Studies on the ascending chain condition for F-pure thresholds

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Preface

In this thesis, we consider some particular kind of singularities of algebraic varieties in positive characteristic, the so-called F-singularities. These singularities include *strongly F-regular* and *sharply F-pure* singularities. Although F-singularities are defined in terms of the Frobenius morphism which exists only in positive characteristic, they have strong connections to some particular kind of singularities in characteristic 0 which arise in the minimal model program. These connections sometimes give a new perspective on birational geometry in characteristic 0, which is one of the reasons why studies on F-singularities are important.

The main object of this thesis is the *F-pure threshold*, which is defined in terms of F-singularities and is an invariant measuring the singularities of an ideal on a germ in positive characteristic. It is expected that F-pure thresholds satisfy several important properties which hold for *log canonical thresholds*, an invariant defined in terms of singularities in MMP. One of such properties is the ascending chain condition (ACC in short). In this thesis, motivated by the ACC for log canonical thresholds in characteristic 0 ([HMX14]), we study the ACC for F-pure thresholds.

In Chapter 1, we state the main results of this thesis. In Chapter 2, as preliminaries of this thesis, we recall some definitions and basic properties about F-singularities and ultraproducts.

Chapter 3 is based on the preprint [Sat17]. In this chapter, we prove that the set of all F-pure thresholds of all ideals on a fixed strongly F-regular germ satisfies the ACC under some mild assumptions (Corollary C), which gives an affirmative answer (Theorem A) to the conjecture given by Blickle, Mustaţă and Smith ([BMS09]).

In Section 3.1, we define some variants of test ideals in terms of the Grothendieck trace map for the Frobenius morphism and q-adic expansions of a real number, where q is a power of the characteristic. In Section 3.2, we consider the rationality of the limit of F-pure thresholds. Blickle, Mustaţă and Smith proved that the limit of any sequence of F-pure thresholds of principal ideals on a regular germ is a rational number ([BMS09]). By using the new ideals defined in Section 3.1, we generalize this result to the case of non-regular germ and non-principal ideals under some mild assumptions (Corollary 3.2.8). In Section 3.3, we verify the ACC for F-jumping numbers with respect to a fixed $\mathfrak{m}$-primary ideal on a fixed germ under some mild assumptions (Theorem B), which implies Corollary C.

Chapter 4 is based on the preprint [Sat18]. In this chapter, as an extension of Corollary C, we verify the ACC for F-pure thresholds on sharply F-pure germs with fixed embedding dimension (Main Theorem).

In Section 4.1, we consider the rationality of F-pure thresholds. Schwede and Tucker proved that the F-pure threshold of any ideal on any strongly F-regular log $\mathbb{Q}$-Gorenstein pair is a rational number ([ST14]). By defining and studying a variant of parameter test modules, we generalize this result to the case of non-strongly F-regular pair under some mild assumptions (Theorem E). In Section 4.2, by combining Corollary C and Theorem E, we prove the main theorem. In Section 4.3, we prove the corollaries of the main theorem (Corollary D), which are positive characteristic analogues of the results in [dFEM10].

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Contents

Chapter 1

Introduction

In characteristic zero, Shokurov ([Sho92]) conjectured that the set of all log canonical thresholds on varieties of any fixed dimension satisfies the ascending chain condition (ACC in short). This conjecture was partially solved by de Fernex, Ein, and Mustaţă in [dFEM10] and [dFEM11] using generic limit, and finally settled by Hacon, M^{c}Kernan, and Xu in [HMX14] using global geometry.

In this thesis, we deal with a positive characteristic analogue of this problem. A scheme X of characteristic $p > 0$ is said to be *F-finite* if the Frobenius morphism $F : X \longrightarrow X$ is finite and is said to be *sharply F-pure* if the morphism $F^{\#} : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ locally splits as an $\mathcal{O}_X$-module homomorphism.

Suppose that X is a sharply F-pure normal variety over an F-finite field k of characteristic $p > 0$. Then, for every coherent ideal sheaf $\mathfrak{a} \subsetneq \mathcal{O}_X$, we can define the *$F$-pure threshold* $\mathrm{fpt}(X; \mathfrak{a}) \in \mathbb{R}_{\geqslant 0}$ in terms of Frobenius splittings (see Definition 2.3.1 below). As seen in [TW04], [MTW05] and [BS15], the F-pure threshold itself is an interesting invariant in both algebraic geometry and commutative algebra in positive characteristic. Moreover, recent studies ([TW04], [Tak13], [HnBWZ16]) reveal that F-pure thresholds have strong connections to log canonical thresholds in characteristic 0. Motivated by the ACC for log canonical thresholds, Blickle, Mustaţă and Smith conjectured the following.

Conjecture ([BMS09, Conjecture 4.4])**.** *Fix an integer $n \geqslant 1$, a prime number $p > 0$ and a set $\mathcal{D}^{\mathrm{reg}}_{n,p}$ such that every element of $\mathcal{D}^{\mathrm{reg}}_{n,p}$ is an n-dimensional F-finite Noetherian regular local ring of characteristic p. The set*

$$\mathcal{T}^{\mathrm{reg}}_{n,p,\mathrm{pr}} := \{\mathrm{fpt}(A; \mathfrak{a}) \mid A \in \mathcal{D}^{\mathrm{reg}}_{n,p}, \mathfrak{a} \subsetneq A \text{ is a principal ideal}\}$$

satisfies the ACC.

This problem has been partially considered by several authors ([BMS09], [HnBWZ16] and [HnBW17]). We give an affirmative answer to this conjecture in full generality.

Theorem A (Corollary 3.3.10)**.** *With the notation above, the set*

$$\mathcal{T}_{n,p}^{\mathrm{reg}} := \{\mathrm{fpt}(A;\mathfrak{a}) \mid A \in \mathcal{D}_{n,p}^{\mathrm{reg}}, \mathfrak{a} \subsetneq A \text{ is an ideal}\}$$

satisfies the ACC.

In order to prove Theorem A, it is enough to show that the set of all F-pure thresholds on a fixed F-finite Noetherian regular local ring satisfies the ACC. We consider this problem in a more general setting.

Let (R,Δ) be a pair, that is, $(R,\mathfrak{m})$ is an F-finite Noetherian normal local ring of characteristic $p>0$ and Δ is an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$. For a proper ideal $\mathfrak{a} \subsetneq R$ and a real number $t \geqslant 0$, we consider the test ideal $\tau(R,\Delta,\mathfrak{a}^t)$, which is defined in terms of the Frobenius morphism (see Definition 2.1.2 below). Since we have $\tau(R,\Delta,\mathfrak{a}^t) \subseteq \tau(R,\Delta,\mathfrak{a}^s)$ for every real numbers $0 \leqslant s \leqslant t$, for a given $\mathfrak{m}$-primary ideal $I \subseteq R$, we define the F-jumping number of $(R,\Delta;\mathfrak{a})$ with respect to I as

$$\mathrm{fjn}^I(R,\Delta;\mathfrak{a}) := \inf\{t \geqslant 0 \mid \tau(R,\Delta,\mathfrak{a}^t) \subseteq I\} \in \mathbb{R}.$$

We also define the set of all F-jumping numbers with respect to I as

$$\mathrm{FJN}^I(R,\Delta) := \{\mathrm{fjn}^I(R,\Delta;\mathfrak{a}) \mid \mathfrak{a} \subsetneq R \text{ is an ideal}\} \subseteq \mathbb{R}_{\geqslant 0}.$$

The following result is the main theorem of Chapter 3.

Theorem B (Theorem 3.3.9)**.** *Let (R,Δ) be a pair such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p, where K_X is a canonical divisor of $X = \operatorname{Spec} R$ and $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Assume that $\tau(R,\Delta)$ is $\mathfrak{m}$-primary or trivial. Then the set $\mathrm{FJN}^I(R,\Delta)$ satisfies the ACC.*

A pair (R,Δ) is said to be *strongly F-regular* if we have $\tau(R,\Delta)=R$. We note that strongly F-regular (resp. sharply F-pure) singularities can be viewed as an F-singularity theoretic counterpart of klt (resp. lc) singularities and that strong F-regularity is a stronger condition than sharp F-purity.

For a strongly F-regular pair (R,Δ), the F-jumping number $\mathrm{fjn}^{\mathfrak{m}}(R,\Delta;\mathfrak{a})$ with respect to $\mathfrak{m}$ coincides with the F-pure threshold $\mathrm{fpt}(R,\Delta;\mathfrak{a})$ and the set $\mathrm{FJN}^{\mathfrak{m}}(R,\Delta)$ coincides with the set of all F-pure thresholds

$$\mathrm{FPT}(R,\Delta) := \{\mathrm{fpt}(R,\Delta;\mathfrak{a}) \mid \mathfrak{a} \subsetneq R \text{ is an ideal}\}.$$

Therefore, as a special case of Theorem B, we have the following result.

Corollary C (Theorem 3.3.9)**.** *Let* (R, Δ) *be a strongly* F*-regular pair such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier with index not divisible by* p*. Then* $\mathrm{FPT}(R, \Delta)$ *satisfies the ACC.*

Since any F-finite regular local ring is strongly F-regular, Theorem A follows from Corollary C.

In Chapter 4, we extend Corollary C to the case of non-strongly F-regular pairs. Since Hacon, M^c^Kernan and Xu verified the ACC for all log canonical thresholds on all log canonical pairs with fixed dimension ([HMX14]), it is natural to ask whether the set of all F-pure thresholds on all sharply F-pure pairs with fixed dimension satisfies the same property. As a partial answer to this question, we verify the property for the set of all F-pure thresholds on all sharply F-pure pairs with fixed embedding dimension.

Main Theorem (Theorem 4.2.5)**.** *Fix a prime number* p *and positive integers* e *and* N*. Suppose that* T *is any set such that every element of* T *is an* F*-finite Noetherian normal local ring* $(R, \mathfrak{m}, k)$ *of characteristic* p *with* $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leqslant N$*. Let* $\mathrm{FPT}(T, e) \subseteq \mathbb{R}_{\geqslant 0}$ *be the set of all* F*-pure thresholds* $\mathrm{fpt}(R, \Delta; \mathfrak{a})$ *such that*

- R *is an element of* T*,*
- $\mathfrak{a}$ *is a proper ideal of* R*, and*
- Δ *is an effective* $\mathbb{Q}$*-Weil divisor on* $X = \operatorname{Spec} R$ *such that* (R, Δ) *is sharply* F*-pure and* $(p^e - 1)(K_X + \Delta)$ *is Cartier.*

Then the set $\mathrm{FPT}(T, e)$ *satisfies the ACC.*

As a corollary of Main Theorem, employing the strategy in [dFEM10], we can also verify the ascending chain condition for F-pure thresholds on tame quotient singularities or on l.c.i. varieties with fixed dimension.

Corollary D (Proposition 4.3.2, Corollary 4.3.4)**.** *Fix an integer* $n \geqslant 1$ *and a prime number* $p > 0$*.*

1. *Suppose that* $\mathcal{D}^{\mathrm{quot}}_{n,p}$ *is a set such that every element of* $\mathcal{D}^{\mathrm{quot}}_{n,p}$ *is an* n*-dimensional* F*-finite Noetherian normal local ring of characteristic* p *with tame quotient singularities. The set*

$$\mathcal{T}^{\mathrm{quot}}_{n,p} := \{\mathrm{fpt}(R; \mathfrak{a}) \mid R \in \mathcal{D}^{\mathrm{quot}}_{n,p}, \mathfrak{a} \subsetneq R \text{ is an ideal}\}$$

   satisfies the ACC.

2. *Suppose that* $\mathcal{D}^{\mathrm{l.c.i.}}_{n,p}$ *is a set such that every element of* $\mathcal{D}^{\mathrm{l.c.i.}}_{n,p}$ *is an* n*-dimensional* F*-finite Noetherian normal local complete intersection ring of characteristic* p *with sharply* F*-pure singularities. The set*

$$\mathcal{T}^{\mathrm{l.c.i.}}_{n,p} := \{\mathrm{fpt}(R;\mathfrak{a}) \mid R \in \mathcal{D}^{\mathrm{quot}}_{n,p}, \mathfrak{a} \subsetneq R \text{ is an ideal}\}$$

satisfies the ACC.

In the process of proving the main theorem, we treat the rationality problem for F-pure thresholds. In characteristic 0, since log canonical thresholds can be computed by a single log resolution, it is obvious that the log canonical threshold of any ideal on any log $\mathbb{Q}$-Gorenstein pair is a rational number. In [dFEM10], they use the rationality to reduce the ascending chain condition for log canonical thresholds on l.c.i. varieties to that on smooth varieties.

However, in positive characteristic, the rationality of F-pure thresholds is a more subtle problem. In [ST14], Schwede and Tucker proved that the F-pure threshold of any ideal on any log $\mathbb{Q}$-Gorenstein strongly F-regular pair is a rational number. In this paper, we generalize their result to the case where the pair is not necessarily strongly F-regular, under the assumption that the Gorenstein index is not divisible by the characteristic.

Theorem E (Corollary 4.1.10)**.** *Suppose that* $(R,\mathfrak{m})$ *is an* F*-finite Noetherian normal local ring of characteristic* $p>0$ *and* Δ *is an effective* $\mathbb{Q}$*-Weil divisor on* $X = \operatorname{Spec} R$ *such that* (R,Δ) *is sharply* F*-pure and* $K_X+\Delta$ *is* $\mathbb{Q}$*-Cartier with index not divisible by* p*. Then the* F*-pure threshold* $\mathrm{fpt}(R,\Delta;\mathfrak{a})$ *is a rational number for every proper ideal* $\mathfrak{a} \subseteq R$*.*

Main Theorem follows from Theorem E and Corollary C.

Chapter 2

Preliminaries

2.1 Test ideals and parameter test modules

In this section, we recall the definitions and basic properties of test ideals and parameter test modules.

A ring R of characteristic $p > 0$ is said to be *F-finite* if the Frobenius morphism $F : R \longrightarrow R$ is a finite ring homomorphism. A scheme X is said to be *F-finite* if for every open affine subscheme $U \subseteq X$, the ring $H^0(U, \mathcal{O}_U)$ is F-finite. If R is an F-finite Noetherian normal domain, then R is excellent ([Kun76]) and $X = \operatorname{Spec} R$ has a *dualizing complex* $\omega_X^\bullet$, a *canonical module* ω_X and a *canonical divisor* K_X (see for example [ST17, p.4]).

Through this paper, all rings will be assumed to be F-finite of characteristic $p > 0$.

Definition 2.1.1. A *pair* (R, Δ) consists of an F-finite Noetherian normal local ring $(R, \mathfrak{m})$ and an effective $\mathbb{Q}$-Weil divisor Δ on X. A *triple* $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$, consists of a pair (R, Δ) and a symbol $\mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$, where $m > 0$ is an integer, $\mathfrak{a}_1, \ldots, \mathfrak{a}_m \subseteq R$ are ideals, and $t_1, \ldots, t_m \geqslant 0$ are real numbers. When $m = 1$ (resp. $m = 2$), we simply denote the triple by $(R, \Delta, \mathfrak{a}_1^{t_1})$ (resp. $(R, \Delta, \mathfrak{a}_1^{t_1}\mathfrak{a}_2^{t_2})$).

Definition 2.1.2. Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ is a triple. The *test ideal* $\tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ (resp. the *parameter test module* $\tau(\omega_X, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$) is the unique smallest non-zero ideal $J \subseteq R$ (resp. non-zero submodule $J \subseteq \omega_X$) such that

$$\varphi(F_*^e(\prod_i \mathfrak{a}_i^{\lceil t_i(p^e-1)\rceil} J)) \subseteq J$$

for every integer $e \geqslant 0$ and every morphism $\varphi \in \operatorname{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta\rceil), R)$ (resp. $\varphi \in \operatorname{Hom}_R(F_*^e \omega_X(\lceil (p^e-1)\Delta\rceil), \omega_X)$).

The test ideal and the parameter test module always exist ([Sch10, Theorem 6.3] and [ST14, Lemma 4.2]). If $\mathfrak{a}_i = R$ for every i, then we write $\tau(R, \Delta)$ (resp. $\tau(\omega_X, \Delta)$). If $\mathfrak{a}_i = 0$ for some i, then we define $\tau(\omega_X, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) := (0)$.

Remark 2.1.3. Suppose that X is an F-finite Noetherian normal scheme, Δ is an effective $\mathbb{Q}$-Weil divisor on X, $\mathfrak{a}_i \subseteq \mathcal{O}_X$ are coherent ideals and $t_i \geqslant 0$ are real numbers.

1. ([HT04, Proposition 3.1]) Since test ideals and parameter test modules are compatible with localization ([HT04, Proposition 3.1]), we can define the test ideal $\tau(X, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) \subseteq \mathcal{O}_X$ and the parameter test module $\tau(\omega_X, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) \subseteq \omega_X$.

2. ([Tak04, p.9 Basic Property (ii)]) We can extend the definitions of test ideals and parameter test modules to the case where Δ is not effective.

Proposition 2.1.4 ([ST14, Lemma 4.2])**.** *Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ be a triple and fix a canonical divisor K_X. We consider ω_X as the submodule $\mathcal{O}_X(K_X)$ of the quotient field of R. Then we have the equation*

$$\tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau(\omega_X, K_X + \Delta, \mathfrak{a}_\bullet^{t_\bullet})$$

as submodules of the quotient field of R.

Lemma 2.1.5. *Let $(R, \Delta, \mathfrak{a}^t)$ be a triple such that Δ is $\mathbb{Q}$-Cartier. Then the following hold.*

1. *If $t \leqslant t'$, $\mathfrak{a}' \subseteq \mathfrak{a}$ and $\Delta \leqslant \Delta'$, then $\tau(\omega_X, \Delta', (\mathfrak{a}')^{t'}) \subseteq \tau(\omega_X, \Delta, \mathfrak{a}^t)$.*

2. ([ST14, Lemma 6.1]) *There exists a real number $\varepsilon > 0$ such that if $t \leqslant t' \leqslant t + \varepsilon$, then $\tau(\omega_X, \Delta, \mathfrak{a}^{t'}) = \tau(\omega_X, \Delta, \mathfrak{a}^t)$.*

3. ([ST14, Lemma 6.2]) *There exists a real number $\varepsilon > 0$ such that if $t - \varepsilon \leqslant t' < t$, then $\tau(\omega_X, \Delta, \mathfrak{a}^{t'}) = \tau(\omega_X, \Delta, \mathfrak{a}^{t-\varepsilon})$.*

4. ([ST14, Lemma 4.4]) *Suppose that $\mathrm{Tr}_R : F_*\omega_X \longrightarrow \omega_X$ is the Grothendieck trace map* ([BST15, Proposition 2.18])*. Then we have*

$$\mathrm{Tr}_R(F_*\tau(\omega_X, \Delta, \mathfrak{a}^t)) = \tau(\omega_X, \Delta/p, \mathfrak{a}^{t/p}).$$

5. ([HT04, Theorem 4.2], cf. [BSTZ10, Lemma 3.26]) *If $\mathfrak{a}$ is generated by l elements and $l \leqslant t$, then $\tau(\omega_X, \Delta, \mathfrak{a}^t\mathfrak{b}^s) = \mathfrak{a}\tau(\omega_X, \Delta, \mathfrak{a}^{t-1}\mathfrak{b}^s)$.*

6. ([Sch11, Lemma 3.1]) *If $\mathfrak{b} = (f)$ is a non-zero principal ideal, then we have $\tau(\omega_X, \Delta, \mathfrak{a}^t\mathfrak{b}^s) = \tau(\omega_X, \Delta + s\,\mathrm{div}(f), \mathfrak{a}^t)$.*

7. *For an integer $r \geqslant 1$, we have $\tau(\omega_X, \Delta, \mathfrak{a}^{rt}) = \tau(\omega_X, \Delta, (\mathfrak{a}^r)^t)$.*

Proof. By Proposition 2.1.4, the assertions in (5) and (6) follow from the same assertions for test ideals. The proof of (7) is similar to the proof of (6). □

Definition 2.1.6. Let $(X = \operatorname{Spec} R, \Delta)$ be a pair and $e \geqslant 0$ be an integer. Assume that $(p^e - 1)(K_X + \Delta)$ is Cartier. Then there exists an isomorphism

$$\operatorname{Hom}_R(F_*^e(R((p^e-1)\Delta)), R) \cong F_*^e R$$

as $F_*^e R$-modules (see for example [Sch09, Lemma 3.1]). We denote by φ_Δ^e a generator of $\operatorname{Hom}_R(F_*^e(R((p^e-1)\Delta)), R)$ as an $F_*^e R$-module.

Remark 2.1.7. Although a map $\varphi_\Delta^e : F_*^e R \longrightarrow R$ is not uniquely determined, it is unique up to multiplication by $F_*^e R^\times$. When we consider this map, we only need the information about the image of this map. Hence we ignore the multiplication by $F_*^e R^\times$.

The following proposition seems to be well-known to experts, but difficult to find a proof in the literature.

Proposition 2.1.8. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be F-finite Noetherian normal local rings with residue fields k and l, respectively. Let $R \longrightarrow S$ be a flat local homomorphism, Δ_X be an effective $\mathbb{Q}$-Weil divisor on $X = \operatorname{Spec} R$ and Δ_Y be the flat pullback of Δ_X to $Y = \operatorname{Spec} S$. Assume that $\mathfrak{m}S = \mathfrak{n}$ and that the relative Frobenius morphism $F_{l/k}^e : F_*^e k \otimes_k l \longrightarrow F_*^e l$ is an isomorphism for every $e \geqslant 0$. Then the following hold.*

1. *The morphism $R \longrightarrow S$ is a regular morphism, that is, every fiber is geometrically regular.*

2. *The relative Frobenius morphism $F_{S/R}^e : F_*^e R \otimes_R S \longrightarrow F_*^e S$ is an isomorphism for every $e \geqslant 0$.*

3. *For every $e \geqslant 0$, we have*

$$\operatorname{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta_X \rceil), R) \otimes_R S \cong \operatorname{Hom}_S(F_*^e S(\lceil (p^e-1)\Delta_Y \rceil), S).$$

4. *Let $(R, \Delta_X, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$ be a triple. We write $(\mathfrak{a}_\bullet \cdot S)^{t_\bullet} := \prod_i (\mathfrak{a}_i S)^{t_i}$. Then we have*

$$\tau(R, \Delta_X, \mathfrak{a}_\bullet^{t_\bullet}) \cdot S = \tau(S, \Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}).$$

5. *If $(p^e-1)(K_X+\Delta_X)$ is Cartier for some $e>0$, then $(p^e-1)(K_Y+\Delta_Y)$ is also Cartier and $\varphi^e_{\Delta_Y} : F^e_*S \longrightarrow S$ coincides with the morphism $\varphi^e_{\Delta_X} \otimes_R S : F^e_*R \otimes_R S \longrightarrow S$ via the isomorphism $F^e_{S/R} : F^e_*R \otimes_R S \longrightarrow F^e_*S$.*

Proof. Since the relative Frobenius morphism $F_{l/k} : F_*k \otimes_k l \longrightarrow F_*l$ is injective, the field extension $k \subseteq l$ is separable by [Mat89, Theorem 26.4]. Then (1) follows from [Mat89, Theorem 28.10] and [And74].

We will prove the assertion in (2). Fix an integer $e \geqslant 0$. By (1), the morphism $R \longrightarrow S$ is generically separable. It follows from [Mat89, Theorem 26.4] that the relative Frobenius morphism $F^e_{S/R} : F^e_*R \otimes_R S \longrightarrow F^e_*S$ is injective.

We next consider the surjectivity of the map $F^e_{S/R}$. We denote the ring $F^e_*R \otimes_R S$ by R'. We consider the following commutative diagram:

$$\begin{array}{ccc} & & F^e_*S \\ & \overset{F^e_S}{\nearrow} & \uparrow{\scriptstyle F^e_{S/R}} \\ S & \longrightarrow & R' \\ \uparrow & & \uparrow \\ R & \xrightarrow[F^e_R]{} & F^e_*R \end{array}$$

Since the morphisms $F^e_R : R \longrightarrow F^e_*R$ and $S \longrightarrow R'$ are both finite and $\mathfrak{n} \cap R = \mathfrak{m}$, every maximal ideal of R' contains the maximal ideal $F^e_*\mathfrak{m}$ of F^e_*R. Therefore, $I := (F^e_*\mathfrak{m}) \cdot R' \subseteq R'$ is contained in the Jacobson radical of R'. On the other hand, since the finite morphism $F^e_S : F^e_*S \longrightarrow S$ factors through $F^e_{S/R}$, the morphism $F^e_{S/R}$ is also finite. Then the morphism

$$F^e_{S/R} \otimes_{R'} (R'/I) : R'/I \longrightarrow (F^e_*S) \otimes_{R'} (R'/I)$$

coincides with the relative Frobenius morphism $F^e_{l/k} : F^e_*k \otimes_k l \longrightarrow F^e_*l$, and hence it is surjective. Therefore, the map $F^e_{S/R}$ is surjective by Nakayama.

We next prove the assertion in (3). Since S is flat over R and $F^e_*R(\lceil (p^e-1)\Delta_X \rceil)$ is a finite R-module, we have

$$\mathrm{Hom}_R(F^e_*R(\lceil (p^e-1)\Delta_X \rceil), R) \otimes_R S \cong \mathrm{Hom}_S(F^e_*R(\lceil (p^e-1)\Delta_X \rceil) \otimes_R S, S).$$

By (1), the flat pullback of a prime divisor on X to Y is a reduced divisor. Therefore, the Weil divisor $\lceil (p^e-1)\Delta_Y \rceil$ coincides with the flat pullback of $\lceil (p^e-1)\Delta_X \rceil$. It follows from (2) that $F^e_*R(\lceil (p^e-1)\Delta_X \rceil) \otimes_R S \cong F^e_*S(\lceil p^e - 1 \rceil \Delta_Y)$, which completes the proof of (3).

For (4), it follows from (3) that the test ideal $\tau(R, \Delta_X, \mathfrak{a}_\bullet^{t_\bullet}) \cdot S$ is uniformly $(\Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}, F)$-compatible and $\tau(S, \Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}) \cap R$ is uniformly $(\Delta_X, \mathfrak{a}_\bullet^{t_\bullet}, F)$-compatible. Therefore, we have

$$\begin{aligned} \tau(S, \Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}) &\subseteq \tau(R, \Delta_X, \mathfrak{a}_\bullet^{t_\bullet}) \cdot S \text{ and} \\ \tau(S, \Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}) \cap R &\supseteq \tau(R, \Delta_X, \mathfrak{a}_\bullet^{t_\bullet}), \end{aligned}$$

which complete the proof of (4).

For (5), we assume that $(p^e - 1)(K_X + \Delta_X)$ is Cartier. Since the canonical divisor K_Y coincides with the flat pullback of K_X ([Aoy83, Proposition 4.1], see also [Sta, Lemma 45.22.1]), the Weil divisor $(p^e - 1)(K_Y + \Delta_Y)$ is also Cartier. The second assertion in (5) follows from (3). □

2.2 F-singularities

In this section, we recall the definitions and some basic properties of F-singularities.

Definition 2.2.1. Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ be a triple.

1. $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ is said to be *sharply F-pure* if there exist an integer $e > 0$ and a morphism $\varphi \in \mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R)$ such that

$$\varphi(F^e_* \prod_i \mathfrak{a}_i^{\lceil t_i(p^e-1) \rceil}) = R.$$

2. $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ is said to be *strongly F-regular* if for every non-zero element $c \in R$, there exist an integer $e > 0$ and a morphism $\varphi \in \mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R)$ such that

$$\varphi(F^e_* (c \prod_i \mathfrak{a}_i^{\lceil t_i(p^e-1) \rceil})) = R.$$

Remark 2.2.2 ([Tak04, Corollary 2.10]). $\tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \mathcal{O}_X$ if and only if $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ is strongly F-regular.

Lemma 2.2.3. *Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ be a triple. Then the following hold.*

1. *If $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ is strongly F-regular, then it is sharply F-pure.*

2. *Suppose that $0 \leqslant \Delta' \leqslant \Delta$ is an $\mathbb{Q}$-Weil divisor and $0 \leqslant t'_i \leqslant t_i$ are real numbers. If $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ is strongly F-regular (resp. sharply F-pure), then so is $(R, \Delta', \mathfrak{a}_\bullet^{t'_\bullet})$.*

3. *If (R, Δ) is strongly F-regular and $(R, \Delta, \mathfrak{a}^t)$ is sharply F-pure, then $(R, \Delta, \mathfrak{a}^s)$ is strongly F-regular for every $0 \leqslant s < t$.*

4. *Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion and $\widehat{\Delta}$ the flat pullback of Δ to $\operatorname{Spec} \widehat{R}$. Then, $(R, \Delta, \prod_i \mathfrak{a}_i^{t_i})$ is sharply F-pure (resp. strongly F-regular) if and only if so is $(\widehat{R}, \widehat{\Delta}, \prod_i (\mathfrak{a}_i \widehat{R})^{t_i})$.*

Proof. (1) and (2) follow from definitions. The proof of (3) is similar to that of [TW04, Proposition 2.2 (5)].

For (4), the case of strongly F-regular triples follows from Proposition 2.1.8 (4). We consider the case of sharply F-pure triples. , we define

$$\begin{aligned} I &:= \sum_{e,\varphi} \varphi(F^e_* \prod_i \mathfrak{a}_i^{\lceil t_i(p^e-1)\rceil}) \subseteq R \\ I' &:= \sum_{e,\psi} \psi(F^e_* (\prod_i \mathfrak{a}_i \widehat{R})^{\lceil t_i(p^e-1)\rceil}) \subseteq \widehat{R}, \end{aligned}$$

where e runs through all positive integers and φ (resp. ψ) runs through all elements in $\operatorname{Hom}(F^e_* R(\lceil (p^e-1)\Delta \rceil), R)$ (resp. in $\operatorname{Hom}_{\widehat{R}}(F^e_* \widehat{R}(\lceil (p^e-1)\widehat{\Delta}\rceil), \widehat{R})$). Then the triple $(R, \Delta, \prod_i \mathfrak{a}_i^{t_i})$ (resp. the triple $(\widehat{R}, \widehat{\Delta}, \prod_i (\mathfrak{a}_i \widehat{R})^{t_i})$) is sharply F-pure if and only if $I = R$ (resp. $I' = \widehat{R}$). Since

$$\operatorname{Hom}_{\widehat{R}}(F^e_* \widehat{R}(\lceil (p^e-1)\widehat{\Delta}\rceil), \widehat{R}) \cong \operatorname{Hom}(F^e_* R(\lceil (p^e-1)\Delta\rceil), R) \otimes_R \widehat{R},$$

we have $I' = I\widehat{R}$, which completes the proof. □

Suppose that R is a ring of characteristic $p > 0$, $e > 0$ is a positive integer and $\mathfrak{a} \subseteq R$ is an ideal. Then we denote by $\mathfrak{a}^{[p^e]}$ the ideal of R generated by $\{f^{p^e} \in R \mid f \in \mathfrak{a}\}$. The following lemma is a variant of Fedder-type criteria.

Lemma 2.2.4 (cf. [Fed83], [HW02, Proposition 2.6])**.** *Suppose that $(A, \mathfrak{m})$ is an F-finite regular local ring of characteristic $p > 0$, $\mathfrak{a} \subseteq A$ is an ideal and $\Delta = \operatorname{div}_A(f)/(p^e - 1)$ is an effective $\mathbb{Q}$-divisor with $f \in A$ and $e > 0$. Then, the triple $(A, \Delta, \mathfrak{a}^t)$ is sharply F-pure if and only if there exists an integer $n > 0$ such that*

$$f^{\frac{p^{en}-1}{p^e-1}} \mathfrak{a}^{\lceil t(p^{en}-1)\rceil} \not\subseteq \mathfrak{m}^{[p^{en}]}.$$

Proof. By the proof of [Sch08, Proposition 3.3], the triple $(A, \Delta, \mathfrak{a}^t)$ is sharply F-pure if and only if there exists an integer $n > 0$ and $\varphi \in \operatorname{Hom}_R(F^{en}_* A((p^{en}-1)\Delta), A)$ such that $\varphi(F^{en}_* \mathfrak{a}^{\lceil t(p^{en}-1)\rceil}) = A$. Since

$$(p^{en}-1)\Delta = \operatorname{div}_A(f^{\frac{p^{en}-1}{p-1}}),$$

the assertion follows from [Fed83, Lemma 1.6]. □

Lemma 2.2.5. *Let $(A, \Delta, \mathfrak{a}^t)$ be a triple such that A is a regular local ring.*

1. *If $(A, \Delta, \mathfrak{a}^t)$ is sharply F-pure, then for any rational numbers $0 < \varepsilon, \varepsilon' < 1$, the triple $(A, (1-\varepsilon)\Delta, \mathfrak{a}^{t(1-\varepsilon')})$ is strongly F-regular.*

2. *If $(p^e - 1)\Delta$ is Cartier for an integer $e > 0$ and $(A, (1-\varepsilon)\Delta, \mathfrak{a}^t)$ is strongly F-regular for every $0 < \varepsilon < 1$, then the triple $(A, \Delta, \mathfrak{a}^{t(1-\varepsilon')})$ is sharply F-pure for every $0 < \varepsilon' < 1$.*

Proof. For (1), we assume that the triple $(A, \Delta, \mathfrak{a}^t)$ is sharply F-pure. Since A is strongly F-regular ([HH89]), it follows from Lemma 2.2.3 (2) and (3) that $(A, (1-\varepsilon)\Delta)$ is strongly F-regular for every $0 < \varepsilon < 1$. Then applying Lemma 2.2.3 (2) and (3) again, we see that $(A, (1-\varepsilon)\Delta, \mathfrak{a}^{t(1-\varepsilon')})$ is strongly F-regular for every $0 < \varepsilon, \varepsilon' < 1$.

For (2), set $q := p^e$ and suppose that $\Delta = \mathrm{div}(f)/(q-1)$ for some nonzero element $f \in A$. Take an integer $l > t$ such that $\mathfrak{a}$ is generated by at most l elements and set $a_n := (l-t)/(q^n - 1)$ for every integer $n \geqslant 0$. Since for any triple, it is strongly F-regular if and only if the test ideal is trivial, we have $\tau(A, ((q^n-1)/q^n)\Delta/, \mathfrak{a}^t) = A$ for every integer $n \geqslant 0$.

Then it follows from Proposition 2.1.4 and Lemma 2.1.5 (4), (5) and (6) that

$$\begin{aligned} A &= \tau(A, ((q^n-1)/q^n)\Delta, \mathfrak{a}^t) \\ &= \mathrm{Tr}_A^{en}(F_*^{en}\tau(A, f^{(q^n-1)/(q-1)}\mathfrak{a}^{tq^n})) \\ &= \varphi_\Delta^{en}(F_*^{en}\tau(A, \mathfrak{a}^{tq^n})) \\ &\subseteq \varphi_\Delta^{en}(F_*^{en}\mathfrak{a}^{\lceil tq^n - l\rceil}) \\ &\subseteq \varphi_\Delta^{en}(F_*^{en}\mathfrak{a}^{\lceil (t-a_n)(q^n-1)\rceil}), \end{aligned}$$

which proves that $(A, \Delta, \mathfrak{a}^{t-a_n})$ is sharply F-pure for every integer $n \geqslant 0$. Since $\lim_{n\longrightarrow\infty} a_n = 0$, the triple $(A, \Delta, \mathfrak{a}^{t(1-\varepsilon')})$ is sharply F-pure for every $0 < \varepsilon' < 1$. □

Suppose that X is an F-finite Noetherian normal connected scheme, Δ is an effective $\mathbb{Q}$-Weil divisor on X, $\mathfrak{a} \subseteq \mathcal{O}_X$ is a coherent ideal sheaf and $t \geqslant 0$ is a real number. For any point $x \in X$, we denote by Δ_x the flat pullback of Δ to $\operatorname{Spec}\mathcal{O}_{X,x}$.

Definition 2.2.6. With the notation above, we say that $(X, \Delta, \mathfrak{a}^t)$ is *sharply F-pure* (resp. *strongly F-regular*) if $(\mathcal{O}_{X,x}, \Delta_x, \mathfrak{a}_x^t)$ is sharply F-pure (resp. strongly F-regular) for every point $x \in X$.

Remark 2.2.7. Suppose that $X = \operatorname{Spec} R$ is an affine scheme. Then, the above definition differs from the one given in [Sch08]. See [Sch10b].

2.3 F-pure thresholds and F-jumping numbers

In this section, we recall the definitions and basic properties of F-pure thresholds and F-jumping numbers.

Definition 2.3.1. Suppose that X is an F-finite Noetherian normal connected scheme and Δ is an effective $\mathbb{Q}$-Weil divisor and $\mathfrak{a} \subseteq \mathcal{O}_X$ is a coherent ideal sheaf. Assume that (X, Δ) is sharply F-pure. We define the *F-pure threshold* of $(X, \Delta; \mathfrak{a})$ by

$$\mathrm{fpt}(X, \Delta; \mathfrak{a}) := \inf \left\{ t \geqslant 0 \mid (X, \Delta, \mathfrak{a}^t) \text{ is not sharply } F\text{-pure} \right\} \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}.$$

When $\Delta = 0$, we denote it by $\mathrm{fpt}(X; \mathfrak{a})$. When $X = \operatorname{Spec} R$ is an affine scheme, we denote it by $\mathrm{fpt}(R, \Delta; \mathfrak{a})$.

Lemma 2.3.2. *With the notation above, assume that $\mathfrak{a} \neq \mathcal{O}_X$. We have* $\mathrm{fpt}(X, \Delta; \mathfrak{a}) = \min \{\mathrm{fpt}(\mathcal{O}_{X,x}, \Delta_x; \mathfrak{a}_x) \mid x \in X\}$.

Proof. We may assume that $X = \operatorname{Spec} R$. For every $t \geqslant 0$, we consider $I_t := \sum_{e,\varphi} \varphi(F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil}) \subseteq R$ as in the proof of Lemma 2.2.3 (4). Then, the set

$$Z_t := \{x \in X \mid (\mathcal{O}_{X,x}, \Delta_x, \mathfrak{a}_x^t) \text{ is not sharply } F\text{-pure}\} \subseteq X$$

is a closed set defined by the ideal I_t. Since R is Noetherian, there exists a real number $\varepsilon > 0$ such that Z_t is constant for all $\mathrm{fpt}(X, \Delta; \mathfrak{a}) < t < \mathrm{fpt}(X, \Delta; \mathfrak{a}) + \varepsilon$. Take a point $x \in Z_t$ for such t. Then we have $\mathrm{fpt}(X, \Delta; \mathfrak{a}) = \mathrm{fpt}(\mathcal{O}_{X,x}, \Delta_x; \mathfrak{a}_x)$, which completes the proof. $\square$

Proposition 2.3.3 (F-adjunction, [Sch09, Theorem 5.5])**.** *Suppose that A is an F-finite regular local ring, $R = A/I$ is a normal ring and Δ_R is an effective $\mathbb{Q}$-Weil divisor on* $\operatorname{Spec} R$. *Assume that the pair (R, Δ_R) is sharply F-pure and there exists an integer $e > 0$ such that $(p^e - 1)(K_X + \Delta_R)$ is Cartier. Then, there exists an effective $\mathbb{Q}$-Weil divisor Δ_A on* $\operatorname{Spec} A$ *with the following properties:*

1. *$(p^e - 1)\Delta_A$ is Cartier, and*
2. *Suppose that $\mathfrak{a} \subseteq R$ is an ideal and $\widetilde{\mathfrak{a}} \subseteq A$ is the lift of $\mathfrak{a}$. Then we have* $\mathrm{fpt}(R, \Delta_R; \mathfrak{a}) = \mathrm{fpt}(A, \Delta_A; \widetilde{\mathfrak{a}})$.

Definition 2.3.4. Let (R, Δ) be a pair, $\mathfrak{a} \subseteq R$ be a proper ideal and $I \subseteq R$ be an $\mathfrak{m}$-primary ideal.

1. A real number $t > 0$ is called a *F-jumping number* of $(R, \Delta; \mathfrak{a})$ if

$$\tau(R, \Delta, \mathfrak{a}^{t-\varepsilon}) \neq \tau(R, \Delta, \mathfrak{a}^t),$$

   for all $\varepsilon > 0$.

2. We define the *F-jumping number* of $(R, \Delta; \mathfrak{a})$ with respect to I as

$$\mathrm{fjn}^I(R, \Delta; \mathfrak{a}) := \inf\{t \in \mathbb{R}_{\geqslant 0} \mid \tau(R, \Delta, \mathfrak{a}^t) \subseteq I\} \in \mathbb{R}_{\geqslant 0}.$$

We note that if (R, Δ) is strongly F-regular, then we have $\mathrm{fpt}(R, \Delta; \mathfrak{a}) = \mathrm{fjn}^{\mathfrak{m}}(R, \Delta; \mathfrak{a})$.

Proposition 2.3.5 ([ST14, Theorem B])**.** *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a})$ be a triple such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then the set of all F-jumping numbers of $(R, \Delta; \mathfrak{a})$ is a discrete set of rational numbers. In particular, if (R, Δ) is strongly F-regular, then the F-pure threshold of $(R, \Delta; \mathfrak{a})$ is a rational number.*

2.4 Ultraproduct

In this section, we define the ultraproduct of a family of sets and recall some properties. We also define the catapower of a Noetherian local ring and prove some properties. The reader is referred to [Scho10] for details.

Definition 2.4.1. Let $\mathfrak{U}$ be a collection of subsets of $\mathbb{N}$. $\mathfrak{U}$ is called an *ultrafilter* if the following properties hold:

1. $\emptyset \notin \mathfrak{U}$.
2. For every subsets $A, B \subseteq \mathbb{N}$, if $A \in \mathfrak{U}$ and $A \subseteq B$, then $B \in \mathfrak{U}$.
3. For every subsets $A, B \subseteq \mathbb{N}$, if $A, B \in \mathfrak{U}$, then $A \cap B \in \mathfrak{U}$.
4. For every subset $A \subseteq \mathbb{N}$, if $A \notin \mathfrak{U}$, then $\mathbb{N} \setminus A \in \mathfrak{U}$.

An ultrafilter $\mathfrak{U}$ is called *non-principal* if the following holds:

5. If A is a finite subset of $\mathbb{N}$, then $A \notin \mathfrak{U}$.

By Zorn's Lemma, there exists a non-principal ultrafilter. From now on, we fix a non-principal ultrafilter $\mathfrak{U}$.

Definition 2.4.2. Let $\{T_m\}_{m\in\mathbb{N}}$ be a family of sets. We define the equivalence relation $\sim$ on the set $\prod_{m\in\mathbb{N}} T_m$ by

$$(a_m)_m \sim (b_m)_m \text{ if and only if } \{m \in \mathbb{N} \mid a_m = b_m\} \in \mathfrak{U}.$$

We define the *ultraproduct* of $\{T_m\}_{m\in\mathbb{N}}$ as

$$\mathrm{ulim}_{m\in\mathbb{N}} T_m := \left(\prod_{m\in\mathbb{N}} T_m\right) / \sim .$$

If T is a set and $T_m = T$ for all m, then we denote $\mathrm{ulim}_m T_m$ by ${}^*T$ and call it the *ultrapower* of T.

Let $\{T_m\}_{m\in\mathbb{N}}$ be a family of sets and $a_m \in T_m$ for every m. We denote by $\mathrm{ulim}_m a_m$ the class of $(a_m)_m$ in $\mathrm{ulim}_m T_m$. Let $\{S_m\}_m$ be another family of sets and $f_m : T_m \longrightarrow S_m$ be a map for every m. We can define the map

$$\mathrm{ulim}_m f_m : \mathrm{ulim}_m T_m \longrightarrow \mathrm{ulim}_m S_m$$

by sending $\mathrm{ulim}_m a_m \in \mathrm{ulim}_m T_m$ to $\mathrm{ulim}_m f_m(a_m) \in \mathrm{ulim}_m S_m$. If $T_m = T$, $S_m = S$, and $f_m = f$ for every $m \in \mathbb{N}$, then we denote the map $\mathrm{ulim}_m f_m$ by ${}^*f : {}^*T \longrightarrow {}^*S$.

Let $\{R_m\}_{m\in\mathbb{N}}$ be a family of rings and M_m be an R_m-module for every m. Then $\mathrm{ulim}_m R_m$ has the ring structure induced by that of $\prod_m R_m$ and $\mathrm{ulim}_m M_m$ has the structure of $\mathrm{ulim}\, R_m$-module induced by the structure of $\prod_m R_m$-module on $\prod_m M_m$. Moreover, if k_m is a field for every m, then $\mathrm{ulim}_m k_m$ is a field.

Proposition 2.4.3. *We have the following properties.*

1. *Let R be a Noetherian ring and M be a finitely generated R-module. Then we have ${}^*M \cong M \otimes_R {}^*R$*

2. *Let k be an F-finite field of positive characteristic. Then the relative Frobenius morphism $F^e_*(k) \otimes_k {}^*k \longrightarrow F^e_*({}^*k)$ is an isomorphism. In particular, ${}^*k$ is an F-finite field.*

Proof. For (1), we consider the natural homomorphism $M \otimes_R {}^*R \longrightarrow {}^*M$. Since the functors ${}^*(-)$ and $(-) \otimes_R {}^*R$ are both right exact, we may assume that M is a free R-module of finite rank. In this case, the assertion is obvious.

For (2), we consider the natural bijection ${}^*(F^e_* k) \cong F^e_*({}^*k)$. Combining with (1), the relative Frobenius morphism $F^e_*(k) \otimes_k {}^*k \longrightarrow F^e_*({}^*k)$ is an isomorphism. □

Let $\mathfrak{a}_m \subseteq R_m$ be an ideal for every m. Then the natural map

$$\mathrm{ulim}_m\, \mathfrak{a}_m \longrightarrow \mathrm{ulim}_m\, R_m$$

is injective, and hence we can consider $\mathrm{ulim}_m\, \mathfrak{a}_m$ as an ideal of the ring $\mathrm{ulim}_m\, R_m$. Let $\mathfrak{b}_m \subseteq R_m$ be another ideals. Then $\mathrm{ulim}_m\, \mathfrak{b}_m \subseteq \mathrm{ulim}_m\, \mathfrak{a}_m$ if and only if

$$\{m \in \mathbb{N} \mid \mathfrak{b}_m \subseteq \mathfrak{a}_m\} \in \mathfrak{U}.$$

Moreover, we have the equation

$$(\mathrm{ulim}_m\, \mathfrak{a}_m) + (\mathrm{ulim}_m\, \mathfrak{b}_m) = \mathrm{ulim}_m(\mathfrak{a}_m + \mathfrak{b}_m).$$

Lemma 2.4.4. *Let $\{R_m\}_{m\in\mathbb{N}}$ be a family of rings, $\mathfrak{a}_m, \mathfrak{b}_m \subseteq R_m$ be ideals for every m. Assume that there exists an integer $l > 0$ such that the number $\mu(\mathfrak{a}_m)$ of minimal generator of the ideal $\mathfrak{a}_m$ satisfies $\mu(\mathfrak{a}_m) \leqslant l$ for every m. Then we have*

$$(\mathrm{ulim}_m\, \mathfrak{a}_m) \cdot (\mathrm{ulim}_m\, \mathfrak{b}_m) = \mathrm{ulim}_m(\mathfrak{a}_m \cdot \mathfrak{b}_m).$$

Proof. Let $\alpha = \mathrm{ulim}_m\, a_m \in \mathrm{ulim}_m\, \mathfrak{a}_m$ and $\beta = \mathrm{ulim}_m\, b_m \in \mathrm{ulim}_m\, \mathfrak{b}$. Then we have $\alpha \cdot \beta = \mathrm{ulim}_m(a_m b_m) \in \mathrm{ulim}_m(\mathfrak{a}_m \cdot \mathfrak{b}_m)$. This shows the inclusion $(\mathrm{ulim}_m\, \mathfrak{a}_m) \cdot (\mathrm{ulim}_m\, \mathfrak{b}_m) \subseteq \mathrm{ulim}_m(\mathfrak{a}_m \cdot \mathfrak{b}_m)$.

We consider the converse inclusion. By the assumption, there exist $f_{m,1}, \dots, f_{m,l} \in \mathfrak{a}_m$ such that $\mathfrak{a}_m = (f_{m,1}, \dots, f_{m,l})$. Then we have $\mathfrak{a}_m \cdot \mathfrak{b}_m = \sum_i f_{m,i} \cdot \mathfrak{b}_m$, and hence we have

$$\mathrm{ulim}_m(\mathfrak{a}_m \cdot \mathfrak{b}_m) = \sum_i f_{\infty,i} \cdot (\mathrm{ulim}_m\, \mathfrak{b}_m),$$

where $f_{\infty,i} := \mathrm{ulim}_m\, f_{m,i} \in \mathrm{ulim}_m\, \mathfrak{a}_m$ for every i, which complete the proof of the lemma. $\square$

Proposition-Definition 2.4.5 ([Gol98, Theorem 5.6.1])**.** Let $\{a_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers such that there exist real numbers M_1, M_2 which satisfies $M_1 < a_m < M_2$ for every $m \in \mathbb{N}$. Then there exists an unique real number $w \in \mathbb{R}$ such that for every real number $\varepsilon > 0$, we have

$$\{m \in \mathbb{N} \mid |w - a_m| < \varepsilon\} \in \mathfrak{U}.$$

We denote this number w by $\mathrm{sh}(\mathrm{ulim}_m\, a_m)$ and call it the *shadow* of $\mathrm{ulim}_m\, a_m$.

Let $(R, \mathfrak{m}, k)$ be a local ring. Then, one can show that $({}^*R, {}^*\mathfrak{m}, {}^*k)$ is a local ring. However, even if R is Noetherian, the ultrapower ${}^*R$ may not be Noetherian because we do not have the equation $\cap_{n\in\mathbb{N}}({}^*\mathfrak{m})^n = 0$ in general.

Definition 2.4.6 ([Scho10]). Let $(R, \mathfrak{m})$ be a Noetherian local ring and $({}^*R, {}^*\mathfrak{m})$ be the ultrapower. We define the *catapower* $R_\#$ as the quotient ring

$$R_\# := {}^*R/(\cap_n ({}^*\mathfrak{m})^n).$$

Proposition 2.4.7 ([Scho10, Theorem 8.1.19]). *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of equicharacteristic and $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. We fix a coefficient field $k \subseteq \widehat{R}$. Then we have*

$$R_\# \cong \widehat{R} \mathbin{\widehat{\otimes}_k} ({}^*k).$$

In particular, if $(R, \mathfrak{m})$ is an F-finite Noetherian normal local ring, then so is $R_\#$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring, $R_\#$ be the catapower and $a_m \in R$ for every m. We denote by $[a_m]_m \in R_\#$ the image of $\mathrm{ulim}_m\, a_m \in {}^*R$ by the natural projection ${}^*R \longrightarrow R_\#$. Let $\mathfrak{a}_m \subseteq R$ be an ideal for every $m \in \mathbb{N}$. We denote by $[\mathfrak{a}_m]_m \subseteq R_\#$ the image of the ideal $\mathrm{ulim}_m\, \mathfrak{a}_m \subseteq {}^*R$ by the projection ${}^*R \longrightarrow R_\#$.

Lemma 2.4.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathfrak{a}_m, \mathfrak{b}_m \subseteq R$ be ideals for every $m \in \mathbb{N}$. If we have $[\mathfrak{a}_m]_m \subseteq [\mathfrak{b}_m]_m$, then for every $\mathfrak{m}$-primary ideal $\mathfrak{q} \subseteq R$, we have*

$$\{m \in \mathbb{N} \mid \mathfrak{a}_m \subseteq \mathfrak{b}_m + \mathfrak{q}\} \in \mathfrak{U}.$$

Proof. By the definition of the catapower, if $[\mathfrak{a}_m]_m \subseteq [\mathfrak{b}_m]_m$, then we have

$$\mathrm{ulim}_m\, \mathfrak{a}_m \subseteq \mathrm{ulim}_m\, \mathfrak{b}_m + ({}^*\mathfrak{m})^n.$$

for every n.

On the other hand, it follows from Lemma 2.4.4 that $({}^*\mathfrak{m})^n = {}^*(\mathfrak{m}^n)$. Therefore we have

$$\begin{aligned} \mathrm{ulim}\, \mathfrak{a}_m &\subseteq (\mathrm{ulim}\, \mathfrak{b}_m) + {}^*(\mathfrak{m}^n) \\ &= \mathrm{ulim}(\mathfrak{b}_m + \mathfrak{m}^n), \end{aligned}$$

which is equivalent to

$$\{m \in \mathbb{N} \mid \mathfrak{a}_m \subseteq \mathfrak{b} + \mathfrak{m}^n\} \in \mathfrak{U}.$$

This implies the assertion in the lemma. □

Chapter 3

ACC for F-jumping numbers on a fixed germ

3.1 Variants of test ideals

In this section, we introduce some variants of test ideals by using the trace maps for the Frobenius morphisms and the q-adic expansion of a real number (Definition 3.1.3 and 3.1.10). We also introduce the stabilization exponent (Definition 3.1.7).

Definition 3.1.1 (cf. [HnBWZ16, Definition 2.1, 2.2])**.** Let $q \geqslant 2$ be an integer, $t > 0$ be a real number and $n \in \mathbb{Z}$ be an integer. We define the *n-th digit* of t in base q by

$$t^{(n)} := \lceil tq^n - 1 \rceil - q \lceil tq^{n-1} - 1 \rceil \in \mathbb{Z}.$$

We define the *n-th round up* and the *n-th truncation* of t in base q by

$$\begin{aligned} \langle t \rangle^{n,q} &:= \lceil tq^n \rceil / q^n \in \mathbb{Q}, \text{ and} \\ \langle t \rangle_{n,q} &:= \lceil tq^n - 1 \rceil / q^n \in \mathbb{Q}, \end{aligned}$$

respectively.

Lemma 3.1.2. *Let $q \geqslant 2$ be an integer, $t > 0$ be a real number and $n \in \mathbb{Z}$ be an integer. Then the following hold.*

1. $0 \leqslant t^{(n)} < q$.
2. $t^{(n)}$ *is eventually zero for $n \ll 0$ and is not eventually zero for $n \gg 0$.*
3. $t = \sum_{m \in \mathbb{Z}} t^{(m)} \cdot q^{-m}$.

4. $\langle t\rangle_{n,q} = \sum_{m\leqslant n} t^{(m)} \cdot q^{-m}$.

5. *The sequence* $\{\langle t\rangle^{n,q}\}_{n\in\mathbb{Z}}$ *is a descending chain which convergences to* t.

6. *The sequence* $\{\langle t\rangle_{n,q}\}_{n\in\mathbb{Z}}$ *is an ascending chain which converges to* t.

Proof. These all follow easily from the definitions. For the assertion in (2), we note that if $t = s/q^m$ for some integers s and m, then we have $t^{(n)} = q-1$ for all $n > m$. $\square$

Definition 3.1.3. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_i \mathfrak{a}_i^{t_i})$ be a triple such that $t_i > 0$ for all i and $e > 0$ be an integer such that $(p^e-1)(K_X+\Delta)$ is Cartier. For every integer $n \geqslant 0$, we define

$$\begin{aligned} \tau_+^{en}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) &:= \varphi_\Delta^{en}(F_*^{en}(\mathfrak{a}_1^{\lceil t_1 p^{en}\rceil}\cdots\mathfrak{a}_m^{\lceil t_m p^{en}\rceil}\cdot\tau(R,\Delta))) \subseteq R \text{ and} \\ \tau_-^{en}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) &:= \varphi_\Delta^{en}(F_*^{en}(\mathfrak{a}_1^{\lceil t_1 p^{en}-1\rceil}\cdots\mathfrak{a}_m^{\lceil t_m p^{en}-1\rceil}\cdot\tau(R,\Delta))) \subseteq R. \end{aligned}$$

Example 3.1.4. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t > 0$ and that $\mathfrak{a}$ is a principal ideal and let e be a positive integer such that $(p^e-1)(K_X+\Delta)$ is Cartier. Then it follows from [BSTZ10, Lemma 5.4] that

$$\begin{aligned} \tau_+^{en}(R,\Delta,\mathfrak{a}^t) &= \tau(R,\Delta,\mathfrak{a}^{\langle t\rangle^{n,q}}), \text{ and} \\ \tau_-^{en}(R,\Delta,\mathfrak{a}^t) &= \tau(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n,q}}). \end{aligned}$$

By Proposition 2.3.5, the sequence $\{\tau_+^{en}(R,\Delta,\mathfrak{a}^t)\}_n$ is an ascending chain of ideals which converges to $\tau(R,\Delta,\mathfrak{a}^t)$ and the sequence $\{\tau_-^{en}(R,\Delta,\mathfrak{a}^t)\}_n$ is a descending chain of ideals which eventually stabilizes.

The following lemma is well-known to experts, but we prove it for convenience.

Lemma 3.1.5. *Let R be a Noetherian ring of characteristic $p > 0$, let $\mathfrak{a} \subseteq R$ be an ideal, and let a, b, n and e be non-negative integers.*

1. *If $n > p^e(\mu_R(\mathfrak{a})-1)$, then we have*

$$\mathfrak{a}^n = (\mathfrak{a}^{\lceil n/p^e\rceil - \mu_R(\mathfrak{a})})^{[p^e]} \cdot \mathfrak{a}^{n-p^e(\lceil n/p^e\rceil-\mu_R(\mathfrak{a}))}.$$

   In particular, if $b > p^e(\mu_R(\mathfrak{a})-1)$, then we have $\mathfrak{a}^{ap^e+b} = (\mathfrak{a}^a)^{[p^e]}\cdot\mathfrak{a}^b$.

2. *Assume that there exist ideals $\mathfrak{a}_1,\dots,\mathfrak{a}_m \subseteq R$ and integers $M_1,\dots,M_m$ such that $\mathfrak{a} = \mathfrak{a}_1^{M_1}+\cdots+\mathfrak{a}_m^{M_m}$. Set $l := \sum_i \mu_R(\mathfrak{a}_i)$. If $n > p^e(l-1)$, then we have*

$$\mathfrak{a}^n = (\mathfrak{a}^{\lceil n/p^e\rceil - l})^{[p^e]} \cdot \mathfrak{a}^{n-p^e(\lceil n/p^e\rceil - l)}.$$

   In particular, if $b > p^e((\sum_i \mu_R(\mathfrak{a}_i))-1)$, then we have $\mathfrak{a}^{ap^e+b} = (\mathfrak{a}^a)^{[p^e]}\cdot\mathfrak{a}^b$.

Proof. The proof of (1) is straightforward by taking a minimal generator of $\mathfrak{a}$. For (2), we first consider the case when $m = 1$. If $M_1 = 1$, then the assertion in (2) is same as that in (1). If $l = \mu_R(\mathfrak{a}_1) = 1$, then the assertion holds because $\mathfrak{a}$ is a principal ideal. Therefore, we may assume that $M_1 \geqslant 2$ and $l \geqslant 2$. In this case, it follows from (1) that

$$\begin{aligned}\mathfrak{a}^n = \mathfrak{a}_1^{nM_1} &= (\mathfrak{a}_1^{\lceil nM_1/p^e\rceil - l})^{[p^e]} \cdot \mathfrak{a}_1^{nM_1 - p^e(\lceil nM_1/p^e\rceil - l)} \\ &\subseteq (\mathfrak{a}_1^{M_1(\lceil n/p^e\rceil - l)})^{[p^e]} \cdot \mathfrak{a}_1^{nM_1 - p^eM_1(\lceil n/p^e\rceil - l)} \\ &= (\mathfrak{a}^{\lceil n/p^e\rceil - l})^{[p^e]} \cdot \mathfrak{a}^{n - p^e(\lceil n/p^e\rceil - l)}.\end{aligned}$$

We next consider the case when $m \geqslant 2$. Set $\mathfrak{b}_i := \mathfrak{a}_i^{M_i}$ and $l_i := \mu_R(\mathfrak{a}_i)$. Then we have

$$\mathfrak{a}^n = \sum_{n_1,\dots,n_m} \prod_{i=1}^m \mathfrak{b}_i^{n_i},$$

where n_i runs through all non-negative integers such that $\sum_i n_i = n$. Fix such integers n_i and set $s_i := \max\{0, \lceil n_i/p^e\rceil - l_i\}$. Then it follows from the first case that $\mathfrak{b}_i^{n_i} = (\mathfrak{b}_i^{s_i})^{[p^e]} \cdot \mathfrak{b}_i^{n_i - p^e s_i}$ for every integer i. Therefore, we have

$$\begin{aligned}\prod_i \mathfrak{b}_i^{n_i} &= (\prod_i \mathfrak{b}_i^{s_i})^{[p^e]} \cdot \prod_i \mathfrak{b}_i^{n_i - p^e s_i} \\ &\subseteq (\mathfrak{a}^{\sum_i s_i})^{[p^e]} \cdot \mathfrak{a}^{\sum_i (n_i - p^e s_i)}, \\ &\subseteq (\mathfrak{a}^{\lceil n/p^e\rceil - l})^{[p^e]} \cdot \mathfrak{a}^{n - p^e(\lceil n/p^e\rceil - l)},\end{aligned}$$

which completes the proof of (2). □

Proposition 3.1.6 (basic properties)**.** *Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ and e be as in Definition 3.1.3. Then the following hold.*

1. ([BSTZ10, Lemma 3.21]) *The sequence $\{\tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})\}_{n \geqslant 0}$ is an ascending chain which converges to the test ideal $\tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$.*

2. *If $t_1 > 1$, then we have*

$$\tau_+^{en}(R, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}) \supseteq \mathfrak{a}_1 \cdot \tau_+^{en}(R, \Delta, \mathfrak{a}_1^{t_1 - 1} \cdots \mathfrak{a}_m^{t_m}).$$

Moreover, if $t_1 > \mu_R(\mathfrak{a}_1)$, then we have

$$\tau_+^{en}(R, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}) = \mathfrak{a}_1 \cdot \tau_+^{en}(R, \Delta, \mathfrak{a}_1^{t_1 - 1} \cdots \mathfrak{a}_m^{t_m}).$$

3. *$\varphi_\Delta^e(F_*^e(\tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{p^e \cdot t_\bullet}))) = \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$, where we set $\mathfrak{a}_\bullet^{p^e \cdot t_\bullet} := \prod_i \mathfrak{a}_i^{p^e t_i}$.*

Proof. The proof of (1) follows as in the case when $m = 1$, see [BSTZ10, Lemma 3.21]. If $t_1 > \mu_R(\mathfrak{a}_1)$, then by Lemma 3.1.5 (1), we have $\mathfrak{a}_1^{\lceil t_1 p^{en} \rceil} = \mathfrak{a}_1^{[p^{en}]} \cdot \mathfrak{a}_1^{\lceil (t_1-1)p^{en} \rceil}$, which proves (2). The assertion in (3) follows from the fact that $\varphi_\Delta^{e(n+1)} = \varphi_\Delta^e \circ F_*^e \varphi_\Delta^{en}$ ([Sch09, Theorem 3.11 (e)]). □

Definition 3.1.7. Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ and e be as in Definition 3.1.3. We define the *stabilization exponent* of $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e)$ by

$$\mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e) := \min\{n \geqslant 0 \mid \tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})\}.$$

Proposition 3.1.8 (basic properties)**.** *Let* $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$ *and* e *be as in Definition 3.1.3. Then the following hold.*

1. *If* $t_1 > \mu_R(\mathfrak{a}_1)$*, then we have*
$$\mathrm{stab}(R, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}; e) \leqslant \mathrm{stab}(R, \Delta, \mathfrak{a}_1^{t_1-1} \cdots \mathfrak{a}_m^{t_m}; e).$$

2. *We have*
$$\mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e) \leqslant \mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{p^e \cdot t_\bullet}; e) + 1.$$

3. *If* $t_i > \mu_R(\mathfrak{a}_i)$ *and* $(p^e - 1)t_i \in \mathbb{N}$ *for every* i*, then for any integer* $n \geqslant 0$*, the inequality* $n \geqslant \mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e)$ *holds if and only if*
$$\tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}).$$

Proof. The assertions in (1) and (2) follow from Proposition 3.1.6 (2) and (3), respectively.

For (3), it follows from Proposition 3.1.6 (2) and (3) that

$$\begin{aligned} \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) &= \varphi_\Delta^e(F_*^e(\tau_+^{en}(R, \Delta, \mathfrak{a}_1^{p^e t_1} \cdots \mathfrak{a}_m^{p^e t_m}))) \\ &= \varphi_\Delta^e(F_*^e(\mathfrak{a}_1^{(p^e-1)t_1} \cdots \mathfrak{a}_m^{(p^e-1)t_m} \cdot \tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}))). \end{aligned}$$

Therefore, if $\tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$, then we have

$$\tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau_+^{e(n+2)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}),$$

which completes the proof. □

Proposition 3.1.9. *Let* $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}_\bullet = \prod_i \mathfrak{a}_i)$ *be a triple,* e *be a positive integer such that* $(p^e - 1)(K_X + \Delta)$ *is Cartier. We define*

$$\widetilde{\mathrm{stab}}(R, \Delta, \mathfrak{a}_\bullet; e) := \sup_{t_1, \ldots, t_m} \{\mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e)\},$$

where every t_i runs through all positive rational numbers such that $(p^e-1)t_i \in \mathbb{N}$. Then we have $\widetilde{\operatorname{stab}}(R, \Delta, \mathfrak{a}_\bullet; e) < \infty$. Moreover, for every integer $l \geqslant 0$ and rational numbers $t_1, \dots, t_m > 0$ such that $p^{el}(p^e-1)t_i \in \mathbb{N}$, we have

$$\operatorname{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e) \leqslant \widetilde{\operatorname{stab}}(R, \Delta, \mathfrak{a}_\bullet; e) + l.$$

Proof. By Proposition 3.1.8 (1), we have

$$\widetilde{\operatorname{stab}}(R, \Delta, \mathfrak{a}_\bullet; e) = \sup_{t_1, \dots, t_m} \{\operatorname{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e)\},$$

where every t_i runs through all positive rational numbers such that $(p^e-1)t_i \in \mathbb{N}$ and $t_i \leqslant \mu_R(\mathfrak{a}_i)$. Hence we have $\widetilde{\operatorname{stab}}(R, \Delta, \mathfrak{a}_\bullet; e) < \infty$.

The second statement follows from Proposition 3.1.8 (2). $\square$

We next consider the sequence of ideals $\{\tau_-^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})\}_n$. In general, the sequence $\{\tau_-^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})\}_n$ may not be a descending chain. In order to make a descending chain, we mix the definitions of τ_+ and τ_-, and define the new variants of test ideals as below. In fact, we later see that we can make a descending chain by using these ideals under some mild assumptions (Proposition 3.1.12).

Definition 3.1.10. Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_i \mathfrak{a}_i^{t_i})$ and e be as in Definition 3.1.3, $\mathfrak{q} \subseteq R$ be an ideal, and $n, u \geqslant 0$ be integers. We define

$$\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) := \varphi_\Delta^{e(n+u)}(F_*^{e(n+u)}(\mathfrak{a}_1^{p^{eu}\lceil t_1 p^{en}-1\rceil} \cdots \mathfrak{a}_m^{p^{eu}\lceil t_m p^{en}-1\rceil} \cdot \mathfrak{q})).$$

When $\mathfrak{q} = \tau(R, \Delta)$, we denote it by $\tau_e^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$.

Proposition 3.1.11 (basic properties)**.** *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$ be a triple such that $t_i > 0$ for every i and $(q-1)(K_X + \Delta)$ is Cartier for some $q = p^e$, $\mathfrak{q} \subseteq R$ be an ideal and $n, u \geqslant 0$ be integers. Then the following hold.*

1. *For real numbers $0 < s_i \leqslant t_i$, we have $\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{s_\bullet}) \supseteq \tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$. Moreover, if $\langle t_i \rangle_{n,q} < s_i \leqslant t_i$ for every i, then we have $\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{s_\bullet}) = \tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$.*
2. *For ideals $\mathfrak{b}_i \subseteq \mathfrak{a}_i$ and $\mathfrak{q}' \subseteq \mathfrak{q}$, we have $\tau_{e,\mathfrak{q}'}^{n,u}(R, \Delta, \mathfrak{b}_\bullet^{t_\bullet}) \subseteq \tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$.*
3. *If $\mathfrak{a}_1 \equiv \mathfrak{b}_1 \bmod J$ for some ideal J and $\mathfrak{a}_i = \mathfrak{b}_i$ for every $i \geqslant 2$, then we have*

   $$\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) \equiv \tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{b}_\bullet^{t_\bullet}) \bmod \tau_{e,J\cdot\mathfrak{q}}^{n,u}(R, \Delta, \prod_{i=2}^m \mathfrak{a}_i^{t_i}).$$

   If $\mathfrak{q} \equiv \mathfrak{q}' \bmod J$ for some ideals $\mathfrak{q}'$ and J, then we have

   $$\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) \equiv \tau_{e,\mathfrak{q}'}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) \bmod \tau_{e,J}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}).$$

4. *If* $\mathfrak{q} = \mathfrak{a}_{m+1}^{q^u \lceil t_{m+1} q^n - 1 \rceil} \tau(R, \Delta)$, *then we have*

$$\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau_e^{n,u}(R, \Delta, \prod_{i=1}^{m+1} \mathfrak{a}_i^{t_i}).$$

5. *If* $t_1 > 1$, *then we have* $\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) \supseteq \mathfrak{a}_1 \cdot \tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_1^{t_1 - 1} \cdots \mathfrak{a}_m^{t_m})$. *Moreover, if* $t_1 > \mu_R(\mathfrak{a}) + (1/q^n)$, *then we have*

$$\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \mathfrak{a}_1 \cdot \tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_1^{t_1 - 1} \cdots \mathfrak{a}_m^{t_m}).$$

6. $\varphi_\Delta^e(F_*^e(\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{p^e \cdot t_\bullet}))) = \tau_{e,\mathfrak{q}}^{n+1,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$.

7. *The sequence* $\{\tau_e^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})\}_{u \in \mathbb{N}}$ *is an ascending chain of ideals which converges to* $\tau(R, \Delta, \prod_i \mathfrak{a}_i^{\langle t_i \rangle_{n,q}})$.

8. *If* $u \geqslant \widetilde{\mathrm{stab}}(R, \Delta, \mathfrak{a}_\bullet; e)$, *then we have*

$$\tau_e^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau(R, \Delta, \prod_i \mathfrak{a}_i^{\langle t_i \rangle_{n,q}})$$

   for every n.

9. *Assume that* $q^{u-1} \geqslant \mu_R(\mathfrak{a}_i)$ *and the* n*-th digit* ${t_i}^{(n)}$ *of* t_i *in base* q *is non-zero for every* i. *Then we have* $\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau_{e,\mathfrak{q}'}^{n-1,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$, *where* $\mathfrak{q}' := \varphi_\Delta^e(F_*^e(\prod_i \mathfrak{a}_i^{q^u \cdot {t_i}^{(n)}} \mathfrak{q}))$.

Proof. The assertions in (1), (2), (3), (4) and (8) follow easily from the definitions. The assertions in (5), (6) and (7) follow from Proposition 3.1.6. The assertion in (9) follows from Lemma 3.1.5 (1). □

Proposition 3.1.12. *Let* $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ *be a triple such that* $t_i > 0$ *for every* i *and* $(q-1)(K_X + \Delta)$ *is Cartier for some* $q = p^e$, *and* $u > 0$ *be an integer such that* $q^{u-1} \geqslant \max_i \mu_R(\mathfrak{a}_i)$. *Assume that* $q(q-1)t_i \in \mathbb{N}$ *for every* i. *Then the sequence* $\{\tau_e^{n,u}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})\}_{n \geqslant 1}$ *is a descending chain of ideals.*

Proof. Since $q(q-1)t_i \in \mathbb{N}$, the n-th digit $t_i^{(n)}$ of t_i in base q is constant for $n \geqslant 2$. By Lemma 3.1.2 (2), it is non-zero. Therefore, the assertion follows from Proposition 3.1.11 (2) and (9). □

Definition 3.1.13. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple with $t > 0$, let I be an $\mathfrak{m}$-primary ideal, $\mathfrak{b} \subseteq R$ be a proper ideal, and let e be a positive integer such that $(p^e - 1)(K_X + \Delta)$ is Cartier. Then we define

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{b}) := \inf\{s > 0 \mid \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{b}^s) \subseteq I\} \in \mathbb{R}_{\geqslant 0}.$$

Proposition 3.1.14. *With the above notation, the following hold.*

1. $0 \leqslant \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{b}) \leqslant \ell\ell_R(R/I) + \mu_R(\mathfrak{b})$.
2. $p^{en} \cdot \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{b}) \in \mathbb{Z}$.

Proof. By Proposition 3.1.11 (5), we have

$$\begin{aligned} \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{b}^{\ell\ell_R(R/I)+\mu_R(\mathfrak{b})}) &= \mathfrak{b}^{\ell\ell_R(R/I)} \cdot \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{b}^{\mu_R(\mathfrak{b})}) \\ &\subseteq \mathfrak{b}^{\ell\ell_R(R/I)} \subseteq I, \end{aligned}$$

which proves the assertion in (1).

The assertion in (2) follows from Proposition 3.1.11 (1). □

Let $(R, \mathfrak{m})$ be a Noetherian local ring. For an R-module M with finite length, we denote by $\ell_R(M)$ the length of M as an R-module and define

$$\ell\ell_R(M) := \min\{n \geqslant 0 \mid \mathfrak{m}^n M = 0\}.$$

Proposition 3.1.15. *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $(p^e - 1)(K_X + \Delta)$ is Cartier for some positive integer e, let $t > 0$ be a rational number, and let $M, \mu > 0$ and $u \geqslant 2$ be positive integers. Assume that*

1. *$q > \mu + \mathrm{emb}(R)$, and*
2. *$q^m(q-1)t \in \mathbb{N}$ for some integer m.*

Then, there exists a positive integer n_1 such that for every ideal $\mathfrak{b} \subseteq R$, if $\mathfrak{b} = \mathfrak{a} + \mathfrak{m}^M$ for some ideal $\mathfrak{a} \subseteq R$ with $\mu_R(\mathfrak{a}) \leqslant \mu$, then we have $\tau_e^{n,u}(R, \Delta, \mathfrak{b}^t) = \tau_e^{n_1,u}(R, \Delta, \mathfrak{b}^t)$ for every $n \geqslant n_1$.

Proof. By Proposition 3.1.11 (6), it is enough to show the assertion in the case when $t > \mu + \mathrm{emb}(R)$ and $(p^e - 1)t \in \mathbb{N}$. Set $n_1 := \ell_R(\tau(R, \Delta)/(\mathfrak{m}^{M\lceil t \rceil} \cdot \tau(R, \Delta)))$. We will prove that the assertion holds for this constant n_1.

Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mu_R(\mathfrak{a}) \leqslant \mu$ and set $\mathfrak{b} := \mathfrak{a} + \mathfrak{m}^M$. We consider the sequence of ideals $\{\tau_e^{n,u}(R, \Delta, \mathfrak{b}^t)\}_{n \geqslant 1}$. As in the proof of Proposition 3.1.12, by using Lemma 3.1.5 (2) instead of Lemma 3.1.5 (1), the sequence $\{\tau_e^{n,u}(R, \Delta, \mathfrak{b}^t)\}_n$ is a descending chain. Moreover, since $\mathfrak{b} \supseteq \mathfrak{m}^M$, we have

$$\begin{aligned} \tau_e^{n,u}(R, \Delta, \mathfrak{b}^t) &\supseteq \tau_e^{n,u}(R, \Delta, (\mathfrak{m}^M)^t) \\ &\supseteq \tau_e^{n,u}(R, \Delta, (\mathfrak{m}^M)^t) \\ &\supseteq \tau_e^{n,0}(R, \Delta, (\mathfrak{m}^M)^t) \\ &\supseteq \mathfrak{m}^{M\lceil t \rceil} \cdot \tau(R, \Delta). \end{aligned}$$

Since we have

$$\tau(R,\Delta) \supseteq \tau_e^{1,u}(R,\Delta,\mathfrak{b}^t) \supseteq \tau_e^{2,u}(R,\Delta,\mathfrak{b}^t) \supseteq \cdots \supseteq \mathfrak{m}^{M\lceil t\rceil} \cdot \tau(R,\Delta),$$

there exists an integer $1 \leqslant m \leqslant n_1$ such that

$$\tau_e^{m,u}(R,\Delta,\mathfrak{b}^t) = \tau_e^{m+1,u}(R,\Delta,\mathfrak{b}^t).$$

On the other hand, as in the proof of Proposition 3.1.11 (5), by using Lemma 3.1.5 (2) instead of Lemma 3.1.5 (1), we have

$$\tau_e^{m+1,u}(R,\Delta,\mathfrak{b}^{t'+1}) = \mathfrak{b} \cdot \tau_e^{m,u}(R,\Delta,\mathfrak{b}^{t'})$$

for any real number $t' > \mu + \mathrm{emb}(R)$. Then, as in the proof of Proposition 3.1.8 (3), we have $\tau_e^{m+1,u}(R,\Delta,\mathfrak{a}^t) = \tau_e^{m+2,u}(R,\Delta,\mathfrak{a}^t)$, which completes the proof. □

3.2 Rationality of the limit of F-jumping numbers

In this section, we give uniform bounds for the denominators of F-jumping numbers (Proposition 3.2.1) and for the stabilization exponents (Proposition 3.2.3) of $\mathfrak{m}$-primary ideals with fixed colength. By using these bounds, we will verify the rationality of the limit of any sequence of F-pure thresholds (Corollary 3.2.8).

Proposition 3.2.1. *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $(p^e-1)(K_X+\Delta)$ is Cartier for some integer $e > 0$ and $M > 0$ be an integer. Then there exists an integer $N > 0$ such that for any ideal $\mathfrak{a} \subseteq R$, if $\mathfrak{a} \supseteq \mathfrak{m}^M$, then any F-jumping number of $(R,\Delta;\mathfrak{a})$ is contained in $(1/N)\cdot\mathbb{Z}$.*

Proof. Set $l := \ell_R(R/\mathfrak{m}^M) + \mu_R(\mathfrak{m}^M)$ and $n := \ell_R(\tau(R,\Delta)/\tau(R,\Delta,\mathfrak{m}^{Ml}))$. We note that the module $\tau(R,\Delta)/\tau(R,\Delta,\mathfrak{m}^{Ml})$ has finite length because the test ideals commute with localization ([HT04, Proposition 3.1]). Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{m}^M \subseteq \mathfrak{a}$ and let $B \subseteq \mathbb{R}_{>0}$ be the set of all F-jumping numbers of $(R,\Delta;\mathfrak{a})$.

Since we have $\mu(\mathfrak{a}) \leqslant l$, it follows from [BSTZ10, Corollary 3.27] that for every element $b \in B \cap \mathbb{R}_{>l}$, we have $b-1 \in B$. It also follows from [BSTZ10, Lemma 3.25] that for every element $b \in B$, we have $p^e b \in B$. Moreover, since $\tau(R,\Delta) \supseteq \tau(R,\Delta,\mathfrak{a}^t) \supseteq \tau(R,\Delta,\mathfrak{m}^{Ml})$ for every $t \leqslant l$, the number of the set $B \cap [0,l]$ is at most n. Then the assertion follows from the lemma below. □

Lemma 3.2.2. *Let $l, n > 0$ and $q \geqslant 2$ be integers. Then there exists an integer $N > 0$ with the following property: if $B \subseteq \mathbb{R}_{\geqslant 0}$ is a subset such that*

1. *for every element $b \in B$, if $b > l$, then we have $b - 1 \in B$,*
2. *if $b \in B$, then $q \cdot b \in B$, and*
3. *the number of the set $B \cap [0, l]$ is at most n,*

then we have $B \subseteq (1/N) \cdot \mathbb{Z}$.

Proof. The proof is essentially the same as that of [BMS08, Proposition 3.8]. Set $N := q^n(q^{n!} - 1)$, where $n!$ is the factorial of n.

For every element $b \in B$ and every integer $m \geqslant 0$, we define $b_m \in B \cap [0, l]$ by

$$b_m := (q^m b - \lfloor q^m b \rfloor) + \min\{l - 1, \lfloor q^m b \rfloor\}.$$

If $b \notin (1/N) \cdot \mathbb{Z}$, then $b_0, b_1, \ldots, b_n$ are all distinct and hence contradiction. $\square$

Proposition 3.2.3. *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$ and $M > 0$ be an integer. Then there exists $u_0 > 0$ such that for every ideals $\mathfrak{a} \supseteq \mathfrak{m}^M$, we have*

$$\widetilde{\operatorname{stab}}(R, \Delta, \mathfrak{a}; e) \leqslant u_0.$$

Proof. Set $l := \ell_R(R/\mathfrak{m}^M) + \mu_R(\mathfrak{m}^M)$ and take an integer $n_0 > 0$ such that $p^{e(n_0 - 1)} > l$. Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \supseteq \mathfrak{m}^M$ and $t > 0$ be a rational number such that $(p^e - 1)t \in \mathbb{N}$.

We first consider the case when $l < t \leqslant lp^{en_0}$. In this case, by Proposition 3.1.6 (1), the sequence $\{\tau_+^{en}(R, \Delta, \mathfrak{a}^t)\}_{n \geqslant 0}$ is an ascending chain such that

$$\tau(R, \Delta) \supseteq \tau_+^{en}(R, \Delta, \mathfrak{a}^t) \supseteq \tau_+^0(R, \Delta, \mathfrak{a}^t) = \mathfrak{a}^{\lceil t \rceil} \cdot \tau(R, \Delta) \supseteq \mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta)$$

for every n. Therefore, there exists an integer $0 \leqslant n < \ell_R(\tau(R, \Delta)/(\mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta)))$ such that

$$\tau_+^{en}(R, \Delta, \mathfrak{a}^t) = \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}^t).$$

By Proposition 3.1.8 (3), we have

$$\operatorname{stab}(R, \Delta, \mathfrak{a}^t; e) \leqslant n \leqslant \ell_R(\tau(R, \Delta)/(\mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta))).$$

We next consider the case when $t \leqslant l$. Since $l < tp^{en_0} \leqslant lp^{en_0}$, it follows from Proposition 3.1.8 (2) that

$$\begin{aligned} \operatorname{stab}(R, \Delta, \mathfrak{a}^t; e) &\leqslant \operatorname{stab}(R, \Delta, \mathfrak{a}^{tp^{en_0}}; e) + n_0 \\ &\leqslant \ell_R(\tau(R, \Delta)/(\mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta))) + n_0. \end{aligned}$$

Therefore, $u_0 := \ell_R(\tau(R,\Delta)/(\mathfrak{m}^{lMp^{en_0}} \cdot \tau(R,\Delta))) + n_0$ satisfies the property. $\square$

Proposition 3.2.4. *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $(p^e-1)(K_X+\Delta)$ is Cartier for some integer $e > 0$, $\{\mathfrak{a}_m\}_{m\in\mathbb{N}}$ be a family of ideals of R and $t > 0$ be a real number. Fix a non-principal ultrafilter $\mathfrak{U}$. Let $(R_\#, \mathfrak{m}_\#)$ be the catapower of the local ring $(R, \mathfrak{m})$, $\Delta_\#$ be the flat pullback of Δ to $\operatorname{Spec} R_\#$ and $\mathfrak{a}_\infty := [\mathfrak{a}_m]_m \subseteq R_\#$. If there exists a positive integer M such that $\mathfrak{a}_m \supseteq \mathfrak{m}^M$ for every m, then we have*

$$\tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) = [\tau(R, \Delta, \mathfrak{a}_m^t)]_m \subseteq R_\#.$$

Proof. We first consider the case when t is a rational number. By enlarging e, we may assume that $p^{en}(p^e-1)t \in \mathbb{Z}$ for some integer $n \geqslant 0$. Take a positive integer u as in Proposition 3.2.3. Then we have

$$\tau(R, \Delta, \mathfrak{a}_m^t) = \tau_+^{e(n+u)}(R, \Delta, \mathfrak{a}_m^t),$$

for every m. By enlarging u, we may assume that

$$\tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) = \tau_+^{e(n+u)}(R_\#, \Delta_\#, \mathfrak{a}_\infty^t).$$

Since $\mu_R(\mathfrak{a}_m) \leqslant \ell_R(R/\mathfrak{m}^M) + \mu_R(\mathfrak{m}^M)$ for every m, it follows from Lemma 2.4.4 that

$$(\mathfrak{a}_\infty)^s = [(\mathfrak{a}_m)^s]_m$$

for every integer $s > 0$. Combining with Proposition 2.1.8 and 2.4.3, we have

$$\begin{aligned}
\tau_+^{el}(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) &= \varphi_{\Delta_\#}^{el}(F_*^{el}(\mathfrak{a}_\infty^{\lceil tp^{el}\rceil} \cdot \tau(R_\#, \Delta_\#))) \\
&= \varphi_{\Delta_\#}^{el}(F_*^{el}[\mathfrak{a}_m^{\lceil tp^{el}\rceil} \cdot \tau(R, \Delta)]_m) \\
&= [\varphi_{\Delta}^{el}(F_*^{el}(\mathfrak{a}_m^{\lceil tp^{el}\rceil} \cdot \tau(R, \Delta)))]_m \\
&= [\tau_+^{el}(R, \Delta, \mathfrak{a}_m^t)]_m \subseteq R_\#
\end{aligned}$$

for every integer l. Therefore, we have

$$\tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) = [\tau(R, \Delta, \mathfrak{a}_m^t)]_m \subseteq R_\#.$$

We next consider the case when t is not a rational number. For sufficiently large integer n, we have

$$\begin{aligned}
\tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) &= \tau_+^{en}(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) \\
&= [\tau_+^{en}(R, \Delta, \mathfrak{a}_m^t)]_m \\
&\subseteq [\tau(R, \Delta, \mathfrak{a}_m^t)]_m \subseteq R_\#.
\end{aligned}$$

For the converse inclusion, by Proposition 2.3.5, we can take a rational number t' such that $t' < t$ and $\tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) = \tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^{t'})$. Then, we have

$$\begin{aligned} \tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) &= \tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^{t'}) \\ &= [\tau(R, \Delta, \mathfrak{a}_m^{t'})]_m \\ &\supseteq [\tau(R, \Delta, \mathfrak{a}_m^{t})]_m, \end{aligned}$$

which completes the proof. □

Proposition 3.2.5. *With the notation above, let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Assume that $\mathfrak{m}^M \subseteq \mathfrak{a}_m \subseteq \mathfrak{m}$ for every m. Then there exists $T \in \mathfrak{U}$ such that for all $m \in T$, we have*

$$\operatorname{fjn}^I(R, \Delta; \mathfrak{a}_m) = \operatorname{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#, \mathfrak{a}_\infty).$$

Proof. Set $t := \operatorname{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{a}_\infty) \in \mathbb{R}_{\geqslant 0}$. If $\tau(R, \Delta) \subseteq I$, then we have $\operatorname{fjn}^I(R, \Delta; \mathfrak{a}_m) = 0$ for every $m \in \mathbb{N}$ and $\operatorname{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#, \mathfrak{a}_\infty) = 0$. Therefore, we may assume that $\tau(R, \Delta) \not\subseteq I$. Since $\mathfrak{a}_\infty \neq (0)$, it follows from Lemma 2.1.5 (2) that $t > 0$.

It follows from Proposition 3.2.4 that we have

$$[\tau(R, \Delta, \mathfrak{a}_m^t)]_m = \tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) \subseteq I \cdot R_\#.$$

Since I is $\mathfrak{m}$-primary, it follows from Lemma 2.4.8 that there exists $S_1 \in \mathfrak{U}$ such that $\tau(R, \Delta, \mathfrak{a}_m^t) \subseteq I$ for every $m \in S_1$. Therefore $\operatorname{fjn}^I(R, \Delta; \mathfrak{a}_m) \leqslant \operatorname{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#, \mathfrak{a}_\infty)$ for every $m \in S_1$.

On the other hand, by Proposition 3.2.1, there exists $0 < t' < t$ such that for every ideal $\mathfrak{b} \supseteq \mathfrak{m}^M$, if $t' < \operatorname{fjn}^I(R, \Delta; \mathfrak{b})$, then $t \leqslant \operatorname{fjn}^I(R, \Delta; \mathfrak{b})$. Since $t' < t$, we have

$$[\tau(R, \Delta, \mathfrak{a}_m^{t'})]_m = \tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^{t'}) \not\subseteq I \cdot R_\#.$$

Hence, we have

$$\operatorname{ulim}_m \tau(R, \Delta, \mathfrak{a}_m^{t'}) \not\subseteq {}^*I.$$

Therefore, there exists $S_2 \in \mathfrak{U}$ such that $\tau(R, \Delta, \mathfrak{a}_m^{t'}) \not\subseteq I$ for every $m \in S_2$. Then $T := S_1 \cap S_2$ satisfies the assertion. □

Lemma 3.2.6 ([BMS09, Lemma 3.3])**.** *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Carter, I be an $\mathfrak{m}$-primary ideal, $\mathfrak{a}, \mathfrak{b} \subseteq R$ be proper ideals. Then we have*

$$\operatorname{fjn}^I(R, \Delta; \mathfrak{a} + \mathfrak{b}) \leqslant \operatorname{fjn}^I(R, \Delta; \mathfrak{a}) + \operatorname{fjn}^I(R, \Delta; \mathfrak{b}).$$

Proof. As in the proof of [Tak06, Theorem 3.1], for every real number $c \geqslant 0$, we can show that

$$\tau(R, \Delta, (\mathfrak{a} + \mathfrak{b})^c) = \sum_{u,v \geqslant 0, u+v=c} \tau(R, \Delta, \mathfrak{a}^u \mathfrak{b}^v).$$

Set $t := \mathrm{fjn}^I(R, \Delta; \mathfrak{a})$ and $s := \mathrm{fjn}^I(R, \Delta; \mathfrak{b})$. Then we have

$$\begin{aligned} \tau(R, \Delta, (\mathfrak{a} + \mathfrak{b})^{t+s}) &= \sum_{u,v \geqslant 0, u+v=s+t} \tau(R, \Delta, \mathfrak{a}^u \mathfrak{b}^v) \\ &\subseteq \tau(R, \Delta, \mathfrak{a}^t) + \tau(R, \Delta, \mathfrak{b}^s) \subseteq I. \end{aligned}$$

□

Theorem 3.2.7. *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$, $(R_\#, \mathfrak{m}_\#)$ be the catapower of $(R, \mathfrak{m})$, $\Delta_\#$ be the flat pullback of Δ to $\operatorname{Spec} R_\#$, $I \subseteq R$ be an $\mathfrak{m}$-primary ideal, $\{\mathfrak{a}_m\}_{m \in \mathbb{N}}$ be a family of proper ideals and $\mathfrak{a}_\infty := [\mathfrak{a}_m]_m \subseteq R_\#$. Then we have*

$$\mathrm{sh}(\mathrm{ulim}_m \, \mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m)) = \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#, \mathfrak{a}_\infty) \in \mathbb{Q}.$$

In particular, if the limit $\lim_{m \longrightarrow \infty} \mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m)$ exists, then we have

$$\lim_{m \longrightarrow \infty} \mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m) = \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#, \mathfrak{a}_\infty).$$

Proof. The proof is essentially the same as the proof of [BMS09, Theorem 1.2]. If $\tau(R, \Delta) \subseteq I$, then the assertion in the theorem is trivial. Therefore, we may assume that $\tau(R, \Delta) \not\subseteq I$.

For every integer $M > 0$, we set $\mathfrak{b}_{\infty,M} := \mathfrak{a}_\infty + (\mathfrak{m}_\#)^M$ and $\mathfrak{b}_{m,M} := \mathfrak{a}_m + \mathfrak{m}^M$ for every integer m. We write $s := \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{m}_\#)$

By Lemma 3.2.6, we have

$$|\mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{a}_\infty) - \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{b}_{\infty,M})| \leqslant s/M \tag{3.1}$$

for every M.

By Proposition 2.1.8 (4), we have $s = \mathrm{fjn}^I(R, \Delta; \mathfrak{m})$. Therefore, it follows from Lemma 3.2.6 that

$$|\mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m) - \mathrm{fjn}^I(R, \Delta; \mathfrak{b}_{m,M})| \leqslant s/M \tag{3.2}$$

for every m and M.

On the other hand, since $\mathfrak{b}_{\infty,M} = [\mathfrak{b}_{m,M}]_m$, it follows from Proposition 3.2.5 that there exists $T_M \in \mathfrak{U}$ such that

$$\mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{b}_{\infty,M}) = \mathrm{fjn}^I(R, \Delta; \mathfrak{b}_{m,M}) \tag{3.3}$$

for every $m \in T_M$.

By combining the equations (3.1), (3.2), and (3.3), we have

$$|\mathrm{fjn}^{I\cdot R_\#}(R_\#, \Delta_\#; \mathfrak{a}_\infty) - \mathrm{fjn}^{I}(R, \Delta; \mathfrak{a}_m)| \leqslant 2s/M$$

for every $m \in T_M$.

It follows from the definition of the shadow that

$$\mathrm{sh}(\mathrm{ulim}_m \mathrm{fjn}^{I}(R, \Delta; \mathfrak{a}_m)) = \mathrm{fjn}^{I\cdot R_\#}(R_\#, \Delta_\#; \mathfrak{a}_\infty),$$

which completes the proof. □

Since we have $\mathrm{fpt}(R, \Delta; \mathfrak{a}) = \mathrm{fjn}^{\mathfrak{m}}(R, \Delta; \mathfrak{a})$ for a strongly F-regular pair (R, Δ), the following is a special case of Theorem 3.2.7.

Corollary 3.2.8 (cf. [BMS09, Theorem 1.2]). *With the notation above, we further assume that (R, Δ) is strongly F-regular. Then we have*

$$\mathrm{sh}(\mathrm{ulim}_m \mathrm{fpt}(R, \Delta_m; \mathfrak{a}_m)) = \mathrm{fpt}(R_\#, \Delta_\infty, \mathfrak{a}_\infty) \in \mathbb{Q}.$$

In particular, if the limit $\lim_{m\longrightarrow\infty} \mathrm{fpt}(R, \Delta_m; \mathfrak{a}_m)$ exists, then we have

$$\lim_{m\longrightarrow\infty} \mathrm{fpt}(R, \Delta_m; \mathfrak{a}_m) = \mathrm{fpt}(R_\#, \Delta_\infty, \mathfrak{a}_\infty).$$

3.3 ACC for F-jumping numbers

In this section, we introduce Condition ($\star$) (Definition 3.3.2) which plays the key role in the proof of the main theorem and we prove some properties of Condition ($\star$) (Proposition 3.3.4 and Proposition 3.3.6). By combining them with Proposition 3.1.15 and Theorem 3.2.7, we give the proof of the main theorem of Chapter 3 (Theorem 3.3.9).

Observation 3.3.1. Let X be a normal variety over a field k of characteristic zero, Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero coherent ideal sheaf, $t \geqslant 0$ be a rational number, $x \in X$ be a closed point and $\mathfrak{m}_x \subseteq \mathcal{O}_X$ be the maximal ideal at x. We consider the *log canonical threshold*

$$\mathrm{lct}_x(X, \Delta, \mathfrak{a}^t; \mathfrak{m}) := \inf\{s \geqslant 0 \mid (X, \Delta, \mathfrak{a}^t\mathfrak{m}^s) \text{ is not log canonical at } x\}.$$

By considering a log resolution of (X, Δ), $\mathfrak{a}$ and $\mathfrak{m}$, we can show that there exist a real number $t' < t$ and rational numbers a, b such that

$$\mathrm{lct}_x(X, \Delta, \mathfrak{a}^s; \mathfrak{m}) = as + b \tag{3.4}$$

for every $t' < s < t$.

Assume that there exist integers $q \geqslant 2$ and $m \geqslant 0$ such that $q^m(q-1)t \in \mathbb{N}$. Then for every $n > m$, the n-th digit of t in base q satisfies $t^{(n)} = l$ for some constant $l > 0$. Set $N := -al/q$. Then we have

$$\mathrm{lct}_x(X, \Delta, \mathfrak{a}^{\langle t \rangle_{n+1,q}}; \mathfrak{m}) = \mathrm{lct}_x(X, \Delta, \mathfrak{a}^{\langle t \rangle_{n,q}}; \mathfrak{m}) - N/q^n \tag{3.5}$$

for sufficiently large n.

Motivated by the observation above, we define the following condition.

Definition 3.3.2. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t > 0$ and $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$, $I \subseteq R$ be an $\mathfrak{m}$-primary ideal and $u, N \geqslant 0$ be integers. We say that $(R, \Delta, \mathfrak{a}^t, I, e, u, N)$ satisfies *Condition* $(\star)$ if for every $n \geqslant 0$, we have

$$\mathrm{fjn}_e^{I,n+1,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \geqslant \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - N/p^{en}.$$

Remark 3.3.3. If we have $u \geqslant \widetilde{\mathrm{stab}}(R, \Delta, \mathfrak{a}, \mathfrak{m}; e)$, then we have

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) = \langle \mathrm{fjn}^I(R, \Delta, \mathfrak{a}^{\langle t \rangle_{n,q}}; \mathfrak{m}) \rangle^{n,q},$$

where we write $q := p^e$. Therefore, Condition $(\star)$ can be regarded as an analogue of the equation 3.5 in Observation 3.3.1. See also Corollary 3.3.5 below.

We also note that the equation 3.4 in Observation 3.3.1 may not hold for F-pure thresholds (cf. [Pér13, Example 5.3]).

We first give a sufficient condition for Condition $(\star)$.

Proposition 3.3.4. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t > 0$ and $(p^e - 1)(K_X + \Delta)$ is Cartier for some $e > 0$, let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal, let $0 < l < p^e$ be a positive integer and let $n_0 \geqslant 0$ and $u \geqslant 2$ be integers. Set*

$$q = p^e, \ N := q^{n_0+3}\mathrm{emb}(R), \ t_0 := \frac{q^2}{q-1}, \text{ and } M_0 := \frac{(q^{n_0+6} - 1)\mathrm{emb}(R)}{q-1}.$$

Assume that

1. *$q > \mu_R(\mathfrak{a})$,*
2. *$q > \ell\ell_R(R/I)$,*
3. *the n-th digit of t in base q satisfies $t^{(n)} = l$ for every $n \geqslant 2$, and*
4. *$\tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}^{lt_0}) + \mathfrak{m}^{M_0} \cdot \tau(R, \Delta) \supseteq \tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{lt_0})$.*

Then, $(R, \Delta, \mathfrak{a}^t, I, e, u, N)$ *satisfies Condition* $(\star)$

Proof. By induction on $n \geqslant 0$, we will show the inequality

$$\mathrm{fjn}_e^{I,n+1,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \geqslant \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - N/q^n. \tag{3.6}$$

Step.1 We consider the case when $n \leqslant n_0 + 2$. In this case, we have

$$N/q^n \geqslant q \cdot \mathrm{emb}(R) \geqslant \ell\ell_R(R/I) + \mathrm{emb}(R).$$

By Proposition 3.1.14 (1), we have

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \leqslant \ell\ell_R(R/I) + \mathrm{emb}(R).$$

Hence we have

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - N/q^n \leqslant 0,$$

which implies the inequality 3.6.

Step.2 From now on, we assume $n \geqslant n_0 + 3$. Set

$$r := q^n \cdot \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}).$$

By Proposition 3.1.14, we have $r \in \mathbb{Z}$. We first consider the case when

$$r \leqslant q^{n_0} \cdot \mathrm{emb}(R).$$

In this case, we have

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - N/q^n \leqslant 0,$$

which shows the inequality 3.6. Therefore, we may assume $r > q^{n_0} \cdot \mathrm{emb}(R)$.

Step.3 Set $s := \lceil r/q^{n_0} \rceil - \mathrm{emb}(R) - 1$ and $s' := \lceil (s + M_0)/q^2 \rceil$.

In this step, we will show the inclusion

$$\begin{aligned} \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{r/q^n}) \subseteq \tau_e^{n+1,u}&(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s/q^{n-n_0}}) \\ &+ \tau_e^{n-n_0-2,2}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s'/q^{n-n_0+2}}). \end{aligned} \tag{3.7}$$

By the assumption (3), $\alpha := tq^{n-n_0} - lt_0 = q^2 \lceil tq^{n-n_0-2} - 1 \rceil$ is an integer. It follows from Proposition 3.1.11 (1), (5), and (6) that

$$\begin{aligned} \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{r/q^n}) &= \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{tq^{n-n_0}} \mathfrak{m}^{r/q^{n_0}}))) \\ &\subseteq \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\mathfrak{a}^\alpha \mathfrak{m}^s \tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{lt_0}))). \end{aligned}$$

Similarly, we have

$$\begin{aligned}\tau_e^{n+1,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s/q^{n-n_0}}) &= \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\tau_e^{n_0+1,u}(R,\Delta,\mathfrak{a}^{tq^{n-n_0}}\mathfrak{m}^s)))\\ &\supseteq \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\mathfrak{a}^\alpha\mathfrak{m}^s\tau_e^{n_0+1,u}(R,\Delta,\mathfrak{a}^{lt_0}))).\end{aligned}$$

On the other hand, it follows from the definitions that

$$\begin{aligned}\tau_e^{n-n_0-2,2}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s'/q^{n-n_0-2}}) &= \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\mathfrak{a}^\alpha\mathfrak{m}^{q^2(s'-1)}\tau(R,\Delta)))\\ &\supseteq \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\mathfrak{a}^\alpha\mathfrak{m}^{s+M_0}\tau(R,\Delta))).\end{aligned}$$

By combining them with the assumption (4), we have the inclusion 3.7.
Step.4 In this step, we will show the inclusion

$$\tau_e^{n-n_0-2,2}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s'/q^{n-n_0-2}}) \subseteq I. \tag{3.8}$$

It follows from the induction hypothesis that

$$\mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) \geqslant \mathrm{fjn}_e^{I,n-n_0-2,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) - (\sum_{i=n-n_0-2}^{n-1}\frac{N}{q^i}).$$

Therefore, we have the inequality

$$\begin{aligned}&\frac{s'}{q^{n-n_0-2}} \geqslant \frac{s+M_0}{q^{n-n_0}} \geqslant \frac{r/q^{n_0}-\mathrm{emb}(R)-1+M_0}{q^{n-n_0}}\\ =\ &\mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) + \frac{-\mathrm{emb}(R)-1+M_0}{q^{n-n_0}}\\ \geqslant\ &\mathrm{fjn}_e^{I,n-n_0-2,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) - (\sum_{i=n-n_0-2}^{n-1}\frac{N}{q^i}) + \frac{-\mathrm{emb}(R)-1+M_0}{q^{n-n_0}}\\ >\ &\mathrm{fjn}_e^{I,n-n_0-2,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}).\end{aligned}$$

Since we have $u \geqslant 2$, It follows from Proposition 3.1.11 (7) that

$$\tau_e^{n-n_0-2,2}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s'/q^{n-n_0-2}}) \subseteq \tau_e^{n-n_0-2,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s'/q^{n-n_0-2}}) \subseteq I.$$

Step.5 It follows from Proposition 3.1.11 (1) that

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{r/q^n}) \nsubseteq I.$$

Combining it with the inclusions 3.7 and 3.8, we have

$$\tau_e^{n+1,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s/q^{n-n_0}}) \nsubseteq I.$$

Hence, we have

$$\begin{aligned}
\mathrm{fjn}_e^{I,n+1,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) &\geqslant \frac{s}{q^{n-n_0}} \\
&\geqslant \frac{r/q^{n_0}-\mathrm{emb}(R)-1}{q^{n-n_0}} \\
&= \mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) - \frac{\mathrm{emb}(R)+1}{q^{n-n_0}} \\
&> \mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) - \frac{N}{q^n},
\end{aligned}$$

which completes the proof of the proposition. □

Corollary 3.3.5. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t > 0$ is a rational number and $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$ and $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then, there exist integers $e', u_0, N > 0$ such that for every $u \geqslant u_0$, $(R, \Delta, \mathfrak{a}^t, I, e', u, N)$ satisfies Condition $(\star)$. In particular, there exists an integer $N' > 0$ such that if we write $q := p^{e'}$, then*

$$\mathrm{fjn}^I(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n+1,q}};\mathfrak{m}) \geqslant \mathrm{fjn}^I(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n,q}};\mathfrak{m}) - N'/q^n$$

for every integer $n \geqslant 0$.

Proof. Take an integer $m > 0$ such that $q := p^{em}$ satisfies the assumptions (1), (2), and (3) in Proposition 3.3.4.

Set $l = t^{(2)}$ and $t_0 := q^2/(q-1)$. Then it follows from Proposition 2.3.5 that there exists an integer $n_0 > 0$ such that

$$\tau(R,\Delta,\mathfrak{a}^{\langle lt_0\rangle_{n_0,q}}) = \tau(R,\Delta,\mathfrak{a}^{\langle lt_0\rangle_{(n_0+1),q}}).$$

Set $e' := em$, $u_0 := \widetilde{\mathrm{stab}}(R,\Delta,\mathfrak{a};e')$ and $N := q^{n_0+3}\cdot\mathrm{emb}(R)$. Then the first assertion follows from Proposition 3.3.4.

Set $N' := N + 1$. Then the second assertion follows from Remark 3.3.3. □

Proposition 3.3.6. *Suppose that $(R, \Delta, \mathfrak{a}^t)$, $q = p^e$, u, and N satisfies the conditions of Proposition 3.3.4. We further assume that $q > \ell\ell_R(R/I) + \mu_R(\mathfrak{a}) + \mathrm{emb}(R)$. Then for every $n \geqslant 1$, we have*

$$\mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) = \mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{b}^t;\mathfrak{m}),$$

where $\mathfrak{b} := \mathfrak{a} + \mathfrak{m}^{q^{u+2}\cdot N}$. In particular, for every n, we have

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}^t) \subseteq I \text{ if and only if } \tau_e^{n,u}(R,\Delta,\mathfrak{b}^t) \subseteq I.$$

Proof. Set $M := q^{u+2} \cdot N$, $M' := q^{u+1} \cdot N$, $s_n := \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m})$, and $\delta_n := q^n s_n$ for every integer n. By Proposition 3.1.14 (2), we have $\delta_n \in \mathbb{N}$. It is enough to show the following claim.

Claim. For every $n \geqslant 1$ and every ideal $\mathfrak{q} \subseteq \mathfrak{m}^{\max\{0, q^u \cdot \delta_n - M'\}} \cdot \tau(R, \Delta)$, we have

$$\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}^t) \equiv \tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{b}^t) \pmod{I}.$$

In fact, if the claim holds, then it follows from Proposition 3.1.11 (4) that

$$\begin{aligned} \tau_e^{n,u}(R, \Delta, \mathfrak{b}^t \mathfrak{m}^{s_n+\varepsilon}) &\equiv \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s_n+\varepsilon}) \pmod{I} \\ &\subseteq I \end{aligned}$$

for every real number $0 < \varepsilon \leqslant 1/q^n$. Therefore we have

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \geqslant \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{b}^t; \mathfrak{m}).$$

Similarly, if $s_n > 0$, then we have

$$\begin{aligned} \tau_e^{n,u}(R, \Delta, \mathfrak{b}^t \mathfrak{m}^{s_n}) &\equiv \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s_n}) \pmod{I} \\ &\not\subseteq I, \end{aligned}$$

which shows $\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \leqslant \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{b}^t; \mathfrak{m})$. Since this inequality also holds when $s_n = 0$, we complete the proof of the proposition.

□

Proof of Claim. We use induction on n.

Step.1 We first consider the case when $n = 1$. It follows from Proposition 3.1.11 (3) that

$$\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}^t) \equiv \tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{b}^t) \pmod{\tau_{e,\mathfrak{q}\cdot\mathfrak{m}^M}^{n,u}(R, \Delta)}.$$

Since we have $\mathfrak{q} \cdot \mathfrak{m}^M \subseteq \mathfrak{m}^{q^u \lceil q(\ell\ell_R(R/I) + \mathrm{emb}(R)) - 1 \rceil} \cdot \tau(R, \Delta)$, it follows from Proposition 3.1.11 (2), (4) and (5) that

$$\tau_{e,\mathfrak{q}\cdot\mathfrak{m}^M}^{n,u}(R, \Delta) \subseteq \mathfrak{m}^{\ell\ell_R(R/I)} \subseteq I.$$

Therefore, the assertion holds when $n = 1$.

Step.2 From now on, we consider the case when $n \geqslant 2$. Set $\mathfrak{q}' := \varphi_\Delta^e(F_*^e(\mathfrak{a}^{t(n) \cdot q^u} \mathfrak{q}))$ and $\mathfrak{q}'' := \varphi_\Delta^e(F_*^e(\mathfrak{b}^{t(n) \cdot q^u} \mathfrak{q}))$.

Then it follows from Proposition 3.1.11 (9) that

$$\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}^t) = \tau_{e,\mathfrak{q}'}^{n-1,u}(R, \Delta, \mathfrak{a}^t). \tag{3.9}$$

Similarly, by using Lemma 3.1.5 (2) instead of (1), we have

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{b}^t)=\tau_{e,\mathfrak{q}''}^{n-1,u}(R,\Delta,\mathfrak{b}^t). \tag{3.10}$$

Step.3 In this step, we will show the equation

$$\tau_{e,\mathfrak{q}'}^{n-1,u}(R,\Delta,\mathfrak{a}^t)\equiv\tau_{e,\mathfrak{q}''}^{n-1,u}(R,\Delta,\mathfrak{a}^t)\ (\operatorname{mod} I). \tag{3.11}$$

Set $J:=\varphi_\Delta^e(F_*^e(\mathfrak{m}^M\mathfrak{q}))$, then we have $\mathfrak{q}'\equiv\mathfrak{q}''(\operatorname{mod} J)$. By Proposition 3.1.11 (3), it is enough to show that

$$\tau_{e,J}^{n-1,u}(R,\Delta,\mathfrak{a}^t)\subseteq I.$$

Since we have $\delta_n\geqslant q\delta_{n-1}-qN$, it follows from Lemma 3.1.5 that

$$\begin{aligned} J &\subseteq \varphi_\Delta^e(\mathfrak{m}^{q^u\delta_n+M-M'}\cdot\tau(R,\Delta)) \\ &\subseteq \mathfrak{m}^{(q^u\delta_n+M-M')/q-\mathrm{emb}(R)}\cdot\tau(R,\Delta) \\ &\subseteq \mathfrak{m}^{q^u\delta_{n-1}}\cdot\tau(R,\Delta). \end{aligned}$$

Therefore, it follows from Proposition 3.1.11 (2) and (4) that

$$\tau_{e,J}^{n-1,u}(R,\Delta,\mathfrak{a}^t)\subseteq\tau_e^{n-1,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s_{n-1}+(1/q^{n-1})})\subseteq I,$$

which shows the equation 3.11.

Step.4 In this step, we will show the equation

$$\tau_{e,\mathfrak{q}''}^{n-1,u}(R,\Delta,\mathfrak{a}^t)\equiv\tau_{e,\mathfrak{q}''}^{n-1,u}(R,\Delta,\mathfrak{b}^t)\ (\operatorname{mod} I). \tag{3.12}$$

As in Step 3, we have

$$\begin{aligned} \mathfrak{q}'' &\subseteq \varphi_\Delta^e(F_*^e(\mathfrak{m}^{\max\{0,q^u\delta_n-M'\}}\cdot\tau(R,\Delta))) \\ &\subseteq \mathfrak{m}^{\max\{0,q^{u-1}\delta_n-(M'/q)-\mathrm{emb}(R)\}}\cdot\tau(R,\Delta) \\ &\subseteq \mathfrak{m}^{\max\{0,q^u\delta_{n-1}-M'\}}\cdot\tau(R,\Delta). \end{aligned}$$

By induction hypothesis, we get the equation 3.12.

By combining the equations 3.9, 3.10, 3.11 and 3.12, we complete the proof of the claim.

Corollary 3.3.7. *Let $(X=\operatorname{Spec}R,\Delta)$ be a pair such that $(p^e-1)(K_X+\Delta)$ is Cartier for some integer $e>0$, $I\subseteq R$ be an $\mathfrak{m}$-primary ideal, $l,n_0\geqslant 0$ and $u\geqslant 2$ be integers and $t>0$ be a rational number such that $p^e(p^e-1)t\in\mathbb{N}$. We set $l:=t^{(2)}$, $t_0:=p^{2e}/(p^e-1)$ and $M_0=(p^{e(n_0+6)}-1)\cdot\mathrm{emb}(R)/(p^e-1)$. Then there exists an integer $n_1>0$ with the following property: for any ideal $\mathfrak{a}\subseteq R$ such that*

1. $p^e > \mu_R(\mathfrak{a}) + \ell\ell_R(R/I) + \mathrm{emb}(R)$, *and*

2. $\tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}^{lt_0}) + \mathfrak{m}^{M_0} \cdot \tau(R, \Delta) \supseteq \tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{lt_0})$

we have

$$\tau_e^{n,u}(R, \Delta, \mathfrak{a}^t) \subseteq I \text{ if and only if } \tau_e^{n_1,u}(R, \Delta, \mathfrak{a}^t) \subseteq I$$

for every integer $n \geqslant n_1$.

Proof. By Proposition 3.3.4 and Proposition 3.3.6, $\mathfrak{b} := \mathfrak{a} + \mathfrak{m}^{q^{u+n_0+5}\mathrm{emb}(R)}$ satisfies

$$\tau_e^{n,u}(R, \Delta, \mathfrak{a}^t) \subseteq I \text{ if and only if } \tau_e^{n,u}(R, \Delta, \mathfrak{b}^t) \subseteq I.$$

for every integer n.

On the other hand, it follows from Proposition 3.1.15 that there exists an integer $n_1 > 0$ which depends only on $\mu := q - \mathrm{emb}(R) - 1$, $M := q^{u+n_0+5}\mathrm{emb}(R)$, e, u, and t such that for every integer $n > n_1$, we have

$$\tau_e^{n,u}(R, \Delta, \mathfrak{b}^t) \subseteq I \text{ if and only if } \tau_e^{n_1,u}(R, \Delta, \mathfrak{b}^t) \subseteq I,$$

which completes the proof. □

By using the method of ultraproduct, we can apply Corollary 3.3.7 to infinitely many ideals simultaneously.

Proposition 3.3.8. *Let* $(X = \mathrm{Spec}\, R, \Delta)$ *be a pair such that* $(p^e-1)(K_X+\Delta)$ *is Cartier for some integer* $e > 0$, $I \subseteq R$ *be an* $\mathfrak{m}$*-primary ideal,* $\{\mathfrak{a}_m\}_{m \in \mathbb{N}}$ *be a family of ideals of* R, $t > 0$ *be a rational number, and* $\mathfrak{U}$ *be a non-principal ultrafilter. Assume that*

1. $\tau(R, \Delta)$ *is* $\mathfrak{m}$*-primary or trivial,*

2. $p^e > \mu_R(\mathfrak{a}_m) + \ell\ell_R(R/I) + \mathrm{emb}(R)$ *for every* m, *and*

3. $p^e(p^e - 1)t \in \mathbb{N}$.

Then for any sufficiently large integer $u > 0$, *there exist an integer* n_1 *and* $T \in \mathfrak{U}$ *such that*

$$\tau_e^{n,u}(R, \Delta, \mathfrak{a}_m^t) \subseteq I \text{ if and only if } \tau_e^{n_1,u}(R, \Delta, \mathfrak{a}_m^t) \subseteq I$$

for every integer $n \geqslant n_1$ *and* $m \in T$.

Proof. Set $t_0 := p^{2e}/(p^e-1)$. Since $p^e(p^e-1)t \in \mathbb{N}$, there exists an integer $0 < l < p^e$ such that $t^{(n)} = l$ for every $n \geqslant 2$. By Corollary 3.3.7, it is enough to show that for any sufficiently large integer $u > 0$, there exist an integer n_0 and $T \in \mathfrak{U}$ such that for every $m \in T$, we have

$$\tau_e^{n_0+1,u}(R,\Delta,\mathfrak{a}^{lt_0}) + \mathfrak{m}^{M_0}\cdot\tau(R,\Delta) \supseteq \tau_e^{n_0,u}(R,\Delta,\mathfrak{a}^{lt_0}),$$

where $M_0 := (p^{e(n_0+6)}-1)\mathrm{emb}(R)/(p^e-1)$.

Let $(R_\#, \mathfrak{m}_\#)$ be the catapower of $(R,\mathfrak{m})$, $\Delta_\#$ be the flat pullback of Δ to $\operatorname{Spec} R_\#$ and $\mathfrak{a}_\infty$ be the ideal $[\mathfrak{a}_m]_m \subseteq R_\#$. It follows from Lemma 2.4.4 that for every integers $u, n \geqslant 0$ we have

$$\tau_e^{n,u}(R_\#,\Delta_\#,\mathfrak{a}_\infty^{l\cdot t_0}) = [\tau_e^{n,u}(R,\Delta,\mathfrak{a}_m^{l\cdot t_0})]_m.$$

By Proposition 2.3.5, there exists an integer $n_0 \geqslant 0$ such that

$$\tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^{\langle l\cdot t_0\rangle_{n_0,q}}) = \tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^{\langle l\cdot t_0\rangle_{(n_0+1),q}})$$

On the other hand, by Proposition 3.1.11 (8), there exists an integer u_0 such that for every integers $u \geqslant u_0$ and $n \geqslant 0$, we have

$$\tau_e^{n,u}(R_\#,\Delta_\#,\mathfrak{a}_\infty^{l\cdot t_0}) = \tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^{\langle l\cdot t_0\rangle_{n,q}}).$$

Therefore, we have

$$[\tau_e^{n_0,u}(R,\Delta,\mathfrak{a}_m^{l\cdot t_0})]_m = [\tau_e^{n_0+1,u}(R,\Delta,\mathfrak{a}_m^{l\cdot t_0})]_m \subseteq R_\#.$$

Since $\mathfrak{m}^{M_0}\cdot\tau(R,\Delta) \subseteq R$ is an $\mathfrak{m}$-primary ideal, it follows from Lemma 2.4.8 that there exists $T \in \mathfrak{U}$ such that for every $m \in T$, we have

$$\tau_e^{n_0,u}(R,\Delta,\mathfrak{a}_m^{l\cdot t_0}) \subseteq \tau_e^{n_0+1,u}(R,\Delta,\mathfrak{a}_m^{l\cdot t_0}) + \mathfrak{m}^{M_0}\cdot\tau(R,\Delta),$$

which completes the proof.

□

Theorem 3.3.9 (Theorem B, Corollary C)**.** *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $\tau(R,\Delta)$ is $\mathfrak{m}$-primary or trivial and that $(p^e-1)(K_X+\Delta)$ is Cartier for some integer $e > 0$, and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then, the set*

$$\mathrm{FJN}^I(R,\Delta) := \left\{\mathrm{fjn}^I(R,\Delta;\mathfrak{a}) \mid \mathfrak{a} \subsetneq R\right\}$$

satisfies the ascending chain condition. In particular, if (R,Δ) is strongly F-regular, then the set

$$\mathrm{FPT}(R,\Delta) := \{\mathrm{fpt}(R,\Delta;\mathfrak{a}) \mid \mathfrak{a} \subsetneq R\}$$

satisfies the ascending chain condition.

Proof. We assume the contrary. Then there exists a family of ideals $\{\mathfrak{a}_m\}_{m\in\mathbb{N}}$ such that $\{\mathrm{fjn}^I(R,\Delta;\mathfrak{a}_m)\}_{m\in\mathbb{N}}$ is a strictly ascending chain. Set

$$t := \lim_{m\longrightarrow\infty} \mathrm{fjn}^I(R,\Delta;\mathfrak{a}_m).$$

It follows from Proposition 2.3.5 and Theorem 3.2.7 that $t \in \mathbb{Q}_{>0}$.

Let $\mathfrak{U}$ be a non-principal ultrafilter, $R_\#$ be the catapower of R, $\Delta_\#$ be the flat pullback of Δ to $\operatorname{Spec} R_\#$, and $\mathfrak{a}_\infty := [\mathfrak{a}_m]_m \subseteq R_\#$. Take elements $f_1,\dots,f_l \in R_\#$ such that $\mathfrak{a}_\infty = (f_1,\dots,f_l)$. Since the natural map $\prod_{m\in\mathbb{N}} \mathfrak{a}_m \longrightarrow [\mathfrak{a}_m]_m$ is surjective, there exists $f_{m,i} \in \mathfrak{a}_m$ for every $m \in \mathbb{N}$ such that $f_i = [f_{m,i}]_m$.

Set $\mathfrak{a}'_m := (f_{m,1},\dots,f_{m,l}) \subseteq \mathfrak{a}_m$. Since we have $[\mathfrak{a}'_m]_m = \mathfrak{a}_\infty$, it follows from Theorem 3.2.7 that $\mathrm{sh}(\mathrm{ulim}_m \mathrm{fjn}^I(R,\Delta;\mathfrak{a}'_m)) = t$. On the other hand, since we have $\mathrm{fjn}^I(R,\Delta;\mathfrak{a}'_m) \leqslant \mathrm{fjn}^I(R,\Delta;\mathfrak{a}_m) < t$, by replacing by a subsequence, we may assume that the sequence $\{\mathrm{fjn}^I(R,\Delta;\mathfrak{a}'_m)\}$ is a strictly ascending chain. By replacing $\mathfrak{a}_m$ by $\mathfrak{a}'_m$, we may assume $\mu_R(\mathfrak{a}_m) \leqslant l$ for every m.

By enlarging e, we may assume that $q = p^e$ satisfies the following properties:

1. $q(q-1)t \in \mathbb{N}$ and
2. $q > \ell\ell_R(R/I) + l + \mathrm{emb}(R)$.

It follows from Proposition 3.3.8 that there exist integers $u, n_1 > 0$ and $T \in \mathfrak{U}$ such that

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}_m^t) \subseteq I \text{ if and only if } \tau_e^{n_1,u}(R,\Delta,\mathfrak{a}_m^t) \subseteq I$$

for every integer $n \geqslant n_1$ and $m \in T$. By enlarging u, we may further assume that $u \geqslant \widetilde{\mathrm{stab}}(R_\#,\Delta_\#,\mathfrak{a}_\infty;e)$

For every $m \in \mathbb{N}$ and for every sufficiently large $n \gg 0$ we have

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}_m^t) \subseteq \tau(R,\Delta,\mathfrak{a}_m^{\langle t\rangle_{n,q}}) \subseteq I.$$

Therefore we have $\tau_e^{n_1,u}(R,\Delta,\mathfrak{a}_m^t) \subseteq I$ for every $m \in T$.

On the other hand, since $\langle t\rangle_{n_1,q} < t = \mathrm{fjn}^{I\cdot R_\#}(R_\#,\Delta_\#;\mathfrak{a}_\infty)$, we have

$$\begin{aligned} [\tau_e^{n_1,u}(R,\Delta,\mathfrak{a}_m^t)]_m &= \tau_e^{n_1,u}(R_\#,\Delta_\#,\mathfrak{a}_\infty^t) \\ &= \tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^{\langle t\rangle_{n_1,q}}) \\ &\not\subseteq I\cdot R_\#. \end{aligned}$$

Therefore, there exists a set $S \in \mathfrak{U}$ such that

$$\tau_e^{n_1,u}(R,\Delta,\mathfrak{a}_m^t) \not\subseteq I$$

for every $m \in S$. Since $S \cap T \neq \emptyset$, we have contradiction.

□

Corollary 3.3.10 (Theorem A)**.** *Fix an integer $n \geqslant 1$, a prime number $p > 0$ and a set $\mathcal{D}_{n,p}^{\mathrm{reg}}$ such that every element of $\mathcal{D}_{n,p}^{\mathrm{reg}}$ is an n-dimensional F-finite Noetherian regular local ring of characteristic p. The set*

$$\mathcal{T}_{n,p}^{\mathrm{reg}} := \{\mathrm{fpt}(A; \mathfrak{a}) \mid A \in \mathcal{D}_{n,p}^{\mathrm{reg}}, \mathfrak{a} \subsetneq A\},$$

satisfies the ascending chain condition.

Proof. We assume the contrary. Then there exists a sequence $\{A_m\}_{m \in \mathbb{N}}$ in $\mathcal{T}_{n,p}^{\mathrm{reg}}$ and ideals $\mathfrak{a}_m \subsetneq A_m$ such that the sequence $\{\mathrm{fpt}(A_m; \mathfrak{a}_m)\}$ is a strictly ascending chain.

Since test ideals commute with completion([HT04, Proposition 3.2]), we may assume that $A_m = k_m[[x_1, \ldots, x_n]]$ for some F-finite field k_m. Take an F-finite field k such that $k_m \subseteq k$ for every m. Let $(A, \mathfrak{m}_A)$ be the local ring $k[[x_1, \ldots, x_n]]$. Then it follows as in the proof of [BMS09, Theorem 3.5 (i)] that $\mathrm{fpt}(A; (\mathfrak{a}_m A)) = \mathrm{fpt}(A_m; \mathfrak{a}_m)$. Therefore, we have $\mathrm{fpt}(A_m; \mathfrak{a}_m) \in \mathrm{FJN}^{\mathfrak{m}_A}(A, 0)$ for every m, which contradicts to Theorem 3.3.9. □

We conclude Chapter 3 with a natural question as below.

Question 3.3.11. Does Theorem A give an alternative proof of [dFEM10, Theorem 1.1]? Moreover, does Theorem B imply that the set of all jumping numbers of multiplier ideals with respect to a fixed $\mathfrak{m}$-primary ideal on a log $\mathbb{Q}$-Gorenstein pair over $\mathbb{C}$ satisfies the ascending chain condition?

We hope to consider this question at a later time.

Chapter 4

ACC for F-pure thresholds with fixed embedding dimension

4.1 Rationality of F-pure thresholds

In this section, we define a variant of parameter test modules (Definition 4.1.5). By considering the jumping numbers associated to these new modules, we prove the rationality of F-pure thresholds (Corollary 4.1.10).

Proposition 4.1.1. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $\Delta = sD$ for some Cartier divisor D and $t = s = 1/(p^e - 1)$ for some integer $e > 0$. Then $\tau(\omega_X, (s-\varepsilon)D, \mathfrak{a}^t)$ is constant for all sufficiently small rational numbers $0 < \varepsilon \ll 1$.*

Proof. The proof is essentially the same as that of [ST14, Lemma 6.2]. We may assume that $\mathfrak{a} \neq 0$. Set $q = p^e$. For every integer $l \geqslant 0$, we define the l-th truncation of s in the base q by

$$\langle s \rangle_{l,q} := \frac{q^l - 1}{q^l(q-1)} \in \mathbb{Q}.$$

Since the sequence $\{\langle s \rangle_{l,q}\}_{l \in \mathbb{N}}$ is a strictly ascending chain which converges to s, it is enough to prove that $\tau(\omega_X, \langle s \rangle_{l,q} \cdot D, \mathfrak{a}^t)$ is constant for all sufficiently large l.

Take the normalized blowup $\pi : Y \longrightarrow X$ along $\mathfrak{a}$. Let G be the Cartier divisor on Y such that $\mathcal{O}_Y(-G) = \mathfrak{a} \cdot \mathcal{O}_Y$. Take the Grothendieck trace maps $\operatorname{Tr}_\pi : \pi_*\omega_Y \longrightarrow \omega_X$, $\operatorname{Tr}_X : F_*\omega_X \longrightarrow \omega_X$ and $\operatorname{Tr}_Y : F_*\omega_Y \longrightarrow \omega_Y$. As in

[BST15, p.4], we have $\mathrm{Tr}_X \circ F_*(\mathrm{Tr}_\pi) = \mathrm{Tr}_\pi \circ \pi_*(\mathrm{Tr}_Y)$ and Tr_π is injective. In particular, we may consider $\pi_*\omega_Y$ as a submodule of ω_X.

By [ST14, Theorem 5.1], for every integer $l \geqslant 0$, there exists an integer m_l such that

$$\tau(\omega_X, \langle s\rangle_{l,q} \cdot D, \mathfrak{a}^t) = \mathrm{Tr}_X^{em}(F_*^{em}\pi_*(\tau(\omega_Y, q^m(\langle s\rangle_{l,q} \cdot \pi^*D + tG)))) \tag{4.1}$$

for all $m \geqslant m_l$.

By Lemma 2.1.5 (3) and (6), there exists l_0 such that $\tau(\omega_Y, \langle s\rangle_{l,q} \cdot \pi^*D + tG)$ is constant for all $l \geqslant l_0$. For every integer $l \geqslant 0$, it follows from Lemma 2.1.5 (4) that the morphism

$$\beta_l := \mathrm{Tr}_Y^e : F_*^e(\tau(\omega_Y, q(\langle s\rangle_{l,q} \cdot \pi^*D + tG)))) \longrightarrow \tau(\omega_Y, \langle s\rangle_{l,q} \cdot \pi^*D + tG)$$

is surjective. We denote the kernel by $\mathcal{N}_l$. Since $\mathcal{N}_l$ is constant for all $l \geqslant l_0$ and $-G$ is π-ample, there exists an integer m' such that

$$R^1\pi_*(\mathcal{N}_l \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-MG)) = 0$$

for all integers $l \geqslant 0$ and $M \geqslant (q^{m'} - 1)/(q-1)$.

Take integers $m, n \geqslant 1$ and consider the surjection

$$\begin{aligned}\gamma_{n,m} := \mathrm{Tr}_Y^e : F_*^e(\tau(\omega_X,&q^m(\langle s\rangle_{n,q}\pi^*D + tG))) \\ &\longrightarrow \tau(\omega_X, q^{m-1}(\langle s\rangle_{n,q}\pi^*D + tG)).\end{aligned}$$

By Lemma 2.1.5 (5) and (6), $\gamma_{n,m}$ coincides with $\beta_{n-m} \otimes \mathcal{O}_Y(-(q^m-1)/(q-1)\cdot(\pi^*D+G))$ if $m < n$ and with $\beta_0 \otimes \mathcal{O}_Y(-q^m\langle s\rangle_{n,q}\pi^*D - (q^m-1)/(q-1)\cdot G)$ if $m \geqslant n$. Therefore, $\pi_*\gamma_{n,m}$ is surjective if $m \geqslant m'$.

Combining with the equation (4.1), we have

$$\tau(\omega_X, \langle s\rangle_{l,q} \cdot D, \mathfrak{a}^t) = \mathrm{Tr}_X^{em'}(F_*^{em'}\pi_*(\tau(\omega_Y, q^{m'}(\langle s\rangle_{l,q} \cdot \pi^*D + tG))))$$

for every l. By the definition of l_0, the right hand side is constant for all $l \geqslant l_0 + m'$. □

Corollary 4.1.2. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t \in \mathbb{Q}$ and Δ is $\mathbb{Q}$-Cartier. Then $\tau(\omega_X, (1-\varepsilon)\Delta, \mathfrak{a}^t)$ is constant for all $0 < \varepsilon \ll 1$.*

Proof. By Lemma 2.1.5 (4) and (7), we may assume that there exists an integer $e > 0$ such that $(p^e - 1)\Delta$ is Cartier and $t = 1/(p^e-1)$. Then the assertion follows from Proposition 4.1.1. □

We define the new variant of the parameter test module as the left limit of the map $s \longmapsto \tau(\omega_X, s\Delta, \mathfrak{a}^t)$ at $s = 1$.

Definition 4.1.3. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t \in \mathbb{Q}$ and Δ is $\mathbb{Q}$-Cartier. Then we define the submodule $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \subseteq \omega_X$ by $\tau(\omega_X, (1-\varepsilon)\Delta, \mathfrak{a}^t)$ for sufficiently small $0 < \varepsilon \ll 1$.

Lemma 4.1.4. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t \in \mathbb{Q}$ and Δ is $\mathbb{Q}$-Cartier. Then the following hold.*

1. $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t'}) \subseteq \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t)$ *for any rational number $t < t'$.*
2. *For any real number $s \geqslant 0$, there exists $0 < \varepsilon$ such that the module $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{s'})$ is constant for every rational number $s < s' < s + \varepsilon$.*
3. *For any rational number $s > 0$, there exists $0 < \varepsilon$ such that the module $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{s'})$ is constant for every rational number $s - \varepsilon < s' < s$.*
4. *If $\mathfrak{a}$ is generated by l elements and $t \geqslant l$, then we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) = \mathfrak{a}\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t-1})$.*
5. $\operatorname{Tr}_X(F_*(\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t))) = \tau(\omega_X, (\Delta/p)_{-0}, \mathfrak{a}^{t/p})$.
6. *If $r\Delta$ is Cartier, then we have*

$$\tau(\omega_X, (r+1)\Delta_{-0}, \mathfrak{a}^t) = \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \otimes \mathcal{O}_X(-r\Delta).$$

Proof. (1), (4), (5) and (6) follow from Lemma 2.1.5. (2) follows from (1) and the ascending chain condition for the set of ideals in R.

For (3), we take a positive integer r such that rs is integer and $r\Delta$ is Cartier. By Lemma 2.1.5 (3), there exists $\delta > 0$ such that the parameter test module $\tau(\omega_X, (\mathfrak{a}^{rs}\mathcal{O}_X(-r\Delta))^{(1-\varepsilon)/r})$ is constant for all rational numbers $0 < \varepsilon < \delta$. We denote this module by M.

It follows from Lemma 2.1.5 (6) and (7) that for every rational number $0 < \varepsilon < \delta$, we have

$$\begin{aligned} \tau(\omega_X, (1-\varepsilon)\Delta, \mathfrak{a}^{s(1-\varepsilon)}) &= \tau(\omega_X, \mathfrak{a}^{s(1-\varepsilon)}\mathcal{O}_X(-r\Delta)^{(1-\varepsilon)/r}) \\ &= \tau(\omega_X, (\mathfrak{a}^{rs}\mathcal{O}_X(-r\Delta))^{(1-\varepsilon)/r}) \\ &= M. \end{aligned}$$

By Lemma 2.1.5 (1), $\tau(\omega_X, (1-\varepsilon)\Delta, \mathfrak{a}^{s(1-\varepsilon')}) = M$ for every $0 < \varepsilon, \varepsilon' < \delta$. Therefore, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{s(1-\varepsilon)}) = M$ for every rational number $0 < \varepsilon < \delta$. □

Definition 4.1.5. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that t is not a rational number and Δ is $\mathbb{Q}$-Cartier. By Lemma 4.1.4 (2), there exists $\varepsilon > 0$ such that the submodule $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^s) \subseteq \omega_X$ is constant for every rational number $t < s < t + \varepsilon$. We denote this submodule of ω_X by $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t)$.

We note that even if t, t', s, and s' are not rational, the same assertions as in Lemma 4.1.4 (1), (2), (4), (5) and (6) hold.

Definition 4.1.6. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a})$ be a triple such that Δ is $\mathbb{Q}$-Cartier. A real number $t \geqslant 0$ is called an *F-jumping number* of $(\omega_X, \Delta_{-0}; \mathfrak{a})$ if one of the following hold:

1. for every $\varepsilon > 0$, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \subsetneq \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t-\varepsilon})$, or
2. for every $\varepsilon > 0$, we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^t) \supsetneq \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t+\varepsilon})$.

Lemma 4.1.7. *Let $q \geqslant 2$ and $l \geqslant 1$ be integers and $B \subseteq \mathbb{R}_{\geqslant 0}$ a subset. B is a discrete set of rational numbers if the following four properties hold:*

1. *For any $x \in B$, $qx \in B$.*
2. *For any $x \in B$, if $x > l$, then $x - 1 \in B$.*
3. *For any real number $t \in \mathbb{R}_{\geqslant 0}$, there exists $\varepsilon > 0$ such that $B \cap (t, t+\varepsilon) = \emptyset$.*
4. *For any rational number $t \in \mathbb{Q}_{>0}$, there exists $\varepsilon > 0$ such that $B \cap (t - \varepsilon, t) = \emptyset$.*

Proof. Let D be the set of all accumulation points of B. By [BSTZ10, Proposition 5.5], we have $D = \emptyset$. This proves that B is a discrete set. If B contains a non-rational number, then by the assumptions (1) and (2), we have infinitely many elements in $B \cap [l-1, l]$, which contradicts to the discreteness of B. $\square$

Corollary 4.1.8. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a})$ is a triple such that Δ is $\mathbb{Q}$-Cartier. Then the set of all F-jumping numbers of $(\omega_X, \Delta_{-0}; \mathfrak{a})$ is a discrete set of rational numbers.*

Proof. It follows from Lemma 4.1.4 (5) that if t is an F-jumping number of $(\omega_X, \Delta_{-0}; \mathfrak{a})$, then pt is an F-jumping number of $(\omega_X, (p\Delta)_{-0}; \mathfrak{a})$. Therefore, we may assume that there exists an integer $e > 0$ such that $(p^e - 1)\Delta$ is Cartier.

Let l be the number of minimal generators of $\mathfrak{a}$ and B be the set of all F-jumping numbers of $(\omega_X, \Delta_{-0}; \mathfrak{a})$. Then it follows from Lemma 4.1.4 that B, $q = p^e$ and l satisfy the assumptions in Lemma 4.1.7. $\square$

Proposition 4.1.9. *Suppose that $(X = \operatorname{Spec} A, \Delta)$ is a sharply F-pure pair such that A is regular and $(p^e - 1)\Delta$ is Cartier for some $e > 0$, and $\mathfrak{a} \subseteq A$ is a non-zero proper ideal. Then the F-pure threshold $\operatorname{fpt}(A, \Delta; \mathfrak{a})$ coincides with the first jumping number of $(\omega_X, \Delta_{-0}; \mathfrak{a})$. In particular, it is a rational number.*

Proof. It is enough to show the equation

$$\mathrm{fpt}(A, \Delta; \mathfrak{a}) = \sup\{s \geqslant 0 \mid \tau(\omega_X, \Delta_{-0}, \mathfrak{a}^s) = \omega_X\}. \tag{4.2}$$

Set $t := \mathrm{fpt}(A, \Delta; \mathfrak{a})$. Since A is regular local, we may identify ω_X with A. By Lemma 2.2.5 (1), we have $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^{t(1-\varepsilon)}) = \omega_X$ for every $0 < \varepsilon < 1$.

On the other hand, take any rational number s such that $\tau(\omega_X, \Delta_{-0}, \mathfrak{a}^s) = \omega_X$. It follows from Lemma 2.2.5 (2) that $(A, \Delta, \mathfrak{a}^{s(1-\varepsilon)})$ is sharply F-pure for every $0 < \varepsilon < 1$, which proves the equation (4.2). □

Corollary 4.1.10 (Theorem E)**.** *Suppose that (R, Δ) is a sharply F-pure pair such that $(p^e - 1)(K_R + \Delta)$ is Cartier for some integer $e > 0$ and $\mathfrak{a} \subseteq R$ is an ideal. Then the F-pure threshold $\mathrm{fpt}(R, \Delta; \mathfrak{a})$ is a rational number.*

Proof. By Lemma 2.2.3 (4), we may assume that R is a complete local ring. By Proposition 2.3.3, we may assume that R is a regular local ring. Hence, the assertion follows from Proposition 4.1.9. □

4.2 Proof of Main Theorem

In this section, as a variant of Corollary 3.2.8, we prove that the shadow of any sequence of F-pure thresholds of ideals on sharply F-pure pairs coincides with the F-pure threshold on the catapower under some assumptions (Theorem 4.2.3). By combining the rationality of F-jumping numbers, we give the proof of the main theorem (Theorem 4.2.5).

Lemma 4.2.1. *Suppose that A is an F-finite regular local ring, $f \in A$ is a non-zero element, $\mathfrak{a} \subseteq A$ is an ideal, $e > 0$ is an integer and $t = u/v > 0$ is a rational number with integers $u, v > 0$. Set $\mathfrak{b} := f^v \cdot \mathfrak{a}^{(p^e-1)u} \subseteq A$ and $\Delta := \mathrm{div}_A(f)/(p^e - 1)$. Assume that (A, Δ) is sharply F-pure. Then $t \leqslant \mathrm{fpt}(A, \Delta; \mathfrak{a})$ if and only if $1/(v(p^e - 1)) \leqslant \mathrm{fpt}(A; \mathfrak{b})$.*

Proof. We may assume that $\mathfrak{a} \neq (0)$. First, we assume that $t \leqslant \mathrm{fpt}(A, \Delta; \mathfrak{a})$. By Lemma 2.2.5 (1), the triple $(A, (1-\varepsilon)\Delta, \mathfrak{a}^{(1-\varepsilon)t})$ is strongly F-regular for every $0 < \varepsilon < 1$. It follows from Lemma 2.1.5 (6) that $(A, \mathfrak{b}^{(1-\varepsilon)/(v(p^e-1))})$ is strongly F-regular, which proves the inequality $1/(v(p^e - 1)) \leqslant \mathrm{fpt}(A; \mathfrak{b})$.

On the other hand, we assume that $1/(v(p^e-1)) \leqslant \mathrm{fpt}(A; \mathfrak{b})$. By Lemma 2.2.3 (3) and (4), the triple $(A, (1 - \varepsilon)\Delta, \mathfrak{a}^{(1-\varepsilon)t})$ is strongly F-regular for every $0 < \varepsilon < 1$. It follows from 2.2.3 (2) that the triple $(A, (1-\varepsilon)\Delta, \mathfrak{a}^{(1-\varepsilon')t})$ is strongly F-regular for every $0 < \varepsilon < \varepsilon' < 1$. By Lemma 2.2.5 (2), we have $t \leqslant \mathrm{fpt}(A, \Delta; \mathfrak{a})$. □

Proposition 4.2.2. *Suppose that A is an F-finite regular local ring, $e > 0$ is an integer, $\Delta_m = \mathrm{div}_A(f_m)/(p^e - 1)$ is an effective $\mathbb{Q}$-divisor on* $\mathrm{Spec}\, A$ *for every $m \in \mathbb{N}$ and $\mathfrak{a}_m \subseteq A$ is a proper ideal for every $m \in \mathbb{N}$. Fix a non-principal ultrafilter $\mathfrak{U}$. Let $A_\#$ be the catapower of A and $\mathfrak{a}_\infty := [\mathfrak{a}_m]_m \subseteq A_\#$. Assume that (A, Δ_m) is sharply F-pure for every integer m. Then the following hold.*

1. *$f_\infty := [f_m]_m \in A_\#$ is a non-zero element.*
2. *Set $\Delta_\infty := \mathrm{div}_{A_\#}(f_\infty)/(p^e - 1)$. Then, $(A_\#, \Delta_\infty)$ is sharply F-pure.*
3. *For every rational number $t > 0$, we have $t \leqslant \mathrm{fpt}(A_\#, \Delta_\infty; \mathfrak{a}_\infty)$ if and only if $\{m \in \mathbb{N} \mid t \leqslant \mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m)\} \in \mathfrak{U}$.*

Proof. By Lemma 2.2.4, we have $f_m \notin \mathfrak{m}^{[p^e]}$ for every m. It follows from Lemma 2.4.8 that $f_\infty \notin \mathfrak{m}_\infty^{[p^e]}$, which proves (1) and (2). For (3), take integers $u, v > 0$ such that $t = u/v$ and set $\mathfrak{b}_m := f_m^v \cdot \mathfrak{a}_m^{u(p^e-1)}$ for every $m \in \mathbb{N} \cup \{\infty\}$. It follows from Lemma 4.2.1 that $\{m \in \mathbb{N} \mid t \leqslant \mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m)\} \in \mathfrak{U}$ if and only if $\{m \in \mathbb{N} \mid 1/(v(p^e-1)) \leqslant \mathrm{fpt}(A; \mathfrak{b}_m)\} \in \mathfrak{U}$. We first assume that $\{m \in \mathbb{N} \mid 1/(v(p^e - 1)) \leqslant \mathrm{fpt}(A; \mathfrak{b}_m)\} \in \mathfrak{U}$. Since we have $\mathrm{sh}(\mathrm{ulim}_m \mathrm{fpt}(A; \mathfrak{b}_m)) = \mathrm{fpt}(A_\#; \mathfrak{b}_\infty)$ (Corollary 3.2.8), we have $1/(v(p^e-1)) \leqslant \mathrm{fpt}(A_\#; \mathfrak{b}_\infty)$. Applying Lemma 4.2.1 again, we have $t \leqslant \mathrm{fpt}(A_\#, \Delta_\infty; \mathfrak{a}_\infty)$.

For the converse implication, we assume that $\{m \in \mathbb{N} \mid 1/(v(p^e - 1)) \leqslant \mathrm{fpt}(A; \mathfrak{b}_m)\} \notin \mathfrak{U}$. In this case, we have $\{m \in \mathbb{N} \mid 1/(v(p^e-1)) > \mathrm{fpt}(A; \mathfrak{b}_m)\} \in \mathfrak{U}$ and hence we have $1/(v(p^e - 1)) \geqslant \mathrm{fpt}(A_\#; \mathfrak{b}_\infty)$. If $1/(v(p^e - 1)) = \mathrm{fpt}(A_\#; \mathfrak{b}_\infty) = \mathrm{sh}(\mathrm{ulim}_m \mathrm{fpt}(A; \mathfrak{b}_m))$, then by replacing by a subsequence, we may assume that the sequence $\{\mathrm{fpt}(A; \mathfrak{b}_m)\}_m$ is a strictly ascending chain, which is contradiction to Theorem 3.3.9. Therefore, we have $1/(v(p^e - 1)) > \mathrm{fpt}(A_\#; \mathfrak{b}_\infty)$, which proves $t > \mathrm{fpt}(A_\#, \Delta_\infty; \mathfrak{a}_\infty)$. □

The following result is a generalization of Corollary 3.2.8 to the case where the pair is not necessarily strongly F-regular.

Theorem 4.2.3. *With the notation above, we have*

$$\mathrm{sh}(\mathrm{ulim}_m \mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m)) = \mathrm{fpt}(A_\#, \Delta_\infty, \mathfrak{a}_\infty) \in \mathbb{Q}.$$

In particular, if the limit $\lim_{m \longrightarrow \infty} \mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m)$ exists, then we have

$$\lim_{m \longrightarrow \infty} \mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m) = \mathrm{fpt}(A_\#, \Delta_\infty, \mathfrak{a}_\infty).$$

Proof. We first note that the shadow always exists because we have

$$\mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m) \leqslant \mathrm{fpt}(A; \mathfrak{m}) = \dim A$$

for all m. For any rational number $t > 0$, it follows from Proposition 4.2.2 that $t \leqslant \mathrm{sh}(\mathrm{ulim}_m \mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m))$ if and only if $t \leqslant \mathrm{fpt}(A_\#, \Delta_\infty; \mathfrak{a}_\infty)$, which completes the proof. □

Corollary 4.2.4. *Suppose that $e > 0$ is an integer and $(A, \mathfrak{m})$ is an F-finite regular local ring of characteristic $p > 0$. Then the set*

$$\mathrm{FPT}(A, e) := \left\{ \mathrm{fpt}(A, \Delta; \mathfrak{a}) \;\middle|\; \begin{array}{l} (A, \Delta) \text{ is sharply } F\text{-pure,} \\ (p^e - 1)\Delta \text{ is Cartier, } \mathfrak{a} \subsetneq A \end{array} \right\}$$

satisfies the ascending chain condition.

Proof. We assume the contrary. Then there exist sequences $\{\Delta_m\}_m$ and $\{\mathfrak{a}_m\}$ such that $\{\mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m)\}_{m \in \mathbb{N}}$ is a strictly ascending chain. Set $t := \lim_m \mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m)$. By Corollary 4.1.10 and Corollary 4.2.3, we have $t = \mathrm{fpt}(A_\#, \Delta_\infty; \mathfrak{a}_\infty) \in \mathbb{Q}$.

Since t is rational and $\mathrm{fpt}(A, \Delta_m; \mathfrak{a}_m) < t$ for all m, it follows from Proposition 4.2.2 that $\mathrm{fpt}(A, \Delta_\infty; \mathfrak{a}_\infty) < t$, which is contradiction. □

For a Noetherian local ring $(R, \mathfrak{m})$, we denote by $\mathrm{emb}(R)$ the embedding dimension of R.

Theorem 4.2.5 (Main Theorem)**.** *Fix a prime number p and positive integers e and N. Suppose that T is any set such that every element of T is an F-finite Noetherian normal local ring $(R, \mathfrak{m})$ of characteristic p with* $\mathrm{emb}(R) \leqslant N$*. Let* $\mathrm{FPT}(T, e)$ *be the set of all F-pure thresholds* $\mathrm{fpt}(R, \Delta; \mathfrak{a})$ *such that*

- *R is an element of T,*
- *$\mathfrak{a}$ is a proper ideal of R, and*
- *Δ is an effective $\mathbb{Q}$-Weil divisor on $X = \mathrm{Spec}\, R$ such that (R, Δ) is sharply F-pure and $(p^e - 1)(K_X + \Delta)$ is Cartier.*

Then the set $\mathrm{FPT}(T, e)$ *satisfies the ascending chain condition.*

Proof. Take an F-finite field k such that for every $(R, \mathfrak{m}) \in T$, there exists a field extension $R/\mathfrak{m} \subseteq k$. Set $A := k[[x_1, \ldots, x_N]]$. Then it follows from Lemma 2.2.3 (4), Lemma 2.2.4 and Proposition 2.3.3 that we have the inclusion $\mathrm{FPT}(T, e) \subseteq \mathrm{FPT}(A, e)$, which proves that the set $\mathrm{FPT}(T, e)$ satisfies the ascending chain condition. □

Corollary 4.2.6. *Suppose that X is a normal variety over an F-finite field. Fix an integer $e > 0$. Let* $\mathrm{FPT}(X, e)$ *be the set of all* $\mathrm{fpt}(X, \Delta; \mathfrak{a})$ *such that*

- $\mathfrak{a}$ *is a proper coherent ideal sheaf on* X *and*
- Δ *is an effective* $\mathbb{Q}$*-Weil divisor on* X *such that* (X, Δ) *is sharply* F*-pure and* $(p^e - 1)(K_X + \Delta)$ *is Cartier.*

The set $\mathrm{FPT}(X, e)$ *satisfies the ascending chain condition.*

Proof. Set $T := \{\mathcal{O}_{X,x} \mid x \in X\}$. It follows from Lemma 2.3.2 that we have $\mathrm{FPT}(X, e) \subseteq \mathrm{FPT}(T, e)$, which completes the proof. □

4.3 Proof of Corollary D

In this section, as a corollary of Main Theorem, we verify the ascending chain condition for F-pure thresholds on tame quotient singularities or on l.c.i. varieties with fixed dimension.

First, we recall the definition and properties of tame quotient singularities. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of equicharacteristic. Then $(R, \mathfrak{m})$ is said to be a *quotient singularity* if there exist a regular affine variety $U = \operatorname{Spec} A$ over k, a finite group G with a group homomorphism $G \longrightarrow \mathrm{Aut}_k(U)$, and a point x of the quotient $V = U/G := \operatorname{Spec}(A^G)$ such that there exists an isomorphism $\widehat{R} \cong \widehat{\mathcal{O}_{V,x}}$ as rings. Moreover, if $|G|$ is coprime to $\mathrm{char}(k)$, then we say that $(R, \mathfrak{m})$ is a *tame quotient singularity*.

Lemma 4.3.1. *Let* $(R, \mathfrak{m}, k)$ *be a tame quotient singularity of dimension* n*. Then, there exists a finite group* $G \subseteq \mathrm{GL}_n(k)$ *with the following properties.*

1. $|G|$ *is coprime to* $\mathrm{char}(k)$*.*
2. *The natural action of* G *on the affine space* $\mathbb{A}^n_k$ *has no fixed points in codimension 1.*
3. *Let* $V := \mathbb{A}^n_k/G$ *be the quotient and* $x \in V$ *be the image of the origin of* $\mathbb{A}^n_k$*. Then we have* $\widehat{R} \cong \widehat{\mathcal{O}_{V,x}}$*.*

Proof. The proof follows as in the case when $\mathrm{char}(k) = 0$ (see [dFEM10, p.15]), but for the convenience of reader we sketch it here.

Since R is a tame quotient singularity, there exists a regular affine variety U, a finite group G which acts on U such that $|G|$ is coprime to $\mathrm{char}(k)$, and a point $x \in V$ such that $\widehat{R} \cong \widehat{\mathcal{O}_{V,x}}$.

Take a point $y \in U$ with image x. By replacing G by the stabilizer subgroup $G_y \subseteq G$, we may assume that G acts on the regular local ring $(A, \mathfrak{m}_A) := (\mathcal{O}_{U,y}, \mathfrak{m}_y)$. Since $|G|$ is coprime to $\mathrm{char}(k)$, it follows from

Maschke's theorem that the natural projection $\mathfrak{m}_A \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2$ has a section as $k[G]$-modules. This section induces $k[G]$-algebra homomorphism $\mathrm{Gr}_{\mathfrak{m}_A}(A) \longrightarrow A$, where $\mathrm{Gr}_{\mathfrak{m}_A}(A)$ is the associated graded ring of $(A, \mathfrak{m}_A)$. Therefore, by replacing U by $\mathrm{Spec}(\mathrm{Gr}_{\mathfrak{m}_A}(A))$, we may assume that $U = \mathbb{A}^n_k$ and $G \subseteq \mathrm{GL}_n(k)$.

Let $H \subseteq G$ be the subgroup generated by elements $g \in G$ which fixes some codimension one point of U. Since $|G|$ is coprime to $\mathrm{char}(k)$, it follows from Chevalley-Shephard-Todd theorem (see for example [Ben93, Theorem 7.2.1]) that $U/H \cong \mathbb{A}^n_k$. By replacing U by U/H and G by G/H, we complete the proof of the lemma. □

Proposition 4.3.2 (Corollary D (1))**.** *Fix an integer $n \geqslant 1$, a prime number $p > 0$ and a set $\mathcal{D}^{\mathrm{quot}}_{n,p}$ such that every element of $\mathcal{D}^{\mathrm{quot}}_{n,p}$ is an n-dimensional F-finite Noetherian normal local ring of characteristic p with tame quotient singularities. The set*

$$\mathcal{T}^{\mathrm{quot}}_{n,p} := \{\mathrm{fpt}(R; \mathfrak{a}) \mid R \in \mathcal{D}^{\mathrm{quot}}_{n,p}, \mathfrak{a} \subsetneq R \text{ is an ideal}\}$$

satisfies the ascending chain condition.

Proof. The proof is essentially the same as [dFEM10, Proposition 5.3]. Let $(R, \mathfrak{m}, k)$ be a local ring such that $R \in \mathcal{D}^{\mathrm{quot}}_{n,p}$ and $\mathfrak{a} \subsetneq R$ be an ideal of R. Let G, V, and x be as in Lemma 4.3.1. Consider the natural morphism $\pi : U := \mathbb{A}^n_k \longrightarrow V$. Since G is a finite group, the morphism π is a finite surjective morphism with $\deg(\pi)$ coprime to $\mathrm{char}(k)$. Since G acts on U with no fixed points in codimension one, the morphism π is étale in codimension one.

Set $W := \mathrm{Spec}(\widehat{R})$ and $U' := U \times_V W$. Since U is a regular scheme and $W \longrightarrow V$ is a regular morphism, each connected component of U' is a regular scheme. Fix a connected component $U'' \subseteq U'$.

Since the morphism $\widehat{\pi} : U'' \longrightarrow W$ is finite surjective, étale in codimension 1 and $\deg \widehat{\pi}$ is coprime to p, it follows from [HT04, Theorem 3.3] that

$$\mathrm{fpt}(W; \mathfrak{a}\mathcal{O}_W) = \mathrm{fpt}(U''; \mathfrak{a}\mathcal{O}_{U''}).$$

On the other hand, since the test ideals commute with completion ([HT04, Proposition 3.2]), we have

$$\mathrm{fpt}(R; \mathfrak{a}) = \mathrm{fpt}(W; \mathfrak{a}\mathcal{O}_W).$$

Therefore, it follows from Corollary 3.3.10 that the set $\mathcal{T}^{\mathrm{quot}}_{n,p}$ satisfies the ascending chain condition.

□

We also verify the ascending chain condition for F-pure thresholds on l.c.i. varieties with fixed dimension.

Lemma 4.3.3 (cf. [dFEM10, Proposition 6.3]). *Let $(R, \mathfrak{m})$ be an F-finite Noetherian normal local ring of dimension d. If R is a complete intersection and sharply F-pure, then* $\mathrm{emb}(R) \leqslant 2d$.

Proof. Set $N := \mathrm{emb}(R)$ and $c := N - d$. There exists an F-finite regular local ring A and a regular sequence $f_1, \ldots, f_c \in A$ with $f_i \in \mathfrak{m}^2$ such that $R \cong A/(f_1, \ldots, f_c)$. By [HW02, Proposition 2.6], we have $(f_1 \cdots f_c)^{p-1} \notin \mathfrak{m}^{[p]}$.

Since $f_i \in \mathfrak{m}^2$ for every i, we have $(f_1 \cdots f_c)^{p-1} \in \mathfrak{m}^{2c(p-1)}$. It follows from the inclusion $\mathfrak{m}^{N(p-1)+1} \subseteq \mathfrak{m}^{[p]}$ that we have $2c \leqslant N$, which proves $N \leqslant 2d$. □

Corollary 4.3.4 (Corollary D). *Let p be a prime number and $n \geqslant 1$ be an integer. Suppose that T is any set such that every element of T is an n-dimensional Noetherian normal connected l.c.i. scheme of characteristic p which is sharply F-pure. Then, the set*

$$\{\mathrm{fpt}(X; \mathfrak{a}) \mid X \in T, \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

satisfies the ascending chain condition.

Proof. It follows from Lemma 4.3.3 that $\mathrm{emb}(\mathcal{O}_{X,x}) \leqslant 2n$ for every $X \in T$ and every $x \in X$. Since every $X \in T$ is Gorenstein, we apply the main theorem. □

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