,(\mathfrak{S},\mathfrak{S};C_2,C_2)\ ,(\mathfrak{H},\mathfrak{H};C_2,C_2)\ .\end{gathered}$$

It turns out that the sixth and the seventh do not produce crossed interval subgroups while the others do. Hence, we finally obtain the list of crossed interval subgroups of $\mathfrak{W}^{\nabla}$ (Table 2.3).

name	**symbol**	**group at $\langle\langle n\rangle\rangle$**	**associated quadruple**
Trivial	$*$	1	$(*,*;*,*)$
Reflexive	C_2	C_2	$(*,C_2;*,C_2)$
Symmetric	$\mathfrak{S}$	$\mathfrak{S}_n$	$(\mathfrak{S},\mathfrak{S};*,*)$
Reflexosymmetric	$\widetilde{\mathfrak{S}}$	$\mathfrak{S}_n \times C_2$	$(\mathfrak{S},\widetilde{\mathfrak{S}};*,C_2)$
Hyperoctahedral	$\mathfrak{H}$	H_n	$(\mathfrak{H},\mathfrak{H};*,*)$
Weyl	$\mathfrak{W}^{\nabla}$	$H_n \times C_2$	$(\mathfrak{H},\mathfrak{H};C_2,C_2)$

Table 2.3: The crossed interval subgroups of $\mathfrak{W}^{\nabla}$

Chapter 3

Group operads and the embedding

In this chapter, we will establish and investigate a fully faithful embedding of the category of group operads into that of crossed interval groups. For this, we introduce a monoidal structure on the slice of the category of operads over the operad of symmetric groups. Comparing with the monoidal structure on the category of interval sets discussed in Chapter 2, we obtain a monoidal functor connecting these two categories. It will be shown that this actually induces a fully faithful functor on monoid objects and does not change the underlying sets, so we obtain a required embedding. The conditions for crossed interval groups to belong to the essential image will be proposed; namely in terms of commutativity of certain elements. As a result, it will turn out that the group operads form a reflective subcategory of the category of crossed interval groups. Finally, we will discuss monoid objects in symmetric monoidal category and Hochschild homologies on them.

3.1 Group operads

We first recall the formal definition of group operads.

Notation. (1) For each natural number $n \in \mathbb{N}$, write

$$\langle n \rangle := \{1, \dots, n\} \ .$$

We often regard it as the linearly ordered set with the canonical order.

(2) If P and Q are poset, i.e. partially ordered set, then we denote by $P \star Q$ the join of them. In other words, $P \star Q$ is the set $P \amalg Q$ together with the ordering so that for $x, y \in P \star Q$,

$$x \leq y \iff \begin{cases} (x, y) \in (P \times P) \amalg (Q \times Q) \text{ with } x \leq y \ , \text{ or} \\ x \in P \text{ and } y \in Q \ . \end{cases}$$

Hence, we have a unique isomorphism $\langle m \rangle \star \langle n \rangle \cong \langle m + n \rangle$ of posets.

For each natural number $n \in \mathbb{N}$, we put $\mathfrak{S}(n)$ the n-th permutation group, or the permutation group on the set $\langle n \rangle = \{1, \dots, n\}$. We begin with the observation that the family $\mathfrak{S} = \{\mathfrak{S}(n)\}_n$ admits a canonical structure of operads. Namely, if $\sigma \in \mathfrak{S}(n)$ and $\sigma_i \in \mathfrak{S}(k_i)$ for $1 \le i \le n$, we write $\gamma(\sigma; \sigma_1, \dots, \sigma_n)$ the permutation on $\langle k_1 + \dots + k_n \rangle$ given below:

$$\begin{aligned}\langle k_1 + \dots + k_n \rangle &\cong \langle k_1 \rangle \star \dots \star \langle k_n \rangle \\ &\xrightarrow{\sigma_1 \amalg \dots \amalg \sigma_n} \langle k_1 \rangle \star \dots \star \langle k_n \rangle \\ &\xrightarrow{\sigma_*} \langle k_{\sigma^{-1}(1)} \rangle \star \dots \star \langle k_{\sigma^{-1}(n)} \rangle \\ &\cong \langle k_1 + \dots + k_n \rangle \ .\end{aligned}$$

This defines a map

$$\gamma = \gamma_{\mathfrak{S}} : \mathfrak{S}(n) \times \prod_{i=1}^{n} \mathfrak{S}(k_1) \to \mathfrak{S}(k_1 + \dots + k_n) \ .$$

It is tedious but not difficult to see it makes $\mathfrak{S}$ into an operad.

Group operads are likely "generalizations" of the operad $\mathfrak{S}$. For the definition, we follow [15] for conventions except the terminology.

Definition. A *group operad* is an operad $\mathcal{G}$ together with data

- a group structure on each $\mathcal{G}(n)$;
- a map $\mathcal{G} \to \mathfrak{S}$ of operads so that each $\mathcal{G}(n) \to \mathfrak{S}(n)$ is a group homomorphism, which gives rise to a left $\mathcal{G}(n)$-action on $\langle n \rangle$;

which satisfy the identity

$$\gamma_{\mathcal{G}}(xy; x_1 y_1, \dots, x_n y_n) = \gamma_{\mathcal{G}}(x; x_{y^{-1}(1)}, \dots, x_{y^{-1}(n)}) \gamma_{\mathcal{G}}(y; y_1, \dots, y_n) \qquad (3.1.1)$$

for every $x, y \in \mathcal{G}(n)$ and $x_i, y_i \in \mathcal{G}(k_i)$ for $1 \le i \le n$.

Remark 3.1.1. Note that, in the paper [15] the terminology "action operads" was used there. This is probably because the name "group operads" may be confusing with *group-enriched operads* or *group objects in operads*. Nevertheless, we stick to the terminology in a certain reason, which will turn out later.

Example 3.1.2. The operad $\mathfrak{S}$ is a group operad with the identity map $\mathfrak{S} \xrightarrow{=} \mathfrak{S}$.

Example 3.1.3. For each $n \in \mathbb{N}$, denote by $\mathcal{B}(n)$ the braid group of n-strands. In a similar manner to $\mathfrak{S}$, one can endow the family $\mathcal{B} = \{\mathcal{B}(n)\}_n$ with the structure of operads. Then, the canonical quotient map $\mathcal{B} \to \mathfrak{S}$ exhibits $\mathcal{B}$ as a group operad. The similar argument works for pure braids, ribbon braids, and so on.

The reader can find more interesting examples in [76] and [31]. We here mention basic properties of group operads.

Proposition 3.1.4. *For a group operad $\mathcal{G}$, the following hold.*

(1) *The composition map*

$$\gamma : \mathcal{G}(1) \times \mathcal{G}(1) \to \mathcal{G}(1)$$

in the operad structure on $\mathcal{G}$ coincides with the multiplication in the group structure. In particular, the unit $e_1 \in \mathcal{G}(1)$ in the group structure is exactly the identity of the operad $\mathcal{G}$. Moreover, $\mathcal{G}(1)$ is an abelian group.

(2) *For each* $n \in \mathbb{N}$*, write* $e_n \in \mathcal{G}(n)$ *the unit in the group structure. Then, for* $k_1, \dots, k_n \in \mathbb{N}$*, we have*

$$\gamma(e_n; e_{k_1}, \dots, e_{k_n}) = e_{k_1+\dots+k_n} \ .$$

In other words, the family $\{e_n\}_n$ *determines a map* $* \to \mathcal{G}$ *of operads from the terminal (or trivial) operad* $*$.

(3) *For each* $n \in \mathbb{N}$*, the map*

$$\mathcal{G}(1) \to \mathcal{G}(n) \ ; \quad x \mapsto \gamma(x; e_n)$$

is a group homomorphism.

Proof. Notice that, for $x, x', y, y' \in \mathcal{G}(1)$, the condition (3.1.1) on group operads implies

$$\gamma(xx'; yy') = \gamma(x; y)\gamma(x'; y') \ . \tag{3.1.2}$$

Hence, the part (1) follows from the *Eckmann-Hilton argument.*

To see (2), since $\mathcal{G}(k_1 + \dots + k_n)$ is a group, it suffices to see the element $\gamma(e_n; e_{k_1}, \dots, e_{k_n})$ is idempotent. By the condition on group operads again, we have

$$\gamma(e_n; e_{k_1}, \dots, e_{k_n})^2 = \gamma(e_n^2; e_{k_1}^2, \dots, e_{k_n}^2) = \gamma(e_n; e_{k_1}, \dots, e_{k_n}) \ .$$

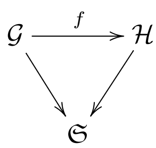
This implies $\gamma(e_n; e_{k_1}, \dots, e_{k_n}) = e_{k_1+\dots+k_n}$.

The last assertion (3) directly follows from the condition (3.1.1) and the part (1). □

Definition. Let $\mathcal{G}$ and $\mathcal{H}$ be group operads. Then, a *map of group operads* is a map $f : \mathcal{G} \to \mathcal{H}$ of operads such that

(i) for each $n \in \mathbb{N}$, the map $f : \mathcal{G}(n) \to \mathcal{H}(n)$ is a group homomorphism;

(ii) the triangle below is commutative:



,

where the vertical arrows are the structure maps.

Clearly maps of group operads compose so as to form a category, which we will denote by **GrpOp**. One of the important results proved in [31] is the presentability of the category.

Theorem 3.1.5 (Theorem 3.5 and Theorem 3.8 in [31])**.** *The category* **GrpOp** *is locally finitely presentable. Moreover, the forgetful functor* U : **GrpOp** $\to$ $\mathbf{Set}_{\mathbb{N}}^{/\mathfrak{S}}$ *into the slice category of* $\mathbb{N}$*-indexed family of sets over the family* $\{\mathfrak{S}(n)\}_n$ *creates limits and filtered colimits.*

We are here not going further with the theory of group operads. In the rest of the section, rather than that, we aim to recover the notion of group operads from another aspect, namely a monoidal structure.

Definition. Let $\mathcal{X}$ and $\mathcal{Y}$ be two operads, and suppose we are given a map $\rho : \mathcal{Y} \to \mathfrak{S}$ of operads. We define an operad $\mathcal{X} \rtimes \mathcal{Y}$ as follows:

- for each $n \in \mathbb{N}$, put $(\mathcal{X} \rtimes \mathcal{Y})(n) := \mathcal{X}(n) \times \mathcal{Y}(n)$;
- for $(x, y) \in (\mathcal{X} \rtimes \mathcal{Y})(n)$ and $(x_i, y_i) \in (\mathcal{X} \rtimes \mathcal{Y})(k_i)$ for $1 \leq i \leq n$, the composition is given by

$$\begin{aligned}&\gamma_{\mathcal{X}\rtimes\mathcal{Y}}((x, y); (x_1, y_1), \dots, (x_n, y_n))\\ &\quad := (\gamma_{\mathcal{X}}(x; x_{\rho(y)^{-1}(1)}, \dots, x_{\rho(y)^{-1}(n)}), \gamma_{\mathcal{Y}}(y; y_1, \dots, y_n)) \ ;\end{aligned}$$

It is easily verified that the above data actually define an operad $\mathcal{X} \rtimes \mathcal{Y}$ so that the identity is the pair $\mathrm{id}_{\mathcal{X}\rtimes\mathcal{Y}} = (\mathrm{id}_{\mathcal{X}}, \mathrm{id}_{\mathcal{Y}})$. Moreover, the assignment $(\mathcal{X}, \mathcal{Y}, \rho) \mapsto \mathcal{X} \rtimes \mathcal{Y}$ is functorial; indeed, if we have a map $f : \mathcal{X} \to \mathcal{X}'$ and a triangle

$$\begin{array}{ccc}\mathcal{Y} & \xrightarrow{g} & \mathcal{Y}' \\ & {}_{\rho}\searrow \quad \swarrow_{\rho'} & \\ & \mathfrak{S} & \end{array}$$

of operads, the maps

$$f \rtimes g : (\mathcal{X} \rtimes \mathcal{Y})(n) \to (\mathcal{X}' \rtimes \mathcal{Y}')(n) \ ; \quad (x, y) \mapsto (f(x), g(y))$$

form a map $\mathcal{X} \rtimes \mathcal{Y} \to \mathcal{X}' \rtimes \mathcal{Y}'$ of operads. In other words, if we denote by **Op** the category of operads and by $\mathbf{Op}^{/\mathfrak{S}}$ the slice category over $\mathfrak{S}$, then we obtain a functor

$$\rtimes : \mathbf{Op} \times \mathbf{Op}^{/\mathfrak{S}} \to \mathbf{Op} \ . \tag{3.1.3}$$

The following result is a direct consequence of the operad structure of $\mathfrak{S}$.

Lemma 3.1.6. *The multiplication maps*

$$\mathtt{mul} : \mathfrak{S}(n) \times \mathfrak{S}(n) \to \mathfrak{S}(n) \ ; \quad (\sigma, \tau) \mapsto \sigma\tau$$

define a map $\mathfrak{S} \rtimes \mathfrak{S} \to \mathfrak{S}$ of operads, here we take $\mathfrak{S} \rtimes \mathfrak{S}$ with respect to the identity map $\mathfrak{S} \xrightarrow{=} \mathfrak{S}$.

Lemma 3.1.6 offers a lift of the functor (3.1.3) to a *binary operation* on $\mathbf{Op}^{/\mathfrak{S}}$; indeed, we have the following composition

$$\rtimes : \mathbf{Op}^{/\mathfrak{S}} \times \mathbf{Op}^{/\mathfrak{S}} \cong (\mathbf{Op} \times \mathbf{Op}^{/\mathfrak{S}})^{/(\mathfrak{S},\mathfrak{S})} \xrightarrow{\rtimes} \mathbf{Op}^{/\mathfrak{S}\rtimes\mathfrak{S}} \xrightarrow{\mathtt{mul}_!} \mathbf{Op}^{/\mathfrak{S}} \ . \tag{3.1.4}$$

Proposition 3.1.7. *The functor* (3.1.4) *defines a monoidal structure on the category $\mathbf{Op}^{/\mathfrak{S}}$ so that the trivial operad $*$ is the unit object.*

Proof. Since the last statement is obvious, we have to give an associativity isomorphism. It suffices to show that, for operads $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ over $\mathfrak{S}$, the identification

$$((\mathcal{X} \rtimes \mathcal{Y}) \rtimes \mathcal{Z})(n) = \mathcal{X}(n) \times \mathcal{Y}(n) \times \mathcal{Z}(n) = (\mathcal{X} \rtimes (\mathcal{Y} \rtimes \mathcal{Z}))(n)$$

is an isomorphism $(\mathcal{X} \rtimes \mathcal{Y}) \rtimes \mathcal{Z} \cong \mathcal{X} \rtimes (\mathcal{Y} \rtimes \mathcal{Z})$. Actually, in either case, the composition operation is given by

$$\begin{aligned}
&\gamma((x,y,z);(x_1,y_1,z_1),\dots,(x_n,y_n,z_n)) \\
&= \big(\gamma_{\mathcal{X}}(x; x_{\pi(z)^{-1}\rho(y)^{-1}(1)},\dots,x_{\pi(z)^{-1}\rho(y)^{-1}(n)}), \\
&\qquad \gamma_{\mathcal{Y}}(y; y_{\pi(z)^{-1}(1)},\dots,y_{\pi(z)^{-1}(n)}), \gamma_{\mathcal{Z}}(z; z_1,\dots,z_n)\big) \ ,
\end{aligned}$$

where $\rho : \mathcal{Y} \to \mathfrak{S}$ and $\pi : \mathcal{Z} \to \mathfrak{S}$ are the structure map. □

Definition. A *monoid operad* is a monoid object in the category $\mathbf{Op}^{/\mathfrak{S}}$ with respect to the monoidal structure $\rtimes$.

We denote by **MonOp** the category of monoid operads and monoid homomorphisms in $\mathbf{Op}^{/\mathfrak{S}}$. Note that a monoid operad $\mathcal{X}$ consists of an operad together with data

- a monoid structure on each $\mathcal{X}(n)$, and
- a map $\mathcal{X} \to \mathfrak{S}$ of operads so that $\mathcal{X}(n) \to \mathfrak{S}$ is a monoid homomorphism;

which satisfy appropriate conditions. Comparing it with the definition of group operads, one may notice that a group operad $\mathcal{G}$ determines a monoid operad and that it gives rise to a functor $\mathbf{GrpOp} \to \mathbf{MonOp}$. The following result is an easy exercise.

Proposition 3.1.8. *The functor* $\mathbf{GrpOp} \to \mathbf{MonOp}$ *is fully faithful. Moreover, a monoid operad* $\mathcal{X}$ *belongs to the essential image if and only if for each* $n \in \mathbb{N}$*, the monoid* $\mathcal{X}(n)$ *is a group.*

3.2 Symmetries on multicategories

One of the important aspects on group operads is their actions on multicategories. In this section, we see that each group operad $\mathcal{G}$ gives rise to the notion of *$\mathcal{G}$-symmetric structure* on multicategories.

The argument begin with the observation that the functor $\rtimes : \mathbf{Op} \times \mathbf{Op}^{/\mathfrak{S}} \to \mathbf{Op}$ extends to a functor

$$\rtimes : \mathbf{MultCat} \times \mathbf{Op}^{/\mathfrak{S}} \to \mathbf{MultCat}. \tag{3.2.1}$$

Indeed, for a multicategory $\mathcal{M}$ and an operad $\rho : \mathcal{X} \to \mathfrak{S}$ over $\mathfrak{S}$, we define a multicategory $\mathcal{M} \rtimes \mathcal{X}$ as follows:

- objects are those of $\mathcal{M}$;
- for $a_1,\dots,a_n,a \in \mathcal{M}$, we set

  $$\mathcal{M}(a_1\dots a_n; a) := \big\{(f,x) \bigm| x \in \mathcal{X}(n),\ f \in \mathcal{M}(a_{\rho(x)^{-1}(1)}\dots a_{\rho(x)^{-1}(n)}; a)\big\} \,;$$

- the composition operation is defined so that

  $$\begin{aligned}
  &\gamma_{\mathcal{M}\rtimes\mathcal{X}}((f,x);(f_1,x_1),\dots,(f_n,x_n)) \\
  &:= \big(\gamma_{\mathcal{M}}(f; f_{\rho(x)^{-1}(1)},\dots,f_{\rho(x)^{-1}(n)}), \gamma_{\mathcal{X}}(x; x_1,\dots,x_n)\big) \ .
  \end{aligned}$$

It is easily checked that the composition makes sense and is associative. Note that we have a canonical identification

$$(\mathcal{M} \rtimes \mathcal{X}) \rtimes \mathcal{Y} = \mathcal{M} \rtimes (\mathcal{X} \rtimes \mathcal{Y}) \ .$$

Hence, the functor (3.2.1) exhibits the category **MultCat** as a right $\mathbf{Op}^{/\mathfrak{S}}$-module with respect to the monoidal structure $\rtimes$ on $\mathbf{Op}^{/\mathfrak{S}}$ defined in Section 3.1. As a consequence, if $\mathcal{X}$ is a monoid operad, it gives rise to a functor

$$(-) \rtimes \mathcal{X} : \mathbf{MultCat} \to \mathbf{MultCat} \ .$$

It is actually a 2-functor: it sends a multinatural transformation $\alpha : F \to G : \mathcal{M} \to \mathcal{N}$ to the multinatural transformation consisting of

$$(\alpha_a, e_1) \in (\mathcal{N} \rtimes \mathcal{G})(F(a); G(a)) = \mathcal{N}(F(a); G(a)) \times \mathcal{G}(1) \tag{3.2.2}$$

for each $a \in \mathcal{M}$. Furthermore, it is easily verified that the monoid operad structure on $\mathcal{X}$ makes the 2-functor $(-) \rtimes \mathcal{X}$ into a 2-monad: we have an obvious 2-natural isomorphism

$$((-) \rtimes \mathcal{X}) \rtimes \mathcal{X} \cong (-) \rtimes (\mathcal{X} \rtimes \mathcal{X})$$

so that we can define two 2-natural transformations

$$\begin{gathered} ((-) \rtimes \mathcal{X}) \rtimes \mathcal{X} \cong (-) \rtimes (\mathcal{X} \rtimes \mathcal{X}) \xrightarrow{\mathrm{Id} \times \mathtt{mul}} (-) \rtimes \mathcal{X} \ , \\ \mathrm{Id} \cong (-) \rtimes * \xrightarrow{\mathrm{Id} \times \mathtt{unit}} (-) \rtimes \mathcal{X} \ . \end{gathered}$$

Definition. Let $\mathcal{X}$ be a monoid operad. Then, an *$\mathcal{X}$-symmetric structure* on a multicategory $\mathcal{M}$ is nothing but a structure of a strict 2-algebra over the 2-monad $(-) \rtimes \mathcal{X}$; i.e. a multifunctor

$$\mathtt{sym} : \mathcal{M} \rtimes \mathcal{X} \to \mathcal{M}$$

which makes the following diagrams commute:

$$\begin{array}{ccc} \mathcal{M} \rtimes \mathcal{X} \rtimes \mathcal{X} & \xrightarrow{\mathrm{Id} \rtimes \mathtt{mul}} & \mathcal{M} \rtimes \mathcal{X} \\ {\scriptstyle \mathtt{sym} \rtimes \mathrm{Id}} \downarrow & & \downarrow {\scriptstyle \mathtt{sym}} \\ \mathcal{M} \rtimes \mathcal{X} & \xrightarrow{\mathtt{sym}} & \mathcal{M} \end{array} \quad , \quad \begin{array}{ccc} \mathcal{M} \rtimes * & \xrightarrow{\mathrm{Id} \times \mathtt{unit}} & \mathcal{M} \rtimes \mathcal{X} \\ & \searrow\!\!\!\!= \quad \swarrow {\scriptstyle \mathtt{sym}} & \\ & \mathcal{M} & \end{array} \quad . \tag{3.2.3}$$

We say $\mathcal{M}$ is *$\mathcal{X}$-symmetric* if it is equipped with an $\mathcal{X}$-symmetric structure.

Example 3.2.1. If $\mathcal{X} = *$ is the trivial group operad, then $*$-symmetric multicategories are just multicategories.

Example 3.2.2. In the case $\mathcal{X} = \mathfrak{S}$ is the group operad of symmetric groups, $\mathfrak{S}$-symmetric multicategories are precisely *symmetric multicategories* in the usual sense.

Example 3.2.3. We say a monoidal category $\mathcal{C}$ is $\mathcal{X}$-symmetric if the multicategory $\mathcal{C}^{\otimes}$ (see Example 1.3.4) is equipped with an $\mathcal{X}$-symmetric structure. For example, $\mathfrak{S}$-symmetric (resp. $\mathcal{B}$-symmetric) monoidal categories are nothing but symmetric (resp. braided) monoidal categories Indeed, in this case, for each $x \in \mathcal{G}(n)$ and for objects $X_1, \dots, X_n \in \mathcal{C}$, we set

$$\Theta^x_{X_1 \dots X_n} : X_1 \otimes \cdots \otimes X_n \to X_{x^{-1}(1)} \otimes X_{x^{-1}(n)}$$

to be the image of the pair (id, x) by the map

$$\begin{aligned}
&(\mathcal{C}^{\otimes} \rtimes \mathcal{X})(X_1 \dots X_n; X_{x^{-1}(1)} \otimes \cdots \otimes X_{x^{-1}(n)}) \\
&\xrightarrow{\mathtt{sym}} \mathcal{C}^{\otimes}(X_1 \dots X_n; X_{x^{-1}(1)} \otimes \cdots \otimes X_{x^{-1}(n)}) \\
&= \mathcal{C}(X_1 \otimes \cdots \otimes X_n; X_{x^{-1}(1)} \otimes \cdots \otimes X_{x^{-1}(n)}) \ .
\end{aligned}$$

It is verified that the family $\Theta^x := \{\Theta^x_{X_1 \dots X_n}\}_{X_1, \dots, X_n}$ forms a natural transformation such that $\Theta^{e_n} = \mathrm{id}$ and $\Theta^x \Theta^y = \Theta^{xy}$. If $\mathcal{X} = \mathfrak{S}$ (resp. $\mathcal{B}$), this natural transformation Θ^x is nothing but the appropriate composition of the *braidings* in the symmetric (resp. braided) structure.

We also consider multifunctors respecting symmetries.

Definition. Let $\mathcal{X}$ be a monoid operad, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{X}$-symmetric multicategory. Then, a multifunctor $F : \mathcal{M} \to \mathcal{N}$ is said to be *$\mathcal{X}$-symmetric* if it is a morphism of algebras over the 2-monad $(-) \rtimes \mathcal{X}$; i.e. the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{M} \rtimes \mathcal{X} & \xrightarrow{F \rtimes \mathrm{Id}} & \mathcal{N} \rtimes \mathcal{X} \\
{\scriptstyle \mathtt{sym}} \downarrow & & \downarrow {\scriptstyle \mathtt{sym}} \\
\mathcal{M} & \xrightarrow{F} & \mathcal{N}
\end{array} \ .$$

We denote by $\mathbf{MultCat}_{\mathcal{X}}$ the 2-category of $\mathcal{X}$-symmetric multicategories, $\mathcal{X}$-symmetric multifunctors, and transformations of morphisms. We also define a 2-category $\mathbf{MonCat}_{\mathcal{X}}$ to be the pullback

$$\mathbf{MonCat}_{\mathcal{X}} := \mathbf{MonCat} \times_{\mathbf{MultCat}} \mathbf{MultCat}_{\mathcal{X}}$$

and call its morphisms *$\mathcal{G}$-symmetric monoidal functors*. Note that we do not provide any special terminologies for 2-morphisms in $\mathbf{MultCat}_{\mathcal{X}}$ or in $\mathbf{MonCat}_{\mathcal{X}}$ because of the following result.

Lemma 3.2.4. *Let $\mathcal{X}$ be a monoid operad. Then, the forgetful 2-functor*

$$\mathbf{MultCat}_{\mathcal{X}} \to \mathbf{MultCat}$$

is locally fully faithful.

Proof. Take two $\mathcal{X}$-symmetric multicategory $\mathcal{M}$ and $\mathcal{N}$. Note that the category $\mathbf{MultCat}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})$ is obtained as the equalizer of the parallel functors

$$\mathbf{MultCat}(\mathcal{M}, \mathcal{N}) \underset{\mathtt{sym}^*}{\overset{\mathtt{sym}_* \circ (-) \rtimes \mathcal{X}}{\rightrightarrows}} \mathbf{MultCat}(\mathcal{M} \rtimes \mathcal{X}, \mathcal{N}) \quad .$$

Thanks to (3.2.2) and (3.2.3), both functors are identities on morphisms, so we get the result. ☐

Remark 3.2.5. We are mainly interested in the case $\mathcal{X}$ is a group operad. In this case, it was proved in [15] that $\mathbf{MultCat}_{\mathcal{X}}$ is biequivalent to the 2-category of *pseudo-algebras* over $(-) \rtimes \mathcal{X}$.

3.3 The embedding of pointed operads

In Section 3.1, we obtained two alternative definitions of group operads; one is the direct definition, and the other is due to Proposition 3.1.8. Comparing them with Proposition 2.4.13, the reader may have a feeling that they can be translated to one another. The goal of this section is to make it clearer and to establish a fully faithful embedding $\mathbf{GrpOp} \hookrightarrow \mathbf{CrsGrp}_{\nabla}$.

We denote by $\mathbf{Op}^{*/}$ the category of pointed operads; i.e. the coslice category, or the under category, on the trivial operad $*$. Since the set $*(n)$ is a singleton for each $n \in \mathbb{N}$, giving a map $* \to \mathcal{X}$ of operads is equivalent to giving a family $\{e_n\}_n$ of elements $e_n \in \mathcal{X}(n)$ satisfying

$$\gamma(e_n; e_{k_1}, \dots, e_{k_n}) = e_{k_1+\dots+k_n} \ . \tag{3.3.1}$$

We first define a functor $\Psi : \mathbf{Op}^{*/} \to \mathbf{Set}_{\nabla}$ as follows: recall that morphisms $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$ correspond in one-to-one to $(n+2)$-tuples $\vec{k} = (k_{-\infty}, k_1, \dots, k_n, k_\infty)$ of non-negative integers with $k_{-\infty}+k_1+\dots+k_n+k_\infty = m$ via the assignment $\varphi \mapsto \vec{k}^{(\varphi)}$ given by (1.2.2). To simplify the notation, for a pointed operad $\mathcal{X}$ with base points $e_n \in \mathcal{X}(n)$, we write $e^{(\varphi)}_j := e_{k^{(\varphi)}_j}$ for each $j \in \langle\!\langle n \rangle\!\rangle$. In this case, we define an interval set $\Psi(\mathcal{X})$ by

- for each $n \in \mathbb{N}$, $\Psi(\mathcal{X})_n := \mathcal{X}(n)$;
- for a morphism $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$, we set

$$\varphi^* : \Psi(\mathcal{X})_n \to \Psi(\mathcal{X})_m \ ; \quad x \mapsto \gamma(e_3; e^{(\varphi)}_{-\infty}, \gamma(x; e^{(\varphi)}_1, \dots, e^{(\varphi)}_n), e^{(\varphi)}_\infty). \tag{3.3.2}$$

Note that, by virtue of the equation (3.3.1), for morphisms $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle$ and $\psi : \langle\!\langle l \rangle\!\rangle \to \langle\!\langle m \rangle\!\rangle$ in ∇, if $\varphi^{-1}\{j\} \setminus \{\pm\infty\} = \{i_1 < \dots < i_{k^{(\varphi)}_j}\} \subset \langle\!\langle m \rangle\!\rangle$, we have

$$e^{(\varphi\psi)}_j = \begin{cases} \gamma(e_2; e^{(\psi)}_{-\infty}, \gamma(e^{(\varphi)}_{-\infty}; e^{(\psi)}_{i_1}, \dots, e^{(\psi)}_{i_{k^{(\varphi)}_j}})) & j = -\infty \ , \\ \gamma(e^{(\varphi)}_j; e^{(\psi)}_{i_1}, \dots, e^{(\psi)}_{i_{k^{(\varphi)}_j}}) & 1 \le j \le n \ , \\ \gamma(e_2; \gamma(e^{(\varphi)}_\infty; e^{(\psi)}_{i_1}, \dots, e^{(\psi)}_{i_{k^{(\varphi)}_j}}), e^{(\psi)}_\infty) & j = \infty \ . \end{cases}$$

This and the associativity of the compositions in operads imply $\psi^*\varphi^* = (\varphi\psi)^*$ so that $\Psi(\mathcal{X})$ is in fact an interval set. On the other hand, if $f : \mathcal{X} \to \mathcal{Y}$ is a map of pointed operads, so we have $f(e_n) = e_n$, then the maps

$$\Psi(f) : \Psi(\mathcal{X})_n \to \Psi(\mathcal{Y})_n \ ; \quad x \mapsto f(x)$$

clearly define a map $\Psi(f) : \Psi(\mathcal{X}) \to \Psi(\mathcal{Y})$ of interval sets. The functoriality is obvious so that we obtain a functor $\Psi : \mathbf{Op}^{*/} \to \mathbf{Set}_{\nabla}$.

Example 3.3.1. Note that every group operad is by definition pointed. In particular, the operad $\mathfrak{S}$ canonically admits the map $* \to \mathfrak{S}$ corresponding to the unit of each $\mathfrak{S}(n)$. The interval set $\Psi(\mathfrak{S})$ is isomorphic to the interval set $\mathfrak{S}$ given in Example 2.1.8..

Thinking of $\mathfrak{S}$ as a pointed object in the category **Op**, we can take the slice category $(\mathbf{Op}^{*/})^{/\mathfrak{S}}$ on $\mathfrak{S}$. Then, one can observe that the monoidal structure $\rtimes$ on $\mathbf{Op}^{/\mathfrak{S}}$ lifts to $(\mathbf{Op}^{*/})^{/\mathfrak{S}}$. Indeed, notice that there is an isomorphism $(\mathbf{Op}^{*/})^{/\mathfrak{S}} \cong (\mathbf{Op}^{/\mathfrak{S}})^{*/}$ so the monoidal structure $\rtimes$ induces a functor

$$\begin{aligned}(\mathbf{Op}^{*/})^{/\mathfrak{S}} \times (\mathbf{Op}^{*/})^{/\mathfrak{S}} &\cong (\mathbf{Op}^{/\mathfrak{S}})^{*/} \times (\mathbf{Op}^{/\mathfrak{S}})^{*/} \\ &\xrightarrow{\rtimes} (\mathbf{Op}^{/\mathfrak{S}})^{(*\rtimes *)/} \cong (\mathbf{Op}^{/\mathfrak{S}})^{*/} \cong (\mathbf{Op}^{*/})^{/\mathfrak{S}} \ .\end{aligned}$$

It is easily verified this defines a monoidal structure on $(\mathbf{Op}^{*/})^{\mathfrak{S}/}$ so that the functor $(\mathbf{Op}^{*/})^{/\mathfrak{S}} \to \mathbf{Op}^{/\mathfrak{S}}$ is strictly monoidal.

Lemma 3.3.2. *The functor*

$$\Psi^{\mathfrak{S}} : (\mathbf{Op}^{*/})^{/\mathfrak{S}} \to \mathbf{Set}_{\nabla}^{/\Psi(\mathfrak{S})} = \mathbf{Set}_{\nabla}^{/\mathfrak{S}}$$

induced by the functor Ψ defined above is strictly monoidal.

Proof. It is obvious that the functor $\Psi^{\mathfrak{S}}$ preserves the unit objects, namely $\Psi^{\mathfrak{S}}(*) = *$ with regard to the maps into $\Psi(\mathfrak{S}) = \mathfrak{S}$. We have to show the equation $\Psi^{\mathfrak{S}}(\mathcal{X} \rtimes \mathcal{Y}) = \Psi^{\mathfrak{S}}(\mathcal{X}) \rtimes \Psi^{\mathfrak{S}}(\mathcal{Y})$ for every pointed operads $\mathcal{X}$ and $\mathcal{Y}$ over $\mathfrak{S}$. It clearly holds degreewisely, so it suffices to verify the structures of interval sets agree with each other. Let $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$ be a morphism. Say $\rho : \mathcal{Y} \to \mathfrak{S}$ is the structure map, then in the interval set $\Psi^{\mathfrak{S}}(\mathcal{X} \rtimes \mathcal{Y})$, the induced map $\varphi^* : \Psi^{\mathfrak{S}}(\mathcal{X} \rtimes \mathcal{Y})_n \to \Psi^{\mathfrak{S}}(\mathcal{X} \rtimes \mathcal{Y})_m$ is given by

$$\begin{aligned}\varphi^*(x,y) &= \gamma_{\mathcal{X}\rtimes\mathcal{Y}}(e_3; e^{(\varphi)}_{-\infty}, \gamma_{\mathcal{X}\rtimes\mathcal{Y}}((x,y); e^{(\varphi)}_1, \dots, e^{(\varphi)}_n), e^{(\varphi)}_{\infty}) \\ &= (\gamma_{\mathcal{X}}(e_3; e^{(\varphi)}_{-\infty}, \gamma_{\mathcal{X}}(x; e^{(\varphi)}_{\rho(y)^{-1}(1)}, \dots, e^{(\varphi)}_{\rho(y)^{-1}(n)}), e^{(\varphi)}_{\infty}), \\ &\qquad\qquad \gamma_{\mathcal{Y}}(e_3; e^{(\varphi)}_{-\infty}, \gamma_{\mathcal{Y}}(y; e^{(\varphi)}_1, \dots, e^{(\varphi)}_n), e^{(\varphi)}_{\infty}))\end{aligned} \tag{3.3.3}$$

Note that for each $\sigma \in \mathfrak{S}_n$, we have

$$e^{(\varphi^\sigma)}_{\pm\infty} = e^{(\varphi)}_{\pm\infty} \ , \quad e^{(\varphi^\sigma)}_j = e^{(\varphi)}_{\sigma^{-1}(j)} \text{ for } 1 \le j \le n \ .$$

Hence, the right hand side of (3.3.3) can be written as $((\varphi^{\rho(y)})^*(x), \varphi^*(y))$, which is exactly the image of the pair (x,y) under the map $\varphi^* : (\Psi^{\mathfrak{S}}(\mathcal{X}) \rtimes \Psi^{\mathfrak{S}}(\mathcal{Y}))_n \to (\Psi^{\mathfrak{S}}(\mathcal{X}) \rtimes \Psi^{\mathfrak{S}}(\mathcal{Y}))_m$. It follows that $\Psi^{\mathfrak{S}}(\mathcal{X} \rtimes \mathcal{Y})$ is identical to $\Psi^{\mathfrak{S}}(\mathcal{X}) \rtimes \Psi^{\mathfrak{S}}(\mathcal{Y})$ as interval sets. The structure maps into $\mathfrak{S}$ obviously coincide, so we obtain the result. □

Theorem 3.3.3. *The functor $\Psi^{\mathfrak{S}}$ induces a fully faithful functor*

$$\widehat{\Psi} : \mathbf{MonOp} \to \mathbf{CrsMon}_{\nabla}^{/\mathfrak{S}} .$$

Moreover, it restricts to a right adjoint functor $\mathbf{GrpOp} \to \mathbf{CrsGrp}_{\nabla}^{/\mathfrak{S}}$.

Proof. Since the functor $\Psi^{\mathfrak{S}}$ is strictly monoidal as proved in Lemma 3.3.2, it induces a functor $\widehat{\Psi}$ between the categories of monoid objects. More precisely, it sends a monoid operad $\mathcal{X} = (\mathcal{X}, \mathtt{mul}, e)$ to the interval set $\Psi^{\mathfrak{S}}(\mathcal{X})$ over $\mathfrak{S}$ together with the structure maps

$$\begin{aligned}\mathtt{mul} : \Psi^{\mathfrak{S}}(\mathcal{X}) \rtimes \Psi^{\mathfrak{S}}(\mathcal{X}) = \Psi^{\mathfrak{S}}(\mathcal{X} \rtimes \mathcal{X}) &\xrightarrow{\Psi(\mathtt{mul})} \Psi^{\mathfrak{S}}(\mathcal{X}) \\ e : * = \Psi^{\mathfrak{S}}(*) &\xrightarrow{\Psi(e)} \Psi^{\mathfrak{S}}(\mathcal{X}) \ .\end{aligned}$$

Hence, the monoid structure on each $\widehat{\Psi}(\mathcal{X})_n$ coincides with that on $\mathcal{X}(n)$. Clearly $\widehat{\Psi}$ is faithful, so we show it is also full.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two monoid operads, and suppose $f : \widehat{\Psi}(\mathcal{X}) \to \Psi(\mathcal{Y})$ be a map of crossed interval monoids. To see f comes from a map of monoid operads, it is enough to show that the maps

$$f : \mathcal{X}(n) = \Psi^{\mathfrak{S}}(\mathcal{X})_n \xrightarrow{f} \Psi^{\mathfrak{S}}(\mathcal{Y})_n = \mathcal{Y}(n)$$

form a map of operads. Since it preserves the unit elements, we have $f(e_n) = e_n$; in particular, f preserves the identities $\mathrm{id} = e_1$ of operads. If $x \in \mathcal{X}(n) = \Psi^{\mathfrak{S}}(\mathcal{X})_n$ and $x_i \in \mathcal{X}(k_i)$ for $1 \le i \le n$, the definition of monoid operads implies

$$\begin{aligned} \gamma_{\mathcal{X}}(x; x_1, \dots, x_n) = \gamma_{\mathcal{X}}&(x; e_{k_1}, \dots, e_{k_n}) \\ &\cdot \gamma_{\mathcal{X}}(e_3; e_0, x_1, e_{k_2+\dots+k_n}) \\ &\cdot \gamma_{\mathcal{X}}(e_3; e_{k_1}, x_2, e_{k_3+\dots+k_n}) \\ &\quad\vdots \\ &\cdot \gamma_{\mathcal{X}}(e_3; e_{k_1+\dots+k_{n-1}}, x_n, e_0)\ . \end{aligned} \tag{3.3.4}$$

Taking the unique morphism $\mu : \langle\langle k_1 + \dots + k_n \rangle\rangle \to \langle\langle n \rangle\rangle$ with $k^{(\mu)}_{\pm\infty} = 0$ and $k^{(\mu)}_j = k_j$ for $j \in \langle n \rangle$ and the morphism $\rho_j : \langle\langle k_1 + \dots + k_n \rangle\rangle \to \langle\langle k_j \rangle\rangle$ defined by

$$\rho_j(i) := \begin{cases} -\infty & i \le k_1 + \dots + k_{j-1}\ , \\ i - (k_1 + \dots + k_{j-1}) & k_1 + \dots + k_{j-1} + 1 \le i \le k_1 + \dots + k_j\ , \\ \infty & i \ge k_1 + \dots + k_j + 1\ , \end{cases}$$

then, by the definition (3.3.2) of the functor Ψ, we can rewrite the formula (3.3.4) as follows:

$$\gamma_{\mathcal{X}}(x; x_1, \dots, x_n) = \mu^*(x)\rho_1^*(x_1)\rho_2^*(x_2)\dots\rho_n^*(x_n)\ .$$

The same formula also holds in $\mathcal{Y}$, and, since f is a map of crossed interval monoids, we obtain

$$\begin{aligned} f(\gamma_{\mathcal{X}}(x; x_1, \dots, x_n)) &= f(\mu^*(x)\rho_1^*(x_1)\dots\rho_n^*(x_n)) \\ &= f(\mu^*(x))f(\rho_1^*(x_1))\dots f(\rho_n^*(x_n)) \\ &= \mu^*(f(x))\rho_1^*(f(x_1))\dots\rho_n^*(f(x_n)) \\ &= \gamma_{\mathcal{Y}}(f(x); f(x_1), \dots, f(x_n))\ . \end{aligned}$$

This implies that the maps $f : \mathcal{X}(n) \to \mathcal{Y}(n)$ form a map of monoid operads.

As for the last statement, it follows from Proposition 3.1.8, Proposition 2.4.13, Theorem 3.1.5, Theorem 2.2.4, Theorem 2.3.13, Example 2.3.17, and General Adjoint Functor Theorem (e.g. see Theorem 1.66 in [1]). □

Note that, the inclusion $\mathfrak{S} \hookrightarrow \mathfrak{W}^{\nabla}$ of crossed interval groups induces a fully faithful functor $\mathbf{CrsGrp}_{\nabla}^{/\mathfrak{S}} \hookrightarrow \mathbf{CrsGrp}_{\nabla}$. Combining it with Theorem 3.3.3, one obtains an embedding $\mathbf{GrpOp} \to \mathbf{CrsGrp}_{\nabla}$. Hence, in the rest of the paper, we identify group operads with their images under this functor. In particular, under this convention, we have $\mathfrak{S} = \widehat{\Psi}(\mathfrak{S})$, which explains the coincidence of the notations.

3.4 Operadic interval groups

In this section, we aim to determine the essential image of the embedding $\mathbf{GrpOp} \hookrightarrow \mathbf{CrsGrp}_\nabla$. Since it is fully faithful, it will provide an alternative definition of group operads. Furthermore, we will see that there is a larger class of crossed interval groups which are associated to operads. This result suggests an extension of the notion of group operads.

First of all, we need to know about the category ∇.

Definition. Let $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle$ be a morphism in ∇.

(1) φ is said to be *active* if we have $\varphi : \varphi^{-1}\{\pm\infty\} \to \{\pm\infty\}$.

(2) φ is said to be *inert* if the restriction $\varphi : \varphi^{-1}\{1,\dots,n\} \to \{1,\dots,n\}$ is bijective.

Remark 3.4.1. In the paper [54], Lurie considered the notions above for morphisms in the category $\mathbf{Fin}_*$ of pointed finite sets. Actually, we have a functor

$$\nabla \to \mathbf{Fin}_* \ ; \quad \langle\langle n\rangle\rangle \mapsto \langle\langle n\rangle\rangle/\{\pm\infty\} \ .$$

A morphism $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$ is active (resp. inert) if and only if so is its image in $\mathbf{Fin}_*$.

The following results are easy to verify.

Lemma 3.4.2. (1) *Every morphism $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$ uniquely factors as $\varphi = \mu\rho$ with ρ inert and μ active.*

(2) *Every inert morphism admits a unique section in ∇.*

In the following arguments, inert morphisms play distinguished roles. One reason is the definition of the functor $\widehat{\Psi}$; if $\mathcal{G}$ is a group operad, then the group $\mathcal{G}(k)$ admits embeddings into $\mathcal{G}(n)$ provided $k \le n$, namely the group homomorphisms of the form

$$\mathcal{G}(k) \to \mathcal{G}(n) \ ; \quad x \mapsto \gamma(e_3; e_p, x, e_q)$$

with $n = k+p+q$. In terms of the category ∇, they are realized as the maps induced by inert morphisms $\rho : \langle\langle n\rangle\rangle \to \langle\langle k\rangle\rangle$. For example if $\rho : \langle\langle n\rangle\rangle \to \langle\langle k\rangle\rangle$ is an inert morphism with the unique section $\delta : \langle\langle k\rangle\rangle \to \langle\langle n\rangle\rangle$, then the map $\rho^* : \mathfrak{S}(k) \to \mathfrak{S}(n)$ exhibits each permutation σ on $\langle k\rangle$ as that on $\{\delta(1),\dots,\delta(k)\} \subset \langle n\rangle$. This *embedding* is one of the characteristic properties of group operads among general crossed interval groups.

Lemma 3.4.3. *For a crossed interval group G, the following are equivalent:*

(a) *the unique map $G \to \mathfrak{W}^\nabla$ of crossed interval groups factors through the hyperoctahedral crossed interval group $\mathfrak{H} \subset \mathfrak{W}^\nabla$.*

(b) *for every inert morphism $\rho : \langle\langle n\rangle\rangle \to \langle\langle k\rangle\rangle$, and for every $x \in G_k$, we have $\rho^x = \rho$.*

In the case the conditions above are satisfied, each inert morphism $\rho : \langle\langle n\rangle\rangle \to \langle\langle k\rangle\rangle \in \nabla$ induces an injective group homomorphism $\rho^* : G_k \to G_l$.

Proof. Note that, for each $n \in \mathbb{N}$, the subgroup $\mathfrak{H}_n \subset (\mathfrak{W}^\nabla)_n$ consists of elements whose actions on inert morphisms are trivial. Since the map $G \to \mathfrak{W}^\nabla$ respects the action on morphisms, this implies the conditions (a) and (b) are equivalent.

The last statement is directly follows from the definition of crossed groups (e.g. see Lemma 2.1.1). □

Remark 3.4.4. The last statement in Lemma 3.4.3 is not equivalent to the two conditions. For example, there is a crossed interval group $\mathfrak{Refl}$ which is the constant presheaf at $\mathbb{Z}/2\mathbb{Z}$ and the action on hom-sets "reverses the order." This has non-trivial actions on inert morphisms while every morphism $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$ induces a group homomorphism $\varphi^* : \mathfrak{Refl}_n \to \mathfrak{Refl}_m$.

The next property we discuss is the commutativity of elements with "distinct supports." Suppose $\mathcal{G}$ is a group operad, and consider two embeddings

$$\begin{aligned} \eta &: \mathcal{G}(k) \to \mathcal{G}(k+l) \ ; \quad x \mapsto \gamma(e_2; x, e_l) \ , \\ \eta' &: \mathcal{G}(l) \to \mathcal{G}(k+l) \ ; \quad y \mapsto \gamma(e_2; e_k, y) \ . \end{aligned}$$

Then, for each $x \in \mathcal{G}(k)$ and $y \in \mathcal{G}(l)$, we have $\eta(x)\eta'(y) = \eta'(y)\eta(x)$ in $G(k+l)$. In other words, they induces an injective group homomorphism $\mathcal{G}(k) \times \mathcal{G}(l) \to \mathcal{G}(k+l)$. To formulate this phenomenon in terms of the category ∇, we introduce the following notion.

Definition. We say two morphisms $\varphi_1 : \langle\langle n\rangle\rangle \to \langle\langle k_1\rangle\rangle$ and $\varphi_2 : \langle\langle n\rangle\rangle \to \langle\langle k_2\rangle\rangle$ in ∇ with the same domain are *dissociated* if, for each morphism $\alpha : \langle\langle 1\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, either of the compositions $\varphi_1\alpha$ or $\varphi_2\alpha$ factors through the object $\langle\langle 0\rangle\rangle \in \nabla$.

Note that, in view of the identification $\nabla(\langle\langle 1\rangle\rangle, \langle\langle n\rangle\rangle) \cong \langle\langle n\rangle\rangle$, two morphisms $\varphi_1 : \langle\langle n\rangle\rangle \to \langle\langle k_1\rangle\rangle$ and $\varphi_2 : \langle\langle n\rangle\rangle \to \langle\langle k_2\rangle\rangle$ in ∇ are dissociated if and only if, for each $i \in \langle\langle n\rangle\rangle$, either $\varphi_1(i)$ or $\varphi_2(i)$ is $\pm\infty$. This observation leads to the result below.

Lemma 3.4.5. *Let $\varphi_i : \langle\langle n\rangle\rangle \to \langle\langle k_i\rangle\rangle$ $(i = 1, 2)$ be morphisms in ∇.*

1. *If $\varphi_1 = \mu_1\rho_1$ and $\varphi_2 = \mu_2\rho_2$ are the unique factorizations into inert morphisms followed by active morphisms, then φ_1 and φ_2 are dissociated if and only if ρ_1 and ρ_2 are so.*
2. *For a morphism $\psi : \langle\langle m\rangle\rangle \to \langle\langle \to \langle\langle n\rangle\rangle \in \nabla$, the compositions $\varphi_1\psi$ and $\varphi_2\psi$ are dissociated provided so are φ_1 and φ_2.*

Lemma 3.4.6. *Let G be a crossed interval group satisfying the conditions in Lemma 3.4.3, and suppose $\varphi_1 : \langle\langle n\rangle\rangle \to \langle\langle k_1\rangle\rangle$ and $\varphi_2 : \langle\langle n\rangle\rangle \to \langle\langle k_2\rangle\rangle$ are dissociated morphisms of ∇. Then, for each element $x \in G(k_1)$ and every morphism $\psi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, we have*

$$\varphi_2\psi^{\varphi_1^*(x)} = \varphi_2\psi \ .$$

Proof. Since the unique map $G \to \mathfrak{W}^\nabla$ factors through $G \to \mathfrak{H}$, and since it respects the actions on the morphisms of ∇, it will suffice to verify the statement only in the case $G = \mathfrak{H}$. Note that, for a morphism $\psi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, and for an element $(\sigma; \vec{\varepsilon}) \in \mathfrak{H}_n$, the compositions $\varphi_2\psi^{(\sigma;\vec{\varepsilon})}$ is, as a map, the composition

$$\langle\langle m\rangle\rangle \xrightarrow{\psi^*(\sigma)^{-1}} \langle\langle m\rangle\rangle \xrightarrow{\psi} \langle\langle n\rangle\rangle \xrightarrow{\sigma} \langle\langle n\rangle\rangle \xrightarrow{\varphi_2} \langle\langle k_2\rangle\rangle \ .$$

If $(\sigma;\vec{\varepsilon})$ belongs to the image of the map $\varphi_1^* : \mathfrak{H}_{k_1} \to \mathfrak{H}_n$, the dissociativity of φ_1 and φ_2 implies the permutation σ is the identity except on exactly one of $\varphi_2^{-1}\{-\infty\}$ or $\varphi_2^{-1}\{\infty\}$. Hence, we have $\varphi_2 \circ \sigma = \varphi_2$ as maps. In the same reason, by virtue of Lemma 3.4.5, we obtain $\varphi_2 \circ \psi \circ \psi^*(\sigma)^{-1} = \varphi_2 \circ \psi$. This completes the proof. □

Now, we formulate the "operad-like" crossed interval groups as below.

Definition. A crossed interval group G is said to be *operadic* if it satisfies the following:

(i) G satisfies the equivalent conditions in Lemma 3.4.3;

(ii) if $\rho_1 : \langle\langle n\rangle\rangle \to \langle\langle k_1\rangle\rangle$ and $\rho_2 : \langle\langle n\rangle\rangle \to \langle\langle k_2\rangle\rangle$ are dissociated inert morphisms in ∇, then elements of the images $\rho_1^*(G_{k_1})$ commute with those of $\rho_2^*(G_{k_2})$; in other words, the commutator $[\rho_1^*(G_{k_1}), \rho_2^*(G_{k_2})] \subset G_n$ is trivial.

Example 3.4.7. As expected, for every group operad $\mathcal{G}$, the crossed interval group $\widehat{\Psi}(\mathcal{G})$ is operadic. To see this, notice that if $\rho_1 : \langle\langle n\rangle\rangle \to \langle\langle k_1\rangle\rangle$ and $\rho_2 : \langle\langle n\rangle\rangle \to \langle\langle k_2\rangle\rangle$ are dissociated inert morphisms, then there are integers $k_{\pm\infty}$ and l such that, for each $x_i \in \mathcal{G}(k_i)$,

$$\rho_{i_1}^*(x_{i_1}) = \gamma(e_5; e_{-\infty}, x_{i_1}, e_l, e_{k_{i_2}}, e_\infty)\ , \quad \rho_{i_2}^*(x_{i_2}) = \gamma(e_5; e_{-\infty}, e_{i_2}, e_l, x_{i_2}, e_\infty)$$

for $\{i_1, i_2\} = \{1, 2\}$. Hence, the condition on group operads implies these elements commute with each other. In view of Lemma 3.4.5, it follows $\widehat{\Psi}(\mathcal{G})$ is operadic.

Example 3.4.8. The crossed interval group $\mathfrak{H}$ is operadic. This follows from the direct computation and that $\mathfrak{S}$ is operadic.

Operadic crossed interval groups are actually associated with operads. To see it, we introduce some notions for simplicity; for a sequence $\vec{k} = (k_1, \dots, k_n)$ of non-negative integers, we put

- $\mu_{\vec{k}} : \langle\langle k_1 + \dots + k_n\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$ to be the unique active morphism with $\#\mu_{\vec{k}}^{-1}\{j\} = k_j$ for each $1 \le j \le n$; and
- $\rho_j^{(\vec{k})} : \langle\langle k_1 + \dots + k_n\rangle\rangle \to \langle\langle k_j\rangle\rangle \in \nabla$ to be the inert morphism given by

$$\begin{aligned}
&\langle\langle k_1 + \dots + k_n\rangle\rangle \\
&\cong \{-\infty, 1, \dots, k_1 + \dots + k_{j-1}\} \star \langle k_j\rangle \\
&\qquad \star \{k_1 + \dots + k_j + 1, \dots, k_1 + \dots + k_n, \infty\} \\
&\xrightarrow{\texttt{const}\star\texttt{id}\star\texttt{const}} \{-\infty\} \star \langle k_j\rangle \star \{\infty\} \\
&\cong \langle\langle k_j\rangle\rangle\ .
\end{aligned}$$

For an operadic crossed interval group G, we define an operad $\mathcal{O}_G$ as follows:

- for each $n \in \mathbb{N}$, $\mathcal{O}_G(n) := G_n$;
- the composition

$$\gamma : \mathcal{O}_G(n) \times \mathcal{O}_G(k_1) \times \dots \times \mathcal{O}_G(k_n) \to \mathcal{O}_G(k_1 + \dots + k_n)$$

is given by

$$\gamma(x;x_1,\dots,x_n) := \mu_{\vec{k}}^*(x)\cdot(\rho_1^{(\vec{k})})^*(x_1)\dots(\rho_n^{(\vec{k})})^*(x_n)\ ,$$

where $\vec{k}=(k_1,\dots,k_n)$.

The associativity is seen as follows: let $x\in G_n$, $x_i\in G_{k_i}$, and $x_s^{(i)}\in G_{k_s^{(i)}}$ for $1\le i\le n$ and $1\le s\le k_i$, and put $\vec{l}=(\sum_s k_s^{(1)},\dots,\sum_s k_s^{(n)})$. Then, we have

$$\begin{aligned}&\gamma\big(x;\gamma(x_1;x_1^{(1)},\dots,x_{k_1}^{(1)}),\dots,\gamma(x_n;x_1^{(n)},\dots,x_{k_n}^{(1)})\big)\\ &=\mu_{\vec{l}}^*(x)\cdot(\rho_1^{(\vec{l})})^*\big(\mu_{\vec{k}^{(1)}}^*(x_1)\cdot(\rho_1^{(\vec{k}^{(1)})})^*(x_1^{(1)})\dots(\rho_{k_1}^{(\vec{k}^{(1)})})^*(x_{k_1}^{(1)})\big)\\ &\qquad\dots(\rho_n^{(\vec{l})})^*\big(\mu_{\vec{k}^{(n)}}^*(x_n)\cdot(\rho_1^{(\vec{k}^{(n)})})^*(x_1^{(n)})\dots(\rho_{k_n}^{(\vec{k}^{(n)})})^*(x_{k_n}^{(n)})\big)\ .\end{aligned}\tag{3.4.1}$$

Since G is operadic, each $(\rho_i^{(\vec{l})})^*$ is a group homomorphism. In addition, $\rho_i^{(\vec{l})}$ and $\rho_j^{(\vec{l})}$ are dissociated provided $i\neq j$, so by virtue of Lemma 3.4.6, the right hand side of (3.4.1) is written as

$$\begin{aligned}&\mu_{\vec{l}}^*(x)(\mu_{\vec{k}^{(1)}}\rho_1^{(\vec{l})})^*(x_1)\dots(\mu_{\vec{k}^{(n)}}\rho_n^{(\vec{l})})^*(x_n)\\ &\quad\cdot(\rho_1^{(\vec{k}^{(1)})}\rho_1^{(\vec{l})})^*(x_1^{(1)})\dots(\rho_{k_1}^{(\vec{k}^{(1)})}\rho_1^{(\vec{l})})^*(x_{k_1}^{(1)})\dots(\rho_{k_n}^{(\vec{k}^{(n)})}\rho_n^{(\vec{l})})^*(x_{k_n}^{(n)})\ .\end{aligned}\tag{3.4.2}$$

Finally, using the formulas

$$\begin{aligned}\mu_{\vec{l}}&=\mu_{\vec{k}}\mu_{\vec{k}^{(1)}\dots\vec{k}^{(n)}}\\ \mu_{\vec{k}^{(i)}}\rho_i^{(\vec{l})}&=\rho_i^{(\vec{k})}\mu_{\vec{k}^{(1)}\dots\vec{k}^{(n)}}\\ \rho_s^{(\vec{k}^{(i)})}\rho_i^{(\vec{l})}&=\rho_{k_1+\dots+k_{i-1}+s}^{(\vec{k}^{(1)}\dots\vec{k}^{(n)})}\end{aligned}$$

with $\vec{k}^{(1)}\dots\vec{k}^{(n)}=(k_1^{(1)},\dots,k_{k_1}^{(1)},\dots,k_{k_n}^{(n)})$, (3.4.2) is furthermore equal to

$$\begin{aligned}&\mu_{\vec{k}^{(1)}\dots\vec{k}^{(n)}}^*\big(\mu_{\vec{k}}^*(x)(\rho_1^{(\vec{k})})^*(x_1)\dots(\rho_n^{(\vec{k})})^*(x_n)\big)\\ &\quad\cdot(\rho_1^{(\vec{k}^{(1)}\dots\vec{k}^{(n)})})^*(x_1^{(1)})\dots(\rho_{k_1}^{(\vec{k}^{(1)}\dots\vec{k}^{(n)})})^*(x_{k_1}^{(1)})\dots(\rho_{k_1+\dots+k_n}^{(\vec{k}^{(1)}\dots\vec{k}^{(n)})})^*(x_{k_n}^{(n)})\\ &=\gamma(\gamma(x;x_1,\dots,x_n);x_1^{(1)},\dots,x_{k_1}^{(1)},\dots,x_{k_n}^{(n)})\ .\end{aligned}$$

Clearly, the unit $e_1\in G_1=\mathcal{O}_G(1)$ behaves as the identity, so $\mathcal{O}_G$ is in fact an operad.

Example 3.4.9. If $\mathcal{G}$ is a group operad, we have a strict identification

$$\mathcal{O}_{\widehat{\Psi}(\mathcal{G})}=\mathcal{G}\ .$$

In particular, $\mathcal{O}_{\mathfrak{S}}=\mathfrak{S}$.

We denote by $\mathbf{CrsOpGrp}\subset\mathbf{CrsGrp}_{\nabla}^{/\mathfrak{H}}$ the full subcategory spanned by operadic crossed interval groups. On the other hand, we define a category $\mathbf{Op}_{\mathrm{gr}}$ whose objects are operads $\mathcal{O}$ equipped with a group structure on each $\mathcal{O}(n)$ and whose morphisms are maps of operads which are group homomorphisms levelwisely. Then, the assignment $G \mapsto \mathcal{O}_G$ clearly extends to a functor $\mathcal{O}_{(-)} : \mathbf{CrsOpGrp} \to \mathbf{Op}_{\mathrm{gr}}$. By abuse of notation, we write $\mathfrak{H} = \mathcal{O}_{\mathfrak{H}}$, so we have the induced functor

$$\mathcal{O}^{\mathfrak{H}}_{(-)} : \mathbf{CrsOpGrp} \to \mathbf{Op}^{/\mathfrak{H}}_{\mathrm{gr}} \ .$$

Theorem 3.4.10. *The functor $\mathcal{O}^{\mathfrak{H}}_{(-)}$ is a fully faithful functor.*

Proof. To obtain the result, it suffices to show that, for operadic crossed interval groups G and H, a family $\{f : G_n \to H_n\}_{n\in\mathbb{N}}$ of group homomorphisms forms a map of crossed interval group if and only if it forms a map of operads. This is verified almost identically to the first part of Theorem 3.3.3. ☐

Theorem 3.4.11. *The subcategory $\mathbf{CrsOpGrp} \subset \mathbf{CrsGrp}^{/\mathfrak{H}}_{\nabla}$ is reflective; i.e. the inclusion functor admits a left adjoint.*

Proof. For a crossed interval group G over $\mathfrak{H}$, define $K(G)_n \subset G_n$ to be the subgroup generated by the commutators

$$\begin{aligned}[\rho_1^*(x_1), \rho_2^*(x_2)] &= \rho_1^*(x_1)\rho_2^*(x_2)\rho_1^*(x_1)^{-1}\rho_2^*(x_2)^{-1} \\ &= \rho_1^*(x_1)\rho_2^*(x_2)\rho_1^*(x_1^{-1})\rho_2^*(x_2^{-1})\end{aligned}$$

for dissociated inert morphisms $\rho_i : \langle\langle n\rangle\rangle \to \langle\langle k_i\rangle\rangle$ and $x_i \in G_{k_i}$ for $i = 1, 2$. We assert $K(G) = \{K(G)_n\}_n$ forms a crossed interval subgroup of G. Indeed, for a morphism $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, in view of Lemma 3.4.6, we have

$$\begin{aligned}&\varphi^*([\rho_1^*(x_1), \rho_2^*(x_2)]) \\ &= (\rho_1\varphi^{\rho_2^*(x_2)\rho_1^*(x_1^{-1})\rho_2^*(x_2^{-1})})^*(x_1)(\rho_2\varphi^{\rho_1^*(x_1^{-1})\rho_2^*(x_2^{-1})})^*(x_2) \\ &\qquad\qquad\qquad \cdot (\rho_1\varphi^{\rho_2^*(x_2^{-1})})^*(x_1^{-1})(\rho_2\varphi)^*(x_2^{-1}) \\ &= (\rho_1\varphi^{\rho_1^*(x_1^{-1})\rho_2^*(x_2^{-1})})^*(x_1)(\rho_2\varphi^{\rho_2^*(x_2^{-1})})^*(x_2)(\rho_1\varphi)^*(x_1^{-1})(\rho_2\varphi)^*(x_2^{-1}) \\ &= ((\rho_1\varphi)^{x_1^{-1}})^*(x_1)((\rho_2\varphi)^{x_2^{-1}})^*(x_2)(\rho_1\varphi)^*(x_1^{-1})(\rho_2\varphi)^*(x_2^{-1}) \\ &= [((\rho_1\varphi)^{x_1^{-1}})^*(x_1), ((\rho_2\varphi)^{x_2^{-1}})^*(x_2)] \ .\end{aligned} \tag{3.4.3}$$

Using Lemma 3.4.5, one can see morphisms $(\rho_1\varphi)^{x_1^{-1}}$ and $(\rho_2\varphi)^{x_2^{-1}}$ are dissociated, so (3.4.3) is one of generators of $K(G)_m$. Hence, $K(G)$ is a crossed interval subgroup of G.

Now, if H is an operadic crossed interval group, then every map $f : G \to H$ of crossed interval groups sends the generators of $K(G)$ to the unit. This in particular implies the map $K(G) \to \mathfrak{H}$ factors through the initial crossed interval group $* \subset \mathfrak{H}$, and the unique map $K(G) \to *$ of interval sets is actually a map of crossed interval groups. We define a crossed interval group $\widetilde{G}$ by the following pushout square in $\mathbf{CrsGrp}^{/\mathfrak{H}}_{\nabla}$:

$$\begin{array}{ccc} K(G) & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ G & \longrightarrow & \widetilde{G} \end{array} \ .$$

It is obvious that $\widetilde{G}$ is operadic. Moreover, the observation above shows that every map $G \to H$ of crossed interval groups uniquely factors through $G \to \widetilde{G}$

provided H is operadic. In other words, the assignment $G \mapsto \widetilde{G}$ gives the left adjoint of the inclusion $\mathbf{CrsOpGrp} \hookrightarrow \mathbf{CrsGrp}_{\nabla}^{/\mathfrak{H}}$, which is exactly the required result. ☐

Example 3.4.12. Recall that we have a functor

$$\mathrm{j} : \Delta \to \nabla \ ; \quad [n] \mapsto \{-\infty\} \star [n] \star \{\infty\} \cong \langle\!\langle n+1 \rangle\!\rangle$$

from the simplex category Δ. By Theorem 2.5.14, it induces a left adjoint functor

$$\mathrm{j}_\flat^{\mathfrak{H}} : \mathbf{CrsGrp}_{\Delta} \to \mathbf{CrsGrp}_{\nabla}^{/\mathfrak{H}} \ .$$

According to the computations in Example 2.5.12 and Example 2.5.13, for a crossed simplicial group G, the crossed interval group $\mathrm{j}_\flat^{\mathfrak{H}} G$ is described as follows: for each $n \in \mathbb{N}$, the group $(\mathrm{j}_\flat^{\mathfrak{H}} G)_n$ is the one generated by pairs (x, ρ) of an inert morphism $\rho : \langle\!\langle n \rangle\!\rangle \to \langle\!\langle k \rangle\!\rangle$ and an element $x \in G_{k-1}$, with assuming $G_{-1} = \pi_0 G$, which are subject to relation

$$(xy, \rho) \sim (x, \rho)(y, \rho) \ . \tag{3.4.4}$$

In particular, $(\mathrm{j}_\flat^{\mathfrak{H}} G)_n$ is the free product of copies of G_{k-1} indexed by inert morphisms $\rho : \langle\!\langle n \rangle\!\rangle \to \langle\!\langle k \rangle\!\rangle$ with k varying. To obtain the "operadification" of $\mathrm{j}_\flat^{\mathfrak{H}} G$, we only have to force the relation

$$(x_1, \rho_1)(x_2, \rho_2) \sim (x_2, \rho_2)(x_1, \rho_1)$$

for dissociated inert morphisms $\rho_i : \langle\!\langle n \rangle\!\rangle \to \langle\!\langle k_i \rangle\!\rangle$ and elements $x_i \in G_{k_i - 1}$ in addition to (3.4.4).

As seen in Example 3.4.7, the embedding $\widehat{\Psi} : \mathbf{GrpOp} \to \mathbf{CrsGrp}_{\nabla}^{/\mathfrak{H}}$ factors through the subcategory $\mathbf{CrsOpGrp} \hookrightarrow \mathbf{CrsGrp}_{\nabla}^{/\mathfrak{H}}$. To conclude the section, we compute the essential image of $\widehat{\Psi}$. This is essentially achieved by interpreting the condition (3.1.1) in terms of crossed interval groups.

Definition. A crossed interval group G is said to be *tame* if it satisfies the following condition: for each sequence $\vec{k} = (k_1, \dots, k_n)$ and for each $x \in G_n$ and $x_i \in G_{k_i}$ for $1 \le i \le n$, one has

$$\mu_{\vec{k}}^*(x) \cdot (\rho_i^{(\vec{k})})^*(x_i) = (\rho_{x(i)}^{(x_*(\vec{k}))})^*(x_i) \cdot \mu_{\vec{k}}^*(x) \ ,$$

where we put $x_*(\vec{k}) = (k_{x^{-1}(1)}, \dots, k_{x^{-1}(n)})$.

Theorem 3.4.13. *A crossed interval group G belongs to the essential image of the functor $\widehat{\Psi} : \mathbf{GrpOp} \to \mathbf{CrsGrp}_{\nabla}$ if and only if it is operadic and tame and lies over $\mathfrak{S} \subset \mathfrak{W}^{\nabla}$.*

Proof. If $G = \widehat{\Psi}(\mathcal{G})$ for a group operad $\mathcal{G}$, then for $\vec{k} = (k_1, \dots, k_n)$, $x \in G_n$, and $x_i \in G_{k_i}$ for $1 \le i \le n$, we have

$$\begin{aligned}
\mu_{\vec{k}}^*(x) \cdot (\rho_i^{(\vec{k})})^*(x_i) &= \gamma(x; e_{k_1}, \dots, \overset{i}{\breve{x_i}}, \dots, e_{k_n}) \\
&= \gamma(e_n; e_{k_{x^{-1}(1)}}, \dots, \overset{x(i)}{\breve{x_i}}, \dots, e_{k_{x^{-1}(n)}}) \cdot \gamma(x; e_{k_1}, \dots, e_{k_n}) \\
&= (\rho_{x(i)}^{(x_*(\vec{k}))})^*(x_i) \cdot \mu_{\vec{k}}^*(x) \ .
\end{aligned}$$

Hence, $\widehat{\Psi}(\mathcal{G})$ is tame as well as operadic.

Conversely, suppose G is an operadic and tame crossed interval group over $\mathfrak{S}$. We assert $\mathcal{O}_G$ is a group operad. Indeed, in view of Example 3.4.9, the operad $\mathcal{O}_G$ admits a canonical map $\mathcal{O}_G \to \mathcal{O}_{\mathfrak{S}} = \mathfrak{S}$ of operads which is levelwise group homomorphism. In addition, since G is tame, for $x, y \in G_n$ and $x_i, y_i \in G_{k_i}$, we have

$$\begin{aligned}
\gamma(xy; x_1y_1, \dots, x_ny_n) &= \mu^*_{\vec{k}}(xy)(\rho_1^{(\vec{k})})^*(x_1y_1)\dots(\rho_n^{(\vec{k})})^*(x_ny_n) \\
&= \mu^*_{y_*(\vec{k})}(x)\mu^*_{\vec{k}}(y) \\
&\quad \cdot (\rho_1^{(\vec{k})})^*(x_1)(\rho_1^{(\vec{k})})^*(y_1)\dots(\rho_n^{(\vec{k})})^*(x_n)(\rho_n^{(\vec{k})})^*(y_n) \\
&= \mu^*_{y_*(\vec{k})}(x)(\rho_{y(1)}^{(y_*(\vec{k}))})^*(x_1)\dots(\rho_{y(n)}^{(y_*(\vec{k}))})^*(x_n) \\
&\qquad \cdot \mu^*_{\vec{k}}(y)(\rho_1^{(\vec{k})})^*(y_1)\dots(\rho_n^{(\vec{k})})^*(y_n) \\
&= \gamma(x; x_{y^{-1}(1)}, \dots, x_{y^{-1}(n)})\gamma(y; y_1, \dots, y_n)\ .
\end{aligned}$$

This implies $\mathcal{O}_G$ satisfies the condition (3.1.1), and it is a group operad. Now, it is clear that $\widehat{\Psi}(\mathcal{O}_G) \cong G$, and this completes the proof. $\square$

Similarly to the operadicity, for each crossed interval group G over $\mathfrak{S}$, one can find a crossed interval subgroup $L(G) \subset G$ so that

(i) $L(G)$ is non-crossed; i.e. there is a map $L(G) \to *$ of crossed interval groups;

(ii) $G \mapsto L(G)$ is functorial; i.e. every map $f : G \to H$ of crossed interval groups over $\mathfrak{S}$ restricts to $L(G) \to L(H)$;

(iii) for each $n \in \mathbb{N}$, $L(G)_n \subset G_n$ contains all the elements of the form

$$\mu^*_{\vec{k}}(x) \cdot (\rho_i^{(\vec{k})})^*(x_i) \cdot \mu^*_{\vec{k}}(x)^{-1} \cdot (\rho_{x(i)}^{(x_*(\vec{k}))})^*(x_i)^{-1}$$

for $\vec{k} = (k_1, \dots, k_m)$ with $\sum k_i = n$, $x \in G_m$, and $x_i \in G_{k_i}$;

(iv) $L(G)$ is trivial provided G is tame.

Then, the "taming" G^{t} of G is obtained by the following pushout square in $\mathbf{CrsGrp}_{\nabla}^{/\mathfrak{S}}$

$$\begin{array}{ccc}
L(G) & \longrightarrow & * \\
\big\downarrow & & \big\downarrow \\
G & \longrightarrow & G^{\mathsf{t}}
\end{array}\quad ,$$

and it gives rise to a left adjoint to the inclusion of the full subcategory $\mathbf{CrsGrp}_{\nabla}^{\mathsf{tame}} \subset \mathbf{CrsGrp}_{\nabla}^{/\mathfrak{S}}$ spanned by tame crossed interval groups. Moreover, since the operadification and the taming commute with each other, the latter is restricted so as to induce the left adjoint to the functor $\widehat{\Psi}$:

$$(-)^{\mathsf{t}} : \mathbf{CrsOpGrp} \rightleftarrows \mathbf{CrsOpGrp}^{\mathsf{tame}} : \widehat{\Psi}\ ,$$

where $\mathbf{CrsOpGrp}^{\mathsf{tame}} = \mathbf{CrsOpGrp} \cap \mathbf{CrsGrp}_{\nabla}^{\mathsf{tame}}$. Note that, in view of Theorem 3.4.13, the functor $\widehat{\Psi}$ induces an equivalence (actually an isomorphism) $\mathbf{GrpOp} \simeq \mathbf{CrsOpGrp}^{\mathsf{tame}}$. Hence, we obtain an explicit description of the left adjoint to $\widehat{\Psi}$, which has been proved to exist in Theorem 3.3.3.

Remark 3.4.14. It was proved in Theorem 3.5 in [31] that the category **GrpOp** is locally presentable. The observation above gives us an alternative proof of this fact: in view of Theorem 3.3.3, we may regard **GrpOp** as a reflective subcategory of $\mathbf{CrsGrp}_{\nabla}^{/\mathfrak{S}}$. It is verified that operadic crossed interval groups and tame ones are closed under filtered colimits respectively. Then, **GrpOp** is locally presentable thanks to Corollary to Theorem 2.48 in [1].

3.5 Associative algebras

In this final section, we give an application of the results established in the previous sections.

We begin with the following observation. Let $\mathcal{C}$ be a monoidal category. Then, the category of monoid objects of $\mathcal{C}$ is equivalent to the category $\mathbf{MultCat}(*, \mathcal{C}^{\otimes})$ of multifunctors from the terminal operad to the multicategory $\mathcal{C}^{\otimes}$ associated to $\mathcal{C}$ (seeExample 1.3.4). We write $\mathbf{Alg}(\mathcal{C}) := \mathbf{Alg}_*(\mathcal{C})$ (cf. Example 1.3.7). The assignment gives rise to a 2-functor

$$\mathbf{Alg}(-) : \mathbf{MonCat} \to \mathbf{Cat} .$$

It was shown in Section VII.5 in [55] that the 2-functor is *represented* by the category $\widetilde{\Delta}$ with the join $\star$ as the monoidal structure. Indeed, as easily checked, the following data defines a multifunctor $M : * \to \widetilde{\Delta}^{\otimes}$:

- for the unique object $*$ of $*$, we set $M(*) := \langle 1 \rangle$;
- for the unique operations $\mu_n \in *(n)$, we set

  $$M(\mu_n) \in \widetilde{\Delta}^{\otimes}(\langle 1 \rangle \dots \langle 1 \rangle; \langle 1 \rangle) = \widetilde{\Delta}(\langle 1 \rangle \star \dots \star \langle 1 \rangle, \langle 1 \rangle) = \widetilde{\Delta}(\langle n \rangle, \langle 1 \rangle)$$

  to be the unique morphism to the terminal object $\langle 1 \rangle$.

Then, the precomposition with M gives rise to a functor

$$\mathbf{MonCat}(\widetilde{\Delta}, \mathcal{C}) \xrightarrow{(-)^{\otimes}} \mathbf{MultCat}(\widetilde{\Delta}^{\otimes}, \mathcal{C}^{\otimes}) \xrightarrow{M^*} \mathbf{Alg}(\mathcal{C})$$

for each monoidal category $\mathcal{C}$, which is claimed to be an equivalence.

On the other hand, if $\mathcal{G}$ is a group operad, then we can consider the composition

$$\mathbf{MonCat}_{\mathcal{G}} \xrightarrow{\textit{forget}} \mathbf{MonCat} \xrightarrow{\mathbf{Alg}(-)} \mathbf{Cat} . \tag{3.5.1}$$

We discuss the representability of this 2-functor. We first have to construct a candidate of the representing object. In view of Theorem 3.3.3, we regard the category **CrsGrp** as a full subcategory of $\mathbf{CrsGrp}_{\nabla}^{/\mathfrak{S}}$, so we may identify $\mathcal{G}$ with its image $\Psi^{\mathfrak{S}}(\mathcal{G})$ in $\mathbf{CrsGrp}_{\nabla}^{/\mathfrak{S}}$. Recall that we have a functor $\mathfrak{J} : \widetilde{\Delta} \to \nabla$ which appeared in (2.1.4). Hence, pulling back along $\mathfrak{J}$, we obtain an augmented crossed simplicial group $\mathfrak{J}^{\natural}\mathcal{G}$ by virtue of Theorem 2.5.14 so as to form the total category $\widetilde{\Delta}_{\mathfrak{J}^{\natural}\mathcal{G}}$.

Lemma 3.5.1. *Let $\mathcal{G}$ be a group operad. Then, the monoidal structure on $\widetilde{\Delta}$ extends to the total category $\widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}$ so that it is $\mathcal{G}$-symmetric.*

Proof. Since the inclusion $\widetilde{\Delta} \to \widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}$ is bijective on objects, we have to extends the monoidal product on morphisms. Recall that morphisms of $\widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}$ are of the form $(\varphi, x) : \langle m\rangle \to \langle n\rangle$ with $\varphi : \langle m\rangle \to \langle n\rangle \in \widetilde{\Delta}$ and $x \in \mathcal{G}(m)$. For morphisms $(\varphi_i, x_i) : \langle m_i\rangle \to \langle n_i\rangle$ for $1 \le i \le n$, we set

$$(\varphi_1, x_1) \star \cdots \star (\varphi_n, x_n) := \bigl(\varphi_1 \star \cdots \star \varphi_n, \gamma(e_n; x_1, \dots, x_n)\bigr)\ .$$

The functoriality is obvious, and the strict associativity follows from those of the join and the composition operation in the operad $\mathcal{G}$.

To introduce a $\mathcal{G}$-symmetric structure, note that we have

$$(\widetilde{\Delta}^{\otimes}_{\mathfrak{J}\natural\mathcal{G}} \rtimes \mathcal{G})(\langle k_1\rangle \dots \langle k_n\rangle; \langle m\rangle) = \widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}(\langle k_1 + \cdots + k_n\rangle, \langle m\rangle) \times \mathcal{G}(n).$$

Then, we consider the map

$$\begin{array}{ccc} (\widetilde{\Delta}^{\otimes}_{\mathfrak{J}\natural\mathcal{G}} \rtimes \mathcal{G})(\langle k_1\rangle \dots \langle k_n\rangle; \langle m\rangle) & \to & \widetilde{\Delta}^{\otimes}_{\mathfrak{J}\natural\mathcal{G}}(\langle k_1\rangle \dots \langle k_n\rangle; \langle m\rangle) \\ ((\varphi, x), u) & \mapsto & \bigl(\varphi, x \cdot \gamma(u; e_{k_1}, \dots, e_{k_n}\bigr)\ . \end{array} \tag{3.5.2}$$

Using the description of the crossed interval group $\mathfrak{S}$ given in Example 2.1.8, one can check the following formula:

$$\begin{aligned} &\bigl(\varphi, x \cdot \gamma(u; e_{k_1}, \dots, e_{k_n})\bigr) \circ \bigl(\varphi_1 \star \cdots \star \varphi_n, \gamma(e_n; x_1, \dots, x_n)\bigr) \\ &= \bigl(\varphi \circ (\varphi_{u^{-1}(1)} \star \cdots \star \varphi_{u^{-1}(n)})^x, \\ &\qquad (\varphi_{u^{-1}(1)} \star \cdots \star \varphi_{u^{-1}(n)})^*(x) \cdot \gamma(u; x_1, \dots, x_n)\bigr)\ , \end{aligned}$$

which implies (3.5.2) actually defines an identity-on-object multifunctor $\widetilde{\Delta}^{\otimes}_{\mathfrak{J}\natural\mathcal{G}} \rtimes \mathcal{G} \to \widetilde{\Delta}^{\otimes}_{\mathfrak{J}\natural\mathcal{G}}$. It is obvious that it exhibits $\widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}$ as a $\mathcal{G}$-symmetric monoidal category. □

We assert that the $\mathcal{G}$-symmetric monoidal category $\widetilde{\Delta}^{\otimes}_{\mathfrak{J}\natural\mathcal{G}}$ in fact represents the 2-functor (3.5.1). To see this, notice that, since the 2-category $\mathbf{MultCat}_{\mathcal{G}}$ is the category of 2-algebras over the 2-monad $(-) \rtimes \mathcal{G}$, we have an isomorphism

$$\mathbf{MultCat}(\mathcal{M}, \mathcal{N}) \cong \mathbf{MultCat}_{\mathcal{G}}(\mathcal{M} \rtimes \mathcal{G}, \mathcal{N})$$

of categories for each $\mathcal{M} \in \mathbf{MultCat}$ and $\mathcal{N} \in \mathbf{MultCat}_{\mathcal{G}}$. Hence, for $\mathcal{G}$-symmetric monoidal category $\mathcal{C}$, we have a canonical isomorphism

$$\mathbf{Alg}(\mathcal{C}) = \mathbf{MultCat}(*, \mathcal{C}^{\otimes}) \cong \mathbf{MultCat}_{\mathcal{G}}(\mathcal{G}, \mathcal{C}^{\otimes})\ ,$$

here we use the isomorphism $* \rtimes \mathcal{G} \cong \mathcal{G}$. We define a multifunctor $M_{\mathcal{G}} : \mathcal{G} \to \widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}$ as follows:

- for the unique object $*$ of $\mathcal{G}$, set $M_{\mathcal{G}}(*) := \langle 1\rangle$;
- for each $x \in \mathcal{G}(n)$, we put $M_{\mathcal{G}}(x) := (\mu_n, x) \in \widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}(\langle n\rangle, \langle 1\rangle)$, where $\mu_n : \langle n\rangle \to \langle 1\rangle \in \widetilde{\Delta}$ is the unique map into the terminal object in $\widetilde{\Delta}$.

It is straightforward from the description (3.5.2) that $M_{\mathcal{G}}$ is even $\mathcal{G}$-symmetric.

Proposition 3.5.2. *Let $\mathcal{G}$ be a group operad. Then, for every $\mathcal{G}$-symmetric monoidal category $\mathcal{C}$, the functor*

$$\begin{aligned} \mathbf{MonCat}_{\mathcal{G}}(\widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}, \mathcal{C}) &\xrightarrow{(-)^{\otimes}} \mathbf{MultCat}_{\mathcal{G}}(\widetilde{\Delta}^{\otimes}_{\mathfrak{J}\natural\mathcal{G}}, \mathcal{C}^{\otimes}) \\ &\xrightarrow{M^*_{\mathcal{G}}} \mathbf{MultCat}_{\mathcal{G}}(\mathcal{G}, \mathcal{C}^{\otimes}) \cong \mathbf{Alg}(\mathcal{C}) \ , \end{aligned} \tag{3.5.3}$$

where the second functor is induced by the precomposition with $M_{\mathcal{G}} : \mathcal{G} \to \widetilde{\Delta}^{\otimes}_{\mathfrak{J}\natural\mathcal{G}}$ is an equivalence of categories.

Proof. We actually construct the inverse to the composition, say $M^{\otimes}_{\mathcal{G}}$, of the first two functors in (3.5.3). Notice that every morphism $\langle m \rangle \to \langle n \rangle \in \widetilde{\Delta}$ can be uniquely written in the form $\mu_{k_1} \star \cdots \star \mu_{k_n}$ for a sequence $\vec{k} = (k_1, \dots, k_n)$ with $k_1 + \cdots + k_n = m$ (cf. Eq. (1.2.2)), where $\mu_k : \langle k \rangle \to \langle 1 \rangle$ is the unique map into the terminal object $\langle 1 \rangle$ in $\widetilde{\Delta}$. For a $\mathcal{G}$-symmetric multifunctor $F : \mathcal{G} \to \mathcal{C}^{\otimes}$, we define a functor $F_{\otimes} : \widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}} \to \mathcal{C}$ as follows:

- for each $n \in \mathbb{N}$, we set $F_{\otimes}(\langle n \rangle) := F(*)^{\otimes n}$, where $*$ is the unique object of the operad $\mathcal{G}$;
- for each morphism $(\varphi, x) : \langle m \rangle \to \langle n \rangle \in \widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}$, say $\varphi = \mu_{k_1} \star \cdots \star \mu_{k_n}$, we define a morphism $F_{\otimes}(\varphi, x) : F(*)^{\otimes m} \to F(*)^{\otimes n}$ to be the composition

  $$F(*)^{\otimes m} \xrightarrow{\Theta^x} F(*)^{\otimes m} \cong F(*)^{\otimes k_1} \otimes \cdots \otimes F(*)^{\otimes k_n} \xrightarrow{F(e_{k_1}) \otimes \cdots \otimes F(e_{k_1})} F(*)^{\otimes n} \ ,$$

  where Θ^x is the natural transformation given in Example 3.2.3.

The naturality of Θ^x implies that, if we have a sequence $(l_1, \dots, l_m)$, the square below commutes in $\mathcal{C}$:

$$\begin{array}{ccc} F(*)^{\otimes l_1} \otimes \cdots \otimes F(*)^{\otimes l_m} & \xrightarrow{F(e_{l_1}) \otimes \cdots \otimes F(e_{l_m})} & F(*)^{\otimes m} \\ {\scriptstyle \Theta^x} \downarrow & & \downarrow {\scriptstyle \Theta^x} \\ F(*)^{\otimes l_{x^{-1}(l_1)}} \otimes \cdots \otimes F(*)^{\otimes l_{x^{-1}(m)}} & \xrightarrow{F(e_{l_{x^{-1}(1)}}) \otimes \cdots \otimes F(e_{l_{x^{-1}(m)}})} & F(*)^{\otimes m} \end{array} \ .$$

Since the left vertical arrow agrees with the morphism $\Theta^{\gamma(x; e_{l_1}, \dots, e_{l_m})}$ under the isomorphisms

$$\begin{aligned} F(*)^{\otimes l_1} \otimes \cdots \otimes F(*)^{\otimes l_m} &\cong F(*)^{\otimes (l_1 + \cdots + l_m)} \\ &\cong F(*)^{\otimes l_{x^{-1}(l_1)}} \otimes \cdots \otimes F(*)^{\otimes l_{x^{-1}(m)}} \ , \end{aligned}$$

the functoriality of $F_{\otimes}$ follows. In addition, it is straightforward from the definition that $F_{\otimes}$ is monoidal, with the comparison isomorphism

$$\begin{aligned} F(\langle k_1 \rangle) \otimes \cdots \otimes F(\langle k_n \rangle) &= F(*)^{\otimes k_1} \otimes \cdots \otimes F(*)^{\otimes k_n} \\ &\cong F(*)^{\otimes (k_1 + \cdots + k_n)} \\ &= F(\langle k_1 \rangle \star \cdots \star \langle k_n \rangle) \ , \end{aligned}$$

and $\mathcal{G}$-symmetric. On the other hand, note that a multinatural transformation $\alpha : F \to G : \mathcal{G} \to \mathcal{C}^{\otimes}$ consists of a morphism $\alpha : F(*) \to G(*) \in \mathcal{C}$ such that, for each $k \in \mathbb{N}$, the square below is commutative:

$$\begin{array}{ccc} F(*)^{\otimes k} & \xrightarrow{\alpha^{\otimes k}} & G(*)^{\otimes k} \\ {\scriptstyle F(e_k)}\downarrow & & \downarrow{\scriptstyle G(e_k)} \\ F(*) & \xrightarrow{\alpha} & G(*) \end{array} .$$

It immediately follows that, setting $(\alpha_{\otimes})_{\langle k\rangle} := \alpha^{\otimes k}$, we get a natural transformation $\alpha_{\otimes} : F_{\otimes} \to G_{\otimes}$. Hence, the construction above defines a functor

$$(-)_{\otimes} : \mathbf{MultCat}_{\mathcal{G}}(\mathcal{G}, \mathcal{C}^{\otimes}) \to \mathbf{MonCat}_{\mathcal{G}}(\widetilde{\Delta}_{\mathfrak{J}^{\natural}\mathcal{G}}, \mathcal{C}) .$$

The composition $M_{\mathcal{G}}^{\otimes} \circ (-)_{\otimes}$ is clearly identified with the identity functor on $\mathbf{MultCat}_{\mathcal{G}}(\mathcal{G}, \mathcal{C}^{\otimes})$. On the other hand, if $F : \widetilde{\Delta}_{\mathfrak{J}^{\natural}\mathcal{G}} \to \mathcal{C}$ is a $\mathcal{G}$-symmetric monoidal functor, it is equipped with an isomorphism

$$\lambda_n : (M_{\mathcal{G}}^{\otimes}(F))_{\otimes}(\langle n\rangle) = M_{\mathcal{G}}^{\otimes}(F)(*)^{\otimes n} = F(\langle 1\rangle)^{\otimes n} \xrightarrow{\cong} F(\langle n\rangle)$$

for each $n \in \mathbb{N}$. We assert $\lambda = \{\lambda_n\}_n$ forms a natural isomorphism $M_{\mathcal{G}}^{\otimes}(-)_{\otimes} \cong \mathrm{Id}$. Indeed, for each sequence $k_1, \dots, k_n$ of non-negative integers, the coherence diagram (1.2.1) for λ guarantees the following diagram to commute:

$$\begin{array}{ccc} F(\langle 1\rangle)^{\otimes k_1+\dots+k_n} & \xrightarrow{\cong} & F(\langle 1\rangle)^{\otimes k_1} \otimes \dots \otimes F(\langle 1\rangle)^{\otimes k_n} \\ {\scriptstyle \lambda_{k_1+\dots+k_n}}\downarrow & & \downarrow{\scriptstyle \lambda_{k_1}\otimes\dots\otimes\lambda_{k_n}} \\ F(\langle k_1+\dots+k_n\rangle) & \xleftarrow{\lambda_n} & F(\langle k_1\rangle) \otimes \dots \otimes F(\langle k_n\rangle) \\ {\scriptstyle F(\mu_{k_1}\star\dots\star\mu_{k_n})}\downarrow & & \downarrow{\scriptstyle F(\mu_{k_1})\otimes\dots\otimes F(\mu_{k_n})} \\ F(\langle n\rangle) & \xleftarrow{\lambda_n} & F(\langle 1\rangle)^{\otimes n} \end{array} , \qquad (3.5.4)$$

where $\mu_k : \langle k\rangle \to \langle 1\rangle$ is the unique morphism to the terminal object. Notice that, according to the description of the multifunctor $F^{\otimes} : \widetilde{\Delta}^{\otimes}_{\mathfrak{J}^{\natural}\mathcal{G}} \to \mathcal{C}^{\otimes}$ given in Example 1.3.9, the composition of the top arrow and the right vertical arrows exactly gives the morphism

$$M_{\mathcal{G}}^{\otimes}(F)_{\otimes}(\mu_{k_1} \star \dots \star \mu_{k_n}) : M_{\mathcal{G}}^{\otimes}(F)_{\otimes}(\langle k_1+\dots+k_n\rangle) \to M_{\mathcal{G}}^{\otimes}(F)_{\otimes}(\langle n\rangle) .$$

Hence, it follows from the commutativity of (3.5.4) that λ is a natural isomorphism. Moreover, the upper half square in (3.5.4) is precisely the coherence diagram for λ to be monoidal. In other words, we obtain an isomorphism $\lambda : M_{\mathcal{G}}^{\otimes}(F)_{\otimes} \cong F$ in the category $\mathbf{MonCat}_{\mathcal{G}}(\widetilde{\Delta}_{\mathfrak{J}^{\natural}\mathcal{G}}, \mathcal{C})$. If $\alpha : F \to G : \widetilde{\Delta}_{\mathfrak{J}^{\natural}\mathcal{G}} \to \mathcal{C}$ is a monoidal natural transformation, then the coherence of α gives us the equation

$$\lambda \circ M_{\mathcal{G}}^{\otimes}(\alpha)_{\otimes} = \alpha \circ \lambda ,$$

so that λ actually defines a natural isomorphism $M_{\mathcal{G}}^{\otimes}(-)_{\otimes} \cong \mathrm{Id}$, which exhibits the functor $(-)_{\otimes}$ as an inverse to $M_{\mathcal{G}}^{\otimes}$. □

To conclude the chapter, we mention the relation of the category $\widetilde{\Delta}_{\mathfrak{J}\natural\mathcal{G}}$ to the Hochschild homologies for algebras. For this, we need to recall the *paracyclic category* Λ_∞ defined in Example 2.1.14, which is the total category of the crossed simplicial group $\boldsymbol{\mathcal{Z}}$. The following result is due to Elmendorf.

Proposition 3.5.3 (Proposition 1 in [23])**.** *There is an identity-on-object functor* $\overline{(-)} : \Lambda_\infty^{\mathrm{op}} \to \Lambda_\infty$ *such that, for* $f \in \lambda_\infty(m,n)$*,*

$$\overline{f} : \mathbb{Z} \to \mathbb{Z} \ ; \quad j \mapsto \min\{i \in \mathbb{Z} \mid j \le f(i)\} \ .$$

Proof. We first verify $\overline{f}$ actually belongs to $\Lambda_\infty(n,m)$. Clearly, it is order-preserving. Note that, since f is order-preserving, the map $\overline{f}$ is characterized by the following property: for every $i \in \mathbb{Z}$,

$$\overline{f}(j) \le i \iff j \le f(i) \ . \tag{3.5.5}$$

Consequently, we obtain

$$\begin{aligned} \overline{f}(j+n+1) \le i &\iff j+n+1 \le f(i) \\ &\iff j \le f(i-m-1) \\ &\iff \overline{f}(j)+m+1 \le i \ , \end{aligned}$$

which implies $\overline{f}(j+n+1) = \overline{f}(j)+m+1$ so that $\overline{f} \in \Lambda_\infty(n,m)$.

The functoriality of $\overline{(-)}$ is another consequence of the property (3.5.5). To see it is an isomorphism, it suffices to show it is fully faithful. In fact, the inverse of the map $\overline{(-)} : \Lambda_\infty(m,n) \to \Lambda_\infty(n,m)$ is given by

$$g \mapsto \left[i \mapsto \max\{j \in \mathbb{Z} \mid g(j) \le i\}\right] \ .$$

□

Example 3.5.4. Let $\varphi : [m] \to [n] \in \Delta$ be a map in the simplex category. Under the identification of it with the image in Λ_∞, the composition

$$\widehat{\varphi} : \{0,\dots,n\} \hookrightarrow \mathbb{Z} \xrightarrow{\overline{\varphi}} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/(m+1)\mathbb{Z} \cong \{0,\dots,m\}$$

is described as follows:

$$\widehat{\varphi}^{-1}\{i\} = \begin{cases} \{0,1,\dots,\varphi(0),\varphi(m)+1,\dots,n\} & i = 0 \ , \\ \{\varphi(i-1),\varphi(i-1)+1,\dots,\varphi(i)\} & 1 \le i \le m \ . \end{cases}$$

Hence, seeing $\overline{\varphi}$ as a morphism in the total category $\Delta_{\boldsymbol{\mathcal{Z}}}$, we can write it as a pair

$$\overline{\varphi} = \left(\mu_{\varphi(0)+n-\varphi(m)} \star \mu_{\varphi(1)-\varphi(0)} \star \cdots \star \mu_{\varphi(m)-\varphi(m-1)}, \tau_n^{n-\varphi(m)}\right),$$

where $\star$ is the join of totally ordered sets, $\mu_k : [k-1] \to [0]$ is the unique morphism to the terminal object in Δ, and $\tau_n \in \boldsymbol{\mathcal{Z}}_n$ is the element corresponding to $1 \in \mathbb{Z}$ under the canonical isomorphism $\boldsymbol{\mathcal{Z}}_n \cong \mathbb{Z}$.

Now, suppose $\mathcal{C}$ is a $\mathcal{G}$-symmetric monoidal abelian category; i.e. it is abelian and equipped with a $\mathcal{G}$-symmetric monoidal structure so that the functor

$$\otimes_n : \mathcal{C}^{\times n} \to \mathcal{C}$$

is right exact in each variable for each $n \in \mathbb{N}$. We denote by $\mathbf{Ch}(\mathcal{C})$ the category of chain complexes in $\mathcal{C}$. For a monoid object $A \in \mathcal{C}$, the Hochschild complex $C_\bullet(A) \in \mathbf{Ch}(\mathcal{C})$ (with coefficient A) has the following construction, which is based on the one given in [62]. By virtue of Proposition 3.5.2, A gives rise to a $\mathcal{G}$-symmetric monoidal functor $A_\otimes : \widetilde{\Delta}_{\mathfrak{J}^\natural\mathcal{G}} \to \mathcal{C}$. On the other hand, the embedding described in Example 2.5.5 enables us to see $\boldsymbol{\mathcal{Z}}$ as an augmented crossed simplicial group, so we can also form the total category $\widetilde{\Delta}_{\boldsymbol{\mathcal{Z}}}$. Note that the embedding $\Lambda_\infty \cong \Delta_{\boldsymbol{\mathcal{Z}}} \hookrightarrow \widetilde{\Delta}_{\boldsymbol{\mathcal{Z}}}$ identifies Λ_∞ with the full subcategory of $\widetilde{\Delta}_{\boldsymbol{\mathcal{Z}}}$ spanned by all but the initial object. Hence, choosing a map $\boldsymbol{\mathcal{Z}} \to \mathfrak{J}^\natural\mathcal{G}$ of augmented crossed simplicial groups, we obtain a *paracyclic object* in $\mathcal{C}$:

$$A^{\circledcirc}_\bullet : \Lambda_\infty^{\mathrm{op}} \cong \Lambda_\infty \hookrightarrow \widetilde{\Delta}_{\boldsymbol{\mathcal{Z}}} \to \widetilde{\Delta}_{\mathfrak{J}^\natural\mathcal{G}} \xrightarrow{A_\otimes} \mathcal{C}\ ,$$

where the first isomorphism is the one given in Proposition 3.5.3. Then, the computations in Example 3.5.4 shows that the Hochschild complex $C_\bullet(A) \in \mathbf{Ch}(\mathcal{C})$ is isomorphic to the associated chain complex of the simplicial object $A^{\circledcirc}|_{\Delta^{\mathrm{op}}}$.

Remark 3.5.5. In the above construction, we chose a map $\boldsymbol{\mathcal{Z}} \to \mathfrak{J}^\natural\mathcal{G}$ of augmented crossed simplicial groups. Note that, since $\boldsymbol{\mathcal{Z}}$ is connected as a simplicial set, so the computation in (2.5.12) shows that for every augmented crossed simplicial group G, maps $\boldsymbol{\mathcal{Z}} \to G$ of crossed simplicial groups correspond in one-to-one to those of augmented ones. It follows that the Hochschild chain $C_\bullet(A)$ is actually indexed by the following isomorphic sets:

$$\mathbf{CrsGrp}_\Delta(\boldsymbol{\mathcal{Z}}, \mathfrak{J}^\natural\mathcal{G}) \cong \mathbf{CrsGrp}_{\widetilde{\Delta}}(\boldsymbol{\mathcal{Z}}, \mathfrak{J}^\natural\mathcal{G}) \cong \mathbf{CrsGrp}_\nabla(\mathfrak{J}_\flat\boldsymbol{\mathcal{Z}}, \mathcal{G})\ .$$

Example 3.5.6. Take $\mathcal{G} = \mathcal{B}$ the group operad of braid groups. In this case, there is the following canonical pullback square

$$\begin{array}{ccc} \boldsymbol{\mathcal{Z}} & \longrightarrow & \mathfrak{J}^\natural\mathcal{B} \\ \downarrow & \lrcorner & \downarrow \\ \boldsymbol{\mathcal{C}} & \longrightarrow & \boldsymbol{\mathcal{S}} \end{array}$$

in the category $\mathbf{CrsGrp}_\Delta$. Hence, there is a canonical choice of a Hochschild chain for a monoid object in a braided monoidal abelian category.

Chapter 4

Symmetries in terms of internal presheaves

For a group operad $\mathcal{G}$, a $\mathcal{G}$-symmetric structure on $\mathcal{M}$ is by definition a multifunctor $\mathcal{M} \rtimes \mathcal{G} \to \mathcal{M}$. On the other hand, there is a way to exhibit multicategories as fibrations over the category ∇. Indeed, as for a *symmetric* multicategories, or colored operads, $\mathcal{C}$, May and Thomason [58] constructed a fibration $\widehat{\mathcal{C}} \to \mathbf{Fin}_*$ over the category of pointed finite sets so that (topological) categories over $\widehat{\mathcal{C}}$ satisfying certain conditions corresponds in one-to-one to algebras over $\mathcal{C}$. The construction has a straightforward non-symmetric analogue; if $\mathcal{M}$ is a multicategory, it gives rise to a fibration $\mathcal{M}^\nabla \to \nabla$. In this section, we aim at encoding $\mathcal{G}$-symmetric structures in this language. More precisely, we will construct a counterpart of the 2-functor $(-) \rtimes \mathcal{G}$ on the fibration side.

4.1 Quotients of the total category

We begin with an observation that a sort of symmetries on an algebraic structure are presented by quotients of the total category of crossed interval groups. Recall that, as pointed out in [68], monoids are associated with functor $M^\nabla : \nabla \to \mathbf{Set}$ satisfying so-called the *Segal condition* with $M^\nabla(\langle\langle 0\rangle\rangle) \cong *$. Namely, if M is a monoid, then

- for each $n \in \mathbb{N}$, $M^\nabla(\langle\langle n\rangle\rangle) = M^{\times n}$;
- for a morphism $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, the map $\varphi_* : M^{\times m} \to M^{\times n}$ is given by

$$\varphi_*(x_1, \dots, x_m) = \left(\prod_{\varphi(i)=1} x_i, \dots, \prod_{\varphi(i)=n} x_i \right) ,$$

  where the products are taken in the obvious orders.

Let G be a crossed interval group. Then, the functor M^∇ canonically extends to the total category ∇_G. Indeed, notice that a functor $X : \nabla_G \to \mathbf{Set}$ determines and determined by the data

- the restriction $X|_\nabla : \nabla \to \mathbf{Set}$;

- for each $n \in \mathbb{N}$, a left G_n-action on the set $X(\langle\langle n\rangle\rangle)$, say $G_n \times X(\langle\langle n\rangle\rangle) \to X(\langle\langle n\rangle\rangle)$; $(u, x) \mapsto x^u$;

such that, for each $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, $x \in X(\langle\langle m\rangle\rangle)$, and $v \in G_n$,

$$\varphi_*(x)^v = (\varphi^v)_*(x^{\varphi^*(v)}) .$$

Each G_n now acts on $M^\nabla(\langle\langle n\rangle\rangle) = M^{\times n}$ through the canonical map $G_n \to \mathfrak{W}_n^\nabla \to \mathfrak{S}_n$, and the action gives an extension $M_G^\nabla : \nabla_G \to \mathbf{Set}$. In particular, when $G = \mathfrak{S}$ (see Example 2.1.8), we assert that the monoid M is commutative if and only if the functor $M_{\mathfrak{S}}^\nabla : \nabla_{\mathfrak{S}} \to \mathbf{Set}$ factors through an appropriate quotient category of $\nabla_{\mathfrak{S}}$. We set $\mu_n : \langle\langle n\rangle\rangle \to \langle\langle 1\rangle\rangle \in \nabla$ with $\mu_n(i) = 1$ for $1 \le i \le n$ and $\mu_n(\pm\infty) = \pm\infty$. Then, for each $(x_1, \dots, x_n) \in M^{\times n}$, and for each $\sigma \in \mathfrak{S}_n$, we have

$$(\mu_n, \sigma)_*(x_1, \dots, x_n) = (\mu_n)_*(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}) = x_{\sigma^{-1}(1)} \dots x_{\sigma^{-1}(n)} .$$

It follows that M is commutative if and only if the two induced maps $(\mu_n, \sigma)_*$ and $(\mu_n)_*$ agrees with each other for every n and σ; in other words, $M_{\mathfrak{S}}^\nabla$ factors through a quotient $q : \nabla_{\mathfrak{S}} \to \mathcal{Q}$ such that $q(\mu_n, \sigma) = q(\mu_n, e_n)$, where $e_n \in \mathfrak{S}_n$ is the unit.

More generally, we can regard any quotients of ∇_G may present a kind of symmetries on monoids or higher variants, and this is the main theme of this section. In particular, we focus on the classification of quotients of the following form.

Definition. Let G be a crossed interval group. Then, a *G-quotal category* is a category $\mathcal{Q}$ equipped with a functor $q : \nabla_G \to \mathcal{Q}$ satisfying the following conditions:

(i) q is full and bijective on objects, so we may assume $\operatorname{Ob}\mathcal{Q} = \operatorname{Ob}\nabla$;

(ii) for $\varphi, \varphi' \in \nabla(\langle\langle m\rangle\rangle, \langle\langle n\rangle\rangle)$ and $x, x' \in G_m$, the equality of morphisms

$$q(\varphi, x) = q(\varphi', x') : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \mathcal{Q}$$

in $\mathcal{Q}$ implies $\varphi = \varphi'$.

Lemma 4.1.1. *Let G be a crossed interval group, and let $\mathcal{Q}$ be a G-quotal category with $q : \nabla_G \to \mathcal{Q}$. Then, the composition*

$$\nabla \hookrightarrow \nabla_G \xrightarrow{q} \mathcal{Q} \tag{4.1.1}$$

is faithful and conservative.

Proof. Let us denote by $e_m \in G_m$ the unit of the group. Then, the composition (4.1.1) sends a morphism $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$ to $q(\varphi, e_m)$. Hence, the condition on G-quotal categories directly implies (4.1.1) is faithful.

Next, suppose $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$ is a morphism such that $q(\varphi, e_m)$ is an isomorphism. Since q is full, the inverse of $q(\varphi, e_m)$ can be written in the form $q(\psi, y)$ with $\psi : \langle\langle n\rangle\rangle \to \langle\langle m\rangle\rangle \in \nabla$ and $y \in G_n$. We have

$$\begin{aligned} \mathrm{id}_{\langle\langle m\rangle\rangle} &= q(\psi, y) \circ q(\varphi, e_m) = q(\psi\varphi^y, \varphi^*(y)) \\ \mathrm{id}_{\langle\langle n\rangle\rangle} &= q(\varphi, e_m) \circ q(\psi, y) = q(\varphi\psi, y) . \end{aligned}$$

By virtue of the condition on G-quotal categories, the former equation implies $\mathrm{id}_{\langle\langle m\rangle\rangle} = \psi\varphi^y$ while the latter implies $\mathrm{id}_{\langle\langle n\rangle\rangle} = \varphi\psi$. It follows that ψ is an inverse of φ in ∇, so (4.1.1) is conservative. □

Thanks to Lemma 4.1.1, we can identify morphisms in ∇ with their images in a G-quotal category; namely, if $q : \nabla_G \to \mathcal{Q}$ is a G-quotal category, then by abuse of notation, we write $\varphi = q(\varphi, e_m)$ for every $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$.

There is a general recipe to construct G-quotal categories. For this, we introduce some notions.

Definition. Let G be a crossed interval group, and let $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$ be a morphism. Then, an element $x \in G_m$ is called a *right stabilizer of* φ if for every morphism $\psi : \langle\!\langle l \rangle\!\rangle \to \langle\!\langle m \rangle\!\rangle \in \nabla$, we have $\varphi\psi^* = \varphi\psi$. We denote by $\mathrm{RSt}^G_\varphi \subset G_m$ the subset of right stabilizers of φ.

It is obvious that, for $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$, the subset $\mathrm{RSt}^G_\varphi \subset G_m$ is a subgroup.

Definition. Let G be a crossed interval group. A *congruence family* on G is a family $K = \{K_\varphi\}_\varphi$ indexed by morphisms in ∇ such that, for every $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$,

(i) K_φ is a subgroup of RSt^G_φ;

(ii) for every morphism $\chi : \langle\!\langle n \rangle\!\rangle \to \langle\!\langle k \rangle\!\rangle \in \nabla$, $K_\varphi \subset K_{\chi\varphi}$;

(iii) for every morphism $\psi : \langle\!\langle l \rangle\!\rangle \to \langle\!\langle m \rangle\!\rangle \in \nabla$, the map $\psi^* : G_m \to G_l$ restricts to a map $K_\varphi \to K_{\varphi\psi}$;

(iv) for every element $y \in G_n$, we have

$$\varphi^*(y) \cdot K_\varphi \cdot \varphi^*(y)^{-1} = K_{\varphi^y} \ . \tag{4.1.2}$$

Remark 4.1.2. The first three conditions above implies $K = \{K_\varphi\}_\varphi$ forms a crossed group over $\mathrm{opTw}(\nabla) := \mathrm{Tw}(\nabla)^{\mathrm{op}}$ the opposite of the twisted arrow category of ∇; $\mathrm{opTw}(\nabla)$ is the category such that

- the objects are morphisms of ∇;
- for morphisms $\varphi_i : \langle\!\langle m_i \rangle\!\rangle \to \langle\!\langle n_i \rangle\!\rangle \in \nabla$ for $i = 1, 2$, morphisms $\varphi_1 \to \varphi_2$ in $\mathrm{Tw}(\nabla)$ are pairs (α, β) of morphisms in commutative squares of the form

$$\begin{array}{ccc} \langle\!\langle m_1 \rangle\!\rangle & \xrightarrow{\ \alpha\ } & \langle\!\langle m_2 \rangle\!\rangle \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ \langle\!\langle n_1 \rangle\!\rangle & \xleftarrow{\ \beta\ } & \langle\!\langle n_2 \rangle\!\rangle \end{array} \ ;$$

- the composition is given by

$$(\gamma, \delta) \circ (\alpha, \beta) = (\gamma\alpha, \beta\delta) \ .$$

The second and the third conditions imply each morphism $(\alpha, \beta) : \varphi_1 \to \varphi_2$ in $\mathrm{opTw}(\nabla)$ induces a map

$$(\alpha, \beta)^* : K_{\varphi_2} \xrightarrow{\alpha^*} K_{\varphi_2\alpha} \hookrightarrow K_{\beta\varphi_2\alpha} = K_{\varphi_1} \ .$$

Moreover, for a morphism $(\alpha,\beta):\varphi_1\to\varphi_2\in\mathrm{opTw}(\nabla)$ as above, if $x\in G_{m_2}$ is a right stabilizer of φ_2, then the pair (α^x,β) is again a morphism $\varphi_1\to\varphi_2$ in $\mathrm{opTw}(\nabla)$. Hence, by virtue of the first condition, this defines a left action of K_{φ_2} on the set $\mathrm{opTw}(\nabla)(\varphi_1,\varphi_2)$. It is easily verified that these data actually form a crossed $\mathrm{opTw}(\nabla)$-group structure on the family $K=\{K_\varphi\}_\varphi$.

Example 4.1.3. The family $\mathrm{RSt}^G=\{\mathrm{RSt}^G_\varphi\}_\varphi$ is itself a congruence family. Indeed, the first three conditions for congruence families are obvious. To verify (4.1.2), observe that, for $\varphi:\langle\!\langle m\rangle\!\rangle\to\langle\!\langle n\rangle\!\rangle\in\nabla$, $\psi:\langle\!\langle l\rangle\!\rangle\to\langle\!\langle l\rangle\!\rangle\nabla$, $x\in\mathrm{RSt}^G_\varphi$, and $y\in G_n$, we have

$$\varphi\psi^{\varphi^*(y)x\varphi^*(y)^{-1}}=(\varphi\psi^{x\varphi^*(y)^{-1}})^y=(\varphi\psi^{\varphi^*(y)^{-1}})^y=\varphi\psi\ .$$

Note that, in view of Remark 4.1.2, a congruence family on G is nothing but a crossed $\mathrm{opTw}(\nabla)$-subgroup of RSt^G satisfying (4.1.2).

Example 4.1.4. Let $\mathcal{G}$ be a group operad. Recall that, as pointed out in Section 1.2, morphisms $\varphi:\langle\!\langle m\rangle\!\rangle\to\langle\!\langle n\rangle\!\rangle\in\nabla$ correspond in one-to-one to tuples $\vec{k}^{(\varphi)}=(k^{(\varphi)}_{-\infty},k^{(\varphi)}_1,\dots,k^{(\varphi)}_n,k^{(\varphi)}_\infty)$ with $k^{(\varphi)}_{-\infty}+k^{(\varphi)}_1+\cdots+k^{(\varphi)}_n+k^{(\varphi)}_\infty=m$. We set a subset $\mathrm{Dec}^{\mathcal{G}}_\varphi\subset G_m$ to be the image of the map

$$\begin{array}{ccc}\mathcal{G}(k^{(\varphi)}_{-\infty})\times\mathcal{G}(k^{(\varphi)}_1)\times\cdots\times\mathcal{G}(k^{(\varphi)}_n)\times\mathcal{G}(k^{(\varphi)}_\infty) & \to & \mathcal{G}(m)\\ (x_{-\infty},x_1,\dots,x_n,x_\infty) & \mapsto & \gamma(e_{n+2};x_{-\infty},x_1,\dots,x_n,x_\infty)\end{array}\ . \tag{4.1.3}$$

As easily verified, the map (4.1.3) is actually a group homomorphism, and $\mathrm{Dec}^G=\{\mathrm{Dec}^G_\varphi\}_\varphi$ forms a crossed $\mathrm{opTw}(\nabla)$-subgroup. Moreover, for $y\in\mathcal{G}(n)$, we have

$$\begin{aligned}&\varphi^*(y)\cdot\gamma(e_{n+2};x_{-\infty},x_1,\dots,x_n,x_\infty)\\&=\gamma(e_3;x_{-\infty},\gamma(y;e_{k_1^{(\varphi)}},\dots,e_{k_n^{(\varphi)}})\gamma(e_n;x_1,\dots,x_n),x_\infty)\\&=\gamma(e_3;x_{-\infty},\gamma(e_n;x_{y^{-1}(1)},\dots,x_{y^{-1}(n)})\gamma(y;e_{k_1^{(\varphi)}},\dots,e_{k_n^{(\varphi)}}),x_\infty)\\&=\gamma(e_{n+2};x_{-\infty},x_{y^{-1}(1)},\dots,x_{y^{-1}(n)})\cdot\varphi^*(y)\ ,\end{aligned}$$

which implies

$$\varphi^*(y)\cdot\mathrm{Dec}^G_\varphi\cdot\varphi^*(y)^{-1}=\mathrm{Dec}^G_{\varphi^y}\ .$$

Thus, Dec^G is a congruence family on $\mathcal{G}$.

Example 4.1.5. For each $\varphi:\langle\!\langle m\rangle\!\rangle\to\langle\!\langle n\rangle\!\rangle\in\nabla$, we set Triv_φ to consists of a single element e_m which is seen as the unit of G_m for any crossed interval group G. Then, clearly $\mathrm{Triv}=\{\mathrm{Triv}_\varphi\}_\varphi$ is a congruence family on G. In view of Remark 4.1.2, Triv is the trivial crossed $\mathrm{opTw}(\nabla)$-group.

Example 4.1.6. Let K be a congruence family on a crossed interval group G. For each $\varphi:\langle\!\langle m\rangle\!\rangle\to\langle\!\langle n\rangle\!\rangle\in\nabla$, we have a canonical group homomorphism

$$K_\varphi\hookrightarrow\mathrm{RSt}^G_\varphi\hookrightarrow G_m\to\mathfrak{W}^\nabla_m\ .$$

We put K'_φ the kernel and claim that $K'=\{K'_\varphi\}_\varphi$ forms a congruence family. Namely, it is an uncrossed $\mathrm{opTw}(\nabla)$-subgroup of RSt^G since it is the kernel of the map $\mathrm{RSt}^G\to\mathrm{RSt}^{\mathfrak{W}^\nabla}$ of crossed $\mathrm{opTw}(\nabla)$-groups. This observation also leads to the equation (4.1.2).

We see congruence families on a crossed interval group G are associated to G-quotal categories. We use the following lemma.

Lemma 4.1.7. *Let G be a crossed interval group, and let $K = \{K_\varphi\}_\varphi$ be a congruence family on G. Suppose $\varphi : \langle\langle l\rangle\rangle \to \langle\langle m\rangle\rangle$ and $\psi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle$ are morphisms in ∇. Then, for each $y \in G_n$, the composition*

$$\begin{array}{ccccc} (K_\psi \cdot y) \times G_m & \hookrightarrow & G_n \times G_m & \to & G_m \\ & & (y', x) & \mapsto & \varphi^*(y') \cdot x \end{array}$$

induces a maps

$$\{K_\varphi \cdot y\} \times (K_\varphi \backslash G_m) \to (K_{\psi\varphi^y} \backslash G_m) \ .$$

Proof. Take $x \in G_m$ and $y \in G_n$. Then, for $u \in K_\varphi$ and $v \in K_\psi$, we have

$$\varphi^*(vy) \cdot ux = (\varphi^y)^*(v) \cdot (\varphi^*(y) \cdot u \cdot \varphi^*(y)^{-1}) \cdot \varphi^*(y)x \ . \tag{4.1.4}$$

By virtue of the conditions on the congruence family K, the first term in the right hand side of (4.1.4) belongs to $K_{\psi\varphi^y}$ while the second to $K_{\varphi^y} \subset K_{\psi\varphi^y}$. Hence, the result follows. □

Now, for a congruence family K on a crossed interval group G, we define a category $\mathcal{Q}_K$ as follows: the objects are the same as ∇, and for $m, n \in \mathbb{N}$,

$$\mathcal{Q}_K(\langle\langle m\rangle\rangle, \langle\langle n\rangle\rangle) = \{(\varphi, [x]) \mid \varphi \in \nabla(\langle\langle m\rangle\rangle, \langle\langle n\rangle\rangle), [x] \in K_\varphi \backslash G_m\} \ .$$

There is an obvious map $q : \nabla_G(\langle\langle m\rangle\rangle, \langle\langle n\rangle\rangle) \to \mathcal{Q}(\langle\langle m\rangle\rangle, \langle\langle n\rangle\rangle)$. Using Lemma 4.1.7 and the inclusion $K_\varphi \subset \mathrm{RSt}^G_\varphi$, one can see the composition in the total category ∇_G induces a composition operation in $\mathcal{Q}_K$ so that q is a functor.

Proposition 4.1.8. *For every congruence family K on a crossed interval group G, the functor*

$$q : \nabla_G \to \mathcal{Q}_K$$

given above exhibits $\mathcal{Q}_K$ as a G-quotal category. Moreover, the assignment $K \mapsto \mathcal{Q}_K$ gives a one-to-one correspondence between congruence families on G and (isomorphism classes of) G-quotal categories respecting the orders.

Proof. The first statement is obvious. To see the assignment is one-to-one, suppose $\mathcal{Q}$ is a G-quotal category with $q : \nabla_G \to \mathcal{Q}$. For each $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, we put

$$K^{\mathcal{Q}}_\varphi := \{x \in G_m \mid q(\varphi, x) = \varphi\} \ .$$

We assert $K^{\mathcal{Q}} = \{K^{\mathcal{Q}}_\varphi\}_\varphi$ forms a congruence family on G. First, clearly we have $K^{\mathcal{Q}}_\varphi \subset K^{\mathcal{Q}}_{\chi\varphi}$ and $\psi^*(K^{\mathcal{Q}}_\varphi) \subset K^{\mathcal{Q}}_{\varphi\psi}$ whenever the compositions make sense. Next, for $\varphi : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle$ and $\psi : \langle\langle l\rangle\rangle \to \langle\langle m\rangle\rangle$, and for each $x \in K^{\mathcal{Q}}_\varphi$, we have

$$\varphi\psi = q(\varphi, x)\psi = q(\varphi\psi^x, \psi^*(x)) \ , \tag{4.1.5}$$

here we identify morphisms in ∇ with their images in $\mathcal{Q}$. Since $\mathcal{Q}$ is G-quotal, (4.1.5) implies $\varphi\psi = \varphi\psi^x$, so we obtain $K^{\mathcal{Q}}_\varphi \subset \mathrm{RSt}^G_\varphi$. In addition, for $y \in G_n$ and $x \in K^{\mathcal{Q}}_\varphi$, we have

$$\begin{aligned} & q(\varphi^y, \varphi^*(y)x\varphi^*(y)^{-1}) \\ & \quad = q(\mathrm{id}, y)q(\varphi, x)q(\mathrm{id}, \varphi^*(y)^{-1}) = q(\mathrm{id}, y)q(\varphi, \varphi^*(y)^{-1}) = \varphi^y \ , \end{aligned}$$

which implies

$$\varphi^*(y) \cdot K^{\mathcal{Q}}_{\varphi} \cdot \varphi^*(y)^{-1} = K^{\mathcal{Q}}_{\varphi^y} \ .$$

Therefore, $K^{\mathcal{Q}}$ is a congruence family.

It is straightforward that $\mathcal{Q} \mapsto K^{\mathcal{Q}}$ is an inverse assignment to $K \mapsto \mathcal{Q}_K$. Furthermore, if we have an inclusion $K \subset K'$ between congruence families, i.e. $K_\varphi \subset K'_\varphi$ for each morphism φ in ∇, then there is a functor $\mathcal{Q}_K \to \mathcal{Q}_{K'}$ which makes the following diagram commutes:

$$\begin{array}{ccc} & \nabla_G & \\ \swarrow & & \searrow \\ \mathcal{Q}_K & \longrightarrow & \mathcal{Q}_{K'} \end{array}$$

In other words, the assignment $K \mapsto K'$ respects the orders, and this completes the proof. □

To end the section, we mention a closure operator on the partially ordered set of congruence families. Recall that we mentioned two classes of morphisms in ∇ in Section 3.4; namely *inert* and *active* morphisms, and every morphism in ∇ uniquely factors through an inert morphism followed by an active one according to Lemma 3.4.2. In other words, if we put I and A the classes of inert and active morphisms in ∇ respectively, then (I, A) is an *orthogonal factorization system*; i.e. it satisfies the following conditions

(i) the classes I and A are closed under compositions and contains all the isomorphisms;

(ii) every morphisms in ∇ is of the form $\mu\rho$ with $\rho \in \mathsf{I}$ and $\mu \in \mathsf{A}$;

(iii) for every commutative square

$$\begin{array}{ccc} \langle\!\langle k \rangle\!\rangle & \xrightarrow{\varphi} & \langle\!\langle m \rangle\!\rangle \\ {\scriptstyle\rho}\downarrow & \overset{\exists!\chi}{\nearrow} & \downarrow{\scriptstyle\mu} \\ \langle\!\langle l \rangle\!\rangle & \xrightarrow{\psi} & \langle\!\langle n \rangle\!\rangle \end{array}$$

with $\rho \in \mathsf{I}$ and $\mu \in \mathsf{A}$, there is a *unique* diagonal χ so that $\chi\rho = \varphi$ and $\mu\chi = \psi$.

Remark 4.1.9. The notion was first introduced by Freyd and Kelly in [25] under the name *factorization*. We instead use the name above to emphasize the *unique* lifting property and to distinguish it from *weak factorization systems*.

Let G be a crossed interval group, and let K be a congruence family on G. Using Lemma 3.4.2, we construct another congruence family $\overline{K}$ as follows: for each active morphism μ in ∇, we put $\overline{K}_\mu = K_\mu$. For a general morphism φ in ∇, we set $\overline{K}_\varphi \subset \mathrm{RSt}^G_\varphi$ to consist $x \in \mathrm{RSt}^G_\varphi$ such that, for every morphism ψ with $\varphi\psi$ making sense and active, $\psi^*(x) \in \mathrm{RSt}^G_{\varphi\psi}$ belongs to $\overline{K}_{\varphi\psi}$. Thanks to Lemma 3.4.2, this extension does not change $\overline{K}_\mu$ for active μ.

Lemma 4.1.10. *In the situation above, the family $\overline{K} = \{\overline{K}_\varphi\}_\varphi$ forms a congruence family on G.*

Proof. We first verify $\overline{K}_\varphi \subset \mathrm{RSt}^G_\varphi$ forms a subgroup. Suppose ψ is a morphism in ∇ with $\varphi\psi$ making sense and active. For two elements $x, y \in \overline{K}_\varphi$, we have

$$\psi^*(x^{-1}y) = (\psi^{x^{-1}y})^*(x)^{-1}\psi^*(y) \ . \tag{4.1.6}$$

Since $x, y \in \mathrm{RSt}^G_\varphi$, the composition $\varphi\psi^{x^{-1}y}$ equals to $\varphi\psi$, which is active. Hence, both terms in the right hand side of (4.1.6) belongs to K_φ, which implies $x^{-1}y \in \overline{K}_\varphi$.

Using Lemma 3.4.2, one can verify $\overline{K} = \{\overline{K}_\varphi\}_\varphi$ forms a crossed $\mathrm{opTw}(\nabla)$-subgroup of RSt^G. It remains to verify the formula (4.1.2). Clearly, the inclusion in one direction will suffice, so we show

$$\varphi^*(y) \cdot \overline{K}_\varphi \cdot \varphi^*(y)^{-1} \subset \overline{K}_{\varphi^y} \tag{4.1.7}$$

for each $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$ and $y \in G_n$. Suppose $\psi : \langle\!\langle l \rangle\!\rangle \to \langle\!\langle m \rangle\!\rangle \in \nabla$ is a morphism with $\varphi^y\psi$ active. For $x \in \overline{K}_\varphi$, we have

$$\begin{aligned} \psi^*(\varphi^*(y)x\varphi^*(y)^{-1}) &= (\varphi\psi^{x\varphi^*(y)^{-1}})^*(y) \cdot (\psi^{\varphi^*(y)^{-1}})^*(x) \cdot \psi^*(\varphi^*(y)^{-1}) \\ &= (\varphi\psi^{\varphi^*(y)^{-1}})^*(y) \cdot (\psi^{\varphi^*(y)^{-1}})^*(x) \cdot (\varphi^y\psi)^*(y^{-1}) \\ &= (\varphi^y\psi)^*(y^{-1})^{-1} \cdot (\psi^{\varphi^*(y)^{-1}})^*(x) \cdot (\varphi^y\psi)^*(y^{-1}) \end{aligned} \tag{4.1.8}$$

Note that, since $\varphi\psi^{\varphi^*(y)^{-1}} = (\varphi^y\psi)^{y^{-1}}$ is active, the middle term in the right hand side of (4.1.8) belongs to $K_{(\varphi^y\psi)^{y^{-1}}}$. Using the formula (4.1.2) for the congruence family K, one gets

$$(\varphi^y\psi)^*(y^{-1})^{-1} \cdot K_{(\varphi^y\psi)^{y^{-1}}} \cdot (\varphi^y\psi)^*(y^{-1}) = K_{\varphi^y\psi} \ .$$

This implies that $\psi^*(\varphi^*(y)x\varphi^*(y)^{-1}) \in K_{\varphi^y\psi}$, and the inclusion (4.1.7) follows. □

Lemma 4.1.11. *Let G be a crossed interval group. Then, the assignment $K \mapsto \overline{K}$ defines a closure operator on the ordered set of congruence families on G.*

Proof. The assignment $K \mapsto \overline{K}$ clearly respects the inclusions. Moreover, since $\overline{K}_\mu = K_\mu$ for active morphisms μ, we also have $\overline{\overline{K}}_\varphi = \overline{K}_\varphi$ for every morphism φ. On the other hand, we have $K_\varphi \subset \mathrm{RSt}^G_\varphi$, and for every morphism ψ with $\varphi\psi$ making sense and active, $\psi^*(K_\varphi) \subset K_{\varphi\psi}$. This implies $K_\varphi \subset \overline{K}_\varphi$. Thus, we obtain the result. □

Definition. A congruence family K on a crossed interval group G is said to be *proper* if it is closed with respect to the closure operator $\overline{(-)}$ defined above in the ordered set of congruence families on G; i.e. $\overline{K} = K$.

Every crossed interval group G admits the minimum proper congruence family; namely the closure $\overline{\mathrm{Triv}}$ of the trivial congruence family given in Example 4.1.5. We write $\mathrm{Inr}^G := \overline{\mathrm{Triv}}$. Hence, every proper congruence family on G contains Inr^G. Moreover, it satisfies the following properties.

Lemma 4.1.12. *Let G be a crossed interval group.*

(1) *If a composition* $\varphi\psi$ *in* ∇ *is active, then the action of* $\mathrm{Inr}_{\varphi}^{G}$ *stabilizes* ψ*.*

(2) *For every morphism* φ *in* ∇*, the subgroup* $\mathrm{Inr}_{\varphi}^{G} \subset \mathrm{RSt}_{\varphi}^{G}$ *is normal.*

(3) *Let* K *be a proper congruence family on* G*. Then, for an active morphism* μ *and for an inert morphism* ρ *with* $\varphi\rho$ *making sense, the composition*

$$K_{\mu} \xrightarrow{\rho^{*}} K_{\mu\rho} \twoheadrightarrow K_{\mu\rho}/\,\mathrm{Inr}_{\mu\rho}^{G} \tag{4.1.9}$$

is bijective.

Proof. We first show (1). Take the factorization $\varphi = \mu\rho$ with ρ inert and μ active, and let δ be the unique section of ρ. Notice that, in view of Lemma 3.4.2, δ is characterized by the following two properties:

(i) $\varphi\delta$ is active;

(ii) every morphism ψ with $\varphi\psi$ making sense and active uniquely factors as $\psi = \delta\psi'$ for a morphism ψ'.

It follows that δ is fixed by the action of $\mathrm{RSt}_{\varphi}^{G}$. Moreover, if $\varphi\psi$ is active, the property above implies there is a morphism ψ' with $\psi = \delta\psi$. Then, for each $x \in \mathrm{Inr}_{\varphi}^{G}$, we have

$$\psi^{x} = (\delta\psi')^{x} = \delta^{x}\psi'^{\delta^{*}(x)} = \delta\psi' = \psi \ ,$$

so that (1) follows.

Next, suppose $u \in \mathrm{Inr}_{\varphi}^{G}$ and $x \in \mathrm{RSt}_{\varphi}^{G}$. For every morphism ψ with $\varphi\psi$ making sense and active, we have

$$\psi^{*}(xux^{-1}) = (\psi^{ux^{-1}})^{*}(x)(\psi^{x^{-1}})^{*}(u)\psi^{*}(x^{-1}) = (\psi^{ux^{-1}})^{*}(x)$$

Note that $\varphi\psi^{x^{-1}} = \varphi\psi$ is active, so $(\psi^{x^{-1}})^{*}(u)$ is the unit. Moreover, the part (1) implies $\psi^{ux^{-1}} = \psi^{x^{-1}}$. It follows that $\psi^{*}(xux^{-1})$ vanishes, and we obtain (2).

Finally, we show (3). Let $\delta : \langle\langle m\rangle\rangle \to \langle\langle l\rangle\rangle \in \nabla$ be the unique section of ρ. We assert that the map $\delta^{*} : K_{\mu\rho} \to K_{\mu}$ induces the inverse of (4.1.9). Indeed, for each $u \in \mathrm{Inv}_{\mu\rho}^{G}$, the definition of $\mathrm{Inv}_{\mu\rho}^{G}$ and the part (1) imply $\delta^{*}(u) = e$ and $\delta^{u} = \delta$. Hence, for every $x \in K_{\mu\rho}$, $\delta^{*}(xu) = \delta^{*}(x)$. In other words, δ^{*} is $\mathrm{Inv}_{\mu\rho}^{G}$-invariant so that it induces a map

$$\delta^{\dagger} : K_{\mu\rho}/\,\mathrm{Inr}_{\mu\rho}^{G} \to K_{\mu} \ .$$

Since δ is a section of ρ, the map is clearly a left inverse of the map (4.1.9). To see it is also a right inverse, it is enough to see that, for each $x \in K_{\mu\rho}$, we have $\rho^{*}(\delta^{*}(x))x^{-1} \in \mathrm{Inr}_{\mu\rho}^{G}$. Note that, by virtue of the characterization of δ above, this holds if and only if the map δ^{*} vanishes the element. We have

$$\delta^{*}(\rho^{*}(\delta^{*}(x)) \cdot x^{-1}) = (\delta\rho\delta^{x^{-1}})^{*}(x) \cdot \delta^{*}(x^{-1}) \ .$$

As mentioned above, δ is fixed by the action of $\mathrm{RSt}_{\mu\rho}^{G}$, so we have $\delta\rho\delta^{x^{-1}} = \delta$ and $\delta^{*}(x^{-1}) = \delta^{*}(x)^{-1}$. Thus, $\delta^{*}(\rho^{*}(\delta^{*}(x)) \cdot x^{-1}) = e$, and we conclude $\delta^{\dagger}$ is a right inverse of (4.1.9), which completes the proof. $\square$

4.2 Associated double categories

In the previous section, we see that congruence families are associated with quotal categories by taking the quotients of the total category. Our problem is, on the other hand, higher categorical so we need "higher categorical quotients" in some sense.

Recall that a *double category* is a category internal to the category **Cat**; i.e. a diagram

$$\mathfrak{C} \overset{s,t}{\rightrightarrows} \mathcal{B}$$

in the category **Cat** of small categories together with functors

$$\gamma : \mathfrak{C} \times_{\mathcal{B}} \mathfrak{C} \to \mathfrak{C} \quad \text{and} \qquad \iota : \mathcal{B} \to \mathfrak{C}$$

satisfying the appropriate conditions which is imposed on ordinary categories, where the domain of γ is the pullback of the cospan $\mathfrak{C} \xrightarrow{s} \mathcal{B} \xleftarrow{t} \mathfrak{C}$. Hence, a double category has two kinds of compositions; namely the *horizontal composition* $\circ_{\mathsf{H}}$, the compositions in the categorical structures of $\mathfrak{C}$ and $\mathcal{B}$, and the *vertical composition* $\circ_{\mathsf{V}}$ given by the functor γ. For morphisms f, g, h, k in $\mathfrak{C}$, we have

$$(f \circ_{\mathsf{H}} g) \circ_{\mathsf{V}} (h \circ_{\mathsf{H}} k) = (f \circ_{\mathsf{V}} h) \circ_{\mathsf{H}} (g \circ_{\mathsf{V}} k)$$

whenever both sides make sense.

Remark 4.2.1. In what follows, we will often drop the structure functors γ and ι from the notation. For example, in the case above, we say $\mathfrak{C} \rightrightarrows \mathcal{B}$ is a double category.

Let G be a crossed interval group, and suppose we have a pair (K, L) of proper congruence families satisfying the following conditions:

(♠1) for each morphism $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$, the subgroup $K_\varphi \subset G_m$ is contained in the normalizer subgroup $N(L_\varphi)$ of L_φ; i.e.

$$N(L_\varphi) = \left\{ x \in G_m \mid xL_\varphi x^{-1} = L_\varphi \right\} \ ;$$

(♠2) if $\psi\varphi$ is a composition of morphisms in ∇, for every $u \in L_\psi$ and for each $x \in K_\varphi$,

$$[\varphi^*(u)x\varphi^*(u)^{-1}] = [x] \in \mathrm{Inr}^G_{\psi\varphi} \backslash K_{\psi\varphi} \ .$$

In this case, we define a category $\mathcal{Q}_{L/\!/K}$ as follows:

- the objects are the same as ∇;
- for $m, n \in \mathbb{N}$, morphisms $\langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle$ in $\mathcal{Q}_{L/\!/K}$ are triples $(\varphi, [u], [x])$ with $\varphi \in \nabla(\langle\!\langle m \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle)$, $[u] \in \mathrm{Inr}^G_\varphi \backslash K_\varphi$, and $[x] \in L_\varphi \backslash G_m$;
- the composition is given by

$$(\psi, [v], [y]) \circ (\varphi, [u], [x]) = (\psi\varphi^y, [\varphi^*(vy)u\varphi^*(y)^{-1}], [\varphi^*(y)x]) \ .$$

Note that we have

$$\varphi^*(vy)u\varphi^*(y)^{-1} = (\varphi^y)^*(v) \cdot \varphi^*(y)u\varphi^*(y)^{-1} \ ,$$

so the conditions on congruence families imply the element belongs to $K_{\psi\varphi^y}$. In addition, (2) in Lemma 4.1.12 and the condition (♠2) guarantee that the composition does not depend on the choice of representatives. If we are given another morphism $(\chi, [w], [z])$ postcomposable with $(\psi, [v], [y])$, the second component of the composition

$$((\chi, [w], [z]) \circ (\psi, [v], [y])) \circ (\varphi, [u], [x])$$

is represented by the element

$$\begin{aligned}
&\varphi^*(\psi^*(wz)v\psi^*(z)^{-1}\psi^*(z)y)u\varphi^*(\psi^*(z)y)^{-1} \\
&= (\psi\varphi^{vy})^*(wz)\varphi^*(vy)u\left((\psi\varphi^y)^*(z)\varphi^*(y)\right)^{-1} \\
&= (\psi\varphi^y)^*(wz)\varphi^*(vy)u\varphi^*(y)^{-1}(\psi\varphi^y)^*(z)^{-1} \ ,
\end{aligned}$$

which also represents the second component of the other composition. Thanks to this and the associativity of morphisms in $\mathbb{E}_G$, one obtains the associativity of the composition in $\mathbb{G}_{L/\!/K}$ so that it is actually a category.

Example 4.2.2. For every proper congruence family L, the pair (Inr^G, L) satisfies the conditions (♠1) and (♠2). One can verify that there is a canonical isomorphism $\mathcal{Q}_{L/\!/\mathrm{Inr}^G} \cong \mathcal{Q}_L$.

Example 4.2.3. Let $\mathcal{G}$ be a group operad, so we have the congruence family $\mathrm{Dec}^{\mathcal{G}}$ given in Example 4.1.4. For each morphism $\varphi : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$, we set $\mathrm{Kec}^{\mathcal{G}}_\varphi \subset \mathrm{Dec}^{\mathcal{G}}$ to be the kernel of the composition

$$\mathrm{Dec}^{\mathcal{G}}_\varphi \hookrightarrow \mathcal{G}(m) \to \mathfrak{S}(m) \ .$$

In view of Example 4.1.6, the family $\mathrm{Kec}^{\mathcal{G}} = \{\mathrm{Kec}^{\mathcal{G}}_\varphi\}_\varphi$ forms a congruence family on $\mathcal{G}$. Taking the closure in the sense of Lemma 4.1.11, we obtain proper congruence families $\overline{\mathrm{Dec}}^{\mathcal{G}}$ and $\overline{\mathrm{Kec}}^{\mathcal{G}}$. One can verify that the pair $(\overline{\mathrm{Dec}}^{\mathcal{G}}, \overline{\mathrm{Kec}}^{\mathcal{G}})$ satisfies the conditions (♠1) and (♠2) so that they give rise to a category $\mathbb{G}_{\overline{\mathrm{Kec}}^{\mathcal{G}}/\!/\overline{\mathrm{Dec}}^{\mathcal{G}}}$.

The category $\mathcal{Q}_{L/\!/K}$ comes equipped with two canonical functors

$$s, t : \mathcal{Q}_{L/\!/K} \rightrightarrows \mathcal{Q}_L \ , \tag{4.2.1}$$

here $\mathcal{Q}_L$ is the G-quotal category associated with L, such that

- they are the identities on objects;
- for each morphism $(\varphi, [u], [x]) \in \mathcal{Q}_{L/\!/K}(\langle\!\langle m\rangle\!\rangle, \langle\!\langle n\rangle\!\rangle)$,

$$s(\varphi, [u], [x]) = (\varphi, [x]) \ , \quad t(\varphi, [u], [x]) = (\varphi, [ux]) \ .$$

Note that the assignment t does not depend on the choice of representatives by virtue of the condition (♠1). Then, the functorialities are easily verified. We assert that the diagram (4.2.1) canonically admits a structure of a double category: define functors $\gamma : \mathcal{Q}_{L/\!/K} \times_{\mathcal{Q}_L} \mathcal{Q}_{L/\!/K} \to \mathcal{Q}_{L/\!/K}$ and $\iota : \mathcal{Q}_L \to \mathcal{Q}_{L/\!/K}$ by

$$\gamma\left((\varphi, [u], [u'x]), (\varphi, [u'], [x])\right) := (\varphi, [uu'], [x]) \ , \quad \iota(\varphi, [x]) := (\varphi, [e], [x]) \ ,$$

where e is the unit in the group K_φ. These actually define functors thanks to (2) in Lemma 4.1.12, and the associativity and the unitality are obvious. We call the double category (4.2.1) the *double category associated to the pair* (K, L).

Remark 4.2.4. In the case $L \subset K$, the double category $\mathcal{Q}_{L/\!/K} \rightrightarrows \mathcal{Q}_L$ looks like a "homotopy quotient" of the category $\mathcal{Q}_L$ with respect to the congruence family K in the following sense: since the functors (4.2.1) are the identities on objects, one can see the double category as a 2-category, say $\mathbf{Q}_{L/\!/K}$. For each $m, n \in \mathbb{N}$, the category $\mathbf{Q}_{L/\!/K}(\langle\!\langle m\rangle\!\rangle, \langle\!\langle n\rangle\!\rangle)$ is a groupoid whose isomorphism classes corresponds in one-to-one to morphisms $\langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle$ in the G-quotal category $\mathcal{Q}_K$ associated with K. We will make further discussions in this point of view in Section 5.1.

We further take a quotient of the double category $\mathcal{Q}_{L/\!/K} \rightrightarrows \mathcal{Q}_L$ using the following general construction.

Proposition 4.2.5. *Let $\mathcal{A}$ be a category, and let M be a left cancellative class of morphisms in $\mathcal{A}$; i.e. if a composition $\delta\varepsilon$ belongs to M, so does δ. For each $a, b \in \mathcal{A}$, define a relation $\sim_{\mathsf{M}}$ on the set $\mathcal{A}(a,b)$ such that $\alpha \sim_{\mathsf{M}} \alpha'$ if and only if, for each morphism β in $\mathcal{A}$ with codomain a, one has $\alpha\beta = \alpha'\beta$ as soon as either of the sides belongs to M. Then, the relation $\sim_{\mathsf{M}}$ is a congruence on $\mathcal{A}$ in the sense in II.8 of [55]. Consequently, taking the quotient of each hom-set of $\mathcal{A}$ by $\sim_{\mathsf{M}}$, one gets a quotient category $\mathcal{A} \to \mathcal{A}/{\sim_{\mathsf{M}}}$.*

Proof. It is obvious that $\sim_{\mathsf{M}}$ is an equivalence relation on each hom-set $\mathcal{A}(a,b)$. We have to show, for compositions $\alpha\beta\gamma$ and $\alpha\beta'\gamma$ with $\beta \sim_{\mathsf{M}} \beta'$, we have $\alpha\beta\gamma \sim_{\mathsf{M}} \alpha\beta\gamma$. Suppose a composition $\alpha\beta\gamma\delta$ belongs to M. By virtue of the observation above, we have $\beta\gamma\delta \in \mathsf{M}$, so $\beta \sim_{\mathsf{M}} \beta'$ implies $\beta\gamma\delta = \beta'\gamma\delta$. Thus, we obtain $\alpha\beta\gamma\delta = \alpha\beta'\gamma\delta$ and conclude $\alpha\beta\gamma \sim_{\mathsf{M}} \alpha\beta'\gamma$. ☐

Remark 4.2.6. Typical examples of left cancellative class of morphisms come from orthogonal factorization systems (see the discussion in page 96). Suppose (E, M) is an orthogonal factorization system on a category $\mathcal{A}$. One can see that the class M is left cancellative provided every morphisms in E is an epimorphism. Indeed, if $\varepsilon = \mu\rho$ and $\delta\mu = \nu\sigma$ are factorizations with $\mu, \nu \in \mathsf{M}$ and $\rho, \sigma \in \mathsf{E}$, then the equation $\delta\varepsilon \circ \mathrm{id} = \nu \circ \sigma\rho$ and the unique lifting property implies $\sigma\rho$ is an isomorphism. Since σ is an epimorphism by the assumption on E, ρ is an isomorphism so that $\varepsilon \in \mathsf{M}$.

We apply Proposition 4.2.5 to the category $\mathcal{Q}_{L/\!/K}$. To obtain a left cancellative classes on it, in view of Remark 4.2.6, we construct orthogonal factorization system. We define two classes $\mathsf{I}_{L/\!/K}$ and $\mathsf{A}_{L/\!/}$ of morphisms in $\mathcal{Q}_{L/\!/K}$ as follows:

$$\mathsf{I}_{L/\!/K} := \{(\varphi, [u], [x]) \mid \varphi\colon \text{inert}\} \ ,$$
$$\mathsf{A}_{L/\!/K} := \{(\varphi, [u], [x]) \mid \varphi\colon \text{active}\} \ .$$

We assert that $(\mathsf{I}_{L/\!/K}, \mathsf{A}_{L/\!/K})$ forms an orthogonal factorization system on $\mathbb{Q}_{L/\!/K}$. In fact, if we have a composition $\mu\rho : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$ with μ active and ρ inert, then for each $u \in K_{\mu\rho}$ and $x \in G_m$, one can use (3) in Lemma 4.1.12 to find a unique element $\bar{u} \in K_\mu$ so that

$$(\mu\rho, [u], [x]) = (\mu, [\bar{u}], [e]) \circ (\rho, [e_m], [x]) \ .$$

As easily verified, the factorization is unique up to a unique isomorphism, and we conclude $(\mathsf{I}_{L/\!/K}, \mathsf{A}_{L/\!/K})$ is an orthogonal factorization. Since every member of $\mathsf{I}_{L/\!/K}$ is a split epimorphism, Lemma 3.4.2 implies $\mathsf{A}_{L/\!/K}$ is left cancellative.

We denote by $\widetilde{\mathcal{Q}}_{L/\!\!/K}$ the quotient category of $\mathcal{Q}_{L/\!\!/K}$ obtained by Proposition 4.2.5 with the class $\mathsf{A}_{L/\!\!/K}$. In particular, as mentioned in Example 4.2.2, there is an isomorphism $\mathcal{Q}_L \cong \mathcal{Q}_{L/\!\!/\mathrm{Inr}^G}$. We take a quotient category $\mathcal{Q}_L \twoheadrightarrow \widetilde{\mathcal{Q}}_L$ which corresponding to $\widetilde{\mathcal{Q}}_{L/\!\!/\mathrm{Inr}^G}$ through the isomorphism. To simplify the notation, we set $(\mathsf{I}_L, \mathsf{A}_L)$ the orthogonal factorization system corresponds to $(\mathsf{I}_{L/\!\!/\mathrm{Inr}^G}, \mathsf{A}_{L/\!\!/\mathrm{Inr}^G})$.

Remark 4.2.7. We have a convenient criterion for the congruence $\sim_{\mathsf{A}_{L/\!\!/K}}$. Suppose $\varphi, \varphi' : \langle\!\langle m\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle \in \nabla$, $x, x' \in G_m$, $u \in K_\varphi$, and $u' \in K_{\varphi'}$. Then, we have $(\varphi, [u], [x]) \sim_{\mathsf{A}_{L/\!\!/K}} (\varphi', [u'], [x'])$ if and only if for every $\psi : \langle\!\langle l\rangle\!\rangle \to \langle\!\langle m\rangle\!\rangle$ with either $\varphi\psi^x$ or $\varphi'\psi^{x'}$ active, the following equations hold:

$$\begin{aligned}\varphi\psi^x = \varphi'\psi^{x'} &\in \nabla(\langle\!\langle l\rangle\!\rangle, \langle\!\langle n\rangle\!\rangle)\\ [\psi^*(x)] = [\psi^*(x')] &\in L_{\varphi\psi^x}\backslash G_l\\ [(\psi^x)^*(u)] = [(\psi^{x'})^*(u')] &\in \mathrm{Inr}^G_{\varphi\psi^x}\backslash K_{\varphi\psi^x} \ .\end{aligned}$$

Furthermore, let $\varphi = \mu\rho$ and $\varphi' = \mu'\rho'$ be the factorization with μ, μ' active and ρ, ρ' inert, and say δ and δ' are the unique sections of ρ and ρ' respectively. Then, it turns out that we only have to test the conditions above in the cases $\psi = \delta^{x^{-1}}$ and $\psi = \delta'^{x'^{-1}}$.

Example 4.2.8. Let G be a crossed interval group and L a proper congruence family. For $1 \le i \le n$, define $\rho_i : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle 1\rangle\!\rangle$ to be the inert morphism such that

$$\rho_i(j) = \begin{cases} -\infty & j < i \ , \\ 1 & j = i \ , \\ \infty & j > i \ , \end{cases}$$

and put δ_i the unique section of ρ_i. Then, for each $x \in G_n$, we have

$$(\rho_{x(i)}, [x]) \sim_{\mathsf{A}_L} (\rho_i, [\rho_i^*\delta_i^*(x)]) \in \mathcal{Q}_L(\langle\!\langle n\rangle\!\rangle, \langle\!\langle 1\rangle\!\rangle) \ . \tag{4.2.2}$$

Indeed, since $\rho_i^*\delta_i^*(x) \in G_n$ acts trivially on active morphisms, we have

$$\begin{aligned}\rho_{x(i)}\delta_{x(i)} = \mathrm{id}_{\langle\!\langle 1\rangle\!\rangle} = \rho_i\delta_i = \rho_i\delta_{x(i)}^{x^{-1}} = \rho_i\delta_{x(i)}^{\rho_i^*(x)\delta_i^*(x)x^{-1}}\\ (\delta_{x(i)}^{x^{-1}})^*(x) = \delta_i^*(x) = \delta_i^*\left((\rho_i\delta_i)^*(x)\right) \ .\end{aligned}$$

Hence, (4.2.2) follows from the argument in Remark 4.2.7.

Example 4.2.9. Let G be a crossed interval group with G_0 trivial, and let (K, L) be a pair of proper congruence family on G satisfying (♠1) and (♠2). We assert that the object $\langle\!\langle 0\rangle\!\rangle \in \widetilde{\mathcal{Q}}_{L/\!\!/K}$ is a terminal object. Indeed, it is obvious that, for each $n \in \mathbb{N}$, there is at least one morphism $\langle\!\langle n\rangle\!\rangle \to \langle\!\langle 0\rangle\!\rangle \in \widetilde{\mathcal{Q}}_{L/\!\!/K}$. On the other hand, for a morphism $[\varphi, u, x] : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle 0\rangle\!\rangle \in \widetilde{\mathcal{Q}}_{L/\!\!/K}$, a composition $\varphi\psi$ in ∇ is active exactly when ψ is the unique morphism $\psi : \langle\!\langle 0\rangle\!\rangle \to \langle\!\langle n\rangle\!\rangle$. Since ψ is fixed by the action of G_n, we have

$$\begin{aligned}\varphi\psi = \mathrm{id}_{\langle\!\langle 0\rangle\!\rangle}\\ \psi^*(u) = \psi^*(x) = e_0 \ .\end{aligned}$$

The right hand sides depend neither on φ, u, nor x. This implies that there is at most one morphism $\langle\!\langle n\rangle\!\rangle \to \langle\!\langle 0\rangle\!\rangle \in \widetilde{\mathcal{Q}}_{L/\!\!/K}$.

We can see the quotient functor $\mathcal{Q}_{L/\!/K} \to \widetilde{\mathcal{Q}}_{L/\!/K}$ transfers the orthogonal factorization $(\mathsf{I}_{L/\!/K}, \mathsf{A}_{L/\!/K})$. Indeed, writing $\widetilde{\mathsf{I}}_{L/\!/K}$ and $\widetilde{\mathsf{A}}_{L/\!/K}$ the images of $\mathsf{I}_{L/\!/K}$ and $\mathsf{A}_{L/\!/K}$ in $\widetilde{\mathcal{Q}}_{L/\!/K}$ respectively, we obtain the following result.

Lemma 4.2.10. *Let G be a crossed interval group, and suppose (K, L) is a pair of proper congruence families satisfying the property (♠1) and (♠2). Then, as for morphisms in $\widetilde{\mathcal{Q}}_{L/\!/K}$, the following hold.*

(1) *Every morphism can be written as a composition of the form*

$$[\mu, u, e] \circ [\rho, e, x]$$

with μ active and ρ inert.

(2) *We have an equation*

$$[\mu, u, e] \circ [\rho, e, x] = [\nu, v, e] \circ [\pi, e, y] \tag{4.2.3}$$

with μ, ν active and ρ, π inert precisely when the following two are satisfied:

(i) *$(\mu, [u], [e]) = (\nu, [v], [e])$ as morphisms in $\mathcal{Q}_{L/\!/K}$;*

(ii) *there is an element $w \in L_\mu$ such that*

$$[\rho, e, \rho^*(w)x] = [\pi, e, y] \ .$$

Consequently, the pair $\left(\widetilde{\mathsf{I}}_{K/\!/L}, \widetilde{\mathsf{A}}_{K/\!/L}\right)$ forms an orthogonal factorization system on $\widetilde{\mathcal{Q}}_{K/\!/L}$.

Proof. The assertion (1) directly follows from the definition of $\widetilde{\mathcal{Q}}_{L/\!/K}$ and (3) in Lemma 4.1.12.

As for the part (2), the two conditions (i) and (ii) clearly implies 4.2.3. Conversely, suppose 4.2.3 holds; so we have

$$[\mu\rho, \rho^*(u), x] = [\nu\pi, \pi^*(v), y] \ .$$

Put δ and ε the sections of ρ and π respectively, and we obtain

$$\begin{aligned} \mu &= \nu\pi\delta^{yx^{-1}} \ , \quad \delta^*(yx^{-1}) \in L_\mu \ , \quad [u] = [(\delta^{yx^{-1}})^*\pi^*(v)] \in \operatorname{Inr}^G_\mu \backslash K_\mu \ , \\ \nu &= \mu\rho\varepsilon^{xy^{-1}} \ , \quad \varepsilon^*(xy^{-1}) \in L_\nu \ , \quad [v] = [(\varepsilon^{xy^{-1}})^*\rho^*(u)] \in \operatorname{Inr}^G_\nu \backslash K_\nu \ . \end{aligned}$$

By virtue of the unique factorization in ∇ and the uniqueness of the sections of ρ and π, the leftmost equations imply $\delta^{x^{-1}} = \varepsilon^{y^{-1}}$ and $\mu = \nu$. In this case, the rightmost equations just say $[u] = [v] \in \operatorname{Inr}^G_\mu \backslash K_\mu$, and the condition (i) follows. On the other hand, in view of Remark 4.2.7, one can see

$$[\rho, e, \rho^*\delta^*(yx^{-1})x] = [\pi, e, y] \ ,$$

which supplies the element w in the condition (ii) thanks to the middle membership relations.

We finally show the last assertion. The part (1) precisely implies the possibility of the factorization. Since the classes $\widetilde{\mathsf{I}}_{L/\!/K}$ and $\widetilde{\mathsf{A}}_{L/\!/K}$ are clearly closed under compositions, it remains to show the unique lifting property. Note that since every morphism in $\widetilde{\mathsf{I}}_{L/\!/K}$ is isomorphic to a morphism of the form

$$\rho = [\rho, e, e]$$

with ρ being an inert morphism in ∇. Similarly, every morphism in $\widetilde{\mathsf{A}}_{L/\!/K}$ is isomorphic to a morphism of the form

$$[\mu, u, e]$$

with μ active. Hence, it suffices to solve the unique lifting problem on the commutative squares of the form

$$\begin{array}{ccc} \langle\!\langle k \rangle\!\rangle & \xrightarrow{\boldsymbol{\varphi}} & \langle\!\langle m \rangle\!\rangle \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle[\mu,u,e]} \\ \langle\!\langle l \rangle\!\rangle & \xrightarrow{\boldsymbol{\psi}} & \langle\!\langle n \rangle\!\rangle \end{array} \tag{4.2.4}$$

in $\widetilde{\mathcal{Q}}_{L/\!/K}$. By the part (1), we may put

$$\boldsymbol{\varphi} = [\kappa, s, e] \circ [\sigma, e, x] \ , \quad \boldsymbol{\psi} = [\lambda, t, e] \circ [\tau, e, y]$$

with κ, λ active and σ, τ inert. Then, the commutative square (4.2.4) implies

$$[\mu\kappa, \kappa^*(u)s, e] \circ [\sigma, e, x] = [\lambda, t, e] \circ [\tau, e, y] \circ \rho \ .$$

By virtue of the part (2), we have

$$\mu\kappa = \lambda \ , \quad [\kappa^*(u)s] = [t] \in \mathrm{Inr}^G_\lambda \backslash K_\lambda \ , \tag{4.2.5}$$

and there is an element $w \in L_\lambda$ such that

$$[\sigma, e, x] = [\tau, e, \tau^*(w)y] \circ \rho \ . \tag{4.2.6}$$

We set

$$\boldsymbol{\chi} := [\kappa, s, w] \circ [\tau, e, y] : \langle\!\langle l \rangle\!\rangle \to \langle\!\langle m \rangle\!\rangle \ .$$

Then, thanks to the equations (4.2.5) and (4.2.6), one can verify $\boldsymbol{\chi}$ is actually a diagonal filler in (4.2.4); i.e.

$$\boldsymbol{\chi} \circ \rho = \boldsymbol{\varphi} \ , \quad [\mu, u, e] \circ \boldsymbol{\chi} = \boldsymbol{\psi} \ .$$

In addition, since ρ is a split epimorphism, $\boldsymbol{\chi}$ is unique. It follows that the pair $(\widetilde{\mathsf{I}}_{L/\!/K}, \widetilde{\mathsf{A}}_{L/\!/})$ satisfies the unique lifting property so it is an orthogonal factorization system on $\widetilde{\mathcal{Q}}_{L/\!/K}$, as required. □

Example 4.2.11. As a consequence of Lemma 4.2.10, for active morphisms $\mu, \mu' : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle$, we have $(\mu, [u], [x]) \sim_{\mathsf{A}_{L/\!/K}} (\mu', [u'], [x'])$ for two morphisms in $\mathcal{Q}_{L/\!/K}$ if and only if they are in fact equal.

Lemma 4.2.12. *Let G be a crossed interval group, and let (K, L) be a pair of proper congruence families on G satisfying (♠1) and (♠2). Then, the quotient* *functors* $\mathcal{Q}_{L/\!/K} \to \widetilde{\mathcal{Q}}_{L/\!/K}$ *and* $\mathcal{Q}_L \to \widetilde{\mathcal{Q}}_L$ *derive functors* $s, t : \widetilde{\mathcal{Q}}_{L/\!/K} \rightrightarrows \widetilde{\mathcal{Q}}_L$ *from the functors* (4.2.1) *so that the diagram below is commutative:*

$$\begin{array}{ccccc} \mathcal{Q}_{L/\!/K} & \xrightarrow{s} & \mathcal{Q}_L & \xleftarrow{t} & \mathcal{Q}_{L/\!/K} \\ \downarrow & \lrcorner & \downarrow & \llcorner & \downarrow \\ \widetilde{\mathcal{Q}}_{L/\!/K} & \xrightarrow{s} & \widetilde{\mathcal{Q}}_L & \xleftarrow{t} & \widetilde{\mathcal{Q}}_{L/\!/K} \end{array} . \tag{4.2.7}$$

Moreover, each square in (4.2.7) *is a pullback of categories.*

Proof. The first statement is straightforward. To see the last, we show that when we fix a morphism $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$ and an element $x \in G_m$, for two elements $u, u' \in K_\varphi$, the following three are all equivalent:

(a) $(\varphi, [u], [x]) \sim_{\mathsf{A}_{L/\!/K}} (\varphi, [u'], [x]) \in \mathcal{Q}_{L/\!/K}(\langle\!\langle m \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle)$;

(b) $(\varphi, [u], [u^{-1}x]) \sim_{\mathsf{A}_{L/\!/K}} (\varphi, [u'], [u'^{-1}x]) \in \mathcal{Q}_{L/\!/K}(\langle\!\langle m \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle)$;

(c) $u^{-1}u' \in \mathrm{Inr}^G_\varphi$.

Note the equivalence of (a) and (c) implies the left square in (4.2.7) is a pullback while the equivalence of (b) and (c) implies the other.

Let $\varphi = \mu\rho$ be the factorization with μ active and ρ inert. In view of Remark 4.2.7, the condition (a) is satisfied if and only if $\delta^*(u) = \delta^*(u')$ since $\mathrm{Inr}^G_{\varphi\delta} = \mathrm{Inr}^G_\mu$ is trivial. The latter is equivalent to (c) in view of (3) in Lemma 4.1.12. The same argument also completely goes well for the condition (b), and this completes the proof. ☐

Proposition 4.2.13. *Let* G *be a crossed interval group, and let* K *be a proper congruence family on* G*. Then, the diagram*

$$s, t : \widetilde{\mathcal{Q}}_{L/\!/K} \rightrightarrows \widetilde{\mathcal{Q}}_L \tag{4.2.8}$$

admits a structure of a double category inherited from (4.2.1).

Proof. It suffices to see that the vertical composition functor

$$\gamma : \mathcal{Q}_{L/\!/K} \times_{\mathcal{Q}_L} \mathcal{Q}_{L/\!/K} \to \mathcal{Q}_{L/\!/K}$$

induces a functor

$$\gamma : \widetilde{\mathcal{Q}}_{L/\!/K} \times_{\widetilde{\mathcal{Q}}_L} \widetilde{\mathcal{Q}}_{L/\!/K} \to \widetilde{\mathcal{Q}}_{L/\!/K} .$$

Note that, in view of the pullback squares in (4.2.7), we have a canonical isomorphism

$$\mathcal{Q}_{L/\!/K} \times_{\mathcal{Q}_L} \mathcal{Q}_{L/\!/K} \cong \mathcal{Q}_L \times_{\widetilde{\mathcal{Q}}_L} \left(\widetilde{\mathcal{Q}}_{L/\!/K} \times_{\widetilde{\mathcal{Q}}_L} \widetilde{\mathcal{Q}}_{L/\!/K} \right)$$

of categories, where the right hand side is the limit of the diagram

$$\begin{array}{ccccc} & & \mathcal{Q}_L & & \\ & & \downarrow & & \\ \widetilde{\mathcal{Q}}_{L/\!/K} & \xrightarrow{s} & \widetilde{\mathcal{Q}}_L & \xleftarrow{t} & \widetilde{\mathcal{Q}}_{L/\!/K} \end{array} .$$

Thus, we have to show the composition

$$\widetilde{\gamma} : \mathcal{Q}_L \times_{\widetilde{\mathcal{Q}}_L} \left(\widetilde{\mathcal{Q}}_{L/\!\!/K} \times_{\widetilde{\mathcal{Q}}_L} \widetilde{\mathcal{Q}}_{L/\!\!/K} \right) \cong \mathcal{Q}_{L/\!\!/K} \times_{\mathcal{Q}_L} \mathcal{Q}_{L/\!\!/K} \xrightarrow{\gamma} \mathcal{Q}_{L/\!\!/K} \to \widetilde{\mathcal{Q}}_{L/\!\!/K}$$

depends, with respect to the first parameter, only on the images under the functor $\mathcal{Q}_L \to \widetilde{\mathcal{Q}}_L$. Since $\widetilde{\gamma}$ is clearly the identity on objects, we concentrate on morphisms. Note that, for a morphism $(\varphi, [x])$ of $\mathcal{Q}_L$ and for morphisms $\boldsymbol{\zeta}$ and $\boldsymbol{\theta}$ of $\widetilde{\mathcal{Q}}_{L/\!\!/K}$ with

$$t(\boldsymbol{\zeta}) = s(\boldsymbol{\theta}) = [\varphi, x] \in \widetilde{\mathcal{Q}}_L \ , \tag{4.2.9}$$

the image $\widetilde{\gamma}((\varphi, [x]), \boldsymbol{\theta}, \boldsymbol{\zeta})$ is given as follows: by virtue of Lemma 4.2.12, (4.2.9) implies there are elements $u, v \in K_\varphi$ so that

$$\boldsymbol{\zeta} = [\varphi, u, u^{-1}x] \ , \quad \boldsymbol{\theta} = [\varphi, v, x] \ .$$

Then we have

$$\widetilde{\gamma}((\varphi, [x]), \boldsymbol{\theta}, \boldsymbol{\zeta}) = [\varphi, vu, u^{-1}x] \ .$$

Now, suppose $(\varphi, [x]) \sim_{\mathsf{A}_L} (\varphi', [x'])$, and take $u', v' \in K_\varphi$ so that

$$\boldsymbol{\zeta} = [\varphi', u', u'^{-1}x'] \ , \quad \boldsymbol{\theta} = [\varphi', v', x'] \ .$$

We show the congruence

$$(\varphi, [vu], [u^{-1}x]) \sim_{\mathsf{A}_{L/\!\!/K}} (\varphi', [v'u'], [u'^{-1}x']) \ . \tag{4.2.10}$$

Let ψ be a morphism precomposable with φ such that either $\varphi\psi^{u^{-1}x}$ or $\varphi'\psi^{u'^{-1}x'}$ is active. Since u and u' are right stabilizers of φ, this implies either $\varphi\psi^x$ or $\varphi'\psi^{x'}$ is also active. Then, in view of Remark 4.2.7, the congruences

$$(\varphi, [u], [u^{-1}x]) \sim_{\mathsf{A}_{L/\!\!/K}} (\varphi', [u'], [u'^{-1}x']) \ , \quad (\varphi, [v], [x]) \sim_{\mathsf{A}_{L/\!\!/K}} (\varphi', [v'], [x'])$$

imply

$$\varphi\psi^{u^{-1}x} = \varphi\psi^{u'^{-1}x'} \ , \quad [\psi^*(u^{-1}x)] = [\psi^*(u'^{-1}x')]$$

and

$$\begin{aligned} [(\psi^{u^{-1}x})^*(vu)] &= [(\psi^x)^*(v)(\psi^{u^{-1}x})^*(u)] \\ &= [(\psi^x)^*(v')(\psi^{u'^{-1}x'})^*(u')] \\ &= [(\psi^{u'^{-1}x'})^*(v'u')] \ . \end{aligned}$$

Thus, (4.2.10) follows, and we obtain $\widetilde{\gamma}((\varphi, [x]), \boldsymbol{\theta}, \boldsymbol{\zeta}) = \widetilde{\gamma}((\varphi', [x']), \boldsymbol{\theta}, \boldsymbol{\zeta})$ as required. □

4.3 Internal presheaves over the associated double categories

We constructed double categories $\mathcal{Q}_{L/\!\!/K} \rightrightarrows \mathcal{Q}_L$ and $\widetilde{\mathcal{Q}}_{L/\!\!/K} \rightrightarrows \widetilde{\mathcal{Q}}_L$ for a sort of pairs (K, L) of proper congruence families on crossed interval groups G. Recall that, as they are internal categories in the category **Cat** of small categories, we can consider the following notion on them.

Definition (cf. [37], Definition 2.14)**.** Let $\mathfrak{C} \overset{s,t}{\rightrightarrows} \mathcal{B}$ be a double category. Then an *internal presheaf* over it consists of the data

- a category $(\mathcal{X} \to \mathcal{B}) \in \mathbf{Cat}^{/\mathcal{B}}$ over $\mathcal{B}$;
- a functor
$$\mathcal{A}_{\mathcal{X}} : \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \to \mathcal{X}$$
in $\mathbf{Cat}^{/\mathcal{B}}$, where the domain is the pullback of the cospan $\mathcal{X} \to \mathcal{B} \overset{t}{\leftarrow} \mathfrak{C}$ and seen as a category over $\mathcal{B}$ with the composition
$$\mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \xrightarrow{\text{proj.}} \mathfrak{C} \overset{s}{\to} \mathcal{B} \ ;$$

such that the diagrams below are commutative:

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \times_{\mathcal{B}} \mathfrak{C} & \xrightarrow{\mathcal{A}_{\mathcal{X}} \times \mathrm{Id}_{\mathfrak{C}}} & \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \\ {\scriptstyle \mathrm{Id}_{\mathcal{X}} \times \gamma_{\mathfrak{C}}} \downarrow & & \downarrow {\scriptstyle \mathcal{A}_{\mathcal{X}}} \\ \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} & \xrightarrow{\mathcal{A}_{\mathcal{X}}} & \mathcal{X} \end{array} \quad , \quad \begin{array}{ccc} \mathcal{X} \times_{\mathcal{B}} \mathcal{B} & \xrightarrow{\mathrm{Id}_{\mathcal{X}} \times \iota} & \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \\ & \searrow\!\!\!\!= & \downarrow {\scriptstyle \mathcal{A}_{\mathcal{X}}} \\ & & \mathcal{X} \end{array} \quad .$$

A double category $\mathfrak{C} \rightrightarrows \mathcal{B}$ gives rise to a 2-monad
$$\mathbf{Cat}^{/\mathcal{B}} \to \mathbf{Cat}^{/\mathcal{B}} \ ; \quad \mathcal{X} \mapsto \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \ .$$

Actually, internal presheaves over $\mathfrak{C} \rightrightarrows \mathcal{B}$ are precisely (strict) 2-algebras on it. In particular, they form a 2-category, which we denote by $\mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B})$. The 2-morphisms in $\mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B})$ are, by definition, natural transformations $\alpha : H \to K : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{B}$ such that the following two horizontal compositions coincide:

$$\mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \ \underset{K \times \mathrm{Id}}{\overset{H \times \mathrm{Id}}{\rightrightarrows}}\ {\scriptstyle (\Downarrow \alpha \times \mathrm{id})}\ \mathcal{Y} \times_{\mathcal{B}} \mathfrak{C} \xrightarrow{\mathcal{A}_{\mathcal{Y}}} \mathcal{Y} \quad ,$$

$$\mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \xrightarrow{\mathcal{A}_{\mathcal{X}}} \mathcal{X} \ \underset{K}{\overset{H}{\rightrightarrows}}\ {\scriptstyle (\Downarrow \alpha)}\ \mathcal{Y} \quad .$$

One can easily prove the following result.

Lemma 4.3.1. *Let $\mathfrak{C} \rightrightarrows \mathcal{B}$ be a double category such that the functors $s, t : \mathfrak{C} \rightrightarrows \mathcal{B}$ are the identity on objects. Then, the forgetful functor*
$$\mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B}) \to \mathbf{Cat}^{/\mathcal{B}}$$
is locally fully faithful; i.e. for internal presheaves $\mathcal{X}$ and $\mathcal{Y}$, the functor
$$\mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B})(\mathcal{X}, \mathcal{Y}) \to \mathbf{Cat}^{/\mathcal{B}}(\mathcal{X}, \mathcal{Y})$$
is fully faithful.

Suppose we are given a group operad $\mathcal{G}$ and a $\mathcal{G}$-symmetric multicategory $\mathcal{M}$. In view of categories of operators of $\mathcal{M}$ with regard to $\mathcal{G}$, the pair $(\overline{\mathrm{Dec}}^{\mathcal{G}}, \overline{\mathrm{Kec}}^{\mathcal{G}})$ plays the fundamental role. We will write
$$\mathbb{G}_{\mathcal{G}} := \mathcal{Q}_{\overline{\mathrm{Kec}}^{\mathcal{G}} /\!/ \overline{\mathrm{Dec}}^{\mathcal{G}}} \ , \quad \mathbb{E}_{\mathcal{G}} := \mathcal{Q}_{\overline{\mathrm{Kec}}^{\mathcal{G}}} \ , \quad \widetilde{\mathbb{G}}_{\mathcal{G}} := \widetilde{\mathcal{Q}}_{\overline{\mathrm{Kec}}^{\mathcal{G}} /\!/ \overline{\mathrm{Dec}}^{\mathcal{G}}} \ , \quad \widetilde{\mathbb{E}}_{\mathcal{G}} := \widetilde{\mathcal{Q}}_{\overline{\mathrm{Kec}}^{\mathcal{G}}} \ .$$

In this section, we see $\mathcal{M}$ gives rise to internal presheaves over the double categories $\mathbb{G}_{\mathcal{G}} \rightrightarrows \mathbb{E}_{\mathcal{G}}$ and $\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}}$ given in (4.2.1) and Proposition 4.2.13. We need some kinds of *word calculus*, and the following notations are convenient.

Notation. Let S be a set and $\vec{a} = a_1 \dots a_n$ a word in S; i.e. $a_i \in S$.

(1) If G is a crossed interval group, then for $x \in G_n$, we write

$$x_*\vec{a} := a_{x^{-1}(1)} \dots a_{x^{-1}(n)} \ .$$

Note that it coincides with the canonical left G_n-action on $S^{\times n}$ induced by the map $G_n \to \mathfrak{W}^{\nabla}_n \to \mathfrak{S}_n$.

(2) Suppose $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$ is an arbitrary morphism, and say $\varphi^{-1}\{j\} = \{i_1 < \dots < i_{k^{(\varphi)}_j}\}$ for each $1 \le j \le n$. Then, we write

$$\vec{a}^{\varphi}_j = a_{i_1} \dots a_{i_{k^{(\varphi)}_j}} \ .$$

Hence, the concatenated word $\vec{a}^{\varphi}_1 \dots \vec{a}^{\varphi}_n$ is a subword of the original $\vec{a}$.

Lemma 4.3.2. *Let S be a set and $\vec{a} = a_1 \dots a_m$ a word in S.*

(1) *Suppose we are given morphisms $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle$ and $\psi : \langle\!\langle n \rangle\!\rangle \to \langle\!\langle p \rangle\!\rangle$ in ∇, and say*

$$\psi^{-1}\{s\} := \{j^s_1 < \dots < j^s_r\} \ .$$

Then, we have

$$\vec{a}^{\psi\varphi}_s = \vec{a}^{\varphi}_{j^s_1} \dots \vec{a}^{\varphi}_{j^s_r} \ .$$

(2) *Let G be a crossed interval group, and write the canonical map $G_n \to \mathfrak{W}^{\nabla}_n \cong (\mathfrak{S}_n \ltimes \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ in the form*

$$y \mapsto (\sigma^y; \varepsilon^y_1, \dots, \varepsilon^y_n; \theta^y) \ .$$

Then, for every morphism $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$ and every $y \in G_n$,

$$(\varphi^*(y)_*\vec{a})^{\varphi^y}_j = \beta_*^{\varepsilon^y_{y^{-1}(j)}} \vec{a}^{\varphi}_{y^{-1}(j)} \ .$$

where β is the order-reversing permutation.

(3) *Let G be a crossed interval group. Suppose we have two morphisms $\varphi, \varphi' : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$ and two elements $x, x' \in G_m$. If two morphisms $[\varphi, x], [\varphi', x']$ in $\mathbb{E}_{\mathcal{G}}$ coincide with each other, then, for each $1 \le j \le n$,*

$$(x_*\vec{a})^{\varphi}_j = (x'_*\vec{a})^{\varphi'}_j \ .$$

Proof. The parts (1) is obvious. On the other hand, the part (2) follows from the following characterization of the permutation on $\langle\!\langle m \rangle\!\rangle$ associated with $\varphi^*(y)$:

(i) the square below is commutative

$$\begin{array}{ccc} \langle\!\langle m \rangle\!\rangle & \xrightarrow{\varphi} & \langle\!\langle n \rangle\!\rangle \\ {\scriptstyle \psi^*(y)} \downarrow & & \downarrow {\scriptstyle y} \\ \langle\!\langle m \rangle\!\rangle & \xrightarrow{\varphi^y} & \langle\!\langle n \rangle\!\rangle \end{array} \quad ;$$

(ii) for each $j \in \langle\!\langle n \rangle\!\rangle$, the bijection

$$\varphi^{-1}\{j\} \to (\varphi^y)^{-1}\{y(j)\}$$

restricting the permutation $\varphi^*(y)$ either preserves or reverses the order depending on ε^y_j.

We show (3). Under the identification $\langle\!\langle k \rangle\!\rangle \cong \nabla(\langle\!\langle 1 \rangle\!\rangle, \langle\!\langle k \rangle\!\rangle)$, the data induces maps

$$\begin{aligned} \langle\!\langle m \rangle\!\rangle \xrightarrow{x} \langle\!\langle m \rangle\!\rangle \xrightarrow{\varphi} \langle\!\langle n \rangle\!\rangle \ , \\ \langle\!\langle m \rangle\!\rangle \xrightarrow{x'} \langle\!\langle m \rangle\!\rangle \xrightarrow{\varphi'} \langle\!\langle n \rangle\!\rangle \ . \end{aligned} \tag{4.3.1}$$

It is observed that if $[\varphi, x] = [\varphi', x']$, the two maps (4.3.1) have the same inverse image of $\langle n \rangle = \{1, \dots, n\} \subset \langle\!\langle n \rangle\!\rangle$ where they agree with each other. Hence, the required equation $(x_* \vec{a})^{\varphi}_j = (x'_* \vec{a})^{\varphi'}_j$ follows for each $1 \le j \le n$. □

We begin the main construction. For a multicategory $\mathcal{M}$, we define a category $\mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$ as follows:

- objects are finite sequences $\vec{a} = a_1 \dots a_n$ of objects of $\mathcal{M}$;
- for $\vec{a} = a_1 \dots a_m$ and $\vec{b} = b_1 \dots b_n$, the hom-set $(\mathcal{M} \wr \mathbb{E}_{\mathcal{G}})(\vec{a}, \vec{b})$ consists of tuples $(\varphi; f_1, \dots, f_n; [x])$ of
  - $\varphi : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \nabla$,
  - $[x] \in \overline{\mathrm{Kec}}^{\mathcal{G}}_{\varphi} \backslash \mathcal{G}(m)$ represented by $x \in \mathcal{G}(m)$, and
  - $f_j \in \mathcal{M}((x_* \vec{a})^{\varphi}_j; b_j)$ for each $1 \le j \le n$ (see (3) in Lemma 4.3.2 and Lemma 4.2.12);
- for morphisms $(\varphi; \vec{f}; [x]) : a_1 \dots a_l \to b_1 \dots b_m$ and $(\psi; \vec{g}; [y]) : b_1 \dots b_m \to c_1 \dots c_n$, the composition is given by

$$(\psi; \vec{g}; [y]) \circ (\varphi; \vec{f}; [x]) := \Big(\psi\varphi^y; \gamma(g_1; (y_* \vec{f})^{\psi}_1), \dots, \gamma(g_n; (y_* \vec{f})^{\psi}_n); [\varphi^*(y)x]\Big) \ .$$

The composition is in fact associative; indeed, suppose we have another morphism $(\chi; \vec{h}; [z]) : \vec{c} \to d_1 \dots d_p$, and consider the equation

$$\Big((\chi; \vec{h}; [z]) \circ (\psi; \vec{g}; [y])\Big) \circ (\varphi; \vec{f}; [x]) = (\chi; \vec{h}; [z]) \circ \Big((\psi; \vec{g}; [y]) \circ (\chi; \vec{h}; [z])\Big) \ . \tag{4.3.2}$$

In terms of the first and the third components of the tuples, the equation clearly holds. If we put $\chi^{-1}\{s\} = \{j^s_1 < \cdots < j^s_r\}$, then, in view of Lemma 4.3.2, each term of the second component in the left hand side of (4.3.2) is given by

$$\begin{aligned} &\gamma\left(\gamma(h_s; (z_* \vec{g})^{\chi}_s); (\psi^*(z)_* y_* \vec{f})^{\chi\psi^z}_s\right) \\ &= \gamma\left(\gamma(h_s; g_{z^{-1}(j^s_1)}, \dots, g_{z^{-1}(j^s_r)});\ (y_* \vec{f})^{\psi}_{z^{-1}(j^s_1)} \dots (y_* \vec{f})^{\psi}_{z^{-1}(j^s_r)}\right) \\ &= \gamma\left(h_s; \gamma(g_{z^{-1}(j^s_1)}; (y_* \vec{f})^{\psi}_{z^{-1}(j^s_1)}), \dots, \gamma(g_{z^{-1}(j^s_r)}; (y_* \vec{f})^{\psi}_{z^{-1}(j^s_r)})\right) \end{aligned}$$

Clearly, the last term is precisely the one appearing as a component in the left hand side of (4.3.2), so that the composition is associative. Note that the identity on the object $a_1 \dots a_n \in \mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$ is the tuple

$$(\mathrm{id}_{\langle\!\langle n \rangle\!\rangle}; \mathrm{id}_{a_1}, \dots, \mathrm{id}_{a_n}; [e_n]) \ .$$

Example 4.3.3. In the case $\mathcal{M} = *$ is the terminal operad, the resulting category $* \wr \mathbb{E}_{\mathcal{G}}$ is nothing but the category $\mathbb{E}_{\mathcal{G}}$ itself.

We extend the constructions $\mathcal{M} \to \mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$ to 2-functors. If $F : \mathcal{M} \to \mathcal{N}$ be a $\mathcal{G}$-symmetric multifunctor, then we define a functor $F^{\mathcal{G}} : \mathcal{M} \wr \mathbb{E}_{\mathcal{G}} \to \mathcal{N} \wr \mathbb{E}_{\mathcal{G}}$ so that

- for each objects $a_1 \dots a_m \in \mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$, we put

$$F^{\mathcal{G}}(a_1 \dots a_m) := F(a_1) \dots F(a_m) \ ;$$

- for $\vec{a} = a_1 \dots a_m, \vec{b} = b_1 \dots b_n \in \mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$, define

$$\begin{array}{rccc} F^{\mathcal{G}} : & (\mathcal{M} \wr \mathbb{E}_{\mathcal{G}})(\vec{a}, \vec{b}) & \to & (\mathcal{N} \wr \mathbb{E}_{\mathcal{G}})(F^{\mathcal{G}}(\vec{a}), F^{\mathcal{G}}(\vec{b})) \\ & (\varphi; f_1, \dots, f_n; [x]) & \mapsto & (\varphi; F(f_1), \dots, F(f_n); [x]) \ . \end{array}$$

The functoriality is easily verified. In addition, if $\alpha : F \to G : \mathcal{M} \to \mathcal{N}$ is a multinatural transformation of multinatural functors, then one can check that the morphisms

$$\alpha^{\mathcal{G}}_{a_1 \dots a_m} = (\mathrm{id}_{\langle\!\langle m \rangle\!\rangle}; \alpha_{a_1}, \dots, \alpha_{a_m}; e_m) : F^{\mathcal{G}}(a_1 \dots a_m) \to G^{\mathcal{G}}(a_1 \dots a_m)$$

for $a_1 \dots a_m \in \mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$ form a natural transformation $\alpha^{\mathcal{G}} : F^{\mathcal{G}} \to G^{\mathcal{G}}$. Combining with Example 4.3.3, we obtain a 2-functor

$$(-) \wr \mathbb{E}_{\mathcal{G}} : \mathbf{MultCat} \to \mathbf{Cat}^{/\mathbb{E}_{\mathcal{G}}} \ . \tag{4.3.3}$$

We furthermore consider a quotient of the category $\mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$. For two morphisms

$$(\varphi; f_1, \dots, f_n; [x]), (\varphi'; f'_1, \dots, f'_n; [x']) : a_1 \dots a_m \to b_1 \dots b_n \in \mathcal{M} \wr \mathbb{E}_{\mathcal{G}} \ ,$$

we write $(\varphi; f_1, \dots, f_n; [x]) \sim_{\mathsf{A}_{\mathcal{G}}} (\varphi'; f'_1, \dots, f'_n; [x'])$ precisely when we have $[\varphi, x] = [\varphi', x']$ in $\widetilde{\mathbb{E}}_{\mathcal{G}}(\langle\!\langle m \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle)$ and $f_j = f'_j$ for each $1 \le j \le n$. Note that, thanks to (3) in Lemma 4.3.2, the first equation implies

$$\mathcal{M}((x_* \vec{a})^{\varphi}_j; b_j) = \mathcal{M}((x'_* \vec{a})^{\varphi'}_j; b_j)$$

so that the latter comparison makes sense. It is straightforward that the relation $\sim_{\mathsf{A}_{\mathcal{G}}}$ is a congruence on the category $\mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$. We denote by $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ the resulting quotient category. For each morphism $(\varphi; f_1, \dots, f_n; [x])$ of $\mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$, we write $[\varphi; f_1, \dots, f_n; x]$ its image in $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$. It is easily verified that the assignment $\mathcal{M} \mapsto \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ also extends to a 2-functor so that the functor $\mathcal{M} \wr \mathbb{E}_{\mathcal{G}} \to \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ forms a 2-natural transformation. More explicitly, a multifunctor $F : \mathcal{M} \to \mathcal{N}$ induces a functor $\widetilde{F}^{\mathcal{G}} : \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \to \mathcal{N} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ such that

- for each object $\vec{a} = a_1 \dots a_m \in \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, $\widetilde{F}^{\mathcal{G}}(\vec{a}) = F(a_1) \dots F(a_m)$;
- as for morphisms, we have

$$\widetilde{F}^{\mathcal{G}}([\varphi; f_1, \dots, f_n; x]) = [\varphi; F(f_1), \dots, F(f_n); x] \ .$$

On the other hand, if $\alpha : F \to G : \mathcal{M} \to \mathcal{N}$ is a multinatural transformation, we have a natural transformation $\widetilde{\alpha}^{\mathcal{G}} : \widetilde{F}^{\mathcal{G}} \to \widetilde{G}^{\mathcal{G}}$ with

$$\widetilde{\alpha}^{\mathcal{G}}_{a_1 \dots a_m} = [\mathrm{id}_{\langle\langle m \rangle\rangle}; \alpha_{a_1}, \dots, \alpha_{a_m}; e_m]$$

for each $a_1 \dots a_m \in \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$. Observing the canonical identification $* \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \cong \widetilde{\mathbb{E}}_{\mathcal{G}}$, we obtain a 2-functor

$$(-) \wr \widetilde{\mathbb{E}}_{\mathcal{G}} : \mathbf{MultCat} \to \mathbf{Cat}^{/\widetilde{\mathbb{E}}_{\mathcal{G}}} \ . \tag{4.3.4}$$

Lemma 4.3.4. *Let $\mathcal{M}$ be a multicategory. Then, for every group operad $\mathcal{G}$, the square below is a pullback:*

$$\begin{array}{ccc} \mathcal{M} \wr \mathbb{E}_{\mathcal{G}} & \longrightarrow & \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{E}_{\mathcal{G}} & \longrightarrow & \widetilde{\mathbb{E}}_{\mathcal{G}} \end{array} \ .$$

Proof. The result is straightforward from the definition of the category $\widetilde{\mathbb{E}}_{\mathcal{G}}$. □

We now take $\mathcal{G}$-symmetries into account and see that they give rise to internal presheaf structures on $\mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$ (resp. of $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$) over the double category $\mathbb{G}_{\mathcal{G}} \rightrightarrows \mathbb{E}_{\mathcal{G}}$ (resp. on $\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}}$). To simplify the notation, we define categories $\mathcal{M} \wr \mathbb{G}_{\mathcal{G}}$ and $\mathcal{M} \wr \widetilde{\mathbb{G}}_{\mathcal{G}}$ by the pullback squares

$$\begin{array}{ccc} \mathcal{M} \wr \mathbb{G}_{\mathcal{G}} & \longrightarrow & \mathcal{M} \wr \mathbb{E}_{\mathcal{G}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{G}_{\mathcal{G}} & \xrightarrow{t} & \mathbb{E}_{\mathcal{G}} \end{array} \ , \qquad \begin{array}{ccc} \mathcal{M} \wr \widetilde{\mathbb{G}}_{\mathcal{G}} & \longrightarrow & \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \widetilde{\mathbb{G}}_{\mathcal{G}} & \xrightarrow{t} & \widetilde{\mathbb{E}}_{\mathcal{G}} \end{array} \ .$$

Hence, the required internal presheaf structures are functors

$$\gamma : \mathcal{M} \wr \mathbb{G}_{\mathcal{G}} \to \mathcal{M} \wr \mathbb{E}_{\mathcal{G}} \ , \quad \gamma : \mathcal{M} \wr \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \ , \tag{4.3.5}$$

over $\mathbb{E}_{\mathcal{G}}$ and $\widetilde{\mathbb{E}}_{\mathcal{G}}$ respectively which satisfy appropriate conditions. Since the latter may be induced from the first, we mainly discuss $\mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$. Note that the category $\mathcal{M} \wr \mathbb{G}_{\mathcal{G}}$ is described explicitly as follows: for each objects $\vec{a} = a_1 \dots a_m, \vec{b} = b_1 \dots b_n \in \mathcal{M} \wr \mathbb{G}_{\mathcal{G}}$, the hom-set $(\mathcal{M} \wr \mathbb{G}_{\mathcal{G}})(\vec{a}, \vec{b})$ consists of tuples $(\varphi; f_1, \dots, f_n; [u], [x])$ such that

- $[u] \in \mathrm{Inr}^{\mathcal{G}}_{\varphi} \backslash \overline{\mathrm{Dec}}^{\mathcal{G}}_{\varphi}$ and $[x] \in \overline{\mathrm{Kec}}^{\mathcal{G}}_{\varphi} \backslash G_m$;
- $(\varphi; f_1, \dots, f_n; [ux]) : \vec{a} \to \vec{b}$ makes sense as a morphism in $\mathcal{M} \wr \mathbb{E}_{\mathcal{G}}$;

The composition is given by

$$\begin{aligned} &(\psi; g_1, \dots, g_l; [v], [y]) \circ (\varphi; f_1, \dots, f_n; [u], [x]) \\ &= (\psi\varphi^y; \gamma(g_1; ((vy)_* \vec{f})^{\psi}_1), \dots, \gamma(g_l; (vy)_* \vec{f})^{\psi}_l; [\varphi^*(vy) u \varphi^*(y)^{-1}], [\varphi^*(y) x]) \ , \end{aligned} \tag{4.3.6}$$

and the structure functor $\mathcal{M}\wr\mathbb{G}_{\mathcal{G}}\to\mathbb{E}_{\mathcal{G}}$ is an identity-on-object functor with

$$\begin{array}{ccc}(\mathcal{M}\wr\mathbb{G}_{\mathcal{G}})(a_1\dots a_m,b_1\dots b_n) & \to & \mathbb{E}_{\mathcal{G}}(\langle\langle m\rangle\rangle,\langle\langle n\rangle\rangle) \\ (\varphi;f_1,\dots,f_n;[u],[x]) & \mapsto & (\varphi,[x])\end{array} .$$

For the construction of a internal presheaf structure, the key is a comparison of the category $\mathcal{M}\wr\mathbb{G}_{\mathcal{G}}$ with $(\mathcal{M}\rtimes\mathcal{G})\wr\mathbb{E}_{\mathcal{G}}$ (see the construction in Section 1.3).

Notation. For a morphism $\varphi:\langle\langle m\rangle\rangle\to\langle\langle n\rangle\rangle\in\nabla$, and for each $1\le j\le n$, suppose

$$\varphi^{-1}\{j\}=\left\{i_1<\dots<i_{k_j^{(\varphi)}}\right\} .$$

In this case, we set

$$\delta_j^{(\varphi)}:\langle\langle k_j^{(\varphi)}\rangle\rangle\to\langle\langle m\rangle\rangle\ ;\quad s\mapsto\begin{cases}-\infty & s=-\infty\ ,\\ i_s & 1\le s\le k_j^{(\varphi)}\ ,\\ \infty & s=\infty\ .\end{cases} \tag{4.3.7}$$

Hence, the composition $\varphi\delta_j^{(\varphi)}$ factors through the map $\langle\langle 1\rangle\rangle\to\langle\langle n\rangle\rangle$ corresponding to the element $j\in\langle\langle n\rangle\rangle$.

Remark 4.3.5. The morphism $\delta_j^{(\varphi)}$ defined above is characterized by the following two properties:

(i) the composition $\varphi\delta_j^{(\varphi)}$ factors through the map $\langle\langle 1\rangle\rangle\to\langle\langle n\rangle\rangle$ corresponding to the element $j\in\langle\langle n\rangle\rangle$;

(ii) if ψ is a morphism with the previous property, then there is a unique morphism ψ' such that $\psi=\delta_j^{(\varphi)}\psi'$.

Lemma 4.3.6. *Let G be a crossed interval group. Then, for every morphism $\varphi:\langle\langle m\rangle\rangle\to\langle\langle n\rangle\rangle$, the map*

$$\vec{\delta}^{(\varphi)*}:\mathrm{RSt}_{\varphi}^{G}\to G_{k_1^{(\varphi)}}\times\dots\times G_{k_n^{(\varphi)}}\ ;\quad x\mapsto\left(\delta_1^{(\varphi)*}(x),\dots,\delta_n^{(\varphi)*}(x)\right)$$

is a group homomorphism. Moreover, its kernel contains the subgroup $\mathrm{Inr}_{\varphi}^{G}\subset\mathrm{RSt}_{\varphi}^{G}$.

Proof. To see each map $\delta_j^{(\varphi)*}:\mathrm{RSt}_{\varphi}^{G}\to G_{k_j^{(\varphi)}}$ is a group homomorphism, it suffices to show $\delta^{(\varphi)}$ is invariant under the left action of $\mathrm{RSt}_{\varphi}^{G}$. This follows from the characterization in Remark 4.3.5. The last assertion is straightforward. □

In the case $G=\mathcal{G}$ is a group operad, if $\varphi=\mu\rho$ is the unique factorization with μ active and ρ inert, then there are canonical identifications

$$\mathcal{G}(k_1^{(\varphi)})\times\dots\times\mathcal{G}(k_n^{(\varphi)})\cong\mathrm{Dec}_{\mu}^{\mathcal{G}}=\overline{\mathrm{Dec}}_{\mu}^{\mathcal{G}} .$$

Put δ the unique section of ρ, then one can see the both squares in the diagram below are commutative:

$$\begin{array}{ccc}\overline{\mathrm{Dec}}_{\varphi}^{\mathcal{G}} & \underset{\rho^*}{\overset{\delta^*}{\rightleftarrows}} & \overline{\mathrm{Dec}}_{\mu}^{\mathcal{G}} \\ \downarrow & & \downarrow\cong \\ \mathrm{RSt}_{\varphi}^{\mathcal{G}} & \xrightarrow{\vec{\delta}^{(\varphi)*}} & \mathcal{G}(k_1^{(\varphi)})\times\dots\times\mathcal{G}(k_n^{(\varphi)})\end{array} \tag{4.3.8}$$

In other words, the composition of the left and the bottom arrows in (4.3.8) induces the inverse of the map (4.1.9) in the case $K = \overline{\mathrm{Dec}}^{\mathcal{G}}$.

Theorem 4.3.7. *Let $\mathcal{G}$ be a group operad, and let $\mathcal{M}$ be a multicategory. Then, the family of maps*

$$\begin{array}{rccc} \Phi\colon & (\mathcal{M}\wr\mathbb{G}_{\mathcal{G}})(\vec{a},\vec{b}) & \to & ((\mathcal{M}\rtimes\mathcal{G})\wr\mathbb{E}_{\mathcal{G}})(\vec{a},\vec{b}) \\ & (\varphi;f_1,\dots,f_n;[u],[x]) & \mapsto & \Big(\varphi;(f_1,\delta_1^{(\varphi)*}(u)),\dots,(f_n,\delta_n^{(\varphi)*}(u));[x]\Big) \end{array}$$

for $\vec{a}=a_1\dots a_m,\vec{b}=b_1\dots b_n\in\mathcal{M}\wr\mathbb{G}_{\mathcal{G}}$ form an identity-on-objects functor

$$\Phi:\mathcal{M}\wr\mathbb{G}_{\mathcal{G}}\to(\mathcal{M}\rtimes\mathcal{G})\wr\mathbb{E}_{\mathcal{G}}\ .$$

*Moreover, Φ is an isomorphism of categories which is 2-natural with respect to $\mathcal{M}\in$ **MultCat**.*

Proof. First notice that, by virtue of Lemma 4.3.6, for each class $[u]\in\mathrm{Inr}^{\mathcal{G}}_{\varphi}\backslash\overline{\mathrm{Dec}}^{\mathcal{G}}_{\varphi}$, the element $\delta_j^{(\varphi)*}(u)$ does not depend on the choice of the representative $u\in\overline{\mathrm{Dec}}^{\mathcal{G}}_{\varphi}$ for every $1\le j\le n$. In particular, in view of Lemma 4.1.12, we may take u of the form

$$u=\gamma(e_{m+2};e^{(\varphi)}_{-\infty},u_1,\dots,u_m,e^{(\varphi)}_{\infty})\in\overline{\mathrm{Dec}}^{\mathcal{G}}_{\varphi}\subset\mathcal{G}(m) \tag{4.3.9}$$

with $u_i\in\mathcal{G}(k_i^{(\varphi)})$, where $e^{(\varphi)}_{\pm\infty}:=e_{k^{(\varphi)}_{\pm\infty}}$. In this case, we have $\delta_i^{(\varphi)*}(u)=u_i$ so that, for each morphism in $\mathcal{M}\wr\mathbb{G}_{\mathcal{G}}$ of the form $(\varphi;f_1,\dots,f_m;[u],[x])$, we have

$$\Phi(\varphi;f_1,\dots,f_m;[u],[x])=(\varphi;(f_1,u_1),\dots,(f_m,u_m);[x])\ .$$

Now, for every morphism $(\psi;g_1,\dots,g_n;[v],[y])$ in $\mathcal{M}\wr\mathbb{G}_{\mathcal{G}}$ postcomposable with $(\varphi;\vec{f};[u],[x])$ above, the explicit formula of the composition operation in $\mathcal{M}\rtimes\mathcal{G}$ and the formulas in Lemma 4.3.2 give the equation

$$\begin{aligned} &\Phi(\psi;\vec{g};[v],[y])\circ\Phi(\varphi;\vec{f};[u],[x])\\ &=\Big(\psi\varphi^y;\Big(\gamma_{\mathcal{M}}(g_1;\delta_1^{(\psi)*}(v)_*(y_*\vec{f})_1^{\psi}),\gamma_{\mathcal{G}}(\delta_1^{(\psi)*}(v);(y_*\vec{u})_1^{\psi})\Big),\\ &\qquad\dots,\Big(\gamma_{\mathcal{M}}(g_n;\delta_n^{(\psi)*}(v)_*(y_*\vec{f})_n^{\psi}),\gamma_{\mathcal{G}}(\delta_n^{(\psi)*}(v);(y_*\vec{u})_n^{\psi})\Big);[\varphi^*(y)x]\Big)\\ &=\Big(\psi\varphi^y;\Big(\gamma_{\mathcal{M}}(g_1;((vy)_*\vec{f})_1^{\psi}),\gamma_{\mathcal{G}}(\delta_1^{(\psi)*}(v);(y_*\vec{u})_1^{\psi})\Big),\\ &\qquad\dots,\Big(\gamma_{\mathcal{M}}(g_n;((vy)_*\vec{f})_n^{\psi}),\gamma_{\mathcal{G}}(\delta_n^{(\psi)*}(v);(y_*\vec{u})_n^{\psi})\Big);[\varphi^*(y)x]\Big)\ . \end{aligned} \tag{4.3.10}$$

The comparison of (4.3.10) with the formula (4.3.6) tells us that, in order to have the functoriality of Φ, we only have to verify the equation

$$\delta_j^{(\psi\varphi^y)*}(\varphi^*(vy)u\varphi^*(y)^{-1})=\gamma(\delta_j^{(\psi)*}(v);(y_*\vec{u})_j^{\psi}) \tag{4.3.11}$$

for each $1\le j\le n$. By virtue of Lemma 4.3.6, the left hand side of (4.3.11) equals

$$(\varphi^y\delta_j^{(\psi\varphi^y)})^*(v)\cdot\delta_j^{(\psi\varphi^y)*}\left(\varphi^*(y)u\varphi^*(y)^{-1}\right)\ . \tag{4.3.12}$$

Notice that there is a unique active morphism $\varphi'_j : \langle\!\langle k_j^{(\psi\varphi^y)} \rangle\!\rangle \to \langle\!\langle k_j^{(\psi)} \rangle\!\rangle$ which makes the square below commute:

$$\begin{array}{ccccc} \langle\!\langle k_j^{(\psi\varphi^y)} \rangle\!\rangle & \xrightarrow{\varphi'_j} & \langle\!\langle k_j^{(\psi)} \rangle\!\rangle & \longrightarrow & \langle\!\langle 1 \rangle\!\rangle \\ {\scriptstyle \delta_j^{(\psi\varphi^y)}}\big\downarrow & & \big\downarrow{\scriptstyle \delta_j^{(\psi)}} & & \big\downarrow{\scriptstyle \{j\}} \\ \langle\!\langle l \rangle\!\rangle & \xrightarrow{\varphi^y} & \langle\!\langle m \rangle\!\rangle & \xrightarrow{\psi} & \langle\!\langle n \rangle\!\rangle \end{array} \ . \tag{4.3.13}$$

It turns out that each square in (4.3.13) forms a pullback square of (ordinary) maps, so one has

$$k_s^{(\varphi'_j)} = k^{(\varphi^y)}_{\delta_j^{(\psi)}(s)} = k^{(\varphi)}_{y^{-1}(\delta_j^{(\psi)}(s))}$$

for each $1 \le s \le k_j^{(\psi)}$. Thus, we obtain

$$\begin{aligned} (\varphi^y \delta_j^{(\psi\varphi^y)})^*(v) &= (\delta_j^{(\psi)} \varphi'_j)^*(v) \\ &= \gamma_{\mathcal{G}}\big(\delta_j^{(\psi)*}(v); e^{(\varphi)}_{y^{-1}(\delta_j^{(\psi)}(1))}, \dots, e^{(\varphi)}_{y^{-1}(\delta_j^{(\psi)}(k_j^{(\psi)}))}\big) \\ &= \gamma_{\mathcal{G}}\left(\delta_j^{(\psi)*}(v); (y_* \vec{e}^{(\varphi)})_j^{\psi}\right) \end{aligned} \tag{4.3.14}$$

where $e_i^{(\varphi)} = e_{k_i^{(\varphi)}}$. On the other hand, in view of the presentation (4.3.9), we have

$$\begin{aligned} \delta_j^{(\psi\varphi^y)*}\left(\varphi^*(y) u \varphi^*(y)^{-1}\right) &= \delta_j^{(\psi\varphi^y)*}\left(\gamma_{\mathcal{G}}(e_{m+2}; e_{-\infty}^{(\varphi)}, u_{y^{-1}(1)}, \dots, u_{y^{-1}(m)}, e_{\infty}^{(\varphi)})\right) \\ &= \gamma_{\mathcal{G}}\big(e_j^{(\psi)}; u_{y^{-1}(\delta_j^{(\psi)}(1))}, \dots, u_{y^{-1}(\delta_j^{(\psi)}(k_j^{(\psi)}))}\big) \\ &= \gamma_{\mathcal{G}}\left(e_j^{(\psi)}; (y_* \vec{u})_j^{\psi}\right) \end{aligned} \tag{4.3.15}$$

Substituting (4.3.14) and (4.3.15) into (4.3.12), we obtain (4.3.11), which implies Φ is actually a functor.

The 2-naturality of Φ immediately follows from definition. We verify Φ is an isomorphism of categories. Since it is the identity on objects, it suffices to show Φ is bijective on each hom-sets. This is actually a consequence of (3) in Lemma 4.1.12. ☐

Corollary 4.3.8. *For every group operad $\mathcal{G}$, the 2-functor $(-) \wr \mathbb{E}_{\mathcal{G}}$ admits a lift depicted as the dashed arrow in the diagram below:*

$$\begin{array}{ccc} \mathbf{MultCat}_{\mathcal{G}} & \dashrightarrow & \mathbf{PSh}(\mathbb{G}_{\mathcal{G}} \rightrightarrows \mathbb{E}_{\mathcal{G}}) \\ {\scriptstyle forget}\big\downarrow & & \big\downarrow{\scriptstyle forget} \\ \mathbf{MultCat} & \xrightarrow{(-)\wr\mathbb{E}_{\mathcal{G}}} & \mathbf{Cat}^{/\mathbb{E}_{\mathcal{G}}} \end{array} \ .$$

Proof. In view of Theorem 4.3.7, each $\mathcal{G}$-symmetric multicategory $\mathcal{M}$ admits a canonical functor

$$(\mathcal{M} \wr \mathbb{E}_{\mathcal{G}}) \times_{\mathbb{E}_{\mathcal{G}}} \mathbb{G}_{\mathcal{G}} = \mathcal{M} \wr \mathbb{G}_{\mathcal{G}} \xrightarrow[\cong]{\Phi} (\mathcal{M} \rtimes \mathcal{G}) \wr \mathbb{E}_{\mathcal{G}} \to \mathcal{M} \wr \mathbb{E}_{\mathcal{G}} \ . \tag{4.3.16}$$

Lemma 4.3.6 and the direct computation shows that it is in fact a structure of an internal presheaf over the double category $\mathbb{G}_\mathcal{G} \rightrightarrows \mathbb{E}_\mathcal{G}$. Moreover, since the isomorphism Φ is 2-natural, the structure functor (4.3.16) is also 2-natural with respect to $\mathcal{G}$-symmetric multicategories $\mathcal{M}$. Therefore, we obtain the result. □

We finally obtain an analogues on quotients.

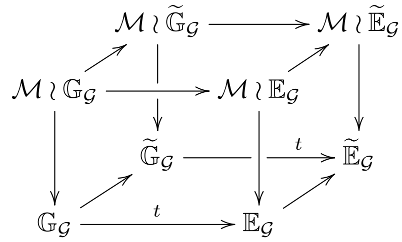
Theorem 4.3.9. *Let $\mathcal{G}$ be a group operad, and let $\mathcal{M}$ be a multicategory. Then, there is an isomorphism $\widetilde{\Phi} : \mathcal{M} \wr \widetilde{\mathbb{G}}_\mathcal{G} \cong (\mathcal{M} \rtimes \mathcal{G}) \wr \widetilde{\mathbb{E}}_\mathcal{G}$ which is the identity on objects and, on each hom-set, described as*

$$\widetilde{\Phi}([\varphi; f_1, \dots, f_n; u, x]) = \left[\varphi; (f_1, \delta_1^{(\varphi)*}(u)), \dots, (f_n, \delta_n^{(\varphi)*}(u)); x\right] . \qquad (4.3.17)$$

*Moreover, $\widetilde{\Phi}$ is a 2-natural transformation with respect to $\mathcal{M} \in$ **MultCat** such that the diagram below is commutative:*

$$\begin{array}{ccc} \mathcal{M} \wr \mathbb{G}_\mathcal{G} & \xrightarrow[\cong]{\Phi} & (\mathcal{M} \rtimes \mathcal{G}) \wr \mathbb{E}_\mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{M} \wr \widetilde{\mathbb{G}}_\mathcal{G} & \xrightarrow[\cong]{\widetilde{\Phi}} & (\mathcal{M} \rtimes \mathcal{G}) \wr \widetilde{\mathbb{E}}_\mathcal{G} \end{array}$$

Proof. We have the following commutative diagram of functors:



(4.3.18)

Note that, Lemmas 4.3.4 and 4.2.12 assert that the bottom and the right faces, as well as the front and the back, are pullbacks. Hence, the "associativity property" of pullbacks (e.g. see Proposition 2.5.9 in [8]) implies the other faces are also pullbacks. In particular, we obtain isomorphisms of categories:

$$(\mathcal{M}\wr\widetilde{\mathbb{G}}_\mathcal{G})\times_{\widetilde{\mathbb{E}}_\mathcal{G}}\mathbb{E}_\mathcal{G} \cong \mathcal{M}\wr\mathbb{G}_{\overline{\mathrm{Dec}}^\mathcal{G}} \xrightarrow[\cong]{\Phi} (\mathcal{M}\rtimes\mathcal{G})\wr\mathbb{E}_\mathcal{G} \cong ((\mathcal{M}\rtimes\mathcal{G})\wr\widetilde{\mathbb{E}}_\mathcal{G})\times_{\widetilde{\mathbb{E}}_\mathcal{G}}\mathbb{E}_\mathcal{G} \qquad (4.3.19)$$

The explicit computation shows that the isomorphism (4.3.19) is induced by the identity on $\mathbb{E}_\mathcal{G}$ and an identity-on-object functor $\widetilde{\Phi} : \mathcal{M} \wr \widetilde{\mathbb{G}}_\mathcal{G} \to (\mathcal{M} \rtimes \mathcal{G}) \wr \widetilde{\mathbb{E}}_\mathcal{G}$ described as (4.3.17). Moreover, since the functor $\mathbb{E}_\mathcal{G} \to \widetilde{\mathbb{E}}_\mathcal{G}$ is full and the identity on objects, the pullback along it preserves and reflects fully-faithfulness. Thus, we conclude $\widetilde{\Phi}$ is an isomorphism of categories. The 2-naturality and the compatibility with Φ are obvious. □

Corollary 4.3.10. *For every group operad $\mathcal{G}$, the 2-functor $(-) \wr \widetilde{\mathbb{E}}_\mathcal{G}$ admits a lift depicted as the dashed arrow in the diagram below:*

$$\begin{array}{ccc} \mathbf{MultCat}_\mathcal{G} & \dashrightarrow & \mathbf{PSh}(\widetilde{\mathbb{G}}_\mathcal{G} \rightrightarrows \widetilde{\mathbb{E}}_\mathcal{G}) \\ {\scriptstyle forget}\downarrow & & \downarrow{\scriptstyle forget} \\ \mathbf{MultCat} & \xrightarrow{(-)\wr\widetilde{\mathbb{E}}_\mathcal{G}} & \mathbf{Cat}^{/\widetilde{\mathbb{E}}_\mathcal{G}} \end{array} .$$

4.4 CoCartesian lifting properties

We investigate the image of the functor $\mathbf{MultCat}_{\mathcal{G}} \to \mathbf{PSh}(\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}})$ given in Corollary 4.3.10.

Definition. For a crossed interval group G, a morphism in $\widetilde{\mathbb{E}}_G$ is called *active* (resp. *inert*) if it is of the form $[\mu, x]$ for $\mu : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$ active (resp. *inert*) and arbitrary $x \in G_m$.

In particular, the functor $\nabla \to \widetilde{\mathbb{E}}_G$ preserves active morphisms and inert morphisms respectively. Throughout the section, the following inert morphisms in ∇ play important roles: for each $1 \le i \le n$, we define a morphism $\rho_i : \langle\langle n\rangle\rangle \to \langle\langle 1\rangle\rangle$ by

$$\rho_i(j) = \begin{cases} -\infty & j < i \ , \\ 1 & j = i \ , \\ \infty & j > i \ . \end{cases}$$

By abuse of notation, we use the same notation ρ_i to denote its image in $\widetilde{\mathbb{E}}_G$.

Proposition 4.4.1. *Let $\mathcal{G}$ be a group operad, and let $\mathcal{M}$ be a multicategory. Then, the canonical functor $p_{\mathcal{M}} : \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$ satisfies the following properties.*

(1) *Every inert morphism $[\rho, x] : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \widetilde{\mathbb{E}}_{\mathcal{G}}$ admits $p_{\mathcal{M}}$-coCartesian lifts along any object in the fiber $(\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle m\rangle\rangle} := p_{\mathcal{M}}^{-1}\{\langle\langle m\rangle\rangle\}$. More precisely, if $\delta : \langle\langle n\rangle\rangle \to \langle\langle m\rangle\rangle$ is the section of ρ, then for each $\vec{a} = a_1 \dots a_m \in (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle}$, the morphism*

$$\begin{aligned} \widehat{[\rho, x]}_{\vec{a}} &:= [\rho; \mathrm{id}_{a_{x^{-1}(\delta(1))}}, \dots, \mathrm{id}_{a_{x^{-1}(\delta(n))}}; x] \\ &\quad : a_1 \dots a_m \to a_{x^{-1}(\delta(1))} \dots a_{x^{-1}(\delta(n))} \end{aligned} \tag{4.4.1}$$

is $p_{\mathcal{M}}$-coCartesian.

(2) *For an object $\vec{a} = a_1 \dots a_n \in (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle}$, choose a $p_{\mathcal{M}}$-coCartesian lift $\widehat{\rho}_j : \vec{a} \to a'_j$ of ρ_j along $\vec{a}$ for each $1 \le i \le n$. Then, for every object $\vec{b} \in \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, the square below is a pullback:*

$$\begin{array}{ccc} (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})(\vec{b}, \vec{a}) & \xrightarrow{((\widehat{\rho}_1)_*, \dots, (\widehat{\rho}_n)_*)} & \prod_{i=1}^n (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})(\vec{b}, a'_i) \\ {\scriptstyle p_{\mathcal{M}}} \downarrow & \lrcorner & \downarrow {\scriptstyle p_{\mathcal{M}}} \\ \widetilde{\mathbb{E}}_{\mathcal{G}}(p_{\mathcal{M}}(\vec{b}), \langle\langle n\rangle\rangle) & \xrightarrow{((\rho_1)_*, \dots, (\rho_n)_*)} & \widetilde{\mathbb{E}}_{\mathcal{G}}(p_{\mathcal{M}}(\vec{b}), \langle\langle 1\rangle\rangle)^{\times n} \end{array} \ . \tag{4.4.2}$$

(3) *For each $1 \le i \le n$, take a functor $(\rho_i)_! : (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \to (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle 1\rangle\rangle}$ together with a natural transformation $\widehat{\rho}_i : \vec{a} \to (\rho_i)_!\vec{a}$ which is (componentwisely) $p_{\mathcal{M}}$-coCartesian. Then the functor*

$$((\rho_1)_!, \dots, (\rho_n)_!) : (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \to (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})^{\times n}_{\langle\langle 1\rangle\rangle}$$

is an equivalence of categories:

Remark 4.4.2. The condition (2) actually does not depend on the choice of coCartesian lifts $\widehat{\rho}_i$ of ρ_i. Indeed, if one choose another coCartesian lift $\widehat{\rho}'_i : \vec{a} \to a''_i$, then the uniqueness of the coCartesian lifts implies there is a unique isomorphism $a'_i \cong a''_i$ so that $\widehat{\rho}'_i$ factors through $\widehat{\rho}_i$ followed by the isomorphism. Moreover, it also gives rise to an isomorphism of squares (4.4.2). Thus, if (2) satisfied for one family of coCartesian lifts, then it is also for the other.

A similar argument shows that the condition (3) does not depend on the choice of the functors $(\rho_i)_!$.

Proof of Proposition 4.4.1. In order to verify (1), it clearly suffices to consider only the case $x \in G_m$ is the unit. For an inert morphism $\rho : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \nabla$, set δ to be the unique section, and suppose we have a morphism in $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ of the form

$$[\varphi\rho; f_1, \dots, f_l; \rho^*(y)] : a_1 \dots a_m \to \vec{b}\ .$$

We show it uniquely factors through the morphism

$$\widehat{\rho}_{\vec{a}} = [\rho; \mathrm{id}_{a_{\delta(1)}}, \dots, \mathrm{id}_{a_{\delta(n)}}; e_m] : \vec{a} \to a_{\delta(1)} \dots a_{\delta(n)}$$

Thanks to the unique factorization in ∇, we have

$$[\varphi\rho; f_1, \dots, f_l; \rho^*(y)] = [\varphi; f_1, \dots, f_l; y] \circ \widehat{\rho}_{\vec{a}}\ , \tag{4.4.3}$$

so there in fact exists a factorization. Moreover, since the morphism $[\varphi; f_1, \dots, f_l; y]$ is uniquely determined by the underlying morphism $[\varphi, y]$ in $\widetilde{\mathbb{E}}_{\mathcal{G}}$ and the tuple $(f_1, \dots, f_l)$, which is determined by the left hand side. This implies the factorization (4.4.3) is unique, so $\widehat{\rho}_{\vec{a}}$ is $p_{\mathcal{M}}$-coCartesian.

We next see (2). For an object $\vec{a} = a_1 \dots a_n \in (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle}$, in view of Remark 4.4.2, we may assume the lift $\widehat{\rho}_i = (\widehat{\rho}_i)_{\vec{a}}$ is the one given in the part (1). Suppose $\vec{b} = b_1 \dots b_m \in \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, and $[\varphi, x] : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \widetilde{\mathbb{E}}_{\mathcal{G}}$. Then, if one has a morphism of the form

$$[\varphi; f_1, \dots, f_n; x] : \vec{b} \to \vec{a}\ , \tag{4.4.4}$$

then $f_i \in \mathcal{M}((x_*\vec{b})^{\varphi}_i; a_i)$. On the other hand, we have $(x_*\vec{b})^{\rho_i\varphi}_1 = (x_*\vec{b})^{\varphi}_i$ so that (4.4.4) makes sense if and only if we have morphisms

$$[\rho_i\varphi; f_i; x] : \vec{b} \to a_i \tag{4.4.5}$$

for $1 \le i \le n$. When we fix a morphism $[\varphi, x]$ in $\widetilde{\mathbb{E}}_{\mathcal{G}}$, the two data (4.4.4) and (4.4.5) clearly correspond in one-to-one to each other. It follows that the square (4.4.2) is a pullback.

We finally show (3). Note that, in view of Example 4.2.11, every morphism in $(\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle}$ is of the form

$$[\mathrm{id}_{\langle\langle n\rangle\rangle}; f_1, \dots, f_n; e_n] : a_1 \dots a_n \to b_1 \dots b_n$$

with $f_i \in \mathcal{M}(a_i; b_i) = \underline{\mathcal{M}}(a_i, b_i)$, here $\underline{\mathcal{M}}$ is the underlying category of $\mathcal{M}$. In other words, we have a canonical isomorphism

$$(\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \cong \underline{\mathcal{M}}^{\times n} \cong (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})^{\times n}_{\langle\langle 1\rangle\rangle}\ . \tag{4.4.6}$$

Hence, it suffices to show the functor $(\rho_i)_! : (\mathcal{M}\wr\widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \to (\mathcal{M}\wr\widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle 1\rangle\rangle}$ coincides with the projection under the isomorphism (4.4.6). If $(\rho_i)_!$ is the one induced by the $p_{\mathcal{M}}$-coCartesian lifts in the part (1), this follows from the correspondence of (4.4.4) to (4.4.5) and the unique factorization (4.4.3). In view of Remark 4.4.2, this completes the proof. □

We define a 2-subcategory $\mathbf{Oper}'_{\mathcal{G}} \subset \mathbf{Cat}^{/\widetilde{\mathbb{E}}_{\mathcal{G}}}$ as follows:

- objects of $\mathbf{Oper}'_{\mathcal{G}}$ are those categories $\mathcal{C}$ over $\widetilde{\mathbb{E}}_{\mathcal{G}}$ that satisfy three properties in Proposition 4.4.1;
- for $\mathcal{C},\mathcal{D} \in \mathbf{Oper}'_{\mathcal{G}}$, the hom-category $\mathbf{Oper}'_{\mathcal{G}}(\mathcal{C},\mathcal{D})$ is the full subcategory of $\mathbf{Cat}^{/\widetilde{\mathbb{E}}_{\mathcal{G}}}$ spanned by functors $\mathcal{C} \to \mathcal{D}$ over $\widetilde{\mathbb{E}}_{\mathcal{G}}$ which preserve coCartesian lifts of inert morphisms in $\widetilde{\mathbb{E}}_{\mathcal{G}}$.

Furthermore, we put

$$\mathbf{Oper}^{\mathsf{alg}}_{\mathcal{G}} := \mathbf{PSh}(\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}}) \times_{\mathbf{Cat}^{/\widetilde{\mathbb{E}}_{\mathcal{G}}}} \mathbf{Oper}'_{\mathcal{G}}\ ,$$

whose objects are called *categories of algebraic $\mathcal{G}$-operators*, and whose morphisms *maps of algebraic $\mathcal{G}$-operators*. Thanks to Corollary 4.3.10 and Proposition 4.4.1, the 2-functor $(-)\wr\widetilde{\mathbb{E}}_{\mathcal{G}}$ induces a 2-functor $\mathbf{MultCat}_{\mathcal{G}} \to \mathbf{Oper}^{\mathsf{alg}}_{\mathcal{G}}$, which we also denote by $(-)\wr\widetilde{\mathbb{E}}_{\mathcal{G}}$ by abuse of notation. Thanks to Lemma 4.3.1, the forgetful functor

$$\mathbf{Oper}^{\mathsf{alg}}_{\mathcal{G}} \to \mathbf{Oper}'_{\mathcal{G}}$$

is locally fully faithful.

Example 4.4.3. As we have $*\wr\widetilde{\mathbb{E}}_{\mathcal{G}} \cong \widetilde{\mathbb{E}}_{\mathcal{G}}$, the identity functor $\widetilde{\mathbb{E}}_{\mathcal{G}} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$ exhibits $\widetilde{\mathbb{E}}_{\mathcal{G}}$ as a category of algebraic $\mathcal{G}$-operators.

Example 4.4.4. Recall that every group operad $\mathcal{G}$ is itself a $\mathcal{G}$-symmetric multicategory with the multiplication map $\mathcal{G}\rtimes\mathcal{G} \to \mathcal{G}$ (see Proposition 3.1.8). On the other hand, in view of Theorem 4.3.9, we have isomorphisms

$$\widetilde{\mathbb{G}}_{\mathcal{G}} \cong *\wr\widetilde{\mathbb{G}}_{\mathcal{G}} \cong (*\rtimes\mathcal{G})\wr\widetilde{\mathbb{E}}_{\mathcal{G}} \cong \mathcal{G}\wr\widetilde{\mathbb{E}}_{\mathcal{G}}\ .$$

It follows that the functor $s : \widetilde{\mathbb{G}}_{\mathcal{G}} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$ exhibits $\widetilde{\mathbb{G}}_{\mathcal{G}}$ as a category of algebraic $\mathcal{G}$-operators.

It turns out that there are *free $\mathcal{G}$-symmetrizations* of objects in $\mathbf{Oper}'_{\mathcal{G}}$. Indeed, we have the following property on the free construction.

Lemma 4.4.5. *For every $\mathcal{C} \in \mathbf{Oper}'_{\mathcal{G}}$, the functor*

$$\mathcal{C} \cong \mathcal{C}\times_{\widetilde{\mathbb{E}}_{\mathcal{G}}}\widetilde{\mathbb{E}}_{\mathcal{G}} \xrightarrow{\mathrm{Id}\times\iota} \mathcal{C}\times_{\widetilde{\mathbb{E}}_{\mathcal{G}}}\widetilde{\mathbb{G}}_{\mathcal{G}} \qquad (4.4.7)$$

preserves coCartesian lifts of inert morphisms.

Proof. Note that the category $\mathcal{C}\times_{\widetilde{\mathbb{E}}_{\mathcal{G}}}\widetilde{\mathbb{G}}_{\mathcal{G}}$ is described as follows:

- objects are the same as $\mathcal{C}$;
- for $X,Y \in \mathcal{C}$, the hom-set $(\mathcal{C}\times_{\widetilde{\mathbb{E}}_{\mathcal{G}}}\widetilde{\mathbb{G}}_{\mathcal{G}})(X,Y)$ consists of tuples $[\varphi;f;u,x]$ so that $[\varphi,u,x] : q(X) \to q(Y) \in \widetilde{\mathbb{G}}_{\mathcal{G}}$ makes sense and $f : X \to Y \in \mathcal{C}$ with $q(f) = [\varphi,ux]$;

- the composition is given by

$$[\psi; g; v, y] \circ [\varphi; f; u, x] = [\psi\varphi^y; gf; \varphi^*(vy)u\varphi^*(y)^{-1}, \varphi^*(y)x] \ ;$$

- the structure functor $\mathcal{C} \times_{\widetilde{\mathbb{E}}_\mathcal{G}} \widetilde{\mathbb{G}}_\mathcal{G} \to \widetilde{\mathbb{E}}_\mathcal{G}$ is given by

$$[\varphi; f; u, x] \mapsto [\varphi, x] \ .$$

Suppose $[\rho, x] : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \widetilde{\mathbb{E}}_\mathcal{G}$ is an inert morphism, and take a coCartesian lift

$$\widehat{[\rho, x]}_X : X \to X' \in \mathcal{C}$$

along $X \in \mathcal{C}$. The functor (4.4.7) sends it to

$$[\rho; \widehat{[\rho, x]}_X; e, x] : X \to X' \in \mathcal{C} \times_{\widetilde{\mathbb{E}}_\mathcal{G}} \widetilde{\mathbb{G}}_\mathcal{G} \ . \tag{4.4.8}$$

To see (4.4.8) is coCartesian, consider a morphism in $\mathcal{C} \times_{\widetilde{\mathbb{E}}_\mathcal{G}} \widetilde{\mathbb{G}}_\mathcal{G}$ of the form $[\varphi\rho; f; u, x] : X \to Y$. In view of (3) in Lemma 4.1.12, there is a unique element $\bar{u} \in \overline{\mathrm{Dec}}^\mathcal{G}_\varphi$ such that $[u] = [\rho^*(\bar{u})] \in \overline{\mathrm{Inr}}^\mathcal{G}_{\varphi\rho} \backslash \overline{\mathrm{Dec}}^\mathcal{G}_{\varphi\rho}$, which implies

$$[\varphi\rho, ux] = [\varphi, \bar{u}] \circ [\rho, x] \ .$$

On the other hand, since $\widehat{[\rho, x]}$ is coCartesian, there is a unique factorization $f = f' \circ \widehat{[\rho, x]}_X$ with $f' : X' \to Y$ covering $[\varphi, \bar{u}]$. One obtains

$$[\varphi\rho; f; u, x] = [\varphi; f'; \bar{u}, e] \circ [\rho; \widehat{[\rho, x]}_X; e, x] \ . \tag{4.4.9}$$

Since the morphisms f' and $\bar{u}$ are uniquely determined by the other data, the factorization (4.4.9) is unique. It follows that the morphism (4.4.8) is coCartesian. □

Proposition 4.4.6. *The free* 2*-functor*

$$\mathbf{Cat}^{/\widetilde{\mathbb{E}}_\mathcal{G}} \to \mathbf{PSh}(\widetilde{\mathbb{G}}_\mathcal{G} \rightrightarrows \widetilde{\mathbb{E}}_\mathcal{G}) \ ; \quad \mathcal{C} \mapsto \mathcal{C} \times_{\widetilde{\mathbb{E}}_\mathcal{G}} \widetilde{\mathbb{G}}_\mathcal{G}$$

associated to the 2*-monad of internal presheaves over the double category* $\widetilde{\mathbb{G}} \rightrightarrows \widetilde{\mathbb{E}}_\mathcal{G}$ *restricts to a* 2*-functor*

$$\mathbf{Oper}'_\mathcal{G} \to \mathbf{Oper}^{\mathsf{alg}}_\mathcal{G} \ .$$

Proof. It clearly suffices to show the composition

$$\mathbf{Oper}'_\mathcal{G} \hookrightarrow \mathbf{Cat}^{/\widetilde{\mathbb{E}}_\mathcal{G}} \xrightarrow{(-)\times_{\widetilde{\mathbb{E}}_\mathcal{G}}\widetilde{\mathbb{G}}_\mathcal{G}} \mathbf{PSh}(\widetilde{\mathbb{G}}_\mathcal{G} \rightrightarrows \widetilde{\mathbb{E}}_\mathcal{G}) \xrightarrow{\textit{forget}} \mathbf{Cat}^{/\widetilde{\mathbb{E}}_\mathcal{G}}$$

factors through the subcategory $\mathbf{Oper}'_\mathcal{G}$ at the end. We have to verify it regarding objects and 1-morphisms.

Let $\mathcal{C} \in \mathbf{Oper}'_\mathcal{G}$ with $q : \mathcal{C} \to \widetilde{\mathbb{E}}_\mathcal{G}$. We verify the three conditions in Proposition 4.4.1 for $\mathcal{C} \times_{\widetilde{\mathbb{E}}_\mathcal{G}} \widetilde{\mathbb{G}}_\mathcal{G}$. Since the unit $\mathcal{C} \to \mathcal{C} \times_{\widetilde{\mathbb{E}}_\mathcal{G}} \widetilde{\mathbb{G}}_\mathcal{G}$ is the identity on objects, Lemma 4.4.5 implies $\mathcal{C} \times_{\widetilde{\mathbb{E}}_\mathcal{G}} \widetilde{\mathbb{G}}_\mathcal{G}$ admits all the coCartesian lifts of inert morphisms. On the other hand, according to the description of $\mathcal{C} \times_{\widetilde{\mathbb{E}}_\mathcal{G}} \widetilde{\mathbb{G}}_\mathcal{G}$ in the proof of Lemma 4.4.5, one easily verify the property (2) in Proposition 4.4.1. To see the property (3), observe that the category $(\widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle}$ consists of automorphisms on $\langle\langle n\rangle\rangle$ in $\widetilde{\mathbb{G}}_{\mathcal{G}}$ of the form

$$[\mathrm{id}_{\langle\langle n\rangle\rangle}, u, e_n]$$

for $u \in \overline{\mathrm{Dec}}^{\mathcal{G}}_{\mathrm{id}_{\langle\langle n\rangle\rangle}}$. It turns out that such morphisms vanished by the functor $t : \widetilde{\mathbb{G}}_{\mathcal{G}} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$, so we obtain isomorphisms

$$(\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \cong \mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} (\widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \cong \mathcal{C}_{\langle\langle n\rangle\rangle} \times (\widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle}$$

Under the identification, it is easily verified that, for each inert morphism $\rho_i : \langle\langle n\rangle\rangle \to \langle\langle 1\rangle\rangle$, the induced functor

$$(\rho_i)_! : (\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \to (\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle 1\rangle\rangle}$$

coincides with the one induced by

$$(\rho_i)_! : \mathcal{C}_{\langle\langle n\rangle\rangle} \to \mathcal{C}_{\langle\langle 1\rangle\rangle}\ , \quad (\rho_i)_! : (\widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \to (\widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle 1\rangle\rangle}\ .$$

Thus, the functor

$$((\rho_1)_!, \dots, (\rho_n)_!) : (\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\langle n\rangle\rangle} \to (\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})^{\times n}_{\langle\langle 1\rangle\rangle}$$

is an equivalence.

As for 1-morphisms, suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor over $\widetilde{\mathbb{E}}_{\mathcal{G}}$ for $\mathcal{C}, \mathcal{D} \in \mathbf{Oper}'_{\mathcal{G}}$. We have the following commutative square:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ \mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} & \xrightarrow{F\times\mathrm{Id}} & \mathcal{D} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \end{array} \quad . \tag{4.4.10}$$

In view of Lemma 4.4.5, all the coCartesian lifts of inert morphisms in $\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$ and $\mathcal{D} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$ are isomorphic to the images of ones in $\mathcal{C}$ and in $\mathcal{D}$ respectively by the vertical functors. It follows that the bottom arrow in (4.4.10) preserves coCartesian lifts of inert morphisms as soon as so does the top. The required result now follows immediately. □

Corollary 4.4.7. *Let $\mathcal{G}$ be a group operad, and let $\mathcal{C}$ be a category of algebraic $\mathcal{G}$-operators. Then, the functor*

$$\mathcal{A}_{\mathcal{C}} : \mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathcal{C}$$

in the internal presheaf structure on $\mathcal{C}$ is a map of algebraic $\mathcal{G}$-operators.

Proof. By virtue of Lemma 4.4.5 and Proposition 4.4.6, $\mathcal{A}_{\mathcal{C}}$ is a 1-morphism in the 2-category $\mathbf{Oper}'_{\mathcal{G}}$. In addition, it is straightforward from the definition of internal presheaves that $\mathcal{A}_{\mathcal{C}}$ is a map of internal presheaves. Combining them, one obtains the result. □

4.5 The equivalence of notions

We continue the discussion on the 2-functor $(-) \wr \widetilde{\mathbb{E}}_{\mathcal{G}} : \mathbf{MultCat}_{\mathcal{G}} \to \mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}$ given in Section 4.3 (precisely Corollary 4.3.10). The goal of this section is to prove the following result.

Theorem 4.5.1. *Let $\mathcal{G}$ be a group operad. Then, the 2-functor*

$$(-) \wr \widetilde{\mathbb{E}}_{\mathcal{G}} : \mathbf{MultCat}_{\mathcal{G}} \to \mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}$$

is a biequivalence of 2-categories. In other words, the following hold.

(1) *It is essentially fully faithful; i.e. for every pair $(\mathcal{M}, \mathcal{N})$ of $\mathcal{G}$-symmetric multicategories, the functor*

$$\begin{array}{ccc} \mathbf{MultCat}_{\mathcal{G}}(\mathcal{M}, \mathcal{N}) & \to & \mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}(\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}, \mathcal{N} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}) \\ F, \alpha & \mapsto & \widetilde{F}^{\mathcal{G}}, \widetilde{\alpha}^{\mathcal{G}} \end{array} \tag{4.5.1}$$

is an equivalence of categories.

(2) *It is essentially surjective; i.e. for every category of algebraic $\mathcal{G}$-operators $\mathcal{C}$, there is a $\mathcal{G}$-symmetric multicategory $\mathcal{M}$ together with an equivalence $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \simeq \mathcal{C}$ in $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}$.*

Remark 4.5.2. It is known that a pseudofunctor $\mathcal{K} \to \mathcal{L}$ between 2-categories admits a *pseudoinverse*, i.e. a pseudofunctor $\mathcal{L} \to \mathcal{K}$ which is the inverse up to natural isomorphisms, provided it is essentially fully faithful and essentially surjective in the sense in Theorem 4.5.1. The reader can find a sketch in Section 3.2 in [49].

In order to prove Theorem 4.5.1, we need to observe that coCartesian lifts of inert morphisms in $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ are preserved *coherently* by arbitrary maps of algebraic $\mathcal{G}$-operators. To simplify the notation, we use the following convention: let $[\rho, x] : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \widetilde{\mathbb{E}}_{\mathcal{G}}$ be an inert morphism. Although we may denote by $\widehat{[\rho, x]}_X$ an arbitrary coCartesian lift ρ along an object X in a general category of algebraic $\mathcal{G}$-operators $\mathcal{C}$, we always assume $\widehat{[\rho, x]}_{\vec{a}}$ is the one in (1) in Proposition 4.4.1 in the special case $\mathcal{C} = \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$. In particular, we write

$$\widehat{\rho}_{\vec{a}} := \widehat{[\rho, e_m]}_{\vec{a}} \ , \quad \widehat{x}_{\vec{a}} := \widehat{[\mathrm{id}, x]}_{\vec{a}} \ .$$

Hence, if δ is the section of ρ, the induced functor

$$\rho_! : (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle m \rangle\rangle} \to (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle n \rangle\rangle}$$

coincides with the canonical projection so that $\rho_!(a_1 \dots a_m) = a_{\delta(1)} \dots a_{\delta(n)}$.

Lemma 4.5.3. *Let $\mathcal{G}$ be a group operad, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{G}$-symmetric multicategories. Suppose $H : \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \to \mathcal{N} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ is a map of algebraic $\mathcal{G}$-operators. Then, for each $\vec{a} = a_1 \dots a_m$ and each $1 \le i \le m$, there is a unique isomorphism*

$$\lambda_{\vec{a},i} : H(a_i) \xrightarrow{\cong} (\rho_i)_! H(\vec{a}) \in (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})_{\langle\langle 1 \rangle\rangle} = \underline{\mathcal{M}}$$

such that $(\widehat{\rho}_i)_{H(\vec{a})} = \lambda_{\vec{a},i} \circ H((\widehat{\rho}_i)_{\vec{a}})$. Moreover, the family

$$\left\{ \lambda_{\vec{a}} = [\mathrm{id}_{\langle\langle m \rangle\rangle}; \lambda_{\vec{a},1}, \dots, \lambda_{\vec{a},m}; e_m] \;\middle|\; \vec{a} = a_1 \dots a_m \in \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \right\}$$

enjoys the property that, for every inert morphism $[\rho, x] : \langle\langle m\rangle\rangle \to \langle\langle n\rangle\rangle \in \widetilde{\mathbb{E}}_{\mathcal{G}}$, say δ is the section of ρ, and for each $\vec{a} = a_1 \dots a_m \in \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, the square below commutes in $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$:

$$\begin{array}{ccc} H(a_1)\dots H(a_m) & \xrightarrow{\lambda_{\vec{a}}} & H(a_1\dots a_m) \\ {\scriptstyle \widehat{[\rho,x]}_{H(a_1)\dots H(a_m)}}\downarrow & & \downarrow{\scriptstyle H(\widehat{[\rho,x]}_{\vec{a}})} \\ H(a_{x^{-1}\delta(1)})\dots H(a_{x^{-1}\delta(n)}) & \xrightarrow{\lambda_{\rho_! x_* \vec{a}}} & H(a_{x^{-1}\delta(1)}\dots a_{x^{-1}\delta(n)}) \end{array} \tag{4.5.2}$$

Proof. Since the functor $H : \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \to \mathcal{N} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ preserves coCartesian lifts of inert morphisms, the morphism $H((\widehat{\rho}_i)_{\vec{a}}) : H(\vec{a}) \to H(a_i)$ is coCartesian. Thus, the first statement is obvious. As for the second, it turns out that we only have to verify the commutativity of (4.5.2) for inert morphisms of the forms $[\rho, e_m]$ and $[\mathrm{id}, x]$. In the first case, for each $1 \le j \le n$, we have

$$\begin{aligned} (\widehat{\rho}_j)_{H(\rho_!\vec{a})} \circ H(\widehat{\rho}_{\vec{a}}) \circ \lambda_{\vec{a}} &= \lambda_{\rho_!\vec{a},j} \circ H((\widehat{\rho}_j)_{\rho_!\vec{a}} \circ \widehat{\rho}_{\vec{a}}) \circ \lambda_{\vec{a}} \\ &= \lambda_{\rho_!\vec{a},j} \circ H((\widehat{\rho}_{\delta(j)})_{\vec{a}}) \circ \lambda_{\vec{a}} \\ &= \lambda_{\rho_!\vec{a},j} \circ \lambda^{-1}_{\vec{a},\delta(j)} \circ (\widehat{\rho}_{\delta(j)})_{H(\vec{a})} \circ \lambda_{\vec{a}} \\ &= \lambda_{\rho_!\vec{a},j} \circ (\widehat{\rho}_{\delta(j)})_{H(a_1)\dots H(a_m)} \\ &= (\widehat{\rho}_j)_{H(\rho_!\vec{a})} \circ \lambda_{\rho_!\vec{a}} \circ \widehat{\rho}_{H(a_1)\dots H(a_m)} \ . \end{aligned}$$

In view of (2) in Proposition 4.4.1, this implies $H(\widehat{\rho}_{\vec{a}})\lambda_{\vec{a}} = \lambda_{\rho_!\vec{a}}\widehat{\rho}_{H(a_1)\dots H(a_m)}$, and (4.5.2) is commutative.

It remains to show the commutativity of (4.5.2) in the case ρ is the identity. Similarly to the case above, we have

$$\begin{aligned} (\widehat{\rho}_j)_{H(x_*\vec{a})} \circ H(\widehat{x}_{\vec{a}}) \circ \lambda_{\vec{a}} &= \lambda_{x_*\vec{a},j} \circ H((\widehat{\rho}_j)_{x_*\vec{a}} \circ \widehat{x}_{\vec{a}}) \circ \lambda_{\vec{a}} \\ &= \lambda_{x_*\vec{a},j} \circ H(\widehat{[\rho_j, x]}_{\vec{a}}) \circ \lambda_{\vec{a}} \ . \end{aligned} \tag{4.5.3}$$

In view of , setting δ_j to be the section of ρ_j for each $1 \le j \le m$, we have

$$\begin{aligned} [\rho_j, x] &= [\rho_{x^{-1}(j)}, \rho^*_{x^{-1}(j)}\delta^*_{x^{-1}(j)}(x)] \\ &= [\mathrm{id}_{\langle\langle n\rangle\rangle}, \delta^*_{x^{-1}(j)}(x)] \circ \rho_{x^{-1}(j)} \\ &= \rho_{x^{-1}(j)} \end{aligned}$$

as morphisms in $\widetilde{\mathbb{E}}_{\mathcal{G}}$ since $\overline{\mathrm{Kec}}^{\mathcal{G}}_{\mathrm{id}_{\langle\langle 1\rangle\rangle}} = \mathrm{Kec}^{\mathcal{G}}_{\mathrm{id}_{\langle\langle 1\rangle\rangle}} = \mathcal{G}(1)$. Substituting it to (4.5.3), we get

$$\begin{aligned} (\widehat{\rho}_j)_{H(x_*\vec{a})} \circ H(\widehat{x}_{\vec{a}}) \circ \lambda_{\vec{a}} &= \lambda_{x_*\vec{a},j} \circ H((\widehat{\rho}_{x^{-1}(j)})_{\vec{a}}) \circ \lambda_{\vec{a}} \\ &= \lambda_{x_*\vec{a},j} \circ \lambda^{-1}_{\vec{a},x^{-1}(j)} \circ (\widehat{\rho}_{x^{-1}(j)})_{H(\vec{a})} \circ \lambda_{\vec{a}} \\ &= \lambda_{x_*\vec{a},j} \circ (\widehat{\rho}_{x^{-1}(j)})_{H(a_1)\dots H(a_m)} \\ &= \lambda_{x_*\vec{a},j} \circ \widehat{[\rho_j, x]}_{H(a_1)\dots H(a_m)} \\ &= (\widehat{\rho}_j)_{H(x_*\vec{a})} \circ \lambda_{x_*\vec{a}} \circ \widehat{x}_{H(a_1)\dots H(a_m)} \ . \end{aligned}$$

Hence, (2) in Proposition 4.4.1 again implies the commutativity of (4.5.2). □

On the other hand, on the construction of a $\mathcal{G}$-symmetric multicategories from a category of algebraic $\mathcal{G}$-operators, we need to observe how coCartesian lifts of inert morphisms determine composition operations. Notice that, if we denote by $\mu_n : \langle\langle n\rangle\rangle \to \langle\langle 1\rangle\rangle$ the active morphism with $\mu_n(\pm\infty) = \pm\infty$ and $\mu_n(i) = 1$ for $1 \le i \le n$, then, for a multicategory $\mathcal{M}$, the multihom-set $\mathcal{M}(a_1 \dots a_n; a)$ is recovered from the category $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ as

$$\mathcal{M}(a_1 \dots a_n; a) \cong (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})(a_1 \dots a_n, a)_{\mu_n} \ ,$$

here the right hand side is the set of morphisms $a_1 \dots a_n \to a$ in $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ covering μ_n.

Notation. Given active morphisms $\nu_1 : \langle\langle k_i\rangle\rangle \to \langle\langle l_i\rangle\rangle \in \nabla$ for $1 \le i \le n$, we define an active morphism $\nu_1 \diamond \cdots \diamond \nu_n : \langle\langle k_1 + \cdots + k_n\rangle\rangle \to \langle\langle l_1 + \cdots + l_n\rangle\rangle \in \nabla$ to be the map

$$\begin{aligned}&\langle\langle k_1 + \cdots + k_n\rangle\rangle \cong \{-\infty\} \star \langle k_1\rangle \star \cdots \star \langle k_n\rangle \star \{\infty\} \\ &\xrightarrow{\mathrm{id} \amalg \nu_1 \amalg \cdots \amalg \nu_n \amalg \mathrm{id}} \{-\infty\} \star \langle l_1\rangle \star \cdots \star \langle l_n\rangle \star \{\infty\} \cong \langle\langle l_1 + \cdots + l_n\rangle\rangle \ ,\end{aligned}$$

here $\star$ is the join of ordered sets and all maps are order-preserving. In particular, we write

$$\mu_{\vec{k}} := \mu_{k_1} \diamond \cdots \diamond \mu_{k_n} \ .$$

On the other hand, we set $\rho_i^{(\vec{k})} : \langle\langle k_1 + \cdots + k_n\rangle\rangle \to \langle\langle k_i\rangle\rangle \in \nabla$ to be the inert morphism with

$$\rho_i^{(\vec{k})}(j) = \begin{cases} -\infty & j \le \sum_{s<i} k_s \ , \\ j - \sum_{s<i} k_s & \sum_{s<i} k_s < j \le \sum_{s\le i} k_s \ , \\ \infty & j > \sum_{s\le i} k_s \ . \end{cases} \tag{4.5.4}$$

We will identify the morphisms above with their images in $\widetilde{\mathbb{E}}_{\mathcal{G}}$.

Using the notation above, one can immediately see

$$(\nu_1'\nu_1 \diamond \cdots \diamond \nu_n'\nu_n) = (\nu_1' \diamond \cdots \diamond \nu_n') \circ (\nu_1 \diamond \cdots \diamond \nu_n) \ , \tag{4.5.5}$$

$$\rho_i^{(\vec{l})} \circ (\nu_1 \diamond \cdots \diamond \nu_n) = \nu_i \circ \rho_i^{(\vec{k})} \ , \tag{4.5.6}$$

for active morphisms $\nu_i : \langle\langle k_i\rangle\rangle \to \langle\langle l_i\rangle\rangle$ and $\nu_i' : \langle\langle l_i\rangle\rangle \to \langle\langle m_i\rangle\rangle$ with $\vec{k} = (k_1, \dots, k_n)$ and $\vec{l} = (l_1, \dots, l_n)$.

Lemma 4.5.4. *Let $\mathcal{C} \in \mathbf{Oper}'_{\mathcal{G}}$, and suppose we are given active morphisms $\nu_i : \langle\langle k_i\rangle\rangle \to \langle\langle l_i\rangle\rangle \in \nabla$ for $1 \le i \le n$. Put $\vec{k} = (k_1, \dots, k_n)$ and $\vec{l} = (l_1, \dots, l_n)$, and suppose in addition $(\widehat{\rho}_i^{(\vec{k})})_X : X \to X_i$ and $(\widehat{\rho}_i^{(\vec{l})})_Y : Y \to Y_i$ are coCartesian morphisms in $\mathcal{C}$ covering $\rho_i^{(\vec{k})}$ and $\rho_i^{(\vec{l})}$ respectively. Then, there is a unique bijection*

$$\varpi : \prod_{i=1}^{n} \mathcal{C}(X_i, Y_i)_{\nu_i} \to \mathcal{C}(X, Y)_{\nu_1 \diamond \cdots \diamond \nu_n}$$

such that $(\widehat{\rho}_i^{(\vec{l})})_Y \circ \varpi(f_1, \dots, f_n) = f_i \circ (\widehat{\rho}_i^{(\vec{k})})$, where $\mathcal{C}(V, W)_\nu$ is the set of morphisms $V \to W \in \mathcal{C}$ covering ν. Moreover, if other active morphisms

$\nu'_i : \langle\langle l_i \rangle\rangle \to \langle\langle m_i \rangle\rangle \in \nabla$ *and coCartesian morphisms* $(\widehat{\rho}^{(\vec{m})})_Z : Z \to Z_i \in \mathcal{C}$ *covering* $\rho^{(\vec{m})}$ *are given for* $1 \le i \le n$*, with* $\vec{m} = (m_1, \dots, m_n)$*, then the square below is commutative:*

$$\begin{array}{ccc} \prod_{i=1}^{n} \mathcal{C}(Y_i, Z_i)_{\nu'_i} \times \mathcal{C}(X_i, Y_i)_{\nu_i} & \xrightarrow{\prod \mathrm{comp}} & \prod_{i=1}^{n} \mathcal{C}(X_i, Z_i)_{\nu'_i \nu_i} \\ \Big\downarrow{\scriptstyle \varpi \times \varpi} & & \Big\downarrow{\scriptstyle \varpi} \\ \mathcal{C}(Y, Z)_{\nu'_1 \diamond \cdots \diamond \nu'_n} \times \mathcal{C}(X, Y)_{\nu_1 \diamond \cdots \diamond \nu_n} & \xrightarrow{\mathrm{comp}} & \mathcal{C}(X, Z)_{(\nu'_1 \nu_1) \diamond \cdots \diamond (\nu'_n \nu_n)} \end{array} \quad . \tag{4.5.7}$$

Proof. The existence of ϖ immediately follows from the equation $\rho_i \mu_{\vec{k}} = \mu_{k_i} \rho_i^{(\vec{k})}$ and the universal property of the coCartesian morphisms. The uniqueness is guaranteed by the property (2) of objects of $\mathbf{Oper}'_{\mathcal{G}}$. To see the last statement, take morphisms $f_i : X_i \to Y_i$ and $g_i : Y_i \to Z_i$ covering ν_i and ν'_i respectively for each $1 \le i \le n$. Then, we have

$$\begin{aligned} (\widehat{\rho}_i^{(\vec{m})})_Z \circ \varpi(g_1, \dots, g_n) \circ \varpi(f_1, \dots, f_n) &= g_i \circ (\widehat{\rho}_i^{(\vec{l})})_Y \circ \varpi(f_1, \dots, f_n) \\ &= g_i f_i \circ (\widehat{\rho}_i^{(\vec{k})})_X \ , \end{aligned}$$

so the uniqueness of ϖ implies

$$\varpi(g_1 f_1, \dots, g_n f_n) = \varpi(g_1, \dots, g_n) \circ \varpi(f_1, \dots, f_n) \ .$$

Hence, the commutativity of (4.5.7) follows. □

Lemma 4.5.5. *Let* $\mathcal{C}$ *be a category of algebraic* $\mathcal{G}$*-operators, and let* $X \in \mathcal{C}_{\langle\langle m \rangle\rangle}$ *is an object together with coCartesian morphisms*

$$(\widehat{\rho}_i)_X : X \to X_i$$

covering the inert morphism $\rho_i : \langle\langle m \rangle\rangle \to \langle\langle 1 \rangle\rangle \in \nabla$ *for* $1 \le i \le m$*. For an inert morphism* $[\rho, x] : \langle\langle n \rangle\rangle \to \langle\langle n \rangle\rangle \in \widetilde{\mathbb{E}}_{\mathcal{G}}$*, say* δ *is the section of* ρ *in* ∇*, suppose we are given an object* $X' \in \mathcal{C}_{\langle\langle n \rangle\rangle}$ *together with coCartesian morphisms*

$$(\widehat{\rho}_j)_{X'} : X' \to X_{x^{-1}(\delta(n))}$$

covering the inert morphism $\rho_j : \langle\langle n \rangle\rangle \to \langle\langle 1 \rangle\rangle$ *for* $1 \le j \le n$*. Then, there is a unique isomorphism* $\widehat{[\rho, x]}_X : X \to X' \in \mathcal{C}$ *covering the morphism* $[\rho, x]$ *which makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\widehat{[\rho,x]}_X} & X' \\ & {\scriptstyle (\widehat{\rho}_{x^{-1}(\delta(j))})_X} \searrow \quad \swarrow {\scriptstyle (\widehat{\rho}_j)_{X'}} & \\ & X_{x^{-1}(\delta(j))} & \end{array} \tag{4.5.8}$$

commutes for each $1 \le j \le n$*. Moreover,* $\widehat{[\rho, x]}_X$ *is coCartesian.*

Proof. According to the computation in Example 4.2.8, we have

$$\rho_j \circ [\rho, x] = [\rho_{\delta(j)}, x] = \rho_{x^{-1}(\delta(j))}$$

so that the first statement directly follows from the property (2) of categories of algebraic $\mathcal{G}$-operators. To prove the last, take coCartesian lifts $\widehat{[\rho,x]}'_X : X \to X''$ of $[\rho,x]$ along X and $(\widehat{\rho}_j)_{X''} : X'' \to X''_j$ of ρ_j along X'' for $1 \le j \le n$. The computation above shows the composition $(\widehat{\rho}_j)_{X''} \circ \widehat{[\rho,x]}'_X$ is a coCartesian lift of $\rho_{x^{-1}(\delta(j))}$, so the uniqueness of coCartesian lifts enables us to assume $X''_j = X_{x^{-1}(\delta(j))}$ and the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\widehat{[\rho,x]}'_X} & X'' \\ & {\scriptstyle (\widehat{\rho}_{x^{-1}(\delta(j))})_X} \searrow \quad \swarrow {\scriptstyle (\widehat{\rho}_j)_{X''}} & \\ & X_{x^{-1}(\delta(j))} & \end{array} \quad . \tag{4.5.9}$$

We have two cones in $\mathcal{C}$ below

$$\left\{(\widehat{\rho}_j)_{X'} : X' \to X_{x^{-1}(\delta(j))}\right\}_{j=1}^{n} \ ,$$
$$\left\{(\widehat{\rho}_j)_{X''} : X'' \to X_{x^{-1}(\delta(j))}\right\}_{j=1}^{n} \ ,$$

both of which consist of coCartesian morphisms and lie over the cone

$$\{\rho_j : \langle\langle n\rangle\rangle \to \langle\langle 1\rangle\rangle\}_{j=1}^{n}$$

in $\widetilde{\mathbb{E}}_{\mathcal{G}}$. Then, the property (2) of categories of algebraic $\mathcal{G}$-operators implies there is a unique isomorphism $\theta : X'' \to X' \in \mathcal{C}_{\langle\langle n\rangle\rangle}$ such that $(\widehat{\rho}_j)_{X'}\theta = (\widehat{\rho}_j)_{X''}$. In view of the uniqueness of the morphism $\widehat{[\rho,x]}_X$, we obtain

$$\theta \circ \widehat{[\rho,x]}_X = \widehat{[\rho,x]}'_X \ .$$

In particular, $\widehat{[\rho,x]}_X$ is isomorphic to a coCartesian morphism, so it is itself coCartesian. ☐

When we endow a $\mathcal{G}$-symmetric structure, it is good to have transfer.

Lemma 4.5.6. *Let $\mathcal{G}$ be a group operad. Suppose we are given a multicategory $\mathcal{M}$ and a category of algebraic $\mathcal{G}$-operators $\mathcal{C}$ together with an equivalence*

$$H : \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \xrightarrow{\sim} \mathcal{C}$$

in the 2-category $\mathbf{Oper}'_{\mathcal{G}}$. Then, there is a unique $\mathcal{G}$-symmetric structure on $\mathcal{M}$ which makes H into an equivalence in $\mathbf{Oper}^{\mathsf{alg}}_{\mathcal{G}}$.

Proof. For each $\mathcal{D} \in \mathbf{Oper}'_{\mathcal{G}}$ with $q' : \mathcal{D} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$, for $X, Y \in \mathcal{D}$, and for $[\varphi, x] : q'(X) \to q'(Y) \in \widetilde{\mathbb{E}}_{\mathcal{G}}$, we write $\mathcal{D}(X,Y)_{[\varphi,x]}$ the set of morphisms of $\mathcal{D}$ lying over $[\varphi, x]$. Hence, in view of Theorem 4.3.9, we have canonical bijections

$$\begin{aligned} \mathcal{M}(a_1 \dots a_n; a) &\cong (\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}})(a_1 \dots a_n, a)_{\mu_n} \\ &\xrightarrow[\cong]{H} \mathcal{C}(H(a_1 \dots a_n), H(a))_{\mu_n} \ . \end{aligned} \tag{4.5.10}$$

for objects $a, a_i \in \mathcal{M}$. On the other hand, since $\widetilde{\mathbb{G}}_{\mathcal{G}} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$ is full and the identity on objects, the induced functor $H_{\widetilde{\mathbb{G}}_{\mathcal{G}}} : \mathcal{M} \wr \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$ is also an equivalence in $\mathbf{Oper}'_{\mathcal{G}}$ in view of Proposition 4.4.6. Thus, we also have bijections

$$\begin{aligned}(\mathcal{M} \rtimes \mathcal{G})(a_1 \dots a_n; a) &\cong (\mathcal{M} \wr \widetilde{\mathbb{G}}_{\mathcal{G}})(a_1 \dots a_n, a)_{\mu_n} \\ &\xrightarrow[\cong]{H_{\widetilde{\mathbb{G}}_{\mathcal{G}}}} (\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(H(a_1 \dots a_n), H(a))_{\mu_n} \ .\end{aligned} \tag{4.5.11}$$

Note that the internal presheaf structure $\mathcal{A}_{\mathcal{C}} : \mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathcal{C}$ is the identity on objects; indeed, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{E}}_{\mathcal{G}} & \xrightarrow{\mathrm{Id} \times \iota} & \mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \\ & \searrow\!\!= \quad \swarrow{\scriptstyle \mathcal{A}_{\mathcal{C}}} & \\ & \mathcal{C} & \end{array} \ . \tag{4.5.12}$$

Combining with the isomorphisms in (4.5.10) and (4.5.11), we now obtain a map

$$\begin{aligned}\mathcal{A}_{\mathcal{M}} : (\mathcal{M} \rtimes \mathcal{G})(a_1 \dots a_n; a) &\cong (\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(H(a_1 \dots a_n), H(a))_{\mu_n} \\ &\xrightarrow{\mathcal{A}_{\mathcal{C}}} \mathcal{C}(H(a_1 \dots a_n), H(a))_{\mu_n} \cong \mathcal{M}(a_1 \dots a_n; a) \ .\end{aligned} \tag{4.5.13}$$

We assert that the map (4.5.13) gives a $\mathcal{G}$-symmetric structure on $\mathcal{M}$. Notice that, the composition operation in $\mathcal{M}$ is recovered from $\mathcal{C}$ as follows: for each $a, a_i \in \operatorname{Ob}\mathcal{M}$ and $\vec{a}^{(i)} = a_1^{(i)} \dots a_{k_i}^{(i)}$, the composition operation is given by

$$\begin{aligned}&\mathcal{M}(a_1 \dots a_n; a) \times \prod_{i=1}^n \mathcal{M}(\vec{a}^{(i)}; a_i) \\ &\cong \mathcal{C}(H(a_1 \dots a_n), H(a))_{\mu_n} \times \prod_{i=1}^n \mathcal{C}(H(\vec{a}^{(i)}), H(a_i))_{\mu_{k_i}} \\ &\xrightarrow[\cong]{\mathrm{id} \times \varpi} \mathcal{C}(H(a_1 \dots a_n), H(a))_{\mu_n} \times \mathcal{C}(H(\vec{a}^{(1)} \dots \vec{a}^{(n)}), H(a_1 \dots a_n))_{\mu_{\vec{k}}} \\ &\xrightarrow{\mathrm{comp.}} \mathcal{C}(H(\vec{a}^{(1)} \dots \vec{a}^{(n)}), H(a))_{\mu_{k_1 + \dots + k_n}} \\ &\cong \mathcal{M}(\vec{a}^{(1)} \dots \vec{a}^{(n)}; a) \ ,\end{aligned} \tag{4.5.14}$$

where ϖ is the bijection in Lemma 4.5.4 with respect to the image by H of the standard coCartesian lifts

$$(\widehat{\rho}_i)_{\vec{a}} : \vec{a} \to a_i \ , \quad (\widehat{\rho}_i^{(\vec{k})})_{\vec{a}^{(1)} \dots \vec{a}^{(n)}} : \vec{a}^{(1)} \dots \vec{a}^{(n)} \to \vec{a}^{(i)} \ \in \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \ .$$

Similarly, the composition in $\mathcal{M} \rtimes \mathcal{G}$ is also recovered from $\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$. Moreover, thanks to the choice of the coCartesian lifts of ρ_i and $\rho_i^{(\vec{k})}$, the square below is commutative:

$$\begin{array}{ccc} \prod_{i=1}^n (\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(H(\vec{a}^{(i)}), H(a_i))_{\mu_{k_i}} & \xrightarrow{\prod \mathcal{A}_{\mathcal{C}}} & \prod_{i=1}^n \mathcal{C}(H(\vec{a}^{(i)}), H(a_i))_{\mu_{k_i}} \\ {\scriptstyle \varpi}\downarrow & & \downarrow{\scriptstyle \varpi} \\ (\mathcal{C} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(H(\vec{a}^{(1)} \dots \vec{a}^{(n)}), H(\vec{a}))_{\mu_{\vec{k}}} & \xrightarrow{\mathcal{A}_{\mathcal{C}}} & \mathcal{C}(H(\vec{a}^{(1)} \dots \vec{a}^{(n)}), H(\vec{a}))_{\mu_{\vec{k}}} \end{array}$$

This together with the functoriality of $\mathcal{A}_{\mathcal{C}}$ implies that the map (4.5.13) actually defines a multifunctor $\mathcal{A}_{\mathcal{M}} : \mathcal{M} \rtimes \mathcal{G} \to \mathcal{M}$. It furthermore turns out that $\mathcal{A}_{\mathcal{M}}$ is actually a $\mathcal{G}$-symmetric structure on $\mathcal{M}$; the unitality and the associativity follow from the corresponding axioms for the internal presheaf structure on $\mathcal{C}$.

Finally, we see H respects the internal presheaf structures over $\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}}$. In view of Theorem 4.3.9 and the property (2) of categories of algebraic $\mathcal{G}$-symmetric operators, it suffices to show that, for each morphism $[\varphi; f_1, \dots, f_n; u, x] : \vec{a} \to \vec{b} = b_1 \dots b_n$ in $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, we have

$$\begin{aligned} H((\widehat{\rho}_j)_{\vec{b}}) \circ H([\varphi; \mathcal{A}_{\mathcal{M}}(f_1, \delta_1^{(\varphi)*}(u)), \dots \mathcal{A}_{\mathcal{M}}(f_n, \delta_n^{(\varphi)*}(u)); x]) \\ = H((\widehat{\rho}_j)_{\vec{b}}) \circ \mathcal{A}_{\mathcal{C}} H_{\widetilde{\mathbb{G}}_{\mathcal{G}}}([\varphi; f_1, \dots, f_n; u, x]) \end{aligned} \tag{4.5.15}$$

for each $1 \le j \le n$. Let $\varphi = \mu\rho$ be the factorization with μ active and ρ inert, so (3) in Lemma 4.1.12 allows us to assume $u = \rho^*(\bar{u})$ with $\bar{u} \in \overline{\mathrm{Dec}}_{\mu}^{\mathcal{G}}$. If we put $\mu = \mu_{\vec{k}}$, then the left hand side of (4.5.15) is computed as

$$\begin{aligned} & H((\widehat{\rho}_j)_{\vec{b}}) \circ H([\varphi; \mathcal{A}_{\mathcal{M}}(f_1, \delta_1^{(\varphi)*}(u)), \dots \mathcal{A}_{\mathcal{M}}(f_n, \delta_n^{(\varphi)*}(u)); x]) \\ & = H([\mu_{k_j}; \mathcal{A}_{\mathcal{M}}(f_j, \delta_j^{(\mu)*}(\bar{u})); e]) \circ H(\widehat{\rho}_j^{(\vec{k})} \widehat{[\rho, x]}_{\vec{a}}) \\ & = \mathcal{A}_{\mathcal{C}} H_{\widetilde{\mathbb{G}}_{\mathcal{G}}}([\mu_{k_j}; f_j; \delta_j^{(\mu)*}(\bar{u}), e]) \circ H(\widehat{\rho}_j^{(\vec{k})} \widehat{[\rho, x]}_{\vec{a}}) \ . \end{aligned} \tag{4.5.16}$$

On the other hand, according to (4.5.12), for every standard coCartesian lift $\widehat{[\rho', x']}$ of an inert morphism in $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, one has

$$H(\widehat{[\rho', x']}) = \mathcal{A}_{\mathcal{C}} H_{\widetilde{\mathbb{G}}_{\mathcal{G}}}(\widehat{[\rho', x']}) \ ,$$

here we identify $\widehat{[\rho', x']}$ with its image in $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ using Lemma 4.4.5. Thus, the right hand side of (4.5.15) is given by

$$\begin{aligned} & H((\widehat{\rho}_j)_{\vec{b}}) \circ \mathcal{A}_{\mathcal{C}} H_{\widetilde{\mathbb{G}}_{\mathcal{G}}}([\varphi; f_1, \dots, f_n; u, x]) \\ & = \mathcal{A}_{\mathcal{C}} H_{\widetilde{\mathbb{G}}_{\mathcal{G}}}(\widehat{\rho}_{j\vec{b}} \circ [\mu_{\vec{k}}; f_1, \dots, f_n; \bar{u}, e] \circ \widehat{[\rho, x]}_{\vec{a}}) \\ & = \mathcal{A}_{\mathcal{C}} H_{\widetilde{\mathbb{G}}_{\mathcal{G}}}([\mu_{k_j}; f_j; \delta_j^{(\mu)*}(\bar{u}), e]) \circ H(\widehat{\rho}_j^{(\vec{k})} \widehat{[\rho, x]}_{\vec{a}}) \ . \end{aligned} \tag{4.5.17}$$

Now, (4.5.16) and (4.5.17) give rise tot the equation (4.5.15), and it shows H is a 1-morphism in $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}$. The uniqueness is obvious by construction. □

Proof of Theorem 4.5.1. In order to show (1), we construct an inverse Θ of (4.5.1). Fix $\mathcal{G}$-symmetric multicategories $\mathcal{M}$ and $\mathcal{N}$. Suppose $H : \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \to \mathcal{N} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ is a map of algebraic $\mathcal{G}$-operators, and take the family

$$\{\lambda_{\vec{a}} = [\mathrm{id}; \lambda_{\vec{a},1}, \dots, \lambda_{\vec{a},m}; e_m]\}_{\vec{a} = a_1 \dots a_m}$$

of morphisms in $\mathcal{N} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ as in Lemma 4.5.3. Note that H induces a functor $\underline{H} : \underline{\mathcal{M}} \to \underline{\mathcal{N}}$ between underlying categories via the restriction to the fibers over $\langle\langle 1 \rangle\rangle \in \widetilde{\mathbb{E}}_{\mathcal{G}}$. On the other hand, if $f \in \mathcal{M}(\vec{a}; b)$ is a multimorphism, we can take a unique multimorphism $H^{\circ}(f) \in \mathcal{N}(H(\vec{a}); H(b))$ so that

$$H([\mu_m; f; e_m]) = [\mu_m; H^{\circ}(f); e_m] \ .$$

We define a multifunctor $\Theta(H) : \mathcal{M} \to \mathcal{N}$ as follows: for each $a \in \operatorname{Ob}\mathcal{M}$, we set $\Theta(H)(a) := \underline{H}(a)$. For each $\vec{a} = a_1 \dots a_m$ and each b, define

$$\begin{array}{rccl} \Theta(H): & \mathcal{M}(\vec{a}; b) & \to & \mathcal{N}(\underline{H}(a_1) \dots \underline{H}(a_m); \underline{H}(b)) \\ & f & \mapsto & \gamma_{\mathcal{N}}(H^{\circ}(f); \lambda_{\vec{a},1}, \dots, \lambda_{\vec{a},m}) \end{array} .$$

Since $\lambda_{a,1}$ is the identity for $a \in \underline{\mathcal{M}}$, $\Theta(H)$ preserves the identities, so we show the multifunctoriality. For $f_i \in \mathcal{M}(a_1^{(i)} \dots a_{k_i}^{(i)}; a_i)$ for $1 \leq i \leq m$, thanks to the commutative square (4.5.2), we have

$$\begin{aligned} &(\widehat{\rho}_i)_{H(\vec{a})} \circ H([\mu_{\vec{k}}; f_1, \dots, f_m; e]) \circ \lambda_{\vec{a}^{(1)} \dots \vec{a}^{(m)}} \\ &= \lambda_{\vec{a},i} \circ H((\widehat{\rho}_i)_{\vec{a}} \circ [\mu_{\vec{k}}; f_1, \dots, f_m; e]) \circ \lambda_{\vec{a}^{(1)} \dots \vec{a}^{(m)}} \\ &= \lambda_{\vec{a},i} \circ H([\mu_{k_i}; f_i; e] \circ (\widehat{\rho}_i^{(\vec{k})})_{\vec{a}^{(1)} \dots \vec{a}^{(m)}}) \circ \lambda_{\vec{a}^{(1)} \dots \vec{a}^{(m)}} \\ &= \lambda_{\vec{a},i} \circ [\mu_{k_i}; H^{\circ}(f_i); e] \circ \lambda_{\vec{a}^{(i)}} \circ (\widehat{\rho}_i^{(\vec{k})})_{H(\vec{a}^{(1)}) \dots H(\vec{a}^{(m)})} \\ &= \lambda_{\vec{a},i} \circ [\mu_{k_i}; \Theta(H)(f_i); e] \\ &= (\widehat{\rho}_i)_{H(\vec{a})} \circ \lambda_{\vec{a}} \circ [\mu_{\vec{k}}; \Theta(H)(f_1), \dots, \Theta(H)(f_m); e] \ , \end{aligned}$$

which, by virtue of the property (2) in Proposition 4.4.1, implies the square below is commutative:

$$\begin{array}{ccc} H(a_1^{(1)}) \dots H(a_{k_1}^{(1)}) \dots H(a_{k_m}^{(m)}) & \xrightarrow{\lambda_{\vec{a}^{(1)} \dots \vec{a}^{(m)}}} & H(\vec{a}^{(1)} \dots \vec{a}^{(m)}) \\ {\scriptstyle [\mu_{\vec{k}}; \Theta(H)(f_1), \dots, \Theta(H)(f_m); e]} \downarrow & & \downarrow {\scriptstyle H([\mu_{\vec{k}}; f_1, \dots, f_m; e])} \\ H(a_1) \dots H(a_m) & \xrightarrow{\lambda_{\vec{a}}} & H(a_1 \dots a_m) \end{array} \tag{4.5.18}$$

Therefore we obtain

$$\begin{aligned} &[\mu_{k_1 + \dots + k_m}; \gamma_{\mathcal{N}}(\Theta(H)(f); \Theta(H)(f_1), \dots, \Theta(H)(f_m)); e] \\ &= [\mu_m; \Theta(H)(f); e] \circ [\mu_{\vec{k}}; \Theta(H)(f_1), \dots, \Theta(H)(f_m); e] \\ &= [\mu_m; H^{\circ}(f); e] \circ \lambda_{\vec{a}} \circ [\mu_{\vec{k}}; \Theta(H)(f_1), \dots, \Theta(H)(f_m); e] \\ &= H([\mu_m; f; e] \circ [\mu_{\vec{k}}; f_1, \dots, f_m; e]) \circ \lambda_{\vec{a}^{(1)} \dots \vec{a}^{(m)}} \\ &= H([\mu_{k_1 + \dots + k_m}; \gamma_{\mathcal{M}}(f; f_1, \dots, f_m); e] \circ \lambda_{\vec{a}^{(1)} \dots \vec{a}^{(m)}} \\ &= [\mu_{k_1 + \dots + k_m}; H^{\circ}(\gamma_{\mathcal{M}}(f; f_1, \dots, f_m)); e] \circ \lambda_{\vec{a}^{(1)} \dots \vec{a}^{(m)}} \\ &= [\mu_{k_1 + \dots + k_m}; \Theta(H)(\gamma_{\mathcal{M}}(f; f_1, \dots, f_m)); e] \ , \end{aligned}$$

and the multifunctoriality of $\Theta(H)$ follows. Furthermore, $\Theta(H) : \mathcal{M} \to \mathcal{N}$ is $\mathcal{G}$-symmetric. To see this, notice that, for $f \in \mathcal{M}(x_* \vec{a}, b)$, we have

$$\begin{aligned} H([\mu_m; f; x]) &= [\mu_m; H^{\circ}(f); e] \circ H(\widehat{x}_{\vec{a}}) \\ &= [\mu_m; H^{\circ}(f); e] \circ \lambda_{x_* \vec{a}} \circ \widehat{x}_{H(\vec{a})} \circ \lambda_{\vec{a}}^{-1} \\ &= [\mu_m; \Theta(H)(f); x] \circ \lambda_{\vec{a}}^{-1} \\ &= \left[\mu_m; \gamma_{\mathcal{N}} \left(\Theta(H)(f); \lambda_{\vec{a}, x^{-1}(1)}^{-1}, \dots, \lambda_{\vec{a}, x^{-1}(m)}\right); x\right] \ . \end{aligned}$$

It follows that, for the induced functor $H_{\mathbb{G}} : \mathcal{M} \wr \mathbb{G}_{\mathcal{G}} \to \mathcal{N} \wr \mathbb{G}_{\mathcal{G}}$, we have

$$H_{\mathbb{G}}([\mu_m; f; x, e_m]) = \left[\mu_m; \gamma_{\mathcal{N}} \left(\Theta(H)(f); \lambda_{\vec{a}, x^{-1}(1)}^{-1}, \dots, \lambda_{\vec{a}, x^{-1}(m)}^{-1}\right); x, e_m\right] \ .$$

Since H is a map of internal presheaves, we obtain

$$\begin{aligned} H([\mu_m; f^x; e_m]) &= \left[\mu_m; \gamma_{\mathcal{N}}\left(\Theta(H)(f); \lambda^{-1}_{\vec{a},x^{-1}(1)}, \dots, \lambda^{-1}_{\vec{a},x^{-1}(m)}\right)^x; e_m\right] \\ &= \left[\mu_m; \gamma_{\mathcal{N}}\left(\Theta(H)(f)^x; \lambda^{-1}_{\vec{a},1}, \dots, \lambda^{-1}_{\vec{a},m}\right); e_m\right] \\ &= [\mu_m; \Theta(H)(f)^x; e_m] \circ \lambda^{-1}_{\vec{a}} \end{aligned}$$

and so $\Theta(H)(f^x) = \Theta(H)(f)^x$.

We extend Θ to an actual functor

$$\Theta : \mathbf{Oper}^{\mathsf{alg}}_{\mathcal{G}}(\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}, \mathcal{N} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}) \to \mathbf{MultCat}_{\mathcal{G}}(\mathcal{M}, \mathcal{N})$$

as follows: note that, in view of Lemma 4.3.1, for 1-morphisms $H, K : \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \to \mathcal{N} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \in \mathbf{Oper}^{\mathsf{alg}}_{\mathcal{G}}$, a 2-morphism $\xi : H \to K$ is nothing but a natural transformation over $\widetilde{\mathbb{E}}_{\mathcal{G}}$. We set

$$\Theta(\xi) := \{\xi_a : H(a) \to K(a)\}_{a \in \mathcal{M}} \ . \tag{4.5.19}$$

To see (4.5.19) forms a multinatural transformation $\Theta(H) \to \Theta(K)$, notice that, for each $\vec{a} = a_1 \dots a_n \in \mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, the naturality of ξ implies the square

$$\begin{array}{ccc} H(a_1 \dots a_n) & \xrightarrow{\xi_{\vec{a}}} & K(a_1 \dots a_n) \\ {\scriptstyle H((\widehat{\rho}_i)_{\vec{a}})}\downarrow & & \downarrow{\scriptstyle K((\widehat{\rho}_i)_{\vec{a}})} \\ H(a_i) & \xrightarrow{\xi_{a_i}} & K(a_i) \end{array}$$

is commutative for each $1 \le i \le n$. Computing the compositions, one obtains

$$\xi_{\vec{a}} = \lambda^{(K)}_{\vec{a}} \circ [\mathrm{id}; \xi_{a_1}, \dots, \xi_{a_n}; e_n] \circ \lambda^{(H)-1}_{\vec{a}} \ ,$$

where $\lambda^{(H)}$ and $\lambda^{(K)}$ are the ones in Lemma 4.5.3 for functors H and K respectively. Then, the multinaturality of (4.5.19) is straightforward.

We verify Θ is actually an inverse to the functor (4.5.1). If $F : \mathcal{M} \to \mathcal{N}$ is a $\mathcal{G}$-symmetric multifunctor, then the $\mathcal{G}$-symmetric multifunctor $\Theta(\widetilde{F}^{\mathcal{G}})$ is exactly F itself since $\lambda_{\vec{a}}$ is trivial in this case. On the other hand, in view of (4.5.2) and (4.5.18), the family $\lambda = \{\lambda_{\vec{a}}\}_{\vec{a}}$ forms a natural isomorphism $\widetilde{\Theta(H)}^{\mathcal{G}} \cong H$. The uniqueness of λ implies it is natural with respect to H. Hence, (4.5.1) is an equivalence of categories, and we have finished the proof of the part (1).

Finally, we show the part (2). Let $q : \mathcal{C} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$ be a category of algebraic $\mathcal{G}$-operators. By virtue of Lemma 4.5.6, in order to see $\mathcal{C}$ lies in the essential image of $(-) \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, it suffices to show there is a multicategory $\mathcal{M}$ together with an equivalence $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \xrightarrow{\sim} \mathcal{C}$ in the 2-category $\mathbf{Oper}'_{\mathcal{G}}$. For each finite word $\vec{W} = W_1 \dots W_n$ of objects in $\mathcal{C}$ with, say, $q(W_i) = \langle\langle k_i \rangle\rangle$, the property (3) of categories of algebraic $\mathcal{G}$-operators allows us to take an object $\varpi(\vec{W}) \in \mathcal{C}_{\langle\langle k_1 + \dots + k_n \rangle\rangle}$ together with coCartesian morphisms

$$(\widehat{\rho}^{(\vec{k})}_i)_{\vec{W}} : \varpi(\vec{W}) \to W_i$$

covering the inert morphism $\rho^{(\vec{k})}_i : \langle\langle k_1 + \dots + k_n \rangle\rangle \to \langle\langle k_i \rangle\rangle \in \nabla$. In the following argument, we fix such data for each $\vec{W}$. Note that the coincidence of the symbol

ϖ here and in Lemma 4.5.4 is intentional; for $\vec{V} = V_1 \dots V_n$ with $V_i \in \mathcal{C}$ and $q(V_i) = \langle\!\langle l_i \rangle\!\rangle$, and for active morphisms $\nu_i : \langle\!\langle k_i \rangle\!\rangle \to \langle\!\langle l_i \rangle\!\rangle \in \nabla$, we have a map

$$\varpi : \prod_{i=1}^{n} \mathcal{C}(W_i, V_i)_{\nu_i} \to \mathcal{C}(\varpi(W_1 \dots W_n), \varpi(V_1 \dots V_n))_{\nu_1 \diamond \cdots \diamond \nu_n}$$

so that

$$(\widehat{\rho}_i^{(\vec{l})})_{\vec{V}} \circ \varpi(f_1, \dots, f_n) = f_i \circ (\widehat{\rho}_i^{(\vec{k})})_{\vec{W}} \ .$$

Put $\underline{\mathcal{C}} := \mathcal{C}_{\langle\!\langle 1 \rangle\!\rangle}$. Note that, for $\vec{X}^{(i)} = X_1^{(i)} \dots X_{k_i}^{(i)}$ with $X_j^{(i)} \in \underline{\mathcal{C}}$, Lemma 4.5.5 asserts that there is a unique isomorphism

$$\theta : \varpi(\varpi(\vec{X}^{(1)}) \dots \varpi(\vec{X}^{(n)})) \cong \varpi(\vec{X}^{(1)} \dots \vec{X}^{(n)}) \in \mathcal{C}_{\langle\!\langle k_1 + \cdots + k_n \rangle\!\rangle}$$

which makes the square below commutes:

$$\begin{array}{ccc} \varpi(\varpi(\vec{X}^{(1)}) \dots \varpi(\vec{X}^{(n)})) & \xrightarrow[\cong]{\theta} & \varpi(\vec{X}^{(1)} \dots \vec{X}^{(n)}) \\ {\scriptstyle \widehat{\rho}_i^{(\vec{k})}} \downarrow & & \downarrow {\scriptstyle \widehat{\rho}_{k_1 + \cdots + k_{i-1} + j}} \\ \varpi(\vec{X}^{(i)}) & \xrightarrow{\widehat{\rho}_j} & X_j^{(i)} \end{array} \ .$$

The property (2) also implies that, for morphisms $f_j^{(i)} : X_j^{(i)} \to Y_j^{(i)} \in \underline{\mathcal{C}}$ for $1 \le i \le n$ and $1 \le j \le k_n$, the following square is also commutative:

$$\begin{array}{ccc} \varpi(\varpi(\vec{X}^{(1)}), \dots, \varpi(\vec{X}^{(n)})) & \xrightarrow{\theta} & \varpi(\vec{X}^{(1)} \dots \vec{X}^{(n)}) \\ {\scriptstyle \varpi(\varpi(\vec{f}^{(1)}), \dots, \varpi(\vec{f}^{(n)}))} \downarrow & & \downarrow {\scriptstyle \varpi(f_1^{(1)}, \dots, f_{k_1}^{(1)}, \dots, f_{k_n}^{(n)})} \\ \varpi(\varpi(\vec{Y}^{(1)}), \dots, \varpi(\vec{Y}^{(n)})) & \xrightarrow{\theta} & \varpi(\vec{Y}^{(1)} \dots \vec{Y}^{(n)}) \end{array} \tag{4.5.20}$$

In addition, the uniqueness of θ guarantees that it makes the diagram below commute:

$$\begin{array}{ccc} & \varpi(\varpi(\varpi(\vec{X}^{(1,1)}) \dots \varpi(\vec{X}^{(1,r_1)})) \dots \varpi(\varpi(\vec{X}^{(n,1)}) \dots \varpi(\vec{X}^{(n,r_n)}))) & \\ \swarrow^{\theta} & & \searrow^{\varpi(\theta, \dots, \theta)} \\ \varpi(\varpi(\vec{X}^{(1,1)}) \dots \varpi(\vec{X}^{(1,r_1)}) \dots \varpi(\vec{X}^{(n,r_n)})) & & \varpi(\varpi(\vec{X}^{(1,1)} \dots \vec{X}^{(1,r_1)}) \dots \varpi(\vec{X}^{(n,r_n)})) \\ \searrow_{\theta} & & \swarrow_{\theta} \\ & \varpi(\vec{X}^{(1,1)} \dots \vec{X}^{(1,r_1)} \dots \vec{X}^{(n,r_n)}) & \end{array} \tag{4.5.21}$$

We now define a multicategory $\mathcal{M}_{\mathcal{C}}$ so that

- objects are those in $\underline{\mathcal{C}}$;
- for $X, X_1, \dots, X_m \in \underline{\mathcal{C}}$, we set

$$\mathcal{M}_{\mathcal{C}}(X_1 \dots X_m; X) := \mathcal{C}(\varpi(X_1 \dots X_m), X)_{\mu_m} \ ;$$

- the composition is given by

$$\begin{aligned}\gamma :&\mathcal{M}_{\mathcal{C}}(X_1 \dots X_n; X) \times \prod_{i=1}^{n} \mathcal{M}_{\mathcal{C}}(\vec{X}^{(i)}; X_i)\\
&\cong \mathcal{C}(\varpi(X_1 \dots X_n), X)_{\mu_n} \times \prod_{i=1}^{n} \mathcal{C}(\varpi(\vec{X}^{(i)}), X_i)_{\mu_{k_i}}\\
&\xrightarrow{\mathrm{id}\times\varpi} \mathcal{C}(\varpi(X_1 \dots X_n), X)_{\mu_n}\\
&\qquad\quad \times \mathcal{C}(\varpi(\varpi(\vec{X}^{(1)}) \dots \varpi(\vec{X}^{(n)})), \varpi(X_1 \dots X_n))_{\mu_{\vec{k}}}\\
&\xrightarrow{\mathrm{comp.}} \mathcal{C}(\varpi(\varpi(\vec{X}^{(1)}) \dots \varpi(\vec{X}^{(n)})), X)_{\mu_{k_1+\dots+k_n}}\\
&\xrightarrow{\theta^{-1*}} \mathcal{C}(\varpi(\vec{X}^{(1)} \dots \vec{X}^{(n)}), X)_{\mu_{k_1+\dots+k_n}} ;\end{aligned}$$

in other words, we have

$$\gamma(f; f_1, \dots, f_n) = f \circ \varpi(f_1, \dots, f_n) \circ \theta^{-1} .$$

The associativity of the composition is verified as follows: take morphisms $f \in \mathcal{M}_{\mathcal{C}}(X_1 \dots X_n; X)$, $f_i \in \mathcal{M}_{\mathcal{C}}(X_1^{(i)} \dots X_{k_i}^{(i)}; X_i)$, and $f_j^{(i)} \in \mathcal{M}_{\mathcal{C}}(\vec{X}^{(i,j)}; X_j^{(i)})$ for $1 \le i \le n$ and $1 \le j \le k_i$. Then, thanks to the commutative squares (4.5.7), (4.5.20), and (4.5.21), we have

$$\begin{aligned}&\gamma(f; \gamma(f_1; f_1^{(1)}, \dots, f_{k_1}^{(1)}), \dots, \gamma(f_n; f_1^{(n)}, \dots, f_{k_n}^{(n)}))\\
&= f \circ \varpi(f_1 \circ \varpi(f_1^{(1)}, \dots, f_{k_1}^{(1)}) \circ \theta^{-1}, \dots, f_n \circ \varpi(f_1^{(n)}, \dots, f_{k_n}^{(n)}) \circ \theta^{-1}) \circ \theta^{-1}\\
&= f \circ \varpi(f_1, \dots, f_n)\\
&\qquad \circ \varpi(\varpi(f_1^{(1)}, \dots, f_{k_1}^{(1)}), \dots, \varpi(f_1^{(n)}, \dots, f_{k_n}^{(n)})) \circ \varpi(\theta, \dots, \theta)^{-1} \circ \theta^{-1}\\
&= \gamma(f; f_1, \dots, f_n) \circ \varpi(f_1^{(1)}, \dots, f_{k_1}^{(1)}, \dots, f_{k_n}^{(n)}) \circ \theta^{-1}\\
&= \gamma(\gamma(f; f_1, \dots, f_n); f_1^{(1)}, \dots, f_{k_1}^{(1)}, \dots, f_{k_n}^{(n)}) ,\end{aligned}$$

which implies the associativity of the composition. The unitality is obvious so that $\mathcal{M}_{\mathcal{C}}$ is actually a multicategory.

We define a functor $P : \mathcal{M}_{\mathcal{C}} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \to \mathcal{C}$ as follows: for each object $X_1 \dots X_m \in \mathcal{M}_{\mathcal{C}} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$, put

$$P(X_1 \dots X_m) := \varpi(X_1 \dots X_m) .$$

As for a morphism $[\varphi; f_1, \dots, f_n; x] : X_1 \dots X_m \to Y_1 \dots Y_n$, taking the factorization $\varphi = \mu\rho$ with μ active and ρ inert, we set

$$P([\varphi; f_1, \dots, f_n; x]) := \varpi(f_1, \dots, f_n) \circ \theta^{-1} \circ \widehat{[\rho, x]}_{X_1 \dots X_m} , \tag{4.5.22}$$

where $\widehat{[\rho, x]}_{X_1 \dots X_m} : \varpi(X_1 \dots X_m) \to \varpi(X_{x^{-1}(1)} \dots X_{x^{-1}(m)})$ is the coCartesian lift of $[\rho, x]$ given in Lemma 4.5.5. The functoriality of P directly follows from the uniqueness of each morphisms in the right hand side of (4.5.22). It is clear that P preserves coCartesian lifts of inert morphisms, so P is a 1-morphism in $\mathbf{Oper}'_{\mathcal{G}}$. In addition, the property (3) of categories of algebraic $\mathcal{G}$-operators implies P is essentially surjective. Hence, in order to see P is an equivalence, it remains to show it is fully faithful. Consider an arbitrary morphism in $\mathcal{C}$ of the form

$$h : \varpi(X_1 \ldots X_m) \to \varpi(Y_1 \ldots Y_n)$$

covering $[\varphi, x] : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \widetilde{\mathbb{E}}_{\mathcal{G}}$. If $\varphi = \mu\rho$ is the factorization with μ active and ρ inert, say δ is the section of ρ, the universal property of the coCartesian morphism $\widehat{[\rho, x]}_{X_1 \ldots X_m}$ given in Lemma 4.5.5 implies that there is a unique morphism $h' : \varpi(X_{x^{-1}(\delta(1))} \ldots X_{x^{-1}(\delta(m))}) \to \varpi(Y_1 \ldots Y_n) \in \mathcal{C}$ covering μ such that

$$h = h' \circ \widehat{[\rho, x]}_{X_1 \ldots X_m}$$

In view of Lemma 4.5.4, h' can be uniquely written as $h' = \varpi(h_1, \ldots, h_m)$. This implies P is fully faithful, and this completes the proof. □

Chapter 5

Categories of operators

It turned out that the notion of categories of operators introduced by May and Thomason [58] give a considerable advantage over the usual algebraic description of operads or multicategories especially when we go into the higher situation. For example, Lurie [54] defined the notion of ∞-operads based on the idea. Namely, he exhibited ∞-operads as fibrations of ∞-categories over the category $\mathbf{Fin}_*$ of pointed finite sets together with several universal lifting properties similar to Proposition 4.4.1. Surprisingly, although he mainly treated *symmetric* operads, there is no algebraic data; everything is encoded into the base category $\mathbf{Fin}_*$ and the lifting properties.

In this chapter, we will discuss a variant of them for $\mathcal{G}$-symmetric multicategories; we will exhibit them as a kind of fibrations over a certain 2-category associated with $\mathcal{G}$.

5.1 Internal Grothendieck construction

We first review the internal analogue of the Grothendieck construction. The essential idea was originally proposed by Meyer [59] [60]. Though the notion is available in general Cartesian categories, we specialize it in the case of the category $\mathbf{Cat}$.

Let $\mathcal{X}$ be an internal presheaf over a double category $\mathfrak{C} \rightrightarrows \mathcal{B}$, say $\gamma : \mathfrak{C}\times_{\mathcal{B}}\mathfrak{C} \to \mathfrak{C}$, $\iota : \mathcal{B} \to \mathfrak{C}$, and $\mathcal{A}_{\mathcal{X}} : \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \to \mathcal{X}$ are the functors in the structures. Notice that, in this case, we have the following diagram

$$
\begin{array}{ccccc}
\mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \times_{\mathcal{B}} \mathfrak{C} & \xrightarrow{\mathcal{A}_{\mathcal{X}}\times \mathrm{Id}} & \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} & \xrightarrow{\mathrm{proj}} & \mathfrak{C} \\
{\scriptstyle \mathrm{proj}}\downarrow & \lrcorner & {\scriptstyle \mathrm{proj}}\downarrow & \lrcorner & \downarrow{\scriptstyle t} \\
\mathcal{X} \times_{\mathcal{B}} \mathfrak{C} & \xrightarrow{\mathcal{A}_{\mathcal{X}}} & \mathcal{X} & \longrightarrow & \mathcal{B}
\end{array}
$$

where each square is a pullback. We write $(\mathcal{X} \times_{\mathcal{B}} \mathfrak{C}) \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{B}} \mathfrak{C})$ the pullback of the cospan

$$\mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \xrightarrow{\mathcal{A}_{\mathcal{X}}} \mathcal{X} \xleftarrow{\mathrm{proj}} \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \,,$$

then we obtain two functors

$$\Gamma : (\mathcal{X} \times_{\mathcal{B}} \mathfrak{C}) \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{B}} \mathfrak{C}) \cong \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \times_{\mathcal{B}} \mathfrak{C} \xrightarrow{\mathrm{Id}\times\gamma} \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \ ,$$
$$I : \mathcal{X} = \mathcal{X} \times_{\mathcal{B}} \mathcal{B} \xrightarrow{\mathrm{Id}\times\iota} \mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \ .$$

It is easily verified that these functors makes the diagram

$$\mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \underset{T:=\mathrm{proj}}{\overset{S:=\mathcal{A}_{\mathcal{X}}}{\rightrightarrows}} \mathcal{X}$$

into a double category, which we call the *internal Grothendieck construction* for $\mathcal{X}$.

Example 5.1.1. If $\mathfrak{C} \rightrightarrows \mathcal{B}$ is a double category, then the category $\mathcal{B}$, seen as the terminal object of $\mathbf{Cat}^{/\mathcal{B}}$, admits a unique structure of an internal presheaf; namely the identity functor

$$\mathcal{B} \times_{\mathcal{B}} \mathfrak{C} \xrightarrow{=} \mathfrak{C} \ .$$

The internal Grothendieck construction for $\mathcal{B}$ is the double category $\mathfrak{C} \rightrightarrows \mathcal{B}$ itself.

We have a special interest in the following case.

Definition. A double category $s, t : \mathfrak{C} \rightrightarrows \mathcal{B}$ is said to have *vertically discrete objects* if s and t are the identity on objects.

If a double category $\mathfrak{C} \rightrightarrows \mathcal{B}$ has vertically discrete objects, the functors $\gamma : \mathfrak{C} \times_{\mathcal{B}} \mathfrak{C} \to \mathfrak{C}$ and $\iota : \mathcal{B} \to \mathfrak{C}$ are also the identity on objects. As mentioned in Remark 4.2.4, in this case, we can see the double category as a 2-category. Indeed, define a 2-category $\mathfrak{C}_{\lrcorner\mathcal{B}}$ as follows:

- objects are those in $\mathcal{B}$, or those in $\mathcal{C}$ equivalently;
- for $a, b \in \mathcal{B}$, 1-morphisms $a \to b$ are morphisms $\varphi : a \to b \in \mathcal{B}$;
- for 1-morphisms $\varphi, \psi : a \to b$, 2-morphisms $\varphi \to \psi$ are morphisms $x : a \to b \in \mathfrak{C}$ with $s(x) = \varphi$ and $t(x) = \psi$;
- the horizontal compositions are given by the compositions in $\mathcal{B}$ and $\mathfrak{C}$;
- the vertical composition is given by the functor $\gamma : \mathfrak{C} \times_{\mathcal{B}} \mathfrak{C} \to \mathfrak{C}$.

Note that if $\mathcal{X}$ is an internal presheaf over $\mathfrak{C} \rightrightarrows \mathcal{B}$, then the internal Grothendieck construction $\mathcal{X} \times_{\mathcal{B}} \mathfrak{C} \rightrightarrows \mathcal{X}$ for $\mathcal{X}$ has horizontally discrete objects as soon as so does $\mathfrak{C} \rightrightarrows \mathcal{B}$. Hence, we can assign the 2-category $(\mathcal{X} \times_{\mathcal{B}} \mathfrak{C})_{\lrcorner\mathcal{X}}$ to each internal presheaf $\mathcal{X}$ over $\mathfrak{C} \rightrightarrows \mathcal{B}$. We write $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} := (\mathcal{X} \times_{\mathcal{B}} \mathfrak{C})_{\lrcorner\mathcal{X}}$. In particular, for the terminal internal presheaf $\mathcal{B}$ over $\mathfrak{C} \rightrightarrows \mathcal{B}$, put $\mathbf{B}_{\mathcal{B}}\mathfrak{C} := \mathcal{B} /\!\!/_{\mathcal{B}} \mathfrak{C}$. Actually, it turns out that the assignment canonically extends to a (strict) 3-functor

$$(-) /\!\!/_{\mathcal{B}} \mathfrak{C} : \mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B}) \to \mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}} \ ,$$

where the codomain is the *strict* slice category of the (strict) 3-category $\mathbf{Cat}_2$ of 2-categories, normalized pseudofunctors, pseudonatural transformations, and modifications; for these materials, the original definitions are found in [4] while there is a lot of literature; e.g. Chapter 7 in [8] and [49]. Note that even though we use weak notions, for each internal presheaf $\mathcal{X}$ over $\mathfrak{C} \rightrightarrows \mathcal{B}$, the canonical pseudofunctor $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}$ is actually a strict 2-functor. The reader might be afraid that the appearance of the 3-category makes things unnecessarily complicated while we have worked on 2-categories. One can, however, see that there are not so many 2-morphisms in $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}$ as we really have to struggle with 3-morphisms. The key is the following result.

Lemma 5.1.2. *Let $\mathcal{X}$ be an internal presheaf over a double category $\mathfrak{C} \rightrightarrows \mathcal{B}$ which has vertically discrete objects. Suppose we are given a 2-morphism $\alpha : \varphi \to \psi$ in the 2-category $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$ together with a 1-morphism f in $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}$ which covers ψ. Then, there is a unique 2-morphism $\alpha^! : f' \to f$ covering α.*

Proof. Let $p : \mathcal{X} \to \mathcal{B}$ be the structure functor of $\mathcal{X} \in \mathbf{Cat}^{/\mathcal{B}}$. For a 2-category $\mathcal{K}$, we denote by $\mathrm{Mor}_k \mathcal{K}$ the set of k-morphisms in $\mathcal{K}$. Then, the canonical 2-functor gives rise to a square

$$\begin{array}{ccc} \mathrm{Mor}_2(\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}) & \longrightarrow & \mathrm{Mor}_2(\mathbf{B}_{\mathcal{B}}\mathfrak{C}) \\ {\scriptstyle \mathrm{cod}}\downarrow & & \downarrow{\scriptstyle \mathrm{cod}} \\ \mathrm{Mor}_1(\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}) & \longrightarrow & \mathrm{Mor}_1(\mathbf{B}_{\mathcal{B}}\mathfrak{C}) \end{array} . \tag{5.1.1}$$

Unwinding the definition, the square (5.1.1) equals to the square

$$\begin{array}{ccc} \coprod_{X,Y \in \mathcal{X}} (\mathcal{X} \times_{\mathcal{B}} \mathfrak{C})(X,Y) & \xrightarrow{\coprod \mathrm{proj}} & \coprod_{a,b \in \mathcal{B}} \mathfrak{C}(a,b) \\ \downarrow & & \downarrow{\scriptstyle t} \\ \coprod_{X,Y \in \mathcal{X}} \mathcal{X}(X,Y) & \longrightarrow & \coprod_{a,b \in \mathcal{B}} \mathcal{B}(a,b) \end{array}$$

which is obviously a pullback. Hence, (5.1.1) is also a pullback, and the result immediately follows. □

In fact, Lemma 5.1.2 is involved with a sort of contraction of higher structures. Following the standard convention, we say a pseudofunctor $F : \mathcal{K} \to \mathcal{L}$ between 2-categories *locally has property P* for a property P on ordinary functors if, for each objects $U, V \in \mathcal{K}$, the functor

$$F : \mathcal{K}(U,V) \to \mathcal{L}(F(U), F(V))$$

has the property P. For example, the following is an immediate consequence of Lemma 5.1.2.

Corollary 5.1.3. *In the situation in Lemma 5.1.2, the 2-functor*

$$\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}$$

is locally faithful and locally conservative.

Note that every locally faithful and locally conservative normalized pseudofunctor $p : \mathcal{K} \to \mathcal{S}$ exhibits $\mathcal{K}$ as a 1-truncated object in the (strict) slice 3-category $\mathbf{Cat}_2^{/\mathcal{S}}$; i.e. for every 2-category $q : \mathcal{L} \to \mathcal{S}$ over $\mathcal{S}$, the 2-category

$$\mathbf{Cat}_2^{/\mathcal{S}}(\mathcal{L}, \mathcal{K}) := \mathbf{Cat}_2(\mathcal{L}, \mathcal{K}) \times_{\mathbf{Cat}_2(\mathcal{L},\mathcal{S})} \{q\}$$

is biequivalent to an ordinary category. To see this, it suffices to show that, for parallel two pseudonatural transformations

$$\alpha, \beta : F \to G : \mathcal{L} \to \mathcal{K}$$

over $\mathcal{S}$, there is at most one modification $\theta : \alpha \to \beta$ over $\mathcal{S}$, which is, if exists, an isomorphism. Expanding the definition, a modification $\theta : \alpha \to \beta$ over $\mathcal{L}$ is an assignment of a 2-morphism

$$\begin{tikzcd} F(M) \arrow[r, bend left, "\alpha_M"] \arrow[r, bend right, "\beta_M"'] \arrow[r, phantom, "\Downarrow\theta_M"] & G(M) \end{tikzcd} \tag{5.1.2}$$

in $\mathcal{K}$ to each object $M \in \mathcal{L}$ such that

(i) for each 1-morphism $h : M \to N \in \mathcal{L}$, the following equation of the compositions of two 2-cells holds:

$$\begin{tikzcd} F(M) \arrow[r, bend left=50, "\alpha_M"] \arrow[r, "\beta_M"'] \arrow[r, phantom, bend left=25, "\Downarrow\theta_M"] \arrow[d, "F(h)"'] & G(M) \arrow[d, "G(h)"] \arrow[dl, Rightarrow, "\beta_h"] \\ F(N) \arrow[r, bend right=50, "\beta_N"'] & G(N) \end{tikzcd} = \begin{tikzcd} F(M) \arrow[r, bend left=50, "\alpha_M"] \arrow[d, "F(h)"'] & G(M) \arrow[d, "G(h)"] \arrow[dl, Rightarrow, "\alpha_h"] \\ F(N) \arrow[r, "\alpha_N"] \arrow[r, bend right=50, "\beta_N"'] \arrow[r, phantom, bend right=25, "\Downarrow\theta_N"] & G(N) \end{tikzcd} ;$$

(ii) for each $M \in \mathcal{L}$, the image of (5.1.2) in $\mathcal{L}$ under $F : \mathcal{K} \to \mathcal{L}$, depicted below, is trivial.

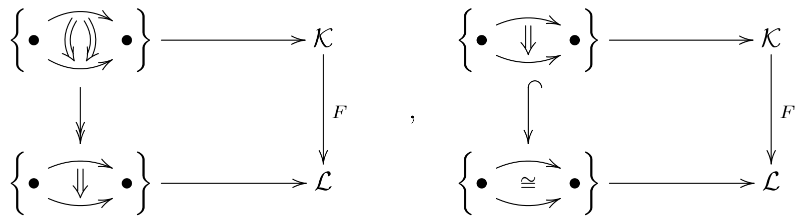
$$\begin{tikzcd} q(M) = pF(M) \arrow[r, bend left, "p(\alpha_S)"] \arrow[r, bend right, "p(\beta_S)"'] \arrow[r, phantom, "\Downarrow p(\theta_S)"] & pG(M) = q(M) \end{tikzcd}$$

In particular, by virtue of the last condition (ii), the local faithfulness of p implies there is only at most one possibility for each θ_M while the local conservativity implies it is an isomorphism.

Corollary 5.1.4. *In the situation in Lemma 5.1.2, the object $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \in \mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}}$ is* 1*-truncated*

We denote by $(\mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}})^{\mathsf{lfc}} \subset \mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}}$ the full 3-subcategory spanned by those objects $\mathcal{K} \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}$ which is locally faithful and locally conservative. Thanks to the argument above, $(\mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}})^{\mathsf{lfc}}$ is triequivalent to a 2-category, that is each hom-2-category is biequivalent to an ordinary category, in spite of its notation.

Remark 5.1.5. From a homotopical aspect of the higher category theory, one can see that the local faithfulness and the local conservativity of a 2-functor $F : \mathcal{K} \to \mathcal{L}$ are equivalent to the solvabilities of the lifting problems of the forms



respectively.

Lemma 5.1.2 has another important consequence.

Corollary 5.1.6. *Let $\mathcal{X}$ be an internal presheaf over a double category $\mathfrak{C} \rightrightarrows \mathcal{B}$ which has vertically discrete objects. Then, the functor*

$$\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}$$

is a local Grothendieck fibration*; i.e. for objects $X, Y \in \mathcal{X}$, the functor*

$$(\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C})(X, Y) \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}(p(X), p(Y))$$

is a Grothendieck fibration, where $p : \mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}$ is the canonical 2-functor.

We denote by $\mathbf{lFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}} \subset \mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}}$ the subcategory consisting of local Grothendieck fibrations over $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$, normalized pseudofunctors over $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$ preserving Cartesian lifts of 2-morphisms of $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$, and all the pseudonatural transformations and modifications over $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$ between such functors. In addition, we set

$$\mathbf{ldFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}} := \mathbf{lFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}} \cap (\mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}})^{\mathsf{lfc}} \subset \mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}} \ .$$

We often think of $\mathbf{ldFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}}$ as a 2-category by taking isomorphism classes of 2-morphisms. By virtue of Corollary 5.1.4 and Corollary 5.1.6, the 2-functor $(-) /\!\!/_{\mathcal{B}} \mathfrak{C}$ factors through the inclusion $\mathbf{ldFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}} \hookrightarrow \mathbf{Cat}_2^{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}}$.

Theorem 5.1.7. *Let $\mathfrak{C} \rightrightarrows \mathcal{B}$ be a double category which has vertically discrete objects. Then, the 2-functor*

$$(-) /\!\!/_{\mathcal{B}} \mathfrak{C} : \mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B}) \to \mathbf{ldFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}} \tag{5.1.3}$$

is a biequivalence of 2-categories.

Proof. We first show (5.1.3) is essentially fully faithful. Let $\mathcal{X}$ and $\mathcal{Y}$ be internal presheaves over $\mathfrak{C} \rightrightarrows \mathcal{B}$. We have to show that the functor

$$\mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B})(\mathcal{X}, \mathcal{Y}) \to \mathbf{ldFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}}(\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}, \mathcal{Y} /\!\!/_{\mathcal{B}} \mathfrak{C}) \tag{5.1.4}$$

is an equivalence of categories. Note that, by virtue of Lemma 5.1.2, every pseudofunctor $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \to \mathcal{Y} /\!\!/_{\mathcal{B}} \mathfrak{C}$ over $\mathbb{B}_{\mathcal{B}}\mathfrak{C}$ is in fact a strict 2-functor. Hence, taking the pullback along the obvious inclusion $\mathcal{B} \hookrightarrow \mathbb{B}_{\mathcal{B}}\mathfrak{C}$, one gets a functor

$$\mathbf{ldFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}}(\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}, \mathcal{Y} /\!\!/_{\mathcal{B}} \mathfrak{C}) \to \mathbf{Cat}^{/\mathcal{B}}(\mathcal{X}, \mathcal{Y}) \ . \tag{5.1.5}$$

Notice that the composition of (5.1.4) and (5.1.5) is exactly the forgetful functor

$$\mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B})(\mathcal{X}, \mathcal{Y}) \to \mathbf{Cat}^{/\mathcal{B}}(\mathcal{X}, \mathcal{Y}) \ ,$$

which is fully faithful by virtue of Lemma 4.3.1. Hence, (5.1.4) is faithful and (5.1.5) is full on the essential image of (5.1.4). Moreover, (5.1.5) is also faithful. Indeed, thanks to Lemma 5.1.2, for functors $F, G : \mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \to \mathcal{B} /\!\!/_{\mathcal{B}} \mathfrak{C}$ over $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$, a 2-natural transformation $\alpha : F \to G$ over $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$ is determined by the family

$$\{\alpha_X : F(X) \to G(X) \in \mathcal{Y} /\!\!/_{\mathcal{B}} \mathfrak{C}\}_{X \in (\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C})} \ .$$

Since $\mathcal{X}$ and $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}$ have the same object set, this implies (5.1.5) is faithful. Using the left cancellativity of fully faithful functors, one can see (5.1.4) is fully faithful. To see it is also essentially surjective, observe that, in view of Lemma 5.1.2, a 1-morphism $F : \mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \to \mathcal{Y} /\!\!/_{\mathcal{B}} \mathfrak{C}$ in $\mathbf{ldFib}_{/\mathbf{B}_{\mathcal{B}}\mathfrak{C}}$ determines and is determined by the data

- for each object $X \in \mathcal{X}$, an object $F(X) \in \mathcal{Y}$;
- for each morphism $f : X \to Y \in \mathcal{X}$, a morphism $F(f) : F(X) \to F(Y)$;

which satisfies the following conditions:

(i) $F(gf) = F(g)F(f)$ for composable morphisms g, f in $\mathcal{X}$;

(ii) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$;

(iii) for each $(f, x) : (\mathcal{X} \times_{\mathcal{B}} \mathfrak{C})(X, Y)$,

$$F(\mathcal{A}_{\mathcal{X}}(f, x)) = \mathcal{A}_{\mathcal{Y}}(F(f), x) \ ,$$

where $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ are the internal presheaf structure on $\mathcal{X}$ and $\mathcal{Y}$ respectively.

The above data, on the other hand, clearly equivalent to a 1-morphism $\mathcal{X} \to \mathcal{Y}$ in $\mathbf{PSh}(\mathfrak{C} \rightrightarrows \mathcal{B})$. Hence, (5.1.4) is essentially surjective.

It remains to show (5.1.3) is essentially surjective. Let $q : \mathcal{K} \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}$ be a locally faithful and locally conservative local Grothendieck fibration. We define a category $\overline{\mathcal{K}}$ as follows:

- objects are those of $\mathcal{K}$;
- for objects $U, V \in \mathcal{K}$, the hom-set $\overline{\mathcal{K}}(U, V)$ is the quotient of the set of 1-morphisms $U \to V$ in $\mathcal{K}$ by the relation $\sim$ such that, for $h, k : a \to b \in \mathcal{K}$, $h \sim k$ if and only if there is a 2-morphism $\alpha : h \to k$ with $q(\alpha)$ trivial in $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$;
- the composition is the one inherited from $\mathcal{K}$.

Thanks to the local faithfulness and the local conservativity, the data above actually define a category $\overline{\mathcal{K}}$, and it is canonically equipped with a functor $\bar{q} : \overline{\mathcal{K}} \to \mathcal{B}$. In addition, since $\mathcal{K} \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}$ is a local Grothendieck fibration, for each 1-morphism $f : U \to V \in \mathcal{K}$, and for each $x : q(U) \to q(V) \in \mathfrak{C}$, we can take a 1-morphism $\mathcal{A}_{\mathcal{K}}(f, x) : U \to V$ together with a 2-morphism $\alpha : \mathcal{A}_{\mathcal{K}}(f, x) \to f$ in $\mathcal{K}$ covering the 2-morphism $(q(f), x) : \gamma(q(f), x) \to q(f)$ in $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$. The local faithfulness and the local conservativity again implies the morphism $\mathcal{A}_{\mathcal{K}}(f,x)$ is unique up to a unique isomorphism whose image is trivial in $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$. Hence, the morphism

$$[\mathcal{A}_{\mathcal{K}}(f,x)] : U \to V$$

in $\overline{\mathcal{K}}$ represented by $\mathcal{A}_{\mathcal{K}}(f,x)$ depends only on the morphism $[f]$. Moreover, by the use of the uniqueness of $\mathcal{A}_{\mathcal{K}}(f,x)$, one can see that it extends to a functor

$$\mathcal{A}_{\mathcal{K}} : \overline{\mathcal{K}} \times_{\mathcal{B}} \mathfrak{C} \to \overline{\mathcal{K}}$$

which gives $\overline{\mathcal{K}}$ a structure of an internal presheaf over $\mathfrak{C} \rightrightarrows \mathcal{B}$. Finally, it is not difficult to see that the canonical functor $\mathcal{K} \to \overline{\mathcal{K}} /\!\!/_{\mathcal{B}} \mathfrak{C}$ is a biequivalence. □

5.2 Review on the canonical model structure on Cat

Before entering the discussion on coCartesian morphisms in 2-categories, we review the canonical model structure on **Cat**. In fact, even when we treat with strict 2-categories, it will turn out that the universal properties on coCartesian morphisms are strongly involved with *weak* notions. Hence, the homotopical aspect in the category theory is essentially important in the following arguments.

There are some interesting model structures constructed on the category **Cat**. The one we are interested in here is the following.

Theorem 5.2.1 (folklore?, Theorem 4 in [43], Theorem 3.1 in [67])**.** *There is a cofibrantly generated simplicial model structure on* **Cat***, in the sense of Quillen, which consists of the following classes:*

- *weak equivalences are equivalences of categories;*
- *cofibrations are functors which is injective on objects;*
- *fibrations are isofibrations.*

Moreover, the model structure is proper.

We call the model structure above the *canonical model structure*. In particular, every object in **Cat** is both fibrant and cofibrant in this model structure.

Remark 5.2.2. The properness of the canonical model structure is a consequence of the results of [42].

Remark 5.2.3. There is another important model structure on **Cat**, namely the *Thomason model structure*, constructed in [74], whose weak equivalences are functors whose geometric realization is a weak homotopy equivalence in the category of (compactly generated) topological spaces.

We are particularly interested in homotopy pullbacks in the canonical model structure. Suppose we have a cospan

$$\mathcal{C} \xrightarrow{F} \mathcal{S} \xleftarrow{G} \mathcal{D} \, . \tag{5.2.1}$$

A recipe to construct homotopy pullbacks is supplied by the *Gluing Lemma* (or the *Cube Lemma* according to Lemma 5.2.6 in [36]). Since every object is fibrant in our case, it asserts that the (strict) pullback on (5.2.1) is a homotopy pullback provided either F or G is an isofibration. We note that the same result was also proved in [42] in the classical category theory. Hence, what we have to do is to take a fibrant replacement of either F or G. Recall that, in **Cat**, we have an excellent factorization of functors.

Definition. Let $F : \mathcal{C} \to \mathcal{S}$ and $G : \mathcal{D} \to \mathcal{S}$ be functors. We define a category $F \downarrow G$, called the *comma category*, as follows:

- objects are triples (X, u, Y) of $X \in \mathcal{C}$, $Y \in \mathcal{G}$, and an arrow $u : F(X) \to G(Y) \in \mathcal{S}$;
- morphisms $(X, u, Y) \to (X', u', Y')$ are pairs (f, g) of $f : X \to X' \in \mathcal{C}$ and $g : Y \to Y' \in \mathcal{D}$ which make the diagram below commute:

$$\begin{array}{ccc} F(X) & \xrightarrow{u} & G(Y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(g)} \\ F(X') & \xrightarrow{u'} & G(Y') \end{array} \quad ;$$

- the composition is defined componentwisely.

The comma category $F \downarrow G$ is canonically equipped with functors

$$\mathrm{dom} : F \downarrow G \to \mathcal{D} \; ; \quad (X, u, Y) \mapsto Y \; ,$$
$$\mathrm{cod} : F \downarrow G \to \mathcal{C} \; ; \quad (X, u, Y) \mapsto X \; ,$$

both of which are isofibrations. In particular, if F is the identity functor, we write $\mathcal{S} \downarrow G := \mathrm{Id}_{\mathcal{S}} \downarrow G$. In this case, we have another canonical functor

$$\mathcal{D} \to \mathcal{S} \downarrow G \; ; \quad Y \mapsto (G(Y), \mathrm{id}_{G(Y)}, Y) \; ,$$

which is obviously fully faithful. Then, it turns out that the functor $G : \mathcal{D} \to \mathcal{S}$ has the following factorization:

$$\mathcal{D} \hookrightarrow \mathcal{S} \downarrow G \xrightarrow{\mathrm{dom}} \mathcal{S} \; .$$

Taking the essential image, say $P(G)$, of the first embedding, one obtains a commutative triangle

$$\begin{array}{ccc} \mathcal{D} & \hookrightarrow & P(G) \\ & {\scriptstyle G}\searrow \quad \swarrow{\scriptstyle \mathrm{dom}} & \\ & \mathcal{S} & \end{array} \quad ,$$

which is a fibrant replacement of $G : \mathcal{D} \to \mathcal{S}$. Note that $P(G)$ coincides with the middle term in the factorization given in part M5 in the proof of Theorem 3.1 in [67]. Finally, we obtain a homotopy pullback on (5.2.1) as the pullback

$$\begin{array}{ccccc} & & \mathcal{C} \times_{\mathcal{S}} P(G) & \longrightarrow & \mathcal{C} \\ & & \downarrow & \lrcorner & \downarrow{\scriptstyle F} \\ \mathcal{D} & \hookrightarrow & P(G) & \xrightarrow{\mathrm{dom}} & \mathcal{S} \end{array}$$

We are interested not only in computing homotopy pullbacks but also in recognizing whether a given square in **Cat** is a homotopy pullback or not. In fact, we will make use of the following criterion.

Lemma 5.2.4. *Suppose we are given a commutative square*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\widetilde{F}} & \mathcal{E} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{B} & \xrightarrow{F} & \mathcal{C} \end{array} \tag{5.2.2}$$

of functors. Then, (5.2.2) *is a homotopy pullback square in the canonical model structure on* **Cat** *if and only if the induced functors*

$$\{b\} \times^{\mathsf{h}}_{\mathcal{B}} \mathcal{D} \to \{F(b)\} \times^{\mathsf{h}}_{\mathcal{C}} \mathcal{E}$$
$$\{b' \xrightarrow{f} b\} \times^{\mathsf{h}}_{\mathcal{B}} \mathcal{D} \to \{F(b') \xrightarrow{F(f)} F(b)\} \times^{\mathsf{h}}_{\mathcal{C}} \mathcal{E}$$

on homotopy pullbacks are equivalences of categories for each object $b \in \mathcal{B}$ and for each morphism $f : b' \to b \in \mathcal{B}$.

Proof. Clearly, the former condition implies the latter, so we show the converse. Note that we only have to prove the statement for specific homotopy pullbacks. Hence, taking a fibrant replacements of functors, we may assume the functors p and q in the diagram (5.2.2) are isofibrations. Moreover, thanks to the base change adjunction $\mathbf{Cat}^{/\mathcal{B}} \rightleftarrows \mathbf{Cat}^{/\mathcal{C}}$, we may also assume the functor $F : \mathcal{B} \to \mathcal{C}$ is the identity functor on $\mathcal{B}$. Then, the latter condition implies we have equivalences of categories

$$\{b\} \times_{\mathcal{B}} \mathcal{D} \to \{b\} \times_{\mathcal{B}} \mathcal{E} \tag{5.2.3}$$
$$\{b' \xrightarrow{f} b\} \times_{\mathcal{B}} \mathcal{D} \to \{b' \xrightarrow{f} b\} \times_{\mathcal{B}} \mathcal{E} \tag{5.2.4}$$

for each $b \in \mathcal{B}$ and each $f : b' \to b \in \mathcal{B}$. We have to show the functor $\widetilde{F} : \mathcal{D} \to \mathcal{E}$ is an equivalence of categories in this case. Considering the equivalences as in (5.2.3) for all the objects in $\mathcal{B}$, one will see $\widetilde{F}$ is an essentially surjective. To see it is also full, suppose we are given a morphism $g : \widetilde{F}(X) \to \widetilde{F}(Y) \in \mathcal{E}$ with $X, Y \in \mathcal{D}$. Since (5.2.4) is essentially surjective for $f = q(g)$, one can find a morphism $\tilde{g}' : X' \to Y' \in \mathcal{D}$ together with a commutative diagram

$$\begin{array}{ccc} \widetilde{F}(X) & \xrightarrow{g} & \widetilde{F}(Y) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ \widetilde{F}(X') & \xrightarrow{\widetilde{F}(\tilde{g}')} & \widetilde{F}(Y') \end{array}$$

in $\mathcal{E}$ with the vertical arrows being isomorphisms covering the identities in $\mathcal{B}$. Since $\widetilde{F}$ is fiberwisely an equivalence of categories by virtue of (5.2.3), there are unique isomorphisms $\tilde{v} : X \to X'$ and $\tilde{w} : Y \to Y'$ in $\mathcal{D}$ with $\widetilde{F}(\tilde{v}) = v$ and $\widetilde{F}(\tilde{w}) = w$, and we obtain $\widetilde{F}(\tilde{w}^{-1}\tilde{g}'\tilde{v}) = \widetilde{F}(g)$. Hence, $\widetilde{F}$ is full. Similarly, It is follows from the faithfulness of (5.2.3) and the fullness of (5.2.4) that $\widetilde{F}$ is also faithful, which completes the proof. ☐

Corollary 5.2.5. *A commutative square*

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{F} & \mathcal{C} \end{array}$$

of functors with $\mathcal{B}$ groupoid is a homotopy pullback square in the canonical model structure on **Cat** *if and only if the induced functor*

$$\{b\} \times^{\mathsf{h}}_{\mathcal{B}} \mathcal{D} \to \{F(b)\} \times^{\mathsf{h}}_{\mathcal{C}} \mathcal{E}$$

on homotopy fibers is an equivalence of categories for each $b \in \mathcal{B}$.

Proof. Clearly the former implies the latter. To see the converse, in view of Lemma 5.2.4, it suffices to show the induced functor

$$\{b' \xrightarrow{f} b\} \times^{\mathsf{h}}_{\mathcal{B}} \mathcal{D} \to \{F(b') \xrightarrow{F(f)} F(b)\} \times^{\mathsf{h}}_{\mathcal{C}} \mathcal{E} \tag{5.2.5}$$

is an equivalence of categories for each morphism $f : b' \to b \in \mathcal{B}$. Since we assumed $\mathcal{B}$ is a groupoid, both morphisms f and $F(f)$ are isomorphisms. It turns out that the functor (5.2.5) is equivalent to the functor

$$\{b\} \times^{\mathsf{h}}_{\mathcal{B}} \mathcal{D} \to \{F(b)\} \times^{\mathsf{h}}_{\mathcal{C}} \mathcal{E} \ .$$

The result is now obvious. □

Remark 5.2.6. Note that the *geometric realization* $|-| : \mathbf{Grpd} \to \mathbf{Top}$, which is a functor from the category of groupoids to that of (compactly generated) topological spaces, exactly exhibits groupoids as models for *homotopy* 1*-types*. In this point of view, Corollary 5.2.5 is also a consequence of Quillen's Theorem A [66].

5.3 CoCartesian morphisms in 2-categories

We begin to discuss the 2-analogue of coCartesian morphisms. Note that there are some different conventions for (co)Cartesian morphisms in 2-categories. For example, the notion appears in Definition 2.1 in [32] with more or less "strict" flavor while we also have 2*-truncated version* for the corresponding notion in $(\infty, 1)$-category theory [53]. To simplify the situation, we consider coCartesian morphisms only for the following kind of pseudofunctors.

Definition (cf. [48])**.** A pseudofunctor $p : \mathcal{K} \to \mathcal{S}$ is called a *pseudo-isofibration* if it satisfies the following two properties:

(i) for every object $U \in \mathcal{K}$, and for every equivalence $u : S \simeq p(U)$ in $\mathcal{S}$, there is an equivalence $\widetilde{u} : \widetilde{S} \to U \in \mathcal{K}$ with $p(\widetilde{u}) = u$;

(ii) p is a local isofibration.

Definition (cf. Definition 3.1.1 in [10] and Proposition 2.4.4.3 in [53])**.** Let $p : \mathcal{K} \to \mathcal{S}$ be a pseudo-isofibration. Then, a 1-morphism $f : U \to V \in \mathcal{K}$ is said to be *p-coCartesian* if, for every object $W \in \mathcal{K}$, the functor

$$\mathcal{K}(V, W) \to \mathcal{K}(U, W) \times_{\mathcal{S}(p(U), p(W))} \mathcal{S}(q(V), q(W))$$

induced by p and f is an equivalence of categories.

Since we assumed p is pseudo-isofibration above, our definition is equivalent to Buckley's one [10] for fibered bicategories. The primary advantage of it over strict variants is the following.

Lemma 5.3.1. *Let $p : \mathcal{K} \to \mathcal{S}$ and $q : \mathcal{L} \to \mathcal{S}$ be pseudo-isofibrations. Suppose we have an equivalence $F : \mathcal{K} \to \mathcal{L}$ in the 3-category $\mathbf{Cat}_2^{/\mathcal{S}}$. Then, a morphism $f : U \to V \in \mathcal{K}$ is p-coCartesian if and only if $F(f)$ is q-coCartesian.*

Proof. Since p is a local isofibration, in view of Theorem 5.2.1, the condition for f to be a coCartesian morphism is equivalent to that, for every object $W \in \mathcal{K}$, the square

$$\begin{array}{ccc} \mathcal{K}(V, W) & \xrightarrow{f^*} & \mathcal{K}(U, W) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{S}(p(V), p(W)) & \xrightarrow{p(f)^*} & \mathcal{S}(p(U), p(W)) \end{array} \tag{5.3.1}$$

is a homotopy pullback in the canonical model structure on **Cat** (or a *bipullback* according to [42]). Note that, since $F : \mathcal{K} \to \mathcal{L}$ is an equivalence in $\mathbf{Cat}_2^{/\mathcal{S}}$, it induces an equivalence from the square (5.3.1) to

$$\begin{array}{ccc} \mathcal{L}(F(V), F(W)) & \xrightarrow{F(f)^*} & \mathcal{L}(F(U), F(W)) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{S}(qF(V), qF(W)) & \xrightarrow{qF(f)^*} & \mathcal{S}(qF(U), qF(W)) \end{array} . \tag{5.3.2}$$

Thus, (5.3.1) is a homotopy pullback if and only if so is (5.3.2). It follows that $f : U \to V$ is p-coCartesian as soon as $F(f)$ is q-coCartesian. To see the converse, for an object $M \in \mathcal{L}$, consider the square below:

$$\begin{array}{ccc} \mathcal{L}(F(V), M) & \xrightarrow{F(f)^*} & \mathcal{L}(F(U), M) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{S}(qF(V), q(M)) & \xrightarrow{qF(f)^*} & \mathcal{S}(qF(U), q(M)) \end{array} . \tag{5.3.3}$$

Since F is an equivalence in $\mathbf{Cat}_2^{/\mathcal{S}}$, one can take an object $W \in \mathcal{K}$ together with an equivalence $h : F(W) \to M$ in $\mathcal{L}$ such that $q(h) : p(W) = qF(W) \to q(M)$ is the identity. The equivalence h clearly gives an equivalence between squares (5.3.2) and (5.3.3). It follows that (5.3.3) is a homotopy pullback for every $M \in \mathcal{L}$ if and only if so is (5.3.2) for every $W \in \mathcal{K}$. The result is now straightforward. □

In the literature [10], the inverse to the bicategorical analogue of the Grothendieck construction is discussed. Although we do not really need the whole construction, we here review the machinery to produce pseudofunctors from the coCartesian lifting, which is well-known in the ordinary category theory.

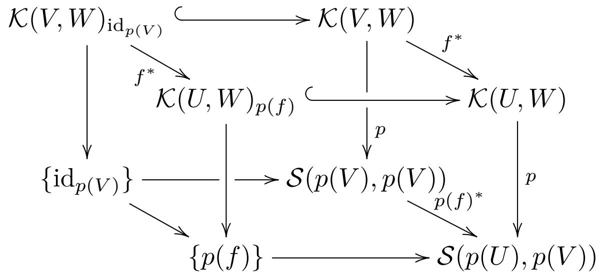
Lemma 5.3.2. *Let $p : \mathcal{K} \to \mathcal{S}$ be a pseudo-isofibration, and suppose $f : U \to V$ is a p-coCartesian morphism in $\mathcal{K}$. For an object $W \in \mathcal{K}$ with $p(W) = p(V)$, define full subcategories $\mathcal{K}(U, W)_{p(f)} \subset \mathcal{K}(U, W)$ and $\mathcal{K}(V, W)_{\mathrm{id}_{p(V)}} \subset \mathcal{K}(V, W)$*

spanned by 1*-morphisms* $U \to W$ *and* $V \to W$ *lying over the* 1*-morphisms* $p(f)$ *and* $\mathrm{id}_{p(V)}$ *in* $\mathcal{S}$ *respectively. Then, the functor*

$$f^* : \mathcal{K}(V,W)_{\mathrm{id}_{p(V)}} \to \mathcal{K}(U,W)_{p(f)}$$

given by the precomposition with f *is an equivalence of categories.*

Proof. We have the following commutative diagram in the category **Cat**:



(5.3.4)

Since f is p-coCartesian, the right face of (5.3.4) is a homotopy pullbacks in the canonical model structure. Moreover, since p is local isofibration, in view of [42], the front and the back faces are also homotopy pullbacks. Then, the "associativity property" of homotopy pullbacks in the right proper model category **Cat** (e.g. see Proposition 13.3.15 in [34]) implies the left face is a homotopy pullback. The required result now follows immediately. □

Let $p : \mathcal{K} \to \mathcal{S}$ be a normalized pseudo-isofibration. Notice that, for each object $A \in \mathcal{S}$, the fiber $\mathcal{K}_A$ of p over A is canonically identified with the hom-2-category $\mathbf{Cat}_2^{/\mathcal{S}}(\{A\}, \mathcal{K})$ of the 3-category $\mathbf{Cat}_2^{/\mathcal{S}}$, where $\{A\}$ is the trivial 2-category together with the 2-functor $\{A\} \hookrightarrow \mathcal{S}$ corresponding to A. Similarly, for a 1-morphism $u : A \to B \in \mathcal{S}$, we denote by $\{u\}$ the 2-category over $\mathcal{S}$ given by

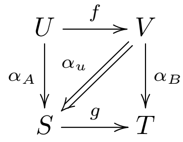
$$\left\{A \xrightarrow{u} B\right\} \hookrightarrow \mathcal{S} \ .$$

The evaluations give rise to 2-functors

$$\mathrm{ev}_A : \mathbf{Cat}_2^{/\mathcal{S}}(\{u\}, \mathcal{K}) \to \mathbf{Cat}_2^{/\mathcal{S}}(\{A\}, \mathcal{K}) \simeq \mathcal{K}_A \ ,$$
$$\mathrm{ev}_B : \mathbf{Cat}_2^{/\mathcal{S}}(\{u\}, \mathcal{K}) \to \mathbf{Cat}_2^{/\mathcal{S}}(\{B\}, \mathcal{K}) \simeq \mathcal{K}_B \ .$$

We see that if u admits all possible p-coCartesian lifts, then there is a pseudo-functor which goes in the inverse direction to ev_A. Define a 2-subcategory $\mathcal{K}_{(u)} \subset \mathbf{Cat}_2^{/\mathcal{S}}(\{u\}, \mathcal{K})$ described as follows:

- objects are coCartesian 1-morphisms $f : U \to V$ in $\mathcal{K}$ covering u;
- the hom-category $\mathcal{K}_{(u)}(f,g)$ is the full subcategory of $\mathbf{Cat}_2^{/\mathcal{S}}(\{u\}, \mathcal{K})(f,g)$ spanned by 2-morphisms in $\mathcal{K}$ of the form



with α_u invertible.

Proposition 5.3.3. *Let $p : \mathcal{K} \to \mathcal{S}$ be a normalized pseudo-isofibration. Suppose $u : A \to B$ is a 1-morphism in $\mathcal{S}$ such that, for each object U in the fiber $\mathcal{K}_A \subset \mathcal{K}$ over A, there is a p-coCartesian morphism $f : U \to V$ with $p(f) = u$. Then, the composition of 2-functors*

$$\mathcal{K}_{(u)} \hookrightarrow \mathbf{Cat}_2^{/\mathcal{S}}(\{u\}, \mathcal{K}) \xrightarrow{\mathrm{ev}_A} \mathcal{K}_A \tag{5.3.5}$$

is a biequivalence.

Proof. The assumption directly implies (5.3.5) is essentially surjective, so it suffices to show it is also essentially fully faithful. More precisely, taking two p-coCartesian morphisms $f : U \to V$ and $g : S \to T$ in $\mathcal{K}$ covering u, we show the functor

$$\mathcal{K}_{(u)}(f, g) \to \mathcal{K}_A(U, S) = \mathcal{K}(U, S)_{\mathrm{id}_A} \tag{5.3.6}$$

is an equivalence of categories. In view of Lemma 5.3.2, it will be enough to see the square

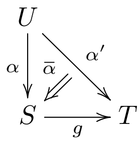
$$\begin{array}{ccc} \mathcal{K}_{(u)}(f,g) & \xrightarrow{\mathrm{ev}_B} & \mathcal{K}(V,T)_{\mathrm{id}_B} \\ {\scriptstyle \mathrm{ev}_A}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{K}(U,S)_{\mathrm{id}_A} & \xrightarrow{g_*} & \mathcal{K}(U,T)_u \end{array} \tag{5.3.7}$$

is a homotopy pullback in the canonical model structure on **Cat**. We follow the recipe described in Section 5.2. Consider the comma category $\mathcal{K}(U,T)_u \downarrow g_*$, and put $P(g_*)$ the essential image of the embedding

$$\mathcal{K}(U,S)_{\mathrm{id}_A} \hookrightarrow \mathcal{K}(U,T)_u \downarrow g_* \; .$$

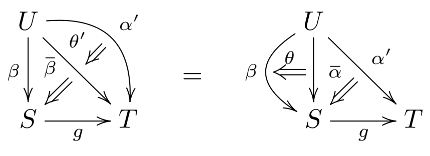
Unwinding the definition, $P(g_*)$ is the category described as follows:

- objects are triples $(\alpha, \bar{\alpha}, \alpha')$ representing invertible 2-morphisms in $\mathcal{K}$ of the form



;

- morphisms $(\alpha, \bar{\alpha}, \alpha') \to (\beta, \bar{\beta}, \beta')$ are pairs (θ, θ') of $\theta : \alpha \to \beta$ and $\theta' : \alpha' \to \beta'$ satisfying the equation of 2-morphisms:



.

We have an obvious functor $\mathcal{K}_{(u)}(f,g) \to P(g_*)$ so that the square below is a pullback:

$$\begin{array}{ccc} \mathcal{K}_{(u)}(f,g) & \xrightarrow{\mathrm{ev}_B} & \mathcal{K}(V,T)_{\mathrm{id}_B} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f^*} \\ P(g_*) & \xrightarrow{\widetilde{g_*}} & \mathcal{K}(U,T)_u \end{array} \quad . \tag{5.3.8}$$

Hence, according to the argument in Section 5.2, we conclude (5.3.7) is a homotopy pullback, and this completes the proof. ☐

In the situation in Proposition 5.3.3, we have a span of 2-functors

$$\mathcal{K}_A \xleftarrow{\mathrm{ev}_A} \mathcal{K}_{(u)} \xrightarrow{\mathrm{ev}_B} \mathcal{K}_B \ ,$$

where the left leg is a biequivalence according to Proposition 5.3.3. Thus, choosing a pseudoinverse to ev_A, we obtains a pseudofunctor

$$u_! : \mathcal{K}_A \to \mathcal{K}_B \ .$$

Although it obviously depends on the choice of a pseudoinverse, we say $u_!$ is a *pseudofunctor induced by* u as long as it is constructed in this way. Fortunately, it is unique up to a pseudonatural isomorphism which is unique up to a unique invertible modification thanks to the uniqueness of the pseudoinverses. In particular, if $\mathcal{K} \to \mathcal{S}$ is 1-truncated in the 3-category $\mathbf{Cat}_2^{/\mathcal{S}}$, the fibers $\mathcal{K}_A$ and $\mathcal{K}_B$ are biequivalent to ordinary categories. Hence, $u_! : \mathcal{K}_A \to \mathcal{K}_B$ can be thought of as an ordinary functor between ordinary categories rather than a pseudofunctor.

To finish the abstract nonsense, we see the inheritance of coCartesian lifting properties through the internal Grothendieck construction. Note that for an internal presheaf $\mathcal{X}$ over a double category $\mathfrak{C} \rightrightarrows \mathcal{B}$ having vertically discrete objects, the functor $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}$ is a local isofibration thanks to Corollary 5.1.6. It is moreover a 2-isofibration if and only if the functor $\mathcal{X} \to \mathcal{B}$ is an isofibration.

Proposition 5.3.4. *Let $\mathcal{X}$ be an internal presheaf over a double category $\mathfrak{C} \rightrightarrows \mathcal{B}$ which has vertically discrete objects, say $\underline{p} : \mathcal{X} \to \mathcal{B}$ and $p_{\mathcal{X}} : \mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \to \mathbf{B}_{\mathcal{B}}\mathfrak{C}$ are the structure functors. If $\underline{p}$ is an isofibration, then for a morphism $f : X \to Y \in \mathcal{X}$, the following are equivalent:*

(a) *f is $\underline{p}$-coCartesian in the ordinary sense;*

(b) *f is $p_{\mathcal{X}}$-coCartesian as a morphism in $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}$ in the sense above.*

Proof. Since we have a pullback square

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C} \\ {\scriptstyle \underline{p}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_{\mathcal{X}}} \\ \mathcal{B} & \hookrightarrow & \mathbf{B}_{\mathcal{B}}\mathfrak{C} \end{array} \quad ,$$

the statement (b) clearly implies (a). Conversely, suppose $f : X \to Y$ is a $\underline{p}$-coCartesian morphism in $\mathcal{X}$. To see it is $p_{\mathcal{X}}$-coCartesian in $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}$, it suffices to show that, for each object $Z \in \mathcal{X}$, the square

$$\begin{array}{ccc} (\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C})(Y,Z) & \xrightarrow{f^*} & (\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C})(X,Z) \\ {\scriptstyle p_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle p_{\mathcal{X}}} \\ \mathbf{B}_{\mathcal{B}}\mathfrak{C}(p_{\mathcal{X}}(Y), p_{\mathcal{X}}(Z)) & \xrightarrow{p_{\mathcal{X}}(f)^*} & \mathbf{B}_{\mathcal{B}}\mathfrak{C}(p_{\mathcal{X}}(Y), p_{\mathcal{X}}(Z)) \end{array} \tag{5.3.9}$$

is actually a pullback. Suppose we have a 2-morphism

$$X \underset{h}{\overset{g}{\underset{\Downarrow\alpha}{\rightrightarrows}}} Z \tag{5.3.10}$$

in $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}$ together with a factorization of its image under $p_{\mathcal{X}}$ depicted as follows:

$$p_{\mathcal{X}}(X) \xrightarrow{p_{\mathcal{X}}(f)} p_{\mathcal{X}}(Y) \underset{\psi}{\overset{\varphi}{\underset{\Downarrow(\psi,x)}{\rightrightarrows}}} p_{\mathcal{X}}(Z) \quad . \tag{5.3.11}$$

Since f is $\underline{p}$-coCartesian in $\mathcal{X}$, there is a unique morphism $h' : Y \to Z \in \mathcal{X}$ covering ψ such that $h = h'f$. In addition, Lemma 5.1.2 implies there is a unique 2-morphism $\alpha' : g' \to h' : Y \to Z$ which covers the 2-morphism (ψ, x) in (5.3.11). Now, we have the 2-morphism

$$X \xrightarrow{f} Y \underset{h'}{\overset{g'}{\underset{\Downarrow\alpha'}{\rightrightarrows}}} Z \tag{5.3.12}$$

which covers (5.3.11). According to Lemma 5.1.2, the lift is unique, so the 2-morphism (5.3.12) equals to (5.3.10). In other words, we obtain a factorization of (5.3.10) through f. Lemma 5.1.2 also guarantees the uniqueness of the factorization. It follows that (5.3.9) is a pullback square, which completes the proof. □

Suppose $\mathcal{X}$ is an internal presheaf over a double category $\mathfrak{C} \rightrightarrows \mathcal{B}$ with vertically discrete objects such that $\mathcal{X} \to \mathcal{B}$ is an isofibration. As a consequence of Proposition 5.3.4, if a morphism $\varphi : a \to b \in \mathcal{B}$ admits coCartesian lifts in $\mathcal{X}$, the corresponding 1-morphism φ in $\mathbf{B}_{\mathcal{B}}\mathfrak{C}$ also admits coCartesian lifts in $\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C}$. Moreover, we can identify the induced functor $\varphi_! : \mathcal{X}_a \to \mathcal{X}_b$ with $\varphi_! : (\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C})_a \to (\mathcal{X} /\!\!/_{\mathcal{B}} \mathfrak{C})_b$ constructed the way above.

5.4 Categories of operators

Let $\mathcal{G}$ be a group operad. We apply the results obtained in the previous sections to the double category $\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}}$, which has vertically discrete objects. Write $\mathbb{B}_{\mathcal{G}} := \mathbf{B}_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$, and call it the *classifying category of* $\mathcal{G}$. We obtain a 3-functor

$$(-) /\!\!/_{\mathbb{B}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} : \mathbf{PSh}(\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}}) \to \mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}} \ . \tag{5.4.1}$$

Since $\mathbb{B}_{\mathcal{G}}$ and $\widetilde{\mathbb{E}}_{\mathcal{G}}$ have the same underlying 1-category, $\mathbb{B}_{\mathcal{G}}$ inherits the orthogonal factorization system described just after Remark 4.2.6. Hence, we say a 1-morphism in $\mathbb{B}_{\mathcal{G}}$ is *inert* (resp. *active*) if it is so in $\widetilde{\mathbb{E}}_{\mathcal{G}}$. On the other hand, it is clear that every 2-morphism in $\mathbb{B}_{\mathcal{G}}$ is invertible, so a pseudofunctor $\mathcal{K} \to \mathbb{B}_{\mathcal{G}}$ is a local Grothendieck fibration if and only if it is a local isofibration.

Definition. Let $\mathcal{G}$ be a group operad. An object $\mathcal{O} \in \mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$ is called a *category of geometric $\mathcal{G}$-operators* if it satisfies the following conditions:

(i) for every inert 1-morphism $[\rho, x] : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \mathbb{B}_{\mathcal{G}}$, and for each object $X \in \mathcal{O}_{\langle\!\langle m \rangle\!\rangle}$ in the fiber, there is a coCartesian morphism $\widehat{[\rho, x]} : X \to X'$ in $\mathcal{O}$ covering $[\rho, x]$;

(ii) if $\widehat{\rho}_i : X \to X_i$ is a coCartesian morphism in $\mathcal{O}$ covering the inert 1-morphism $\rho_i : \langle\!\langle n \rangle\!\rangle \to \langle\!\langle 1 \rangle\!\rangle \in \mathbb{B}_{\mathcal{G}}$ for $1 \leq i \leq n$, then for every object $W \in \mathcal{O}$, say $W \in \mathcal{O}_{\langle\!\langle k \rangle\!\rangle}$ the square

$$\begin{array}{ccc} \mathcal{O}(W, X) & \xrightarrow{((\widehat{\rho}_1)_*, \dots, (\widehat{\rho}_n)_*)} & \prod_{i=1}^{n} \mathcal{O}(W, X_i) \\ \downarrow & & \downarrow \\ \mathbb{B}_{\mathcal{G}}(\langle\!\langle k \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle) & \xrightarrow{((\rho_1)_*, \dots, (\rho_n)_*)} & \mathbb{B}_{\mathcal{G}}(\langle\!\langle k \rangle\!\rangle, \langle\!\langle 1 \rangle\!\rangle)^{\times n} \end{array}$$

is a homotopy pullback in the canonical model structure on **Cat**;

(iii) if $(\rho_i)_! : \mathcal{O}_{\langle\!\langle n \rangle\!\rangle} \to \mathcal{O}_{\langle\!\langle 1 \rangle\!\rangle}$ is a pseudofunctor induced by $\rho_i : \langle\!\langle n \rangle\!\rangle \to \langle\!\langle 1 \rangle\!\rangle$ for $1 \leq i \leq n$, then the pseudofunctor

$$((\rho_1)_!, \dots, (\rho_n)_!) : \mathcal{O}_{\langle\!\langle n \rangle\!\rangle} \to \mathcal{O}_{\langle\!\langle 1 \rangle\!\rangle}^{\times n}$$

is a biequivalence.

Remark 5.4.1. If $\mathcal{O} \in \mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$ satisfies the condition (i), then the pseudofunctor $\mathcal{O} \to \mathbb{B}_{\mathcal{G}}$ is a pseudo-isofibration. Indeed, every equivalence in $\mathbb{B}_{\mathcal{G}}$ is inert, and it is easily seen that a morphism lying over an equivalence is coCartesian if and only if it is an equivalence. This is why the other conditions make sense.

Definition. Let $\mathcal{G}$ be a group operad, and let $\mathcal{O}$ and $\mathcal{P}$ are categories of geometric $\mathcal{G}$-operators. Then, a *map of geometric $\mathcal{G}$-operators* is a normalized pseudofunctor $\mathcal{O} \to \mathcal{P}$ which is a 1-morphism in $\mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$ and preserves coCartesian lifts of inert morphisms.

We denote by $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}} \subset \mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$ the 3-subcategory consisting of categories of geometric $\mathcal{G}$-operators, maps of geometric $\mathcal{G}$-operators, and all 2- and 3-morphisms between them. Similarly to $\mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$, every object of $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$ is 1-truncated, so we can regard $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$ as a 2-category by taking isomorphism classes of 2-morphisms.

Theorem 5.4.2. *Let $\mathcal{G}$ be a group operad. Then, the 2-functor $(-) /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} : \mathbf{PSh}(\widetilde{\mathbb{E}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{G}}_{\mathcal{G}}) \to \mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$ given in* (5.4.1)*restricts to a biequivalence*

$$\mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}} \to \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}} .$$

Proof. In view of Proposition 5.3.4, a pseudofunctor $F : \mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathcal{Y} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$ over $\mathbb{B}_{\mathcal{G}}\mathcal{G}$ preserves coCartesian lifts of inert morphisms if and only if so does the underlying functor $\underline{F} : \mathcal{X} \to \mathcal{Y}$. This implies that, to see the result, it suffices to show the 2-functor $(-) /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$ restricts to $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}} \to \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$ *on objects essentially surjectively.* Notice that, by virtue of Lemma 5.3.1, the subcategory $\mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}} \subset \mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$ is closed under equivalences; i.e. it contains all the (1-)equivalences, and if one has an equivalence $\mathcal{K} \simeq \mathcal{O} \in \mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$ with $\mathcal{O} \in \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$, then $\mathcal{K} \in \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$. Hence, since the 2-functor $(-) /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}$

is a biequivalence, in particular essentially surjective, the problem reduces to showing that, for each $\mathcal{X} \in \mathbf{PSh}(\widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{E}}_{\mathcal{G}})$, we have $\mathcal{X} \in \mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}$ if and only if $\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \in \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$. In particular, using Proposition 5.3.4 and its following argument, one can easily see that the coCartesian lifting problems of inert morphisms are equivalent and that the induced (pseudo)functors

$$\mathcal{X}_{\langle\!\langle n\rangle\!\rangle} \to \mathcal{X}_{\langle\!\langle 1\rangle\!\rangle}^{\times n}, \quad (\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\!\langle n\rangle\!\rangle} \to (\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})_{\langle\!\langle 1\rangle\!\rangle}^{\times n}$$

are canonically identified. Therefore, it remains to show the following statement: if $\widehat{\rho}_i : X \to X_i$ is a coCartesian morphism in $\mathcal{X}$ covering the inert morphism $\rho_i : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle 1\rangle\!\rangle$ for $1 \leq i \leq n$, then for every $W \in \mathcal{X}$, say $W \in \mathcal{X}_{\langle\!\langle k\rangle\!\rangle}$, the following two conditions are equivalent:

(a) the square

$$\begin{array}{ccc} \mathcal{X}(W,X) & \longrightarrow & \prod_{i=1}^{n} \mathcal{X}(W,X_i) \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{E}}_{\mathcal{G}}(\langle\!\langle k\rangle\!\rangle, \langle\!\langle n\rangle\!\rangle) & \longrightarrow & \widetilde{\mathbb{E}}_{\mathcal{G}}(\langle\!\langle k\rangle\!\rangle, \langle\!\langle 1\rangle\!\rangle)^{\times n} \end{array} \tag{5.4.2}$$

is a pullback (of sets and maps).

(b) the square

$$\begin{array}{ccc} (\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(W,X) & \longrightarrow & \prod_{i=1}^{n} (\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(W,X_i) \\ \downarrow & & \downarrow \\ \mathbb{B}_{\mathcal{G}}(\langle\!\langle k\rangle\!\rangle, \langle\!\langle n\rangle\!\rangle) & \longrightarrow & \mathbb{B}_{\mathcal{G}}(\langle\!\langle k\rangle\!\rangle, \langle\!\langle 1\rangle\!\rangle)^{\times n} \end{array} \tag{5.4.3}$$

is a homotopy pullback in the canonical model structure on **Cat**.

We first verify (a) implies (b). In fact, if (5.4.2) is a pullback, then (5.4.3) is a (strict) pullback in **Cat**. Indeed, in this case, using "associativity property" of pullbacks, one can obtain the following pullback square:

$$\begin{array}{ccc} (\mathcal{X} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(W,X) & \longrightarrow & \prod_{i=1}^{n} (\mathcal{X} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(W,X_i) \\ \downarrow & \lrcorner & \downarrow \\ \widetilde{\mathbb{G}}_{\mathcal{G}}(\langle\!\langle k\rangle\!\rangle, \langle\!\langle n\rangle\!\rangle) & \longrightarrow & \widetilde{\mathbb{G}}_{\mathcal{G}}(\langle\!\langle k\rangle\!\rangle, \langle\!\langle 1\rangle\!\rangle)^{\times n} \end{array},$$

which implies (5.4.3) is a pullback on morphisms as well as on objects. Since $\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathbb{B}_{\mathcal{G}}$ is a local isofibration, it follows that (5.4.3) is a homotopy pullback.

Conversely suppose (5.4.3) is a homotopy pullback. To show (a), it suffices to show that the induced equivalence

$$(\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(W,X) \to \mathbb{B}_{\mathcal{G}}(\langle\!\langle k\rangle\!\rangle, \langle\!\langle n\rangle\!\rangle) \times_{\mathbb{B}_{\mathcal{G}}(\langle\!\langle k\rangle\!\rangle, \langle\!\langle 1\rangle\!\rangle)^{\times n}} \prod_{i=1}^{n} (\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})(W,X_i) \tag{5.4.4}$$

of categories is bijective on objects. It is a consequence of Lemma 5.1.2 that (5.4.4) reflects the identities. Since (5.4.4) is an equivalence of categories, in particular fully faithful, this implies (5.4.4) is injective on objects. On the other hand, Lemma 5.1.2 also implies every isomorphism in the right hand side of (5.4.4) is determined by its domain and the image in $\mathbb{B}_{\mathcal{G}}(\langle\langle k\rangle\rangle, \langle\langle n\rangle\rangle)$ under the canonical projection. Since $\mathcal{X} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \to \mathbb{B}_{\mathcal{G}}$ is a local isofibration, it is straightforward that (5.4.4) is an isofibration and, so, surjective objects. Hence, we obtain the part (a). □

Combining Theorem 5.4.2 with Theorem 4.5.1, we obtain a sequence of biequivalences

$$\mathbf{MultCat}_{\mathcal{G}} \xrightarrow{(-)\wr\widetilde{\mathbb{E}}_{\mathcal{G}}} \mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}} \xrightarrow{(-)/\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}}\widetilde{\mathbb{G}}_{\mathcal{G}}} \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}} .$$

We denote the composition by $(-)^{\nabla\mathcal{G}} : \mathbf{MultCat}_{\mathcal{G}} \to \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}$. We here sketch how we can recover the original $\mathcal{G}$-symmetric multicategory $\mathcal{M}$ from the category of geometric $\mathcal{G}$-operators $\mathcal{M}^{\nabla\mathcal{G}} \to \mathbb{B}_{\mathcal{G}}$. Since we have a canonical 2-functor $\widetilde{\mathbb{E}}_{\mathcal{G}} \to \mathbb{B}_{\mathcal{G}}$, we can consider the pullback along it; namely $\mathcal{M}^{\nabla\mathcal{G}} \times_{\mathbb{B}_{\mathcal{G}}} \widetilde{\mathbb{E}}_{\mathcal{G}} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$, which also admits coCartesian lifts of inert morphisms. Hence, one can recover the multicategory $\mathcal{M}$ in the same way as in the latter half of the proof of Theorem 4.5.1. It remains to give the $\mathcal{G}$-symmetric structure on $\mathcal{M}$. For each $n \in \mathbb{N}$, we define a full subcategory $B_n^{\mathcal{G}} \subset \mathbb{B}_{\mathcal{G}}(\langle\langle n\rangle\rangle, \langle\langle 1\rangle\rangle)$ which is the connected component of $\mu_n \in \mathbb{B}_{\mathcal{G}}(\langle\langle n\rangle\rangle, \langle\langle 1\rangle\rangle)$. Notice that, writing $\underline{\mathcal{G}} \subset \mathfrak{S}$ the image of the canonical map $\mathcal{G} \to \mathfrak{S}$ of group operads, we have an exact sequence

$$1 \to \overline{\mathrm{Kec}}_{\mu_n}^{\mathcal{G}} \hookrightarrow \overline{\mathrm{Dec}}_{\mu_n}^{\mathcal{G}} \twoheadrightarrow \underline{\mathcal{G}}_n \to 1$$

since $\overline{\mathrm{Dec}}_{\mu_n}^{\mathcal{G}} \cong \mathcal{G}(n)$. It then turns out that the category $B_n^{\mathcal{G}}$ is canonically isomorphic to the category such that

- objects are permutations $\sigma \in \mathfrak{S}_n$;
- for permutations $\sigma, \tau \in \mathfrak{S}_n$, morphisms $\sigma \to \tau$ are pairs (x, σ) with $x \in \mathcal{G}(n)$ whose underlying permutations are $\tau\sigma^{-1}$;
- the composition is the multiplication in $\mathcal{G}(n)$.

Now, for $a_1, \dots, a_n, b \in \mathcal{M}$, consider the pullback square

$$\begin{array}{ccc} \mathcal{M}^{\nabla\mathcal{G}}(a_1 \dots a_n, b)_{B_n^{\mathcal{G}}} & \longrightarrow & \mathcal{M}^{\nabla\mathcal{G}}(a_1 \dots a_n, b) \\ \downarrow & \lrcorner & \downarrow \\ B_n^{\mathcal{G}} & \hookrightarrow & \mathbb{B}_{\mathcal{G}}(\langle\langle n\rangle\rangle, \langle\langle 1\rangle\rangle) \end{array} .$$

Since the left vertical arrow is a discrete fibration, it is associated with a functor $\widehat{\mathcal{M}}^{\nabla\mathcal{G}}_{a_1 \dots a_n, b} : (B_n^{\mathcal{G}})^{\mathrm{op}} \to \mathbf{Set}$. Expanding the construction, one can see it is given as follows:

- for each permutation $\sigma \in \mathfrak{S}_n$,

$$\widehat{\mathcal{M}}^{\nabla\mathcal{G}}_{a_1 \dots a_n, b}(\sigma) = \mathcal{M}(a_{\sigma^{-1}(1)} \dots a_{\sigma^{-1}(n)}; b) ;$$

- for $(\tau, x) : \sigma \to \tau \in B_n^{\mathcal{G}}$,

$$\widehat{\mathcal{M}}^{\nabla\mathcal{G}}_{a_1\dots a_n,b}(x,\sigma) : \begin{array}{ccc} \mathcal{M}(a_{\tau^{-1}(1)}\dots a_{\tau^{-1}(n)};b) & \to & \mathcal{M}(a_{\sigma^{-1}(1)}\dots a_{\sigma^{-1}(n)};b) \\ f & \mapsto & f^x \end{array},$$

where f^x is the image of f under the action of $x \in \mathcal{G}(n)$.

In other words, we have recovered the $\mathcal{G}$-symmetric structure on $\mathcal{M}$ in terms of the coCartesian lifting.

We finally mention the relation of the notions of categories of geometric $\mathfrak{S}$-operators and of ∞-operads introduced by Lurie [54]. Recall that Segal's category Γ is a category such that

- objects are pointed sets $\langle n\rangle_+ := \{1,\dots,n,*\}$ for $n \in \mathbb{N}$;
- morphisms are, in the opposite direction, arbitrary maps preserving the base-points.

Hence, the opposite category Γ^{op} is a skeleton of the category of finite pointed sets. It admits an orthogonal factorization system: a morphism $\varphi : \langle m\rangle_+ \to \langle n\rangle_+ \in \Gamma^{\mathrm{op}}$ is said to be *inert* if the restriction $\varphi^{-1}\{1,\dots,n\} \to \{1,\dots,n\}$ is bijective while φ *active* if $\varphi^{-1}\{*\} = \{*\}$. It is easily verified that the pair $(\{\text{inert}\},\{\text{active}\})$ forms an orthogonal factorization system. In particular, for each $1 \le i \le n$, we write $\rho_i : \langle n\rangle_+ \to \langle 1\rangle_+$ the inert morphism in Γ^{op} with $\rho_i(j) = 1$ if and only if $j = i$. Then, an ∞-*operad* is an "isofibration" $p : \mathcal{E} \to \Gamma^{\mathrm{op}}$ which satisfies the following three conditions

(i) every inert morphism $\langle m\rangle_+ \to \langle n\rangle_+ \in \Gamma^{\mathrm{op}}$ admits a "coCartesian" lift along each object in the fiber $\mathcal{E}_{\langle m\rangle_+}$;

(ii) if $\widehat{\rho}_i : X \to X_i \in \mathcal{E}$ is a "coCartesian" morphism in $\mathcal{E}$ for each $1 \le i \le n$, then for each $W \in \mathcal{E}$, say $W \in \mathcal{E}_{\langle k\rangle_+}$, the square

$$\begin{array}{ccc} \mathcal{E}(W,X) & \xrightarrow{((\widehat{\rho}_1)_*,\dots,(\widehat{\rho}_n)_*)} & \prod_{i=1}^n \mathcal{E}(W,X_i) \\ \downarrow & & \downarrow \\ \Gamma^{\mathrm{op}}(\langle k\rangle_+,\langle n\rangle_+) & \xrightarrow{((\rho_1)_*,\dots,(\rho_n)_*)} & \Gamma^{\mathrm{op}}(\langle k\rangle_+,\langle 1\rangle_+)^{\times n} \end{array}$$

is a "homotopy pullback";

(iii) the "functor"

$$((\rho_1)_!,\dots,(\rho_n)_!) : \mathcal{E}_{\langle n\rangle_+} \to \mathcal{E}^{\times n}_{\langle 1\rangle_+}$$

induced by coCartesian lifts of $\rho_1,\dots,\rho_n$ is an equivalence of "categories."

Note that although all the double-quoted words above are originally considered in the ∞-categorical context, we here interpret them just literally. In fact, one can regard the resulting notion as 1-*truncated* ∞-*operads*. To compare it with the notion of categories of geometric $\mathfrak{S}$-operators, we consider the following functor:

$$\nabla \to \Gamma^{\mathrm{op}}\ ; \quad \langle\!\langle k\rangle\!\rangle \mapsto \langle\!\langle k\rangle\!\rangle/\{-\infty,\infty\}\ .$$

Lemma 5.4.3. *The functor above extends to a locally fully faithful* 2*-functor* $\mathbb{B}_{\mathfrak{S}} \to \Gamma^{\mathrm{op}}$*. Moreover, a* 1*-morphism in* $\mathbb{B}_{\mathfrak{S}}$ *is inert (resp. active) if and only if so is its image in* Γ^{op}*.*

Proof. To see the first statement, notice that, for two 1-morphisms $[\varphi, \sigma], [\psi, \tau] : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \mathbb{B}_{\mathfrak{S}}$, there is a 2-morphism $[\varphi, \sigma] \to [\psi, \tau]$ if and only if there is an element $\upsilon \in \overline{\mathrm{Dec}}^{\mathfrak{S}}_{\varphi}$ such that

$$[\psi, \tau] = [\varphi, \upsilon\sigma] \ . \tag{5.4.5}$$

Since υ is a right stabilizer of φ, the equation (5.4.5) implies that, for each $\alpha : \langle\!\langle 1 \rangle\!\rangle \to \langle\!\langle m \rangle\!\rangle$ with $\varphi\delta^{\sigma}$ active, we have

$$\varphi\delta^{\sigma} = \varphi\delta^{\upsilon\sigma} = \psi\delta^{\tau} \ .$$

In other words, the induced maps

$$\langle\!\langle m \rangle\!\rangle \cong \nabla(\langle\!\langle 1 \rangle\!\rangle, \langle\!\langle m \rangle\!\rangle) \to \nabla(\langle\!\langle 1 \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle) \cong \langle\!\langle n \rangle\!\rangle \twoheadrightarrow \langle\!\langle n \rangle\!\rangle / \{-\infty, \infty\}$$

coincide with each other. Hence, we obtain an extension $\mathbb{B}_{\mathfrak{S}} \to \Gamma^{\mathrm{op}}$.

Using the equation $\overline{\mathrm{Kec}}^{\mathfrak{S}} = \mathrm{Inr}^{\mathfrak{S}}$, one can see that the 2-category $\mathbb{B}_{\mathfrak{S}}$ allows at most one 2-morphism between a fixed pair of a source and a target, and it immediately implies $\mathbb{B}_{\mathfrak{S}} \to \Gamma^{\mathrm{op}}$ is locally faithful. On the other hand, notice that, for $\upsilon \in \overline{\mathrm{Dec}}^{\mathfrak{S}}_{\varphi}$, (5.4.5) holds if and only if the following two conditions are satisfied:

(i) $(\varphi\upsilon\sigma)^{-1}\{-1, \dots, n\} = (\psi\tau)^{-1}\{1, \dots, n\}$, or equivalently, $(\varphi\upsilon\sigma)^{-1}\{\pm\infty\} = (\psi\tau)^{-1}\{\pm\infty\}$;

(ii) the composition

$$\begin{aligned}\varphi^{-1}\{1, \dots, n\} &\xrightarrow{\upsilon^{-1}} (\varphi\upsilon)^{-1}\{1, \dots, n\} \\ &\xrightarrow{\sigma^{-1}} (\psi\tau)^{-1}\{1, \dots, n\} \xrightarrow{\tau} \psi^{-1}\{1, \dots, n\}\end{aligned}$$

is an order-preserving bijection;

where we identify $\mathfrak{S}_m$ with the subgroup of the permutation group on $\langle\!\langle m \rangle\!\rangle$ consisting of the stabilizers of $\pm\infty$. It follows that one can find $\upsilon \in \overline{\mathrm{Dec}}^{\mathfrak{S}}_{\varphi}$ satisfying (5.4.5) provided the induced maps

$$\langle\!\langle m \rangle\!\rangle / \{\pm\infty\} \to \langle\!\langle n \rangle\!\rangle / \{\pm\infty\}$$

coincide with each other. This completes the proof of the local fully-faithfulness of $\mathbb{B}_{\mathfrak{S}} \to \Gamma^{\mathrm{op}}$.

The last assertion is straightforward. □

Remark 5.4.4. Since the 2-functor $\mathbb{B}_{\mathfrak{S}} \to \Gamma^{\mathrm{op}}$ is bijective on objects, as a consequence of Lemma 5.4.3, $\mathbb{B}_{\mathfrak{S}}$ is biequivalent to a wide subcategory of Γ^{op}. Namely, a morphism $\varphi : \langle m \rangle_+ \to \langle n \rangle_+$ in Γ^{op} belongs to the image of $\mathbb{B}_{\mathfrak{S}}$ if and only if the subset $\varphi^{-1}\{1, \dots, n\} \subset \{1, \dots, m\}$ is consecutive.

Thanks to Lemma 5.4.3, we have a 2-functor

$$\mathbf{ldFib}_{/\Gamma^{\mathrm{op}}} \to \mathbf{ldFib}_{/\mathbb{B}_{\mathfrak{S}}}$$

which sends 1-truncated ∞-operads to categories of geometric $\mathfrak{S}$-operators. In addition, the 2-functor $(-)^{\nabla\mathfrak{S}}$ factors through it and the Lurie's *operadic nerve functor* $N^{\otimes} : \mathbf{MultCat}_{\mathfrak{S}} \to \mathbf{Op}_{\infty}^{\leq 1}$ (see Definition 2.1.1.23 in [54]), where the codomain is the 2-category of 1-truncated ∞-operads, so that the triangle below commutes:

$$\begin{array}{ccc} \mathbf{MultCat}_{\mathfrak{S}} & \xrightarrow{N^{\otimes}} & \mathbf{Op}_{\infty}^{\leq 1} \\ & {\scriptstyle (-)^{\nabla\mathfrak{S}}} \searrow \quad \swarrow & \\ & \mathbf{Oper}_{\mathfrak{S}}^{\mathsf{geom}} & \end{array} .$$

This observation provides us a comparison of two models of categories of operators for symmetric multicategories.

5.5 Representability

We investigate representability of multicategories in terms of categories of operators. Recall that we defined in Section 4.5 the binary operation $\diamond$ on the active morphisms in ∇.

Lemma 5.5.1. *Let $\mathcal{G}$ be a group operad, and let $\mathcal{M}$ be a $\mathcal{G}$-symmetric multicategory. Then, a multimorphism $u \in \mathcal{M}(a_1 \dots a_n; a)$ is strongly universal, in the sense in Section 1.3 if and only if, for every $b_1, \dots, b_k, c_1 \dots c_l \in \mathcal{M}$, the* 1*-morphism*

$$\begin{aligned} &[\mathrm{id}_{\langle\langle k\rangle\rangle} \diamond \mu_n \diamond \mathrm{id}_{\langle\langle l\rangle\rangle}; \mathrm{id}_{b_1}, \dots, \mathrm{id}_{b_k}, u, \mathrm{id}_{c_1}, \dots, \mathrm{id}_{c_l}; e_{k+n+l}] \\ &\qquad : b_1 \dots b_k a_1 \dots a_n c_1 \dots c_l \to b_1 \dots b_k a c_1 \dots c_l \end{aligned} \tag{5.5.1}$$

in $\mathcal{M}^{\nabla\mathcal{G}}$ is coCartesian with respect to the canonical 2*-functor $\mathcal{M}^{\nabla\mathcal{G}} \to \mathbb{B}_{\mathcal{G}}$.*

Proof. It is a straightforward consequence of Proposition 5.3.4 that the multimorphism u is strongly universal provided (5.5.1) is coCartesian for every $b_1, \dots, b_k, c_1, \dots, c_l \in \mathcal{M}$.

Conversely, suppose $u \in \mathcal{M}(a_1 \dots a_n; a)$ is a strongly universal multimorphism. Let us write $\mu = \mathrm{id}_{\langle\langle k\rangle\rangle} \diamond \mu_n \diamond \mathrm{id}_{\langle\langle l\rangle\rangle}$. To prove the morphism (5.5.1) is coCartesian, by virtue of Proposition 5.3.4, it suffices to verify it is a coCartesian morphism in the category $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}}$ with respect to the functor $\mathcal{M} \wr \widetilde{\mathbb{E}}_{\mathcal{G}} \to \widetilde{\mathbb{E}}_{\mathcal{G}}$. For this, take a 1-morphism in $\mathcal{M}^{\nabla\mathcal{G}}$ of the form

$$[\varphi\mu; f_1, \dots, f_s; \mu^*(x)] : b_1 \dots b_k a_1 \dots a_n c_1 \dots c_l \to d_1 \dots d_s .$$

Take the factorization $\varphi = \nu\rho$ in ∇ with ν active and ρ inert. We have

$$\begin{aligned} &\widehat{[\rho, x]} \circ [\mu; \mathrm{id}, \dots, \overset{k+1}{\breve{u}}, \dots, \mathrm{id}; e_{k+n+l}] \\ &\quad = \begin{cases} [\rho\mu^x; \mathrm{id}, \dots, \mathrm{id}; \mu^*(x)] & \rho(x(k+1)) = \pm\infty , \\ [\rho\mu^x; \mathrm{id}, \dots, \overset{i}{\breve{u}}, \dots, \mathrm{id}; \mu^*(x)] & \rho(x(k+1)) = i \text{ with } -\infty < i < \infty . \end{cases} \end{aligned}$$

In the first case, notice that the composition $\rho\mu^x$ is inert so that the presentation $[\nu; f_1, \dots, f_s; e_m]$ makes sense as a morphism in $\mathcal{M}\wr\widetilde{\mathbb{E}}_{\mathcal{G}}$. Thus, we obtain a unique factorization

$$\begin{aligned}
&[\varphi\mu; f_1, \dots, f_s; \mu^*(x)] \\
&= [\nu; f_1, \dots, f_s; e_m] \circ \widehat{[\rho, x]} \circ [\mu; \mathrm{id}, \dots, \overset{k+1}{\breve{u}}, \dots, \mathrm{id}; e_{k+n+l}] \\
&= [\varphi; f_1, \dots, f_s; x] \circ [\mu; \mathrm{id}, \dots, \overset{k+1}{\breve{u}}, \dots, \mathrm{id}; e_{k+n+l}] \ .
\end{aligned}$$

In the other case, if $\varphi(x(k+1)) = j$, the multimorphism f_j belongs to the multihom-set of the form

$$\mathcal{M}(b'_1 \dots b'_{k'} a_1 \dots a_n c'_1 \dots c'_{l'}; d_j)$$

with $k' + n + l' = k_j^{(\varphi\mu)}$. Since u is strongly universal, there is a unique multimorphism $f'_j \in \mathcal{M}(b'_1 \dots b'_{k'} a c'_1 \dots c'_{l'}; d_j)$ such that

$$f_j = \gamma(f'_j; \mathrm{id}, \dots, \overset{k'+1}{\breve{u}}, \mathrm{id}, \dots, \mathrm{id}) \ .$$

Hence, we obtain a factorization

$$\begin{aligned}
[\varphi\mu; f_1, \dots, f_s; \mu^*(x)] &= [\nu; f_1, \dots, f_{j-1}, f'_j, f_{j+1}, \dots, f_s; e] \\
&\qquad \circ [\rho\mu^x; \mathrm{id}, \dots, \overset{i}{\breve{u}}, \dots, \mathrm{id}; \mu^*(x)] \\
&= [\varphi, f_1, \dots, f_{j-1}, f'_j, f_{j+1}, \dots, f_s; x] \\
&\qquad \circ [\mu; \mathrm{id}, \dots, \overset{k+1}{\breve{u}}, \dots, \mathrm{id}] \ ,
\end{aligned}$$

which is clearly unique. It follows that the morphism (5.5.1) is coCartesian. □

A pseudo-isofibration $p : \mathcal{K} \to \mathcal{S}$ is called a *Grothendieck opfibration* if it is a local Grothendieck opfibration such that every 1-morphism in $\mathcal{S}$ admits p-coCartesian lifts, which is the one simply called *opfibration* in the literature [10]. We write $2\text{-}\mathbf{opFib}_{/\mathcal{S}} \subset \mathbf{Cat}_2^{/\mathcal{S}}$ the 3-subcategory consisting of normalized Grothendieck opfibrations over $\mathcal{S}$, normalized pseudofunctors over $\mathcal{S}$ preserving coCartesian 1-morphisms and 2-morphisms, and all the 2-natural transformations and modifications between them over $\mathcal{S}$. In particular, we write

$$\mathbf{RepOper}_{\mathcal{G}}^{\mathsf{geom}} := \mathbf{Oper}^{\mathsf{geom}} \times_{\mathbf{Cat}_2^{/\mathbb{B}_{\mathcal{G}}}} 2\text{-}\mathbf{opFib}_{\mathbb{B}_{\mathcal{G}}} \ .$$

We call an object of $\mathbf{RepOper}_{\mathcal{G}}^{\mathsf{geom}}$ a *representable categories of geometric $\mathcal{G}$-operator*. On the other hand, we have the 2-subcategory $\mathbf{RepMulCat}_{\mathcal{G}} \subset \mathbf{MulCat}_{\mathcal{G}}$ consisting of representable $\mathcal{G}$-symmetric multicategories, $\mathcal{G}$-symmetric multifunctors preserving strongly universal multimorphisms, and multinatural transformations.

Theorem 5.5.2. *Let $\mathcal{G}$ be a group operad. Then, there is a square*

$$\begin{array}{ccc}
\mathbf{RepMulCat}_{\mathcal{G}} & \longrightarrow & \mathbf{RepOper}_{\mathcal{G}}^{\mathsf{geom}} \\
\big\downarrow & & \big\downarrow \\
\mathbf{MultCat}_{\mathcal{G}} & \xrightarrow{(-)^{\nabla\mathcal{G}}} & \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}
\end{array}$$

of 2-functors with horizontal arrows biequivalences.

Proof. The result is a direct consequence of Lemma 5.5.1, Theorem 5.4.2, and Theorem 4.5.1. □

Corollary 5.5.3. *Let $\mathcal{G}$ be a group operad. Then, the composition*

$$\mathbf{MonCat}_{\mathcal{G}} \xrightarrow{(-)^{\otimes}} \mathbf{RepMultCat}_{\mathcal{G}} \xrightarrow{(-)^{\nabla\mathcal{G}}} \mathbf{RepOper}_{\mathcal{G}}^{\mathsf{geom}}$$

is a biequivalence.

Proof. The result immediately follows from Theorem 5.5.2 and Theorem 1.3.10. □

For a $\mathcal{G}$-symmetric monoidal category $\mathcal{C}$, we obtain a normalized Grothendieck opfibration $(\mathcal{C}^{\otimes})^{\nabla\mathcal{G}} \to \mathbb{B}_{\mathcal{G}}$. It turns out that it coincides with the 2-categorical analogue of the Grothendieck construction for the normalized pseudofunctor $\mathcal{C}_{\mathcal{G}}^{\circledS} : \mathbb{B}_{\mathcal{G}} \to \mathbf{Cat}$ given as follows:

- for each $n \in \mathbb{N}$, we set $\mathcal{C}_{\mathcal{G}}^{\circledS}(\langle\!\langle n \rangle\!\rangle) := \mathcal{C}^{\times n}$;
- for each 1-morphism $[\varphi, x] : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \mathbb{B}_{\mathcal{G}}$, we define the functor $\mathcal{C}_{\mathcal{G}}^{\circledS}([\varphi, x])$ to be the composition

  $$\mathcal{C}^{\times m} \xrightarrow{x_*} \mathcal{C}^{\times m} \xrightarrow{\mathcal{C}^{\circledS}(\varphi)} \mathcal{C}^{\times n} ,$$

  where $\mathcal{C}^{\circledS} : \nabla \to \mathbf{Cat}$ is the normalized pseudofunctor associated with the underlying monoidal category of $\mathcal{C}$ (see the construction in the end of Section 1.2);
- for a 2-morphism associated to a morphism $[\varphi, u, x] : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle$ in $\widetilde{\mathbb{G}}_{\mathcal{G}}$, take an element $u_j \in \mathcal{G}(k_j^{(\varphi)})$ for each $1 \leq j \leq n$ so that

  $$[u] = [\gamma_{\mathcal{G}}(e_{n+2}; e_{-\infty}^{(\varphi)}, u_1, \dots, u_n, e_{\infty}^{(\varphi)})] \in \mathrm{Inr}_{\varphi}^{\mathcal{G}} \backslash \overline{\mathrm{Dec}}_{\varphi}^{\mathcal{G}}$$

  using Lemma 4.2.12 and (3) in Lemma 4.1.12, where

  $$e_{\pm\infty}^{(\varphi)} := e_{k_{\pm\infty}^{(\varphi)}} \in \mathcal{G}(k_{\pm\infty}^{(\varphi)}) .$$

  Then, we define the natural transformation $\mathcal{C}_{\mathcal{G}}^{\circledS}([\varphi, u, x]) : \mathcal{C}_{\mathcal{G}}^{\circledS}([\varphi, x]) \to \mathcal{C}_{\mathcal{G}}^{\circledS}([\varphi, ux])$ is the one given by

  $$\begin{aligned}\mathcal{C}_{\mathcal{G}}^{\circledS}([\varphi, x])(X_1, \dots, X_m) &= \big(\otimes_{k_1^{(\varphi)}}((x_*\vec{X})_1^{\varphi}), \dots, \otimes_{k_n^{(\varphi)}}((x_*\vec{X})_n^{\varphi}\big) \\ &\xrightarrow{\Theta^{u_1} \times \cdots \times \Theta^{u_n}} \big(\otimes_{k_1^{(\varphi)}}((u_1)_*(x_*\vec{X})_1^{\varphi}), \dots, \otimes_{k_n^{(\varphi)}}((u_n)_*(x_*\vec{X})_n^{\varphi}\big) \\ &\qquad = \mathcal{C}_{\mathcal{G}}^{\circledS}([\varphi, ux])(X_1, \dots, X_m)\end{aligned}$$

  where Θ^{u_j} is the natural isomorphism given in Example 3.2.3.

Note that the assignments of 1-morphisms and 2-morphisms do not depend on the choice of representatives thanks to Lemma 4.3.2. The pseudofunctor $\mathcal{C}_{\mathcal{G}}^{\circledS} : \mathbb{B}_{\mathcal{G}} \to \mathbf{Cat}$ satisfies the *Segal condition*; namely, for the inert morphisms $\rho_i : \langle\!\langle n \rangle\!\rangle \to \langle\!\langle 1 \rangle\!\rangle \in \mathbb{B}_{\mathcal{G}}$ for $1 \le i \le n$, the functor

$$(\mathcal{C}_{\mathcal{G}}^{\circledS}(\rho_1), \dots, \mathcal{C}_{\mathcal{G}}^{\circledS}(\rho_n)) : \mathcal{C}_{\mathcal{G}}^{\circledS}(\langle\!\langle n \rangle\!\rangle) \to \mathcal{C}_{\mathcal{G}}^{\circledS}(\langle\!\langle 1 \rangle\!\rangle)^{\times n}$$

is an equivalence of categories. More generally, as pointed out in the literature [10], the Grothendieck construction offers a correspondence between pseudofunctors into $\mathbf{Cat}_2$ and Grothendieck opfibrations. In this point of view, it turns out that the Segal condition is the counterpart of the conditions (ii) and (iii) in the definition of categories of geometric $\mathcal{G}$-operators (cf. Proposition 2.1.2.12 in [54] or Lemma 5.5.7 below), and we obtain a correspondence between pseudofunctors $\mathbb{B}_{\mathcal{G}} \to \mathbf{Cat}$ satisfying the Segal condition and representable categories of geometric $\mathcal{G}$-operators.

Finally, we see there is a free construction for representable categories of geometric $\mathcal{G}$-operators. Fortunately, by virtue of Theorem 5.5.2, the traditional recipe to formally add coCartesian lifts works well. We use a factorization system on the 2-category $\mathbb{B}_{\mathcal{G}}$.

Definition. Let $\mathcal{G}$ be a group operad. A 1-morphism in $\mathbb{B}_{\mathcal{G}}$ is said to be *purely active* if it is the image of an active morphism in ∇ under the canonical functor $\nabla \to \widetilde{\mathbb{E}}_{\mathcal{G}} \to \mathbb{B}_{\mathcal{G}}$.

By abuse of notation, we identify morphisms in ∇ with their images in $\mathbb{B}_{\mathcal{G}}$. Hence, purely active morphisms are always written as μ using active morphisms μ in ∇.

Lemma 5.5.4. *Let $\mathcal{G}$ be a group operad.*

(1) *Every 1-morphism in $\mathbb{B}_{\mathcal{G}}$ strictly factors as an inert 1-morphism followed by a strictly unique purely active 1-morphism.*

(2) *the classes of inert morphisms and of active morphisms form an orthogonal factorization system on $\mathbb{B}_{\mathcal{G}}$ in the following sense:*

   (i) *every 1-morphism in $\mathbb{B}_{\mathcal{G}}$ factors as an inert 1-morphism followed by an active 1-morphism;*

   (ii) *if $\boldsymbol{\mu} : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle$ is active and $\boldsymbol{\rho} : \langle\!\langle k \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle$ is inert, then the commutative square*

$$\begin{array}{ccc} \mathbb{B}_{\mathcal{G}}(\langle\!\langle l \rangle\!\rangle, \langle\!\langle m \rangle\!\rangle) & \xrightarrow{\boldsymbol{\mu}_*} & \mathbb{B}_{\mathcal{G}}(\langle\!\langle l \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle) \\ {\scriptstyle \boldsymbol{\rho}^*}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\rho}^*} \\ \mathbb{B}_{\mathcal{G}}(\langle\!\langle k \rangle\!\rangle, \langle\!\langle m \rangle\!\rangle) & \xrightarrow{\boldsymbol{\mu}_*} & \mathbb{B}_{\mathcal{G}}(\langle\!\langle k \rangle\!\rangle, \langle\!\langle n \rangle\!\rangle) \end{array} \tag{5.5.2}$$

   is a homotopy pullback in the canonical model structure on $\mathbf{Cat}$.

Proof. The assertions are direct consequences of Lemma 4.2.10. □

We almost trace the construction described in Section 2.2.4 in [54]. Consider the 2-category $\mathbb{B}_{\mathcal{G}}^{[1]} := \mathbf{Cat}_2([1], \mathbb{B}_{\mathcal{G}})$ of normalized pseudofunctors $[1] = \{0 <$

$1\} \to \mathbb{B}_{\mathcal{G}}$, pseudonatural transformations, and modifications. We denote by $\mathbb{A}_{\mathcal{G}}$ the full 2-subcategory of $\mathbb{B}_{\mathcal{G}}^{[1]}$ spanned by normalized pseudofunctors $[1] \to \mathbb{B}_{\mathcal{G}}$ corresponding to 1-morphisms in $\mathbb{B}_{\mathcal{G}}$ of the form $\mu = [\mu, e]$ for active morphisms μ in ∇. For a normalized 2-isofibration $\mathcal{K} \to \mathbb{B}_{\mathcal{G}}$, we define a 2-category $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ by the pullback square

$$\begin{array}{ccc} \mathrm{Env}_{\mathcal{G}}(\mathcal{K}) & \longrightarrow & \mathcal{K} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{A}_{\mathcal{G}} & \xrightarrow{\mathrm{ev}_0} & \mathbb{B}_{\mathcal{G}} \end{array} ,$$

where the bottom arrow is the evaluation 2-functor at the object $0 \in [1]$. In addition, we regard $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ as a 2-category over $\mathbb{B}_{\mathcal{G}}$ with the composition

$$\mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathbb{A}_{\mathcal{G}} \xrightarrow{\mathrm{ev}_1} \mathbb{B}_{\mathcal{G}} \ .$$

Lemma 5.5.5. *The evaluation* $\mathrm{ev}_1 : \mathbb{A}_{\mathcal{G}} \to \mathbb{B}_{\mathcal{G}}$ *at the object* $1 \in [1]$ *is locally faithful.*

Proof. Let $\mu : \langle\langle m' \rangle\rangle \to \langle\langle n' \rangle\rangle$ and $\nu : \langle\langle n' \rangle\rangle \to \langle\langle n \rangle\rangle$ be two purely active morphisms in $\mathbb{B}_{\mathcal{G}}$, and consider two 1-morphisms $\boldsymbol{\xi}, \boldsymbol{\xi}' : \mu \to \nu \in \mathbb{A}_{\mathcal{G}}$ depicted as the 2-cells below:

$$\begin{array}{ccc} \langle\langle m' \rangle\rangle & \xrightarrow{\boldsymbol{\xi}_0} & \langle\langle n' \rangle\rangle \\ {\scriptstyle \mu}\big\downarrow & \overset{\boldsymbol{\xi}_{01}}{\swarrow} & \big\downarrow{\scriptstyle \nu} \\ \langle\langle m \rangle\rangle & \xrightarrow[\boldsymbol{\xi}_1]{} & \langle\langle n \rangle\rangle \end{array} , \qquad \begin{array}{ccc} \langle\langle m' \rangle\rangle & \xrightarrow{\boldsymbol{\xi}'_0} & \langle\langle n' \rangle\rangle \\ {\scriptstyle \mu}\big\downarrow & \overset{\boldsymbol{\xi}'_{01}}{\swarrow} & \big\downarrow{\scriptstyle \nu} \\ \langle\langle m \rangle\rangle & \xrightarrow[\boldsymbol{\xi}'_1]{} & \langle\langle n \rangle\rangle \end{array}$$

A 2-morphism $\boldsymbol{\theta} : \boldsymbol{\xi}' \to \boldsymbol{\xi} : \mu \to \nu \in \mathbb{A}_{\mathcal{G}}$ is nothing but a pair $(\boldsymbol{\theta}_0, \boldsymbol{\theta}_1)$ of 2-morphisms

$$\boldsymbol{\theta}_0 : \boldsymbol{\xi}'_0 \Rightarrow \boldsymbol{\xi}_0 : \langle\langle m' \rangle\rangle \to \langle\langle n' \rangle\rangle , \qquad \boldsymbol{\theta}_1 : \boldsymbol{\xi}'_1 \Rightarrow \boldsymbol{\xi}_1 : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle$$

satisfying the equation

$$\boldsymbol{\xi}_{01} \circ (\mathrm{id}_\nu \circ_{\mathsf{H}} \boldsymbol{\theta}_0) = (\boldsymbol{\theta}_1 \circ_{\mathsf{H}} \mathrm{id}_\mu) \circ \boldsymbol{\xi}'_{01} \tag{5.5.3}$$

of 2-morphisms in $\mathbb{B}_{\mathcal{G}}$. Since all the 2-morphisms in $\mathbb{B}_{\mathcal{G}}$ are invertible with respect to the vertical composition, the equation (5.5.3) implies that the horizontal composition $\mathrm{id}_\nu \circ_{\mathsf{H}} \boldsymbol{\theta}_0$ is determined by $\boldsymbol{\xi}'$, $\boldsymbol{\xi}$, and $\boldsymbol{\theta}_1 = \mathrm{ev}_1(\boldsymbol{\theta})$. On the other hand, since ν is purely active, in view of Lemma 4.2.10, the functor

$$\mathbb{B}_{\mathcal{G}}(\langle\langle m' \rangle\rangle, \langle\langle n' \rangle\rangle) \xrightarrow{\nu_*} \mathbb{B}_{\mathcal{G}}(\langle\langle m' \rangle\rangle, \langle\langle n \rangle\rangle)$$

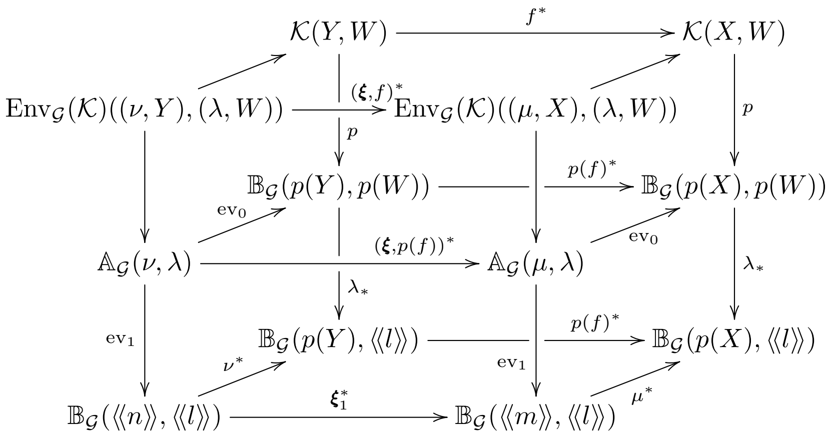
given as the postcomposition with ν is locally faithful. It follows that the horizontal composition $\mathrm{id}_\nu \circ_{\mathsf{H}} \boldsymbol{\theta}_0$ even determines $\boldsymbol{\theta}_0$, and the result follows. □

Lemma 5.5.6. *Let $p : \mathcal{K} \to \mathbb{B}_{\mathcal{G}}$ be a normalized 2-isofibration which admits all coCartesian lifts of inert morphisms, and consider the normalized pseudofunctor $p' : \mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathbb{B}_{\mathcal{G}}$. Then, a morphism in $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ is p'-coCartesian if and only if its image in $\mathcal{K}$ is p-coCartesian and covering an inert morphism.*

Proof. We first show the "if" part of the last statement; i.e. every 1-morphism $(\boldsymbol{\xi}, f) : (\mu, X) \to (\nu, Y) \in \mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ depicted as

$$\begin{array}{ccc} p(X) & \xrightarrow{p(f)} & p(Y) \\ {\scriptstyle\mu}\downarrow & \swarrow_{\boldsymbol{\xi}_{01}} & \downarrow{\scriptstyle\nu} \\ \langle\!\langle m \rangle\!\rangle & \xrightarrow[\boldsymbol{\xi}_1]{} & \langle\!\langle n \rangle\!\rangle \end{array}$$

is p'-coCartesian provided $f : X \to Y \in \mathcal{K}$ is p-coCartesian and $p(f)$ is inert. To see this, for every object $(\lambda, W) \in \mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ with $\lambda : p(W) \to \langle\!\langle l \rangle\!\rangle \in \mathbb{B}_{\mathcal{G}}$, consider the diagram



(5.5.4)

of functors which is commutative up to coherent natural isomorphisms. Since p is a local isofibration, the upper left and right faces in (5.5.4) are homotopy pullbacks by definition. It is also verified that the lower left and right faces are homotopy pullbacks by the explicit computation in Section 5.2. On the other hand, since f is p-coCartesian and $p(f)$ is inert, the back faces in (5.5.4) are homotopy pullbacks by Lemma 5.5.4. Thus, using the "associativity property" for homotopy pullbacks, we conclude that the front faces and their composition are also homotopy pullbacks. Since (λ, W) is arbitrary, It follows that the 1-morphism $(\boldsymbol{\xi}, f)$ is p'-coCartesian.

Next, we show every p'-coCartesian morphism in $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ is of the described form. Consider the set $\mathcal{I}$ of morphisms in $\mathrm{Env}_{\mathcal{G}}$ whose images in $\mathcal{K}$ are inert. Clearly, $\mathcal{I}$ is closed under compositions and isomorphisms. Moreover, it is verified that, for every morphism $\boldsymbol{\varphi} : \langle\!\langle m \rangle\!\rangle \to \langle\!\langle n \rangle\!\rangle \in \mathbb{B}_{\mathcal{G}}$, and for every $(\mu, X) \in \mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ with $p(X) = \langle\!\langle m \rangle\!\rangle$, there is a morphism $(\boldsymbol{\xi}, f) : (\mu, X) \to$

$(\nu, f_! X) \in \mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ which belongs to $\mathcal{I}$. In view of the uniqueness of coCartesian lifts, it follows that $\mathcal{I}$ contains all the p'-coCartesian morphisms, and this completes the proof. □

Before showing $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ is a category of geometric $\mathcal{G}$-operators, we assert that, for Grothendieck opfibrations over $\mathbb{B}_{\mathcal{G}}$, the conditions on categories of geometric $\mathcal{G}$-operators can be relaxed.

Lemma 5.5.7 (cf. Proposition 2.1.2.12 in [54])**.** *Let $p : \mathcal{K} \to \mathbb{B}_{\mathcal{G}}$ be a locally faithful normalized Grothendieck opfibration. Then, to show $\mathcal{K}$ is a category of geometric $\mathcal{G}$-operators, it suffices to verify only the condition* (iii) *in the definition; i.e. the functor*

$$((\rho_1)_!, \dots, (\rho_n)_!) : \mathcal{K}_{\langle\langle n\rangle\rangle} \to \mathcal{K}_{\langle\langle 1\rangle\rangle}^{\times n}$$

induced by the coCartesian lifts of the inert morphisms $\rho_1, \dots, \rho_n : \langle\langle n\rangle\rangle \to \langle\langle 1\rangle\rangle \in \mathbb{B}_{\mathcal{G}}$ is an equivalence of categories.

Proof. Since p is a Grothendieck opfibration, the condition (i) is straightforward. Hence, it suffices to show the condition (ii) follows from (iii) in this case. Suppose we are given a p-coCartesian morphism $\widehat{\rho}_i : X \to X_i$ covering the inert morphism $\rho_i : \langle\langle n\rangle\rangle \to \langle\langle 1\rangle\rangle$ for each $1 \le i \le n$, and consider the commutative square below for an object $W \in \mathcal{K}$:

$$\begin{array}{ccc} \mathcal{K}(W, X) & \xrightarrow{((\widehat{\rho}_1)_*, \dots, (\widehat{\rho}_n)_*)} & \prod_{i=1}^{n} \mathcal{K}(W, X_i) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \prod p} \\ \mathbb{B}_{\mathcal{G}}(p(W), \langle\langle n\rangle\rangle) & \xrightarrow{((\rho_1)_*, \dots, (\rho_n)_*)} & \mathbb{B}_{\mathcal{G}}(p(W), \langle\langle 1\rangle\rangle)^{\times n} \end{array} . \tag{5.5.5}$$

We have to see (5.5.5) is a homotopy pullback square in the canonical model structure on **Cat**. Since p is a Grothendieck opfibration and so a local isofibration, in view of Lemma 5.2.4, it suffices to show that the induced functor

$$\mathcal{K}(W, X)_{\boldsymbol{\varphi}} \to \prod_{i=1}^{n} \mathcal{K}(W, X_i)_{\rho_i \boldsymbol{\varphi}} \tag{5.5.6}$$

is an equivalence of categories for every morphism $\boldsymbol{\varphi} : p(W) \to \langle\langle n\rangle\rangle$, where we write

$$\mathcal{K}(Y, Z)_{\boldsymbol{\psi}} := \{\boldsymbol{\psi}\} \times_{\mathcal{B}_{\mathcal{G}}(p(Y), p(Z))} \mathcal{K}(Y, Z)$$

for objects $Y, Z \in \mathcal{K}$ and morphisms $\boldsymbol{\psi} : p(Y) \to p(Z) \in \mathbb{B}_{\mathcal{G}}$. Since p is a Grothendieck opfibration, we can take a p-coCartesian morphism $\widehat{\boldsymbol{\varphi}} : W \to \boldsymbol{\varphi}_! W$ with $p(\widehat{\boldsymbol{\varphi}}) = \boldsymbol{\varphi}$. Taking also a p-coCartesian lift $\widehat{\rho}_i : \boldsymbol{\varphi} W \to (\rho_i)_! \boldsymbol{\varphi}_! W$ covering

ρ_i for each $1 \leq i \leq n$, we obtain a diagram

$$
\begin{array}{ccc}
\mathcal{K}_{\langle\!\langle n\rangle\!\rangle}(\boldsymbol{\varphi}_! W, X) & \xrightarrow{((\rho_1)_!,\dots,(\rho_n)_!)} & \prod_{i=1}^{n} \mathcal{K}_{\langle\!\langle 1\rangle\!\rangle}((\rho_i)_! \boldsymbol{\varphi}_! W, X_i) \\
\Vert & & \Big\downarrow {\scriptstyle \prod_i \widehat{\rho}_i^*} \\
\mathcal{K}(\boldsymbol{\varphi}_! W, X)_{\mathrm{id}_{\langle\!\langle n\rangle\!\rangle}} & \xrightarrow{((\widehat{\rho}_1)_*,\dots,(\widehat{\rho}_n)_*)} & \prod_{i=1}^{n} \mathcal{K}(\boldsymbol{\varphi}_! W, X_i)_{\rho_i} \\
{\scriptstyle \widehat{\boldsymbol{\varphi}}^*} \Big\downarrow & & \Big\downarrow {\scriptstyle \prod \widehat{\boldsymbol{\varphi}}^*} \\
\mathcal{K}(W, X)_{\boldsymbol{\varphi}} & \xrightarrow{((\widehat{\rho}_1)_*,\dots,(\widehat{\rho}_n)_*)} & \prod_{i=1}^{n} \mathcal{K}(W, X_i)_{\rho_i \boldsymbol{\varphi}}
\end{array}
$$

which is commutative up to coherent natural isomorphisms and with all the vertical arrows being equivalences of categories, where the top horizontal arrow is a part of the functor $\mathcal{K}_{\langle\!\langle n\rangle\!\rangle} \to \mathcal{K}_{\langle\!\langle 1\rangle\!\rangle}^{\times n}$ induced by coCartesian lifts of the inert morphisms $\rho_1, \dots, \rho_n : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle 1\rangle\!\rangle$. It follows that the functor (5.5.6) is an equivalence of categories as soon as $\mathcal{K}_{\langle\!\langle n\rangle\!\rangle} \to \mathcal{K}_{\langle\!\langle 1\rangle\!\rangle}^{\times n}$ is essentially fully faithful. Therefore, we can deduce the condition (ii) from (iii) in the definition of categories of geometric $\mathcal{G}$-operators. $\square$

Theorem 5.5.8 (cf. Proposition 2.2.4.4 in [54])**.** *Let $\mathcal{G}$ be a group operad. Then, for a category of geometric $\mathcal{G}$-operators $\mathcal{K}$, the following hold.*

(1) *The functor $\mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathbb{B}_{\mathcal{G}}$ exhibits $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ as a representable category of geometric $\mathcal{G}$-operators.*

(2) *The normalized pseudofunctor $\mathcal{K} \to \mathrm{Env}_{\mathcal{G}}(\mathcal{K})$; $X \mapsto (\mathrm{id}, X)$ is a map of geometric $\mathcal{G}$-operators such that, for every representable category of geometric $\mathcal{G}$-operators $\mathcal{L}$, the induced 2-functor*

$$
\begin{aligned}
\mathbf{RepOper}_{\mathcal{G}}^{\mathsf{geom}}(\mathrm{Env}_{\mathcal{G}}(\mathcal{K}), \mathcal{L}) \hookrightarrow \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}&(\mathrm{Env}_{\mathcal{G}}(\mathcal{K}), \mathcal{L}) \\
&\to \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}(\mathcal{K}, \mathcal{L})
\end{aligned}
\tag{5.5.7}
$$

is a biequivalence.

Proof. We write $p : \mathcal{K} \to \mathbb{B}_{\mathcal{G}}$ and $p' : \mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathbb{B}_{\mathcal{G}}$ the canonical pseudofunctors. Note first that the functor $\mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathbb{A}_{\mathcal{G}}$ is a pullback of a locally faithful pseudofunctor $\mathcal{K} \to \mathbb{B}_{\mathcal{G}}$, so it is itself locally faithful. Combining with Lemma 5.5.5, one can easily see p' is locally faithful. In addition, it is also a normalized pseudofunctor, so we have $\mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \in \mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$.

Thanks to Lemma 5.5.6, the pseudofunctor p' is a Grothendieck opfibration. Hence, in order to verify $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ is a category of geometric $\mathcal{G}$-operators, in view of Lemma 5.5.7, it suffices to show the pseudofunctor

$$
((\rho_1)_!, \dots, (\rho_n)_!) : \mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\!\langle n\rangle\!\rangle} \to \mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\!\langle 1\rangle\!\rangle}^{\times n} \tag{5.5.8}
$$

induced by the inert morphisms $\rho_1, \dots, \rho_n : \langle\!\langle n\rangle\!\rangle \to \langle\!\langle 1\rangle\!\rangle$ is a biequivalence. In fact, it is essentially surjective: for objects $(\mu_{k_1}, X_1), \dots, (\mu_{k_n}, X_n) \in \mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\!\langle 1\rangle\!\rangle}$

for the purely active morphisms $\mu_{k_i} : \langle\langle k_i \rangle\rangle \to \langle\langle 1 \rangle\rangle$, the condition (iii) on the category of geometric $\mathcal{G}$-operators $\mathcal{K}$ enables us to take an object $X \in \mathcal{K}_{\langle\langle k_1+\cdots+k_n \rangle\rangle}$ together with a p-coCartesian morphism

$$\widehat{\rho}_i^{(\vec{k})} : X \to X_i \in \mathcal{K}$$

covering the inert morphism $\rho_i^{(\vec{k})} : \langle\langle k_1 + \cdots + k_n \rangle\rangle \to \langle\langle k_i \rangle\rangle \in \nabla$ given by (4.5.4) for each $1 \leq i \leq n$. Put $\vec{k} = (k_1, \ldots, k_n)$, and we obtain a commutative square

$$\begin{tikzcd} p(X) \arrow[r, "p(\widehat{\rho}_i^{(\vec{k})})"] \arrow[d, "\mu_{\vec{k}}"'] & p(X_i) \arrow[d, "\mu_{k_i}"] \\ \langle\langle n \rangle\rangle \arrow[r, "\rho_i"] & \langle\langle 1 \rangle\rangle \end{tikzcd} ,$$

which presents a p'-coCartesian morphism in $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ by virtue of Lemma 5.5.6, so the uniqueness of coCartesian lifts implies we have an equivalence

$$(\rho_i)_!(\mu_{\vec{k}}, X) \simeq X_i \in \mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\langle 1 \rangle\rangle}$$

for each $1 \leq i \leq n$. This implies the tuple $(X_1, \ldots, X_n)$ belongs to the essential image of the pseudofunctor (5.5.8), and it is essentially surjective.

To see (5.5.8) is also essentially fully faithful, take tuples $\vec{k} = (k_1, \ldots, k_n), \vec{l} = (l_1, \ldots, l_n)$ of non-negative integers and objects $X \in \mathcal{K}_{\langle\langle k_1+\cdots+k_n \rangle\rangle}$ and $Y \in \mathcal{K}_{\langle\langle l_1+\cdots+l_n \rangle\rangle}$. The category $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\langle n \rangle\rangle}((\mu_{\vec{k}}, X), (\mu_{\vec{l}}, Y))$ is described as follows:

- objects are pairs $(\boldsymbol{\xi}, f)$ of 1-morphism $f : X \to Y$ together with a 2-morphism in $\mathbb{B}_{\mathcal{G}}$ depicted as

$$\begin{tikzcd} p(X) \arrow[rr, "p(f)"] \arrow[dr, "\mu_{\vec{k}}"'] & {} \arrow[d, Rightarrow, "\boldsymbol{\xi}"] & p(Y) \arrow[dl, "\mu_{\vec{l}}"] \\ & \langle\langle n \rangle\rangle & \end{tikzcd} ; \qquad (5.5.9)$$

- morphisms $(\boldsymbol{\xi}, f) \to (\boldsymbol{\xi}', f')$ are 2-morphisms $\boldsymbol{\theta} : f \to f'$ in $\mathcal{K}$ satisfying the equation

$$\begin{tikzcd} p(X) \arrow[rr, bend left, "p(f)"] \arrow[rr, "p(f')"'] \arrow[dr, "\mu_{\vec{k}}"'] & \Downarrow p(\boldsymbol{\theta}) \; \Leftarrow \boldsymbol{\xi}' & p(Y) \arrow[dl, "\mu_{\vec{l}}"] \\ & \langle\langle n \rangle\rangle & \end{tikzcd} = \begin{tikzcd} p(X) \arrow[rr, "p(f)"] \arrow[dr, "\mu_{\vec{k}}"'] & \Leftarrow \boldsymbol{\xi} & p(Y) \arrow[dl, "\mu_{\vec{l}}"] \\ & \langle\langle n \rangle\rangle & \end{tikzcd} .$$

Note that the existence of the 2-morphism (5.5.9) implies that, for every morphism $(\boldsymbol{\xi}, f) : (\mu_{\vec{k}}, X) \to (\mu_{\vec{l}}, Y)$, $\boldsymbol{\xi}$ and $p(f)$ are respectively of the forms

$$\boldsymbol{\xi} = [\mu_{\vec{k}}, \gamma_{\mathcal{G}}(x_1, \ldots, x_n), \gamma_{\mathcal{G}}(x_1, \ldots, x_n)^{-1}]$$
$$p(f) = [\nu_1 \diamond \cdots \diamond \nu_n, \gamma_{\mathcal{G}}(x_1, \ldots, x_n)]$$

with $\nu_i : \langle\langle k_i \rangle\rangle \to \langle\langle l_i \rangle\rangle \in \nabla$ active and $x_i \in \mathcal{G}(k_i)$ for each $1 \le i \le n$, so we have a formula

$$\rho_i^{(\vec{l})} \circ p(f) = [\nu, x_i] \circ \rho_i^{(\vec{k})} \ .$$

Taking p-coCartesian morphisms $\widehat{\rho}_i^{(\vec{k})} : X \to X_i$ and $\widehat{\rho}_i^{(\vec{l})} : Y \to Y_i$ covering $\rho_i^{(\vec{k})}$ and $\rho_i^{(\vec{l})}$ respectively for each $1 \le i \le n$, we obtain a sequence of functors

$$\begin{aligned} \mathcal{K}(X,Y)_{p(f)} &\xrightarrow[\simeq]{((\widehat{\rho}_1^{(\vec{l})})_*, \dots, (\widehat{\rho}_n^{(\vec{l})})_*)} \prod_{i=1}^{n} \mathcal{K}(X, Y_i)_{[\nu,x_i]\circ\rho_i^{(\vec{k})}} \\ &\xleftarrow[\simeq]{((\widehat{\rho}_1^{(\vec{k})})^*, \dots, (\widehat{\rho}_n^{(\vec{k})})^*)} \prod_{i=1}^{n} \mathcal{K}(X_i, Y_i)_{[\nu,x_i]} \ , \end{aligned} \tag{5.5.10}$$

which are equivalences thanks to the condition (ii) on the category of geometric $\mathcal{G}$-operators $\mathcal{K}$. Since one can describe the functor

$$\begin{aligned} ((\rho_i)_!, \dots, (\rho_n)_!) :\ & \mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\langle n \rangle\rangle}((\mu_{\vec{k}}, X), (\mu_{\vec{l}}, Y)) \\ & \to \prod_{i=1}^{n} \mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\langle 1 \rangle\rangle}((\mu_{k_i}, X_i), (\mu_{l_i}, Y_i)) \end{aligned} \tag{5.5.11}$$

in terms of (5.5.10), it turns out (5.5.11) is essentially surjective. Moreover, since $p : \mathcal{K} \to \mathbb{B}_{\mathcal{G}}$ is locally fully faithful, it follows from the direct computation of morphisms in $\mathbb{B}_{\mathcal{G}}$ that (5.5.11) is even fully faithful. Therefore, the pseudo-functor (5.5.8) is essentially surjective and essentially fully faithful, and this completes the proof of the part (1).

We prove the part (2). It is straightforward from Lemma 5.5.6 that $\mathcal{K} \to \mathrm{Env}_{\mathcal{G}}(\mathcal{K})$ is a map of geometric $\mathcal{G}$-operators. For a representable category of geometric $\mathcal{G}$-operators $\mathcal{L}$, consider a map of geometric $\mathcal{G}$-operators

$$F : \mathcal{K} \to \mathcal{L} \ .$$

We can extend it to $\widetilde{F} : \mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathcal{L}$ as follows: for each morphism $\boldsymbol{\varphi} : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \mathbb{B}_{\mathcal{G}}$, choose an induced functor

$$\boldsymbol{\varphi}_! : \mathcal{L}_{\langle\langle m \rangle\rangle} \to \mathcal{L}_{\langle\langle n \rangle\rangle} \ .$$

We set

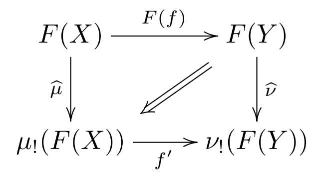
$$\widetilde{F}(\mu, X) := \mu_!(F(X)) \ .$$

As for a morphism $(\boldsymbol{\xi}, f) : (\mu, X) \to (\nu, Y)$ depicted as

$$\begin{array}{ccc} p(X) & \xrightarrow{p(f)} & p(Y) \\ {\scriptstyle \mu}\downarrow & \overset{\boldsymbol{\xi}_{01}}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle \nu} \\ \langle\langle m \rangle\rangle & \xrightarrow[\boldsymbol{\xi}_1]{} & \langle\langle n \rangle\rangle \end{array} \ , \tag{5.5.12}$$

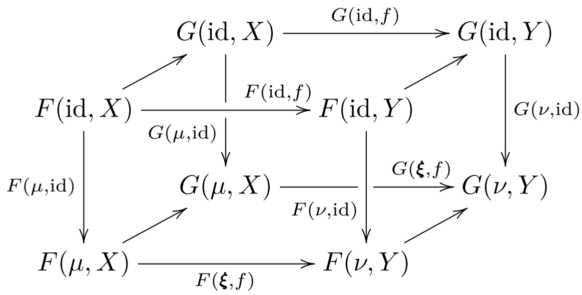
the universal property of the coCartesian morphism $\widehat{\mu} : F(X) \to \mu_!(F(X))$

implies there is an essentially unique 2-morphism



in $\mathcal{L}$ covering (5.5.12), so we put $\widetilde{F}(\boldsymbol{\xi}, f) := f'$. Similarly, we also obtain an assignment on 2-morphisms. The essential uniqueness on each choice guarantees $\widetilde{F} : \mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathcal{L}$ is actually a pseudofunctor. In addition, since F preserves coCartesian morphisms covering the inert morphisms, $\widetilde{F}$ even preserves all the coCartesian morphisms by construction, so $\widetilde{F} \in \mathbf{RepOper}^{\mathsf{geom}}_{\mathcal{G}}(\mathrm{Env}_{\mathcal{G}}(\mathcal{K}), \mathcal{L})$. In other words, $F : \mathcal{K} \to \mathcal{L}$ belongs to the essential image of the pseudofunctor (5.5.7). Since F is arbitrary, it follows that (5.5.7) is essentially surjective.

On the other hand, let $F, G : \mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathcal{L}$ be two map of geometric $\mathcal{G}$-operators preserving all the coCartesian morphisms. For each pseudonatural transformation $F \to G$ over $\mathbb{B}_{\mathcal{G}}$, and for $(\boldsymbol{\xi}, f) : (\mu, X) \to (\nu, Y) \in \mathrm{Env}_{\mathcal{G}}(\mathcal{K})$, consider the induced cube



(5.5.13)

with each faces filled with specific 2-morphisms in $\mathcal{L}$ coherently. Note that since both F and G preserve coCartesian morphisms, the vertical arrows in (5.5.13) are coCartesian. Hence, the universal property implies the top face of (5.5.13) actually determines the other faces essentially uniquely. In other words, the whole pseudonatural transformation $F \to G$ is essentially determined by its restriction on $\mathcal{K} \subset \mathrm{Env}_{\mathcal{G}}(\mathcal{K})$. The same argument clearly works on the modifications, so we conclude that (5.5.7) is also essentially fully faithful. This completes the proof of the part (2). ☐

In view of the observation that we have $\mathcal{C} \cong (\mathcal{C}^{\otimes})^{\triangledown\mathcal{G}}_{\langle\langle 1 \rangle\rangle}$ for every $\mathcal{G}$-symmetric monoidal category $\mathcal{C}$, it follows from Theorem 5.5.8 and Corollary 5.5.3 that a Grothendieck opfibration $\mathcal{L} \to \mathbb{B}_{\mathcal{G}}$ exhibits the fiber $\mathcal{L}_{\langle\langle 1 \rangle\rangle}$ as a $\mathcal{G}$-symmetric monoidal category. In particular, for every category of geometric $\mathcal{G}$-operators $\mathcal{K}$,

$$\mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\langle 1 \rangle\rangle} \cong \mathcal{K} \times_{\mathbb{B}_{\mathcal{G}}} (\mathbb{A}_{\mathcal{G}})_{\langle\langle 1 \rangle\rangle}$$

is a $\mathcal{G}$-symmetric monoidal category.

Remark 5.5.9. Since $\mathrm{Env}_{\mathcal{G}}(\mathcal{K}) \to \mathbb{B}_{\mathcal{G}}$ is locally fully faithful in our setting, the fiber $\mathrm{Env}_{\mathcal{G}}(\mathcal{K})_{\langle\langle 1 \rangle\rangle}$ is equivalent to an ordinary category by taking isomorphism classes of 1-morphisms. This is why we called it a $\mathcal{G}$-symmetric monoidal category instead of a $\mathcal{G}$-symmetric monoidal 2-category in the argument above.

Example 5.5.10. Notice that we have a canonical identification $\mathrm{Env}_{\mathcal{G}}(\mathbb{B}_{\mathcal{G}}) = \mathbb{A}_{\mathcal{G}}$. For each $n \in \mathbb{N}$, there is exactly one purely active morphism $\mu_n : \langle\langle n \rangle\rangle \to \langle\langle 1 \rangle\rangle$ in $\mathbb{B}_{\mathcal{G}}$, so the objects of $(\mathbb{A}_{\mathcal{G}})_{\langle\langle 1 \rangle\rangle}$ are denoted by $\langle\langle n \rangle\rangle$ for $n \in \mathbb{N}$. Moreover, by virtue of Lemma 4.2.10, a 2-morphism in $\mathbb{B}_{\mathcal{G}}$ of the form

$$\begin{array}{ccc} \langle\langle m \rangle\rangle & \longrightarrow & \langle\langle n \rangle\rangle \\ & \swarrow\!\!\!\!\Leftarrow & \\ {\scriptstyle \mu_m} \searrow & & \swarrow {\scriptstyle \mu_n} \\ & \langle\langle 1 \rangle\rangle & \end{array}$$

is determined by the top arrow, while a 1-morphism $[\varphi, x] : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle \in \mathbb{B}_{\mathcal{G}}$ admits such a 2-morphism if and only if it is active. The similar argument makes sense for 2-morphisms, one can see $(\mathbb{A}_{\mathcal{G}})_{\langle\langle 1 \rangle\rangle}$ is equivalent to the quotient category of the total category $\widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}}$ of the augmented crossed simplicial group $\mathfrak{J}^\natural \mathcal{G}$ (see Example 2.5.6) by the congruence

$$(\mu, x) \sim (\nu, y) \iff \mu = \nu \text{ and } xy^{-1} \in \mathrm{Dec}^{\mathcal{G}}_{\mu} \ .$$

In other words, $(\mathbb{A}_{\mathcal{G}})_{\langle\langle 1 \rangle\rangle}$ is equivalent to the augmented simplicial analogue of the $\mathfrak{J}^\natural \mathcal{G}$-quotal category associated with $\mathrm{Dec}^{\mathcal{G}}_{\mu}$.

Example 5.5.11. We assert that, for each group operad $\mathcal{G}$, we have

$$\mathrm{Env}_{\mathcal{G}}(\widetilde{\mathbb{G}}_{\mathcal{G}} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}) \simeq \widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}} \ .$$

We have bijection on objects for the same reason as Example 5.5.10, and it is also seen that the 1-morphisms in the left hand side are identified with active morphisms in $\widetilde{\mathbb{G}}_{\mathcal{G}}$. In addition, it follows from the structure of the double category

$$\widetilde{\mathbb{G}}_{\mathcal{G}} \times_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}} \rightrightarrows \widetilde{\mathbb{G}}_{\mathcal{G}}$$

and Lemma 4.2.10 that two 1-morphisms

$$[\mu, u, x], [\nu, v, y] : \langle\langle m \rangle\rangle \to \langle\langle n \rangle\rangle$$

in $\mathrm{Env}_{\mathcal{G}}(\widetilde{\mathbb{G}} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}})$ are connected by a 2-morphism if and only if we have $\mu = \nu$ and $ux = vy$. In other words, the 2-functor

$$\begin{array}{ccc} \widetilde{\Delta}_{\mathfrak{J}^\natural \mathcal{G}} & \to & \mathrm{Env}_{\mathcal{G}}(\widetilde{\mathbb{G}}_{\mathcal{G}} /\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}} \widetilde{\mathbb{G}}_{\mathcal{G}}) \\ (\mu, x) & \mapsto & [\mu, e, x] \end{array}$$

is bijective on objects and essentially fully faithful, so it is a biequivalence.

As a consequence of Example 5.5.11, we obtain an alternative proof of Proposition 3.5.2. Indeed, let $*$ be the trivial operad, so the category of algebras over $*$ in a monoidal category $\mathcal{C}$ can be written as

$$\mathbf{Alg}(\mathcal{C}) = \mathbf{Alg}_*(\mathcal{C}) = \mathbf{MultCat}(*, \mathcal{C}^{\otimes}) \ .$$

In particular, if $\mathcal{C}$ is a $\mathcal{G}$-symmetric monoidal category, we obtain a sequence of equivalences of categories as below:

$$\begin{aligned}\mathbf{Alg}(\mathcal{C}) &\simeq \mathbf{MultCat}(*,\mathcal{C}^{\otimes})\\ &\simeq \mathbf{MultCat}_{\mathcal{G}}(\mathcal{G},\mathcal{C}^{\otimes})\\ &\simeq \mathbf{Oper}_{\mathcal{G}}^{\mathsf{alg}}(\widetilde{\mathbb{G}}_{\mathcal{G}},\mathcal{C}^{\otimes}\wr\widetilde{\mathbb{E}}_{\mathcal{G}})\\ &\simeq \mathbf{Oper}_{\mathcal{G}}^{\mathsf{geom}}(\widetilde{\mathbb{G}}_{\mathcal{G}}/\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}}\widetilde{\mathbb{G}}_{\mathcal{G}},(\mathcal{C}^{\otimes})^{\triangledown\mathcal{G}})\\ &\simeq \mathbf{RepOper}_{\mathcal{G}}^{\mathsf{geom}}(\mathrm{Env}(\widetilde{\mathbb{G}}_{\mathcal{G}}/\!\!/_{\widetilde{\mathbb{E}}_{\mathcal{G}}}\widetilde{\mathbb{G}}_{\mathcal{G}}),(\mathcal{C}^{\otimes})^{\triangledown\mathcal{G}})\\ &\simeq \mathbf{MonCat}_{\mathcal{G}}(\widetilde{\Delta}_{\mathcal{G}},\mathcal{C})\ .\end{aligned}$$

Note that, in the article [62], Nikolaus and Scholze constructed Hochschild homology and cyclic homology for spectra using an ∞-analogue of the equivalence above.

Remark 5.5.12. Although we have stuck to the lower category theory throughout the paper, the notion of categories of geometric $\mathcal{G}$-operators has a straightforward higher categorical analogues. Indeed, one can consider the category $\mathbf{InFib}_{/\mathbb{B}_{\mathcal{G}}}$ of inner fibrations of ∞-categories over $\mathbb{B}_{\mathcal{G}}$ instead of $\mathbf{ldFib}_{/\mathbb{B}_{\mathcal{G}}}$ in the definition of categories of geometric $\mathcal{G}$-operators. Using the Joyal model structure on the category of simplicial sets, which is a model of the homotopy theory for ∞-categories, instead of the canonical model structure on **Cat**, one would obtain the notion of "∞-categories of $\mathcal{G}$-operators" thanks to the notions introduced in [53]. The only reason we did not do this is because the base category $\mathbb{B}_{\mathcal{G}}$ does not admit satisfactorily higher structures. For example, it is an exciting challenge to formulate the notion of "homotopy group operads."

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