Mathematical analysis of evolution equations in curved thin domains or on moving surfaces

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Contents

Chapter 1

Introduction

1.1 Purpose of the thesis

The purpose of this thesis is to study five problems related to evolution equations in a curved thin domain or on a moving surface. They are completely independent and studied separately in each chapter. References are listed at the end of each chapter and cited only in that chapter. Also, notations are different from chapter to chapter.

In Chapters 2–5 we consider parabolic equations in curved thin domains degenerating into stationary or moving hypersurfaces. Partial differential equations (PDEs) in thin domains appear in many problems of natural sciences such as solid mechanics (thin elastic bodies) and fluid mechanics (lubrication, meteorology, ocean dynamics). In the mathematical study of PDEs in thin domains we are mainly interested in two problems. One problem is to investigate the relation between the existence, uniqueness, and long time behavior of solutions to PDEs and the smallness of the width of thin domains. For example, in the study of the Navier–Stokes equations in a three-dimensional thin domain we expect to show the global-in-time existence of a strong solution for large data, since a three-dimensional thin domain with very small width can be considered almost two-dimensional. Another problem is a singular limit problem for a PDE in a thin domain as it degenerates into a lower dimensional set. We are concerned with derivation of a limit equation on the limit set and comparison of the original and limit equations. Such problems were first studied by Hale and Raugel [6, 7], who considered damped wave and reaction-diffusion equations in a flat thin domain degenerating into a lower dimensional domain. Since then, many researchers have studied PDEs, mainly a reaction-diffusion equation and the Navier–Stokes equations, in flat thin domains. In the case of curved thin domains whose limit sets are a lower dimensional manifold, there are several works on the asymptotic behavior of eigenvalues of the Laplacian on a curved thin domain around a hypersurface (see [9] and the references cited therein). However, evolution equations in curved thin domains have not been studied well, except for a few works on a reaction-diffusion equation in a curved thin domain [17] and the Navier–Stokes equations in a thin spherical shell [20]. It is also important to consider PDEs in moving thin domains for applications in physical problems and a few researchers have studied them. We refer to [5] for the study of a diffuse interface model in a moving thin domain for an advection-diffusion equation on a moving surface and to [16] for the study of a singular limit problem for a reaction-diffusion equation in a moving thin domain degenerating into a lower dimensional stationary domain. However, there is no literature that derives an unknown limit equation of a PDE in a moving thin domain whose limit set also moves in time, even in the case of a linear diffusion equation. The difficulties in the analysis of PDEs in curved or moving thin domains arise from the geometry and motion of the boundary of the thin domains and their limit sets. Our purpose in Chapters 2–5 is to provide mathematical methods for dealing with such difficulties and to investigate the effects of the geometry and motion of the curved thin domains and their limit surfaces on the original and limit equations.

In Chapter 6 we study the first order Hamilton–Jacobi equation on a moving surface. PDEs on moving surfaces arise in many applications in biology, fluid mechanics, and material sciences as frequently as those in thin domains. For example, an advection-diffusion equation on a moving surface describes transport of surfactants on the interface between two fluids. They provide mathematically interesting problems such as the well-posedness, numerical analysis, and interaction between the evolution of a surface and the behavior of a solution to a surface PDE. Many researchers have studied various kinds of PDEs on moving surfaces in recent years. We refer to [4] and the references cited therein for the mathematical and numerical study of PDEs on moving surfaces and its applications. In this thesis we consider the Hamilton–Jacobi equation as a new kind of PDE on a moving surface. Our goal is to establish the existence and uniqueness of a viscosity solution as well as to provide a numerical scheme with an error bound.

1.2 Introduction to Chapter 2

In Chapter 2 we study a singular limit problem for the heat equation in a moving thin domain. We consider the Neumann type problem of the heat equation in a moving thin domain that degenerates into a closed moving hypersurface as the width of the domain tends to zero. Here the Neumann type boundary condition is imposed to make the total amount of heat in the moving thin domain conserved in time.

Our purpose in Chapter 2 is to investigate the behavior of a solution to the heat equation as the width of the thin domain tends to zero and to derive a limit equation on the moving surface. To this end, we use a change of variables formula (a co-area formula) to transform an integral over the thin domain into integrals over the limit surface and its normal direction, and analyze the weighted average in the thin direction of a variational solution to the heat equation. Then, under suitable assumptions, we prove the weak convergence of the weighted average of a solution in an appropriate function space on the limit surface. Moreover, we show that the weak limit is a unique variational solution to a limit equation on the moving limit surface, which is a linear diffusion equation involving the mean curvature and normal velocity of the surface. We also estimate the difference between a solution to the heat equation and a solution to the limit equation. This is the first result that derives an a priori unknown limit equation of a PDE in a thin domain whose limit set evolves in time.

Chapter 2 is based on the work [13].

1.3 Introduction to Chapter 3

The subject of Chapter 3 is nonlinear diffusion equations of porous media type in a moving thin domain and on a moving surface. After a brief review of the transport equations, we consider the thin width limit of a nonlinear diffusion equation in a moving thin domain that consists of the transport equation, Darcy's law, and the boundary condition corresponding to the situation that there is no exchange of mass between the domain and its surroundings. Under the assumption that the moving thin domain shrinks to a moving closed hypersurface as its width tends to zero, we formally derive a limit nonlinear diffusion equation on the moving surface based on calculations of the Taylor series for bulk quantities with respect to the signed distance from the limit surface. In particular, we see that in the thin width limit the transport equation and Darcy's law in the moving thin domain become those on the moving surface. We also show that the thin width limit of the energy law for the nonlinear diffusion equation in the moving thin domain is that for the limit equation on the moving surface. Then we discuss an energetic variational approach to derivation of nonlinear diffusion equations in a moving domain and on a moving surface, and observe that the energetic variation commutes with the thin width limit.

Chapter 3 is based on the joint work [15] with Yoshikazu Giga and Chun Liu.

1.4 Introduction to Chapter 4

Chapter 4 is devoted to formal derivation of and discussions on singular limit equations of the incompressible Euler and Navier–Stokes equations in a three-dimensional moving thin domain. For a given closed moving surface, we define a moving thin domain as its tubular neighborhood of small radius. We consider the Euler equations with impermeable boundary condition and the Navier–Stokes equations with Navier's perfect slip boundary conditions in the moving thin domain. Then we formally derive their limit equations on the moving limit surface by calculations of the Taylor series for the bulk velocity and pressure with respect to the signed distance from the surface. Our limit equations are basically the same as incompressible fluid equations on a moving surface derived from local conservation laws of mass and linear momentum for a surface fluid [10] and by a global energetic variational approach [12]. We also observe that in the thin width limit the energy identities of the Euler and Navier–Stokes equations in the moving thin domain become those of the corresponding limit equations on the moving surface.

The limit equations on the moving surface involve the first and second order derivatives of tangential and normal vector fields on the surface. To understand the structure of the limit equations, we give several formulas on the derivatives of vector fields on an embedded surface in the three-dimensional Euclidean space. We use them to show that our limit equations are the same as the Euler and Navier–Stokes equations on a manifold introduced by Arnold [1,2] and Taylor [19] when the limit surface is stationary.

Chapter 4 is based on the work [14].

1.5 Introduction to Chapter 5

In Chapter 5 we study the three-dimensional Navier–Stokes equations in a stationary curved thin domain, which is defined as a region between two very close parametrized surfaces of a given two-dimensional closed surface. Under the assumption that the curved thin domain degenerates into the given surface as its width tends to zero, we consider the Navier–Stokes equations with Navier's slip boundary conditions.

The Navier–Stokes equations in thin domains have been studied in the case of a flat thin product domain [18], a flat thin domain whose top and bottom boundaries are given by the graph of functions on a two-dimensional domain [8], and a thin spherical shell given as a region between two concentric spheres of near radii [20]. Our goal is not just to generalize the results in the previous works, but to study the effect of the curvatures of a general limit surface on the bulk and limit equations.

The first result in Chapter 5 is the global-in-time existence of a strong solution to the Naiver–Stokes equations. Under suitable assumptions on the limit surface and friction coefficients appearing in the slip boundary conditions, we establish the global existence and uniform estimates of a strong solution for very large data according to the smallness of the width of the thin domain. Main tools for the proof of the global existence are an average operator in the normal direction of the limit surface and an extension of a surface vector field to the thin domain that satisfies the impermeable boundary condition. Using them, the slip boundary conditions, and Sobolev type inequalities on the thin domain and the surface we derive a good product estimate for the inertial and viscous terms in the Navier–Stokes equations. A key idea is to decompose a vector field on the thin domain into the average part, to which we can use a product estimate for a function on the thin domain and that on the surface, and the residual part, to which a good L^∞-estimate is applicable.

The second result is concerning a singular limit problem for the Navier–Stokes equations as the curved thin domain degenerates into the closed surface. We show that, under suitable assumptions on given data, the averaged tangential component of a strong solution to the bulk Navier–Stokes equations converges weakly in an appropriate function space on the limit surface, and that the weak limit is a unique weak solution to limit equations, which are the damped and weighted Navier–Stokes equations on the limit surface with viscous term involving the Gaussian curvature of the surface and the functions that parametrize the boundaries of the thin domain. To prove these results, we approximate a weak formulation for the bulk equations by that for the averaged tangential component of a strong solution and derive its energy estimate. In approximation of the weak formulation, we use the average operator and change of variables formulas for integrals over the thin domain and its boundary. We also employ the impermeable extension of a surface vector field and a uniform estimate for the gradient part of the Helmholtz–Leray decomposition on the thin domain to construct an appropriate test function for the bulk equations from a test function for the limit equations. To derive the energy estimate for the averaged tangential component of a strong solution to the bulk equations, we would like to take it as a test function for its weak formulation. However, it is not allowed since the averaged tangential component is not in the space of test functions for the limit equations, which is a weighted solenoidal space on the surface. To overcome this difficultly, we use the weighted Helmholtz–Leray projection on the surface.

Besides the weak convergence and characterization of the limit, we estimate the difference between a strong solution to the Navier–Stokes equations and a weak solution to the limit equations. It is worth noting that the normal derivative (with respect to the surface) of a strong solution to the Navier–Stokes equations is compared with a surface vector field given by a weak solution to the limit equations and the Weingarten map (or shape operator) of the surface. In particular, it is not necessarily small even though the thin domain and its limit surface are stationary.

1.6 Introduction to Chapter 6

In Chapter 6 we consider the first order Hamilton–Jacobi equation on a moving closed surface in the three-dimensional Euclidean space. One motivation for considering such an equation is to describe the motion of a curve on an evolving surface. The aim of Chapter 6 is to study the well-posedness as well as to provide a numerical scheme and an error bound.

We first extend the definition of viscosity sub- and supersolutions to a moving surface by using the material and tangential derivative operators. Here the material derivative is the time derivative along the total velocity of the surface, which consists of the outward normal velocity and a given tangential velocity. Then we prove a comparison principle by a standard doubling of variables method that yields the uniqueness of a viscosity solution.

To establish the existence of a viscosity solution as well as to give a numerical scheme, we consider discretization of the Hamilton–Jacobi equation in space and time. We approximate the smooth moving surface by a triangulated surface with moving vertices. Then, following the idea of the work [11] in the case of a flat stationary domain, we introduce a finite volume scheme based on the viscous approximation and discretization of surface integrals, and prove its monotonicity and consistency. We point out that to prove the monotonicity and consistency we require only the regularity of the triangulation. In particular, it is not necessary to assume that the triangulation is acute, which is very important for implementation of the numerical scheme.

Using the monotonicity and consistence of our scheme, we prove the existence of a viscosity solution by the half-relaxed limits method. Moreover, by a doubling of variables method we establish an error bound between a viscosity solution and a numerical solution of the same order as in the case of a flat stationary domain.

Chapter 6 is based on the joint work [3] with Klaus Deckelnick and Charles M. Elliott.

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Chapter 2

Zero width limit of the heat equation on moving thin domains

2.1 Introduction

For $t \in [0,T]$, $T>0$, let $\Omega_\varepsilon(t)$ be a moving thin domain in $\mathbb{R}^n$, $n \geq 2$, with width of order $\varepsilon > 0$ that converges to an evolving closed hypersurface $\Gamma(t)$ as $\varepsilon \to 0$. We consider the Neumann type problem of the heat equation of the form

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = 0 & \text{in} \quad Q_{\varepsilon,T}, \\ \partial_{\nu_\varepsilon} u^\varepsilon + v_\varepsilon^N u^\varepsilon = 0 & \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \\ u^\varepsilon(0) = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon(0). \end{cases} \tag{H_ε}$$

Here $Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \Omega_\varepsilon(t) \times \{t\}$, $\partial_\ell Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \partial\Omega_\varepsilon(t) \times \{t\}$, and ν_ε, v_ε^N are the unit outward normal vector field of $\partial\Omega_\varepsilon(t)$ and the outer normal velocity of $\partial\Omega_\varepsilon(t)$, respectively. The term $v_\varepsilon^N u^\varepsilon$ in the boundary condition is added so that the total amount of heat $\int_{\Omega_\varepsilon(t)} u^\varepsilon\, dx$ is conserved, see the beginning of Section 2.3. Also, if u^ε denotes the concentration of some chemicals, the boundary condition says that chemicals near the boundary move along it and do not go into and out of the moving thin domain.

We are interested in the behavior of a solution u^ε to (H_ε) as $\varepsilon \to 0$. Our goal is to characterize its limit as well as its convergence. Let us explain the simplest case when $\Omega_\varepsilon(t)$ is the set of all points in $\mathbb{R}^n$ with distance less than ε from $\Gamma(t)$ so that the width of $\Omega_\varepsilon(t)$ is 2ε. Let ν be the unit outward normal vector field of $\Gamma(t)$ and $V_\Gamma = v_\Gamma^N \nu + V_\Gamma^T$ be the total velocity of $\Gamma(t)$, where v_Γ^N and V_Γ^T are the outer normal velocity of $\Gamma(t)$ and a given tangential velocity field. Then our main result formally implies that, under suitable assumptions on the initial data u_0^ε of (H_ε), the limit v is a solution to

$$\partial^\circ v - v_\Gamma^N H v - \Delta_{\Gamma(t)} v = 0 \quad \text{on} \quad S_T. \tag{2.1.1}$$

Here $S_T := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$ and $\partial^\circ v = \partial_t v + v_\Gamma^N \nu \cdot \nabla v$ is the normal time derivative of v. (The notation ∂° is used in [2,5]. We refer to [3] for the normal time derivative.) Also, $H := -\mathrm{div}_{\Gamma(t)}\nu$ and $\Delta_{\Gamma(t)} := \mathrm{div}_{\Gamma(t)}\nabla_{\Gamma(t)}$ are the mean curvature of $\Gamma(t)$ and the Laplace–Beltrami operator on $\Gamma(t)$, where $\mathrm{div}_{\Gamma(t)}$ and $\nabla_{\Gamma(t)}$ are the surface divergence operator and the tangential gradient on $\Gamma(t)$, respectively (see Section 2.2 for their definitions). We will give a heuristic derivation of the limit equation (2.1.1) in Appendix 2.A. The equation (2.1.1) is equivalent to

$$\partial^\bullet v + (\mathrm{div}_{\Gamma(t)} V_\Gamma) v - \Delta_{\Gamma(t)} v - \mathrm{div}_{\Gamma(t)}(v V_\Gamma^T) = 0 \quad \text{on} \quad S_T, \tag{2.1.2}$$

which we will actually derive in Section 2.6. Here $\partial^\bullet v = \partial^\circ v + V_\Gamma^T \cdot \nabla_{\Gamma(t)} v$ denotes the material derivative of v (see Section 2.4 for its precise definition). Note that the equation (2.1.1) is independent of the tangential velocity V_Γ^T. In other words, the evolution of the limit v is not affected by advection along $\Gamma(t)$. Such a phenomenon does not occur in an advection-diffusion equation widely studied in recent years [2, 4–9, 19, 28]:

$$\partial^\bullet v + (\mathrm{div}_{\Gamma(t)} V_\Gamma) v - \Delta_{\Gamma(t)} v = 0 \quad \text{on} \quad S_T. \tag{2.1.3}$$

This equation is derived from a conservation law such that

$$\frac{d}{dt} \int_{\mathcal{M}(t)} v \, d\mathcal{H}^{n-1} = - \int_{\partial \mathcal{M}(t)} q \cdot \mu \, d\mathcal{H}^{n-2}$$

for an arbitrary portion $\mathcal{M}(t)$ of $\Gamma(t)$, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure for $k \in \mathbb{N}$, μ is the co-normal to the boundary $\partial \mathcal{M}(t)$, and q is the surface flux, see [4, Section 3] and [5, Section 3.1] for details.

Partial differential equations on thin domains are studied over the years [12–16, 20–24, 26, 27], and many researchers deal with a nonmoving thin domain of the form

$$\Omega_\varepsilon = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x' \in \omega, \, \varepsilon g_0(x') < x_n < \varepsilon g_1(x')\}, \quad \varepsilon > 0, \tag{2.1.4}$$

where ω is a domain in $\mathbb{R}^{n-1}$ and g_0, g_1 are functions on ω. In their pioneering works [12, 13], Hale and Raugel compared the dynamics of reaction-diffusion equations and damped wave equations on Ω_ε of the form (2.1.4) (with $g_0 = 0$ and slightly modified g_1) and that of corresponding limit equations on ω by the scaling argument. They transformed the equations on Ω_ε into scaled equations on a fixed reference domain $\Omega_0 = \omega \times (0, 1)$ by the change of variables, and formally derived the limit equations on ω by letting $\varepsilon \to 0$ in the scaled equations on Ω_0 and omitting divergent terms. Then they compared the dynamics of the scaled equations on Ω_0 and that of the limit equations on ω by analyzing weighted bilinear forms that appear in variational formulations of the scaled equations and the limit equations. Their scaling argument is applicable to more general thin domains such as a thin L-shaped domain [14] and a moving thin domain of the form (2.1.4) where $g_0 = 0$ and g_1 depends on time [20]. Prizzi and Rybakowski [22] generalized the scaling argument in [12, 13] to study reaction-diffusion equations on a (nonmoving) thin domain with holes around a lower dimensional domain. The generalized scaling argument in [22] is also valid for a (nonmoving) thin domain with holes around a lower dimensional manifold [21, 23]. We refer to [24] and references therein for other examples of thin domains.

In contrast to the above papers, the limit hypersurface $\Gamma(t)$ of our thin domain $\Omega_\varepsilon(t)$ evolves. Such a situation has been considered only in the paper [8], which deals with a diffuse interface model for the advection-diffusion equation (2.1.3). See also [9] for numerical computations of the advection-diffusion equation (2.1.3) based on the diffuse interface model. In [8], however, the limit equation (2.1.3) on the evolving surface is given and a bulk equation on the moving thin domain involves a weight function that vanishes on the boundary of the domain. Therefore, there is no literature on initial-boundary value problems of partial differential equations on moving thin domains around evolving surfaces whose limit equations are unknown in advance, even in the case of the heat equation.

The difficulty caused by the evolution of the hypersurface $\Gamma(t)$ is in transforming equations on $\Omega_\varepsilon(t)$ and $\Gamma(t)$ into equations on fixed (in time and width) domain and hypersurface. In particular, transformations of differential operators on $\Gamma(t)$ into those on a fixed hypersurface is so complicated that we can hardly find a limit equation on the fixed hypersurface and convert it into an equation on $\Gamma(t)$, see [7] for the actual transformations of differential operators.

To avoid this difficulty, we employ another method that does not require transformations of $\Omega_\varepsilon(t)$ and $\Gamma(t)$. Let us explain our idea of derivation of a limit equation on $\Gamma(t)$. We start from a variational formulation of (H_ε) (see (2.3.2)) that consists of integrals over the noncylindrical domain $Q_{\varepsilon,T}$ of a variational solution u^ε to (H_ε) and a test function defined on $Q_{\varepsilon,T}$. In this variational formulation, we take a test function independent of the normal direction of $\Gamma(t)$ and apply the co-area formula (see (2.5.1)) and a weighted average operator M_ε (see Definition 2.5.1) to get a variational formulation (with some residual term) of the average $M_\varepsilon u^\varepsilon$ (see (2.6.1)) that consists of integrals over the space-time manifold S_T of $M_\varepsilon u^\varepsilon$ and a test function defined on S_T. Then we obtain a variational formulation of a limit equation on $\Gamma(t)$ (see (2.6.13)) by omitting the residual term in the variational formulation of $M_\varepsilon u^\varepsilon$. Moreover, we prove that $M_\varepsilon u^\varepsilon$ converges weakly in a function space on S_T as $\varepsilon \to 0$ and that the limit is a unique variational solution to the limit equation (see Theorem 2.6.9), and estimate the $L^2(Q_{\varepsilon,T})$-norm of the difference between variational solutions to (H_ε) and the limit equation (see Theorem 2.6.12). These results indicate that our limit equation on $\Gamma(t)$ derived as above is indeed the "limit" of (H_ε).

In our derivation of a limit equation, Lemma 2.5.6 and Lemma 2.5.13 play an important role. In Lemma 2.5.6 we approximate an H^1-bilinear form on $\Omega_\varepsilon(t)$ for each $t \in [0,T]$ by that on $\Gamma(t)$ with the tangential gradient of the average $M_\varepsilon u$ of a function u on $\Omega_\varepsilon(t)$. The proof of Lemma 2.5.6 is based on simple representations of the gradient in $\mathbb{R}^n$ and the tangential gradient on $\Gamma(t)$ under a special local coordinate system for each fixed point on $\Gamma(t)$. On the other hand, Lemma 2.5.13 gives an integral formula that formally represents a relation between the weak time derivative of a function u on $Q_{\varepsilon,T}$ and the weak material derivative of its average $M_\varepsilon u$ (in fact, we do not explicitly deal with the time derivative of u). Lemma 2.5.13 essentially follows from Lemma 2.5.11, which gives a relation between the time derivative and the material derivative of functions defined on S_T.

Average operators in the thin direction were originally introduced by Hale and Raugel [12,13], but they took the average of functions on the scaled domain $\Omega_0 = \omega \times (0,1)$. Average operators on actual thin domains Ω_ε appears in the study of the Navier–Stokes equations on three-dimensional thin domains [15,16,26,27]. Temam and Ziane [26,27] first employed them to study the global existence of strong solutions to the Navier–Stokes equations for large initial data and external forces and the behavior of solutions as $\varepsilon \to 0$ when Ω_ε is a three-dimensional thin product domain $\Omega_\varepsilon = \omega \times (0,\varepsilon)$ with a bounded domain ω in $\mathbb{R}^2$ and a thin spherical domain $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid a < |x| < (1+\varepsilon)a\}$ with a constant $a > 0$. In [15,16], average operators were employed to study the dynamics of the Navier–Stokes equations on Ω_ε of the form (2.1.4). In particular, the authors of [16] compared the dynamics of the Navier–Stokes equations with that of limit equations by estimating the difference of the average of solutions to the Navier–Stokes equations and solutions to the limit equations.

We point out that our weighted average operator given in Definition 2.5.1 is a generalization of average operators given in [15,16,26] and that its weight function is different from that of an average operator given in [27]. In fact, the weight function of our average operator is a Jacobian that appears when we change variables of integrals over a tubular neighborhood of $\Gamma(t)$ in terms of the normal coordinate system around $\Gamma(t)$. Our choice of the weight function enables us to avoid including the material derivative of a test function in the estimate for the residual term in the variational formulation of the average of a variational solution to (H_ε), which is essential for derivation of its energy estimate, see Lemma 2.5.13 and Remark 2.5.14. We also note that, contrary to our case, Kublik, Tanushev, and Tsai [17] employed the same

Jacobian and co-area formula to transform integrals over boundaries of domains into those over their tubular neighborhoods. Based on this transformation, they proposed a new approach to numerical computations of boundary integrals without explicit parametrizations of boundaries and a simple formulation for constructing boundary integral methods to solve Poisson's equation. Their method of the numerical computations of boundary integrals is also applicable to integrals over nonclosed manifolds of higher codimension, such as curves in $\mathbb{R}^3$ with different endpoints, see [18] for details.

Finally we mention variational formulations of partial differential equations on evolving surfaces. There are several kinds of variational frameworks for equations on evolving surfaces, mainly the advection-diffusion equation (2.1.3), see [4, 19, 28] for example. In addition, Alphonse, Elliott, and Stinner [1, 2] proposed an abstract variational setting with evolving Hilbert spaces and applied it to some equations on moving domains and evolving surfaces. Among these variational frameworks, we adopt the one introduced by Olshanskii, Reusken, and Xu [19]. Their variational formulation is imposed on function spaces on S_T, which is suitable for our calculation of bilinear forms on function spaces on S_T and $Q_{\varepsilon,T}$ performed in Section 2.5 and Section 2.6.

This chapter is organized as follows. In Section 2.2 we introduce notations related to the evolving surface $\Gamma(t)$ and define the moving thin domain $\Omega_\varepsilon(t)$. In Section 2.3 we define a variational solution to (H_ε) and prove its existence and uniqueness. We also derive an energy estimate of a variational solution to (H_ε) with a constant independent of ε. In Section 2.4 we define function spaces on S_T introduced in [19] and give their properties. In Section 2.5 we define the weighted average operator M_ε and establish estimates and formulas related to M_ε. In Section 2.6 we derive a limit equation on $\Gamma(t)$ of the form (2.1.2) via its variational formulation and prove our main theorems (Theorem 2.6.9 and Theorem 2.6.12). In Appendix 2.A we give a heuristic derivation of the limit equation (2.1.1) when $\Omega_\varepsilon(t)$ is the set of all points in $\mathbb{R}^n$ with distance less than ε from $\Gamma(t)$. In Appendix 2.B we give complete proofs of some results in Section 2.4 related to integrals over $\Gamma(t)$. In Appendix 2.C we show detailed calculations in proofs of some lemmas in Section 2.5 involving the differential geometry of tubular neighborhoods of $\Gamma(t)$.

2.2 Evolving surfaces and moving thin domains

For each $t \in [0,T]$, let $\Gamma(t)$ be a closed (that is, compact and without boundary), connected and oriented smooth hypersurface in $\mathbb{R}^n$. We set $\Gamma_0 := \Gamma(0)$ and define a space-time manifold $S_T \subset \mathbb{R}^{n+1}$ as $S_T := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. We assume that each point y on $\Gamma(t)$ evolves with velocity $V_\Gamma(y,t)$, which is not necessarily normal to $\Gamma(t)$, and the velocity field $V_\Gamma \colon \overline{S_T} \to \mathbb{R}^n$ is smooth. Let $\Phi(\cdot,t)\colon \Gamma_0 \to \Gamma(t)$ be the flow map of V_Γ, that is, $\Phi(\cdot,t)$ is a diffeomorphism from Γ_0 onto $\Gamma(t)$ with its inverse $\Phi^{-1}(\cdot,t)$ for each $t \in [0,T]$ and satisfies

$$\Phi(Y,0) = Y, \quad \frac{\partial \Phi}{\partial t}(Y,t) = V_\Gamma(\Phi(Y,t),t) \quad \text{for all} \quad Y \in \Gamma_0,\, t \in [0,T].$$

We assume that Φ and Φ^{-1} are smooth on $\Gamma_0 \times [0,T]$ and $\overline{S_T}$, respectively. Due to this assumption, $\overline{S_T}$ is a compact smooth manifold in $\mathbb{R}^{n+1}$.

Let $\nu : \overline{S_T} \to \mathbb{R}^n$ be the unit outward normal vector field of $\Gamma(t)$. The velocity V_Γ is decomposed into $V_\Gamma = v_\Gamma^N \nu + V_\Gamma^T$, where $v_\Gamma^N \colon \overline{S_T} \to \mathbb{R}$ is the outer normal velocity and $V_\Gamma^T \colon \overline{S_T} \to \mathbb{R}^n$ is a tangential velocity field. Note that to describe the geometric motion of $\Gamma(t)$ it is sufficient to prescribe the normal velocity. However, to describe a limit equation on $\Gamma(t)$ we will derive in Section 2.6, we also need to consider a tangential velocity, which represents advection along $\Gamma(t)$.

For each $t \in [0,T]$, let $d(\cdot,t)$ be the signed distance function from $\Gamma(t)$ that increases in the direction of $\nu(\cdot,t)$. By the smoothness (in space and time) and compactness of $\Gamma(t)$, there is an open set $N(t)$ in $\mathbb{R}^n$ of the form $N(t) = \{x \in \mathbb{R}^n \mid -\delta < d(x,t) < \delta\}$ for each $t \in [0,T]$, where $\delta > 0$ is a constant independent of t, that satisfies the following conditions:

- The signed distance function d is smooth on $\overline{N_T}$, where

$$N_T := \bigcup_{t\in(0,T)} N(t) \times \{t\} \subset \mathbb{R}^{n+1}.$$

- For each $(x,t) \in \overline{N_T}$, there is a unique point $p(x,t) \in \Gamma(t)$ such that

$$x = p(x,t) + d(x,t)\nu(p(x,t),t), \quad \nabla d(x,t) = \nu(p(x,t),t).$$

The set $N(t)$ is called a tubular neighborhood of $\Gamma(t)$. Based on the above equality, we extend the outward normal ν to $\overline{N_T}$ by setting $\nu(x,t) := \nabla d(x,t)$ for $(x,t) \in \overline{N_T}$. Then, by the smoothness of d, the extended outward normal ν and the projection mapping p are smooth on $\overline{N_T}$. Also, the normal velocity v_Γ^N of $\Gamma(t)$ is given by $v_\Gamma^N = -\partial_t d$ on $\overline{S_T}$.

Next, we give definitions of differential operators on evolving surfaces. For a function v and a vector field F on S_T, we define the tangential gradient of v and the surface divergence of F as

$$\nabla_{\Gamma(t)} v(y,t) := [I_n - \nu(y,t) \otimes \nu(y,t)]\nabla \overline{v}(y,t),$$
$$\mathrm{div}_{\Gamma(t)} F(y,t) := \mathrm{trace}[\{I_n - \nu(y,t) \otimes \nu(y,t)\}\nabla \overline{F}(y,t)]$$

for $(y,t) \in S_T$. Here I_n is the identity matrix of size n and $\nu \otimes \nu := (\nu_i \nu_j)_{i,j}$ is the tensor product of ν. Also, $\overline{v}$ and $\overline{F}$ are the constant extensions of v and F in the normal direction of $\Gamma(t)$ given by

$$\overline{v}(x,t) := v(p(x,t),t), \quad \overline{F}(x,t) := F(p(x,t),t), \quad (x,t) \in N_T.$$

By definition, $\nu \cdot \nabla_{\Gamma(t)} v = 0$ holds. Hereafter we use the same notations for functions and vector fields on $\Gamma(t)$ with each fixed $t \in [0,T]$.

Finally, we define a moving thin domain. Let g_0 and g_1 be smooth functions on $\overline{S_T}$. We assume that there is a constant $c > 0$ such that

$$g(y,t) := g_1(y,t) - g_0(y,t) \geq c \quad \text{for all} \quad (y,t) \in \overline{S_T}. \tag{2.2.1}$$

Then we define a moving thin domain $\Omega_\varepsilon(t) \subset \mathbb{R}^n$ as

$$\Omega_\varepsilon(t) := \{y + \rho\nu(y,t) \mid y \in \Gamma(t),\, \varepsilon g_0(y,t) < \rho < \varepsilon g_1(y,t)\}, \quad t \in [0,T],\, \varepsilon > 0$$

and a space-time noncylindrical domain $Q_{\varepsilon,T} \subset \mathbb{R}^{n+1}$ as $Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \Omega_\varepsilon(t) \times \{t\}$. Note that $\Omega_\varepsilon(t)$ does not necessarily include $\Gamma(t)$, since we do not assume that g_0 is negative and g_1 is positive. Since g_0 and g_1 are smooth and thus bounded on the compact manifold $\overline{S_T}$, there is a positive number ε_0 such that $\overline{\Omega_\varepsilon(t)} \subset N(t)$ for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T]$. Hereafter we assume that $\varepsilon \in (0,\varepsilon_0)$.

2.3 Heat equation on moving thin domains

In this section, we consider the initial-boundary problem (H_ε) of the heat equation on the moving thin domain $\Omega_\varepsilon(t)$. First we show that the boundary condition of (H_ε) yields the conservation of the total amount of heat. Suppose that u^ε satisfies the heat equation in $Q_{\varepsilon,T}$. By the Reynolds transport theorem and Green's formula (see [10, Appendix C]) we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx &= \int_{\Omega_\varepsilon(t)} \partial_t u^\varepsilon\,dx + \int_{\partial\Omega_\varepsilon(t)} v_\varepsilon^N u^\varepsilon\,d\mathcal{H}^{n-1} \\ &= \int_{\Omega_\varepsilon(t)} \Delta u^\varepsilon\,dx + \int_{\partial\Omega_\varepsilon(t)} v_\varepsilon^N u^\varepsilon\,d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega_\varepsilon(t)} (\partial_{\nu_\varepsilon} u^\varepsilon + v_\varepsilon^N u^\varepsilon)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Hence if u^ε additionally satisfies the boundary condition of (H_ε), then $\frac{d}{dt}\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx = 0$ for all $t \in (0,T)$, that is, the total amount of heat $\int_{\Omega_\varepsilon(t)} u^\varepsilon\,dx$ is conserved.

Next, we give a definition of a variational solution to (H_ε). For each $\varepsilon > 0$, we define a function space $L^2_{H^1(\varepsilon)}$ on $Q_{\varepsilon,T}$ and an inner product on $L^2_{H^1(\varepsilon)}$ as

$$\begin{aligned} L^2_{H^1(\varepsilon)} &:= \{u \in L^2(Q_{\varepsilon,T}) \mid \nabla u \in L^2(Q_{\varepsilon,T})\}, \\ (u_1,u_2)_{L^2_{H^1(\varepsilon)}} &:= \int_0^T\int_{\Omega_\varepsilon(t)} (u_1u_2 + \nabla u_1\cdot\nabla u_2)\,dx\,dt. \end{aligned} \tag{2.3.1}$$

The space $L^2_{H^1(\varepsilon)}$ is a Hilbert space endowed with the above inner product. Let $\|\cdot\|_{L^2_{H^1(\varepsilon)}}$ denote the norm of $L^2_{H^1(\varepsilon)}$ induced by the inner product $(\cdot,\cdot)_{L^2_{H^1(\varepsilon)}}$.

Definition 2.3.1. Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$. A function $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ is said to be a *variational solution* to the initial-boundary value problem (H_ε) if it satisfies

$$\int_0^T\int_{\Omega_\varepsilon(t)} (-u^\varepsilon\partial_t w + \nabla u^\varepsilon\cdot\nabla w)\,dx\,dt - \int_{\Omega_\varepsilon(0)} u_0^\varepsilon w(0)\,dx = 0 \tag{2.3.2}$$

for all $w \in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T) = 0$ in $\Omega_\varepsilon(T)$.

The variational formulation (2.3.2) is derived as follows. Suppose that u^ε is a classical solution to (H_ε). We multiply both sides of the heat equation in $Q_{\varepsilon,T}$ by an arbitrary function $w \in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T) = 0$ in $\Omega_\varepsilon(T)$ and integrate them over $Q_{\varepsilon,T}$ to get

$$\int_0^T\int_{\Omega_\varepsilon(t)} (\partial_t u^\varepsilon - \Delta u^\varepsilon) w\,dx\,dt = 0.$$

We calculate the left-hand side of the above equality. By the Reynolds transport theorem and the conditions $u^\varepsilon(0) = u_0^\varepsilon$ in $\Omega_\varepsilon(0)$ and $w(T) = 0$ in $\Omega_\varepsilon(T)$, we have

$$\begin{aligned}\int_0^T\int_{\Omega_\varepsilon(t)} (\partial_t u^\varepsilon) w\,dx\,dt = &-\int_0^T\int_{\Omega_\varepsilon(t)} u^\varepsilon\partial_t w\,dx\,dt \\ &-\int_0^T\int_{\partial\Omega_\varepsilon(t)} v_\varepsilon^N u^\varepsilon w\,d\mathcal{H}^{n-1}\,dt - \int_{\Omega_\varepsilon(0)} u_0^\varepsilon w(0)\,dx.\end{aligned}$$

On the other hand, by integration by parts,

$$-\int_{\Omega_\varepsilon(t)}(\Delta u^\varepsilon)w\,dx\,dt=\int_{\Omega_\varepsilon(t)}\nabla u^\varepsilon\cdot\nabla w\,dx-\int_{\partial\Omega_\varepsilon(t)}(\partial_{\nu_\varepsilon}u^\varepsilon)w\,d\mathcal{H}^{n-1}.$$

Hence it follows that

$$\int_0^T\int_{\Omega_\varepsilon(t)}(-u^\varepsilon\partial_t w+\nabla u^\varepsilon\cdot\nabla w)\,dx\,dt-\int_0^T\int_{\partial\Omega_\varepsilon(t)}(\partial_{\nu_\varepsilon}u^\varepsilon+v_\varepsilon^N u^\varepsilon)w\,d\mathcal{H}^{n-1}\,dt \\ -\int_{\Omega_\varepsilon(0)}u_0^\varepsilon w(0)\,dx=0$$

and we obtain (2.3.2) by applying the boundary condition of (H_ε) to the second term of the left-hand side in the above equality.

Our goal in this section is to obtain a unique variational solution to (H_ε) that satisfies an energy estimate with a constant independent of ε. To this end, we transform (2.3.2) into a variational formulation of some equation on the fixed (in time) domain $\Omega_\varepsilon(0)$ with the aid of a suitable diffeomorphism between $\Omega_\varepsilon(0)$ and $\Omega_\varepsilon(t)$.

Lemma 2.3.2. *For each $t\in[0,T]$, there exists a diffeomorphism $\Psi_\varepsilon(\cdot,t)\colon\Omega_\varepsilon(0)\to\Omega_\varepsilon(t)$ with its inverse $\Psi_\varepsilon^{-1}(\cdot,t)\colon\Omega_\varepsilon(t)\to\Omega_\varepsilon(0)$ such that Ψ_ε and Ψ_ε^{-1} are smooth on $\overline{\Omega_\varepsilon(0)}\times[0,T]$ and $\overline{Q_{\varepsilon,T}}$, respectively, and $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$. Moreover, there exists a constant $c>0$ independent of ε such that*

$$|\partial_X^\alpha\partial_t^k\Psi_\varepsilon(X,t)|\le c,\quad |\partial_x^\alpha\partial_t^k\Psi_\varepsilon^{-1}(x,t)|\le c \tag{2.3.3}$$

for all $(X,t)\in\Omega_\varepsilon(0)\times(0,T)$, $(x,t)\in Q_{\varepsilon,T}$, and $|\alpha|+k\le2$, $k=0,1,2$.

Proof. We observe that for each $X\in\Omega_\varepsilon(0)$ there is a unique $\theta\in(0,1)$ such that

$$X=p(X,0)+\varepsilon\{(1-\theta)g_0(p(X,0),0)+\theta g_1(p(X,0),0)\}\nu(p(X,0),0), \tag{2.3.4}$$

that is, X divides the line segment A_0A_1 internally in the ratio $\theta\colon 1-\theta$, where

$$A_i:=p(X,0)+\varepsilon g_i(p(X,0),0)\nu(p(X,0),0),\quad i=0,1.$$

Based on this observation we define $\Psi_\varepsilon(X,t)\in\Omega_\varepsilon(t)$ as

$$\Psi_\varepsilon(X,t):=\Phi(p(X,0),t) \\ +\varepsilon\{(1-\theta)g_0(\Phi(p(X,0),t),t)+\theta g_1(\Phi(p(X,0),t),t)\}\nu(\Phi(p(X,0),t),t), \tag{2.3.5}$$

that is, $\Psi_\varepsilon(X,t)$ divides the line segment B_0B_1 internally in the ratio $\theta\colon 1-\theta$, where

$$B_i:=\Phi(p(X,0),t)+\varepsilon g_i(\Phi(p(X,0),t),t)\nu(\Phi(p(X,0),t),t),\quad i=0,1.$$

To eliminate θ in (2.3.5), we take the inner product of both sides of (2.3.4) and the vector $\nu(p(X,0),0)$. Then

$$\{X-p(X,0)\}\cdot\nu(p(X,0),0)=\varepsilon\{(1-\theta)g_0(p(X,0),0)+\theta g_1(p(X,0),0)\}.$$

Since $\{X-p(X,0)\}\cdot\nu(p(X,0),0)=d(X,0)$ and $g_1-g_0=g>0$, it follows that

$$\theta=\frac{d(X,0)-\varepsilon g_0(p(X,0),0)}{\varepsilon g(p(X,0),0)}.$$

Hence, by substituting this for θ in (2.3.5), we obtain

$$\Psi_\varepsilon(X,t) = \Phi(p(X,0),t) + \{d(X,0)\phi_1(X,t) + \varepsilon\phi_2(X,t)\}\nu(\Phi(p(X,0),t),t) \tag{2.3.6}$$

for $X \in \Omega_\varepsilon(0)$ and $t \in [0,T]$, where

$$\phi_1(X,t) := \frac{g(\Phi(p(X,0),t),t)}{g(p(X,0),0)}, \quad \phi_2(X,t) := g_0(\Phi(p(X,0),t),t) - \phi_1(X,t)g_0(p(X,0),0).$$

Similarly we define a mapping Ψ_ε^{-1} as

$$\Psi_\varepsilon^{-1}(x,t) := \Phi^{-1}(p(x,t),t) + \{d(x,t)\phi_3(x,t) + \varepsilon\phi_4(x,t)\}\nu(\Phi^{-1}(p(x,t),t),0) \tag{2.3.7}$$

for $(x,t) \in Q_{\varepsilon,T}$, where

$$\phi_3(x,t) := \frac{g(\Phi^{-1}(p(x,t),t),0)}{g(p(x,t),t)}, \quad \phi_4(x,t) := g_0(\Phi^{-1}(p(x,t),t),0) - \phi_3(x,t)g_0(p(x,t),t).$$

By definition, $\Psi_\varepsilon(\cdot,t)\colon \Omega_\varepsilon(0) \to \Omega_\varepsilon(t)$ is a bijection with its inverse $\Psi_\varepsilon^{-1}(\cdot,t)\colon \Omega_\varepsilon(t) \to \Omega_\varepsilon(0)$ for each $t \in [0,T]$. Also, since $\Phi(\cdot,0)$ is the identity mapping on Γ_0, we have $\phi_1(X,0) = 1$, $\phi_2(X,0) = 0$ and thus

$$\Psi_\varepsilon(X,0) = p(X,0) + d(X,0)\nu(p(X,0),0) = X \quad \text{for all} \quad X \in \Omega_\varepsilon(0),$$

that is, $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$. Due to the smoothness of Φ, Φ^{-1}, d, p, g_0, and g_1, the right-hand sides of (2.3.6) and (2.3.7) are smooth on the compact sets $\overline{N(0)} \times [0,T]$ and $\overline{N_T}$, respectively, and thus bounded independently of ε along with their derivatives. From this fact and the inclusion $\overline{\Omega_\varepsilon(t)} \subset N(t)$ for each $t \in [0,T]$, it follows that Ψ_ε and Ψ_ε^{-1} are smooth on $\overline{\Omega_\varepsilon(0)} \times [0,T]$ and $\overline{Q_{\varepsilon,T}}$, respectively, and that the inequalities (2.3.3) hold with a constant $c > 0$ independent of ε. In particular, $\Psi_\varepsilon(\cdot,t)\colon \Omega_\varepsilon(0) \to \Omega_\varepsilon(t)$ is a diffeomorphism for each $t \in [0,T]$. □

Let Ψ_ε and Ψ_ε^{-1} be the mappings given by Lemma 2.3.2. In (2.3.2), we set

$$U^\varepsilon(X,t) := u^\varepsilon(\Psi_\varepsilon(X,t),t), \quad W(X,t) := w(\Psi_\varepsilon(X,t),t), \quad (X,t) \in \Omega_\varepsilon(0) \times (0,T).$$

Then, by the change of variables $x = \Psi_\varepsilon(X,t)$, we transform (2.3.2) into

$$\int_0^T \{-(U^\varepsilon(t), J^\varepsilon(t)\partial_t W(t))_{L^2} + (A^\varepsilon(t)\nabla U^\varepsilon(t) - U^\varepsilon(t)B^\varepsilon(t), \nabla W(t))_{L^2}\}\,dt \\ - (u_0^\varepsilon, W(0))_{L^2} = 0. \tag{2.3.8}$$

Here $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\Omega_\varepsilon(0))$ and

$$\begin{aligned} J^\varepsilon(X,t) &:= |\det \nabla\Psi_\varepsilon(X,t)| \in \mathbb{R}, \\ A^\varepsilon(X,t) &:= J^\varepsilon(X,t)\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T \in \mathbb{R}^{n\times n}, \\ B^\varepsilon(X,t) &:= J^\varepsilon(X,t)\partial_t\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t) \in \mathbb{R}^n \end{aligned}$$

for $(X,t) \in \Omega_\varepsilon(0) \times (0,T)$, where

$$\nabla\Psi_\varepsilon^{-1} := \begin{pmatrix} \partial_1(\Psi_\varepsilon^{-1})_1 & \dots & \partial_n(\Psi_\varepsilon^{-1})_1 \\ \vdots & \ddots & \vdots \\ \partial_1(\Psi_\varepsilon^{-1})_n & \dots & \partial_n(\Psi_\varepsilon^{-1})_n \end{pmatrix} \quad \text{for} \quad \Psi_\varepsilon^{-1} = \begin{pmatrix} (\Psi_\varepsilon^{-1})_1 \\ \vdots \\ (\Psi_\varepsilon^{-1})_n \end{pmatrix}$$

and $[\nabla\Psi_\varepsilon^{-1}]^T$ denotes the transposed matrix of $\nabla\Psi_\varepsilon^{-1}$. Note that the vector field B^ε comes from the differentiation of $w(x,t) = W(\Psi_\varepsilon^{-1}(x,t),t)$ with respect to t holding $x \in \Omega_\varepsilon(t)$ fixed:

$$\partial_t w(x,t) = \partial_t W(\Psi_\varepsilon^{-1}(x,t),t) + \partial_t\Psi_\varepsilon^{-1}(x,t)\cdot\nabla W(\Psi_\varepsilon^{-1}(x,t),t).$$

Since $w(T) = 0$ in $\Omega_\varepsilon(T)$ and $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$, we have $W(T) = 0$ and $J^\varepsilon(0) = 1$ in $\Omega_\varepsilon(0)$. Thus, by integration by parts with respect to t, we further transform (2.3.8) into

$$\begin{aligned}\int_0^T \{{}_{(H^1)'}\langle\partial_t U^\varepsilon(t), J^\varepsilon(t)W(t)\rangle_{H^1} + (U^\varepsilon(t), W(t)\partial_t J^\varepsilon(t))_{L^2} \\ + (A^\varepsilon(t)\nabla U^\varepsilon(t) - U^\varepsilon(t)B^\varepsilon(t), \nabla W(t))_{L^2}\}\,dt = 0.\end{aligned} \tag{2.3.9}$$

Here ${}_{(H^1)'}\langle\cdot,\cdot\rangle_{H^1}$ is the duality product between $H^1(\Omega_\varepsilon(0))$ and its dual space $(H^1(\Omega_\varepsilon(0)))'$.

Theorem 2.3.3. *For every $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$, there exists a unique function*

$$U^\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon(0)))\cap L^2(0,T;H^1(\Omega_\varepsilon(0))) \quad \textit{with} \quad \partial_t U^\varepsilon \in L^2(0,T;(H^1(\Omega_\varepsilon(0)))')$$

that satisfies (2.3.9) for all $W \in L^2(0,T;H^1(\Omega_\varepsilon(0)))$ and $U^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$. Moreover, there exists a constant $c > 0$ independent of u_0^ε, U^ε, and ε such that

$$\sup_{t\in(0,T)}\|U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))} + \int_0^T \|\nabla U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}\,dt \le c\|u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon(0))}. \tag{2.3.10}$$

Proof. For $i,j = 1,\dots,n$, let A^ε_{ij} be the (i,j)-entry of A^ε and B^ε_i be the i-th component of B^ε. Suppose that there is a positive constant C independent of ε such that

$$C^{-1} \le J^\varepsilon(X,t) \le C, \tag{2.3.11}$$

$$|\nabla J^\varepsilon(X,t)| \le C, \quad |\partial_t J^\varepsilon(X,t)| \le C, \quad |A^\varepsilon_{ij}(X,t)| \le C, \quad |B^\varepsilon_i(X,t)| \le C, \tag{2.3.12}$$

$$A^\varepsilon(X,t)\zeta\cdot\zeta \ge C|\zeta|^2 \tag{2.3.13}$$

for all $(X,t) \in \Omega_\varepsilon(0)\times(0,T)$, $\zeta \in \mathbb{R}^n$, and $i,j = 1,\dots,n$. Then the theorem is proved by a standard Galerkin method and Gronwall argument, see [10, Section 7.1] for details. In particular, the constant c in (2.3.10) depends only on the above C and thus it is independent of ε.

Let us prove (2.3.11), (2.3.12), and (2.3.13). The inequalities (2.3.12) and the right-hand inequality of (2.3.11) immediately follow from (2.3.3). For all $(X,t) \in \Omega_\varepsilon(0)\times(0,T)$ since $\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)\nabla\Psi_\varepsilon(X,t) = I_n$ it follows that

$$|\det\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)|J^\varepsilon(X,t) = 1, \quad [\nabla\Psi_\varepsilon(X,t)]^T[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T = I_n.$$

The first equality yields the left-hand inequality of (2.3.11) because $|\det\nabla\Psi_\varepsilon^{-1}|$ is bounded on $Q_{\varepsilon,T}$ independently of ε by (2.3.3). Moreover, the above equality and (2.3.3) imply that, for all $(X,t) \in \Omega_\varepsilon(0)\times(0,T)$ and $\zeta \in \mathbb{R}^n$,

$$\begin{aligned}|\zeta|^2 &= |[\nabla\Psi_\varepsilon(X,t)]^T[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T\zeta|^2 \\ &\le c|[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T\zeta|^2 \\ &= c\{\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T\zeta\}\cdot\zeta \\ &= c|\det\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)|A^\varepsilon(X,t)\zeta\cdot\zeta \le cA^\varepsilon(X,t)\zeta\cdot\zeta\end{aligned}$$

with a constant $c > 0$ independent of ε. Thus (2.3.13) follows. □

Now we can show the existence and uniqueness of a variational solution to (H_ε) and its energy estimate with a constant independent of ε.

Theorem 2.3.4. *For every $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$, there exists a unique variational solution u^ε to (H_ε). Moreover, u^ε satisfies that $u^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$ and*

$$\sup_{t\in(0,T)} \|u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} + \int_0^T \|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))}\,dt \leq c\|u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon(0))} \tag{2.3.14}$$

with a constant $c > 0$ independent of u_0^ε, u^ε, and ε.

Proof. For each $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$, let U^ε be the unique function given by Theorem 2.3.3 and we set

$$u^\varepsilon(x,t) := U^\varepsilon(\Psi_\varepsilon^{-1}(x,t),t), \quad (x,t) \in Q_{\varepsilon,T}.$$

Since $\Psi_\varepsilon(\cdot,0)$ is the identity mapping on $\Omega_\varepsilon(0)$ by Lemma 2.3.2 and $U^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$ by Theorem 2.3.3, we have $u^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$. Let us show that u^ε satisfies (2.3.2) for all $w \in C^1(\overline{Q_{\varepsilon,T}})$ with $w(T) = 0$ in $\Omega_\varepsilon(T)$. Since Ψ_ε is smooth on $\overline{\Omega_\varepsilon(0)} \times [0,T]$, a function

$$W(X,t) := w(\Psi_\varepsilon(X,t),t), \quad (X,t) \in \overline{\Omega_\varepsilon(0)} \times [0,T]$$

is in $C^1(\overline{\Omega_\varepsilon(0)} \times [0,T])$ and satisfies $W(T) = 0$ in $\Omega_\varepsilon(0)$. Hence we can substitute it for W in (2.3.9) and integrate by parts with respect to t to get (2.3.8). By changing variables $X = \Psi_\varepsilon^{-1}(x,t)$ in (2.3.8), we obtain (2.3.2).

Next we prove the energy estimate (2.3.14). By the change of variables $x = \Psi_\varepsilon(X,t)$ we have

$$\int_{\Omega_\varepsilon(t)} |u^\varepsilon(x,t)|^2\,dx = \int_{\Omega_\varepsilon(0)} |U^\varepsilon(X,t)|^2 |\det \nabla\Psi_\varepsilon(X,t)|\,dX,$$

$$\int_{\Omega_\varepsilon(t)} |\nabla u^\varepsilon(x,t)|^2\,dx = \int_{\Omega_\varepsilon(0)} |[\nabla\Psi_\varepsilon^{-1}(\Psi_\varepsilon(X,t),t)]^T \nabla U^\varepsilon(X,t)|^2 |\det \nabla\Psi_\varepsilon(X,t)|\,dX$$

for all $t \in [0,T]$. Hence the inequalities (2.3.3) yield

$$\|u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} \leq c\|U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}, \quad \|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(t))} \leq c\|\nabla U^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon(0))}$$

with a constant $c > 0$ independent of ε. By these inequalities and (2.3.10), we obtain (2.3.14) and thus $u^\varepsilon \in L^2_{H^1(\varepsilon)}$. Hence u^ε is a variational solution to (H_ε).

Finally, the uniqueness of a variational solution to (H_ε) follows from that of a function given by Theorem 2.3.3. The proof is complete. □

Remark 2.3.5. Let u^ε be the unique variational solution to (H_ε) with initial data $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$. Then it immediately follows from (2.3.14) that

$$\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}} \leq c\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}, \tag{2.3.15}$$

where $c > 0$ is a constant independent of u_0^ε, u^ε, and ε. We will use this inequality in Section 2.6.

2.4 Basic function spaces on evolving surfaces

In this section, we define function spaces on the space-time manifold S_T introduced by Olshanskii, Reusken, and Xu [19] and give their properties. These spaces will give an appropriate variational formulation of a limit equation on $\Gamma(t)$ we will derive in Section 2.6. All results in this section are originally obtained in [19] for the three-dimensional case. They can be easily extended for arbitrary dimensions and we give proofs of them for the readers' convenience.

For each fixed $T>0$, we define a function space H_T and an inner product on H_T as

$$\begin{gathered} H_T := \{v \in L^2(S_T) \mid \nabla_{\Gamma(t)} v \in L^2(S_T)\}, \\ (v_1,v_2)_{H_T} := \int_0^T \int_{\Gamma(t)} \{v_1(y,t)v_2(y,t) + \nabla_{\Gamma(t)} v_1(y,t) \cdot \nabla_{\Gamma(t)} v_2(y,t)\}\, d\mathcal{H}^{n-1}(y)\, dt. \end{gathered} \tag{2.4.1}$$

This inner product induces the norm $\|\cdot\|_{H_T}$ that is equivalent to the one induced by the inner product $\int_{S_T}\{v_1(\sigma)v_2(\sigma)+\nabla_{\Gamma(t)}v_1(\sigma)\cdot\nabla_{\Gamma(t)}v_2(\sigma)\}\,d\mathcal{H}^n(\sigma)$, since the identity

$$\int_0^T \int_{\Gamma(t)} f(y,t)\, d\mathcal{H}^{n-1}(y)\, dt = \int_{S_T} f(\sigma)(1+|v_\Gamma^N(\sigma)|^2)^{-1/2}\, d\mathcal{H}^n(\sigma) \tag{2.4.2}$$

holds and v_Γ^N is bounded on S_T. This identity is stated in [19] without proof. We give the proof of (2.4.2) in Appendix 2.B for the readers' convenience. If $T_1<T_2$, then H_{T_2} is continuously embedded into H_{T_1} just by restricting elements of H_{T_2} on S_{T_1}.

Next we define an auxiliary space. Let $H^1(\Gamma_0) := \{V \in L^2(\Gamma_0) \mid \nabla_{\Gamma_0} V \in L^2(\Gamma_0)\}$ with the inner product $(V_1,V_2)_{H^1(\Gamma_0)} := \int_{\Gamma_0}(V_1V_2 + \nabla_{\Gamma_0}V_1\cdot\nabla_{\Gamma_0}V_2)\,d\mathcal{H}^{n-1}$, where ∇_{Γ_0} is the tangential gradient on Γ_0. Then we define a Hilbert space $\widehat{H}_T$ as

$$\widehat{H}_T := L^2(0,T;H^1(\Gamma_0)), \quad (V_1,V_2)_{\widehat{H}_T} := \int_0^T (V_1(t),V_2(t))_{H^1(\Gamma_0)}\, dt$$

and let $\|\cdot\|_{\widehat{H}_T}$ denote the norm of $\widehat{H}_T$ induced by the inner product $(\cdot,\cdot)_{\widehat{H}_T}$.

Let $\Phi(\cdot,t)\colon \Gamma_0 \to \Gamma(t)$ be the flow map of V_Γ and $\Phi^{-1}(\cdot,t)$ be its inverse mapping (see Section 2.2). For a function V on $\Gamma_0\times(0,T)$, we define a function $v=LV$ on S_T as

$$v(y,t) := V(\Phi^{-1}(y,t),t), \quad (y,t)\in S_T. \tag{2.4.3}$$

Also, for a function v on S_T, we define a function $V=L^{-1}v$ on $\Gamma_0\times(0,T)$ as

$$V(Y,t) := v(\Phi(Y,t),t), \quad (Y,t)\in\Gamma_0\times(0,T).$$

Clearly L and L^{-1} are linear mappings and satisfy $L^{-1}(LV)=V$ and $L(L^{-1}v)=v$ for all functions V on $\Gamma_0\times(0,T)$ and v on S_T.

Lemma 2.4.1. *The linear mapping L given by* (2.4.3) *defines an isomorphism between $\widehat{H}_T$ and H_T.*

A short proof is given in [19]. We give a detailed proof in Appendix 2.B for the completeness.

Let $C_0^1(S_T)$ be the space of all functions in $C^1(S_T)$ with compact support in S_T. That is, each function in $C_0^1(S_T)$ vanishes near $t=0$ and $t=T$.

Lemma 2.4.2. *The space H_T is a Hilbert space and $C_0^1(S_T)$ is dense in H_T.*

Proof. Since $\widehat{H}_T$ is a Hilbert space, Lemma 2.4.1 implies that H_T is a Hilbert space. Also, since $C_0^1(\Gamma_0 \times (0,T))$ is dense in $\widehat{H}_T$ (see [19, Lemma 3.1]) and $C_0^1(S_T)$ includes $L[C_0^1(\Gamma_0 \times (0,T))]$, Lemma 2.4.1 again implies that $C_0^1(S_T)$ is dense in H_T. □

The space H_T is continuously embedded into $L^2(S_T)$. Moreover, H_T is dense in $L^2(S_T)$ since it includes a dense subspace $C_0^1(S_T)$ of $L^2(S_T)$. Hence we have continuous and dense embeddings $H_T \hookrightarrow L^2(S_T) \hookrightarrow H_T'$, where H_T' is the dual space of H_T.

For $v \in C^1(S_T)$, we define its (strong) material derivative $\partial^\bullet v$ as

$$\partial^\bullet v(\Phi(Y,t),t) := \frac{d}{dt}(v(\Phi(Y,t),t)), \quad (Y,t) \in \Gamma_0 \times (0,T). \tag{2.4.4}$$

From the Leibniz formula (see [4, Lemma 2.2])

$$\frac{d}{dt}\int_{\Gamma(t)} v\,d\mathcal{H}^{n-1} = \int_{\Gamma(t)} (\partial^\bullet v + v\,\mathrm{div}_{\Gamma(t)} V_\Gamma)\,d\mathcal{H}^{n-1}, \quad v \in C^1(S_T),$$

we have the integration by parts identity

$$\begin{aligned}\int_0^T\int_{\Gamma(t)} &(v_2\partial^\bullet v_1 + v_1\partial^\bullet v_2 + v_1v_2\,\mathrm{div}_{\Gamma(t)} V_\Gamma)\,d\mathcal{H}^{n-1}\,dt \\ &= \int_{\Gamma(T)} v_1(T)v_2(T)\,d\mathcal{H}^{n-1} - \int_{\Gamma(0)} v_1(0)v_2(0)\,d\mathcal{H}^{n-1} \end{aligned}\tag{2.4.5}$$

for all $v_1, v_2 \in C^1(S_T)$. Based on this identity, we define the weak material derivative of $v \in H_T$ as a functional $\partial^\bullet v$ such that

$$\langle \partial^\bullet v, \psi\rangle_T := -\int_0^T\int_{\Gamma(t)} (v\,\partial^\bullet\psi + v\psi\,\mathrm{div}_{\Gamma(t)} V_\Gamma)\,d\mathcal{H}^{n-1}\,dt, \quad \psi \in C_0^1(S_T). \tag{2.4.6}$$

If $v \in C^1(S_T)$, then its weak material derivative agrees with the strong one. We set

$$\|\partial^\bullet v\|_{H_T'} := \sup_{\psi \in C_0^1(S_T)\setminus\{0\}} \frac{\langle \partial^\bullet v, \psi\rangle_T}{\|\psi\|_{H_T}}, \quad v \in H_T.$$

If $\|\partial^\bullet v\|_{H_T'}$ is finite for some $v \in H_T$, then $\partial^\bullet v$ can be extended to a bounded linear functional on H_T because $C_0^1(S_T)$ is dense in H_T (see Lemma 2.4.2). In this case, we say that $\partial^\bullet v$ is in H_T' and we define a function space W_T and its norm as

$$W_T := \{v \in H_T \mid \partial^\bullet v \in H_T'\}, \quad \|v\|_{W_T} := \left(\|v\|_{H_T}^2 + \|\partial^\bullet v\|_{H_T'}^2\right)^{1/2}. \tag{2.4.7}$$

For $T_1 < T_2$, the space W_{T_2} is continuously embedded into W_{T_1} since $C_0^1(S_{T_1}) \subset C_0^1(S_{T_2})$ and H_{T_2} is continuously embedded into H_{T_1}.

To investigate properties of W_T, we define an auxiliary Hilbert space and its norm as

$$\widehat{W}_T := \{V \in \widehat{H}_T \mid \partial_t V \in \widehat{H}_T'\}, \quad \|V\|_{\widehat{W}_T} := \left(\|V\|_{\widehat{H}_T}^2 + \|\partial_t V\|_{\widehat{H}_T'}^2\right)^{1/2}.$$

Here $\widehat{H}_T'$ is the dual space of $\widehat{H}_T$ and $\partial_t V$ is the weak time derivative of $V \in \widehat{H}_T$ defined as

$$[\partial_t V, \Psi]_T := -\int_0^T\int_{\Gamma_0} V\,\partial_t\Psi\,d\mathcal{H}^{n-1}\,dt, \quad \Psi \in C_0^1(\Gamma_0 \times (0,T)),$$

and we say $\partial_t V \in \widehat{H}_T'$ if $\|\partial_t V\|_{\widehat{H}_T'} := \sup_{\Psi \in C_0^1(\Gamma_0\times(0,T))\setminus\{0\}} [\partial_t V, \Psi]_T / \|\Psi\|_{\widehat{H}_T}$ is finite.

Lemma 2.4.3. *The linear mapping L given by* (2.4.3) *defines an isomorphism between $\widehat{W_T}$ and W_T.*

A proof for the three-dimensional case is given in [19] and easily extended for arbitrary dimensions. We give a complete proof in Appendix 2.B for the readers' convenience.

Lemma 2.4.3 shows that W_T has similar properties to those of $\widehat{W_T}$.

Lemma 2.4.4. *The space W_T is a Hilbert space and $C^1(S_T)$ is dense in W_T. Moreover, the trace operator $v \mapsto v(t)$ from $C^1(S_T)$ into $L^2(\Gamma(t))$ for each $t \in [0,T]$ can be extended to a bounded linear operator from W_T to $L^2(\Gamma(t))$ and there exists a constant $c > 0$ such that*

$$\max_{t\in[0,T]} \|v(t)\|_{L^2(\Gamma(t))} \leq c\|v\|_{W_T}$$

for all $v \in W_T$.

Proof. Since $\widehat{W_T}$ is a Hilbert space, Lemma 2.4.3 implies that W_T is a Hilbert space. For the rest of the proof, see [19, Theorem 3.6]. □

Finally we show an integration by parts formula which we will use in Section 2.6.

Lemma 2.4.5. *For all $v_1, v_2 \in W_T$, we have*

$$\begin{aligned}\langle \partial^\bullet v_1, v_2\rangle_T + \langle \partial^\bullet v_2, v_1\rangle_T + \int_0^T \int_{\Gamma(t)} v_1 v_2 \operatorname{div}_{\Gamma(t)} V_\Gamma \, d\mathcal{H}^{n-1}\, dt \\ = \int_{\Gamma(T)} v_1(T)v_2(T)\, d\mathcal{H}^{n-1} - \int_{\Gamma_0} v_1(0)v_2(0)\, d\mathcal{H}^{n-1}. \quad (2.4.8)\end{aligned}$$

Note that, by Lemma 2.4.4, $v_i(0)$ and $v_i(T)$, $i = 1, 2$, are well-defined as functions in $L^2(\Gamma_0)$ and $L^2(\Gamma(T))$, respectively.

Proof. For $v \in C^1(S_T)$, its weak material derivative agrees with the strong one. Thus we have

$$\langle \partial^\bullet v, \psi\rangle_T = \int_0^T \int_{\Gamma(t)} (\partial^\bullet v)\psi \, d\mathcal{H}^{n-1}\, dt, \quad \psi \in C_0^1(S_T).$$

Moreover, since $C_0^1(S_T)$ is dense in H_T (see Lemma 2.4.2), the above equality holds for all $\psi \in H_T$ and thus (2.4.8) follows from (2.4.5) when $v_1, v_2 \in C^1(S_T)$. Since $C^1(S_T)$ is dense in W_T (see Lemma 2.4.4), a density argument shows that (2.4.8) holds for general $v_1, v_2 \in W_T$. □

2.5 Average operator

2.5.1 Definition and basic properties of the average operator

In this section we define and investigate a weighted average operator. Lemma 2.5.6 and Lemma 2.5.13 are fundamental to derivation of a limit equation of (H_ε) in Section 2.6. Other results in this section are also useful themselves.

For $(y,t) \in \overline{S_T}$, let $\kappa_1(y,t), \dots, \kappa_{n-1}(y,t)$ be the principal curvatures of $\Gamma(t)$ at y (see [11, Section 14.6]). We define a function J on $\overline{S_T} \times (-\delta, \delta)$ as

$$J(y,t,\rho) := \prod_{i=1}^{n-1}\{1 - \rho\kappa_i(y,t)\}, \quad (y,t) \in \overline{S_T},\ \rho \in (-\delta, \delta).$$

Here $\delta > 0$ is a half of the width of the tubular neighborhood $N(t)$ of $\Gamma(t)$, which is independent of $t \in [0,T]$ (see Section 2.2). The function J is the Jacobian appearing in the transformation formula

$$\int_{\Omega_\varepsilon(t)} u(x)\,dx = \int_{\Gamma(t)} \int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y+\rho\nu(y,t))J(y,t,\rho)\,d\rho\,d\mathcal{H}^{n-1}(y) \tag{2.5.1}$$

for a function u on $\Omega_\varepsilon(t)$ with each fixed $t \in [0,T]$, see (14.98) in [11]. This formula can be viewed as a co-area formula. Based on this formula, we define a weighted average operator M_ε as follows.

Definition 2.5.1. For a function u on $Q_{\varepsilon,T}$, we define its weighted average $M_\varepsilon u$ as

$$M_\varepsilon u(y,t) := \frac{1}{\varepsilon g(y,t)} \int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y+\rho\nu(y,t),t)J(y,t,\rho)\,d\rho, \quad (y,t) \in S_T. \tag{2.5.2}$$

We use the same notation $M_\varepsilon u$ for the average of a function u on $\Omega_\varepsilon(t)$ with each fixed $t \in [0,T]$.

Before starting to derive properties of the average operator, we give inequalities which we use throughout Section 2.5 and Section 2.6. Since $\kappa_1,\dots,\kappa_{n-1}$ are smooth on $\overline{S_T}$, they are bounded on $\overline{S_T}$ along with their derivatives. Hence, by taking $\delta > 0$ sufficiently small, we may assume that there exists a constant $c > 0$ such that

$$c^{-1} \leq 1-\rho\kappa_i(y,t) \leq c \quad \text{for all} \quad (y,t) \in \overline{S_T},\ \rho \in (-\delta,\delta),\ i=1,\dots,n-1. \tag{2.5.3}$$

Then J is smooth and bounded on $\overline{S_T} \times (-\delta,\delta)$ along with its derivatives and satisfies

$$c^{-1} \leq J(y,t,\rho) \leq c \quad \text{for all} \quad (y,t) \in \overline{S_T},\ \rho \in (-\delta,\delta). \tag{2.5.4}$$

Moreover, since $J(y,t,\rho)$ is of the form

$$J(y,t,\rho) = 1-\rho\sum_{i=1}^{n-1}\kappa_i(y,t) + \rho^2 P(\kappa_1(y,t),\dots,\kappa_{n-1}(y,t),\rho),$$

where $P(z)$ is a polynomial in $z=(z_1,\dots,z_n) \in \mathbb{R}^n$, we have

$$|1-J(y,t,\rho)| \leq c\varepsilon, \quad |\nabla_{\Gamma(t)}J(y,t,\rho)| \leq c\varepsilon, \quad \left|\frac{\partial J}{\partial\rho}(y,t,\rho)\right| \leq c \tag{2.5.5}$$

for all $(y,t) \in \overline{S_T}$ and $\rho \in (\varepsilon g_0(y,t), \varepsilon g_1(y,t))$ with a constant $c > 0$ independent of ε.

Now let us derive properties of the average operator M_ε. For a function u on $Q_{\varepsilon,T}$, we set

$$u^\sharp(y,t,\rho) := u(y+\rho\nu(y,t),t), \quad (y,t) \in S_T,\ \rho \in (\varepsilon g_0(y,t), \varepsilon g_1(y,t)). \tag{2.5.6}$$

For simplicity, we omit arguments of functions unless we need to specify them. For example, the co-area formula (2.5.1) is referred to as

$$\int_{\Omega_\varepsilon(t)} u\,dx = \int_{\Gamma(t)} \int_{\varepsilon g_0}^{\varepsilon g_1} u^\sharp J\,d\rho\,d\mathcal{H}^{n-1}.$$

Throughout the rest of this subsection and the next subsection, we fix $t \in [0,T]$ and omit it. For example, we refer to $\Gamma(t)$ as Γ. Also, let c denote a general positive constant independent of t.

Lemma 2.5.2. *If $v \in L^2(\Gamma)$, then its constant extension $\overline{v}$ in the normal direction of Γ is in $L^2(\Omega_\varepsilon)$. Moreover, there exists a constant $c > 0$ independent of ε such that*

$$\|\overline{v}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{1/2}\|v\|_{L^2(\Gamma)}. \tag{2.5.7}$$

Proof. By the co-area formula (2.5.1) and (2.5.4),

$$\|\overline{v}\|^2_{L^2(\Omega_\varepsilon)} = \int_\Gamma \int_{\varepsilon g_1}^{\varepsilon g_0} |v|^2 J\,d\rho\,d\mathcal{H}^{n-1} \leq c\int_\Gamma \varepsilon g|v|^2\,d\mathcal{H}^{n-1} \leq c\varepsilon\|v\|^2_{L^2(\Gamma)}.$$

Thus (2.5.7) follows. □

Lemma 2.5.3. *If $u \in L^2(\Omega_\varepsilon)$, then $M_\varepsilon u \in L^2(\Gamma)$ and*

$$\|M_\varepsilon u\|_{L^2(\Gamma)} \leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)} \tag{2.5.8}$$

with a constant $c > 0$ independent of ε.

Proof. By Hölder's inequality, (2.5.4), (2.2.1), and the co-area formula (2.5.1),

$$\begin{aligned}\int_\Gamma |M_\varepsilon u|^2\,d\mathcal{H}^{n-1} &\leq \int_\Gamma (\varepsilon g)^{-2}\left(\int_{\varepsilon g_0}^{\varepsilon g_1} J\,d\rho\right)\left(\int_{\varepsilon g_0}^{\varepsilon g_1} |u^\sharp|^2 J\,d\rho\right)d\mathcal{H}^{n-1} \\ &\leq c\int_\Gamma (\varepsilon g)^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1} |u^\sharp|^2 J\,d\rho\,d\mathcal{H}^{n-1} \leq c\varepsilon^{-1}\int_{\Omega_\varepsilon} |u|^2\,dx.\end{aligned}$$

Thus (2.5.8) follows. □

By Lemma 2.5.2 and Lemma 2.5.3, the constant extension of $M_\varepsilon u$ in the normal direction of Γ is in $L^2(\Omega_\varepsilon)$ for all $u \in L^2(\Omega_\varepsilon)$. Let us estimate the difference between u and $M_\varepsilon u$ in $L^2(\Omega_\varepsilon)$.

Lemma 2.5.4. *There exists a constant $c > 0$ independent of ε such that*

$$\left\|u - \overline{M_\varepsilon u}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{H^1(\Omega_\varepsilon)} \tag{2.5.9}$$

for all $u \in H^1(\Omega_\varepsilon)$. Here $\overline{M_\varepsilon u}$ is the constant extension of $M_\varepsilon u$ in the normal direction of Γ.

Proof. For $y \in \Gamma$ and $\rho \in (\varepsilon g_0(y), \varepsilon g_1(y))$, we set

$$\begin{aligned}I_1(y,\rho) &= \frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \{u^\sharp(y,\rho) - u^\sharp(y,r)\}\,dr, \\ I_2(y) &= \frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} u^\sharp(y,r)\{1 - J(y,r)\}\,dr.\end{aligned}$$

Then we have $u^\sharp(y,\rho) - M_\varepsilon u(y) = I_1(y,\rho) + I_2(y)$. Let us estimate I_1 and I_2. Since

$$\begin{aligned}|u^\sharp(y,\rho) - u^\sharp(y,r)| &= \left|\int_r^\rho \frac{d}{d\eta}(u(y + \eta\nu(y)))\,d\eta\right| \\ &= \left|\int_r^\rho \nu(y)\cdot\nabla u(y + \eta\nu(y))\,d\eta\right| \leq \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\nabla u)^\sharp(y,\eta)|\,d\eta\end{aligned}$$

for all $\rho, r \in (\varepsilon g_0(y), \varepsilon g_1(y))$ and the right-hand side of the above inequality is independent of r,

$$|I_1(y,\rho)| \leq \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\nabla u)^\sharp(y,\eta)|\,d\eta.$$

On the other hand, by (2.2.1) and (2.5.5) we have

$$|I_2(y)| \leq c\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |u^\sharp(y,r)|\,dr.$$

These inequalities and Hölder's inequality yield

$$\begin{aligned}
|u^\sharp(y,\rho) - M_\varepsilon u(y)| &\leq |I_1(y,\rho)| + |I_2(y)| \leq c\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u^\sharp(y,r)| + |(\nabla u)^\sharp(y,r)|)\,dr \\
&\leq c\left(\varepsilon g(y)\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u^\sharp(y,r)|^2 + |(\nabla u)^\sharp(y,r)|^2)\,dr\right)^{1/2}.
\end{aligned}$$

Here the last term is independent of ρ. Hence by the co-area formula (2.5.1) and (2.5.4) we obtain

$$\begin{aligned}
\left\|u - \overline{M_\varepsilon u}\right\|_{L^2(\Omega_\varepsilon)}^2 &= \int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |u^\sharp(y,\rho) - M_\varepsilon u(y)|^2 J(y,\rho)\,d\rho\,d\mathcal{H}^{n-1}(y) \\
&\leq c\int_\Gamma \{\varepsilon g(y)\}^2 \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u^\sharp(y,r)|^2 + |(\nabla u)^\sharp(y,r)|^2)\,dr\,d\mathcal{H}^{n-1}(y) \\
&\leq c\varepsilon^2 \int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u^\sharp(y,r)|^2 + |(\nabla u)^\sharp(y,r)|^2) J(y,r)\,dr\,d\mathcal{H}^{n-1}(y) \\
&= c\varepsilon^2 \|u\|_{H^1(\Omega_\varepsilon)}^2.
\end{aligned}$$

Thus (2.5.9) follows. □

2.5.2 Tangential gradient of the average operator

In this subsection, we investigate relations between the usual gradient operator in Ω_ε and the tangential gradient operator on Γ. We first establish estimates for the gradient of the constant extension of a function on Γ in the normal direction of Γ.

Lemma 2.5.5. *If $v \in H^1(\Gamma)$, then its constant extension $\overline{v}$ in the normal direction of Γ is in $H^1(\Omega_\varepsilon)$. Moreover, there exists a constant $c > 0$ independent of ε such that*

$$\|\nabla \overline{v}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{1/2}\|\nabla_\Gamma v\|_{L^2(\Gamma)}, \quad \left\|\nabla \overline{v} - \overline{\nabla_\Gamma v}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{3/2}\|\nabla_\Gamma v\|_{L^2(\Gamma)}. \tag{2.5.10}$$

Proof. The first inequality of (2.5.10) and Lemma 2.5.2 imply $\overline{v} \in H^1(\Omega_\varepsilon)$ for all $v \in H^1(\Gamma)$. The inequalities (2.5.10) follow from the co-area formula (2.5.1), (2.5.4), and the inequalities

$$|\nabla \overline{v}(y + \rho\nu(y))| \leq c|\nabla_\Gamma v(y)|, \quad |\nabla \overline{v}(y + \rho\nu(y)) - \nabla_\Gamma v(y)| \leq c\varepsilon|\nabla_\Gamma v(y)| \tag{2.5.11}$$

for all $y \in \Gamma$ and $\rho \in (\varepsilon g_0(y), \varepsilon g_1(y))$. We prove (2.5.11) in Appendix 2.C. Here we give the main idea for the proof. We fix each $y_0 \in \Gamma$. By a rotation of coordinates, we can take a smooth function $f\colon U \to \mathbb{R}$ with an open set U in $\mathbb{R}^{n-1}$ such that Γ is described as the graph of f near y_0 and

$$\nabla' f(s_0) = 0, \quad (\nabla')^2 f(s_0) = \mathrm{diag}[\kappa_1(y_0), \dots, \kappa_{n-1}(y_0)],$$

where $y_0 = (s_0, f(s_0))$ with $s_0 \in U$ and ∇' is the gradient in $s \in \mathbb{R}^{n-1}$ (see [11, Section 14.6]). Then (2.5.11) at y_0 is proved by direct calculations under this local coordinate system. $\square$

Next we approximate an H^1-bilinear form on Ω_ε by that on Γ with the tangential gradient of the weighted average of a function on Ω_ε.

Lemma 2.5.6. *For $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, let*

$$I_\varepsilon^1(u,\varphi) := \int_{\Omega_\varepsilon} \nabla u \cdot \nabla \overline{\varphi}\, dx - \varepsilon \int_\Gamma g \nabla_\Gamma M_\varepsilon u \cdot \nabla_\Gamma \varphi \, d\mathcal{H}^{n-1}. \tag{2.5.12}$$

Then there exists a constant $c > 0$ independent of u, φ, and ε such that

$$|I_\varepsilon^1(u,\varphi)| \leq c\varepsilon^{3/2} \|u\|_{H^1(\Omega_\varepsilon)} \|\nabla_\Gamma \varphi\|_{L^2(\Gamma)}. \tag{2.5.13}$$

Remark 2.5.7. The bilinear form $I_\varepsilon^1(u,\varphi)$ is well-defined for $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, since $\overline{\varphi} \in H^1(\Omega_\varepsilon)$ by Lemma 2.5.5 and $M_\varepsilon u$ is smooth on Γ and thus in $H^1(\Gamma)$ by the compactness of Γ. We will observe later that $I_\varepsilon^1(u,\varphi)$ is well-defined and (2.5.13) holds for all $u \in H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$, see Remark 2.5.9.

Proof of Lemma 2.5.6. By (2.5.1) we have $I_\varepsilon^1(u,\varphi) = \int_\Gamma I(y)\, d\mathcal{H}^{n-1}(y)$, where

$$I(y) := \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (\nabla u)^\sharp(y,\rho) \cdot (\nabla \overline{\varphi})^\sharp(y,\rho) J(y,\rho)\, d\rho - \varepsilon g(y) \nabla_\Gamma M_\varepsilon u(y) \cdot \nabla_\Gamma \varphi(y).$$

Here we used the notation (2.5.6). Suppose that there is a constant $c > 0$ independent of ε such that

$$|I(y)| \leq c\varepsilon |\nabla_\Gamma \varphi(y)| \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u^\sharp(y,\rho)| + |(\nabla u)^\sharp(y,\rho)|)\, d\rho \tag{2.5.14}$$

for all $y \in \Gamma$. Then, by (2.5.14), Hölder's inequality, and (2.5.4) we have

$$\begin{aligned}
|I_\varepsilon^1(u,\varphi)| &\leq c\varepsilon \int_\Gamma |\nabla_\Gamma \varphi| \int_{\varepsilon g_0}^{\varepsilon g_1} (|u^\sharp| + |(\nabla u)^\sharp|)\, d\rho\, d\mathcal{H}^{n-1} \\
&\leq c\varepsilon \left(\int_\Gamma |\nabla_\Gamma \varphi|^2 \, d\mathcal{H}^{n-1} \right)^{1/2} \left\{ \int_\Gamma \left(\int_{\varepsilon g_0}^{\varepsilon g_1} (|u^\sharp| + |(\nabla u)^\sharp|)\, d\rho \right)^2 d\mathcal{H}^{n-1} \right\}^{1/2} \\
&\leq c\varepsilon \|\nabla_\Gamma \varphi\|_{L^2(\Gamma)} \left(\int_\Gamma \varepsilon g \int_{\varepsilon g_0}^{\varepsilon g_1} (|u^\sharp|^2 + |(\nabla u)^\sharp|^2) J\, d\rho\, d\mathcal{H}^{n-1} \right)^{1/2} \\
&\leq c\varepsilon^{3/2} \|\nabla_\Gamma \varphi\|_{L^2(\Gamma)} \|u\|_{H^1(\Omega_\varepsilon)}.
\end{aligned}$$

Hence (2.5.13) holds. The inequality (2.5.14) is proved by direct calculations under the local coordinate system we took in the proof of Lemma 2.5.5. We give a complete proof in Appendix 2.C. $\square$

Lemma 2.5.6 gives an estimate for the $L^2(\Gamma)$-norm of $\nabla_\Gamma M_\varepsilon u$ for $u \in H^1(\Omega_\varepsilon)$.

Lemma 2.5.8. *If $u \in H^1(\Omega_\varepsilon)$, then $M_\varepsilon u \in H^1(\Gamma)$ and*

$$\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)} \leq c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)} \tag{2.5.15}$$

with a constant $c > 0$ independent of ε.

Proof. First, we show (2.5.15) for all $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$. For such u, its average $M_\varepsilon u$ is smooth on Γ and thus in $H^1(\Gamma)$ by the compactness of Γ. We substitute $M_\varepsilon u$ for φ in (2.5.12), (2.5.13) to get

$$\int_\Gamma g|\nabla_\Gamma M_\varepsilon u|^2 \, d\mathcal{H}^{n-1} = \varepsilon^{-1}\left(\int_{\Omega_\varepsilon} \nabla u \cdot \nabla \overline{M_\varepsilon u} \, dx - I_\varepsilon^1(u, M_\varepsilon u)\right),$$
$$|I_\varepsilon^1(u, M_\varepsilon u)| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}.$$

Hence, by (2.2.1), Hölder's inequality, and (2.5.10) we obtain

$$\begin{aligned}
\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}^2 &\leq c\int_\Gamma g|\nabla_\Gamma M_\varepsilon u|^2 \, d\mathcal{H}^{n-1} \\
&\leq c\varepsilon^{-1}\left(\|\nabla u\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\overline{M_\varepsilon u}\right\|_{L^2(\Omega_\varepsilon)} + |I_\varepsilon^1(u, M_\varepsilon u)|\right) \\
&\leq c\varepsilon^{-1}(\varepsilon^{1/2} + \varepsilon^{3/2})\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)} \\
&\leq c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla_\Gamma M_\varepsilon u\|_{L^2(\Gamma)}
\end{aligned}$$

and thus (2.5.15) follows when $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$. Since Ω_ε is bounded, $C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$, see [10, Section 5.3.2] for the proof. Hence a density argument together with Lemma 2.5.3 yields that $M_\varepsilon u \in H^1(\Gamma)$ and (2.5.15) holds for all $u \in H^1(\Omega_\varepsilon)$. □

Remark 2.5.9. By Lemma 2.5.5 and Lemma 2.5.8, the bilinear form $I_\varepsilon^1(u, \varphi)$ given by (2.5.12) is well-defined for all $u \in H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$. Moreover, since $C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$ is dense in $H^1(\Omega_\varepsilon)$, a density argument implies that (2.5.13) also holds for all $u \in H^1(\Omega_\varepsilon)$ and $\varphi \in H^1(\Gamma)$.

2.5.3 Material derivative of the average operator

Now let us return to the evolving surface $\Gamma(t)$. Recall the function spaces $L^2_{H^1(\varepsilon)}$ and H_T given by (2.3.1) and (2.4.1), respectively. By Lemma 2.5.3 and Lemma 2.5.8 we immediately get the next lemma.

Lemma 2.5.10. *If $u \in L^2_{H^1(\varepsilon)}$, then $M_\varepsilon u \in H_T$ and*

$$\|M_\varepsilon u\|_{H_T} \leq c\varepsilon^{-1/2}\|u\|_{L^2_{H^1(\varepsilon)}}$$

with a constant $c > 0$ independent of ε.

Lemma 2.5.10 enables us to consider the weak material derivative of $M_\varepsilon u \in H_T$ for $u \in L^2_{H^1(\varepsilon)}$. Our goal in this subsection is to give a relation between the weak time derivative of u and the weak material derivative of $M_\varepsilon u$. To this end, we show an auxiliary statement about the material derivative of a function on S_T.

Lemma 2.5.11. *Let $\varphi \in C^1(S_T)$ and $\overline{\varphi}$ be its constant extension in the normal direction of $\Gamma(t)$. Then*

$$\partial^\bullet \varphi(p(x,t),t) = \partial_t \overline{\varphi}(x,t) + \{V_\Gamma(p(x,t),t) + a(x,t)\} \cdot \nabla \overline{\varphi}(x,t) \tag{2.5.16}$$

holds for all $(x,t) \in N_T$ with a vector field $a : N_T \to \mathbb{R}^n$ given by

$$a(x,t) := d(x,t)\{\partial_t \nu(p(x,t),t) + \nabla \nu(p(x,t),t) V_\Gamma(p(x,t),t)\}. \tag{2.5.17}$$

Here $\nabla \nu := (\partial \nu_i / \partial x_j)_{i,j}$ is the gradient matrix of ν.

Proof. For $X \in N(0)$ and $t \in (0,T)$ we set

$$\Psi(X,t) := \Phi(p(X,0),t) + d(X,0)\nu(\Phi(p(X,0),t),t),$$

where $\Phi(\cdot,t) \colon \Gamma_0 \to \Gamma(t)$ is the flow map of V_Γ (see Section 2.2). By the definition of the constant extension $\overline{\varphi}$ and the formula $p(\Psi(X,t),t) = \Phi(p(X,0),t)$ we have

$$\overline{\varphi}(\Psi(X,t),t) = \varphi(\Phi(p(X,0),t),t)$$

for all $X \in N(0)$ and $t \in (0,T)$. Differentiating both sides with respect to t and observing that each $x \in N(t)$ is represented as $x = \Psi(X,t)$ with a unique $X \in N(0)$, we get the formula (2.5.16). For detailed calculations, see Appendix 2.C. $\square$

Remark 2.5.12. Let $\varphi \in C^1(S_T)$. Since $p(y,t) = y$ and $d(y,t) = 0$ for all $(y,t) \in S_T$, we have

$$\partial^\bullet \varphi = \partial_t \overline{\varphi} + V_\Gamma \cdot \nabla \overline{\varphi} = \partial_t \overline{\varphi} + v_\Gamma^N \nu \cdot \nabla \overline{\varphi} + V_\Gamma^T \cdot \nabla_{\Gamma(t)} \overline{\varphi} \quad \text{on} \quad S_T$$

by Lemma 2.5.11. Here the last equality follows from the fact that V_Γ^T is tangent to $\Gamma(t)$. Based on this equality, the material derivative operator acting on functions on $\Gamma(t)$ is formally represented as $\partial^\bullet = \partial_t + V_\Gamma \cdot \nabla = \partial_t + v_\Gamma^N \nu \cdot \nabla + V_\Gamma^T \cdot \nabla_{\Gamma(t)}$.

Using Lemma 2.5.11, we derive an integral formula related to the weak time derivative of a function $u \in L^2_{H^1(\varepsilon)}$ and the weak material derivative of its average $M_\varepsilon u \in H_T$.

Lemma 2.5.13. *Let $u \in L^2_{H^1(\varepsilon)}$, $\varphi \in C_0^1(S_T)$, and $\overline{\varphi}$ be its constant extension. Then we have*

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon(t)} u \, \partial_t \overline{\varphi} \, dx \, dt &= -\varepsilon \langle \partial^\bullet M_\varepsilon u, g\varphi \rangle_T - \varepsilon \int_0^T \int_{\Gamma(t)} (\partial^\bullet g + g \operatorname{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u)\varphi \, d\mathcal{H}^{n-1} \, dt \\ &\quad - \varepsilon \int_0^T \int_{\Gamma(t)} g(M_\varepsilon u) V_\Gamma^T \cdot \nabla_{\Gamma(t)} \varphi \, d\mathcal{H}^{n-1} \, dt + I_\varepsilon^2(u,\varphi;T). \end{aligned} \tag{2.5.18}$$

Here $I_\varepsilon^2(u,\varphi;T)$ is a residual term that satisfies

$$|I_\varepsilon^2(u,\varphi;T)| \leq c\varepsilon^{3/2} \int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))} \|\nabla_{\Gamma(t)} \varphi(t)\|_{L^2(\Gamma(t))} \, dt \tag{2.5.19}$$

with a constant $c > 0$ independent of u, φ, and ε.

Note that the tangential velocity V_Γ^T appears instead of the total velocity V_Γ in the third term of the right-hand side of (2.5.18), see Remark 2.5.15 below.

Proof. By (2.5.16), we have $\overline{\partial^\bullet \varphi} = \partial_t \overline{\varphi} + (\overline{V_\Gamma} + a) \cdot \nabla \overline{\varphi}$ on N_T, where a is the vector field on N_T given by (2.5.17). Hence if we set

$$I_\varepsilon^2(u,\varphi;T) := -\int_0^T \int_{\Omega_\varepsilon(t)} u \left\{ a \cdot \nabla \overline{\varphi} + \overline{V_\Gamma} \cdot \left(\nabla \overline{\varphi} - \overline{\nabla_{\Gamma(t)} \varphi} \right) \right\} dx\, dt,$$

then we have

$$\int_0^T \int_{\Omega_\varepsilon(t)} u\, \partial_t \overline{\varphi}\, dx\, dt = \int_0^T \int_{\Omega_\varepsilon(t)} u \Big(\overline{\partial^\bullet \varphi} - \overline{V_\Gamma} \cdot \overline{\nabla_{\Gamma(t)} \varphi} \Big)\, dx\, dt + I_\varepsilon^2(u,\varphi;T). \tag{2.5.20}$$

Let us compute the first term of the right-hand side of (2.5.20). By the co-area formula (2.5.1) and the definition of the weighted average $M_\varepsilon u$ (see (2.5.2)),

$$\begin{aligned} \int_{\Omega_\varepsilon(t)} u(x,t) \overline{\partial^\bullet \varphi}(x,t)\, dx &= \int_{\Gamma(t)} \int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y+\rho\nu(y,t),t) \partial^\bullet \varphi(y,t) J(y,t,\rho)\, d\rho\, d\mathcal{H}^{n-1}(y) \\ &= \varepsilon \int_{\Gamma(t)} g(y,t) M_\varepsilon u(y,t) \partial^\bullet \varphi(y,t)\, d\mathcal{H}^{n-1}(y) \end{aligned}$$

for all $t \in (0,T)$. On the other hand, since the weak material derivative is given by (2.4.6),

$$\begin{aligned} \langle \partial^\bullet M_\varepsilon u, g\varphi \rangle_T &= -\int_0^T \int_{\Gamma(t)} \{ (M_\varepsilon u) \partial^\bullet (g\varphi) + (M_\varepsilon u) g\varphi \operatorname{div}_{\Gamma(t)} V_\Gamma \}\, d\mathcal{H}^{n-1}\, dt \\ &= -\int_0^T \int_{\Gamma(t)} \{ (\partial^\bullet g + g \operatorname{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u)\varphi + g(M_\varepsilon u)\partial^\bullet \varphi \}\, d\mathcal{H}^{n-1}\, dt. \end{aligned}$$

Thus it follows that

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon(t)} u\, \overline{\partial^\bullet \varphi}\, dx\, dt &= -\varepsilon \langle \partial^\bullet M_\varepsilon u, g\varphi \rangle_T \\ &\qquad - \varepsilon \int_0^T \int_{\Gamma(t)} (\partial^\bullet g + g \operatorname{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u) \varphi\, d\mathcal{H}^{n-1}\, dt. \end{aligned} \tag{2.5.21}$$

Since $V_\Gamma = v_\Gamma^N \nu + V_\Gamma^T$ and $\nu \cdot \nabla_{\Gamma(t)} \varphi = 0$, we have $V_\Gamma \cdot \nabla_{\Gamma(t)} \varphi = V_\Gamma^T \cdot \nabla_{\Gamma(t)} \varphi$ on S_T. This equality together with the co-area formula (2.5.1) yields

$$\begin{aligned} &\int_{\Omega_\varepsilon(t)} u(x,t) \overline{V_\Gamma}(x,t) \cdot \overline{\nabla_{\Gamma(t)} \varphi}(x,t)\, dx \\ &\qquad = \int_{\Gamma(t)} \int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y+\rho\nu(y,t),t) V_\Gamma(y,t) \cdot \nabla_{\Gamma(t)} \varphi(y,t) J(y,t,\rho)\, d\rho\, d\mathcal{H}^{n-1}(y) \\ &\qquad = \varepsilon \int_{\Gamma(t)} g(y,t) M_\varepsilon u(y,t) V_\Gamma^T(y,t) \cdot \nabla_{\Gamma(t)} \varphi(y,t)\, d\mathcal{H}^{n-1}(y) \end{aligned}$$

for all $t \in (0,T)$ and thus

$$\int_0^T \int_{\Omega_\varepsilon(t)} u \Big(\overline{V_\Gamma} \cdot \overline{\nabla_{\Gamma(t)} \varphi} \Big)\, dx\, dt = \varepsilon \int_0^T \int_{\Gamma(t)} g(M_\varepsilon u) V_\Gamma^T \cdot \nabla_{\Gamma(t)} \varphi\, d\mathcal{H}^{n-1}\, dt. \tag{2.5.22}$$

Substituting (2.5.21) and (2.5.22) for (2.5.20), we obtain the equality (2.5.18).

Let us show the inequality (2.5.19). In (2.5.17), the first-order derivatives of ν are bounded on N_T and V_Γ is bounded on S_T. Hence there is a constant $c > 0$ independent of ε such that

$$|a(x,t)| \leq c|d(x,t)| \leq c\varepsilon \max_{i=1,2} \sup_{(y,\tau)\in\overline{S_T}} |g_i(y,\tau)| \leq c\varepsilon$$

for all $(x,t) \in Q_{\varepsilon,T}$. By this inequality, Hölder's inequality, and (2.5.10) we obtain

$$\begin{aligned} |I_\varepsilon^2(u,\varphi;T)| &\leq c\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\left(\varepsilon\|\nabla\overline{\varphi}(t)\|_{L^2(\Omega_\varepsilon(t))} + \left\|\nabla\overline{\varphi}(t) - \overline{\nabla_{\Gamma(t)}\varphi(t)}\right\|_{L^2(\Omega_\varepsilon(t))}\right) dt \\ &\leq c\varepsilon^{3/2}\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt. \end{aligned}$$

Thus (2.5.19) holds. □

Remark 2.5.14. If $M_\varepsilon u$ is in the Hilbert space W_T given by (2.4.7), then the right-hand side of (2.5.18) is well-defined for $\varphi \in H_T$ since $C_0^1(S_T)$ is dense in H_T (see Lemma 2.4.2). In particular, we can substitute $M_\varepsilon u$ for φ in the right-hand side of (2.5.18). This fact is essential for derivation of the energy estimate for the weighted average of a variational solution to (H_ε) (see Lemma 2.6.4). If we replace M_ε in (2.5.18) by a usual unweighted average operator

$$\mathcal{M}_\varepsilon u(y,t) := \frac{1}{\varepsilon g(y,t)}\int_{\varepsilon g_0(y,t)}^{\varepsilon g_1(y,t)} u(y+\rho\nu(y,t),t)\,d\rho,$$

then the estimate for the residual term becomes

$$|I_\varepsilon^2(u,\varphi;T)| \leq c\varepsilon^{3/2}\int_0^T \|u(t)\|_{L^2(\Omega_\varepsilon(t))}\left(\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))} + \|\partial^\bullet\varphi(t)\|_{L^2(\Gamma(t))}\right) dt.$$

Because of the term $\|\partial^\bullet\varphi(t)\|_{L^2(\Gamma(t))}$ in the above inequality, the right-hand side of (2.5.18) with M_ε replaced by $\mathcal{M}_\varepsilon$ is not well-defined for $\varphi \in H_T$. Therefore we can not derive the energy estimate for the unweighted average of a variational solution to (H_ε).

Remark 2.5.15. Let $\Gamma \subset \mathbb{R}^n$ be a closed, connected, and oriented smooth hypersurface. Then, since $\partial\Gamma = \emptyset$, the integral formula (see [25, Section 7.2])

$$\int_\Gamma \mathrm{div}_\Gamma V\,d\mathcal{H}^{n-1} = -\int_\Gamma (V\cdot\nu)H\,d\mathcal{H}^{n-1}$$

holds for smooth vector fields $V\colon \Gamma \to \mathbb{R}^n$. Here ν is the unit outward normal vector of Γ and $H := -\mathrm{div}_\Gamma\nu$ is the mean curvature of Γ. This formula yields the equality

$$\int_\Gamma V\cdot\nabla_\Gamma\varphi\,d\mathcal{H}^{n-1} = -\int_\Gamma \{\mathrm{div}_\Gamma V + (V\cdot\nu)H\}\varphi\,d\mathcal{H}^{n-1}$$

for smooth functions φ on Γ. In this equality we decompose $V = v^N\nu + V^T$ into the normal component $v^N := V\cdot\nu$ and the tangential component $V^T := V - (V\cdot\nu)\nu$. Then, since

$$\nu\cdot\nabla_\Gamma\varphi = 0, \quad \mathrm{div}_\Gamma(v^N\nu) = \nabla_\Gamma v^N\cdot\nu + v^N\mathrm{div}_\Gamma\nu = 0 + v^N\cdot(-H) = -(V\cdot\nu)H,$$

we obtain a usual integration by parts formula

$$\int_\Gamma V^T\cdot\nabla_\Gamma\varphi\,d\mathcal{H}^{n-1} = -\int_\Gamma \varphi\,\mathrm{div}_\Gamma V^T\,d\mathcal{H}^{n-1}, \tag{2.5.23}$$

which we will use to recover a limit equation on $\Gamma(t)$ from its variational formulation. This is the reason the tangential velocity V_Γ^T appears in (2.5.18) instead of the total velocity V_Γ of $\Gamma(t)$.

2.6 Convergence and characterization of the limit

2.6.1 Variational formulations of the average of solutions to the heat equation

Let us return to the initial-boundary value problem (H_ε) of the heat equation. By Theorem 2.3.4, for every $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ there exists a unique variational solution $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ to (H_ε).

Let M_ε be the weighted average operator defined in Definition 2.5.1. Our goal in this subsection is to derive a variational formulation of $M_\varepsilon u^\varepsilon$.

Lemma 2.6.1. *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be the unique variational solution to (H_ε) given by Theorem 2.3.4. Then $M_\varepsilon u^\varepsilon \in H_T$ and it satisfies*

$$\begin{aligned}\langle \partial^\bullet M_\varepsilon u^\varepsilon, g\varphi\rangle_T &+ \int_0^T\int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u^\varepsilon)\varphi\, d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^T\int_{\Gamma(t)} g\{\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon + (M_\varepsilon u^\varepsilon)V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}\varphi\, d\mathcal{H}^{n-1}\,dt = I_\varepsilon(u^\varepsilon,\varphi;T) \quad (2.6.1)\end{aligned}$$

for all $\varphi \in C_0^1(S_T)$. Here $I_\varepsilon(u^\varepsilon,\varphi;T)$ is a residual term that satisfies

$$|I_\varepsilon(u^\varepsilon,\varphi;T)| \le c\varepsilon^{1/2}\int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt \quad (2.6.2)$$

with a constant $c>0$ independent of u^ε, φ, and ε.

Proof. Since $u^\varepsilon \in L^2_{H^1(\varepsilon)}$, we have $M_\varepsilon u^\varepsilon \in H_T$ by Lemma 2.5.10. For each $\varphi \in C_0^1(S_T)$, its constant extension $\overline{\varphi}$ is in $C^1(\overline{Q_{\varepsilon,T}})$ and satisfies $\overline{\varphi}(0)=0$ in $\Omega_\varepsilon(0)$ and $\overline{\varphi}(T)=0$ in $\Omega_\varepsilon(T)$. Thus, by substituting $\overline{\varphi}$ for w in the variational formulation (2.3.2) we obtain

$$\int_0^T\int_{\Omega_\varepsilon(t)} (-u^\varepsilon\partial_t\overline{\varphi} + \nabla u^\varepsilon\cdot\nabla\overline{\varphi})\,dx\,dt = 0. \quad (2.6.3)$$

Moreover, from Lemma 2.5.6 and Lemma 2.5.13 we have

$$\int_0^T\int_{\Omega_\varepsilon(t)} \nabla u^\varepsilon\cdot\nabla\overline{\varphi}\,dx\,dt = \varepsilon\int_0^T\int_{\Gamma(t)} g\nabla_{\Gamma(t)}M_\varepsilon u^\varepsilon\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt + I_\varepsilon^1(u^\varepsilon,\varphi;T) \quad (2.6.4)$$

and

$$\begin{aligned}\int_0^T\int_{\Omega_\varepsilon(t)} u^\varepsilon\partial_t\overline{\varphi}\,dx\,dt = -\varepsilon\langle\partial^\bullet M_\varepsilon u^\varepsilon, g\varphi\rangle_T &- \varepsilon\int_0^T\int_{\Gamma(t)}(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)(M_\varepsilon u^\varepsilon)\varphi\,d\mathcal{H}^{n-1}\,dt \\ &- \varepsilon\int_0^T\int_{\Gamma(t)} g(M_\varepsilon u^\varepsilon)V_\Gamma^T\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt + I_\varepsilon^2(u^\varepsilon,\varphi;T), \quad (2.6.5)\end{aligned}$$

where $I_\varepsilon^1(u^\varepsilon,\varphi;T)$ and $I_\varepsilon^2(u^\varepsilon,\varphi;T)$ satisfy

$$|I_\varepsilon^k(u^\varepsilon,\varphi;T)| \le c\varepsilon^{3/2}\int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt, \quad k=1,2, \quad (2.6.6)$$

with some constant $c>0$ independent of ε. Hence, by substituting (2.6.4) and (2.6.5) for (2.6.3) and dividing both sides by ε, we obtain (2.6.1) with the residual term

$$I_\varepsilon(u^\varepsilon,\varphi;T) := \varepsilon^{-1}\bigl\{I_\varepsilon^2(u^\varepsilon,\varphi;T) - I_\varepsilon^1(u^\varepsilon,\varphi;T)\bigr\},$$

which satisfies (2.6.2) because $I_\varepsilon^1(u^\varepsilon,\varphi;T)$ and $I_\varepsilon^2(u^\varepsilon,\varphi;T)$ satisfy (2.6.6). □

2.6.2 Estimates for the average $M_\varepsilon u^\varepsilon$ in the space W_T

In this subsection, we estimate $M_\varepsilon u^\varepsilon$ in the Hilbert space W_T given by (2.4.7).

Lemma 2.6.2. *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be the unique variational solution to (H_ε) given by Theorem 2.3.4. Then $M_\varepsilon u^\varepsilon \in W_T$ and there exists a constant $c > 0$ independent of u^ε and ε such that*

$$\|\partial^\bullet M_\varepsilon u^\varepsilon\|_{H_T'} \leq c\Big(\|M_\varepsilon u^\varepsilon\|_{H_T} + \varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\Big). \tag{2.6.7}$$

Proof. Let φ be an arbitrary function in $C_0^1(S_T)$. By substituting $g^{-1}\varphi \in C_0^1(S_T)$ for φ in (2.6.1), we obtain $\langle \partial^\bullet M_\varepsilon u^\varepsilon, \varphi\rangle_T = I(u^\varepsilon, \varphi) + I_\varepsilon(u^\varepsilon, g^{-1}\varphi; T)$, where

$$\begin{aligned} I(u^\varepsilon, \varphi) := \int_0^T \int_{\Gamma(t)} &\{g^{-1}(V_\Gamma^T \cdot \nabla_{\Gamma(t)} g - \partial^\bullet g) - \mathrm{div}_{\Gamma(t)} V_\Gamma\}(M_\varepsilon u^\varepsilon)\varphi \, d\mathcal{H}^{n-1}\, dt \\ &- \int_0^T \int_{\Gamma(t)} \{\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon + (M_\varepsilon u^\varepsilon) V_\Gamma^T\} \cdot \nabla_{\Gamma(t)}\varphi \, d\mathcal{H}^{n-1}\, dt \\ &+ \int_0^T \int_{\Gamma(t)} g^{-1}(\nabla_{\Gamma(t)} g \cdot \nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon)\varphi\, d\mathcal{H}^{n-1}\, dt. \end{aligned}$$

Since g and V_Γ are smooth on $\overline{S_T}$, they are bounded on S_T along with their derivatives. Moreover, g^{-1} and V_Γ^T are bounded on S_T. Thus we have $|I(u^\varepsilon, \varphi)| \leq c\|M_\varepsilon u^\varepsilon\|_{H_T}\|\varphi\|_{H_T}$ with a constant $c > 0$ independent of u^ε, φ, and ε. Also, by (2.6.2),

$$\begin{aligned} |I_\varepsilon(u^\varepsilon, g^{-1}\varphi; T)| &\leq c\varepsilon^{1/2}\int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}(g^{-1}\varphi)(t)\|_{L^2(\Gamma(t))}\, dt \\ &\leq c\varepsilon^{1/2}\int_0^T \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\Big(\|\varphi(t)\|_{L^2(\Gamma(t))} + \|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\Big)\, dt \\ &\leq c\varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\|\varphi\|_{H_T} \end{aligned}$$

with some $c > 0$ independent of u^ε, φ, and ε. Hence we obtain

$$|\langle \partial^\bullet M_\varepsilon u^\varepsilon, \varphi\rangle_T| \leq |I(u^\varepsilon, \varphi)| + |I_\varepsilon(u^\varepsilon, g^{-1}\varphi; T)| \leq c\Big(\|M_\varepsilon u^\varepsilon\|_{H_T} + \varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\Big)\|\varphi\|_{H_T}$$

for all $\varphi \in C_0^1(S_T)$, which implies $M_\varepsilon u^\varepsilon \in W_T$ and the inequality (2.6.7). □

Remark 2.6.3. Since $M_\varepsilon u^\varepsilon \in W_T$ and $C_0^1(S_T)$ is dense in H_T (see Lemma 2.4.2), the equality (2.6.1) also holds for all $\varphi \in H_T$. Moreover, since W_{T_1} is continuously embedded into W_{T_2} when $T_1 > T_2$, we have $M_\varepsilon u^\varepsilon \in W_\tau$ for each $\tau \in [0, T]$. Hence (2.6.1) and (2.6.2) with T replaced by each $\tau \in [0, T]$ are also valid for all $\varphi \in H_\tau$.

Lemma 2.6.4. *Let u_0^ε and u^ε be as in Lemma 2.6.2. Then there exists a constant $c > 0$ independent of u_0^ε, u^ε, and ε such that the energy estimate*

$$\begin{aligned} \|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\, dt \\ &\leq c\Big(\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + \varepsilon\|u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon(0))}\Big) \end{aligned} \tag{2.6.8}$$

holds for all $\tau \in [0, T]$.

Proof. As we mentioned in Remark 2.6.3, the equality (2.6.1) holds with T replaced by each $\tau \in [0,T]$. Hence, by substituting $g^{-1}M_\varepsilon u^\varepsilon \in H_\tau$ for φ in (2.6.1) with T replaced by τ, we obtain

$$\begin{aligned}\langle \partial^\bullet M_\varepsilon u^\varepsilon, M_\varepsilon u^\varepsilon\rangle_\tau &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2\,dt \\ &+ \int_0^\tau \int_{\Gamma(t)} \{g^{-1}(\partial^\bullet g - V_\Gamma^T \cdot \nabla_{\Gamma(t)} g) + \mathrm{div}_{\Gamma(t)} V_\Gamma\}|M_\varepsilon u^\varepsilon|^2\,d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^\tau \int_{\Gamma(t)} M_\varepsilon u^\varepsilon (V_\Gamma^T - g^{-1}\nabla_{\Gamma(t)} g)\cdot \nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon\,d\mathcal{H}^{n-1}\,dt = I_\varepsilon(u^\varepsilon, g^{-1}M_\varepsilon u^\varepsilon;\tau).\end{aligned}$$

Moreover, from (2.4.8) with T replaced by τ,

$$\begin{aligned}\langle \partial^\bullet M_\varepsilon u^\varepsilon, M_\varepsilon u^\varepsilon\rangle_\tau = -\frac{1}{2}\int_0^\tau \int_{\Gamma(t)} |M_\varepsilon u^\varepsilon|^2\,\mathrm{div}_{\Gamma(t)} V_\Gamma\,d\mathcal{H}^{n-1}\,dt& \\ + \frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|_{L^2(\Gamma(\tau))}^2 - \frac{1}{2}\|M_\varepsilon u^\varepsilon(0)\|_{L^2(\Gamma_0)}^2&.\end{aligned}$$

Applying this equality and the relation $u^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega_\varepsilon(0))$ (see Theorem 2.3.4) to the above equality, we have

$$\begin{aligned}\frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|_{L^2(\Gamma(\tau))}^2 &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2\,dt \\ &= \frac{1}{2}\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)}^2 + I_1(\tau) + I_2(\tau) + I_\varepsilon(u^\varepsilon, g^{-1}M_\varepsilon u^\varepsilon;\tau),\end{aligned} \tag{2.6.9}$$

where

$$\begin{aligned}I_1(\tau) &:= -\frac{1}{2}\int_0^\tau \int_{\Gamma(t)} \{2g^{-1}(\partial^\bullet g - V_\Gamma^T \cdot \nabla_{\Gamma(t)} g) + \mathrm{div}_{\Gamma(t)} V_\Gamma\}|M_\varepsilon u^\varepsilon|^2\,d\mathcal{H}^{n-1}\,dt, \\ I_2(\tau) &:= -\int_0^\tau \int_{\Gamma(t)} M_\varepsilon u^\varepsilon (V_\Gamma^T - g^{-1}\nabla_{\Gamma(t)} g)\cdot \nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

Since g and V_Γ are smooth on $\overline{S_T}$, they are bounded on S_T along with their derivatives. Also, g^{-1} and V_Γ^T are bounded on S_T. Thus it follows that

$$\begin{aligned}|I_1(\tau)| &\le c\int_0^\tau \|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}^2\,dt, \\ |I_2(\tau)| &\le c\int_0^\tau \|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}\|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}\,dt.\end{aligned} \tag{2.6.10}$$

On the other hand, the inequality (2.6.2) with T replaced by τ yields

$$\begin{aligned}|I_\varepsilon(u^\varepsilon, g^{-1}M_\varepsilon u^\varepsilon;\tau)| &\le c\varepsilon^{1/2}\int_0^\tau \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\|\nabla_{\Gamma(t)}(g^{-1}M_\varepsilon u^\varepsilon)(t)\|_{L^2(\Gamma(t))}\,dt \\ &\le c\varepsilon^{1/2}\int_0^\tau \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon(t))}\Big(\|M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))} + \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|_{L^2(\Gamma(t))}\Big)\,dt.\end{aligned} \tag{2.6.11}$$

Thus, by applying (2.6.10) and (2.6.11) to (2.6.9), we obtain

$$\begin{aligned}\frac{1}{2}\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \\ &\leq \frac{1}{2}\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + \frac{1}{2}\int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \\ &\qquad + c\int_0^\tau \Big(\|M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))} + \varepsilon\|u^\varepsilon(t)\|^2_{H^1(\Omega_\varepsilon(t))}\Big)\,dt.\end{aligned}$$

We multiply both sides by two and subtract $\int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt$ to get

$$\begin{aligned}\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} &+ \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \\ &\leq \|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + c\int_0^\tau \Big(\|M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))} + \varepsilon\|u^\varepsilon(t)\|^2_{H^1(\Omega_\varepsilon(t))}\Big)\,dt.\end{aligned}$$

Hence Gronwall's inequality implies

$$\|M_\varepsilon u^\varepsilon(\tau)\|^2_{L^2(\Gamma(\tau))} + \int_0^\tau \|\nabla_{\Gamma(t)} M_\varepsilon u^\varepsilon(t)\|^2_{L^2(\Gamma(t))}\,dt \leq c\Big(\|M_\varepsilon u_0^\varepsilon\|^2_{L^2(\Gamma_0)} + \varepsilon\|u^\varepsilon\|^2_{L^2_{H^1(\varepsilon)}}\Big)$$

for all $\tau \in [0,T]$, and we obtain (2.6.8) by applying (2.3.15) to the second term of the right-hand side of the above inequality. □

Lemma 2.6.5. *Let u_0^ε and u^ε be as in Lemma 2.6.2. Then there exists a constant $c > 0$ independent of u_0^ε, u^ε, and ε such that*

$$\|M_\varepsilon u^\varepsilon\|_{W_T} \leq c\Big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\Big). \tag{2.6.12}$$

Proof. It follows from (2.6.8) that

$$\|M_\varepsilon u^\varepsilon\|_{H_T} \leq c\Big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\Big).$$

Moreover, by applying this inequality and (2.3.15) to (2.6.7) we have

$$\|\partial^\bullet M_\varepsilon u^\varepsilon\|_{H_T'} \leq c\Big(\|M_\varepsilon u^\varepsilon\|_{H_T} + \varepsilon^{1/2}\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}}\Big) \leq c\Big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\Big).$$

Thus we obtain (2.6.12). □

2.6.3 Limit equation on evolving surfaces and weak convergence of $M_\varepsilon u^\varepsilon$

Assume that $I_\varepsilon(u^\varepsilon,\varphi;T) = 0$ holds for all $\varphi \in C_0^1(S_T)$ and $v = M_\varepsilon u^\varepsilon$ is independent of ε in the variational formulation (2.6.1). Then v satisfies

$$\begin{aligned}\langle \partial^\bullet v, g\varphi\rangle_T &+ \int_0^T\int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)} V_\Gamma)v\,\varphi\,d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^T\int_{\Gamma(t)} g(\nabla_{\Gamma(t)} v + vV_\Gamma^T)\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt = 0\end{aligned} \tag{2.6.13}$$

for all $\varphi \in C_0^1(S_T)$. In addition we assume that v is sufficiently smooth. Since vector fields gvV_Γ^T and $g\nabla_{\Gamma(t)}v$ are tangent to $\Gamma(t)$ for each $t \in [0,T]$, we can apply the integration by parts formula (2.5.23) to obtain

$$\langle \partial^\bullet v, g\varphi \rangle_T + \int_0^T \int_{\Gamma(t)} \Big\{ (\partial^\bullet g + g \operatorname{div}_{\Gamma(t)} V_\Gamma) v - \operatorname{div}_{\Gamma(t)} \big[g(\nabla_{\Gamma(t)} v + vV_\Gamma^T) \big] \Big\} \varphi \, d\mathcal{H}^{n-1} \, dt = 0.$$

Since this equality holds for all $\varphi \in C_0^1(S_T)$, we conclude that v satisfies

$$\partial^\bullet(gv) + (g \operatorname{div}_{\Gamma(t)} V_\Gamma)v - \operatorname{div}_{\Gamma(t)} \big[g(\nabla_{\Gamma(t)} v + vV_\Gamma^T) \big] = 0 \quad \text{on} \quad S_T.$$

This is the limit equation of (H_ε). To justify the above argument, we employ a variational framework introduced by Olshanskii, Reusken, and Xu [19].

Definition 2.6.6. Let $v_0 \in L^2(\Gamma_0)$. A function $v \in W_T$ is said to be a *variational solution* to the initial value problem

$$\begin{cases} \partial^\bullet(gv) + (g \operatorname{div}_{\Gamma(t)} V_\Gamma)v - \operatorname{div}_{\Gamma(t)} \big[g(\nabla_{\Gamma(t)} v + vV_\Gamma^T) \big] = 0 & \text{on} \quad S_T, \\ v(0) = v_0 & \text{on} \quad \Gamma_0, \end{cases} \tag{H_0}$$

if v satisfies (2.6.13) for all $\varphi \in H_T$ and $v(0) = v_0$ in $L^2(\Gamma_0)$.

Note that the condition $v(0) = v_0$ in $L^2(\Gamma_0)$ makes sense for $v \in W_T$ by Lemma 2.4.4.

Remark 2.6.7. Suppose that $v \in W_T$ is a variational solution to (H_0). Then we have $v \in W_\tau$ for each $\tau \in [0,T]$ since W_T is continuously embedded into W_τ. Moreover, by taking test functions φ from $C_0^1(S_\tau)$ we observe that v is a variational solution to (H_0) with T replaced by τ.

We first prove the uniqueness of a variational solution to the initial value problem (H_0).

Lemma 2.6.8. *For each $v_0 \in L^2(\Gamma_0)$, there is at most one variational solution to (H_0).*

Proof. Since (H_0) is linear, it is sufficient to show that if $v \in W_T$ is a variational solution to (H_0) with zero initial data then $v = 0$.

Let v be a variational solution to (H_0) with $v(0) = 0$ in $L^2(\Gamma_0)$. For each $\tau \in [0,T]$, we substitute $g^{-1}v \in H_\tau$ for φ in (2.6.13) with T replaced by τ and compute as in the proof of Lemma 2.6.4 (replace $M_\varepsilon u^\varepsilon$ by v and omit $I_\varepsilon(u^\varepsilon, \varphi; \tau)$). Then we have

$$\|v(\tau)\|_{L^2(\Gamma(\tau))}^2 + \int_0^\tau \|\nabla_{\Gamma(t)} v(t)\|_{L^2(\Gamma(t))}^2 \, dt \le \|v(0)\|_{L^2(\Gamma_0)}^2 + c\int_0^\tau \|v(t)\|_{L^2(\Gamma(t))}^2 \, dt.$$

Since $v(0) = 0$ in $L^2(\Gamma_0)$, the above inequality yields

$$\|v(\tau)\|_{L^2(\Gamma(\tau))}^2 \le \int_0^\tau \|v(t)\|_{L^2(\Gamma(t))}^2 \, dt.$$

Hence by Gronwall's inequality we obtain $v(\tau) = 0$ in $L^2(\Gamma(\tau))$ for all $\tau \in [0,T]$. $\square$

Now let us show that $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges weakly in W_T and that the limit is a unique variational solution to the initial value problem (H_0).

Theorem 2.6.9. *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be the unique variational solution to (H_ε) given by Theorem 2.3.4. Suppose that the following two conditions are satisfied:*

(a) *There exist* $c > 0$ *and* $\gamma \in (0, 1/2)$ *such that* $\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} \le c\varepsilon^{-\gamma}$ *for all* $\varepsilon > 0$.

(b) *There exists* $v_0 \in L^2(\Gamma_0)$ *such that* $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ *converges weakly to* v_0 *in* $L^2(\Gamma_0)$ *as* $\varepsilon \to 0$.

Then $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ *converges weakly in* W_T *as* $\varepsilon \to 0$. *Moreover, the weak limit* $v \in W_T$ *of* $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ *is the unique variational solution to* (H_0) *with initial data* v_0.

Proof. By the condition (b), $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ is bounded in $L^2(\Gamma_0)$. From this fact, the inequality (2.6.12), and the condition (a) it follows that

$$\|M_\varepsilon u^\varepsilon\|_{W_T} \le c\Big(\|M_\varepsilon u_0^\varepsilon\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\Big) \le c(1+\varepsilon^{-\gamma+1/2}) \le c \tag{2.6.14}$$

with some constant $c > 0$ independent of ε. Here the last inequality follows from the condition $\gamma \in (0, 1/2)$. Hence $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ is bounded in the Hilbert space W_T and there exist $v \in W_T$ and a sequence $\{\varepsilon_k\}_k$ of positive numbers with $\lim_{k\to\infty}\varepsilon_k = 0$ such that $\{M_{\varepsilon_k}u^{\varepsilon_k}\}_k$ converges weakly to v in W_T as $k \to \infty$.

Let us show that v is the unique variational solution to (H_0) with initial data v_0. First we show that v satisfies the variational formulation (2.6.13) for all $\varphi \in H_T$. To this end, we return to the variational formulation (2.6.1) of $M_{\varepsilon_k}u^{\varepsilon_k}$:

$$\begin{aligned}\langle \partial^\bullet M_{\varepsilon_k}u^{\varepsilon_k}, g\varphi\rangle_T &+ \int_0^T\int_{\Gamma(t)}(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)(M_{\varepsilon_k}u^{\varepsilon_k})\varphi\,d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^T\int_{\Gamma(t)} g\{\nabla_{\Gamma(t)}M_{\varepsilon_k}u^{\varepsilon_k} + (M_{\varepsilon_k}u^{\varepsilon_k})V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt = I_{\varepsilon_k}(u^{\varepsilon_k},\varphi;T).\end{aligned} \tag{2.6.15}$$

Let $k \to \infty$ in (2.6.15). Since $\{M_{\varepsilon_k}u^{\varepsilon_k}\}_k$ converges weakly to v in W_T as $k \to \infty$ and g, V_Γ are bounded on S_T along with their derivatives, the left-hand side of (2.6.15) converges to

$$\begin{aligned}\langle \partial^\bullet v, g\varphi\rangle_T &+ \int_0^T\int_{\Gamma(t)}(\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)}V_\Gamma)v\varphi\,d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^T\int_{\Gamma(t)} g(\nabla_{\Gamma(t)}v + vV_\Gamma^T)\cdot\nabla_{\Gamma(t)}\varphi\,d\mathcal{H}^{n-1}\,dt.\end{aligned}$$

On the other hand, it follows from (2.6.2) and (2.3.15) that

$$\begin{aligned}|I_{\varepsilon_k}(u^{\varepsilon_k},\varphi;T)| &\le c\varepsilon_k^{1/2}\int_0^T\|u^{\varepsilon_k}(t)\|_{H^1(\Omega_{\varepsilon_k}(t))}\|\nabla_{\Gamma(t)}\varphi(t)\|_{L^2(\Gamma(t))}\,dt \\ &\le c\varepsilon_k^{1/2}\|u^{\varepsilon_k}\|_{L^2_{H^1(\varepsilon_k)}}\|\varphi\|_{H_T} \le c\varepsilon_k^{1/2}\|u_0^{\varepsilon_k}\|_{L^2(\Omega_{\varepsilon_k}(0))}\|\varphi\|_{H_T}\end{aligned}$$

with a constant $c > 0$ independent of ε_k. This inequality and the condition (a) imply that

$$|I_{\varepsilon_k}(u^{\varepsilon_k},\varphi;T)| \le c\varepsilon_k^{-\gamma+1/2}\|\varphi\|_{H_T} \to 0 \quad \text{as} \quad k\to\infty, \tag{2.6.16}$$

since $\gamma \in (0, 1/2)$ and c is independent of ε_k. Hence v satisfies (2.6.13) for all $\varphi \in H_T$.

Next we show that $v(0) = v_0$ in $L^2(\Gamma_0)$. Let $\eta \in C^\infty([0,T])$ satisfy $\eta(0) = 1$ and $\eta(T) = 0$. We take an arbitrary $\varphi_0 \in C^\infty(\Gamma_0)$ and set $\varphi(y,t) := \varphi_0(\Phi^{-1}(y,t))\eta(t)$ for $(y,t) \in \overline{S_T}$, where $\Phi^{-1}(\cdot,t)$ is the inverse mapping of the flow map $\Phi(\cdot,t)\colon \Gamma_0 \to \Gamma(t)$ (see Section 2.2). Due to the smoothness of Φ^{-1}, the function φ is smooth on $\overline{S_T}$ and thus $\varphi \in W_T$. Moreover, it satisfies $\varphi(0)=\varphi_0$ on Γ_0 and $\varphi(T)=0$ on $\Gamma(T)$. Substituting $g^{-1}\varphi$ for φ in (2.6.13) and (2.6.15), we have

$$\begin{aligned}\langle\partial^\bullet v,\varphi\rangle_T &+\int_0^T\int_{\Gamma(t)}(g^{-1}\partial^\bullet g+\mathrm{div}_{\Gamma(t)}V_\Gamma)v\,\varphi\,d\mathcal{H}^{n-1}\,dt\\ &+\int_0^T\int_{\Gamma(t)}g(\nabla_{\Gamma(t)}v+vV_\Gamma^T)\cdot\nabla_{\Gamma(t)}(g^{-1}\varphi)\,d\mathcal{H}^{n-1}\,dt=0\end{aligned}$$

and

$$\begin{aligned}&\langle\partial^\bullet M_{\varepsilon_k}u^{\varepsilon_k},\varphi\rangle_T+\int_0^T\int_{\Gamma(t)}(g^{-1}\partial^\bullet g+\mathrm{div}_{\Gamma(t)}V_\Gamma)(M_{\varepsilon_k}u^{\varepsilon_k})\varphi\,d\mathcal{H}^{n-1}\,dt\\ &+\int_0^T\int_{\Gamma(t)}g\{\nabla_{\Gamma(t)}M_{\varepsilon_k}u^{\varepsilon_k}+(M_{\varepsilon_k}u^{\varepsilon_k})V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}(g^{-1}\varphi)\,d\mathcal{H}^{n-1}\,dt=I_{\varepsilon_k}(u^{\varepsilon_k},g^{-1}\varphi;T).\end{aligned}$$

Since φ, v, and $M_{\varepsilon_k}u^{\varepsilon_k}$ are in W_T, we can apply the identity (2.4.8) to get

$$\langle\partial^\bullet v,\varphi\rangle_T=-\langle\partial^\bullet\varphi,v\rangle_T-\int_{\Gamma_0}v(0)\varphi_0\,d\mathcal{H}^{n-1}-\int_0^T\int_{\Gamma(t)}v\varphi\,\mathrm{div}_{\Gamma(t)}V_\Gamma\,d\mathcal{H}^{n-1}\,dt$$

and the same identity for $M_{\varepsilon_k}u^{\varepsilon_k}$. Here we used the conditions $\varphi(0)=\varphi_0$ on Γ_0 and $\varphi(T)=0$ on $\Gamma(T)$. Thus we have

$$\begin{aligned}&-\langle\partial^\bullet\varphi,v\rangle_T+\int_0^T\int_{\Gamma(t)}g^{-1}(\partial^\bullet g)v\varphi\,d\mathcal{H}^{n-1}\,dt\\ &\qquad+\int_0^T\int_{\Gamma(t)}g(\nabla_{\Gamma(t)}v+vV_\Gamma^T)\cdot\nabla_{\Gamma(t)}(g^{-1}\varphi)\,d\mathcal{H}^{n-1}\,dt=\int_{\Gamma_0}v(0)\varphi_0\,d\mathcal{H}^{n-1}\end{aligned}\tag{2.6.17}$$

and

$$\begin{aligned}&-\langle\partial^\bullet\varphi,M_{\varepsilon_k}u^{\varepsilon_k}\rangle_T+\int_0^T\int_{\Gamma(t)}g^{-1}(\partial^\bullet g)(M_{\varepsilon_k}u^{\varepsilon_k})\varphi\,d\mathcal{H}^{n-1}\,dt\\ &\qquad+\int_0^T\int_{\Gamma(t)}g\{\nabla_{\Gamma(t)}M_{\varepsilon_k}u^{\varepsilon_k}+(M_{\varepsilon_k}u^{\varepsilon_k})V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}(g^{-1}\varphi)\,d\mathcal{H}^{n-1}\,dt\\ &\qquad=\int_{\Gamma_0}(M_{\varepsilon_k}u_0^{\varepsilon_k})\varphi_0\,d\mathcal{H}^{n-1}+I_{\varepsilon_k}(u^{\varepsilon_k},g^{-1}\varphi;T).\end{aligned}\tag{2.6.18}$$

Let $k\to\infty$ in (2.6.18). Since $\{M_\varepsilon u_0^\varepsilon\}_\varepsilon$ converges weakly to v_0 in $L^2(\Gamma_0)$ as $\varepsilon\to0$,

$$\lim_{k\to\infty}\int_{\Gamma_0}M_{\varepsilon_k}u_0^{\varepsilon_k}\,\varphi_0\,d\mathcal{H}^{n-1}=\int_{\Gamma_0}v_0\,\varphi_0\,d\mathcal{H}^{n-1}.$$

Moreover, since $\{M_{\varepsilon_k}u^{\varepsilon_k}\}_k$ converges weakly to v in W_T as $k\to\infty$ and (2.6.16) holds with φ replaced by $g^{-1}\varphi$, both sides of (2.6.18) converge to

$$\begin{aligned}&-\langle\partial^\bullet\varphi,v\rangle_T+\int_0^T\int_{\Gamma(t)}g^{-1}(\partial^\bullet g)v\,\varphi\,d\mathcal{H}^{n-1}\,dt\\ &\qquad+\int_0^T\int_{\Gamma(t)}g(\nabla_{\Gamma(t)}v+vV_\Gamma^T)\cdot\nabla_{\Gamma(t)}(g^{-1}\varphi)\,d\mathcal{H}^{n-1}\,dt=\int_{\Gamma_0}v_0\,\varphi_0\,d\mathcal{H}^{n-1}.\end{aligned}\tag{2.6.19}$$

Comparing (2.6.17) and (2.6.19), we obtain

$$\int_{\Gamma_0} v(0)\varphi_0\, d\mathcal{H}^{n-1} = \int_{\Gamma_0} v_0\, \varphi_0\, d\mathcal{H}^{n-1} \quad \text{for all} \quad \varphi_0 \in C^\infty(\Gamma_0).$$

Since $C^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, it follows that $v(0) = v_0$ in $L^2(\Gamma_0)$. Hence v is the unique variational solution to (H_0) with initial data v_0. Here the uniqueness follows from Lemma 2.6.8.

Finally, using the boundedness of $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ in W_T (see (2.6.14)) and the uniqueness of a variational solution to (H_0) (see Lemma 2.6.8), we can prove by contradiction that the full sequence $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges weakly to v in W_T as $\varepsilon \to 0$. The argument is standard and thus we omit the details. □

Corollary 2.6.10. *For every $v_0 \in L^2(\Gamma_0)$, there exists a unique variational solution to (H_0).*

Proof. For each $\varepsilon > 0$, we define a function u_0^ε on $\Omega_\varepsilon(0)$ as

$$u_0^\varepsilon(X) := \frac{v_0(p(X,0))}{J(p(X,0),0,d(X,0))}, \quad X \in \Omega_\varepsilon(0).$$

Clearly $M_\varepsilon u_0^\varepsilon = v_0$ holds on Γ_0. Moreover, by the co-area formula (2.5.1) and (2.5.4) we have

$$\begin{aligned} \|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} &= \left(\int_{\Gamma_0}\int_{\varepsilon g_0(Y,0)}^{\varepsilon g_1(Y,0)} |v_0(Y)|^2 J(Y,0,\rho)^{-1}\, d\rho\, d\mathcal{H}^{n-1}(Y)\right)^{1/2} \\ &\leq c\left(\int_{\Gamma_0} \varepsilon g(Y,0)|v_0(Y)|^2\, d\mathcal{H}^{n-1}(Y)\right)^{1/2} \leq c\varepsilon^{1/2}\|v_0\|_{L^2(\Gamma_0)} \end{aligned}$$

with a constant $c > 0$ independent of ε. Hence $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and u_0^ε, v_0 satisfy the conditions (a) and (b) of Theorem 2.6.9. Thus the corollary follows from Theorem 2.3.4 and Theorem 2.6.9. □

Remark 2.6.11. Let $H = -\mathrm{div}_{\Gamma(t)}\nu$ be the mean curvature of $\Gamma(t)$. Since the material derivative operator is formally of the form $\partial^\bullet = \partial_t + v_\Gamma^N\nu\cdot\nabla + V_\Gamma^T\cdot\nabla_{\Gamma(t)}$ (see Remark 2.5.12) and the formula $\mathrm{div}_{\Gamma(t)}(v_\Gamma^N\nu) = -v_\Gamma^N H$ holds (see Remark 2.5.15), the limit equation (H_0) is formally equivalent to

$$\partial^\circ(gv) - v_\Gamma^N Hgv - \mathrm{div}_{\Gamma(t)}(g\nabla_{\Gamma(t)}v) = 0 \quad \text{on} \quad S_T.$$

Here $\partial^\circ = \partial_t + v_\Gamma^N\nu\cdot\nabla$ is the normal time derivative (see [2,3,5]). This equation depends on v_Γ^N, ν, and H, which represent the geometric motion of $\Gamma(t)$. On the other hand, it is independent of the tangential velocity V_Γ^T, which represents advection along $\Gamma(t)$. Hence, as we mentioned in Section 2.1, the evolution of the limit v given by Theorem 2.6.9 is not affected by advection along $\Gamma(t)$, but the geometric motion of $\Gamma(t)$.

2.6.4 Estimates for the difference between solutions to the heat equation and the limit equation

Let us estimate the difference between variational solutions to (H_ε) and (H_0). For a function v on S_T, let $\overline{v}$ be its constant extension in the normal direction of $\Gamma(t)$. For a function u on $Q_{\varepsilon,T}$, we set

$$\|u\|_{L^2(Q_{\varepsilon,T})} := \left(\int_0^T\int_{\Omega_\varepsilon(t)} |u|^2\, dx\, dt\right)^{1/2}.$$

Theorem 2.6.12. *Let $u_0^\varepsilon \in L^2(\Omega_\varepsilon(0))$ and $u^\varepsilon \in L^2_{H^1(\varepsilon)}$ be a unique variational solution to (H_ε). Also, let $v_0 \in L^2(\Gamma_0)$ and $v \in W_T$ be a unique variational solution to (H_0). Then there exists a constant $c > 0$ independent of u_0^ε, u^ε, v_0, v, and ε such that*

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leq c\Big(\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon^{3/2}\|v_0\|_{L^2(\Gamma_0)}\Big). \tag{2.6.20}$$

In particular, for each $\alpha \in [0, 3/2)$ we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha}\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} = 0 \quad \textit{provided} \quad \lim_{\varepsilon\to 0} \varepsilon^{-\alpha}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} = 0.$$

We first estimate the difference between $M_\varepsilon u^\varepsilon$ and v in the space W_T.

Lemma 2.6.13. *Let u_0^ε, u^ε, v_0, and v be as in Theorem 2.6.12. Then there exists a constant $c > 0$ independent of u_0^ε, u^ε, v_0, v, and ε such that*

$$\|M_\varepsilon u^\varepsilon - v\|_{W_T} \leq c\Big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}\Big). \tag{2.6.21}$$

In particular, if $\lim_{\varepsilon\to 0}\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} = 0$ and $\lim_{\varepsilon\to 0}\varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} = 0$, then $\{M_\varepsilon u^\varepsilon\}_\varepsilon$ converges strongly to v in W_T.

Proof. For each $\tau \in [0, T]$, we subtract both sides of (2.6.13) with T replaced by τ from those of (2.6.1). Then we have

$$\begin{aligned}\langle \partial^\bullet(M_\varepsilon u^\varepsilon - v), g\varphi\rangle_\tau &+ \int_0^\tau\int_{\Gamma(t)} (\partial^\bullet g + g\,\mathrm{div}_{\Gamma(t)} V_\Gamma)(M_\varepsilon u^\varepsilon - v)\varphi\, d\mathcal{H}^{n-1}\,dt \\ &+ \int_0^\tau\int_{\Gamma(t)} g\{\nabla_{\Gamma(t)}(M_\varepsilon u^\varepsilon - v) + (M_\varepsilon u^\varepsilon - v)V_\Gamma^T\}\cdot\nabla_{\Gamma(t)}\varphi\, d\mathcal{H}^{n-1}\,dt = I_\varepsilon(u^\varepsilon, \varphi; \tau)\end{aligned}$$

for all $\varphi \in H_\tau$. Hence, by calculating as in the proof of Lemma 2.6.2, Lemma 2.6.4, and Lemma 2.6.5 (replace $M_\varepsilon u^\varepsilon$ by $M_\varepsilon u^\varepsilon - v$), we obtain (2.6.21). □

Proof of Theorem 2.6.12. First we show the inequality

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leq c\varepsilon^{1/2}\Big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\Big). \tag{2.6.22}$$

To this end, we use the triangle inequality

$$\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} \leq \Big\|u^\varepsilon - \overline{M_\varepsilon u^\varepsilon}\Big\|_{L^2(Q_{\varepsilon,T})} + \Big\|\overline{M_\varepsilon u^\varepsilon} - \overline{v}\Big\|_{L^2(Q_{\varepsilon,T})}$$

and estimate the right-hand side of the above inequality. By (2.5.9) and (2.3.15), we have

$$\Big\|u^\varepsilon - \overline{M_\varepsilon u^\varepsilon}\Big\|_{L^2(Q_{\varepsilon,T})} \leq c\varepsilon\|u^\varepsilon\|_{L^2_{H^1(\varepsilon)}} \leq c\varepsilon\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}$$

with a constant $c > 0$ independent of ε. On the other hand, by (2.5.7) and (2.6.21),

$$\Big\|\overline{M_\varepsilon u^\varepsilon} - \overline{v}\Big\|_{L^2(Q_{\varepsilon,T})} \leq c\varepsilon^{1/2}\|M_\varepsilon u^\varepsilon - v\|_{H_T} \leq c\varepsilon^{1/2}\Big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}\Big).$$

Hence (2.6.22) follows.

Next we estimate the right-hand side of (2.6.22) to get (2.6.20). We use the notation

$$(u_0^\varepsilon)^\sharp(Y,\rho) := u_0^\varepsilon(Y+\rho\nu(Y,0)), \quad Y\in\Gamma_0,\ \rho\in(\varepsilon g_0(Y,0),\varepsilon g_1(Y,0)),$$

and omit the variables Y, ρ, and $t=0$. We set

$$I_1 := \frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1}\{(u_0^\varepsilon)^\sharp - v_0\}J\,d\rho, \quad I_2 := \frac{v_0}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1}(J-1)\,d\rho.$$

Then $M_\varepsilon u_0^\varepsilon - v_0 = I_1 + I_2$ on Γ_0. By Hölder's inequality and (2.2.1), (2.5.4), we have

$$|I_1|^2 \le \frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1}|(u_0^\varepsilon)^\sharp - v_0|^2 J^2\,d\rho \le c\varepsilon^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1}|(u_0^\varepsilon)^\sharp - v_0|^2 J\,d\rho.$$

On the other hand, (2.5.5) yields $|I_2|\le c\varepsilon|v_0|$. Hence

$$\begin{aligned}
\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)}^2 &\le c\int_{\Gamma_0}(|I_1|^2+|I_2|^2)\,d\mathcal{H}^{n-1} \\
&\le c\int_{\Gamma_0}\left(\varepsilon^{-1}\int_{\varepsilon g_0}^{\varepsilon g_1}|(u_0^\varepsilon)^\sharp - v_0|^2 J\,d\rho + \varepsilon^2|v_0|^2\right)d\mathcal{H}^{n-1} \\
&= c\Big(\varepsilon^{-1}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))}^2 + \varepsilon^2\|v_0\|_{L^2(\Gamma_0)}^2\Big).
\end{aligned}$$

Here we used the co-area formula (2.5.1) in the last equality. The above inequality is equivalent to

$$\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} \le c\Big(\varepsilon^{-1/2}\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon\|v_0\|_{L^2(\Gamma_0)}\Big). \tag{2.6.23}$$

Moreover, by the triangle inequality and (2.5.7),

$$\begin{aligned}
\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))} &\le \|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \|\overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} \\
&\le \|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + c\varepsilon^{1/2}\|v_0\|_{L^2(\Gamma_0)}.
\end{aligned} \tag{2.6.24}$$

Finally, by applying (2.6.23) and (2.6.24) to (2.6.22), we obtain

$$\begin{aligned}
\|u^\varepsilon - \overline{v}\|_{L^2(Q_{\varepsilon,T})} &\le c\varepsilon^{1/2}\Big(\|M_\varepsilon u_0^\varepsilon - v_0\|_{L^2(\Gamma_0)} + \varepsilon^{1/2}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon(0))}\Big) \\
&\le c\varepsilon^{1/2}\Big((\varepsilon^{-1/2}+\varepsilon^{1/2})\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon\|v_0\|_{L^2(\Gamma_0)}\Big) \\
&\le c\Big(\|u_0^\varepsilon - \overline{v_0}\|_{L^2(\Omega_\varepsilon(0))} + \varepsilon^{3/2}\|v_0\|_{L^2(\Gamma_0)}\Big)
\end{aligned}$$

with a constant $c>0$ independent of ε. Hence (2.6.20) holds. □

2.A Heuristic derivation of the limit equation

Let us give a heuristic derivation of the limit equation (2.1.1) from (H_ε) when $\Omega_\varepsilon(t)$ is of the form $\Omega_\varepsilon(t) = \{x\in\mathbb{R}^n \mid -\varepsilon < d(x,t) < \varepsilon\}$. In this case, the unit outward normal vector field ν_ε of $\partial\Omega_\varepsilon(t)$ and the outer normal velocity v_ε^N of $\partial\Omega_\varepsilon(t)$ are of the form

$$\nu_\varepsilon(x,t) = \pm\nu(p(x,t),t), \quad v_\varepsilon^N(x,t) = \pm v_\Gamma^N(p(x,t),t), \quad (x,t)\in\partial_\ell Q_{\varepsilon,T},$$

according to $d(x,t) = \pm\varepsilon$ (double-sign corresponds). Thus we start from the heat equation

$$\partial_t u^\varepsilon(x,t) - \Delta u^\varepsilon(x,t) = 0, \quad (x,t) \in Q_{\varepsilon,T}$$

with the boundary condition

$$\nu(p(x,t),t) \cdot \nabla u^\varepsilon(x,t) + v_\Gamma^N(p(x,t),t) u^\varepsilon(x,t) = 0, \quad (x,t) \in \partial_\ell Q_{\varepsilon,T}. \tag{2.A.1}$$

To derive the limit equation, we make the following assumptions:

(1) The signed distance $d(x,t)$ of $x \in \Omega_\varepsilon(t)$ is negligible ($d(x,t) \approx 0$), although the quantity $\varepsilon^{-1} d(x,t)$ is not negligible.

(2) The relation $v_\Gamma^N(p(x,t),t) \approx -\partial_t d(x,t)$ holds for all $(x,t) \in Q_{\varepsilon,T}$.

(3) The boundary condition (2.A.1) also holds in the noncylindrical domain $Q_{\varepsilon,T}$.

These assumptions come from the smallness of the width 2ε of $\Omega_\varepsilon(t)$. Taking the third assumption into account, we consider the two equations

$$\partial_t u^\varepsilon(x,t) - \Delta u^\varepsilon(x,t) = 0, \tag{2.A.2}$$

$$\nu(p(x,t),t) \cdot \nabla u^\varepsilon(x,t) + v_\Gamma^N(p(x,t),t) u^\varepsilon(x,t) = 0 \tag{2.A.3}$$

for $(x,t) \in Q_{\varepsilon,T}$. Recall that each $x \in \Omega_\varepsilon(t)$ is represented as

$$x = p(x,t) + d(x,t)\nu(p(x,t),t), \quad \nabla d(x,t) = \nu(x,t) = \nu(p(x,t),t).$$

First, we consider the gradient matrix of the projection $p(x,t)$ onto $\Gamma(t)$ given by

$$\nabla p = \begin{pmatrix} \partial_1 p_1 & \dots & \partial_n p_1 \\ \vdots & \ddots & \vdots \\ \partial_1 p_n & \dots & \partial_n p_n \end{pmatrix} \quad \text{for} \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}.$$

By differentiating both sides of $x = p(x,t) + d(x,t)\nu(x,t)$ and using $\nabla d(x,t) = \nu(x,t)$, we have

$$I_n = \nabla p(x,t) + \nu(x,t) \otimes \nu(x,t) + d(x,t) \nabla \nu(x,t).$$

According to the assumption (1), the above equality reads

$$\nabla p(x,t) \approx I_n - \nu(x,t) \otimes \nu(x,t) = I_n - \nu(p(x,t),t) \otimes \nu(p(x,t),t). \tag{2.A.4}$$

We define a function $v \colon S_T \times (-1,1) \to \mathbb{R}$ as

$$v(y,t,r) := u^\varepsilon(y + \varepsilon r \nu(y,t), t), \quad (y,t) \in S_T,\ r \in (-1,1).$$

Then u^ε is represented by v as

$$u^\varepsilon(x,t) = v(p(x,t), t, \varepsilon^{-1} d(x,t)), \quad (x,t) \in Q_{\varepsilon,T}. \tag{2.A.5}$$

For abbreviation, we write p and d for $p(x,t)$ and $d(x,t)$ in arguments of functions unless we would like to emphasize them. For example, we write $\nu(p,t)$ for $\nu(p(x,t),t)$ and $v(p,t,\varepsilon^{-1}d)$ for $v(p(x,t),t,\varepsilon^{-1}d(x,t))$. By the chain rule of differentiation we have

$$\nabla u^\varepsilon(x,t) = [\nabla p(x,t)]^T \nabla v(p,t,\varepsilon^{-1}d) + \varepsilon^{-1} \partial_r v(p,t,\varepsilon^{-1}d) \nabla d(x,t).$$

By (2.A.4) and $\nabla d(x,t)=\nu(p(x,t),t)$, this equality reads

$$\nabla u^\varepsilon(x,t)\approx\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)+\varepsilon^{-1}\partial_r v(p,t,\varepsilon^{-1}d)\nu(p,t). \tag{2.A.6}$$

Here we abused the definition of the tangential gradient $\nabla_{\Gamma(t)}=(I_n-\nu\otimes\nu)\nabla$. Applying (2.A.6) to (2.A.3) and observing that $\nu(p,t)\cdot\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)=0$, we obtain

$$\varepsilon^{-1}\partial_r v(p,t,\varepsilon^{-1}d)\approx -v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d) \tag{2.A.7}$$

and thus (2.A.6) becomes

$$\nabla u^\varepsilon(x,t)\approx\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)-v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t). \tag{2.A.8}$$

Next we compute $\Delta u^\varepsilon=\operatorname{div}\nabla u^\varepsilon$. For a vector field F on $\Omega_\varepsilon(t)$ with each fixed $t\in[0,T]$,

$$\begin{aligned}\operatorname{div}F(x)&=\operatorname{trace}[\nabla F(x)]\\&=\operatorname{trace}[\{I_n-\nu(x,t)\otimes\nu(x,t)\}\nabla F(x)]+\operatorname{trace}[\nu(x,t)\otimes\nu(x,t)\nabla F(x)]\\&=\operatorname{div}_{\Gamma(t)}F(x)+\nu(x,t)\cdot\partial_\nu F(x)\end{aligned}$$

holds since $\nu\otimes\nu$ is a projection matrix onto the ν-direction. Hence we have

$$\operatorname{div}\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big]=\operatorname{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big]+\nu(x,t)\cdot\partial_\nu\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big].$$

Moreover, since $p(x+h\nu(x,t),t)=p(x,t)$ and $d(x+h\nu(x,t),t)=d(x,t)+h$ for sufficiently small $h\in\mathbb{R}$, it follows that

$$\nabla_{\Gamma(t)}v(p(x+h\nu(x,t),t),t,\varepsilon^{-1}d(x+h\nu(x,t),t))=\nabla_{\Gamma(t)}v(p(x,t),t,\varepsilon^{-1}d(x,t)+\varepsilon^{-1}h)$$

and thus

$$\partial_\nu\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big]=\varepsilon^{-1}\partial_r\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big]$$

by the formula $\partial_\nu f(x)=\lim_{h\to0}\{f(x+h\nu(x,t))-f(x)\}/h$ for functions f on $\Omega_\varepsilon(t)$ with fixed $t\in[0,T]$. Hence we obtain

$$\operatorname{div}\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big]=\operatorname{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big]+\varepsilon^{-1}\nu(x,t)\cdot\partial_r\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big].$$

Similarly we have

$$\begin{aligned}&\operatorname{div}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big]\\&\quad=\operatorname{div}_{\Gamma(t)}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big]+\nu(x,t)\cdot\{\varepsilon^{-1}v_\Gamma^N(p,t)\partial_r v(p,t,\varepsilon^{-1}d)\nu(p,t)\}\\&\quad\approx\operatorname{div}_{\Gamma(t)}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big]-\{v_\Gamma^N(p,t)\}^2v(p,t,\varepsilon^{-1}d).\end{aligned}$$

Here the last approximation follows from $\nu(x,t)=\nu(p(x,t),t)$ and (2.A.7). Hence, by (2.A.8),

$$\begin{aligned}\Delta u^\varepsilon(x,t)\approx{}&\operatorname{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big]+\varepsilon^{-1}\nu(x,t)\cdot\partial_r\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big]\\&-\operatorname{div}_{\Gamma(t)}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big]+\{v_\Gamma^N(p,t)\}^2v(p,t,\varepsilon^{-1}d).\end{aligned} \tag{2.A.9}$$

On the other hand, we differentiate both sides of (2.A.5) with respect to t to get

$$\partial_t u^\varepsilon(x,t)=\partial_t p(x,t)\cdot\nabla v(p,t,\varepsilon^{-1}d)+\partial_t v(p,t,\varepsilon^{-1}d)+\varepsilon^{-1}\partial_t d(x,t)\partial_r v(p,t,\varepsilon^{-1}d).$$

To this equality we apply (2.A.7) and

$$\partial_t p(x,t) = -\partial_t d(x,t)\nu(x,t) - d(x,t)\partial_t \nu(x,t) \approx v_\Gamma^N(p,t)\nu(p,t),$$

where the last approximation follows from the assumptions (1), (2), and $\nu(x,t) = \nu(p(x,t),t)$. Then we have

$$\partial_t u^\varepsilon(x,t) \approx v_\Gamma^N(p,t)\nu(p,t)\cdot\nabla v(p,t,\varepsilon^{-1}d) + \partial_t v(p,t,\varepsilon^{-1}d) + \{v_\Gamma^N(p,t)\}^2\, v(p,t,\varepsilon^{-1}d). \tag{2.A.10}$$

Substituting (2.A.9) and (2.A.10) for the equation (2.A.2), we obtain

$$\begin{aligned}\partial_t v(p,t,\varepsilon^{-1}d) + v_\Gamma^N(p,t)\nu(p,t)\cdot\nabla v(p,t,\varepsilon^{-1}d) + \mathrm{div}_{\Gamma(t)}\big[v_\Gamma^N(p,t)v(p,t,\varepsilon^{-1}d)\nu(p,t)\big]& \\ - \mathrm{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big] - \varepsilon^{-1}\nu(x,t)\cdot\partial_r\big[\nabla_{\Gamma(t)}v(p,t,\varepsilon^{-1}d)\big] &= 0.\end{aligned}$$

Now let us make an additional assumption: the function $v(y,t,r)$ is independent of the variable r. Then, the above equation reads

$$\begin{aligned}\partial_t v(y,t) + v_\Gamma^N(y,t)\nu(y,t)\cdot\nabla v(y,t) + \mathrm{div}_{\Gamma(t)}\big[v_\Gamma^N(y,t)v(y,t)\nu(y,t)\big]& \\ - \mathrm{div}_{\Gamma(t)}\big[\nabla_{\Gamma(t)}v(y,t)\big] &= 0\end{aligned}$$

with $y = p(x,t) \in \Gamma(t)$. Finally we observe that

$$\mathrm{div}_{\Gamma(t)}(v_\Gamma^N v\nu) = \nabla_{\Gamma(t)}(v_\Gamma^N v)\cdot\nu + v_\Gamma^N v\,\mathrm{div}_{\Gamma(t)}\nu = 0 + v_\Gamma^N v\cdot(-H) = -v_\Gamma^N Hv,$$

where $H = -\mathrm{div}_{\Gamma(t)}\nu$ is the mean curvature of $\Gamma(t)$, to obtain

$$\partial_t v(y,t) + v_\Gamma^N(y,t)\nu(y,t)\cdot\nabla v(y,t) - v_\Gamma^N(y,t)H(y,t)v(y,t) - \Delta_{\Gamma(t)}v(y,t) = 0$$

for $(y,t)\in S_T$. This is the limit equation (2.1.1) we mentioned in Section 2.1.

2.B Elementary facts on integrals over evolving surfaces

In this appendix we give complete proofs of several facts on integrals over evolving surfaces which are essentially known or easily proved but there is no detailed proof for the readers' convenience. We first show the transformation formula (2.4.2).

Proof of (2.4.2). By a localization argument with a partition of unity of S_T, it is sufficient to show

$$\int_I\int_{\mu_t(U)} f(y,t)\,d\mathcal{H}^{n-1}(y)\,dt = \int_{\zeta(U\times I)} f(\sigma)(1+|v_\Gamma^N(\sigma)|^2)^{-1/2}\,d\mathcal{H}^n(\sigma), \tag{2.B.1}$$

where I is an open interval in $(0,T)$, U is an open set in $\mathbb{R}^{n-1}$, $\mu_t\colon U\to\Gamma(t)$ is a smooth local parametrization of $\Gamma(t)$ for each $t\in I$, and $\zeta\colon U\times I\to S_T$ is given by $\zeta(s,t) = (\mu_t(s),t)$. Moreover, by rotating coordinates and taking I sufficiently small, we may assume that there exists a smooth function h on $U\times I$ such that $\mu_t(s) = (s,h(s,t))$ for all $s\in U$ and $t\in I$. Then $\zeta(s,t) = (s,h(s,t),t)$ and the outward normal velocity v_Γ^N of $\Gamma(t)$ is given by

$$v_\Gamma^N(\mu_t(s),t) = \frac{\partial_t h(s,t)}{\sqrt{1+|\nabla' h(s,t)|^2}}, \quad (s,t)\in U\times I. \tag{2.B.2}$$

Here ∇' is the gradient in $s \in \mathbb{R}^{n-1}$ and we assume that the n-th component of the normal ν is positive on $\zeta(U \times I)$. For $t \in I$ the Riemannian metric on $\Gamma(t)$ is locally given by

$$\frac{\partial \mu_t}{\partial s_i}(s) \cdot \frac{\partial \mu_t}{\partial s_j}(s) = \delta_{ij} + \frac{\partial h}{\partial s_i}(s,t)\frac{\partial h}{\partial s_j}(s,t), \quad s \in U,\, i,j = 1,\dots,n-1,$$

where δ_{ij} is the Kronecker delta. Hence the left-hand side of (2.B.1) is

$$\int_I \int_{\mu_t(U)} f(y,t)\, d\mathcal{H}^{n-1}(y)\, dt = \int_I \int_U f(\mu_t(s),t)\sqrt{1 + |\nabla' h(s,t)|^2}\, ds\, dt. \tag{2.B.3}$$

On the other hand, since the Riemannian metric on S_T is locally given by

$$\frac{\partial \zeta}{\partial s_i}(s,t) \cdot \frac{\partial \zeta}{\partial s_j}(s,t) = \delta_{ij} + \frac{\partial h}{\partial s_i}(s,t)\frac{\partial h}{\partial s_i}(s,t),$$
$$\frac{\partial \zeta}{\partial s_i}(s,t) \cdot \frac{\partial \zeta}{\partial t}(s,t) = \frac{\partial h}{\partial s_i}(s,t)\frac{\partial h}{\partial t}(s,t), \quad \frac{\partial \zeta}{\partial t}(s,t) \cdot \frac{\partial \zeta}{\partial t}(s,t) = 1 + \left|\frac{\partial h}{\partial t}(s,t)\right|^2$$

for $s \in U$, $t \in I$, and $i,j = 1,\dots,n-1$, the right-hand side of (2.B.1) is

$$\begin{aligned} &\int_{\zeta(U\times I)} f(\sigma)(1 + |v_\Gamma^N(\sigma)|^2)^{-1/2}\, d\mathcal{H}^n(\sigma) \\ &\qquad = \int_{U\times I} f(\mu_t(s),t)(1 + |v_\Gamma^N(\mu_t(s),t)|^2)^{-1/2}\sqrt{\det A(s,t)}\, ds\, dt. \end{aligned} \tag{2.B.4}$$

Here A is a matrix of the form

$$A = \begin{pmatrix} I_{n-1} + \nabla' h \otimes \nabla' h & \partial_t h \nabla' h \\ \partial_t h (\nabla' h)^T & 1 + |\partial_t h|^2 \end{pmatrix},$$

where $(\nabla' h)^T$ is the transpose of the column vector $\nabla' h$. By elementary row operations we have

$$\begin{aligned} \det A &= \det \begin{pmatrix} I_{n-1} + \{1 - |\partial_t h|^2/(1 + |\partial_t h|^2)\}\nabla' h \otimes \nabla' h & 0 \\ \partial_t h (\nabla' h)^T & 1 + |\partial_t h|^2 \end{pmatrix} \\ &= (1 + |\partial_t h|^2) \det \left[I_{n-1} + \left(1 - \frac{|\partial_t h|^2}{1 + |\partial_t h|^2}\right) \nabla' h \otimes \nabla' h \right] \\ &= (1 + |\partial_t h|^2) \left\{ 1 + \left(1 - \frac{|\partial_t h|^2}{1 + |\partial_t h|^2}\right) |\nabla' h|^2 \right\} \\ &= \left(1 + \frac{|\partial_t h|^2}{1 + |\nabla' h|^2}\right)(1 + |\nabla' h|^2). \end{aligned}$$

Hence, by (2.B.2),

$$\det A(s,t) = (1 + |v_\Gamma^N(\mu_t(s),t)|^2)(1 + |\nabla' h(s,t)|^2).$$

Substituting this for the right-hand side of (2.B.4) and applying Fubini's theorem, we get the right-hand side of (2.B.3) and thus conclude that (2.B.1) holds. □

Next we give complete proofs of Lemma 2.4.1 and Lemma 2.4.3. Before starting to prove, let us construct a partition of unity of $\Gamma(t)$ by that of Γ_0. Since Γ_0 is compact, we can take a finite family $\{U_k\}_{k=1}^N$ of open sets in $\mathbb{R}^{n-1}$ and smooth local parametrizations $\mu_0^k\colon U_k \to \Gamma_0$,

$k = 1, \dots, N$ such that $\{\mu_0^k(U_k)\}_{k=1}^N$ is an open covering of Γ_0. Let $\{\psi_0^k\}_{k=1}^N$ be a partition of unity of Γ_0 subordinate to the covering $\{\mu_0^k(U_k)\}_{k=1}^N$. For $k = 1, \dots, N$ and $t \in [0, T]$ we set

$$\mu_t^k(s) := \Phi(\mu_0^k(s), t), \quad s \in U_k, \quad \psi_t^k := \psi_0^k \circ \mu_0^k \circ (\mu_t^k)^{-1}, \tag{2.B.5}$$

where $\Phi(\cdot, t) \colon \Gamma_0 \to \Gamma(t)$ is the flow map of V_Γ (see Section 2.2). Then for each $k = 1, \dots, N$ the mapping $\mu_t^k \colon U_k \to \Gamma(t)$ is a local parametrization of $\Gamma(t)$ and $\{\mu_t^k(U_k)\}_{k=1}^N$ is an open covering of $\Gamma(t)$. Moreover, $\{\psi_t^k\}_{k=1}^N$ is a partition of unity of $\Gamma(t)$ subordinate to $\{\mu_t^k(U_k)\}_{k=1}^N$. We use these partitions of unity to localize integrals over $\Gamma(t)$.

Proof of Lemma 2.4.1. Let V be a function on $\Gamma_0 \times (0, T)$ and $v := LV$. Our goal is to show

$$\begin{aligned} c_1 \|V(t)\|_{L^2(\Gamma_0)} \leq \|v(t)\|_{L^2(\Gamma(t))} \leq c_2 \|V(t)\|_{L^2(\Gamma_0)}, \\ c_1 \|\nabla_{\Gamma_0} V(t)\|_{L^2(\Gamma_0)} \leq \|\nabla_{\Gamma(t)} v(t)\|_{L^2(\Gamma(t))} \leq c_2 \|\nabla_{\Gamma_0} V(t)\|_{L^2(\Gamma_0)} \end{aligned}$$

for all $t \in (0, T)$ with some positive constants c_1, c_2 independent of t. These inequalities yield $c_1 \|V\|_{\widehat{H}_T} \leq \|v\|_{H_T} \leq c_2 \|V\|_{\widehat{H}_T}$, which means that L is an isomorphism between $\widehat{H}_T$ and H_T. By a localization argument with the partitions of unity given by (2.B.5), it is sufficient to show that

$$c_1 \int_{\mu_0(Q)} |V(t)|^2 \, d\mathcal{H}^{n-1} \leq \int_{\mu_t(Q)} |v(t)|^2 \, d\mathcal{H}^{n-1} \leq c_2 \int_{\mu_0(Q)} |V(t)|^2 \, d\mathcal{H}^{n-1}, \tag{2.B.6}$$

$$c_1 \int_{\mu_0(Q)} |\nabla_{\Gamma_0} V(t)|^2 \, d\mathcal{H}^{n-1} \leq \int_{\mu_t(Q)} |\nabla_{\Gamma(t)} v(t)|^2 \, d\mathcal{H}^{n-1} \leq c_2 \int_{\mu_0(Q)} |\nabla_{\Gamma_0} V(t)|^2 \, d\mathcal{H}^{n-1} \tag{2.B.7}$$

for all $t \in (0, T)$ and all V supported in $\mu_0(Q) \times (0, T)$. Here $\mu_0 \colon U \to \Gamma_0$ be a smooth local parametrization of Γ_0 with an open set U in $\mathbb{R}^{n-1}$, Q is a compact subset of U, and $\mu_t \colon U \to \Gamma(t)$ is the local parametrization of $\Gamma(t)$ given by $\mu_t(s) := \Phi(\mu_0(s), t)$ for $s \in U$. Note that in this case $v = LV$ is supported in $\bigcup_{t \in (0,T)} \mu_t(Q) \times \{t\}$. Let $\theta_t = (\theta_{t,ij})_{i,j}$ be a matrix given by

$$\theta_{t,ij}(s) := \frac{\partial \mu_t}{\partial s_i}(s) \cdot \frac{\partial \mu_t}{\partial s_j}(s), \quad (s, t) \in U \times [0, T], \ i, j = 1, \dots, n-1, \tag{2.B.8}$$

and $\theta_t^{-1} = (\theta_t^{ij})_{ij}$ be the inverse matrix of θ_t. By the definition of integrals over hypersurfaces,

$$\begin{aligned} \int_{\mu_0(Q)} |V(Y, t)|^2 \, d\mathcal{H}^{n-1}(Y) = \int_Q |V(\mu_0(s), t)|^2 \sqrt{\det \theta_0(s)} \, ds, \\ \int_{\mu_t(Q)} |v(y, t)|^2 \, d\mathcal{H}^{n-1}(y) = \int_Q |v(\mu_t(s), t)|^2 \sqrt{\det \theta_t(s)} \, ds. \end{aligned}$$

Since $\sqrt{\det \theta_t(s)}$ is continuous and does not vanish as a function of (s, t) on the compact set $Q \times [0, T]$, there is a constant $c > 0$ such that

$$c^{-1} \leq \sqrt{\det \theta_t(s)} \leq c \quad \text{for all} \quad (s, t) \in Q \times [0, T]. \tag{2.B.9}$$

Moreover, by the definitions of L and μ_t,

$$v(\mu_t(s), t) = V(\Phi^{-1}(\mu_t(s), t), t) = V(\Phi^{-1}(\Phi(\mu_0(s), t), t), t) = V(\mu_0(s), t) \tag{2.B.10}$$

for all $(s,t) \in U \times [0,T]$. Hence (2.B.6) follows. Similarly, by (2.B.9) and the equality

$$\begin{aligned}\int_{\mu_0(Q)} |\nabla_{\Gamma_0} V(Y,t)|^2\, d\mathcal{H}^{n-1}(Y) &= \int_Q |\nabla_{\Gamma_0} V(\mu_0(s),t)|^2 \sqrt{\det \theta_0(s)}\, ds, \\ \int_{\mu_t(Q)} |\nabla_{\Gamma(t)} v(y,t)|^2\, d\mathcal{H}^{n-1}(y) &= \int_Q |\nabla_{\Gamma(t)} v(\mu_t(s),t)|^2 \sqrt{\det \theta_t(s)}\, ds,\end{aligned}$$

it is sufficient for (2.B.7) to show that

$$c_1 |\nabla_{\Gamma_0} V(\mu_0(s),t)|^2 \leq |\nabla_{\Gamma(t)} v(\mu_t(s),t)|^2 \leq c_2 |\nabla_{\Gamma_0} V(\mu_0(s),t)|^2 \tag{2.B.11}$$

for all $(s,t) \in Q \times [0,T]$. The tangential gradients $\nabla_{\Gamma_0} V$ and $\nabla_{\Gamma(t)} v$ are locally expressed as (see [6, Section 2.1 and Section 2.2] for example)

$$\begin{aligned}\nabla_{\Gamma_0} V(\mu_0(s),t) &= \sum_{i,j=1}^{n-1} \theta_0^{ij}(s) \frac{\partial}{\partial s_j}(V(\mu_0(s),t)) \frac{\partial \mu_0}{\partial s_i}(s), \\ \nabla_{\Gamma(t)} v(\mu_t(s),t) &= \sum_{i,j=1}^{n-1} \theta_t^{ij}(s) \frac{\partial}{\partial s_j}(v(\mu_t(s),t)) \frac{\partial \mu_t}{\partial s_i}(s)\end{aligned}$$

for $(s,t) \in U \times [0,T]$ and their Euclidean norms are

$$\begin{aligned}|\nabla_{\Gamma_0} V(\mu_0(s),t)|^2 &= \sum_{i,j=1}^{n-1} \theta_0^{ij}(s) \frac{\partial}{\partial s_i}(V(\mu_0(s),t)) \frac{\partial}{\partial s_j}(V(\mu_0(s),t)), \\ |\nabla_{\Gamma(t)} v(\mu_t(s),t)|^2 &= \sum_{i,j=1}^{n-1} \theta_t^{ij}(s) \frac{\partial}{\partial s_i}(v(\mu_t(s),t)) \frac{\partial}{\partial s_j}(v(\mu_t(s),t)).\end{aligned}$$

Then, by (2.B.10), it is sufficient for (2.B.11) to show

$$c_1 \theta_0^{-1}(s) a \cdot a \leq \theta_t^{-1}(s) a \cdot a \leq c_2 \theta_0^{-1}(s) a \cdot a \quad \text{for all} \quad (s,t,a) \in Q \times [0,T] \times \mathbb{R}^{n-1}. \tag{2.B.12}$$

To this end, we consider a real-valued function

$$F(s,t,a) := \theta_t^{-1}(s) a \cdot a, \quad (s,t,a) \in Q \times [0,T] \times \mathbb{R}^{n-1}.$$

It is continuous on $Q \times [0,T] \times \mathbb{R}^{n-1}$ and satisfies $F(s,t,a) = |B(s,t,a)|^2$, where

$$B(s,t,a) := \sum_{i=1}^{n-1} b_i(s,t,a) \frac{\partial \mu_t}{\partial s_i}(s), \quad b = (b_1, \dots, b_{n-1}) := \theta_t^{-1}(s) a.$$

For $a \neq 0$ we have $b \neq 0$ and thus $B \neq 0$. Hence F does not vanish on the compact set $Q \times [0,T] \times S^{n-2}$, where S^{n-2} is the unit sphere in $\mathbb{R}^{n-1}$. From this fact and the continuity of F there is a constant $c > 0$ such that $c^{-1} \leq F(s,t,a) \leq c$ for all $(s,t,a) \in Q \times [0,T] \times S^{n-2}$ and thus

$$c^{-1}|a|^2 \leq \theta_t^{-1}(s) a \cdot a \leq c|a|^2 \quad \text{for all} \quad (s,t,a) \in Q \times [0,T] \times \mathbb{R}^{n-1}.$$

This inequality yields (2.B.12) and we conclude that (2.B.7) is valid. □

Proof of Lemma 2.4.3. First we give transformation formulas of integrals over Γ_0 and $\Gamma(t)$. Let U be an open set in $\mathbb{R}^{n-1}$ and $\mu_0\colon U \to \Gamma_0$ be a smooth local parametrization of Γ_0. Moreover, let $\mu_t\colon U \to \Gamma(t)$ be the local parametrization of $\Gamma(t)$ given by $\mu_t(s) := \Phi(\mu_0(s), t)$. We set

$$\Lambda(\mu_0(s), t) := \sqrt{\frac{\det \theta_t(s)}{\det \theta_0(s)}}, \quad \lambda(\mu_t(s), t) := \sqrt{\frac{\det \theta_0(s)}{\det \theta_t(s)}}, \quad (s, t) \in U \times [0, T],$$

where $\theta_t = (\theta_{t,ij})_{ij}$ is given by (2.B.8). We can show that the right-hand sides of the above definitions are independent of the choice of the local parametrization μ_0. From this fact and the smoothness assumption on Φ, the functions Λ and λ are well-defined and smooth on the compact manifolds $\Gamma_0 \times [0, T]$ and $\overline{S_T}$, respectively. In particular, they are bounded on $\Gamma_0 \times [0, T]$ and $\overline{S_T}$ along with their derivatives. Moreover, by a localization argument with the partitions of unity given by (2.B.5), we get the integral transformation formulas

$$\int_{\Gamma(t)} v(y, t)\, d\mathcal{H}^{n-1}(y) = \int_{\Gamma_0} V(Y, t)\Lambda(Y, t)\, d\mathcal{H}^{n-1}(Y), \tag{2.B.13}$$

$$\int_{\Gamma_0} V(Y, t)\, d\mathcal{H}^{n-1}(Y) = \int_{\Gamma(t)} v(y, t)\lambda(y, t)\, d\mathcal{H}^{n-1}(y) \tag{2.B.14}$$

for all functions V on $\Gamma_0 \times (0, T)$ and all $t \in (0, T)$, where $v = LV$.

Now let us prove the statement of Lemma 2.4.3. For $V \in \widehat{W}_T$ we set $v := LV$. Then Lemma 2.4.1 yields $v \in H_T$ and $\|v\|_{H_T} \leq c\|V\|_{\widehat{H}_T}$. We next show that $\partial^\bullet v \in H_T'$ and $\|\partial^\bullet v\|_{H_T'} \leq c\|V\|_{\widehat{W}_T}$. Let $\psi \in C_0^1(S_T)$. Then $\Psi := L^{-1}\psi$ is in $C_0^1(\Gamma_0 \times (0, T))$ and $\partial^\bullet \psi(\Phi(Y, t), t) = \partial_t \Psi(Y, t)$ for all $Y \in \Gamma_0$. Hence (2.B.13) yields

$$\begin{aligned}
\langle \partial^\bullet v, \psi\rangle_T &= -\int_0^T \int_{\Gamma(t)} (v\, \partial^\bullet \psi + v\psi\, \mathrm{div}_{\Gamma(t)} V_\Gamma)\, d\mathcal{H}^{n-1}\, dt \\
&= -\int_0^T \int_{\Gamma_0} (V\, \partial_t \Psi + V\Psi F)\Lambda\, d\mathcal{H}^{n-1}\, dt,
\end{aligned}$$

where $F := L^{-1}(\mathrm{div}_{\Gamma(t)} V_\Gamma) \in C^\infty(\Gamma_0 \times [0, T])$. Moreover, since $\Psi\Lambda \in C_0^1(\Gamma_0 \times (0, T))$,

$$-\int_0^T \int_{\Gamma_0} V\Lambda\, \partial_t \Psi\, d\mathcal{H}^{n-1}\, dt = [\partial_t V, \Psi\Lambda]_T + \int_0^T \int_{\Gamma_0} V\Psi\, \partial_t \Lambda\, d\mathcal{H}^{n-1}\, dt$$

by the definition of the weak time derivative $\partial_t V$. From these formulas and the boundedness of F and Λ on $\Gamma_0 \times (0, t)$ along with their derivatives, it follows that

$$\begin{aligned}
|\langle \partial^\bullet v, \psi\rangle_T| &= \left| [\partial_t V, \Psi\Lambda]_T + \int_0^T \int_{\Gamma_0} (V\Psi\, \partial_t \Lambda - V\Psi\Lambda F)\, d\mathcal{H}^{n-1}\, dt \right| \\
&\leq c(\|\partial_t V\|_{\widehat{H}_T'} \|\Psi\Lambda\|_{\widehat{H}_T} + \|V\|_{\widehat{H}_T} \|\Psi\|_{\widehat{H}_T}) \leq c\|V\|_{\widehat{W}_T} \|\psi\|_{H_T}
\end{aligned}$$

with a constant $c > 0$ independent of V and ψ, which implies $\partial^\bullet v \in H_T'$ and $\|\partial^\bullet v\|_{H_T'} \leq c\|V\|_{\widehat{W}_T}$. Hence $v = LV$ is in W_T and $\|v\|_{W_T} \leq c\|V\|_{\widehat{W}_T}$ for every $V \in \widehat{W}_T$.

Similarly, by (2.B.14) and the smoothness of λ on $\overline{S_T}$ we can show that $V := L^{-1}v$ is in $\widehat{W}_T$ and $\|V\|_{\widehat{W}_T} \leq c\|v\|_{W_T}$ for every $v \in W_T$. Hence L is an isomorphism between $\widehat{W}_T$ and W_T. □

2.C Calculations involving the differential geometry of tubular neighborhoods

The purpose of this appendix is to show detailed calculations in the proofs of Lemma 2.5.5, Lemma 2.5.6, and Lemma 2.5.11. We fix $t \in [0,T]$ and omit it until the end of the proof of Lemma 2.5.6.

The proofs of Lemma 2.5.5 and Lemma 2.5.6 involve calculations of the usual gradient in N and the tangential gradient on Γ under a local coordinate system. Let $\mu\colon U \to \Gamma$ be a local parametrization of Γ with an open set U in $\mathbb{R}^{n-1}$. We set

$$\theta_{ij}(s) := \frac{\partial \mu}{\partial s_i}(s) \cdot \frac{\partial \mu}{\partial s_j}(s), \quad s \in U,\ i,j = 1,\dots,n-1.$$

Then, the tangential gradient of a function v on Γ is locally expressed as

$$\nabla_\Gamma v(y) = \sum_{i,j=1}^{n-1} \theta^{ij}(s) \frac{\partial \widetilde{v}}{\partial s_j}(s) \frac{\partial \mu}{\partial s_i}(s), \quad y = \mu(s) \in \mu(U), \tag{2.C.1}$$

where $\widetilde{v}(s) := v(\mu(s))$ and $\theta^{-1} = (\theta^{ij})_{i,j}$ denotes the inverse matrix of $\theta = (\theta_{ij})_{i,j}$. We define a mapping $M\colon U \times (-\delta,\delta) \to N$ as $M(s,\rho) := \mu(s) + \rho\nu(\mu(s))$ for $(s,\rho) \in U \times (-\delta,\delta)$ and set

$$\Theta_{ij}(s,\rho) := \frac{\partial M}{\partial s_i}(s,\rho) \cdot \frac{\partial M}{\partial s_j}(s,\rho), \quad (s,\rho) \in U \times (-\delta,\delta),\ i,j = 1,\dots,n,$$

where $s_n := \rho$. Then the gradient (in $\mathbb{R}^n$) of a function u on N is locally expressed as

$$\nabla u(x) = \sum_{i,j=1}^{n} \Theta^{ij}(s,\rho) \frac{\partial \widetilde{u}}{\partial s_j}(s,\rho) \frac{\partial M}{\partial s_i}(s,\rho), \quad x = M(s,\rho) \in M(U \times (-\delta,\delta)), \tag{2.C.2}$$

where $\widetilde{u}(s,\rho) := u(M(s,\rho))$ and $\Theta^{-1} = (\Theta^{ij})_{i,j}$ is the inverse matrix of $\Theta = (\Theta_{ij})_{i,j}$.

Let v be a function on Γ and $\overline{v}$ be its constant extension in the normal direction of Γ. Then their local representations $\widetilde{v} := v \circ \mu$ and $\widetilde{\overline{v}} := \overline{v} \circ M$ satisfy

$$\widetilde{\overline{v}}(s,\rho) = \overline{v}(p(M(s,\rho))) = v(\mu(s)) = \widetilde{v}(s), \quad (s,\rho) \in U \times (-\delta,\delta).$$

Hereafter we use this fact without mention.

Proof of Lemma 2.5.5. Let $v \in H^1(\Gamma)$. Our goal is to show the inequalities

$$|\nabla \overline{v}(y + \rho\nu(y))| \leq c|\nabla_\Gamma v(y)|, \quad |\nabla \overline{v}(y + \rho\nu(y)) - \nabla_\Gamma v(y)| \leq c\varepsilon|\nabla_\Gamma v(y)| \tag{2.C.3}$$

for all $y \in \Gamma$ and $\rho \in (\varepsilon g_0(y), \varepsilon g_1(y))$ with a constant $c > 0$ independent of y, ρ, and ε. For each fixed $y_0 \in \Gamma$, by a rotation of coordinates we can take an open set U in $\mathbb{R}^{n-1}$ and a local parametrization $\mu\colon U \to \Gamma$ such that $y_0 = \mu(s_0)$ with $s_0 \in U$ and μ is of the form $\mu(s) = (s, f(s))$ with a smooth function f on U satisfying

$$\nabla' f(s_0) = 0, \quad (\nabla')^2 f(s_0) = \mathrm{diag}[\kappa_1,\dots,\kappa_{n-1}], \tag{2.C.4}$$

where ∇' is the gradient in $s \in \mathbb{R}^{n-1}$ and $\kappa_i := \kappa_i(y_0)$ for $i = 1,\dots,n-1$ (see [11, Section 14.6]). We set the direction of $\nu(y_0)$ in the positive direction of the x_n-axis to get

$$\nu(\mu(s)) = \frac{(-\nabla' f(s), 1)}{\sqrt{1 + |\nabla' f(s)|^2}}, \quad s \in U.$$

Then we have $\nu(y_0) = \nu(\mu(s_0)) = e_n$ and

$$\frac{\partial \mu}{\partial s_i}(s_0) = e_i, \quad \frac{\partial}{\partial s_i}\Big(\nu(\mu(s))\Big)\Big|_{s=s_0} = -\kappa_i e_i, \quad i = 1, \dots, n-1 \tag{2.C.5}$$

by (2.C.4), where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. This equality yields

$$\frac{\partial M}{\partial s_i}(s_0, \rho) = (1 - \rho\kappa_i)e_i, \quad i = 1, \dots, n-1, \quad \frac{\partial M}{\partial \rho}(s_0, \rho) = \nu(\mu(s_0)) = e_n. \tag{2.C.6}$$

Hence we have $\theta(s_0) = I_{n-1}$, $\Theta(s_0, \rho) = \mathrm{diag}[(1 - \rho\kappa_1)^2, \dots, (1 - \rho\kappa_{n-1})^2, 1]$, and

$$\theta^{-1}(s_0) = I_{n-1}, \quad \Theta^{-1}(s_0, \rho) = \mathrm{diag}[(1 - \rho\kappa_1)^{-2}, \dots, (1 - \rho\kappa_{n-1})^{-2}, 1]. \tag{2.C.7}$$

Applying (2.C.5), (2.C.6), and (2.C.7) to (2.C.1) and (2.C.2) with $u = \overline{v}$, we obtain

$$\nabla_\Gamma v(y_0) = \sum_{i=1}^{n-1} \frac{\partial \widetilde{v}}{\partial s_i}(s_0)e_i, \quad \nabla\overline{v}(y_0 + \rho\nu(y_0)) = \sum_{i-1}^{n-1}(1 - \rho\kappa_i)^{-1}\frac{\partial \widetilde{v}}{\partial s_i}(s_0)e_i$$

and thus (2.5.3) implies that

$$|\nabla\overline{v}(y_0 + \rho\nu(y_0))|^2 = \sum_{i=1}^{n-1}(1 - \rho\kappa_i)^{-2}\left(\frac{\partial \widetilde{v}}{\partial s_i}(s_0)\right)^2 \leq c\sum_{i=1}^{n-1}\left(\frac{\partial \widetilde{v}}{\partial s_i}(s_0)\right)^2 = |\nabla_\Gamma v(y_0)|^2,$$

which yields the first inequality of (2.C.3) with y replaced by y_0. Moreover, by (2.5.3) we have

$$|(1 - \rho\kappa_i)^{-1} - 1| = |\rho\kappa_i(1 - \rho\kappa_i)^{-1}| \leq c\varepsilon$$

for all $\rho \in (\varepsilon g_0(y_0), \varepsilon g_1(y_0))$ and $i = 1, \dots, n-1$ and thus

$$|\nabla\overline{v}(y_0 + \rho\nu(y_0)) - \nabla_\Gamma v(y_0)|^2 = \sum_{i=1}^{n-1}\{(1 - \rho\kappa_i)^{-1} - 1\}^2\left(\frac{\partial \widetilde{v}}{\partial s_i}(s_0)\right)^2 \leq c\varepsilon^2|\nabla_\Gamma v(y_0)|^2.$$

Hence the second inequality of (2.C.3) with y replaced by y_0 is valid. $\square$

To prove Lemma 2.5.6, we need a differentiation formula of the average operator under a local coordinate system. Let U be an open set in $\mathbb{R}^{n-1}$ and $\mu\colon U \to \Gamma$ be a local parametrization of Γ. The weighted average of a function u on Ω_ε is locally expressed as

$$\widetilde{M_\varepsilon u}(s) = \frac{1}{\varepsilon\widetilde{g}(s)}\int_{\varepsilon\widetilde{g}_0(s)}^{\varepsilon\widetilde{g}_1(s)} \widetilde{u}(s, \rho)\widetilde{J}(s, \rho)\,d\rho, \quad s \in U, \tag{2.C.8}$$

where $\widetilde{M_\varepsilon u}(s) = M_\varepsilon u(\mu(s))$, $\widetilde{u}(s, \rho) = u(M(s, \rho))$, and

$$\widetilde{J}(s, \rho) := J(\mu(s), \rho) = \prod_{i=1}^{n-1}\{1 - \rho\kappa_i(\mu(s))\}. \tag{2.C.9}$$

Lemma 2.C.1. *Let $u \in H^1(\Omega_\varepsilon)$. Then*

$$\frac{\partial \widetilde{M_\varepsilon u}}{\partial s_i}(s) = \frac{1}{\varepsilon \widetilde{g}(s)} \int_{\varepsilon \widetilde{g}_0(s)}^{\varepsilon \widetilde{g}_1(s)} \left\{ \frac{\partial \widetilde{u}}{\partial s_i}(s,\rho)\widetilde{J}(s,\rho) + \widetilde{u}(s,\rho)\frac{\partial \widetilde{J}}{\partial s_i}(s,\rho) \right\} d\rho$$
$$+ \frac{1}{\varepsilon \widetilde{g}(s)} \int_{\varepsilon \widetilde{g}_0(s)}^{\varepsilon \widetilde{g}_1(s)} \left\{ \frac{\partial \widetilde{u}}{\partial \rho}(s,\rho)\widetilde{J}(s,\rho) + \widetilde{u}(s,\rho)\frac{\partial \widetilde{J}}{\partial \rho}(s,\rho) \right\} \chi_i(s,\rho)\, d\rho \quad (2.C.10)$$

for all $s \in U$ and $i = 1, \dots, n-1$, where

$$\chi_i(s,\rho) := \frac{1}{\widetilde{g}(s)} \left\{ (\rho - \varepsilon \widetilde{g}_0(s)) \frac{\partial \widetilde{g}_1}{\partial s_i}(s) + (\varepsilon \widetilde{g}_1(s) - \rho) \frac{\partial \widetilde{g}_0}{\partial s_i}(s) \right\}. \quad (2.C.11)$$

Proof. For simplicity, we set $\partial_i = \partial/\partial s_i$ and $\partial_\rho = \partial/\partial\rho$. For each $i = 1, \dots, n-1$, we differentiate both sides of (2.C.8) with respect to s_i to get

$$\partial_i \widetilde{M_\varepsilon u} = \frac{I}{\varepsilon \widetilde{g}} - \frac{\partial_i \widetilde{g}}{\varepsilon (\widetilde{g})^2} \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \widetilde{u}\widetilde{J}\, d\rho + \frac{1}{\varepsilon \widetilde{g}} \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \{(\partial_i \widetilde{u})\widetilde{J} + \widetilde{u}(\partial_i \widetilde{J})\}\, d\rho, \quad (2.C.12)$$

where $I = I(s)$ is given by

$$I(s) := \varepsilon \partial_i \widetilde{g}_1(s) \widetilde{u}(s, \varepsilon \widetilde{g}_1(s)) \widetilde{J}(s, \varepsilon \widetilde{g}_1(s)) - \varepsilon \partial_i \widetilde{g}_0(s) \widetilde{u}(s, \varepsilon \widetilde{g}_0(s)) \widetilde{J}(s, \varepsilon \widetilde{g}_0(s)).$$

Since $I = \left[\widetilde{u}(\rho)\widetilde{J}(\rho)\chi_i(\rho)\right]_{\rho=\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1} = \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1} \partial_\rho(\widetilde{u}\widetilde{J}\chi_i)\, d\rho$ and $\partial_\rho \chi_i = \partial_i \widetilde{g}/\widetilde{g}$, we have

$$\frac{I}{\varepsilon \widetilde{g}} = \frac{\partial_i \widetilde{g}}{\varepsilon (\widetilde{g})^2} \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \widetilde{u}\widetilde{J}\, d\rho + \frac{1}{\varepsilon \widetilde{g}} \int_{\varepsilon \widetilde{g}_0}^{\varepsilon \widetilde{g}_1} \{(\partial_\rho \widetilde{u})\widetilde{J} + \widetilde{u}(\partial_\rho \widetilde{J})\}\chi_i\, d\rho. \quad (2.C.13)$$

Substituting (2.C.13) for (2.C.12), we obtain (2.C.10). □

Proof of Lemma 2.5.6. As in the proof of Lemma 2.C.1, we write $\partial_i = \partial/\partial s_i$ and $\partial_\rho = \partial/\partial\rho$. Let $u \in C^\infty(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)$, $\varphi \in H^1(\Gamma)$, and

$$I(y) := \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (\nabla u)^\sharp(y,\rho) \cdot (\nabla \overline{\varphi})^\sharp(y,\rho) J(y,\rho)\, d\rho - \varepsilon g(y) \nabla_\Gamma M_\varepsilon u(y) \cdot \nabla_\Gamma \varphi(y).$$

Here we used the notation (2.5.6). Our goal is to show

$$|I(y)| \leq c\varepsilon |\nabla_\Gamma \varphi(y)| \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} (|u^\sharp(y,\rho)| + |(\nabla u)^\sharp(y,\rho)|)\, d\rho \quad (2.C.14)$$

for all $y \in \Gamma$ with a constant $c > 0$ independent of y and ε. As in the proof of Lemma 2.5.5, we fix $y_0 \in \Gamma$ and take a local parametrization $\mu(s) = (s, f(s))$ of Γ near $y_0 = \mu(s_0)$, $s_0 \in U$, where U is an open set in $\mathbb{R}^{n-1}$ and f is a smooth function on U satisfying (2.C.4). We set the direction of $\nu(y_0)$ in the positive direction of the x_n-axis. Then by (2.C.5), (2.C.6), and (2.C.7) we have

$$(\nabla u)^\sharp(y_0,\rho) = \sum_{i=1}^{n-1} (1 - \rho\kappa_i)^{-1} \partial_i \widetilde{u}(s_0,\rho) e_i + \partial_\rho \widetilde{u}(s_0,\rho) e_n, \quad \nabla_\Gamma M_\varepsilon u(y_0) = \sum_{i=1}^{n-1} \partial_i \widetilde{M_\varepsilon u}(s_0) e_i,$$
$$(\nabla \overline{\varphi})^\sharp(y_0,\rho) = \sum_{i=1}^{n-1} (1 - \rho\kappa_i)^{-1} \partial_i \widetilde{\varphi}(s_0) e_i, \quad \nabla_\Gamma \varphi(y_0) = \sum_{i=1}^{n-1} \partial_i \widetilde{\varphi}(s_0) e_i,$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$ and $\kappa_i := \kappa_i(y_0)$, $i = 1,\dots,n-1$. Hereafter we omit the variables ρ and s_0 unless we need to specify them. The above equality yields

$$\begin{aligned}(\nabla u)^\sharp(y_0,\rho)\cdot(\nabla\overline{\varphi})^\sharp(y_0,\rho) &= \sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-2}\partial_i\widetilde{u}\,\partial_i\widetilde{\varphi}, \\ \varepsilon g(y_0)\nabla_\Gamma M_\varepsilon u(y_0)\cdot\nabla_\Gamma\varphi(y_0) &= \sum_{i=1}^{n-1}\varepsilon\widetilde{g}\Bigl(\partial_i\widetilde{M_\varepsilon u}\Bigr)\partial_i\widetilde{\varphi}.\end{aligned} \tag{2.C.15}$$

Moreover, (2.C.10) implies that

$$\varepsilon\widetilde{g}\Bigl(\partial_i\widetilde{M_\varepsilon u}\Bigr) = \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\{(\partial_i\widetilde{u})\widetilde{J}+\widetilde{u}(\partial_i\widetilde{J})+(\partial_\rho\widetilde{u})\widetilde{J}\chi_i+\widetilde{u}(\partial_\rho\widetilde{J})\chi_i\}\,d\rho,$$

where χ_i is given by (2.C.11), and thus

$$\begin{aligned}\varepsilon g(y_0)\nabla_\Gamma M_\varepsilon u(y_0)\cdot\nabla_\Gamma\varphi(y_0) &= \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{J}\sum_{i=1}^{n-1}\partial_i\widetilde{u}\,\partial_i\widetilde{\varphi}\,d\rho + \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{u}\sum_{i=1}^{n-1}\partial_i\widetilde{J}\,\partial_i\widetilde{\varphi}\,d\rho \\ &\quad + \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\{(\partial_\rho\widetilde{u})\widetilde{J}+\widetilde{u}(\partial_\rho\widetilde{J})\}\sum_{i=1}^{n-1}\chi_i\,\partial_i\widetilde{\varphi}\,d\rho.\end{aligned}$$

From this equality and (2.C.15), we obtain $I(y_0) = I_1 + I_2 + I_3$ with

$$\begin{gathered}I_1 = \int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{J}\sum_{i=1}^{n-1}\{(1-\rho\kappa_i)^{-2}-1\}\partial_i\widetilde{u}\,\partial_i\widetilde{\varphi}\,d\rho, \\ I_2 = -\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\widetilde{u}\sum_{i=1}^{n-1}\partial_i\widetilde{J}\,\partial_i\widetilde{\varphi}\,d\rho, \quad I_3 = -\int_{\varepsilon\widetilde{g}_0}^{\varepsilon\widetilde{g}_1}\{(\partial_\rho\widetilde{u})\widetilde{J}+\widetilde{u}(\partial_\rho\widetilde{J})\}\sum_{i=1}^{n-1}\chi_i\,\partial_i\widetilde{\varphi}\,d\rho.\end{gathered}$$

Let us estimate these integrals. By the definition of $\widetilde{J}$ (see (2.C.9)), we have

$$\nabla_\Gamma J(y_0,\rho) = \sum_{i=1}^{n-1}\partial_i\widetilde{J}(s_0,\rho)e_i, \quad \sum_{i=1}^{n-1}\partial_i\widetilde{J}(s_0,\rho)\partial_i\widetilde{\varphi}(s_0) = \nabla_\Gamma J(y_0,\rho)\cdot\nabla_\Gamma\varphi(y_0).$$

Hence I_2 is of the form

$$I_2 = -\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)} u^\sharp(y_0,\rho)\nabla_\Gamma J(y_0,\rho)\cdot\nabla_\Gamma\varphi(y_0)\,d\rho$$

and by applying (2.5.5) to the right-hand side we obtain

$$|I_2| \leq c\varepsilon|\nabla_\Gamma\varphi(y_0)|\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)}|u^\sharp(y_0,\rho)|\,d\rho. \tag{2.C.16}$$

Next we estimate I_3. By the definitions of $\widetilde{u}$, $\widetilde{J}$, and χ_i (see (2.C.9) and (2.C.11)),

$$\begin{gathered}\partial_\rho\widetilde{u}(s_0,\rho) = \nu(y_0)\cdot(\nabla u)^\sharp(y_0,\rho), \quad \partial_\rho\widetilde{J}(s_0,\rho) = \partial_\rho J(y_0,\rho), \\ \sum_{i=1}^{n-1}\chi_i(s_0,\rho)\partial_i\widetilde{\varphi}(s_0) = \chi_\varepsilon(y_0,\rho)\cdot\nabla_\Gamma\varphi(y_0),\end{gathered}$$

where

$$\chi_\varepsilon(y_0,\rho) := \frac{(\rho-\varepsilon g_0(y_0))\nabla_\Gamma g_1(y_0) + (\varepsilon g_1(y_0)-\rho)\nabla_\Gamma g_0(y_0)}{g(y_0)}.$$

Hence I_3 is of the form

$$I_3 = -\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)} \chi_\varepsilon(y_0,\rho)\cdot\nabla_\Gamma\varphi(y_0)\{\nu(y_0)\cdot(\nabla u)^\sharp(y_0,\rho)J(y_0,\rho)+u^\sharp(y_0,\rho)\partial_\rho J(y_0,\rho)\}\,d\rho.$$

Since $\nabla_\Gamma g_0$, $\nabla_\Gamma g_1$ are bounded and $g_1-g_0=g$,

$$|\chi_\varepsilon(y_0,\rho)| \leq \frac{|\nabla_\Gamma g_0(y_0)|+|\nabla_\Gamma g_1(y_0)|}{g(y_0)}\{(\rho-\varepsilon g_0(y_0))+(\varepsilon g_1(y_0)-\rho)\} \leq c\varepsilon$$

for all $\rho\in(\varepsilon g_0(y_0),\varepsilon g_1(y_0))$. This inequality together with (2.5.4) and (2.5.5) yields

$$|I_3| \leq c\varepsilon|\nabla_\Gamma\varphi(y_0)|\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)}(|u^\sharp(y_0,\rho)|+|(\nabla u)^\sharp(y_0,\rho)|)\,d\rho. \tag{2.C.17}$$

Let us estimate I_1. For all $\rho\in(\varepsilon g_0(y_0),\varepsilon g_1(y_0))$ and $i=1,\dots,n-1$, we have

$$|(1-\rho\kappa_i)^{-2}-1| = |\rho\kappa_i(2-\rho\kappa_i)(1-\rho\kappa_i)^{-2}| \leq c\varepsilon$$

by (2.5.3). From this inequality, Hölder's inequality, and (2.5.3),

$$\begin{aligned}\left|\sum_{i=1}^{n-1}\{(1-\rho\kappa_i)^{-2}-1\}\partial_i\widetilde{u}\,\partial_i\widetilde{\varphi}\right| &\leq c\varepsilon\left(\sum_{i=1}^{n-1}(\partial_i\widetilde{u})^2\right)^{1/2}\left(\sum_{i=1}^{n-1}(\partial_i\widetilde{\varphi})^2\right)^{1/2}\\ &\leq c\varepsilon\left(\sum_{i=1}^{n-1}(1-\rho\kappa_i)^{-2}(\partial_i\widetilde{u})^2\right)^{1/2}\left(\sum_{i=1}^{n-1}(\partial_i\widetilde{\varphi})^2\right)^{1/2}\\ &\leq c\varepsilon|(\nabla u)^\sharp(y_0,\rho)||\nabla_\Gamma\varphi(y_0)|.\end{aligned}$$

Using this inequality and (2.5.4) we obtain

$$|I_1| \leq c\varepsilon|\nabla_\Gamma\varphi(y_0)|\int_{\varepsilon g_0(y_0)}^{\varepsilon g_1(y_0)}|(\nabla u)^\sharp(y_0,\rho)|\,d\rho. \tag{2.C.18}$$

By (2.C.16), (2.C.17), and (2.C.18) we conclude that (2.C.14) with y replaced by y_0 holds. □

Finally we give the complete proof of Lemma 2.5.11.

Proof of Lemma 2.5.11. Let $\Phi(\cdot,t)\colon\Gamma_0\to\Gamma(t)$ be the flow map of V_Γ and $\Phi^{-1}(\cdot,t)$ be its inverse mapping (see Section 2.2). For $X\in N(0)$ and $t\in(0,T)$ we set

$$\Psi(X,t) := \Phi(p(X,0),t)+d(X,0)\nu(\Phi(p(X,0),t),t). \tag{2.C.19}$$

For each $t\in(0,T)$ the mapping $\Psi(\cdot,t)\colon N(0)\to N(t)$ is a bijection whose inverse is given by

$$\Psi^{-1}(x,t) := \Phi^{-1}(p(x,t),t)+d(x,t)\nu(\Phi^{-1}(p(x,t),t),0),\quad (x,t)\in N_T.$$

Let $\varphi \in C^1(S_T)$ and $\overline{\varphi}$ be its constant extension in the normal direction of $\Gamma(t)$. By the definition of $\overline{\varphi}$ and the formula $p(\Psi(X,t),t) = \Phi(p(X,0),t)$ we have

$$\overline{\varphi}(\Psi(X,t),t) = \varphi(\Phi(p(X,0),t),t), \quad (X,t) \in N(0) \times (0,T).$$

We differentiate both sides with respect to t. The time derivative of the left-hand side is

$$\partial_t\overline{\varphi}(\Psi(X,t),t) + \partial_t\Psi(X,t) \cdot \nabla\overline{\varphi}(\Psi(X,t),t).$$

On the other hand, the time derivative of the right-hand side is

$$\partial^\bullet\varphi(\Phi(p(X,0),t),t) = \partial^\bullet\varphi(p(\Psi(X,t),t),t)$$

by the definition of the strong material derivative (see (2.4.4)). Hence

$$\partial^\bullet\varphi(p(\Psi(X,t),t),t) = \partial_t\overline{\varphi}(\Psi(X,t),t) + \partial_t\Psi(X,t) \cdot \nabla\overline{\varphi}(\Psi(X,t),t)$$

for all $(X,t) \in N(0) \times (0,T)$. Substituting $\Psi^{-1}(x,t)$ for X in this equality we further get

$$\partial^\bullet\varphi(p(x,t),t) = \partial_t\overline{\varphi}(x,t) + \partial_t\Psi(\Psi^{-1}(x,t),t) \cdot \nabla\overline{\varphi}(x,t) \tag{2.C.20}$$

for all $(x,t) \in N_T$. Let us show

$$\partial_t\Psi(\Psi^{-1}(x,t),t) = V_\Gamma(p(x,t),t) + a(x,t), \tag{2.C.21}$$

where $a(x,t)$ is given by (2.5.17). We differentiate both sides of (2.C.19) with respect to t to get

$$\begin{aligned}\partial_t\Psi(X,t) &= \partial_t\Phi(p(X,0),t) \\ &\quad + d(X,0)\{\partial_t\nu(\Phi(p(X,0),t),t) + \nabla\nu(\Phi(p(X,0),t),t)\partial_t\Phi(p(X,0),t)\}\end{aligned}$$

for $(X,t) \in N(0) \times (0,T)$. Moreover, since

$$\begin{gathered} d(X,0) = d(\Psi(X,t),t), \quad \Phi(p(X,0),t) = p(\Psi(X,t),t), \\ \partial_t\Phi(p(X,0),t) = V_\Gamma(\Phi(p(X,0),t),t) = V_\Gamma(p(\Psi(X,t),t),t), \end{gathered}$$

it follows that

$$\begin{aligned}\partial_t\Psi(X,t) &= V_\Gamma(p(\Psi(X,t),t),t) \\ &\quad + d(\Psi(X,t),t)\{\partial_t\nu(p(\Psi(X,t),t),t) + \nabla\nu(p(\Psi(X,t),t),t)V_\Gamma(p(\Psi(X,t),t),t)\}.\end{aligned}$$

Substituting $\Psi^{-1}(x,t)$ for X in this equality we obtain (2.C.21). Finally, the formula (2.5.16) follows from (2.C.20) and (2.C.21). $\square$

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Chapter 3

An energetic variational approach for nonlinear diffusion equations in moving thin domains

3.1 Introduction

In this chapter, we are interested in deriving diffusion equations on a moving surface, by regarding it as a thin width limit of the problem in a moving thin domain around the moving surface.

Let us begin with an equation of the conservation of mass ρ with velocity u in a moving domain $\Omega(t)$, $t \in (0,T)$ in $\mathbb{R}^n$, $n \geq 2$ of the form

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in} \quad \Omega(t),\, t \in (0,T), \tag{3.1.1}$$

which represents the local conservation of mass. Considering the situation that there is no exchange of mass on the boundary, i.e.

$$u \cdot \nu_\Omega = V_\Omega^N \quad \text{on} \quad \partial\Omega(t),\, t \in (0,T), \tag{3.1.2}$$

where V_Ω^N is the normal velocity of the boundary $\partial\Omega(t)$ in the direction of the outward normal vector field ν_Ω of $\partial\Omega(t)$. Similar conservation law of mass η with velocity v on a moving surface $\Gamma(t)$ can be derived from the local conservation of mass. It turns out (see Section 3.3) that, when the normal component of v is equal to the outward normal velocity V_Γ^N of the moving surface $\Gamma(t)$, the resulting equation is of the form:

$$\partial^\circ \eta - V_\Gamma^N H \eta + \operatorname{div}_\Gamma(\eta v^T) = 0 \quad \text{on} \quad \Gamma(t),\, t \in (0,T), \tag{3.1.3}$$

where $\partial^\circ = \partial_t + V_\Gamma^N \nu_\Gamma \cdot \nabla$ is the normal time derivative, ν_Γ is the outward normal vector field of $\Gamma(t)$, H is the ($n-1$ times) mean curvature of $\Gamma(t)$, $\operatorname{div}_\Gamma$ is the surface divergence operator on $\Gamma(t)$, and v^T is a tangential vector field satisfying $v = V_\Gamma^N \nu_\Gamma + v^T$. Note that this equation is obtained as the zero width limit of the corresponding equation (3.1.1) in a moving thin domain $\Omega_\varepsilon(t)$ defined as the set of all points in $\mathbb{R}^n$ with distance less than ε from $\Gamma(t)$ (see Remark 3.4.2).

The conventional diffusion equations, or even the porous-media equations, can be viewed as the combination of incompressible fluids with the damping in the form of Darcy's law. Take the usual Darcy's law for the velocity u in the moving thin domain $\Omega_\varepsilon(t)$:

$$-\rho u = \nabla p(\rho) \quad \text{in} \quad \Omega_\varepsilon(t),\, t \in (0,T), \tag{3.1.4}$$

where p is the pressure, then we can prove (see Theorem 3.4.1) that the zero width limit of the diffusion equations (3.1.1) and (3.1.4) yields diffusion equations on the moving surface $\Gamma(t)$: (3.1.3) and Darcy's law

$$-\eta v^T = \nabla_\Gamma p(\eta) \quad \text{on} \quad \Gamma(t),\, t \in (0,T). \tag{3.1.5}$$

Here v^T is the tangential component of the velocity v and ∇_Γ is the tangential gradient operator on $\Gamma(t)$.

The diffusion equations (3.1.1), (3.1.4) and (3.1.3), (3.1.5) possess specific energy identities. It can be easily proven (see Section 3.5) that for ρ and u satisfying (3.1.1) and (3.1.4) the energy identity

$$\frac{d}{dt}\int_{\Omega(t)} \omega(\rho)\,dx = -\int_{\Omega(t)} \rho|u|^2\,dx - \int_{\partial\Omega(t)} p(\rho)V_\Omega^N\,d\mathcal{H}^{n-1} \tag{3.1.6}$$

holds. Here ω is a function satisfying $p(\rho) = \omega'(\rho)\rho - \omega(\rho)$. Similarly, for η and v satisfying (3.1.3) and (3.1.5) we have

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\,d\mathcal{H}^{n-1} = -\int_{\Gamma(t)} \eta|v^T|^2\,d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta)V_\Gamma^N H\,d\mathcal{H}^{n-1}, \tag{3.1.7}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Fortunately, the energy identity (3.1.7) on the moving surface can be derived as the zero width limit of the energy identity (3.1.6) in the moving thin domain (see Theorem 3.5.3).

With the results from this chapter, we can also note that the passing of zero width limit commutes with an energetic variational approach originated from the works of Lord Rayleigh [24] and Onsager [16, 17] and developed by Liu and others [3, 12, 25] (see Section 3.6). In summary, we show that the diagram below is commutative.

$$\begin{tikzcd}
\begin{matrix}\text{Diffusion equation}\\ \text{in } \Omega_\varepsilon(t)\end{matrix} \arrow[r, shift left, "\begin{matrix}\text{integration}\\ \text{by parts}\end{matrix}"] \arrow[d, "\varepsilon \to 0"'] & \begin{matrix}\text{Energy identity}\\ \text{in } \Omega_\varepsilon(t)\end{matrix} \arrow[l, shift left, "\begin{matrix}\text{energetic}\\ \text{variation}\end{matrix}"] \arrow[d, "\varepsilon \to 0"] \\
\begin{matrix}\text{Diffusion equation}\\ \text{on } \Gamma(t)\end{matrix} \arrow[r, shift left, "\begin{matrix}\text{integration}\\ \text{by parts}\end{matrix}"] & \begin{matrix}\text{Energy identity}\\ \text{on } \Gamma(t)\end{matrix} \arrow[l, shift left, "\begin{matrix}\text{energetic}\\ \text{variation}\end{matrix}"]
\end{tikzcd}$$

A standard approach for finding the limit of a thin domain problem is a rescaling argument: one transforms a partial differential equation in a thin domain into that in a fixed in width reference domain by the change of variables and then gets a limit equation by assuming that a rescaled solution is independent of variables in thin directions. In our case where the thin domain and the surface both move, one may transform the moving thin domain into a fixed in time and width reference domain. However, it yields tedious calculations because of the geometry of the limit moving surface and it is difficult to bring a limit equation obtained on a stationary reference surface back to an equation on the original moving surface. One other method is to rescale the width of the moving thin domain without fixing time, which is used in [15] to find the limit of the Neumann type problem of the heat equation (equations

(3.1.1), (3.1.2), and (3.1.4) with $p(\rho) = \rho$) in moving thin domains. However, it is still complicated and requires a questionable assumption that the boundary condition holds in a middle of the moving thin domain. It is also artificial in the sense that we have to make rescaled solutions constant in the thin direction at an "appropriate" point to derive the limit energy identity and if we take a wrong point then we get a wrong limit (see Remarks 3.5.5).

To derive a thin width limit with more straightforward calculations we consider the Taylor series of a function on $\Omega_\varepsilon(t)$ in powers of the signed distance from $\Gamma(t)$. We assume that $\Omega_\varepsilon(t)$ admits the normal coordinate system around $\Gamma(t)$, i.e. for each $x \in \Omega_\varepsilon(t)$ there exists a unique point $\pi(x,t) \in \Gamma(t)$ such that

$$x = \pi(x,t) + d(x,t)\nu_\Gamma(\pi(x,t),t),$$

where d is the signed distance function from $\Gamma(t)$ increasing in the direction of ν_Γ. Based on the normal coordinate system we consider expansions of

$$\begin{aligned}\rho(x,t) &= \rho(\pi(x,t) + d(x,t)\nu_\Gamma(\pi(x,t),t),t),\\ u(x,t) &= u(\pi(x,t) + d(x,t)\nu_\Gamma(\pi(x,t),t),t)\end{aligned}$$

in powers of the signed distance $d(x,t)$:

$$\begin{aligned}\rho(x,t) &= \eta(\pi(x,t),t) + d(x,t)\eta^1(\pi(x,t),t) + d(x,t)^2\eta^2(\pi(x,t),t) + \cdots,\\ u(x,t) &= v(\pi(x,t),t) + d(x,t)v^1(\pi(x,t),t) + d(x,t)^2v^2(\pi(x,t),t) + \cdots.\end{aligned} \tag{3.1.8}$$

In these expansions we assume that η, v, and the coefficients of the powers of $d(x,t)$ are functions on $\Gamma(t)$ and independent of ε (note that the functions ρ and u on $\Omega_\varepsilon(t)$ depend on ε). Under this and other suitable assumptions, we obtain the limit equations (3.1.3) and (3.1.5) as the zeroth order terms of expansions in powers of $d(x,t)$ (or ε) of the bulk equations (3.1.1), (3.1.2), and (3.1.4) by differentiating (3.1.8) and substituting them for (3.1.1), (3.1.2), and (3.1.4) (see Section 3.4). Note that, if we take the average of (3.1.8) in the normal direction of $\Gamma(t)$, then we get

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\rho(y + r\nu_\Gamma(y,t),t)\,dr = \eta(y,r) + (\text{higher order terms in } \varepsilon), \quad y \in \Gamma(t)$$

and a similar equality for u. Thus, formally speaking, we derive the limit equations (3.1.3) and (3.1.5) as equations on $\Gamma(t)$ satisfied by the limit as $\varepsilon \to 0$ of the averages of ρ and u in the thin direction.

The idea mentioned above also applies to derivation of the energy identity (3.1.7) on the moving surface from that in the moving thin domain (3.1.6) (see Section 3.5). To get the limit energy identity we use integral transformation formulas from surface integrals over the level-set surfaces $\{x \in \mathbb{R}^n \mid d(x,t) = r\}$ $(-\varepsilon < r < \varepsilon)$ into that over the zero level-set surface $\Gamma(t)$ (see Lemma 3.5.4).

There is a long history in the study of partial differential equations in thin domains, such as the pioneering work by Hale and Raugel [8,9], where they investigated damped hyperbolic equations and reaction-diffusion equations in a flat stationary thin domain of the form

$$\Omega_\varepsilon = \{(x',x_n) \in \mathbb{R}^n \mid x' \in \omega,\ 0 < x_n < \varepsilon g(x')\}, \tag{3.1.9}$$

where ω is an open set in $\mathbb{R}^{n-1}$ and g is a function on ω. There is also a large number of the literature on reaction-diffusion equations in various types of thin domains such as a thin L-shaped domain [10], a moving flat thin domain of the form (3.1.9) with g time-dependent

[18], and flat and curved thin domains with holes [19–21] (here a curved thin domain is a thin domain degenerating into a lower dimensional manifold). A main subject in the above literature is to compare the dynamics of equations in thin domains with that of limit equations in their limit sets rather than to find the limit equations of the original equations in the thin domains, since their limit sets are stationary and thus the rescaling argument works well for finding the limit equations. The Navier–Stokes equations in thin domains has been also studied well [11, 13, 23, 26, 27] since fluid flows in thin domains often appear in natural sciences like the flow of water in a large lake, geophysical flows, etc. Researchers are especially interested in the relation between the smallness of the width of thin domains and the large time behavior of solutions to the Navier–Stokes equations in thin domains. We refer to [22] and references therein for other types of thin domains degenerating into stationary sets and mathematical analysis of partial differential equations in such thin domains.

In the case where the limit set of a thin domain moves, derivation of the limit of a partial differential equation in the thin domain is more complicated since the geometry of the limit set changes as it moves. Such a problem was first considered in [15] where the author derived both formally and rigorously the limit equation of the Neumann type problem of the heat equation (equations (3.1.1), (3.1.2), and (3.1.4) with $p(\rho) = \rho$) in a moving thin domain degenerating into a closed smooth moving surface. He also found that the normal velocity and the mean curvature of the degenerate moving surface affects the limit equation, which is not observed in the case where the limit set of a thin domain does not move.

The rest of this chapter is organized as follows. In Section 3.2, we fix notations on various quantities related to the moving surface. In Section 3.3, we briefly observe that the transport equations in the moving domain and the moving surface are equivalent to the local mass conservation. In Section 3.4, we derive the limit equations (3.1.3) and (3.1.5) on the moving surface from the diffusion equations (3.1.1), (3.1.2), and (3.1.4) on the moving thin domain by means of expansion in terms of the signed distance. In Section 3.5, we derive the energy identities (3.1.6) and (3.1.7) from corresponding diffusion equations and then show that the energy identity (3.1.7) on the moving surface is the zero width limit of the energy identity (3.1.6) on the moving thin domain. In Section 3.6, we apply an energetic variational approach to the energy identities (3.1.6) and (3.1.7) to obtain Darcy's laws (3.1.4) and (3.1.5).

3.2 Quantities on a moving surface

We start with several notations for a moving surface. Let $\Gamma(t)$, $t \in [0, T]$ be an $(n-1)$-dimensional closed (that is, compact and without boundary), connected, oriented and smooth moving surface in $\mathbb{R}^n$ with $n \geq 2$. Also, let

$$S_T := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\} \subset \mathbb{R}^{n+1}$$

be a space-time hypersurface associated with the moving surface $\Gamma(t)$. For each $t \in [0, T]$ we write $\nu_\Gamma(\cdot, t)$, $V_\Gamma^N(\cdot, t)$, and $d(\cdot, t)$ for the unit outward normal vector field of $\Gamma(t)$, the scalar outward normal velocity of $\Gamma(t)$, and the signed distance function from $\Gamma(t)$, respectively. Note that to describe the evolution of a closed surface it is sufficient to give the normal velocity. Since the smooth closed surface $\Gamma(t)$ varies smoothly in time, the principal curvatures $\kappa_1(\cdot, t), \dots, \kappa_{n-1}(\cdot, t)$ of $\Gamma(t)$ are bounded uniformly in $t \in [0, T]$. Then there exists a constant $\delta > 0$ independent of t such that for each $t \in [0, T]$ the tubular neighborhood of

$\Gamma(t)$ of the form

$$N(t) := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \Gamma(t)) < \delta\}$$

admits the normal coordinate system

$$x = \pi(x,t) + d(x,t)\nu_\Gamma(\pi(x,t),t), \quad x \in N(t), \tag{3.2.1}$$

where $\pi(x,t)$ is the closest point on $\Gamma(t)$ to x (see [6, Section 14.6] for example). For each $t \in [0,T]$ we suppose that $d(\cdot,t)$ increases along the direction of $\nu_\Gamma(\cdot,t)$. Then we have

$$\begin{aligned} \nabla d(x,t) &= \nu_\Gamma(\pi(x,t),t), \quad && (x,t) \in N(t), \\ \partial_t d(y,t) &= -V_\Gamma^N(y,t), && (y,t) \in \Gamma(t), \end{aligned} \tag{3.2.2}$$

Moreover, differentiating both sides of

$$d(x,t) = \{x - \pi(x,t)\} \cdot \nabla d(x,t), \quad d(\pi(x,t),t) = 0$$

with respect to t we easily obtain

$$\partial_t d(x,t) = \partial_t d(\pi(x,t),t) = -V_\Gamma^N(\pi(x,t),t), \quad (x,t) \in N_T, \tag{3.2.3}$$

where $N_T := \bigcup_{t\in(0,T)} N(t) \times \{t\}$.

Next we fix $t \in [0,T]$ and give differential operators on the surface $\Gamma(t)$. We define the orthogonal projection onto the tangent plane of $\Gamma(t)$ by

$$P_\Gamma(y,t) := I_n - \nu_\Gamma(y,t) \otimes \nu_\Gamma(y,t), \quad y \in \Gamma(t).$$

Here I_n denotes the identity matrix of size n and $a \otimes b = (a_i b_j)_{i,j}$ is the tensor product of two vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ in $\mathbb{R}^n$. For a function $f \colon \Gamma(t) \to \mathbb{R}$ and a vector field $F \colon \Gamma(t) \to \mathbb{R}^n$ we define the tangential gradient of f and the surface divergence of F as

$$\nabla_\Gamma f(y) := P_\Gamma(y,t)\nabla\tilde{f}(y), \quad \operatorname{div}_\Gamma F(y) := \operatorname{tr}[P_\Gamma(y,t)\nabla\widetilde{F}(y)], \quad y \in \Gamma(t).$$

Here $\tilde{f}$ and $\widetilde{F}$ are extensions of f and F to $N(t)$ satisfying $\tilde{f} = f$ and $\widetilde{F} = F$ on $\Gamma(t)$. Also, $\operatorname{tr}[M]$ denotes the trace of a square matrix M and we use the notation

$$\nabla G = \begin{pmatrix} \partial_1 G_1 & \dots & \partial_1 G_n \\ \vdots & \ddots & \vdots \\ \partial_n G_1 & \dots & \partial_n G_n \end{pmatrix}$$

for the gradient matrix of a vector field $G = (G_1, \dots, G_n)$. Note that the tangential gradient of f and the surface divergence of F do not depend on a choice of extensions (see e.g. [5, Lemma 2.4]). Moreover, for any function f on $\Gamma(t)$ we easily see that

$$\nabla_\Gamma f(y) \cdot \nu_\Gamma(y,t) = 0, \quad y \in \Gamma(t). \tag{3.2.4}$$

We define the ($n-1$ times) mean curvature H of $\Gamma(t)$ as

$$H(y,t) := -\operatorname{div}_\Gamma \nu_\Gamma(y,t), \quad y \in \Gamma(t). \tag{3.2.5}$$

Note that the mean curvature is equal to the sum of the principal curvatures:

$$H(y,t) = \sum_{i=1}^{n-1} \kappa_i(y,t), \quad y \in \Gamma(t). \tag{3.2.6}$$

Finally, for a function f on the space-time hypersurface S_T we define the normal time derivative (the time derivative along the normal velocity) as

$$\partial^\circ f(y,t) := \partial_t \tilde{f}(y,t) + V_\Gamma(y,t)\nu_\Gamma(y,t) \cdot \nabla \tilde{f}(y,t), \quad (y,t) \in S_T.$$

Here $\tilde{f}$ is an extension of f to N_T satisfying $\tilde{f} = f$ on S_T. Note that the value of $\partial^\circ f$ does not depend on a choice of an extension of f and the formula

$$\partial^\circ f(y,t) = \frac{d}{dt}\bigl(f(\pi(y,t),t)\bigr), \quad (y,t) \in S_T \tag{3.2.7}$$

holds (see [2, Section 3.4] for details).

3.3 Transport equation in a moving domain and on a moving surface

In this section we give the transport equation for a scalar quantity in a moving domain and on a moving surface. We use some of the same terminology and techniques as in [14]. We first consider transportation of a scalar quantity in a bounded moving domain $\Omega(t)$ in $\mathbb{R}^n$. Let $\rho(x,t)$ and $u(x,t)$ be the density and the velocity field of the scalar quantity at $x \in \Omega(t)$, respectively. Our starting point is the local mass conservation

$$\frac{d}{dt}\int_{U(t)} \rho\, dx = 0 \tag{3.3.1}$$

for any portion $U(t)$ (relatively open set) of $\Omega(t)$ moving with velocity $u(\cdot,t)$ and whose closure (in $\mathbb{R}^n$) is contained in $\Omega(t)$. Since the left-hand side is equal to $\int_{U(t)}\{\partial_t\rho + \mathrm{div}(\rho u)\}\, dx$ by the Reynolds transport theorem [7] and the divergence theorem, the condition (3.3.1) for any $U(t)$ is equivalent to the transport equation

$$\partial_t\rho + \mathrm{div}(\rho u) = 0 \quad \text{in} \quad Q_T := \bigcup_{t\in(0,T)} \Omega(t) \times \{t\}. \tag{3.3.2}$$

To make the total mass $\int_{\Omega(t)} \rho\, dx$ conserved, we impose the boundary condition

$$u \cdot \nu_\Omega = V_\Omega^N \quad \text{on} \quad \partial_\ell Q_T := \bigcup_{t\in(0,T)} \partial\Omega(t) \times \{t\}, \tag{3.3.3}$$

where $\nu_\Omega(\cdot,t)$ and $V_\Omega^N(\cdot,t)$ are the unit outward normal vector field and the scalar outward normal velocity of $\partial\Omega(t)$, respectively. The boundary condition (3.3.3) physically means that the quantity in $\Omega(t)$ moves along the boundary of $\Omega(t)$ and it does not go into and out of $\Omega(t)$.

Next we give the transport equation for a scalar quantity on a moving surface. Let $\Gamma(t)$ be a closed, connected, oriented moving surface in $\mathbb{R}^n$. As in Section 3.2, we write $\nu_\Gamma(\cdot,t)$ and $V_\Gamma(\cdot,t)$ for the outward normal vector field and the scalar outward normal velocity of

$\Gamma(t)$, respectively. Suppose that for each $t \in (0,T)$ a scalar quantity on $\Gamma(t)$ has the density $\eta(y,t)$ at $y \in \Gamma(t)$ and moves with velocity

$$v(y,t) = V_\Gamma(y,t)\nu_\Gamma(y,t) + v^T(y,t), \quad y \in \Gamma(t),$$

where $v^T(\cdot,t)$ is a given tangential velocity field on $\Gamma(t)$. Then its local mass conservation is expressed as

$$\int_{U(t)} \eta \, d\mathcal{H}^{n-1} = 0 \tag{3.3.4}$$

for any portion $U(t)$ (relatively open set) of $\Gamma(t)$ moving with velocity $v(\cdot,t)$. The Leibniz formula [4, Lemma 2.2] yields

$$\frac{d}{dt}\int_{U(t)} \eta \, d\mathcal{H}^{n-1} = \int_{U(t)} \{\partial^\circ \eta - V_\Gamma^N H\eta + \mathrm{div}_\Gamma(\eta v^T)\} \, d\mathcal{H}^{n-1}.$$

From this formula, the condition (3.3.4) for any $U(t)$ is equivalent to

$$\partial^\circ \eta - V_\Gamma^N H\eta + \mathrm{div}_\Gamma(\eta v^T) = 0 \quad \text{on} \quad S_T. \tag{3.3.5}$$

This is the transport equation on the moving surface $\Gamma(t)$.

3.4 Zero width limit for nonlinear diffusion equations

Let us consider nonlinear diffusion of a scalar quantity in $\Omega(t)$ with density ρ and velocity u. Suppose that the diffusion process is described by the transport equation (3.3.2) and Darcy's law $-\rho u = \nabla p(\rho)$, where

$$p(\rho) := \omega'(\rho)\rho - \omega(\rho) \tag{3.4.1}$$

is the pressure with a given function $\omega(\rho)$, $\rho \in \mathbb{R}$. We impose the boundary condition (3.3.3). Hence the nonlinear diffusion equations we deal with are

$$\partial_t\rho + \mathrm{div}(\rho u) = 0 \qquad \text{in} \quad Q_T, \tag{3.4.2}$$

$$-\rho u = \nabla p(\rho) \quad \text{in} \quad Q_T, \tag{3.4.3}$$

$$u \cdot \nu_\Omega = V_\Omega^N \qquad \text{on} \quad \partial_\ell Q_T. \tag{3.4.4}$$

We consider these equations in a moving thin domain. For sufficiently small $\varepsilon > 0$, we define a moving thin domain $\Omega_\varepsilon(t)$ as the set of all points in $\mathbb{R}^n$ with distance less than ε from the moving surface $\Gamma(t)$:

$$\Omega_\varepsilon(t) := \{x \in \mathbb{R}^n \mid \mathrm{dist}(x,\Gamma(t)) < \varepsilon\}. \tag{3.4.5}$$

We write $Q_{\varepsilon,T}$ and $\partial_\ell Q_{\varepsilon,T}$ for Q_T and $\partial_\ell Q_T$ with $\Omega(t) = \Omega_\varepsilon(t)$. Our goal in this section is to find the limit equations of (3.4.2)–(3.4.4) in $\Omega_\varepsilon(t)$ as ε goes to zero, that is, the moving thin domain $\Omega_\varepsilon(t)$ degenerates into the moving surface $\Gamma(t)$. According to the normal coordinate system (3.2.1), we expand ρ and u in powers of the signed distance $d(x,t)$ as

$$\rho(x,t) = \eta(\pi(x,t),t) + d(x,t)\eta^1(\pi(x,t),t) + R(d(x,t)^2), \tag{3.4.6}$$

$$u(x,t) = v(\pi(x,t),t) + d(x,t)v^1(\pi(x,t),t) + R(d(x,t)^2) \tag{3.4.7}$$

for $(x,t) \in Q_{\varepsilon,T}$ and assume that η, v, and the coefficients of $d(x,t)^k$ for each $k \in \mathbb{N}$ in (3.4.6) and (3.4.7) are independent of ε. Here $R(d(x,t)^k)$ $(k \in \mathbb{N})$ is the sum of the terms of order equal to or higher than k with respect to small $d(x,t)$. In particular, $R(f(x,t))$ for a function $f(x,t)$ can be of the form

$$R(f(x,t)) = f(x,t)g(x,t)$$

with some (bounded) function $g(x,t)$. Note that we can differentiate $R(d(x,t)^k)$ and its j-th order derivative is of the form $R(d(x,t)^{k-j})$ for $j \leq k$ although we cannot differentiate $O(d(x,t)^k)$ since it only represents a quantity whose absolute value is bounded above by $|d(x,t)|^k$. Also, since $d(x,t)$ is of order ε on $Q_{\varepsilon,T}$, we have $R(d(x,t)^k) = O(\varepsilon^k)$ for $(x,t) \in Q_{\varepsilon,T}$ and $k \in \mathbb{N}$.

Under the expansions (3.4.6) and (3.4.7), the limit equations of (3.4.2)–(3.4.4) in $\Omega_\varepsilon(t)$ as ε goes to zero are given as equations on $\Gamma(t)$ satisfied by η and v.

Theorem 3.4.1. *Let ρ and u satisfy the equations* (3.4.2)–(3.4.4) *in the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$ given by* (3.4.5)*. Also, let η and v be the zeroth order terms in the expansions* (3.4.6) *and* (3.4.7) *of ρ and u, respectively. Then v is of the form*

$$v = V_\Gamma^N \nu_\Gamma + v^T \quad \textit{on} \quad S_T \tag{3.4.8}$$

with some tangential velocity field v^T on $\Gamma(t)$, and η and v satisfy the equations

$$\partial^\circ \eta - V_\Gamma^N H\eta + \mathrm{div}_\Gamma(\eta v^T) = 0 \qquad \textit{on} \quad S_T, \tag{3.4.9}$$

$$-\eta v^T = \nabla_\Gamma p(\eta) \quad \textit{on} \quad S_T. \tag{3.4.10}$$

Proof. For the sake of simplicity, we use the abbreviations

$$f(\pi,t) = f(\pi(x,t),t), \quad R(d^k) = R(d(x,t)^k) \tag{3.4.11}$$

for a function f on S_T, $(x,t) \in Q_{\varepsilon,T}$, and $k \in \mathbb{N}$. We also abbreviate the product of several functions with the same argument like

$$[u_1 \cdot u_2](x,t) = u_1(x,t) \cdot u_2(x,t) \tag{3.4.12}$$

for vector fields u_1 and u_2 on $Q_{\varepsilon,T}$. First we show that v is of the form (3.4.8). By the definition (3.4.5) of the moving thin domain $\Omega_\varepsilon(t)$, the unit outward normal vector and the outward normal velocity of its boundary are given by

$$\nu_\Omega(x,t) = \pm\nu_\Gamma(\pi,t), \quad V_\Omega^N(x,t) = \pm V_\Gamma^N(\pi,t) \tag{3.4.13}$$

for $(x,t) \in \partial_\ell Q_{\varepsilon,T}$ with $d(x,t) = \pm\varepsilon$ (double-sign corresponds). Hence the boundary condition (3.4.4) reads

$$u(x,t) \cdot \nu_\Gamma(\pi,t) = V_\Gamma^N(\pi,t), \quad (x,t) \in \partial_\ell Q_{\varepsilon,T}.$$

We substitute (3.4.7) for u in the above equality. Then

$$[v \cdot \nu](\pi,t) \pm \varepsilon[v^1 \cdot \nu](\pi,t) + O(\varepsilon^2) = V_\Gamma^N(\pi,t).$$

Since v, v^1, ν_Γ, and V_Γ^N are independent of ε, it follows that

$$[v \cdot \nu](\pi(x,t),t) = V_\Gamma(\pi(x,t),t), \quad [v^1 \cdot \nu](\pi(x,t),t) = 0$$

for all $(x,t) \in \partial_\ell Q_{\varepsilon,T}$, which imply that

$$[v \cdot \nu](y,t) = V_\Gamma^N(y,t), \quad (y,t) \in S_T, \tag{3.4.14}$$
$$[v^1 \cdot \nu](y,t) = 0, \quad (y,t) \in S_T. \tag{3.4.15}$$

Hence v is of the form (3.4.8) with some tangential velocity field v^T on $\Gamma(t)$.

Next we derive the equations (3.4.9)–(3.4.10). Let $(x,t) \in Q_{\varepsilon,T}$. We differentiate both sides of (3.4.6) with respect to t and apply (3.2.3) and (3.2.7) to get

$$\partial_t \rho(x,t) = \partial^\circ \eta(\pi,t) - [V_\Gamma^N \eta^1](\pi,t) + R(d). \tag{3.4.16}$$

Let us compute the divergence of ρu. We differentiate $\pi(x,t) = x - d(x,t)\nu_\Gamma(\pi,t)$ with respect to x and apply (3.2.2) to get

$$\nabla \pi(x,t) = P_\Gamma(\pi,t) + R(d). \tag{3.4.17}$$

From the expansions (3.4.6) and (3.4.7),

$$[\rho u](x,t) = V(\pi,t) + d(x,t)V^1(\pi,t) + R(d^2), \tag{3.4.18}$$

where

$$V(\pi,t) := [\eta v](\pi,t), \tag{3.4.19}$$
$$V^1(\pi,t) := [\eta v^1](\pi,t) + [\eta^1 v](\pi,t). \tag{3.4.20}$$

We differentiate both sides of (3.4.18) with respect to x. Then by (3.2.2) and (3.4.17),

$$\begin{aligned}[\nabla(\rho u)](x,t) &= \nabla \pi(x,t) \nabla V(\pi,t) + \nabla d(x,t) \otimes V^1(\pi,t) + R(d) \\ &= [P_\Gamma \nabla V](\pi,t) + [\nu_\Gamma \otimes V^1](\pi,t) + R(d).\end{aligned}$$

From this formula and $\mathrm{tr}[\nu_\Gamma \otimes V^1] = \nu_\Gamma \cdot V^1$, the divergence of ρu is

$$[\mathrm{div}(\rho u)](x,t) = \mathrm{div}_\Gamma V(\pi,t) + [\nu_\Gamma \cdot V^1](\pi,t) + R(d).$$

Since v is of the form (3.4.8) and V is given by (3.4.19),

$$\begin{aligned}\mathrm{div}_\Gamma V = \mathrm{div}_\Gamma[\eta(V_\Gamma^N \nu_\Gamma + v^T)] &= \nabla_\Gamma(\eta V_\Gamma^N) \cdot \nu_\Gamma + \eta V_\Gamma^N \mathrm{div}_\Gamma \nu_\Gamma + \mathrm{div}_\Gamma(\eta v^T) \\ &= -\eta V_\Gamma^N H + \mathrm{div}_\Gamma(\eta v^T)\end{aligned}$$

on S_T by (3.2.4) and (3.2.5). We also have

$$[\nu_\Gamma \cdot V^1](\pi,t) = [\eta^1 V_\Gamma^N](\pi,t)$$

by (3.4.14), (3.4.15), and (3.4.20). Therefore,

$$[\mathrm{div}(\rho u)](x,t) = -[V_\Gamma^N H \eta](\pi,t) + [\mathrm{div}_\Gamma(\eta v^T)](\pi,t) + [\eta^1 V_\Gamma^N](\pi,t) + R(d). \tag{3.4.21}$$

Substituting (3.4.16) and (3.4.21) for (3.4.2), we obtain

$$\partial^\circ \eta(\pi,t) - [V_\Gamma^N H \eta](\pi,t) + [\mathrm{div}_\Gamma(\eta v^T)](\pi,t) = R(d).$$

Here each term on the left-hand side is independent of $d = d(x,t)$. Hence

$$\partial^\circ \eta(\pi(x,t),t) - [V_\Gamma^N H \eta](\pi(x,t),t) + [\mathrm{div}_\Gamma(\eta v^T)](\pi(x,t),t) = 0$$

for all $(x,t) \in Q_{\varepsilon,T}$, which shows that η and $v = V_\Gamma^N \nu_\Gamma + v^T$ satisfy (3.4.9) on S_T.

Let us derive (3.4.10). We expand the pressure $p(\rho)$ in $d(x,t)$ as

$$p(\rho(x,t)) = p^0(\pi,t) + d(x,t)p^1(\pi,t) + R(d^2), \quad (x,t) \in Q_{\varepsilon,T}. \tag{3.4.22}$$

Then it follows form the expansions (3.4.6) and (3.4.22) that

$$p^0(\pi(x,t),t) = p(\eta(\pi(x,t),t))$$

for all $(x,t) \in Q_{\varepsilon,T}$, which implies that

$$p^0(y,t) = p(\eta(y,t)), \quad (y,t) \in S_T. \tag{3.4.23}$$

Moreover, differentiating (3.4.22) in x and applying (3.2.2) and (3.4.17) we get

$$\begin{aligned}\nabla p(\rho(x,t)) &= \nabla \pi(x,t)\nabla p^0(\pi,t) + p^1(\pi,t)\nabla d(x,t) + R(d) \\ &= \nabla_\Gamma p^0(\pi,t) + [p^1\nu_\Gamma](\pi,t) + R(d).\end{aligned}$$

for $(x,t) \in Q_{\varepsilon,T}$. We substitute this for (3.4.3) and apply (3.4.8). Then we have

$$-[\eta v^T](\pi,t) - [\eta V_\Gamma^N \nu_\Gamma](\pi,t) + R(d) = \nabla_\Gamma p^0(\pi,t) + [p^1\nu_\Gamma](\pi,t) + R(d).$$

Since all terms except of $R(d)$ are independent of $d = d(x,t)$ and the vectors v^T and $\nabla_\Gamma p^0$ are tangential to $\Gamma(t)$, it follows that

$$-[\eta v^T](\pi(x,t),t) = \nabla_\Gamma p^0(\pi(x,t),t), \quad -[\eta V_\Gamma^N](\pi(x,t),t) = p^1(\pi(x,t),t)$$

for all $(x,t) \in Q_{\varepsilon,T}$. Therefore, we get

$$-[\eta v^T](y,t) = \nabla_\Gamma p^0(y,t), \quad (y,t) \in S_T, \tag{3.4.24}$$

$$-[\eta V_\Gamma^N](y,t) = p^1(y,t), \qquad (y,t) \in S_T. \tag{3.4.25}$$

By (3.4.23) and (3.4.24) we conclude that η and v satisfy (3.4.10) on S_T. □

Remark 3.4.2. By the proof of Theorem 3.4.1 we observe that the transport equation (3.3.5) on the moving surface $\Gamma(t)$ can be derived as the limit of the transport equation (3.3.2) in the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$ with the boundary condition (3.3.3) as ε goes to zero.

3.5 Energy law

The subject in this section is the energy law for nonlinear diffusion equations (3.4.2)–(3.4.4) and (3.4.9)–(3.4.10). As in Section 3.4, the pressure $p(\rho)$ is given by (3.4.1) with a given function $\omega(\rho)$.

Proposition 3.5.1. *Assume that ρ and u satisfy* (3.4.2)–(3.4.4)*. Then*

$$\frac{d}{dt}\int_{\Omega(t)} \omega(\rho)\,dx = -\int_{\Omega(t)} \rho|u|^2\,dx - \int_{\partial\Omega(t)} p(\rho)V_\Omega^N\,d\mathcal{H}^{n-1}. \tag{3.5.1}$$

Proof. By the Reynolds transport theorem,

$$\frac{d}{dt}\int_{\Omega(t)} \omega(\rho)\, d\mathcal{H}^{n-1} = \int_{\Omega(t)} \partial_t\omega(\rho)\, dx + \int_{\partial\Omega(t)} \omega(\rho) V_\Omega^N\, d\mathcal{H}^{n-1}.$$

Since $\partial_t\omega(\rho) = \omega'(\rho)\partial_t\rho$ and the transport equation (3.4.2) is satisfied,

$$\partial_t\omega(\rho) = -\omega'(\rho)\mathrm{div}(\rho u) = -\mathrm{div}(\omega'(\rho)\rho u) + \nabla\omega'(\rho)\cdot(\rho u).$$

Hence the divergence theorem and (3.4.4) yield

$$\int_{\Omega(t)} \partial_t\omega(\rho)\, dx = -\int_{\partial\Omega(t)} \omega'(\rho)\rho V_\Omega^N\, d\mathcal{H}^{n-1} + \int_{\Omega(t)} \nabla\omega'(\rho)\cdot(\rho u)\, dx.$$

Using this formula we get

$$\frac{d}{dt}\int_{\Omega(t)} \omega(\rho)\, dx = \int_{\Omega(t)} \nabla\omega'(\rho)\cdot(\rho u)\, dx - \int_{\partial\Omega(t)} \{\omega'(\rho)\rho - \omega(\rho)\} V_\Omega^N\, d\mathcal{H}^{n-1}.$$

The energy law (3.5.1) follows from this equality, (3.4.1), and

$$u = -\frac{\nabla p(\rho)}{\rho} = -\nabla\omega'(\rho)$$

by (3.4.1) and (3.4.3). □

Proposition 3.5.2. *Suppose that η and v of the form* (3.4.8) *satisfy* (3.4.9) *and* (3.4.10). *Then*

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^{n-1} = -\int_{\Gamma(t)} \eta|v^T|^2\, d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta) V_\Gamma^N H\, d\mathcal{H}^{n-1}. \tag{3.5.2}$$

Proof. By the Leibniz formula [4, Lemma 2.2],

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^{n-1} = I - \int_{\Gamma(t)} \omega(\eta) V_\Gamma^N H\, d\mathcal{H}^{n-1},$$

where

$$I = \int_{\Gamma(t)} \{\partial^\circ\omega(\eta) + \mathrm{div}_\Gamma(\omega(\eta)v^T)\}\, d\mathcal{H}^{n-1}.$$

Since $\partial^\circ\omega(\eta) = \omega'(\eta)\partial^\circ\eta$, the transport equation (3.4.9) implies that

$$\partial^\circ\omega(\eta) = \omega'(\eta)\{V_\Gamma^N H\eta - \mathrm{div}_\Gamma(\eta v^T)\} = \omega'(\eta)V_\Gamma^N H\eta + \nabla_\Gamma\omega'(\eta)\cdot(\eta v^T) - \mathrm{div}_\Gamma(\omega'(\eta)\eta v^T).$$

Hence

$$I = \int_{\Gamma(t)} \{\omega'(\eta)V_\Gamma^N H\eta + \nabla_\Gamma\omega'(\eta)\cdot(\eta v^T)\}\, d\mathcal{H}^{n-1} + \int_{\Gamma(t)} \mathrm{div}_\Gamma[(\omega(\eta) - \omega'(\eta)\eta)v^T]\, d\mathcal{H}^{n-1}.$$

The second integral on the right-hand side vanishes by the Stokes formula and the fact that v^T is tangential and $\Gamma(t)$ has no boundary. Therefore,

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^{n-1} = \int_{\Gamma(t)} \nabla_\Gamma\omega'(\eta)\cdot(\eta v^T)\, d\mathcal{H}^{n-1} + \int_{\Gamma(t)} \{\omega'(\eta)\eta - \omega(\eta)\} V_\Gamma^N H\, d\mathcal{H}^{n-1}.$$

Applying

$$v^T = -\frac{\nabla_\Gamma p(\eta)}{\eta} = -\nabla_\Gamma\omega'(\eta),$$

which follows from (3.4.1) and (3.4.10), to the first term on the right-hand side and (3.4.1) to the second term, we get the energy identity (3.5.2). □

Next we derive the energy law (3.5.2) as a limit of the energy law (3.5.1) with the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$ when ε goes to zero. As in Section 3.4, we expand ρ and u in powers of the signed distance as (3.4.6) and (3.4.7) and determine an equality satisfied by η and v.

Theorem 3.5.3. *Let ρ and u satisfy the energy law* (3.5.1) *in the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$ given by* (3.4.5)*. Also, let η and v be the zeroth order terms in the expansions* (3.4.6) *and* (3.4.7) *of ρ and u, respectively. Assume that v is of the form* (3.4.8) *with some tangential velocity field v^T on $\Gamma(t)$ and Darcy's law* (3.4.3) *holds in $\Omega_\varepsilon(t)$. Then η and v satisfy the energy law* (3.5.2)*.*

We give change of variables formulas for integrals which we use in the proof of Theorem 3.5.3. For $y \in \Gamma(t)$ and $\rho \in [-\varepsilon, \varepsilon]$ we set

$$J(y,t,r) := \prod_{i=1}^{n-1} \{1 - r\kappa_i(y,t)\}, \tag{3.5.3}$$

where $\kappa_1(\cdot,t), \dots, \kappa_{n-1}(\cdot,t)$ are the principal curvatures of $\Gamma(t)$. It is the Jacobian that appears when we change variables of integrals over a tubular neighborhood $\{x \in \mathbb{R}^n \mid -r < d(x,t) < r\}$ $(r > 0)$ of $\Gamma(t)$ and a level-set surface $\{x \in \mathbb{R}^n \mid d(x,t) = s\}$ $(s \in \mathbb{R})$ in terms of the normal coordinate system around $\Gamma(t)$ (see [6, Section 14.6] for example). The first formula in Lemma 3.5.4 is often called the co-area formula.

Lemma 3.5.4. *For functions f on $\Omega_\varepsilon(t)$ and g on $\partial\Omega_\varepsilon(t)$ we have*

$$\int_{\Omega_\varepsilon(t)} f(x)\,dx = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} f(y + r\nu_\Gamma(y,t)) J(y,t,r)\,dr\,d\mathcal{H}^{n-1}(y) \tag{3.5.4}$$

and

$$\begin{aligned} \int_{\partial\Omega_\varepsilon(t)} g(x)\,d\mathcal{H}^{n-1}(x) &= \int_{\Gamma(t)} g(y + \varepsilon\nu_\Gamma(y,t)) J(y,t,\varepsilon)\,d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\Gamma(t)} g(y - \varepsilon\nu_\Gamma(y,t)) J(y,t,-\varepsilon)\,d\mathcal{H}^{n-1}(y). \end{aligned} \tag{3.5.5}$$

Proof of Theorem 3.5.3. As in the proof of Theorem 3.4.1, we use the abbreviations (3.4.11) and (3.4.12). Let us calculate each term of (3.5.1). We expand $\omega(\rho)$ in powers of the signed distance $d(x,t)$ as

$$\omega(\rho(x,t)) = \omega(\eta(\pi,t)) + d(x,t)\omega^1(\pi,t) + R(d^2), \quad (x,t) \in Q_{\varepsilon,T}.$$

Here the zeroth order term is $\omega(\eta(\pi,t))$ since the zeroth order term of $\rho(x,t)$ is $\eta(\pi,t)$. We divide the integral of $\omega(\rho)$ over $\Omega_\varepsilon(t)$ as

$$\int_{\Omega_\varepsilon(t)} \omega(\rho(x,t))\,dx = I_1 + I_2 + I_3,$$

where

$$I_1 := \int_{\Omega_\varepsilon(t)} \omega(\eta(\pi,t))\,dx, \quad I_2 := \int_{\Omega_\varepsilon(t)} d(x,t)\omega^1(\pi,t)\,dx, \quad I_3 := \int_{\Omega_\varepsilon(t)} R(d(x,t)^2)\,dx.$$

By the co-area formula (3.5.4) and the fact that $J(y,t,r)$ is a polynomial in r with $J(y,t,0) = 1$ whose coefficients are polynomials in the principal curvatures, we have

$$I_1 = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} \omega(\eta(y,t)) J(y,t,r)\, dr\, d\mathcal{H}^{n-1}(y) = 2\varepsilon \int_{\Gamma(t)} \omega(\eta(y,t))\, d\mathcal{H}^{n-1}(y) + \varepsilon^2 f_1(\varepsilon,t),$$

where $f_1(\varepsilon,t)$ is a polynomial in ε with time-dependent coefficients. Therefore,

$$\frac{dI_1}{dt} = 2\varepsilon \frac{d}{dt} \int_{\Gamma(t)} \omega(y,t)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \tag{3.5.6}$$

Similarly we have

$$I_2 = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} r\omega^1(y,t) J(y,t,r)\, dr\, d\mathcal{H}^{n-1}(y) = \varepsilon^2 f_2(\varepsilon,t)$$

with a polynomial $f_2(\varepsilon,t)$ in ε with time-dependent coefficients and thus

$$\frac{dI_2}{dt} = O(\varepsilon^2). \tag{3.5.7}$$

We apply the Reynolds transport theorem to the time derivative of I_3. Then, since the time derivative of $R(d(x,t)^2)$ is $R(d(x,t))$, we have

$$\frac{dI_3}{dt} = \int_{\Omega_\varepsilon(t)} R(d(x,t))\, dx + \int_{\partial\Omega_\varepsilon(t)} R(d(x,t)^2) V_\Omega^N(x,t)\, d\mathcal{H}^{n-1}(x).$$

Since $J(y,t,r)$ is bounded independently of ε, the co-area formula (3.5.4) yields

$$\int_{\Omega_\varepsilon(t)} R(d(x,t))\, dx = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} R(r) J(y,t,r)\, dr\, d\mathcal{H}^{n-1}(y) = O(\varepsilon^2).$$

Moreover, applying (3.4.13) and (3.5.5) to the integral of $R(d(x,t)^2)V_\Omega^N(x,t)$ over $\partial\Omega_\varepsilon(t)$ and observing that $R(d(x,t)^2) = R(\varepsilon^2)$ holds for $x \in \partial\Omega_\varepsilon(t)$ we have

$$\int_{\partial\Omega_\varepsilon(t)} R(d(x,t)^2) V_\Omega^N(x,t)\, d\mathcal{H}^{n-1}(x) = O(\varepsilon^2).$$

Thus, we get the estimate

$$\frac{dI_3}{dt} = O(\varepsilon^2). \tag{3.5.8}$$

Since the integral of $\omega(\rho)$ over $\Omega_\varepsilon(t)$ is the sum of I_1, I_2, and I_3, it follows from (3.5.6), (3.5.7), and (3.5.8) that

$$\frac{d}{dt} \int_{\Omega_\varepsilon(t)} \omega(\rho(x,t))\, dx = 2\varepsilon \frac{d}{dt} \int_{\Gamma(t)} \omega(\eta(y,t))\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \tag{3.5.9}$$

Next we calculate the first term on the right-hand side of (3.5.1). From the expansions (3.4.6) and (3.4.7), the product $\rho|u|^2$ is of the form

$$[\rho|u|^2](x,t) = [\eta|v|^2](\pi,t) + R(d), \quad (x,t) \in Q_{\varepsilon,T}.$$

Hence, by (3.5.4),

$$\begin{aligned} \int_{\Omega_\varepsilon(t)} [\rho|u|^2](x,t)\,dx &= \int_{\Gamma(t)}\int_{-\varepsilon}^{\varepsilon} \{[\eta|v|^2](y,t)+R(r)\}J(y,t,r)\,dr\,d\mathcal{H}^{n-1}(y) \\ &= 2\varepsilon\int_{\Gamma(t)}[\eta|v|^2](y,t)\,d\mathcal{H}^{n-1}(y)+O(\varepsilon^2). \end{aligned} \tag{3.5.10}$$

Let us compute the last term on the right-hand side of (3.5.1). We expand the pressure $p(\rho)$ in $d(x,t)$ as (3.4.22). Then, by the assumption that v is of the form (3.4.8) and Darcy's law (3.4.3) holds, we get (3.4.23) and (3.4.25) as in the proof of Theorem 3.4.1 and thus we can write

$$p(\rho(x,t)) = p(\eta(\pi,t)) - d(x,t)[\eta V_\Gamma^N](\pi,t) + R(d^2), \quad (x,t)\in Q_{\varepsilon,T}.$$

Therefore, by (3.4.13) and (3.5.5),

$$\int_{\partial\Omega_\varepsilon(t)} [p(\rho)V_\Omega^N](x,t)\,d\mathcal{H}^{n-1}(x) = J_1 + J_2 + O(\varepsilon^2),$$

where

$$\begin{aligned} J_1 &:= \int_{\Gamma(t)}[p(\eta)V_\Gamma^N](y,t)\{J(y,t,\varepsilon)-J(y,t,-\varepsilon)\}\,d\mathcal{H}^{n-1}(y), \\ J_2 &:= -\varepsilon\int_{\Gamma(t)}[\eta|V_\Gamma^N|^2](y,t)\{J(y,t,\varepsilon)+J(y,t,-\varepsilon)\}\,d\mathcal{H}^{n-1}(y). \end{aligned}$$

By (3.5.3) and (3.2.6) we have

$$\begin{aligned} J(y,t,\varepsilon)-J(y,t,-\varepsilon) &= -2\varepsilon H(y,t)+O(\varepsilon^2), \\ J(y,t,\varepsilon)+J(y,t,-\varepsilon) &= 2+O(\varepsilon). \end{aligned}$$

Hence it follows that

$$\begin{aligned} J_1 &= -2\varepsilon\int_{\Gamma(t)}[p(\eta)V_\Gamma^N H](y,t)\,d\mathcal{H}^{n-1}(y)+O(\varepsilon^2), \\ J_2 &= -2\varepsilon\int_{\Gamma(t)}[\eta|V_\Gamma^N|^2](y,t)\,d\mathcal{H}^{n-1}(y)+O(\varepsilon^2) \end{aligned}$$

and the integral of $p(\rho)V_\Omega^N$ over $\partial\Omega_\varepsilon(t)$ becomes

$$\begin{aligned} \int_{\partial\Omega_\varepsilon(t)}[p(\rho)V_\Omega^N](x,t)\,d\mathcal{H}^{n-1}(x) &= -2\varepsilon\int_{\Gamma(t)}[p(\eta)V_\Gamma^N H](y,t)\,d\mathcal{H}^{n-1}(y) \\ &\quad - 2\varepsilon\int_{\Gamma(t)}[\eta|V_\Gamma^N|^2](y,t)\,d\mathcal{H}^{n-1}(y)+O(\varepsilon^2). \end{aligned} \tag{3.5.11}$$

Finally, substituting (3.5.9), (3.5.10), and (3.5.11) for (3.5.1), dividing both sides by 2ε, and observing that $|v|^2 = |V_\Gamma^N|^2 + |v^T|^2$ we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\Gamma(t)}\omega(\eta(y,t))\,d\mathcal{H}^{n-1}(y) &= -\int_{\Gamma(t)}[\eta|v^T|^2](y,t)\,d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\Gamma(t)}[p(\eta)V_\Gamma^N H](y,t)\,d\mathcal{H}^{n-1}(y)+O(\varepsilon). \end{aligned}$$

In the above equality all terms except for $O(\varepsilon)$ are independent of ε. Hence we conclude that η and v satisfy the energy law (3.5.2). □

Remark 3.5.5 (A failure of a simple rescaling argument for a moving surface)**.** It is possible to derive the limit energy identity by a rescaling argument. However, derivation by a rescaling argument is somewhat misleading. Let ρ and u satisfy the energy identity (3.5.1) in the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$. We set

$$\eta(y,t,r) := \rho(y+\varepsilon r\nu_\Gamma(y,t),t), \quad v(y,t,r) := u(y+\varepsilon r\nu_\Gamma(y,t),t)$$

for $(y,t) \in S_T$ and $r \in (-1,1)$. Then by (3.4.13) and the integral transformation formulas (3.5.4) and (3.5.5) we can write (3.5.1) in terms of η and v as

$$\begin{aligned}
&\varepsilon\frac{d}{dt}\int_{\Gamma(t)}\int_{-1}^{1}\omega(\eta(y,r))J(y,\varepsilon r)\,dr\,d\mathcal{H}^{n-1}(y) \\
&= -\varepsilon\int_{\Gamma(t)}\int_{-1}^{1}[\eta|v|^2](y,r)J(y,\varepsilon r)\,dr\,d\mathcal{H}^{n-1}(y) - \int_{\Gamma(t)}p(\eta(y,1))V_\Gamma^N(y)J(y,\varepsilon)\,d\mathcal{H}^{n-1}(y) \\
&\qquad + \int_{\Gamma(t)}p(\eta(y,-1))V_\Gamma^N(y)J(y,-\varepsilon)\,d\mathcal{H}^{n-1}(y). \quad (3.5.12)
\end{aligned}$$

Here we used the abbreviation (3.4.12) and suppressed the argument t of functions.

In formal derivation of a thin width limit by a rescaling argument we usually assume that rescaled functions are independent of the thin direction to get limit relations on a limit set. However, making the assumption at an inappropriate point may result in a wrong limit. To see this, let us assume that η and v are independent of the variable r in (3.5.12). Then since

$$J(y,\varepsilon) - J(y,-\varepsilon) = -2\varepsilon H(y) + O(\varepsilon^2)$$

by (3.5.3) and (3.2.6), it follows that

$$\begin{aligned}
2\varepsilon\frac{d}{dt}\int_{\Gamma(t)}\omega(\eta(y))\,d\mathcal{H}^{n-1}(y) &= -2\varepsilon\int_{\Gamma(t)}[\eta|v|^2](y)\,d\mathcal{H}^{n-1}(y) \\
&\qquad + 2\varepsilon\int_{\Gamma(t)}[p(\eta)V_\Gamma^N H](y)\,d\mathcal{H}^{n-1}(y) + O(\varepsilon^2).
\end{aligned}$$

Dividing both sides by 2ε and taking the principal term we obtain

$$\frac{d}{dt}\int_{\Gamma(t)}\omega(\eta)\,d\mathcal{H}^{n-1} = -\int_{\Gamma(t)}\eta|v|^2\,d\mathcal{H}^{n-1} + \int_{\Gamma(t)}p(\eta)V_\Gamma^N H\,d\mathcal{H}^{n-1}.$$

In this equality v should be of the form $v = V_\Gamma^N\nu_\Gamma + v^T$ with some tangential velocity field v^T, since it is the velocity of a substance on the moving surface $\Gamma(t)$ with normal velocity V_Γ^N. Hence we get

$$\frac{d}{dt}\int_{\Gamma(t)}\omega(\eta)\,d\mathcal{H}^{n-1} = -\int_{\Gamma(t)}\eta(|V_\Gamma^N|^2 + |v^T|^2)\,d\mathcal{H}^{n-1} + \int_{\Gamma(t)}p(\eta)V_\Gamma^N H\,d\mathcal{H}^{n-1},$$

which includes an additional term $\int_{\Gamma(t)}\eta|V_\Gamma^N|^2\,d\mathcal{H}^{n-1}$ compared to the limit energy identity (3.5.2). This improper term appears because we ignore the difference between $p(\eta(y,t,1))$ and $p(\eta(y,t,-1))$ in (3.5.12) by assuming that η is independent of the variable r. Of course it vanishes if the shape of the surface does not change, i.e. $V_\Gamma^N = 0$. This is the reason why this simple rescaling argument is popular to derive a thin width limit problem in a formal level when the limit set of a thin domain does not change its shape.

Remark 3.5.6 (Corrected rescaling argument). To obtain the correct limit (3.5.2) we should take into account the difference between $p(\eta(y,t,1))$ and $p(\eta(y,t,-1))$ in (3.5.12). Let us rewrite the sum of the last two terms in the right-hand side of (3.5.12) into the sum of

$$
\begin{aligned}
I_1 &= -\int_{\Gamma(t)} \{p(\eta(y,1)) - p(\eta(y,-1))\} V_\Gamma^N(y)\, d\mathcal{H}^{n-1}(y), \\
I_2 &= \varepsilon \int_{\Gamma(t)} \{p(\eta(y,1)) + p(\eta(y,-1))\} [V_\Gamma^N H](y)\, d\mathcal{H}^{n-1}(y),
\end{aligned}
$$

and a residual term $O(\varepsilon^2)$ and calculate them properly (here we again suppressed the argument t of functions). For I_2 we merely assume that η is independent of r to get

$$
I_2 = 2\varepsilon \int_{\Gamma(t)} [p(\eta) V_\Gamma^N H](y)\, d\mathcal{H}^{n-1}(y). \tag{3.5.13}
$$

For a proper calculation of I_1 we need to impose Darcy's law (3.4.3) in $\Omega_\varepsilon(t)$ and describe it in terms of the rescaled functions. By the definition of η,

$$
p(\rho(x)) = p(\eta(\pi(x), \varepsilon^{-1} d(x))), \quad x \in \Omega_\varepsilon(t).
$$

We differentiate both sides in x and use (3.2.2) and (3.4.17). Then

$$
\nabla p(\rho(x)) = \nabla_\Gamma p(\eta(\pi, \varepsilon^{-1} d)) + \varepsilon^{-1} \partial_r p(\eta(\pi, \varepsilon^{-1} d)) \nu_\Gamma(\pi) + O(\varepsilon),
$$

where we abbreviate $\pi(x)$ and $d(x)$ to π and d in the right-hand side. Substituting this for (3.4.3) and taking the normal component of the resulting equation we obtain

$$
\partial_r p(\eta(y,r)) = -\varepsilon [\eta v](y,r) \cdot \nu_\Gamma(y) + O(\varepsilon^2) \tag{3.5.14}
$$

for $y \in \Gamma(t)$ and $r \in (-1,1)$. We apply the mean value theorem and (3.5.14) to the difference between $p(\eta(y,1))$ and $p(\eta(y,-1))$. Then

$$
p(\eta(y,1)) - p(\eta(y,-1)) = 2\partial_r p(\eta(y,\theta)) = -2\varepsilon [\eta v](y,\theta) \cdot \nu_\Gamma(y) + O(\varepsilon^2)
$$

with some $\theta = \theta(y,t) \in (-1,1)$. Hence I_1 is expressed as

$$
I_1 = 2\varepsilon \int_{\Gamma(t)} [\eta v](y,\theta) \cdot \nu_\Gamma(y) V_\Gamma^N(y)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2).
$$

Now we assume that η and v are independent of the argument r. Then we have

$$
I_1 = 2\varepsilon \int_\Gamma [\eta (v \cdot \nu_\Gamma) V_\Gamma^N](y)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \tag{3.5.15}
$$

We substitute (3.5.13) and (3.5.15) for (3.5.12), assume that the rescaled functions are constant in the variable r for the left-hand side and the first term on the right-hand side, and divide both sides by 2ε after calculations. Then the principal term on the resulting equation is

$$
\begin{aligned}
\frac{d}{dt} \int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^{n-1} = &-\int_{\Gamma(t)} \eta |v|^2\, d\mathcal{H}^{n-1} \\
&+ \int_{\Gamma(t)} \eta (v \cdot \nu_\Gamma) V_\Gamma^N\, d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta) V_\Gamma^N H\, d\mathcal{H}^{n-1}.
\end{aligned}
$$

Finally we suppose that v is of the form $v = V_\Gamma^N \nu_\Gamma + v^T$ with some tangential velocity v^T, which is natural since it is the velocity of a substance on the moving surface $\Gamma(t)$ with normal velocity V_Γ^N as we mentioned in Remark 3.5.5. Then we obtain the proper limit energy identity (3.5.2) from the above equality.

3.6 Energetic variation for derivation of Darcy's law

In this section we discuss the energetic variational approach [3, 12, 25] for nonlinear diffusion equations in a moving domain and on a moving surface. For a general non-equilibrium thermodynamic system, if the system is isothermal, then the combination of the first and second laws of thermodynamics yields

$$\frac{d}{dt}E^{\text{total}} = \dot{W} - \Delta,$$

where $E^{\text{total}} = \mathcal{K} + \mathcal{F}$ is the sum of the kinetic energy $\mathcal{K}$ and the Helmholtz free energy $\mathcal{F}$, Δ is the entropy production, and $\dot{W}$ is the rate of change of work done by the external environment. If the system is closed, i.e. $\dot{W} = 0$, we further get the energy dissipation law

$$\frac{d}{dt}E^{\text{total}} = -2\mathcal{D},$$

where $\mathcal{D} = \Delta/2$ is sometimes called the energy dissipation. For a conservative system ($\Delta = 0$), the principle of least action (LAP) [1] states that the variation of the kinetic and the free energies with respect to the flow map in Lagrangian coordinates yield the internal force F_i and the conservative force F_c. Formally it can be written as

$$\begin{aligned} \delta\left(\int_0^T \mathcal{K}\,dt\right) &= \int_0^T \int (F_i \cdot \delta x)\,dx\,dt, \\ \delta\left(\int_0^T \mathcal{F}\,dt\right) &= \int_0^T \int (F_c \cdot \delta x)\,dx\,dt, \end{aligned}$$

where δ represents the procedure of variation. Sometimes such a calculation is also referred to as the principle of virtual work. Based on the LAP, the equation of motion for a conservative system is described by balance of forces:

$$F_i = F_c.$$

For a dissipative system, we use the maximum dissipation principle (MDP) [16, 17] to the dissipative force F_d: by taking the variation of the dissipation with respect to the velocity in Eulerian coordinates, we have

$$\delta\mathcal{D} = F_d \cdot \delta u.$$

When all forces are derived, the equation of motion for a dissipative system is formulated as balance of forces (Newton's third law):

$$F_i = F_c + F_d.$$

Let us apply the above energetic variational framework to the energy laws (3.5.1) and (3.5.2). For (3.5.1) we have

$$\begin{gathered} \mathcal{K} = 0, \quad \mathcal{F} = \int_{\Omega(t)} \omega(\rho)\,dx, \\ \mathcal{D} = \frac{1}{2}\int_{\Omega(t)} \rho|u|^2\,dx, \quad \dot{W} = -\int_{\partial\Omega(t)} p(\rho)V_\Omega^N\,d\mathcal{H}^{n-1}. \end{gathered}$$

Let $x\colon \Omega(0)\times[0,T]\to\mathbb{R}^n$ be the flow map of the velocity field u, i.e. for each $t\in[0,T]$ the mapping $x(\cdot,t)$ is a diffeomorphism from $\Omega(0)$ onto $\Omega(t)$ and

$$x(X,0)=X,\quad \frac{\partial}{\partial t}x(X,t)=u(x(X,t),t),\quad (X,t)\in\Omega(0)\times(0,T).$$

We write F for the deformation matrix of x:

$$F(X,t)=\frac{\partial x}{\partial X}(X,t),\quad (X,t)\in\Omega(0)\times(0,T).$$

The MDP gives the dissipative force

$$\frac{\delta\mathcal{D}}{\delta u}=\rho u. \tag{3.6.1}$$

On the other hand, the LAP shows that the conservative force is given by the gradient of the pressure.

Lemma 3.6.1. *Suppose that ρ and u satisfy the transport equation* (3.3.2)*. Then*

$$\frac{\delta\mathcal{F}}{\delta x}=\nabla p(\rho), \tag{3.6.2}$$

where $p(\rho)$ is given by (3.4.1)*.*

Proof. Throughout the proof we use the notation

$$f^\sharp(X,t)=f(x(X,t),t),\quad (X,t)\in\Omega(0)\times(0,T)$$

for a function f on Q_T. Since the transport equation (3.3.2) is satisfied, the density ρ is given by

$$\rho(x(X,t),t)=\frac{\rho_0(X)}{\det F(X,t)},\quad (X,t)\in\Omega(0)\times(0,T) \tag{3.6.3}$$

with the initial density ρ_0 and thus the free energy $\mathcal{F}(x)=\mathcal{F}(x(\cdot,t))$ is of the form

$$\mathcal{F}(x)=\int_{\Omega(0)}\omega\left(\frac{\rho_0(X)}{\det F(X,t)}\right)\det F(X,t)\,dX,\quad t\in(0,T). \tag{3.6.4}$$

Let $\{x^\varepsilon\}_\varepsilon$ be a family of flow maps and $u^\varepsilon=\partial x^\varepsilon/\partial t$ such that

$$\begin{gathered}x^\varepsilon(\cdot,0)=x(\cdot,0),\quad x^\varepsilon(\cdot,T)=x(\cdot,T)\quad\text{for all}\quad\varepsilon,\\ x^\varepsilon(\cdot,t)|_{\varepsilon=0}=x(\cdot,t),\quad u^\varepsilon(\cdot,t)|_{\varepsilon=0}=u(\cdot,t),\quad \left.\frac{d}{d\varepsilon}x^\varepsilon(\cdot,t)\right|_{\varepsilon=0}=w(x(\cdot,t),t)\end{gathered}$$

with any given vector field $w\colon Q_T\to\mathbb{R}^n$. We write F^ε for the deformation matrix of x^ε. Suppose that ρ^ε and u^ε satisfy the transport equation (3.3.2) with the same initial density ρ_0. Then the relation (3.6.3) with ρ, x, and F replaced by ρ^ε, x^ε, and F^ε holds and by (3.6.4) the free energy $\mathcal{F}$ with respect to the perturbed flow map x^ε is given by

$$\mathcal{F}(x^\varepsilon)=\int_{\Omega(0)}\omega\left(\frac{\rho_0}{\det F^\varepsilon}\right)\det F^\varepsilon\,dX. \tag{3.6.5}$$

Note that the argument $t \in (0,T)$ is suppressed in the above equality. We differentiate $\int_0^T \mathcal{F}(x^\varepsilon)\,dt$ with respect to ε at $\varepsilon = 0$. Since $F^\varepsilon|_{\varepsilon=0} = F$ and

$$\left.\frac{dF^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial}{\partial X}\frac{dx^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\partial w^\sharp}{\partial X},$$

the derivative of the determinant of F^ε with respect to ε at $\varepsilon = 0$ is

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\det F^\varepsilon\right|_{\varepsilon=0} &= \operatorname{tr}\left((F^\varepsilon)^{-1}\frac{dF^\varepsilon}{d\varepsilon}\right)\left.\det F^\varepsilon\right|_{\varepsilon=0} \\ &= \operatorname{tr}\left(F^{-1}\frac{\partial w^\sharp}{\partial X}\right)\det F = (\operatorname{div} w)^\sharp \det F,\end{aligned} \tag{3.6.6}$$

where F^{-1} and $(F^\varepsilon)^{-1}$ are the inverse matrix of F and F^ε, respectively. We differentiate the integrand of (3.6.5) at $\varepsilon = 0$ and apply (3.6.3), (3.6.6), and $F^\varepsilon|_{\varepsilon=0} = F$ to obtain

$$\left.\frac{d}{d\varepsilon}\left(\omega\left(\frac{\rho_0}{\det F^\varepsilon}\right)\det F^\varepsilon\right)\right|_{\varepsilon=0} = \{-\omega'(\rho^\sharp)\rho^\sharp + \omega(\rho^\sharp)\}(\operatorname{div} w)^\sharp \det F.$$

Therefore,

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\int_0^T \mathcal{F}(x^\varepsilon)\,dt\right|_{\varepsilon=0} &= \int_0^T\int_{\Omega(0)} \{-\omega'(\rho^\sharp)\rho^\sharp + \omega(\rho^\sharp)\}(\operatorname{div} w)^\sharp \det F\,dX\,dt \\ &= \int_0^T\int_{\Omega(t)} \{-\omega'(\rho)\rho + \omega(\rho)\}\operatorname{div} w\,dx\,dt \\ &= \int_0^T\int_{\Omega(t)} \nabla[\omega'(\rho)\rho - \omega(\rho)]\cdot w\,dx\,dt\end{aligned}$$

and (3.6.2) follows. □

By (3.6.1), (3.6.2), and $\mathcal{K} = 0$, the balance of forces

$$\frac{\delta\mathcal{K}}{\delta x} = \frac{\delta\mathcal{F}}{\delta x} + \frac{\delta\mathcal{D}}{\delta u}$$

is of the form

$$0 = \nabla p(\rho) + \rho u, \quad \text{i.e.} \quad -\rho u = \nabla p(\rho),$$

which is exactly Darcy's law in a moving domain. Combining this with the transport equation (3.3.2), we obtain the nonlinear diffusion equations

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \quad -\rho u = \nabla p(\rho)$$

in the moving domain $\Omega(t)$, where $p(\rho)$ is given by (3.4.1).

From the above discussion, we expect that the energetic variational approach for (3.5.2) yields Darcy' law (3.4.10) on a moving surface. For (3.5.2) we have

$$\mathcal{K} = 0, \quad \mathcal{F} = \int_{\Gamma(t)} \omega(\eta)\,d\mathcal{H}^{n-1},$$
$$\mathcal{D} = \frac{1}{2}\int_{\Gamma(t)} \eta|v^T|^2\,d\mathcal{H}^{n-1}, \quad \dot{W} = \int_{\Gamma(t)} p(\eta)HV_\Gamma^N\,d\mathcal{H}^{n-1}.$$

The variation of $\mathcal{D}$ with respect to the total velocity $v = V_\Gamma^N \nu + v^T$ gives

$$\frac{\delta \mathcal{D}}{\delta v} = \eta v^T, \tag{3.6.7}$$

since $v^T = P_\Gamma v$. Let us apply the LAP to the free energy $\mathcal{F}$.

Lemma 3.6.2. *Suppose that η and v of the form* (3.4.8) *satisfy the transport equation* (3.3.5)*. Then*

$$\frac{\delta \mathcal{F}}{\delta y} = \nabla_\Gamma p(\eta), \tag{3.6.8}$$

where $p(\eta)$ is given by (3.4.1)*.*

We localize integrals over $\Gamma(t)$ with a partition of unity of $\Gamma(t)$ as in [14, Section 2.4] and take the variation of $\mathcal{F}$ with respect to a flow map in "local Lagrangian coordinates." Let U be an open set in $\mathbb{R}^{n-1}$. We call a mapping $y\colon U \times [0,T] \to \mathbb{R}^n$ the flow map of the velocity $v = V_\Gamma \nu_\Gamma + v^T$ in local Lagrangian coordinates if $y(\cdot,t)\colon U \to \Gamma(t)$ is a smooth local parametrization of $\Gamma(t)$ for each $t \in [0,T]$ and

$$y(Y,0) \in \Gamma(0), \quad \frac{\partial}{\partial t} y(Y,t) = v(y(Y,t),t), \quad (Y,t) \in U \times (0,T). \tag{3.6.9}$$

We consider a localized surface integral

$$\mathcal{F}(y) = \mathcal{F}(y(\cdot,t)) = \int_{y(U,t)} \omega(\eta)\, d\mathcal{H}^{n-1} \tag{3.6.10}$$

and take its variation with respect to y. Let $\{y^\varepsilon\}_\varepsilon$ be a family of flow maps in local Lagrangian coordinates and $v^\varepsilon = \partial y^\varepsilon/\partial t$ such that

$$\begin{gathered} y^\varepsilon(\cdot,0) = y(\cdot,0), \quad y^\varepsilon(\cdot,T) = y(\cdot,T) \quad \text{for all} \quad \varepsilon, \\ y^\varepsilon(\cdot,t)|_{\varepsilon=0} = y(\cdot,t), \quad v^\varepsilon(\cdot,t)|_{\varepsilon=0} = v(\cdot,t), \quad \left.\frac{d}{d\varepsilon} y^\varepsilon(\cdot,t)\right|_{\varepsilon=0} = w(y(\cdot,t),t) \end{gathered} \tag{3.6.11}$$

with any given vector field $w\colon S_T \to \mathbb{R}^n$ such that $w(\cdot,t)$ is tangential on $\Gamma(t)$ for each $t \in (0,T)$. For a function f on S_T we use the notation

$$f^\sharp(Y,t) = f(y(Y,t),t), \quad (Y,t) \in U \times (0,T). \tag{3.6.12}$$

Lemma 3.6.3. *Let $g = (g_{ij})_{i,j}$ be a matrix given by*

$$g_{ij} = \frac{\partial y}{\partial Y_i} \cdot \frac{\partial y}{\partial Y_j}, \quad i,j = 1,\dots,n-1 \tag{3.6.13}$$

and $g^\varepsilon = (g^\varepsilon_{ij})_{i,j}$ be a matrix given as above with y replaced by y^ε. Then

$$\left.\frac{d}{d\varepsilon}\sqrt{\det g^\varepsilon}\right|_{\varepsilon=0} = (\operatorname{div}_\Gamma w)^\sharp \sqrt{\det g}. \tag{3.6.14}$$

Proof. Since $g^\varepsilon|_{\varepsilon=0} = g$ and

$$\frac{d}{d\varepsilon}\det g^\varepsilon = \operatorname{tr}\left((g^\varepsilon)^{-1}\frac{dg^\varepsilon}{d\varepsilon}\right)\det g^\varepsilon,$$

where $(g^\varepsilon)^{-1}$ is the inverse matrix of g^ε, we have

$$\frac{d}{d\varepsilon}\sqrt{\det g^\varepsilon}\bigg|_{\varepsilon=0} = \frac{1}{2}\mathrm{tr}\left(g^{-1}\frac{dg^\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}\right)\sqrt{\det g}, \tag{3.6.15}$$

where $g^{-1} = (g^{ij})_{i,j}$ is the inverse matrix of g. Moreover, since

$$\frac{dg_{ij}^\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0} = \left(\frac{\partial}{\partial Y_i}\frac{dy^\varepsilon}{d\varepsilon}\cdot\frac{\partial y^\varepsilon}{\partial Y_j} + \frac{\partial y^\varepsilon}{\partial Y_j}\cdot\frac{\partial}{\partial Y_j}\frac{dy^\varepsilon}{d\varepsilon}\right)\bigg|_{\varepsilon=0} = \frac{\partial w^\sharp}{\partial Y_i}\cdot\frac{\partial y}{\partial Y_j} + \frac{\partial y}{\partial Y_i}\cdot\frac{\partial w^\sharp}{\partial Y_j}$$

for each $i, j = 1, \dots, n-1$, where we used the notation (3.6.12), and g^{-1} is symmetric,

$$\begin{aligned}\mathrm{tr}\left(g^{-1}\frac{dg^\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}\right) &= \sum_{i,j=1}^{n-1} g^{ij}\left(\frac{\partial w^\sharp}{\partial Y_i}\cdot\frac{\partial y}{\partial Y_j} + \frac{\partial y}{\partial Y_i}\cdot\frac{\partial w^\sharp}{\partial Y_j}\right) \\ &= 2\sum_{i,j=1}^{n-1} g^{ij}\frac{\partial w^\sharp}{\partial Y_i}\cdot\frac{\partial y}{\partial Y_j} = 2(\mathrm{div}_\Gamma w)^\sharp.\end{aligned}$$

Substituting this for (3.6.15), we get (3.6.14). □

Proof of Lemma 3.6.2. We first express the free energy $\mathcal{F}$ in "local Lagrangian coordinates." Let U be an open set in $\mathbb{R}^{n-1}$ and $y\colon U\times[0,T]\to\mathbb{R}^n$ be the flow map of the velocity $v = V_\Gamma\nu_\Gamma + v^T$ in local Lagrangian coordinates. Also, let $g = (g_{ij})_{i,j}$ be the matrix given by (3.6.13). For every open subset U' of U the integral

$$\int_{U'(t)}\eta(y,t)\,d\mathcal{H}^{n-1}(y) = \int_{U'}\eta(y(Y,t),t)\sqrt{\det g(Y,t)}\,dY \quad (U'(t) := y(U',t))$$

is constant in t, since η and v satisfy the transport equation (3.3.5) and $U'(t)$ moves with velocity v. Hence for $(Y,t)\in U\times(0,T)$ we have

$$\eta(y(Y,t),t)\sqrt{\det g(Y,t)} = \eta(y(Y,0),0)\sqrt{\det g(Y,0)} \tag{3.6.16}$$

and the localized surface integral (3.6.10) is expressed as

$$\begin{aligned}\mathcal{F}(y) &= \int_U \omega(\eta(y(Y,t),t))\sqrt{\det g(Y,t)}\,dY \\ &= \int_U \omega\left(\frac{\eta_0(Y)}{\sqrt{\det g(Y,t)}}\right)\sqrt{\det g(Y,t)}\,dY,\end{aligned} \tag{3.6.17}$$

where $\eta_0(Y)$ is given by the right-hand side of (3.6.16).

Next we take a variation of $\mathcal{F}$ with respect to the flow map y. Let $\{y^\varepsilon\}_\varepsilon$ be a family of flow maps in local Lagrangian coordinates satisfying (3.6.11) with $v^\varepsilon = \partial y^\varepsilon/\partial t$. Also, let $g^\varepsilon = (g_{ij}^\varepsilon)_{i,j}$ be given by (3.6.13) with y replaced by y^ε. Suppose that η^ε and v^ε satisfy the transport equation (3.3.5) and $\eta^\varepsilon|_{t=0} = \eta|_{t=0}$ holds on $y^\varepsilon(U,0) = y(U,0)$. Then the relation (3.6.16) with η^ε, y^ε, and g^ε holds and by (3.6.17) the free energy $\mathcal{F}$ with respect to the perturbed flow map y^ε is given by

$$\mathcal{F}(y^\varepsilon) = \int_U \omega\left(\frac{\eta_0}{\sqrt{\det g^\varepsilon}}\right)\sqrt{\det g^\varepsilon}\,dY.$$

Here the the argument $t \in (0,T)$ is suppressed. Note that the right-hand side of (3.6.16) with η^ε, y^ε, and g^ε is equal to $\eta_0(Y)$ since $\eta^\varepsilon|_{t=0} = \eta|_{t=0}$, $y^\varepsilon|_{t=0} = y|_{t=0}$, and $g^\varepsilon|_{t=0} = g|_{t=0}$. We differentiate the integrand of the right-hand side with respect to ε at $\varepsilon = 0$. Then by (3.6.14), (3.6.17), and $g^\varepsilon|_{\varepsilon=0} = g$ we get

$$\frac{d}{d\varepsilon}\left(\omega\left(\frac{\eta_0}{\sqrt{\det g^\varepsilon}}\right)\sqrt{\det g^\varepsilon}\right)\bigg|_{\varepsilon=0} = \{-\omega'(\eta^\sharp)\eta^\sharp + \omega(\eta^\sharp)\}(\mathrm{div}_\Gamma w)^\sharp\sqrt{\det g}.$$

Here we used the notation (3.6.12). Hence

$$\begin{aligned}\frac{d}{d\varepsilon}\int_0^T \mathcal{F}(y^\varepsilon)\,dt\bigg|_{\varepsilon=0} &= \int_0^T\int_U \{-\omega'(\eta^\sharp)\eta^\sharp + \omega(\eta^\sharp)\}(\mathrm{div}_\Gamma w)^\sharp\sqrt{\det g}\,dY\,dt \\ &= \int_0^T\int_{U(t)} \{-\omega'(\eta)\eta + \omega(\eta)\}\mathrm{div}_\Gamma w\,d\mathcal{H}^{n-1}\,dt \\ &= \int_0^T\int_{U(t)} \nabla_\Gamma[\omega'(\eta)\eta - \omega(\eta)]\cdot w\,d\mathcal{H}^{n-1}\,dt,\end{aligned}$$

where $U(t) = y(U,t)$ and the last equality follows from the Stokes theorem and the fact that the vector field w is tangential on $\Gamma(t)$. (Note that we may assume that $\omega'(\eta)\eta - \omega(\eta)$ has a compact support in $U(t)$ since we localize the surface integral by using a partition of unity of $\Gamma(t)$.) Since w is an arbitrary tangential vector field on $\Gamma(t)$, we conclude from the above equality that (3.6.8) holds. □

By (3.6.7), (3.6.8), and $\mathcal{K} = 0$, the balance of forces

$$\frac{\delta\mathcal{K}}{\delta y} = \frac{\delta\mathcal{F}}{\delta y} + \frac{\delta\mathcal{D}}{\delta v}$$

is of the form

$$0 = \nabla_\Gamma p(\eta) + \eta v^T, \quad \text{i.e.} \quad -\eta v^T = \nabla_\Gamma p(\eta),$$

which is Darcy's law on a moving surface as we expected. Finally, combining this with the transport equation (3.3.5) we obtain the nonlinear diffusion equations

$$\partial^\circ\eta - V_\Gamma^N H\eta + \mathrm{div}_\Gamma(\eta v^T) = 0, \quad -\eta v^T = \nabla_\Gamma p(\eta)$$

on the moving surface $\Gamma(t)$, where $p(\eta)$ is given by (3.4.1).

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Chapter 4

On singular limit equations for incompressible fluids in moving thin domains

4.1 Introduction

Fluid flows in a thin domain appear in many problems of natural sciences, e.g. ocean dynamics, geophysical fluid dynamics, and fluid flows in cell membranes. In the study of the incompressible Navier–Stokes equations in a three-dimensional thin domain mathematical researchers are mainly interested in global existence of a strong solution for large data since a three-dimensional thin domain with sufficiently small width can be considered "almost two-dimensional." It is also important to investigate the behavior of a solution as the width of a thin domain goes to zero. We may naturally ask whether we can derive limit equations as a thin domain degenerates into a two-dimensional set and compare properties of solutions to the original three-dimensional equations and the corresponding two-dimensional limit equations. There are several works studying such problems with a three-dimensional flat thin domain [15, 16, 29, 33] of the form

$$\Omega_\varepsilon = \{x = (x', x_3) \in \mathbb{R}^3 \mid x' \in \omega,\ \varepsilon g_0(x') < x_3 < \varepsilon g_1(x')\}$$

for small $\varepsilon > 0$, where ω is a two-dimensional domain and g_0 and g_1 are functions on ω, and a three-dimensional thin spherical domain [34] which is a region between two concentric spheres of near radii. (We also refer to [28] for the strategy of analysis of the Euler equations in a flat and spherical thin domain and its limit equations.) However, mathematical studies of an incompressible fluid in a thin domain have not been done in the case where a thin domain and its limit set have more complicated geometric structures. (See [27] for the mathematical analysis of a reaction-diffusion equation in a thin domain degenerating into a lower dimensional manifold.)

In this chapter we are concerned with the incompressible Euler and Navier–Stokes equations in a three-dimensional thin domain that moves in time. The purpose of this chapter is to give a heuristic derivation of singular limits of these equations as a moving thin domain degenerates into a two-dimensional moving closed surface. We also investigate relations between the energy structures of the incompressible fluid systems in a moving thin domain and the corresponding limit systems on a moving closed surface.

Here let us explain our results on limit equations and strategy to derive them. Let $\Gamma(t)$ be an evolving closed surface in $\mathbb{R}^3$ and $V_\Gamma^N(\cdot, t)$ and $\nu(\cdot, t)$ its (scalar) outward normal velocity and unit outward normal vector field, respectively. We assume that $\Gamma(t)$ does not change its topology. Also, let $\Omega_\varepsilon(t)$ be a tubular neighborhood of $\Gamma(t)$ of radius ε in $\mathbb{R}^3$ with sufficiently small $\varepsilon > 0$. We consider the Euler equations

$$\partial_t u + (u\cdot\nabla)u + \nabla p = 0 \quad \text{in} \quad \Omega_\varepsilon(t),\ t\in(0,T), \tag{4.1.1}$$
$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega_\varepsilon(t),\ t\in(0,T), \tag{4.1.2}$$
$$u\cdot\nu_\varepsilon = V_\varepsilon^N \quad \text{on} \quad \partial\Omega_\varepsilon(t),\ t\in(0,T) \tag{4.1.3}$$

and the Navier–Stokes equations with (perfect slip) Navier boundary condition

$$\partial_t u + (u\cdot\nabla)u + \nabla p = \mu_0\Delta u \quad \text{in} \quad \Omega_\varepsilon(t),\ t\in(0,T), \tag{4.1.4}$$
$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega_\varepsilon(t),\ t\in(0,T), \tag{4.1.5}$$
$$u\cdot\nu_\varepsilon = V_\varepsilon^N \quad \text{on} \quad \partial\Omega_\varepsilon(t),\ t\in(0,T), \tag{4.1.6}$$
$$[D(u)\nu_\varepsilon]_{\mathrm{tan}} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon(t),\ t\in(0,T). \tag{4.1.7}$$

Here ν_ε and V_ε^N denote the unit outward normal vector field and the (scaler) outward normal velocity of $\partial\Omega_\varepsilon(t)$. Also, $\mu_0 > 0$ is the viscosity coefficient and $D(u) := \{\nabla u + (\nabla u)^T\}/2$ is the strain rate tensor with $(\nabla u)^T$ the transpose of the gradient matrix ∇u. We suppose that $\Omega_\varepsilon(t)$ admits the normal coordinate system $x = \pi(x,t) + d(x,t)\nu(\pi(x,t),t)$ for $x \in \Omega_\varepsilon(t)$, where $\pi(\cdot,t)$ is the closest point mapping onto $\Gamma(t)$ and $d(\cdot,t)$ is the signed distance from $\Gamma(t)$ increasing in the direction of $\nu(\cdot,t)$. Based on the normal coordinates, we expand the velocity field $u(x,t)$ on $\Omega_\varepsilon(t)$ in powers of the signed distance $d(x,t)$ as

$$u(x,t) = v(\pi(x,t),t) + d(x,t)v^1(\pi(x,t),t) + \cdots, \quad x\in\Omega_\varepsilon(t) \tag{4.1.8}$$

and the pressure $p(x,t)$ similarly. We substitute them for the equations in $\Omega_\varepsilon(t)$ and determine equations on $\Gamma(t)$ that the zeroth order term v in (4.1.8) satisfies. Then we obtain limit equations of the Euler equations (4.1.1)–(4.1.3):

$$\partial_v^\bullet v + \nabla_\Gamma q + q^1\nu = 0 \quad \text{on} \quad \Gamma(t),\ t\in(0,T), \tag{4.1.9}$$
$$\operatorname{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma(t),\ t\in(0,T), \tag{4.1.10}$$
$$v\cdot\nu = V_\Gamma^N \quad \text{on} \quad \Gamma(t),\ t\in(0,T). \tag{4.1.11}$$

Here $\partial_v^\bullet = \partial_t + v\cdot\nabla$ is the material derivative along the velocity field v and ∇_Γ and $\operatorname{div}_\Gamma$ denote the tangential gradient and the surface divergence on $\Gamma(t)$, respectively (see Section 4.2 for their definitions). Similarly, we get limit equations of the Navier–Stokes equations (4.1.4)–(4.1.7):

$$\partial_v^\bullet v + \nabla_\Gamma q + q^1\nu = 2\mu_0\operatorname{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma) \quad \text{on} \quad \Gamma(t),\ t\in(0,T), \tag{4.1.12}$$
$$\operatorname{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma(t),\ t\in(0,T), \tag{4.1.13}$$
$$v\cdot\nu = V_\Gamma^N \quad \text{on} \quad \Gamma(t),\ t\in(0,T). \tag{4.1.14}$$

Here $D^{tan}(v) := \{\nabla_\Gamma v + (\nabla_\Gamma v)^T\}/2$ and P_Γ is the orthogonal projection onto the tangent plane of $\Gamma(t)$. Note that if we take the average of (4.1.8) in the normal direction of $\Gamma(t)$ then

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(y+\rho\nu(y,t),t)\,d\rho = v(y,t) + (\text{higher order terms in } \varepsilon), \quad y\in\Gamma(t).$$

Therefore, formally speaking, our limit equations are equations satisfied by the limit of the average in the thin direction of a solution to the original Euler or Navier–Stokes equations in

$\Omega_\varepsilon(t)$ as ε goes to zero. (The above method is also applied in [23] to derive a limit equation of a nonlinear diffusion equation in a moving thin domain.)

In the equations (4.1.9) and (4.1.12) the scalar function q^1, which comes from the normal derivative of the bulk pressure p (see the expansion (4.3.5) of p and (4.3.17) in the proof of Theorem 4.3.1), is determined by the normal component of (4.1.9) and (4.1.12). Therefore, the limit Euler system (4.1.9)–(4.1.11) is intrinsically equivalent to

$$P_\Gamma \partial_v^\bullet v + \nabla_\Gamma q = 0, \quad \mathrm{div}_\Gamma v = 0, \quad v \cdot \nu = V_\Gamma^N \tag{4.1.15}$$

and the limit Navier–Stokes system (4.1.12)–(4.1.14) is equivalent to

$$P_\Gamma \partial_v^\bullet v + \nabla_\Gamma q = 2\mu_0 P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v) P_\Gamma), \quad \mathrm{div}_\Gamma v = 0, \quad v \cdot \nu = V_\Gamma^N. \tag{4.1.16}$$

We note that these tangential surface fluid systems were also derived in [17, 18] recently. The derivation of the Navier–Stokes equations on a moving surface in [17] is based on local conservation laws of mass and linear momentum for a surface fluid. On the other hand, the authors of [18] applied a global energetic variational approach to derive several kinds of equations for an incompressible fluid on an evolving surface.

The viscous term $2\mu_0 \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v) P_\Gamma)$ in the momentum equation (4.1.12) of the limit Navier–Stokes system appears in the Boussinesq–Scriven surface fluid model which was first described by Boussinesq [7] and generalized by Scriven [30] to an arbitrary curved moving surface (see also [1, Chapter 10] for derivation of the Boussinesq–Scriven surface fluid model). In [4] the Boussinesq–Scriven surface fluid model was considered to formulate a continuum model for fluid membranes in a bulk fluid, which contains equations for a viscous fluid on a curved moving surface, and study the effect of membrane viscosity in the dynamics of fluid membranes. It was also studied in the context of two-phase flows [5, 6, 25] in which equations for a surface fluid are considered as the boundary condition on a fluid interface.

Since we consider an incompressible fluid on a moving surface or in its tubular neighborhood, some constraints on the motion of the surface are necessary. For the existence of a surface incompressible fluid it is required that the area of the moving surface is preserved in time. To consider a bulk incompressible fluid in the ε-tubular neighborhood of the moving surface for all $\varepsilon > 0$ sufficiently small, we need another constraint on the moving surface besides the area preserving condition. However, it is automatically satisfied by the Gauss–Bonnet theorem and the assumption that the moving surface does not change its topology. See Remark 4.3.3 for details.

When the surface does not move in time, our tangential limit system (4.1.15) of the Euler equations is the same as the Euler system on a fixed manifold derived by Arnol′d [2, 3], who applied the Lie group of diffeomorphisms of a manifold (see also Ebin and Marsden [12]). Also, for a stationary surface our tangential limit system (4.1.16) of the Navier–Stokes equations is the same as the Navier–Stokes system on a manifold derived by Taylor [31], although the authors of [18] claim that (4.1.16) is different from Taylor's system (see Remark 4.4.3). For detailed comparison of our limit systems and the systems derived in previous works see Remarks 4.3.2 and 4.4.2. We further note that the function q^1 in the limit momentum equations (4.1.9) and (4.1.12), which is determined by the normal component of these equations, does not vanish even if the surface is stationary. See Remarks 4.3.2 and 4.4.2 for details.

Finally we note that our results are based on formal calculations and thus mathematical justification is required. There are a few works that present rigorous derivation of limit equations in the case where a limit set is a hypersurface or a manifold. Temam and Ziane [34] derived limit equations for the Navier–Stokes equations in a thin spherical domain by characterizing the thin width limit of a solution to the original equations as a solution to the limit equations. In [27], Prizzi, Rinaldi, and Rybakowski compared the dynamics of a reaction-diffusion equation in a thin domain and that of a limit equation when a thin domain degenerates into a lower dimensional manifold. Recently, the present author derived a limit equation of the heat equation in a moving thin domain shrinking to a moving closed hypersurface by characterization of the thin width limit of a solution [22]. Although there are several tools and methods introduced in the above papers, it seems that mathematical justification of our results is difficult because of the nonlinearity of the equations and the evolution of the shape of the limit surface, and that we need some new techniques.

This chapter is organized as follows. In Section 4.2 we give notations and formulas on quantities related to a moving surface and a moving thin domain. In Sections 4.3 and 4.4 we derive the limit equations of the Euler and Navier–Stokes equations in a moving thin domain, respectively. In Section 4.5 we derive the energy identities of the Euler and Navier–Stokes equations and the corresponding limit equations and investigate relations between them. In Appendices 4.A and 4.B we give proofs of lemmas in Section 4.2 involving the differential geometry of a surface embedded in the Euclidean space.

4.2 Preliminaries

We fix notations on various quantities of a moving surface and give formulas on them. All functions appearing in this section are assumed to be sufficiently smooth.

Lemmas in this section are proved by straightforward calculations. To avoid making this section too long we give proofs of them in Appendix 4.A, except for the proofs of Lemmas 4.2.4 and 4.2.5. Also, a proof of the formula (4.2.15) in Lemma 4.2.4 is given in Appendix 4.B. Although we are concerned with a two-dimensional surface in this chapter, all notations and formulas in this section apply to hypersurfaces of any dimension with easy modifications.

4.2.1 Moving surfaces and moving thin domains

Let $\Gamma(t)$, $t \in [0,T]$ be a two-dimensional closed (i.e. compact and without boundary), connected, and oriented moving surface in $\mathbb{R}^3$. The unit outward normal vector and the (scalar) outward normal velocity of $\Gamma(t)$ are denoted by $\nu(\cdot,t)$ and $V_\Gamma^N(\cdot,t)$, respectively. Also, let $S_T := \bigcup_{t\in(0,T)} \Gamma(t)\times\{t\}$ be a space-time hypersurface associated with $\Gamma(t)$. We assume that $\Gamma(t)$ is smooth at each $t \in [0,T]$ and moves smoothly in time. In particular, $\Gamma(t)$ does not change its topology. By the smoothness assumption on $\Gamma(t)$, the (outward) principal curvatures $\kappa_1(\cdot,t)$ and $\kappa_2(\cdot,t)$ of $\Gamma(t)$ are bounded uniformly with respect to t. Hence there is a tubular neighborhood

$$N(t) := \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x,\Gamma(t)) < \delta\}$$

of radius $\delta > 0$ independent of t that admits the normal coordinate system

$$x = \pi(x,t) + d(x,t)\nu(\pi(x,t),t), \quad x \in N(t), \tag{4.2.1}$$

where $\pi(\cdot,t)$ is the closest point mapping onto $\Gamma(t)$ and $d(\cdot,t)$ is the signed distance function from $\Gamma(t)$ (see e.g. [11, Lemma 2.8]). Moreover, the mapping π and the signed distance d are smooth in the closure (in $\mathbb{R}^4$) of a space-time noncylindrical domain $N_T := \bigcup_{t\in(0,T)} N(t) \times \{t\}$. We assume that $d(\cdot,t)$ increases in the direction of $\nu(\cdot,t)$. Therefore,

$$\nabla d(x,t) = \nu(\pi(x,t),t), \quad (x,t) \in N_T, \tag{4.2.2}$$

$$\partial_t d(y,t) = -V_\Gamma^N(y,t), \quad (y,t) \in S_T. \tag{4.2.3}$$

Moreover, differentiating both sides of

$$d(x,t) = \{x - \pi(x,t)\} \cdot \nabla d(x,t), \quad d(\pi(x,t),t) = 0$$

with respect to t and using (4.2.2) and (4.2.3) we easily get

$$\partial_t d(x,t) = \partial_t d(\pi(x,t),t) = -V_\Gamma^N(\pi(x,t),t), \quad (x,t) \in N_T. \tag{4.2.4}$$

For a sufficiently small $\varepsilon > 0$ we define a moving thin domain $\Omega_\varepsilon(t)$ in $\mathbb{R}^3$ as

$$\Omega_\varepsilon(t) := \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x, \Gamma(t)) < \varepsilon\}$$

and a space-time noncylindrical domain $Q_{\varepsilon,T}$ and its lateral boundary $\partial_\ell Q_{\varepsilon,T}$ as

$$Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \Omega_\varepsilon(t) \times \{t\}, \quad \partial_\ell Q_{\varepsilon,T} := \bigcup_{t\in(0,T)} \partial\Omega_\varepsilon(t) \times \{t\}.$$

Since $\Omega_\varepsilon(t)$ is a tubular neighborhood of $\Gamma(t)$, the unit outward normal vector $\nu_\varepsilon(\cdot,t)$ and the outward normal velocity $V_\varepsilon^N(\cdot,t)$ of its boundary are given by

$$\nu_\varepsilon(x,t) = \begin{cases} \nu(\pi(x,t),t) & \text{if} \quad d(x,t) = \varepsilon, \\ -\nu(\pi(x,t),t) & \text{if} \quad d(x,t) = -\varepsilon, \end{cases} \tag{4.2.5}$$

$$V_\varepsilon^N(x,t) = \begin{cases} V_\Gamma^N(\pi(x,t),t) & \text{if} \quad d(x,t) = \varepsilon, \\ -V_\Gamma^N(\pi(x,t),t) & \text{if} \quad d(x,t) = -\varepsilon. \end{cases} \tag{4.2.6}$$

4.2.2 Notations and formulas for quantities on fixed surfaces

In this subsection we fix and suppress the time $t \in [0,T]$. Hence Γ denotes a two-dimensional closed, connected, oriented and smooth surface in $\mathbb{R}^3$. Let us give notations and formulas for several quantities on the fixed surface Γ. (In the sequel we use the same notations given in this subsection for the moving surface $\Gamma(t)$.) Let P_Γ be the orthogonal projection onto the tangent plane of Γ at each point on Γ given by

$$P_\Gamma(y) := I_3 - \nu(y) \otimes \nu(y), \quad y \in \Gamma,$$

where I_3 is the identity matrix of three dimension and $a \otimes b$ for $a, b \in \mathbb{R}^3$ denotes the tensor product of a and b given by

$$a \otimes b := \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}, \quad a = (a_1,a_2,a_3),\ b = (b_1,b_2,b_3).$$

For a function f on Γ we define its tangential gradient $\nabla_\Gamma f$ as

$$\nabla_\Gamma f(y) := P_\Gamma(y) \nabla \tilde{f}(y), \quad y \in \Gamma.$$

Here $\tilde{f}$ is an extension of f to N satisfying $\tilde{f}|_\Gamma = f$. Note that the tangential gradient of f is independent of the choice of its extension (see e.g. [11, Lemma 2.4]). Also, it is easy to see that $\nabla_\Gamma f \cdot \nu = 0$ and $P_\Gamma \nabla_\Gamma f = \nabla_\Gamma f$ hold on Γ. The tangential derivative operators are given by

$$\partial_i^{tan} f(y) := \sum_{j=1}^3 \{\delta_{ij} - \nu_i(y)\nu_j(y)\} \partial_j \tilde{f}(y), \quad i = 1,2,3$$

so that $\nabla_\Gamma = (\partial_1^{tan}, \partial_2^{tan}, \partial_3^{tan})$, which are again independent of the choice of an extension $\tilde{f}$ of f. For example, we may take the constant extension in the normal direction of Γ given by $\bar{f}(x) := f(\pi(x))$ for $x \in N$.

For vector fields $F = (F_1, F_2, F_3)$ on N and $G = (G_1, G_2, G_3)$ on Γ, we define the gradient matrix and the divergence of F as

$$\nabla F := \begin{pmatrix} \partial_1 F_1 & \partial_1 F_2 & \partial_1 F_3 \\ \partial_2 F_1 & \partial_2 F_2 & \partial_2 F_3 \\ \partial_3 F_1 & \partial_3 F_2 & \partial_3 F_3 \end{pmatrix}, \quad \operatorname{div} F := \sum_{i=1}^{3} \partial_i F_i$$

and the tangential gradient matrix and the surface divergence of G as

$$\nabla_\Gamma G := \begin{pmatrix} \partial_1^{tan} G_1 & \partial_1^{tan} G_2 & \partial_1^{tan} G_3 \\ \partial_2^{tan} G_1 & \partial_2^{tan} G_2 & \partial_2^{tan} G_3 \\ \partial_3^{tan} G_1 & \partial_3^{tan} G_2 & \partial_3^{tan} G_3 \end{pmatrix}, \quad \operatorname{div}_\Gamma G := \sum_{i=1}^{3} \partial_i^{tan} G_i.$$

These notations are consistent with the formula $\nabla_\Gamma G = P_\Gamma \nabla \widetilde{G}$ on Γ, where $\widetilde{G}$ is an arbitrary extension of G to N with $\widetilde{G}|_\Gamma = G$. For a function f on Γ we denote by $\nabla_\Gamma^2 f$ the tangential Hessian matrix of f whose (i,j)-entry is given by $\partial_i^{tan}\partial_j^{tan} f$ $(i,j = 1,2,3)$. Let M be a 3×3 matrix-valued function defined on N or on Γ of the form

$$M = (M_{ij})_{i,j} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}.$$

We define the divergence $\operatorname{div} M$ on N or the surface divergence $\operatorname{div}_\Gamma M$ on Γ as a vector field whose j-th component is given by

$$[\operatorname{div} M]_j := \sum_{i=1}^{3} \partial_i M_{ij} \quad \text{or} \quad [\operatorname{div}_\Gamma M]_j := \sum_{i=1}^{3} \partial_i^{tan} M_{ij}, \quad j = 1,2,3.$$

Finally we set

$$A := -\nabla_\Gamma \nu = (-\partial_i^{tan} \nu_j)_{i,j}, \quad \Delta_\Gamma := \operatorname{div}_\Gamma \nabla_\Gamma = \sum_{i=1}^{3} (\partial_i^{tan})^2,$$
$$H := -\operatorname{div}_\Gamma \nu = \operatorname{tr}[A], \quad K := \kappa_1 \kappa_2$$

and call them the Weingarten map of Γ, the Laplace–Beltrami operator on Γ, (twice) the mean curvature of Γ, and the Gaussian curvature of Γ, respectively. The usual Laplacian Δ and the Laplace–Beltrami operator Δ_Γ acting on vector fields are understood to be componentwise operators.

Lemma 4.2.1. *For all $y \in \Gamma$ we have*

$$A(y)\nu(y) = 0, \tag{4.2.7}$$
$$A(y)P_\Gamma(y) = P_\Gamma(y)A(y) = A(y), \tag{4.2.8}$$
$$A(y) = -\nabla^2 d(y). \tag{4.2.9}$$

By (4.2.7) we see that A has the eigenvalue 0. Note that the other eigenvalues of A are κ_1 and κ_2 (see e.g. [19, Section VII.5]) and thus

$$H(y) = \kappa_1(y) + \kappa_2(y), \quad y \in \Gamma. \tag{4.2.10}$$

Also, A is symmetric (i.e. $\partial_i^{tan}\nu_j = \partial_j^{tan}\nu_i$) and $H = -\Delta d$ holds on Γ by (4.2.9).

The tangential derivatives ∂_i^{tan} $(i = 1, 2, 3)$ are noncommutative in general. An exchange formula for them includes the unit outward normal of the surface.

Lemma 4.2.2. *Let f be a function on Γ. For each $i, j = 1, 2, 3$ we have*

$$\partial_i^{tan}\partial_j^{tan} f - \partial_j^{tan}\partial_i^{tan} f = [A\nabla_\Gamma f]_i\nu_j - [A\nabla_\Gamma f]_j\nu_i. \tag{4.2.11}$$

Here $[A\nabla_\Gamma f]_i$ denotes the i-th component of the vector field $A\nabla_\Gamma f$.

The next formula is a consequence of (4.2.11), which we use in Section 4.4 to express a viscous term of limit equations of the Navier–Stokes equations in terms of the Laplace–Beltrami operator. For a vector field v on Γ we set

$$D^{tan}(v) := \frac{\nabla_\Gamma v + (\nabla_\Gamma v)^T}{2}. \tag{4.2.12}$$

The matrices $D^{tan}(v)$ and $P_\Gamma D^{tan}(v)P_\Gamma$ are called a tangential strain rate and a projected strain rate in [18], respectively.

Lemma 4.2.3. *Let v be a (not necessarily tangential) vector field on Γ. Then*

$$2\mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma) = 2\mathrm{tr}[A\nabla_\Gamma v]\nu + P_\Gamma(\Delta_\Gamma v) + \nabla_\Gamma(\mathrm{div}_\Gamma v) + H(\nabla_\Gamma v)\nu \tag{4.2.13}$$

holds on Γ (note that $(\nabla_\Gamma v)\nu = P_\Gamma(\nabla_\Gamma v)\nu$ on the right-hand side is tangential).

To compare our limit systems with the incompressible fluid systems on a fixed manifold derived by Arnol′d [2, 3] and Taylor [31] we need formulas on the Levi-Civita connection. Let $\overline{\nabla}$ be the Levi-Civita connection on Γ with respect to the metric on Γ induced by the Euclidean metric of $\mathbb{R}^3$ (see e.g. [9, Section 2.3] and [24, Sections 3.3.1 and 4.1.2] for the definition of the Levi-Civita connection). Hence for tangential vector fields X and Y on Γ the covariant derivative of X along Y is denoted by $\overline{\nabla}_Y X$, which is again a tangential vector field on Γ. The Levi-Civita connection is considered as a mapping

$$\overline{\nabla}\colon C^\infty(T\Gamma) \to C^\infty(T^*\Gamma \otimes T\Gamma), \quad X \mapsto \overline{\nabla}X,$$

where $T\Gamma$ and $T^*\Gamma$ are the tangent and cotangent bundle of Γ, respectively, and for a vector bundle E over Γ we denote by $C^\infty(E)$ the set of all smooth sections of E. (Hence $C^\infty(T\Gamma)$ denotes the set of all smooth tangential vector fields on Γ. We refer to [20, Chapter 10] for the definitions of a vector bundle and a section.) Also, for a tangential vector field X on Γ the notation $\overline{\nabla}X$ stands for a mapping $Y \mapsto \overline{\nabla}_Y X$ from $C^\infty(T\Gamma)$ into itself. Then we write $\overline{\nabla}^*\colon C^\infty(T^*\Gamma\otimes T\Gamma) \to C^\infty(T\Gamma)$ for the formal adjoint operator of $\overline{\nabla}$ (see [24, Section 10.1.3]) and set $\Delta_B := -\overline{\nabla}^*\overline{\nabla}$. The operator $\Delta_B\colon C^\infty(T\Gamma) \to C^\infty(T\Gamma)$ is called the Bochner Laplacian (note that there is another definition of the Bochner Laplacian where the sign is taken opposite).

Lemma 4.2.4. *Let X and Y are tangential vector fields on Γ. Then*

$$(Y\cdot\nabla)\widetilde{X} = \overline{\nabla}_Y X + (AX\cdot Y)\nu, \tag{4.2.14}$$

$$\Delta_B X = P_\Gamma(\Delta_\Gamma X) + A^2 X \tag{4.2.15}$$

hold on Γ. Here $\widetilde{X}$ is an extension of X to N with $\widetilde{X}|_\Gamma = X$ and $(Y\cdot\nabla)\widetilde{X}$ denotes the directional derivative of $\widetilde{X}$ along Y in $\mathbb{R}^3$, i.e.

$$(Y\cdot\nabla)\widetilde{X} = \left(\sum_{i=1}^3 Y_i\partial_i\widetilde{X}_1,\ \sum_{i=1}^3 Y_i\partial_i\widetilde{X}_2,\ \sum_{i=1}^3 Y_i\partial_i\widetilde{X}_3\right).$$

Also, the left-hand side of (4.2.14) is independent of the choice of the extension $\widetilde{X}$.

The formula (4.2.14) is well-known as the Gauss formula (see e.g. [9, Section 4.2] and [19, Section VII.3]) and we omit its proof. Note that $(Y \cdot \nabla)\widetilde{X} = (Y \cdot \nabla_\Gamma)X$ on Γ since Y is tangential. Hence the Gauss formula (4.2.14) is also expressed as

$$(Y \cdot \nabla_\Gamma)X = \overline{\nabla}_Y X + (AX \cdot Y)\nu \quad \text{on} \quad \Gamma \tag{4.2.16}$$

for tangential vector fields X and Y on Γ. We also call (4.2.16) the Gauss formula.

A proof of the formula (4.2.15) is given in Appendix 4.B. Note that (4.2.15) is useful by itself since it gives a global expression under the fixed Cartesian coordinate system of the Bochner Laplacian acting on tangential vector fields on Γ, which is originally defined intrinsically and represented under only local coordinate systems.

Combining Lemmas 4.2.3 and 4.2.4 we get the following formula on the surface divergence of the projected strain rate, which is crucial for comparison of our limit Navier–Stokes system and the incompressible viscous fluid system on a manifold derived by Taylor [31] (see Remark 4.4.2).

Lemma 4.2.5. *For a tangential vector field v on Γ satisfying $\mathrm{div}_\Gamma v = 0$ we have*

$$2P_\Gamma \mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma) = \Delta_B v + Kv \quad \textit{on} \quad \Gamma. \tag{4.2.17}$$

Proof. Let v be a tangential vector field on Γ satisfying $\mathrm{div}_\Gamma v = 0$. Then

$$(\nabla_\Gamma v)\nu = \nabla_\Gamma(v \cdot \nu) - (\nabla_\Gamma \nu)v = Av$$

by $v \cdot \nu = 0$ and $-\nabla_\Gamma \nu = A$. Applying this and

$$P_\Gamma(\mathrm{tr}[A\nabla_\Gamma v]\nu) = \mathrm{tr}[A\nabla_\Gamma v]P_\Gamma \nu = 0, \quad \mathrm{div}_\Gamma v = 0$$

to the formula (4.2.13), and observing that $(\nabla_\Gamma v)\nu = Av$ is tangential, we have

$$2P_\Gamma \mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma) = P_\Gamma(\Delta_\Gamma v) + HAv. \tag{4.2.18}$$

Moreover, since A is symmetric and has the eigenvalues 0, κ_1, and κ_2, where the eigenvector corresponding to the eigenvalue 0 is ν (see Lemma 4.2.1), for each $y \in \Gamma$ we can take an orthonormal basis $\{e_1, e_2\}$ of the tangent plane of Γ at y such that $Ae_i = \kappa_i e_i$, $i = 1, 2$. (The vectors e_1 and e_2 are called the principal directions at y. See e.g. [19, Section VII.5] for details.) Expressing the tangential vector v as a linear combination of e_1 and e_2 and using $H = \kappa_1 + \kappa_2$ and $K = \kappa_1\kappa_2$ we easily obtain $HAv = Kv + A^2v$. Applying this and (4.2.15) to (4.2.18) we obtain (4.2.17). $\square$

Besides derivation of limit equations, we are also interested in thin width limits of energy identities for the Euler and Navier–Stokes equations. To derive limit energy identities we give change of variables formulas for integrals over level-set surfaces and tubular neighborhoods of Γ. For $y \in \Gamma$ and $\rho \in [-\varepsilon, \varepsilon]$ we set

$$J(y, \rho) := \{1 - \rho\kappa_1(y)\}\{1 - \rho\kappa_2(y)\} = 1 - \rho H(y) + \rho^2 K(y). \tag{4.2.19}$$

Here the second equality follows from the definition of the Gaussian curvature and (4.2.10). The function J is the Jacobian appearing in the following change of variables formulas (see [13, Section 14.6] or Appendix 4.A).

Lemma 4.2.6. *For a function f on $\overline{\Omega_\varepsilon}$ we have*

$$\int_{\Omega_\varepsilon} f(x)\,dx = \int_\Gamma \int_{-\varepsilon}^{\varepsilon} f(y+\rho\nu(y))J(y,\rho)\,d\rho\,d\mathcal{H}^2(y) \tag{4.2.20}$$

and

$$\int_{\partial\Omega_\varepsilon} f(x)\,d\mathcal{H}^2(x) = \int_\Gamma f(y+\varepsilon\nu(y))J(y,\varepsilon)\,d\mathcal{H}^2(y) + \int_\Gamma f(y-\varepsilon\nu(y))J(y,-\varepsilon)\,d\mathcal{H}^2(y). \tag{4.2.21}$$

Here $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure.

When we use Lemma 4.2.6 with the moving surface $\Gamma(t)$ we write $J(y,t,\rho)$ for the Jacobian given by (4.2.19).

4.2.3 Material derivatives and differentiation of composite functions with the closest point mapping

Now let us return to the moving surface $\Gamma(t)$. We first give a material time derivative of a function on S_T. Let v be a vector field on S_T with $v\cdot\nu = V_\Gamma^N$. Suppose that there exists the flow map Φ_v of v, i.e. $\Phi_v(\cdot,t)\colon \Gamma(0)\to\mathbb{R}^3$ is a diffeomorphism onto its range for each $t\in[0,T]$ and

$$\Phi_v(Y,0)=Y, \quad \frac{d\Phi_v}{dt}(Y,t) = v(\Phi_v(Y,t),t) \quad \text{for} \quad (Y,t)\in\Gamma(0)\times(0,T).$$

Note that $\Phi_v(\cdot,t)$ is a diffeomorphism from $\Gamma(0)$ onto $\Phi_v(\Gamma(0),t)=\Gamma(t)$ for each $t\in[0,T]$ since the normal component of v is equal to the outward normal velocity V_Γ^N of the moving surface $\Gamma(t)$, which completely determines the change of the shape of $\Gamma(t)$. We define the material derivative of a function f on S_T along the velocity field v as

$$\partial_v^\bullet f(\Phi_v(Y,t),t) := \frac{d}{dt}\bigl(f(\Phi_v(Y,t),t)\bigr), \quad (Y,t)\in\Gamma(0)\times(0,T).$$

By the chain rule of differentiation it is also represented as

$$\partial_v^\bullet f(y,t) = \partial_t\tilde{f}(y,t) + v(y,t)\cdot\nabla\tilde{f}(y,t), \quad (y,t)\in S_T, \tag{4.2.22}$$

where $\tilde{f}$ is an arbitrary extension of f to N_T satisfying $\tilde{f}|_{S_T}=f$. We write ∂° for $\partial_v^\bullet$ with $v=V_\Gamma^N\nu$ and call it the normal time derivative. Note that the normal time derivative of a function f on S_T is equal to the time derivative of its constant extension $\bar{f}$ in the normal direction, i.e.

$$\partial^\circ f(y,t) = \partial_t\bar{f}(y,t) = \frac{d}{dt}\bigl(f(\pi(y,t),t)\bigr), \quad (y,t)\in S_T.$$

Also, for a tangential vector field v^T on S_T the material derivative of f along the velocity field of the form $v=V_\Gamma^N\nu+v^T$ is expressed as

$$\partial_v^\bullet f = \partial^\circ f + v^T\cdot\nabla_\Gamma f \quad \text{on} \quad S_T \tag{4.2.23}$$

by (4.2.22) and $v^T \cdot \nabla \tilde{f} = v^T \cdot \nabla_\Gamma f$ on S_T since v^T is tangential. See also [8, Section 3] for the time derivative of functions on a moving surface.

In the following sections we frequently differentiate the composition of a function on $\Gamma(t)$ and the closest point mapping $\pi(\cdot,t)$. To avoid repetition of the same calculations we give several formulas on derivatives of composite functions with π.

Let $f(x,t)$ be a function on $Q_{\varepsilon,T}$. Based on the normal coordinate system $x = \pi(x,t) + d(x,t)\nu(\pi(x,t),t)$ for $x \in \Omega_\varepsilon(t)$, we expand $f(x,t)$ in powers of the signed distance $d(x,t)$:

$$f(x,t) = g(\pi(x,t),t) + d(x,t)g^1(\pi(x,t),t) + \cdots .$$

Here g, g^1, and the coefficients of higher order terms in $d(x,t)$ are considered as functions on S_T. Also, for $k \in \mathbb{N}$ we write $R(d(x,t)^k)$ for the terms of order higher than $k-1$ with respect to small $d(x,t)$, i.e.

$$\begin{aligned} f(x,t) &= g(\pi(x,t),t) + \cdots + d(x,t)^{k-1}g^{k-1}(\pi(x,t),t) + R(d(x,t)^k), \\ R(d(x,t)^k) &= d(x,t)^k g^k(\pi(x,t),t) + d(x,t)^{k+1}g^{k+1}(\pi(x,t),t) + \cdots . \end{aligned} \tag{4.2.24}$$

In the sequel, we also use Landau's symbol $O(\varepsilon^k)$ (as $\varepsilon \to 0$) for a nonnegative integer k, i.e. $O(\varepsilon^k)$ is a quantity satisfying $|O(\varepsilon^k)| \leq C\varepsilon^k$ for small $\varepsilon > 0$ with a constant $C > 0$ independent of ε. Note that, contrary to $O(\varepsilon^k)$, we may differentiate $R(d(x,t)^k)$ with respect to x and t since it just stands for the higher order terms in the expansion (4.2.24) with respect to small $d(x,t)$, and the l-th order derivative of $R(d(x,t)^k)$ is $R(d(x,t)^{k-l})$ for $l \leq k$. Also, $R(d(x,t)^k) = O(\varepsilon^k)$ for $(x,t) \in Q_{\varepsilon,T}$ and $k \in \mathbb{N}$ by $|d(x,t)| < \varepsilon$ on $Q_{\varepsilon,T}$. We use the same notations on the expansion (4.2.24) for functions on $\Omega_\varepsilon(t)$ with each fixed $t \in [0,T]$.

Lemma 4.2.7. *Let f be a scalar- or vector-valued function on S_T. The derivatives of the composite function $f(\pi(x,t),t)$ with respect to x and t are of the form*

$$\nabla\bigl(f(\pi,t)\bigr) = \nabla_\Gamma f(\pi,t) + d(x,t)[A\nabla_\Gamma f](\pi,t) + R(d(x,t)^2), \tag{4.2.25}$$

$$\partial_t\bigl(f(\pi,t)\bigr) = \partial^\circ f(\pi,t) + d(x,t)[(\nabla_\Gamma V_\Gamma^N \cdot \nabla_\Gamma)f](\pi,t) + R(d(x,t)^2) \tag{4.2.26}$$

for $(x,t) \in Q_{\varepsilon,T}$. Here we abbreviate $\pi(x,t)$ to π.

We also give an expansion formula for the divergence of a matrix-valued function which we need to derive limit equations of the Navier–Stokes equations.

Lemma 4.2.8. *Let S and S^1 be 3×3 matrix-valued functions on $\Gamma(t)$ with each fixed $t \in (0,T)$. For $x \in \Omega_\varepsilon(t)$ we set*

$$D(x) = S(\pi(x,t)) + d(x,t)S^1(\pi(x,t)) + R(d(x,t)^2).$$

Then we have

$$\operatorname{div} D(x) = \operatorname{div}_\Gamma S(\pi(x,t)) + \bigl(S^1(\pi(x,t))\bigr)^T \nu(\pi,t) + R(d(x,t)). \tag{4.2.27}$$

for $x \in \Omega_\varepsilon(t)$. Here $(S^1)^T$ denotes the transpose of the matrix S^1.

4.3 Limit equations of the Euler equations

We consider the incompressible Euler equations in $\Omega_\varepsilon(t)$:

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in} \quad Q_{\varepsilon,T}, \tag{4.3.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad Q_{\varepsilon,T}, \tag{4.3.2}$$

$$u \cdot \nu_\varepsilon = V_\varepsilon^N \quad \text{on} \quad \partial_\ell Q_{\varepsilon,T}. \tag{4.3.3}$$

Here $u = (u_1, u_2, u_3)$ is the velocity of a bulk fluid and p is the pressure. The goal of this section is to derive limit equations of the Euler equations as ε goes to zero. According to the normal coordinate system (4.2.1), we expand u and p with respect to the signed distance $d(x,t)$ as

$$u(x,t) = v(\pi(x,t),t) + d(x,t)v^1(\pi(x,t),t) + R(d(x,t)^2), \tag{4.3.4}$$
$$p(x,t) = q(\pi(x,t),t) + d(x,t)q^1(\pi(x,t),t) + R(d(x,t)^2). \tag{4.3.5}$$

Here we used the notation (4.2.24). The limit equations are given as the principal term in the expansion with respect to $d(x,t)$ of the Euler equations in $\Omega_\varepsilon(t)$.

Theorem 4.3.1. *Let u and p satisfy the Euler equations* (4.3.1)–(4.3.3) *in the moving thin domain $\Omega_\varepsilon(t)$. Then the normal component of the zeroth order term v in the expansion* (4.3.4) *is equal to the outward normal velocity of the moving surface $\Gamma(t)$, i.e. $v \cdot \nu = V_\Gamma^N$. Moreover, v and the zeroth order term q and the first order term q^1 in the expansion* (4.3.5) *satisfy*

$$\partial_v^\bullet v + \nabla_\Gamma q + q^1\nu = 0 \quad \text{on} \quad S_T, \tag{4.3.6}$$
$$\mathrm{div}_\Gamma v = 0 \quad \text{on} \quad S_T. \tag{4.3.7}$$

Before starting to prove Theorem 4.3.1 we give remarks on the limit equations (4.3.6)–(4.3.7) and necessary conditions on the motion of $\Gamma(t)$ for the existence of incompressible fluids in $\Gamma(t)$ and $\Omega_\varepsilon(t)$ for all $\varepsilon > 0$.

Remark 4.3.2. Let us explain how the limit equations (4.3.6) and (4.3.7) determine v, q, and q^1. As stated in Theorem 4.3.1, the normal component of v is equal to the outward normal velocity of the moving surface. The tangential component of v and the scalar function q are determined by the equations

$$P_\Gamma\partial_v^\bullet v + \nabla_\Gamma q = 0, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad S_T. \tag{4.3.8}$$

Finally the scalar function q^1 is given just by the inner product of (4.3.6) and ν:

$$q^1 = -\partial_v^\bullet v \cdot \nu \quad \text{on} \quad S_T. \tag{4.3.9}$$

Note that q^1 comes from the normal derivative of the pressure p of the bulk fluid in the moving thin domain (see (4.3.17) below).

The system (4.3.8) is the same as the incompressible Euler system (II) in [18] with the constant density. When the surface $\Gamma(t) = \Gamma$ is stationary, the limit velocity v is tangential ($v \cdot \nu = V_\Gamma^N = 0$) and $P_\Gamma\{(v\cdot\nabla)v\} = \overline{\nabla}_v v$ holds on Γ by the Gauss formula (4.2.14), where $\overline{\nabla}_v v$ is the covariant derivative. From this and the fact that P_Γ is independent of the time it follows that

$$P_\Gamma\partial_v^\bullet v = P_\Gamma\partial_t v + P_\Gamma\{(v\cdot\nabla)v\} = \partial_t v + \overline{\nabla}_v v \quad \text{on} \quad \Gamma. \tag{4.3.10}$$

Hence the tangential limit system (4.3.8) becomes

$$\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = 0, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma \times (0,T),$$

which is the same as the Euler system on a manifold derived by Arnol′d [2,3] (see also Ebin and Marsden [12]). Also, applying $v \cdot \nu = 0$, (4.2.14), and the fact that ν is independent of time to (4.3.9) we obtain

$$q^1 = -\partial_v^\bullet v \cdot \nu = -\partial_t(v\cdot\nu) - \{(v\cdot\nabla)v\}\cdot\nu = -Av\cdot v, \tag{4.3.11}$$

which does not vanish in general even if the surface is stationary.

Remark 4.3.3. For the existence of a surface incompressible fluid obeying (4.3.7) it is required that the area of the moving surface $\Gamma(t)$ is preserved in time. Indeed, by the Leibniz formula (see [10, Lemma 2.2]) with a velocity field v on S_T satisfying $v\cdot\nu = V_\Gamma^N$ and (4.3.7) we have

$$\frac{d}{dt}|\Gamma(t)| = \frac{d}{dt}\int_{\Gamma(t)} 1\,d\mathcal{H}^2 = \int_{\Gamma(t)} \mathrm{div}_\Gamma v\,d\mathcal{H}^2 = 0, \tag{4.3.12}$$

where $|\Gamma(t)|$ is the area of $\Gamma(t)$. Similarly, when the moving thin domain $\Omega_\varepsilon(t)$ is filled with an incompressible fluid satisfying (4.3.2) and the impermeable boundary condition (4.3.3), its volume $|\Omega_\varepsilon(t)|$ must remain constant by the Reynolds transport theorem (see e.g. [14]):

$$\frac{d}{dt}|\Omega_\varepsilon(t)| = \frac{d}{dt}\int_{\Omega_\varepsilon(t)} 1\,dx = \int_{\partial\Omega_\varepsilon(t)} V_\varepsilon^N\,d\mathcal{H}^2 = \int_{\partial\Omega_\varepsilon(t)} u\cdot\nu_\varepsilon\,d\mathcal{H}^2 = \int_{\Omega_\varepsilon(t)} \mathrm{div}\,u\,dx = 0.$$

By the change of variables formula (4.2.20) the volume of $\Omega_\varepsilon(t)$ is expressed as

$$|\Omega_\varepsilon(t)| = \int_{\Omega_\varepsilon(t)} 1\,dx = \int_{\Gamma(t)}\int_{-\varepsilon}^{\varepsilon} J(y,t,\rho)\,d\rho\,d\mathcal{H}^2 = 2\varepsilon|\Gamma(t)| + \frac{2}{3}\varepsilon^3\int_{\Gamma(t)} K\,d\mathcal{H}^2.$$

Hence we need to assume

$$\frac{d}{dt}|\Gamma(t)| = 0, \quad \frac{d}{dt}\int_{\Gamma(t)} K\,d\mathcal{H}^2 = 0$$

for the existence of an incompressible fluid in the ε-tubular neighborhood $\Omega_\varepsilon(t)$ of $\Gamma(t)$ for all $\varepsilon > 0$. However, by the Gauss–Bonnet theorem we have

$$\int_{\Gamma(t)} K\,d\mathcal{H}^2 = 2\pi\chi(\Gamma(t)),$$

where $\chi(\Gamma(t))$ is the Euler characteristic of $\Gamma(t)$ (see e.g. [32, Section C.5]). Since the Euler characteristic is a topological invariant and the moving surface $\Gamma(t)$ does not change its topology, the integral of the Gaussian curvature K over $\Gamma(t)$ is constant in time. Therefore, only the area preserving condition (4.3.12) on $\Gamma(t)$ is necessary for the existence of incompressible fluids on $\Gamma(t)$ and in $\Omega_\varepsilon(t)$ for all $\varepsilon > 0$. Note that this assertion is valid only for a moving surface in $\mathbb{R}^3$ or a moving hypersurface in $\mathbb{R}^4$. Indeed, when $\Gamma(t)$ is a moving hypersurface in $\mathbb{R}^n$ with $n > 4$, the Jacobian $J(y,t,\rho)$ is a polynomial in ρ of degree greater than three (see e.g. [13, Section 14.6] and [22, Section 5.1]) and thus we need more constraints on the motion of $\Gamma(t)$.

Proof of Theorem 4.3.1. For the sake of simplicity, we use the abbreviations

$$f(\pi,t) = f(\pi(x,t),t), \quad R(d^k) = R(d(x,t)^k) \tag{4.3.13}$$

for a function f on S_T and $k\in\mathbb{N}$. Since ν_ε and V_ε^N are given by (4.2.5) and (4.2.6), the boundary condition (4.3.3) reads

$$u(x,t)\cdot\nu(\pi,t) = V_\Gamma^N(\pi,t), \quad x\in\partial\Omega_\varepsilon(t).$$

We substitute (4.3.4) for u in the above equality. Then

$$v(\pi,t)\cdot\nu(\pi,t) \pm \varepsilon v^1(\pi,t)\cdot\nu(\pi,t) + O(\varepsilon^2) = V_\Gamma^N(\pi,t)$$

when $d(x,t) = \pm\varepsilon$ (double-sign corresponds). Since $v(\pi,t)$, $v^1(\pi,t)$, $\nu(\pi,t)$, and $V_\Gamma^N(\pi,t)$ are independent of ε, it follows from the above equation that

$$v(\pi,t)\cdot\nu(\pi,t) = V_\Gamma^N(\pi,t), \tag{4.3.14}$$

$$v^1(\pi,t)\cdot\nu(\pi,t) = 0. \tag{4.3.15}$$

The first statement of the theorem follows from the equality (4.3.14). Let us write $v = V_\Gamma^N\nu + v^T$ with a tangential velocity field v^T on $\Gamma(t)$ and derive the equations (4.3.6) and (4.3.7). By (4.2.2) and (4.2.25) we have

$$\begin{aligned}\nabla u(x,t) &= \nabla\bigl(v(\pi,t)\bigr) + \nabla d(x,t)\otimes v^1(\pi,t) + R(d)\\ &= \nabla_\Gamma v(\pi,t) + \nu(\pi,t)\otimes v^1(\pi,t) + R(d)\end{aligned} \tag{4.3.16}$$

and

$$\nabla p(x,t) = \nabla_\Gamma q(\pi,t) + q^1(\pi,t)\nu(\pi,t) + R(d). \tag{4.3.17}$$

Also, by (4.2.4) and (4.2.26),

$$\begin{aligned}\partial_t u(x,t) &= \partial_t\bigl(v(\pi,t)\bigr) + \partial_t d(x,t)v^1(\pi,t) + R(d)\\ &= \partial^\circ v(\pi,t) - V_\Gamma^N(\pi,t)v^1(\pi,t) + R(d).\end{aligned} \tag{4.3.18}$$

From (4.3.16) the gradient of the j-th component of u is

$$\nabla u_j(x,t) = \nabla_\Gamma v_j(\pi,t) + v_j^1(\pi,t)\nu(\pi,t) + R(d).$$

We take the inner product of this equation and (4.3.4), and then apply (4.3.14) and $v\cdot\nabla_\Gamma v_j = v^T\cdot\nabla_\Gamma v_j$ to get the j-th component of the inertia term

$$u(x,t)\cdot\nabla u_j(x,t) = v^T(\pi,t)\cdot\nabla_\Gamma v_j(\pi,t) + V_\Gamma^N(\pi,t)v_j^1(\pi,t) + R(d).$$

Hence the inertia term $(u\cdot\nabla)u$ is of the form

$$[(u\cdot\nabla)u](x,t) = [(v^T\cdot\nabla_\Gamma)v](\pi,t) + V_\Gamma^N(\pi,t)v^1(\pi,t) + R(d). \tag{4.3.19}$$

Substituting (4.3.17), (4.3.18), and (4.3.19) for (4.3.1) and applying (4.2.23) we obtain

$$\partial_v^\bullet v(\pi,t) + \nabla_\Gamma q(\pi,t) + q^1(\pi,t)\nu(\pi,t) = R(d).$$

In this equation, each term on the left-hand side is independent of d. Therefore, the equation (4.3.6) should be satisfied.

Finally, by (4.3.15) and (4.3.16) we have

$$\operatorname{div} u(x,t) = \operatorname{tr}[\nabla u(x,t)] = \operatorname{div}_\Gamma v(\pi,t) + \nu(\pi,t)\cdot v^1(\pi,t) + R(d) = \operatorname{div}_\Gamma v(\pi,t) + R(d)$$

and thus the equation (4.3.2) reads $\operatorname{div}_\Gamma v(\pi,t) = R(d)$. Since the left-hand side is independent of d, we conclude that v satisfies the equation (4.3.7). $\square$

4.4 Limit equations of the Navier–Stokes equations

In this section, we consider the incompressible Navier–Stokes equations in $\Omega_\varepsilon(t)$:

$$\partial_t u + (u\cdot\nabla)u + \nabla p = \mu_0\Delta u \quad \text{in} \quad Q_{\varepsilon,T}, \tag{4.4.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad Q_{\varepsilon,T}. \tag{4.4.2}$$

Here $u = (u_1, u_2, u_3)$ is the velocity of a bulk fluid, p is the pressure, and $\mu_0 > 0$ is the viscosity coefficient. On these equations we impose the (perfect slip) Navier boundary condition of the form

$$u\cdot\nu_\varepsilon = V_\varepsilon^N \quad \text{on} \quad \partial_\ell Q_{\varepsilon,T}, \tag{4.4.3}$$

$$[D(u)\nu_\varepsilon]_{\tan} = 0 \quad \text{on} \quad \partial_\ell Q_{\varepsilon,T}. \tag{4.4.4}$$

Here $[a]_{\tan}$ denotes the tangential component to $\partial\Omega_\varepsilon(t)$ of a vector $a\in\mathbb{R}^3$ and $D(u)$ is the strain rate tensor given by

$$D(u) := \frac{\nabla u + (\nabla u)^T}{2},$$

where $(\nabla u)^T$ is the transposed matrix of ∇u.

In order to derive limit equations of the Navier–Stokes equations (4.4.1)–(4.4.4) we expand the velocity field u with respect to the signed distance $d(x,t)$ as

$$u(x,t) = v(\pi(x,t),t) + d(x,t)v^1(\pi(x,t),t) + d(x,t)^2v^2(\pi(x,t),t) + R(d(x,t)^3) \tag{4.4.5}$$

and the pressure p as (4.3.5). We need to expand u up to the second order term in $d(x,t)$ since the momentum equation (4.4.1) has the second order derivatives of u.

Theorem 4.4.1. *Let u and p satisfy the Navier–Stokes equations* (4.4.1)–(4.4.4) *in the moving thin domain $\Omega_\varepsilon(t)$. Then the normal component of the zeroth order term v in the expansion* (4.4.5) *is equal to the outward normal velocity of the moving surface $\Gamma(t)$, i.e. $v\cdot\nu = V_\Gamma^N$. Moreover, the velocity field v and the zeroth and first order terms q and q^1 in the expansion* (4.3.5) *satisfy*

$$\partial_v^\bullet v + \nabla_\Gamma q + q^1\nu = 2\mu_0\operatorname{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma) \quad \text{on} \quad S_T, \tag{4.4.6}$$

$$\operatorname{div}_\Gamma v = 0 \quad \text{on} \quad S_T. \tag{4.4.7}$$

Here $D^{tan}(v)$ is the tangential strain rate given by (4.2.12).

Remark 4.4.2. As in Remark 4.3.2, the normal component of v is equal to V_Γ^N, the tangential component of v and the scalar function q are determined by

$$P_\Gamma\partial_v^\bullet v + \nabla_\Gamma q = 2\mu_0 P_\Gamma\operatorname{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma), \quad \operatorname{div}_\Gamma v = 0 \quad \text{on} \quad S_T, \tag{4.4.8}$$

and the scalar function q^1 is given by the normal component of (4.4.6). The tangential system (4.4.8) is the same as the tangential incompressible Navier–Stokes–Scriven–Koba (NSSK) system in [18] with constant density (see (4.4) in [18]).

When $\Gamma(t) = \Gamma$ is fixed in time, the tangential system (4.4.8) is the same as the incompressible Navier–Stokes system on a fixed manifold derived by Taylor [31]

$$\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = \mu_0(\Delta_B v + Kv), \quad \operatorname{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma\times(0,T) \tag{4.4.9}$$

for a tangential velocity field v on Γ, although the authors of [18] claim that the system (4.4.8) on the stationary surface Γ is different from Taylor's model (4.4.9) (see Remark 4.4.3 below). Indeed, when the surface Γ is stationary, i.e. $V_\Gamma^N = 0$, the velocity field v in the system (4.4.8) is tangential and by applying (4.3.10) to the left-hand side of the first equation in (4.4.8) we obtain

$$\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = 2\mu_0 P_\Gamma \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v) P_\Gamma), \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma \times (0, T).$$

Moreover, since v is tangential and satisfies $\mathrm{div}_\Gamma v = 0$, the right-hand side of the first equation in the above system is the same as that in Taylor's system (4.4.9) by (4.2.17). Hence the tangential incompressible Navier–Stokes system (4.4.8) on the stationary surface Γ agrees with the system (4.4.9) given by Taylor.

As in the case of the Euler equations (see Remark 4.3.2), when the surface is stationary the function q^1 in (4.4.6) is given by

$$q^1 = \{-\partial_v^\bullet v + 2\mu_0 \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v) P_\Gamma)\} \cdot \nu = -Av \cdot v + 2\mu_0 \mathrm{tr}[A \nabla_\Gamma v],$$

where the second equality follows from (4.2.13) and (4.3.11). From this formula we observe that q^1 does not vanish in general even if the surface is stationary.

Remark 4.4.3. The authors of [18] argue that the tangential incompressible Navier–Stokes system (4.4.8) on a stationary surface Γ is different from the Navier–Stokes system (4.4.9) on a manifold given by Taylor [31], which is inconsistent with our argument in Remark 4.4.2. Unfortunately, there seems to be a flaw in derivation of Taylor's system (4.4.9) in [18, Section 5]. The authors of [18] applied an energetic variational approach with the dissipation energy given by the tangential strain rate $D^{tan}(v) = \{\nabla_\Gamma v + (\nabla_\Gamma v)^T\}/2$ to obtain (4.4.9). In their derivation of (4.4.9) they claim that $P_\Gamma \mathrm{div}_\Gamma\big(P_\Gamma D^{tan}(v)\big) = \Delta_B v + Kv$ holds on Γ when Γ is stationary and v is tangential and satisfies $\mathrm{div}_\Gamma v = 0$ (see the argument after [18, Theorem 5.1]). However, we have

$$2P_\Gamma \mathrm{div}_\Gamma\big(P_\Gamma D^{tan}(v)\big) = \Delta_B v + Kv - A^2 v$$

for any tangential vector field v on Γ satisfying $\mathrm{div}_\Gamma v = 0$, since the sum of the first two terms on the right-hand side is equal to $2P_\Gamma \mathrm{div}_\Gamma(P_\Gamma D^{tan}(v) P_\Gamma)$ by (4.2.17) and

$$2P_\Gamma \mathrm{div}_\Gamma\big(P_\Gamma D^{tan}(v)\big) - 2P_\Gamma \mathrm{div}_\Gamma(P_\Gamma D^{tan}(v) P_\Gamma) = 2P_\Gamma \mathrm{div}_\Gamma\big(P_\Gamma D^{tan}(v)(\nu \otimes \nu)\big) = -A^2 v$$

holds by the same calculations as in the proof of Lemma 4.2.3 (see Appendix 4.A).

It seems that their choice of the dissipation energy for derivation of (4.4.9) comes from a subtle misunderstanding of the strain rate tensor in Taylor's model, which is called the deformation tensor in [21, 31]. Taylor [31] defined the deformation tensor $\mathrm{Def}\, v$ for a tangential vector field v on Γ as a symmetric tensor field of type $(0,2)$ on the manifold Γ (see e.g. [20, Chapter 12] for tensor fields) satisfying

$$(\mathrm{Def}\, v)(X, Y) = \frac{1}{2}\Big(\overline{\nabla}_X v \cdot Y + X \cdot \overline{\nabla}_Y v\Big), \quad X, Y \in C^\infty(T\Gamma), \tag{4.4.10}$$

where $C^\infty(T\Gamma)$ is the set of all smooth tangential vector fields on Γ. (See also (2.3) in [21]. Note that (2.3) in [21] is a formula for one-forms on Γ and here we identify tangential vector fields on Γ with one-forms on Γ via raising and lowering indices.) Let us show that the right-hand side of (4.4.10) is equal to $\{D^{tan}(v)X\} \cdot Y$. By the Gauss formula (4.2.16) and the fact that the covariant derivative $\overline{\nabla}_X v$ is tangential,

$$\overline{\nabla}_X v = P_\Gamma\{(X \cdot \nabla_\Gamma) v\} = P_\Gamma (\nabla_\Gamma v)^T X \quad \text{on} \quad \Gamma,$$

where the second equality just follows from our notation on the tangential gradient matrix (see Section 4.2). From this formula and the facts that P_Γ is symmetric and that Y is tangential it follows that

$$\overline{\nabla}_X v \cdot Y = \{P_\Gamma(\nabla_\Gamma v)^T X\} \cdot Y = \{(\nabla_\Gamma v)^T X\} \cdot (P_\Gamma Y) = \{(\nabla_\Gamma v)^T X\} \cdot Y.$$

Similarly we have $X \cdot \overline{\nabla}_Y v = X \cdot \{(\nabla_\Gamma v)^T Y\} = \{(\nabla_\Gamma v)X\} \cdot Y$ and thus

$$\frac{1}{2}\Big(\overline{\nabla}_X v \cdot Y + X \cdot \overline{\nabla}_Y v\Big) = \frac{1}{2}\Big(\{\nabla_\Gamma v + (\nabla_\Gamma v)^T\}X\Big) \cdot Y = \{D^{tan}(v)X\} \cdot Y.$$

Therefore, for any X, $Y \in C^\infty(T\Gamma)$ the equality

$$(\mathrm{Def}\, v)(X,Y) = \{D^{tan}(v)X\} \cdot Y \tag{4.4.11}$$

holds. Therefore, the deformation tensor $\mathrm{Def}\, v$ can be identified with the restriction on $C^\infty(T\Gamma) \times C^\infty(T\Gamma)$ of the symmetric bilinear map

$$\mathcal{T}_{D^{tan}(v)} \colon C^\infty(\Gamma)^3 \times C^\infty(\Gamma)^3 \to C^\infty(\Gamma), \quad (F,G) \mapsto \{D^{tan}(v)F\} \cdot G.$$

Here $C^\infty(\Gamma)$ denotes the set of all smooth functions on Γ and $C^\infty(\Gamma)^3$ is the set of all smooth three-dimensional vector fields on Γ not necessarily tangential. However, it does not mean that $\mathrm{Def}\, v$ can be identified with the matrix $D^{tan}(v)$. Since $\mathrm{Def}\, v$ is a tensor field of type $(0,2)$ on the manifold Γ, for any $X \in C^\infty(T\Gamma)$ the mapping

$$(\mathrm{Def}\, v)(X,\cdot) \colon C^\infty(T\Gamma) \to C^\infty(\Gamma), \quad Y \mapsto (\mathrm{Def}\, v)(X,Y)$$

is a linear map from $C^\infty(T\Gamma)$ into $C^\infty(\Gamma)$, i.e. a one-form on Γ. By identifying one-forms on Γ with tangential vector fields on Γ via raising and lowering indices, we may consider $(\mathrm{Def}\, v)(X,\cdot) = (\mathrm{Def}\, v)X$ as a tangential vector field on Γ. On the other hand, for a tangential vector field X on Γ the vector field $D^{tan}(v)X$ is not tangential in general, even if v is tangential to Γ. Indeed, since $(\nabla_\Gamma v)^T\nu = (\nabla_\Gamma v)^T P_\Gamma \nu = 0$ and $(\nabla_\Gamma v)\nu = -(\nabla_\Gamma \nu)v = Av$, where the second relation follows from the fact that v is tangential, we have

$$D^{tan}(v)\nu = \frac{1}{2}\{(\nabla_\Gamma v)\nu + (\nabla_\Gamma v)^T\nu\} = \frac{1}{2}Av.$$

From this equality and the symmetry of the matrix $D^{tan}(v)$ it follows that

$$D^{tan}(v)X \cdot \nu = X \cdot D^{tan}(v)\nu = \frac{1}{2}X \cdot Av$$

for any tangential vector field X on Γ. The last term does not vanish and thus the vector field $D^{tan}(v)X$ is not tangential on Γ in general.

To give a proper interpretation of the deformation tensor as a matrix, we observe that in (4.4.11) the vector fields X and Y are tangential to Γ and thus

$$\{D^{tan}(v)X\} \cdot Y = \{D^{tan}(v)P_\Gamma X\} \cdot (P_\Gamma Y) = \{P_\Gamma D^{tan}(v)P_\Gamma X\} \cdot Y$$

by the symmetry of the orthogonal projection P_Γ. Then (4.4.11) becomes

$$(\mathrm{Def}\, v)(X,Y) = \{P_\Gamma D^{tan}(v)P_\Gamma X\} \cdot Y$$

for all tangential vector fields X and Y on Γ. Moreover, the matrix $P_\Gamma D^{tan}(v)P_\Gamma$ is symmetric and for any $X \in C^\infty(T\Gamma)$ the vector field $P_\Gamma D^{tan}(v)P_\Gamma X$ is tangential to Γ. Therefore, we may identify the deformation tensor

$$\mathrm{Def}\, v = \mathcal{T}_{D^{tan}(v)}|_{C^\infty(T\Gamma)\times C^\infty(T\Gamma)} \colon C^\infty(T\Gamma) \times C^\infty(T\Gamma) \to C^\infty(\Gamma)$$

with the symmetric matrix $P_\Gamma D^{tan}(v)P_\Gamma$.

The matrix $P_\Gamma D^{tan}(v)P_\Gamma$ is called a projected strain rate in [18] and employed to define the dissipation energy in their energetic variational method for derivation of the incompressible NSSK system on the moving surface (see [18, Lemma 3.4 and Section 4]). Therefore, the strain rate tensor in Taylor's system (4.4.9) is the same as that in the tangential incompressible Navier–Stokes system (4.4.8).

Proof of Theorem 4.4.1. As in the proof of Theorem 4.3.1 we use the abbreviations (4.3.13). Due to the first boundary condition (4.4.3) we have

$$v(\pi,t)\cdot\nu(\pi,t) = V_\Gamma^N(\pi,t), \tag{4.4.12}$$
$$v^1(\pi,t)\cdot\nu(\pi,t) = 0, \tag{4.4.13}$$
$$v^2(\pi,t)\cdot\nu(\pi,t) = 0 \tag{4.4.14}$$

and the surface divergence-free condition (4.4.7) for v by the same argument as in the proof of Theorem 4.3.1. Moreover, we already calculated the expansion of the left-hand side of (4.4.1) in the proof of Theorem 4.3.1:

$$\begin{aligned}\partial_t u(x,t) + [(u\cdot\nabla)u](x,t) + \nabla p(x,t)&\\ = \partial_v^\bullet v(\pi,t) + \nabla_\Gamma q(\pi,t) + q^1(\pi,t)\nu(\pi,t) + R(d).&\end{aligned} \tag{4.4.15}$$

Let us show that the expansion of the viscous term Δu is of the form

$$\Delta u(x,t) = 2[\mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma)](\pi,t) + R(d). \tag{4.4.16}$$

Since $\Delta u = 2\mathrm{div}\, D(u)$ holds by the divergence-free condition (4.4.2), we consider the expansion in powers of d of the strain rate tensor $D(u)$. We differentiate both sides of (4.4.5) with respect to x and apply (4.2.2) and (4.2.25) to get

$$\begin{aligned}\nabla u(x,t) = \nabla_\Gamma v(\pi,t) + [\nu\otimes v^1](\pi,t)&\\ + d(x,t)\{[A\nabla_\Gamma v](\pi,t) + \nabla_\Gamma v^1(\pi,t) + 2[\nu\otimes v^2](\pi,t)\} + R(d^2).&\end{aligned} \tag{4.4.17}$$

Hence the strain rate tensor of u is expressed as

$$D(u)(x,t) = S(\pi,t) + d(x,t)S^1(\pi,t) + R(d^2), \tag{4.4.18}$$

where

$$S := D^{tan}(v) + \frac{\nu\otimes v^1 + v^1\otimes\nu}{2}, \tag{4.4.19}$$
$$S^1 := \frac{A\nabla_\Gamma v + (A\nabla_\Gamma v)^T}{2} + D^{tan}(v^1) + \nu\otimes v^2 + v^2\otimes\nu. \tag{4.4.20}$$

Let us write the second boundary condition (4.4.4) in terms of S and S^1. By (4.2.5) and (4.2.6) the boundary condition (4.4.4) reads

$$P_\Gamma(\pi,t)D(u)(x,t)\nu(\pi,t) = 0, \quad x\in\partial\Omega_\varepsilon(t).$$

We substitute (4.4.18) for the above $D(u)(x,t)$ to obtain

$$P_\Gamma(\pi,t)S(\pi,t)\nu(\pi,t) \pm \varepsilon P_\Gamma(\pi,t)S^1(\pi,t)\nu(\pi,t) + O(\varepsilon^2) = 0$$

according to $d(x,t) = \pm\varepsilon$ (double-sign corresponds). Since the matrices $S(\pi,t)$, $S^1(\pi,t)$, $P_\Gamma(\pi,t)$, and the vector $\nu(\pi,t)$ are independent of ε, we have

$$P_\Gamma(\pi,t)S(\pi,t)\nu(\pi,t) = 0 \tag{4.4.21}$$

$$P_\Gamma(\pi,t)S^1(\pi,t)\nu(\pi,t) = 0. \tag{4.4.22}$$

Substituting (4.4.19) for S in (4.4.21) and observing

$$(\nu \otimes v^1)\nu = (v^1 \cdot \nu)\nu = 0, \quad (v^1 \otimes \nu)\nu = (\nu \cdot \nu)v^1 = v^1, \quad P_\Gamma v^1 = v^1$$

by (4.4.13) we get

$$v^1(\pi,t) = -2P_\Gamma(\pi,t)D^{tan}(v)(\pi,t)\nu(\pi,t). \tag{4.4.23}$$

Moreover, we multiply ν by S^1 given by (4.4.20) and apply

$$(A\nabla_\Gamma v)^T\nu = (\nabla_\Gamma v)^T A\nu = 0, \quad (\nabla_\Gamma v^1)^T\nu = (\nabla_\Gamma v^1)^T P_\Gamma \nu = 0$$

by the symmetry of A and P_Γ, $\nabla_\Gamma = P_\Gamma\nabla_\Gamma$, and (4.2.7), and then use $(\nu \otimes v^2)\nu = 0$ and $(v^2 \otimes \nu)\nu = v^2$ by (4.4.14) to obtain

$$S^1\nu = \frac{1}{2}(A\nabla_\Gamma v + \nabla_\Gamma v^1)\nu + v^2. \tag{4.4.24}$$

It is tangential to $\Gamma(t)$ by $\nabla_\Gamma = P_\Gamma\nabla$, (4.2.7) and (4.4.14). Hence (4.4.22) yields

$$S^1(\pi,t)\nu(\pi,t) = 0. \tag{4.4.25}$$

Now we apply the formula (4.2.27) to the expansion (4.4.18). Then by the symmetry of S^1 (see (4.4.20)) and the equality (4.4.25) we get

$$\operatorname{div} D(u)(x,t) = \operatorname{div}_\Gamma S(\pi,t) + R(d). \tag{4.4.26}$$

Let us write S in terms of v. Substituting (4.4.23) for (4.4.19), using the formulas

$$(Ma) \otimes b = M(a \otimes b), \quad a \otimes (Mb) = (a \otimes b)M^T$$

for a square matrix M of order three and three-dimensional vectors a and b, and observing $\big(P_\Gamma D^{tan}(v)\big)^T = D^{tan}(v)P_\Gamma$ by the symmetry of P_Γ and $D^{tan}(v)$, we have

$$\begin{aligned} S &= D^{tan}(v) - (\nu \otimes \nu)D^{tan}(v)P_\Gamma - P_\Gamma D^{tan}(v)(\nu \otimes \nu) \\ &= P_\Gamma D^{tan}(v)P_\Gamma + (\nu \otimes \nu)D^{tan}(v)(\nu \otimes \nu). \end{aligned}$$

Here the second term on the last line vanishes by $(\nu \otimes \nu)\nabla_\Gamma v = (\nabla_\Gamma v)^T(\nu \otimes \nu) = 0$. Hence it follows that

$$S(\pi,t) = P_\Gamma(\pi,t)D^{tan}(v)(\pi,t)P_\Gamma(\pi,t) \tag{4.4.27}$$

and we obtain (4.4.16) by applying (4.4.26) and (4.4.27) to $\Delta u = 2\operatorname{div} D(u)$. Finally, we substitute (4.4.15) and (4.4.16) for the momentum equation (4.4.1) to get

$$\partial_v^\bullet v(\pi,t) + \nabla_\Gamma q(\pi,t) + q^1(\pi,t)\nu(\pi,t) + R(d) = 2\mu_0[\operatorname{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma)](\pi,t) + R(d).$$

Since all terms except for $R(d)$ are independent of d, we conclude that the equation (4.4.6) should be satisfied. $\square$

Remark 4.4.4. We may replace the perfect slip condition (4.4.4) by the partial slip condition

$$[D(u)\nu_\varepsilon]_{\mathrm{tan}} + k(u^T - v_\Omega^T) = 0 \quad \text{on} \quad \partial_\ell Q_{\varepsilon,T},$$

where $u^T = (I_3 - \nu_\varepsilon \otimes \nu_\varepsilon)u$, $k > 0$ is a constant, and $v_\Omega^T(\cdot,t)$ is a given tangential velocity field on $\partial\Omega_\varepsilon(t)$. However, it makes the limit velocity overdetermined. Indeed, suppose that v_Ω^T is given by

$$v_\Omega^T(x,t) = \begin{cases} v_{\mathrm{outer}}(\pi(x,t),t) & \text{if} \quad d(x,t) = \varepsilon, \\ v_{\mathrm{inner}}(\pi(x,t),t) & \text{if} \quad d(x,t) = -\varepsilon, \end{cases}$$

where $v_{\mathrm{outer}}(\cdot,t)$ and $v_{\mathrm{inner}}(\cdot,t)$ are given tangential velocity fields on $\Gamma(t)$. Then the same calculations as in the proof of Theorem 4.4.1 yield

$$v = V_\Gamma^N \nu + \frac{v_{\mathrm{outer}} + v_{\mathrm{inner}}}{2}.$$

Hence the limit velocity v is completely determined by given velocities while it should satisfy similar equations to (4.4.6) and (4.4.7).

Remark 4.4.5. In the proof of Theorem 4.4.1 we obtained the expansion (4.4.16) of the viscous term Δu by using the expansion of the strain rate tensor $D(u)$. Here let us expand Δu by direct calculations. In what follows, we abbreviate $\pi(x,t)$ and $R(d(x,t))$ to π and $R(d)$ for $x \in \Omega_\varepsilon(t)$ and suppress the argument t. By (4.4.17) the gradient of the j-th component of u $(j = 1,2,3)$ is

$$\nabla u_j(x) = \nabla_\Gamma v_j(\pi) + v_j^1(\pi)\nu(\pi) + d(x)F_j(\pi) + R(d^2), \tag{4.4.28}$$

where $F_j = A\nabla_\Gamma v_j + \nabla_\Gamma v_j^1 + 2v_j^2\nu$. We differentiate both sides of (4.4.28) with respect to x and apply (4.2.2), (4.2.25), and $\nabla_\Gamma \nu = -A$ to get

$$\nabla^2 u_j(x) = \nabla_\Gamma^2 v_j(\pi) + [\nabla_\Gamma v_j^1 \otimes \nu](\pi) - v_j^1(\pi)A(\pi) + [\nu \otimes F_j](\pi) + R(d).$$

Taking the trace of both sides and observing $A\nabla_\Gamma v_j \cdot \nu = \nabla_\Gamma v_j^1 \cdot \nu = 0$ we obtain

$$\Delta u_j(x) = \Delta_\Gamma v_j(\pi) - v_j^1(\pi)H(\pi) + 2v_j^2(\pi) + R(d)$$

for each $j = 1,2,3$ and thus

$$\Delta u_j(x) = \Delta_\Gamma v(\pi) - H(\pi)v^1(\pi) + 2v^2(\pi) + R(d).$$

Let us express v^1 and v^2 in terms of v. The first order term v^1 is given by (4.4.23), $\nabla_\Gamma = P_\Gamma \nabla_\Gamma$, and $(\nabla_\Gamma v)^T\nu = (\nabla_\Gamma v)^T P_\Gamma \nu = 0$:

$$v^1 = -2P_\Gamma D^{tan}(v)\nu = -(\nabla_\Gamma v)\nu.$$

By (4.4.24) and (4.4.25) we can represent v^2 in terms of v and v^1 as

$$v^2 = -\frac{1}{2}(A\nabla_\Gamma v + \nabla_\Gamma v^1)\nu.$$

From this it follows that $v^2 = 0$ since $v^1 = -(\nabla_\Gamma v)\nu$ is tangential and thus

$$(\nabla_\Gamma v^1)\nu = \nabla_\Gamma(v^1 \cdot \nu) - (\nabla_\Gamma \nu)v^1 = Av^1 = -A(\nabla_\Gamma v)\nu.$$

Hence we obtain another expansion formula of the viscous term

$$\Delta u(x) = \Delta_\Gamma v(\pi) + [H(\nabla_\Gamma v)\nu](\pi) + R(d). \tag{4.4.29}$$

Comparing the expansions (4.4.16) and (4.4.29) we expect that the equality

$$2\mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma) = \Delta_\Gamma v + H(\nabla_\Gamma v)\nu \tag{4.4.30}$$

holds for the limit velocity v. Let us prove this equality. By the formula (4.2.13) for the left-hand side, the proof of (4.4.30) reduces to showing

$$\nabla_\Gamma(\mathrm{div}_\Gamma v) = 0, \quad 2\mathrm{tr}[A\nabla_\Gamma v] = (\Delta_\Gamma v)\cdot\nu. \tag{4.4.31}$$

The first equality follows from the surface divergence-free condition (4.4.7) for the limit velocity v. To obtain the second equality we need to observe the expansion of the divergence-free condition (4.4.2) in powers of the signed distance d up to the first order term. Taking the trace of (4.4.17) and using $v^1\cdot\nu = 0$ and $v^2 = 0$ we have

$$\mathrm{div}\, u(x) = \mathrm{div}_\Gamma v(\pi) + d(x)\{\mathrm{tr}[A\nabla_\Gamma v](\pi) + \mathrm{div}_\Gamma v^1(\pi)\} + R(d^2).$$

Since the left-hand side vanishes for all $x\in\Omega_\varepsilon(t)$ by (4.4.2), observing the first order term in $d(x)$ on the right-hand side we obtain

$$\mathrm{tr}[A\nabla_\Gamma v] + \mathrm{div}_\Gamma v^1 = 0. \tag{4.4.32}$$

To the second term on the left-hand side we apply $v^1 = -(\nabla_\Gamma v)\nu$. Then since

$$\begin{aligned}\mathrm{div}_\Gamma[(\nabla_\Gamma v)\nu] &= (\mathrm{div}_\Gamma \nabla_\Gamma v)\cdot\nu + \mathrm{tr}[(\nabla_\Gamma \nu)^T\nabla_\Gamma v],\\ &= (\Delta_\Gamma v)\cdot\nu - \mathrm{tr}[A^T\nabla_\Gamma v]\end{aligned}$$

and the Weingarten map A is symmetric, the equality (4.4.32) becomes

$$2\mathrm{tr}[A\nabla_\Gamma v] - (\Delta_\Gamma v)\cdot\nu = 0.$$

Hence the second equality in (4.4.31) holds and (4.4.30) follows.

4.5 Energy identities

The purpose of this section is to find a relation between energy identities of the Euler and Navier–Stokes equations in the moving thin domains and those of the limit equations on the moving surface. We first derive the energy identities from the equations and then show that the energy identities of the limit surface equations are also derived as thin width limits of those of the original bulk equations.

4.5.1 Euler equations

Lemma 4.5.1. *Let u and p satisfy the Euler equations* (4.3.1)–(4.3.3) *in the moving thin domain $\Omega_\varepsilon(t)$. Then we have*

$$\frac{d}{dt}\int_{\Omega_\varepsilon(t)}\frac{|u|^2}{2}\,dx = -\int_{\partial\Omega_\varepsilon(t)} pV_\varepsilon^N\,d\mathcal{H}^2. \tag{4.5.1}$$

The identity (4.5.1) means that the rate of change of the kinetic energy of the incompressible perfect fluid in a moving domain is equal to the rate of work done by the pressure caused by the motion of the boundary.

Proof. By the Reynolds transport theorem (see [14]) and (4.3.1),

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_\varepsilon(t)}\frac{|u|^2}{2}\,dx &= \int_{\Omega_\varepsilon(t)} u\cdot\partial_t u\,dx + \int_{\partial\Omega_\varepsilon(t)}\frac{|u|^2}{2}V_\varepsilon^N\,d\mathcal{H}^2 \\ &= \int_{\Omega_\varepsilon(t)} u\cdot\{-(u\cdot\nabla)u-\nabla p\}\,dx + \int_{\partial\Omega_\varepsilon(t)}\frac{|u|^2}{2}V_\varepsilon^N\,d\mathcal{H}^2.\end{aligned} \tag{4.5.2}$$

By integration by parts and the equations (4.3.2) and (4.3.3) we have

$$\begin{aligned}\int_{\Omega_\varepsilon(t)} u\cdot(u\cdot\nabla)u\,dx &= \int_{\partial\Omega_\varepsilon(t)}|u|^2(u\cdot\nu_\varepsilon)\,d\mathcal{H}^2 - \int_{\Omega_\varepsilon(t)}\{u\cdot(u\cdot\nabla)u+|u|^2\mathrm{div}\,u\}\,dx \\ &= \int_{\partial\Omega_\varepsilon(t)}|u|^2V_\varepsilon^N\,d\mathcal{H}^2 - \int_{\Omega_\varepsilon(t)} u\cdot(u\cdot\nabla)u\,dx.\end{aligned}$$

Therefore,

$$\int_{\Omega_\varepsilon(t)} u\cdot(u\cdot\nabla)u\,dx = \int_{\partial\Omega_\varepsilon(t)}\frac{|u|^2}{2}V_\varepsilon^N\,d\mathcal{H}^2. \tag{4.5.3}$$

On the other hand, by integration by parts

$$\int_{\Omega_\varepsilon(t)} u\cdot\nabla p\,dx = \int_{\partial\Omega_\varepsilon(t)}(u\cdot\nu_\varepsilon)p\,d\mathcal{H}^2 - \int_{\Omega_\varepsilon(t)}(\mathrm{div}\,u)p\,dx$$

and we apply (4.3.2) and (4.3.3) to the right-hand side to get

$$\int_{\Omega_\varepsilon(t)} u\cdot\nabla p\,dx = \int_{\partial\Omega_\varepsilon(t)} pV_\varepsilon^N\,d\mathcal{H}^2. \tag{4.5.4}$$

Substituting (4.5.3) and (4.5.4) for (4.5.2) we obtain the energy identity (4.5.1). □

Lemma 4.5.2. *Let v, q, and q^1 satisfy the limit equations* (4.3.6) *and* (4.3.7) *of the Euler equations. Suppose that the normal component of v is equal to the outward normal velocity of $\Gamma(t)$, i.e. $v\cdot\nu = V_\Gamma^N$. Then we have*

$$\frac{d}{dt}\int_{\Gamma(t)}\frac{|v|^2}{2}\,d\mathcal{H}^2 = \int_{\Gamma(t)}(qH-q^1)V_\Gamma^N\,d\mathcal{H}^2. \tag{4.5.5}$$

The right-hand side of (4.5.5) represents the rate of work done by the moving surface to the fluid. Note that it contains the scalar function q^1, which corresponds to the normal derivative of the surface pressure.

Proof. By the assumption we can write $v = V_\Gamma^N\nu + v^T$ with a tangential velocity field v^T on $\Gamma(t)$. We apply the Leibniz formula (see [10, Lemma 2.2]) with $v = V_\Gamma^N\nu + v^T$ to the integral of $|v|^2/2$ over $\Gamma(t)$. (Note that the tangential velocity v^T does not affect the change of the shape of $\Gamma(t)$.) Then we have

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma(t)}\frac{|v|^2}{2}\,d\mathcal{H}^2 &= \int_{\Gamma(t)}\left\{\partial_v^\bullet\left(\frac{|v|^2}{2}\right)+\frac{|v|^2}{2}\,\mathrm{div}_\Gamma v\right\}d\mathcal{H}^2 \\ &= \int_{\Gamma(t)} v\cdot\partial_v^\bullet v\,d\mathcal{H}^2 + \int_{\Gamma(t)}\frac{|v|^2}{2}\,\mathrm{div}_\Gamma v\,d\mathcal{H}^2.\end{aligned}$$

To the last line we apply the equations (4.3.6) and (4.3.7). Then

$$\frac{d}{dt}\int_{\Gamma(t)} \frac{|v|^2}{2}\,d\mathcal{H}^2 = -\int_{\Gamma(t)} v\cdot(\nabla_\Gamma q + q^1\nu)\,d\mathcal{H}^2. \tag{4.5.6}$$

For the first term on the right-hand side,

$$v\cdot\nabla_\Gamma q = \mathrm{div}_\Gamma(qv) + q\,\mathrm{div}_\Gamma v = -qHV_\Gamma^N + \mathrm{div}_\Gamma(qv^T)$$

by $v = V_\Gamma^N\nu + v^T$, $\nabla_\Gamma(qV_\Gamma^N)\cdot\nu = 0$, $\mathrm{div}_\Gamma\nu = -H$, and (4.3.7). Moreover, the integral of the surface divergence of the tangential vector field qv^T over $\Gamma(t)$ vanishes by Stokes' theorem since $\Gamma(t)$ is closed. Hence we have

$$\int_{\Gamma(t)} v\cdot\nabla_\Gamma q\,d\mathcal{H}^2 = -\int_{\Gamma(t)} qHV_\Gamma^N\,d\mathcal{H}^2. \tag{4.5.7}$$

For the second term we have

$$\int_{\Gamma(t)} v\cdot(q^1\nu)\,d\mathcal{H}^2 = \int_{\Gamma(t)} q^1V_\Gamma^N\,d\mathcal{H}^2 \tag{4.5.8}$$

by $v\cdot\nu = V_\Gamma^N$. The energy identity (4.5.5) follows from (4.5.6), (4.5.7), and (4.5.8). □

Let us show that the energy identity (4.5.5) on the moving surface can be derived as a thin width limit of that in the moving thin domain (4.5.1). As in Section 4.3 we expand the velocity u and the pressure p in powers of the signed distance d as (4.3.4) and (4.3.5) and determine the zeroth order term in ε of the energy identity (4.5.1).

Theorem 4.5.3. *Let u and p satisfy the energy identity* (4.5.1). *Then the zeroth order term v in the expansion* (4.3.4) *and the zeroth and first order terms q and q^1 in the expansion* (4.3.5) *satisfy the energy identity* (4.5.5).

Proof. From the expansion (4.3.4) we have

$$\frac{|u(x,t)|^2}{2} = \frac{|v(\pi(x,t),t)|^2}{2} + d(x,t)V(\pi(x,t),t) + R(d(x,t)^2)$$

for $x\in\Omega_\varepsilon(t)$, where $V := v\cdot v^1$. Using this expansion we write

$$\int_{\Omega_\varepsilon(t)} \frac{|u(x,t)|^2}{2}\,dx = I_1 + I_2 + I_3,$$

where

$$I_1 := \int_{\Omega_\varepsilon(t)} \frac{|v(\pi(x,t),t)|^2}{2}\,dx,$$

$$I_2 := \int_{\Omega_\varepsilon(t)} d(x,t)V(\pi(x,t),t)\,dx, \quad I_3 := \int_{\Omega_\varepsilon(t)} R(d(x,t)^2)\,dx.$$

To I_1 and I_2 we apply the change of variables formula (4.2.20) to get

$$\begin{aligned} I_1 &= \int_{\Gamma(t)}\int_{-\varepsilon}^{\varepsilon} \frac{|v(y,t)|^2}{2}J(y,t,\rho)\,d\rho\,d\mathcal{H}^2(y) = 2\varepsilon\int_{\Gamma(t)} \frac{|v(y,t)|^2}{2}\,d\mathcal{H}^2(y) + \varepsilon^2 f_1(\varepsilon,t), \\ I_2 &= \int_{\Gamma(t)}\int_{-\varepsilon}^{\varepsilon} \rho V(y,t)J(y,t,\rho)\,d\rho\,d\mathcal{H}^2(y) = \varepsilon^2 f_2(\varepsilon,t), \end{aligned}$$

where f_1 and f_2 are polynomials in ε with time-dependent coefficients. (Note that the Jacobian $J(y,t,\rho)$ given by (4.2.19) is a polynomial in ρ and the principal curvatures of $\Gamma(t)$.) Hence

$$\frac{dI_1}{dt} = 2\varepsilon\frac{d}{dt}\int_{\Gamma(t)}\frac{|v(y,t)|^2}{2}\,d\mathcal{H}^2(y) + O(\varepsilon^2), \quad \frac{dI_2}{dt} = O(\varepsilon^2). \tag{4.5.9}$$

For I_3, using the Reynolds transport theorem and observing that the first order time derivative of $R(d(x,t)^2)$ is $R(d(x,t))$ we have

$$\frac{dI_3}{dt} = \int_{\Omega_\varepsilon(t)} R(d(x,t))\,dx + \int_{\partial\Omega_\varepsilon(t)} R(d(x,t)^2)V_\varepsilon^N(x,t)\,d\mathcal{H}^2(x).$$

We apply the change of variables formula (4.2.20) to the first term on the right-hand side of the above equality. Then by $R(d(x,t)) = R(\rho) = O(\varepsilon)$ and $J(y,t,\rho) = O(1)$ for $d(x,t) = \rho \in (-\varepsilon,\varepsilon)$ with $x \in \Omega_\varepsilon(t)$ to get

$$\int_{\Omega_\varepsilon(t)} R(d(x,t))\,dx = \int_{\Gamma(t)}\int_{-\varepsilon}^{\varepsilon} R(\rho)J(y,t,\rho)\,d\rho\,d\mathcal{H}^2(y) = O(\varepsilon^2).$$

Moreover, by $R(d(x,t)^2) = O(\varepsilon^2)$ for $x \in \partial\Omega_\varepsilon(t)$ and

$$|V_\varepsilon^N(x,t)| = |V_\Gamma^N(\pi(x,t),t)| = O(1), \quad x \in \partial\Omega_\varepsilon(t),$$

which follows from (4.2.6) and the fact that V_Γ^N is independent of ε, and the change of variables formula (4.2.21) and $J(y,t,\pm\varepsilon) = O(1)$, we have

$$\int_{\partial\Omega_\varepsilon(t)} R(d(x,t)^2)V_\varepsilon^N(x,t)\,d\mathcal{H}^2(x) = \sum_{\rho=\pm\varepsilon}\int_{\Gamma(t)} O(\varepsilon^2)J(y,t,\rho)\,d\mathcal{H}^2(y) = O(\varepsilon^2).$$

Hence $dI_3/dt = O(\varepsilon^2)$. From this estimate and (4.5.9) it follows that

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_\varepsilon(t)}\frac{|u(x,t)|^2}{2}\,dx &= \frac{d}{dt}(I_1+I_2+I_3) \\ &= 2\varepsilon\frac{d}{dt}\int_{\Gamma(t)}\frac{|v(y,t)|^2}{2}\,d\mathcal{H}^2(y) + O(\varepsilon^2).\end{aligned} \tag{4.5.10}$$

Let us expand the right-hand side of the energy identity (4.5.1) in ε. By the expansion (4.3.5) of the pressure p, the relation (4.2.6), and the formula (4.2.21),

$$\int_{\partial\Omega_\varepsilon(t)} p(x,t)V_\varepsilon^N(x,t)\,d\mathcal{H}^2(x) = J_1 + \varepsilon J_2 + O(\varepsilon^2), \tag{4.5.11}$$

where

$$\begin{aligned} J_1 &:= \int_{\Gamma(t)} q(y,t)V_\Gamma^N(y,t)\{J(y,t,\varepsilon) - J(y,t,-\varepsilon)\}\,d\mathcal{H}^2(y), \\ J_2 &:= \int_{\Gamma(t)} q^1(y,t)V_\Gamma^N(y,t)\{J(y,t,\varepsilon) + J(y,t,-\varepsilon)\}\,d\mathcal{H}^2(y). \end{aligned}$$

From (4.2.19) we have

$$\begin{aligned} &J(y,t,\varepsilon) - J(y,t,-\varepsilon) = -2\varepsilon H(y,t) + O(\varepsilon^2), \\ &J(y,t,\varepsilon) + J(y,t,-\varepsilon) = 2 + O(\varepsilon^2). \end{aligned}$$

Hence

$$J_1 = -2\varepsilon \int_{\Gamma(t)} q(y,t)H(y,t)V_\Gamma^N(y,t)\,d\mathcal{H}^2(y),$$
$$J_2 = 2\int_{\Gamma(t)} q^1(y,t)V_\Gamma^N(y,t)\,d\mathcal{H}^2(y)$$

and applying them to the right-hand side of (4.5.11) we get

$$\int_{\partial\Omega_\varepsilon(t)} p(x,t)V_\varepsilon^N(x,t)\,d\mathcal{H}^2(x) = -2\varepsilon\int_{\Gamma(t)}\{q(y,t)H(y,t)-q^1(y,t)\}V_\Gamma^N(y,t)\,d\mathcal{H}^2(y)+O(\varepsilon^2). \quad (4.5.12)$$

Finally, we substitute (4.5.10) and (4.5.12) for (4.5.1) and divide both sides by 2ε to obtain

$$\frac{d}{dt}\int_{\Gamma(t)}\frac{|v(y,t)|^2}{2}\,d\mathcal{H}^2(y) = \int_{\Gamma(t)}\{q(y,t)H(y,t)-q^1(y,t)\}V_\Gamma^N(y,t)\,d\mathcal{H}^2(y)+O(\varepsilon).$$

Since the left-hand side and the first term on the right-hand side are independent of ε, we conclude that the identity (4.5.5) should be satisfied. □

4.5.2 Navier–Stokes equations

Lemma 4.5.4. *Let u and p satisfy the Navier–Stokes equations* (4.4.1)–(4.4.4) *in the moving thin domain $\Omega_\varepsilon(t)$. Then we have*

$$\frac{d}{dt}\int_{\Omega_\varepsilon(t)}\frac{|u|^2}{2}\,dx = -2\mu_0\int_{\Omega_\varepsilon(t)}|D(u)|^2\,dx + \int_{\partial\Omega_\varepsilon(t)}(\sigma\nu_\varepsilon\cdot\nu_\varepsilon)V_\varepsilon^N\,d\mathcal{H}^2. \quad (4.5.13)$$

Here $\sigma := 2\mu_0 D(u) - pI_3$ denotes the Cauchy stress tensor.

The first term on the right-hand side of (4.5.13) represents the energy dissipation by viscosity and the second term stands for the rate of work done by the normal component of the stress vector $\sigma\nu_\varepsilon$ on the moving boundary.

Proof. By the Reynolds transport theorem (see [14]) and the equation (4.4.1),

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_\varepsilon(t)}\frac{|u|^2}{2}\,dx &= \int_{\Omega_\varepsilon(t)} u\cdot\partial_t u\,dx + \int_{\partial\Omega_\varepsilon(t)}\frac{|u|^2}{2}V_\varepsilon^N\,d\mathcal{H}^2 \\ &= \int_{\Omega_\varepsilon(t)} u\cdot\{-(u\cdot\nabla)u - \nabla p + \mu_0\Delta u\}\,dx + \int_{\partial\Omega_\varepsilon(t)}\frac{|u|^2}{2}V_\varepsilon^N\,d\mathcal{H}^2.\end{aligned} \quad (4.5.14)$$

We already computed the integrals of $u\cdot(u\cdot\nabla)u$ and $u\cdot\nabla p$ over $\Omega_\varepsilon(t)$ in the proof of Lemma 4.5.1, see (4.5.3) and (4.5.4). Let us calculate the integral of $u\cdot\Delta u$. Since $\Delta u = 2\mathrm{div}\,D(u)$ by the divergence-free condition (4.4.2),

$$\begin{aligned}\int_{\Omega_\varepsilon(t)} u\cdot\Delta u\,dx &= 2\int_{\Omega_\varepsilon(t)} u\cdot\mathrm{div}\,D(u)\,dx \\ &= 2\int_{\partial\Omega_\varepsilon(t)} u\cdot D(u)^T\nu_\varepsilon\,d\mathcal{H}^2 - 2\int_{\Omega_\varepsilon(t)}\nabla u : D(u)\,dx,\end{aligned}$$

where $F : G := \mathrm{tr}[F^T G]$ for square matrices F and G of order three. In the last line we use the symmetry of the strain rate tensor $D(u)$ and the boundary conditions (4.4.3) and (4.4.4) to get

$$u \cdot D(u)^T \nu_\varepsilon = (u \cdot \nu_\varepsilon)(D(u)\nu_\varepsilon \cdot \nu_\varepsilon) = V_\varepsilon^N (D(u)\nu_\varepsilon \cdot \nu_\varepsilon)$$

on $\partial\Omega_\varepsilon(t)$. Also, we easily observe that

$$\nabla u : D(u) = (\nabla u)^T : D(u) = |D(u)|^2 \left(= \sum_{i,j=1}^{3} [D(u)]_{ij}^2 \right).$$

Here $[D(u)]_{ij}$ is the (i,j)-entry of $D(u)$, i.e. $[D(u)]_{ij} = (\partial_i u_j + \partial_j u_i)/2$. Hence

$$\int_{\Omega_\varepsilon(t)} u \cdot \Delta u \, dx = 2 \int_{\partial\Omega_\varepsilon(t)} (D(u)\nu_\varepsilon \cdot \nu_\varepsilon) V_\varepsilon^N \, d\mathcal{H}^2 - 2 \int_{\Omega_\varepsilon(t)} |D(u)|^2 \, dx. \tag{4.5.15}$$

Finally we substitute (4.5.3), (4.5.4), and (4.5.15) for (4.5.14) to obtain (4.5.13). □

Lemma 4.5.5. *Let v, q, and q^1 satisfy the limit equations* (4.4.6) *and* (4.4.7) *of the Navier–Stokes equations. Suppose that the normal component of v is equal to the outward normal velocity of $\Gamma(t)$, i.e. $v \cdot \nu = V_\Gamma^N$. Then we have*

$$\frac{d}{dt} \int_{\Gamma(t)} \frac{|v|^2}{2} \, d\mathcal{H}^2 = -2\mu_0 \int_{\Gamma(t)} |P_\Gamma D^{tan}(v) P_\Gamma|^2 \, d\mathcal{H}^2 + \int_{\Gamma(t)} (qH - q^1) V_\Gamma^N \, d\mathcal{H}^2. \tag{4.5.16}$$

The first and second terms on the right-hand side of (4.5.16) correspond to the energy dissipation of the surface fluid by viscosity and the rate of work done by the moving surface, respectively.

Proof. As in the proof of Lemma 4.5.2 we use the Leibniz formula [10, Lemma 2.2] with velocity field v and the equations (4.4.6) and (4.4.7):

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma(t)} \frac{|v|^2}{2} \, d\mathcal{H}^2 &= \int_{\Gamma(t)} v \cdot \partial_v^\bullet v \, d\mathcal{H}^2 + \int_{\Gamma(t)} \frac{|v|^2}{2} \mathrm{div}_\Gamma v \, d\mathcal{H}^2 \\ &= \int_{\Gamma(t)} v \cdot \{-\nabla_\Gamma q - q^1 \nu + 2\mu_0 \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v) P_\Gamma)\} \, d\mathcal{H}^2. \end{aligned} \tag{4.5.17}$$

The first two terms in the last line were calculated in the proof of Lemma 4.5.2, see (4.5.7) and (4.5.8). For the viscous term,

$$v \cdot \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v) P_\Gamma) = \mathrm{div}_\Gamma (P_\Gamma D^{tan}(v) P_\Gamma v) - \nabla_\Gamma v : P_\Gamma D^{tan}(v) P_\Gamma.$$

The integral of the first term on the right-hand side over $\Gamma(t)$ vanishes by Stokes' theorem since $\Gamma(t)$ is closed and $P_\Gamma D^{tan}(v) P_\Gamma v$ is a tangential vector field on $\Gamma(t)$. Also, since the matrix $P_\Gamma D^{tan}(v) P_\Gamma$ is symmetric,

$$\nabla_\Gamma v : P_\Gamma D^{tan}(v) P_\Gamma = (\nabla_\Gamma v)^T : P_\Gamma D^{tan}(v) P_\Gamma = D^{tan}(v) : P_\Gamma D^{tan}(v) P_\Gamma.$$

Moreover, by the formulas $P_\Gamma^2 = P_\Gamma^T = P_\Gamma$ and $E : FG = F^T E : G = E G^T : F$ for square matrices E, F, and G of order three we obtain

$$\nabla_\Gamma v : P_\Gamma D^{tan}(v) P_\Gamma = D^{tan}(v) : P_\Gamma D^{tan}(v) P_\Gamma = |P_\Gamma D^{tan}(v) P_\Gamma|^2.$$

Hence the integral of the inner product of v and $\mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma)$ is

$$\int_{\Gamma(t)} v\cdot\mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma)\,d\mathcal{H}^2=-\int_{\Gamma(t)}|P_\Gamma D^{tan}(v)P_\Gamma|^2\,d\mathcal{H}^2. \tag{4.5.18}$$

Applying (4.5.7), (4.5.8), and (4.5.18) to (4.5.17) we obtain (4.5.16). □

As in the case of the Euler equations, the energy identity (4.5.16) on the moving surface can be derived as a thin width limit of that in the moving thin domain (4.5.13). Let us expand u and p in powers of d as (4.4.5) and (4.3.5) and determine the zeroth order term in ε of the energy identity (4.5.13).

Theorem 4.5.6. *Let u and p satisfy the energy identity* (4.5.13)*. Suppose that the velocity field u satisfies the boundary conditions* (4.4.3) *and* (4.4.4)*. Then the zeroth order term v in the expansion* (4.4.5) *and the zeroth and first order terms q and q^1 in the expansion* (4.3.5) *satisfy the energy identity* (4.5.16)*.*

Proof. The remaining part of the proof is to show that

$$\int_{\Omega_\varepsilon(t)}|D(u)(x,t)|^2\,dx=2\varepsilon\int_{\Gamma(t)}|(P_\Gamma D^{tan}(v)P_\Gamma)(y,t)|^2\,d\mathcal{H}^2(y)+O(\varepsilon^2) \tag{4.5.19}$$

and

$$\int_{\partial\Omega_\varepsilon(t)}[(D(u)\nu_\varepsilon\cdot\nu_\varepsilon)V_\varepsilon^N](x,t)\,d\mathcal{H}^2(x)=O(\varepsilon^2) \tag{4.5.20}$$

since we already computed other terms in the proof of Theorem 4.5.3, see (4.5.10) and (4.5.12). By (4.4.18) and (4.4.27) in the proof of Theorem 4.4.1 we have

$$D(u)(x,t)=(P_\Gamma D^{tan}(v)P_\Gamma)(\pi(x,t),t)+d(x,t)S^1(\pi(x,t),t)+R(d(x,t)^2)$$

for $x\in\Omega_\varepsilon(t)$ (Note that to get (4.4.27) we only need the boundary conditions (4.4.3) and (4.4.4) for the Navier–Stokes equations. See the proof of Theorem 4.4.1.) Using this expansion and the change of variable formula (4.2.20) we obtain (4.5.19) as

$$\begin{aligned}\int_{\Omega_\varepsilon(t)}|D(u)(x,t)|^2\,dx&=\int_{\Omega_\varepsilon(t)}\{|(P_\Gamma D^{tan}(v)P_\Gamma)(\pi(x,t),t)|^2+R(d(x,t))\}\,dx\\&=\int_{\Gamma(t)}\int_{-\varepsilon}^{\varepsilon}\{|(P_\Gamma D^{tan}(v)P_\Gamma)(y,t)|^2+R(\rho)\}J(y,t,\rho)\,d\rho\,d\mathcal{H}^2(y)\\&=2\varepsilon\int_{\Gamma(t)}|(P_\Gamma D^{tan}(v)P_\Gamma)(y,t)|^2\,d\mathcal{H}^2(y)+O(\varepsilon^2).\end{aligned}$$

Let us show (4.5.20). By (4.2.5) we have

$$(D(u)\nu_\varepsilon)(x,t)=\pm(P_\Gamma D^{tan}(v)P_\Gamma\nu)(\pi(x,t),t)+\varepsilon(S^1\nu)(\pi(x,t),t)+O(\varepsilon^2).$$

for $x\in\partial\Omega_\varepsilon(t)$ according to $d(x,t)=\pm\varepsilon$ (double-sign corresponds). Moreover, the first two terms on the right-hand side vanishes since $P_\Gamma\nu=0$ and $S^1\nu=0$ on $\Gamma(t)$ by (4.4.25). (Note that, similarly to the proof of (4.4.27), only the boundary conditions (4.4.3) and (4.4.4) are necessary to show (4.4.25). See the proof of Theorem 4.4.1.) Hence $D(u)\nu_\varepsilon=O(\varepsilon^2)$ on $\partial\Omega_\varepsilon(t)$. Applying this estimate and

$$|\nu_\varepsilon(x,t)|=1,\quad |V_\varepsilon^N(x,t)|=|V_\Gamma^N(\pi(x,t),t)|=O(1),\quad x\in\partial\Omega_\varepsilon(t),$$

where the second relation follows from (4.2.6) and the fact that V_Γ^N is independent of ε, to the left-hand side of (4.5.20), and then using the change of variables formula (4.2.21) and $J(y,t,\pm\varepsilon) = O(1)$, we obtain (4.5.20) as

$$\int_{\partial\Omega_\varepsilon(t)} [(D(u)\nu_\varepsilon \cdot \nu_\varepsilon)V_\varepsilon^N](x,t)\,d\mathcal{H}^2(x) = \sum_{\rho=\pm\varepsilon} \int_{\Gamma(t)} O(\varepsilon^2)J(y,t,\rho)\,d\mathcal{H}^2(y) = O(\varepsilon^2).$$

Now we substitute (4.5.10), (4.5.12), (4.5.19), and (4.5.20) for the energy identity (4.5.13) and divide both sides by 2ε to obtain

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma(t)} \frac{|v(y,t)|^2}{2}\,d\mathcal{H}^2(y) = -2\mu_0 \int_{\Gamma(t)} |(P_\Gamma D^{tan}(v)P_\Gamma)(y,t)|^2\,d\mathcal{H}^2(y) \\ + \int_{\Gamma(t)} (qH - q^1)(y,t)V_\Gamma^N(y,t)\,d\mathcal{H}^2(y) + O(\varepsilon).\end{aligned}$$

Since all terms except for $O(\varepsilon)$ are independent of ε, we conclude that the energy identity (4.5.16) must be satisfied. □

Remark 4.5.7. The assumption in Theorem 4.5.6 that the boundary conditions (4.4.3) and (4.4.4) are satisfied is necessary to deal with integrals including the strain rate tensor $D(u)$. Note that, contrary to the case of the Navier–Stokes equations (Theorem 4.5.6), we do not need even the impermeable boundary condition (4.3.3) to derive the thin width limit of the energy identity of the Euler equations in the moving thin domain, see Theorem 4.5.3.

4.A Elementary calculations of various quantities on surfaces

In this appendix we prove elementary facts on various quantities and differential operators on a surface given in Section 4.2. Until the end of the proof of Lemma 4.2.6 we fix and suppress $t \in [0,T]$.

Proof of Lemma 4.2.1. Since $|\nu|^2 = 1$ on Γ, we have

$$0 = \nabla_\Gamma|\nu|^2 = 2(\nabla_\Gamma\nu)\nu = -2A\nu \quad \text{on} \quad \Gamma,$$

which implies (4.2.7). The formula (4.2.8) is an immediate consequence of (4.2.7). Now let us prove (4.2.9). Let $\tilde{\nu}$ be an extension of ν to N. By (4.2.2) and $\tilde{\nu}|_\Gamma = \nu$ we have

$$\nabla^2 d(x) = \nabla\pi(x)(\nabla\tilde{\nu})(\pi(x)), \quad x \in N. \tag{4.A.1}$$

Moreover, we differentiate both sides of (4.2.1) and apply (4.2.2) to get

$$\nabla\pi(x) = P_\Gamma(\pi(x)) - d(x)\nabla\pi(x)(\nabla\tilde{\nu})(\pi(x)), \quad x \in N.$$

In particular, if $x = y \in \Gamma$ then $d(x) = 0$, $\pi(x) = y$ and thus $\nabla\pi(y) = P_\Gamma(y)$. Applying this formula to (4.A.1) with $x = y \in \Gamma$ we obtain (4.2.9). □

Proof of Lemma 4.2.2. Let f be a function on Γ and $\tilde{f}$ its extension to N satisfying $\tilde{f}|_\Gamma = f$. For $j = 1,2,3$, by (4.2.2) and the definition of the tangential derivative operators we have

$$\partial_j^{tan} f(y) = \sum_{l=1}^{3}\{\delta_{jl} - \partial_j d(y)\partial_l d(y)\}\partial_l\tilde{f}(y), \quad y \in \Gamma.$$

From now on we suppress the argument y. By the above formula we have

$$\partial_i^{tan}\partial_j^{tan} f = \sum_{k,l=1}^{3}\{\delta_{ik}-(\partial_i d)(\partial_k d)\}\partial_k\Big[\{\delta_{jl}-(\partial_j d)(\partial_l d)\}\partial_l\tilde{f}\Big] = \alpha_1+\alpha_2+\alpha_3$$

for $i,j=1,2,3$, where

$$\begin{aligned}
\alpha_1 &:= \sum_{k,l=1}^{3}\{\delta_{ik}-(\partial_i d)(\partial_k d)\}\{\delta_{jl}-(\partial_j d)(\partial_l d)\}\partial_k\partial_l\tilde{f},\\
\alpha_2 &:= -\sum_{k,l=1}^{3}\{\delta_{ik}-(\partial_i d)(\partial_k d)\}(\partial_k\partial_j d)(\partial_l d)\partial_l\tilde{f},\\
\alpha_3 &:= -\sum_{k,l=1}^{3}\{\delta_{ik}-(\partial_i d)(\partial_k d)\}(\partial_j d)(\partial_k\partial_l d)\partial_l\tilde{f}.
\end{aligned}$$

Similarly, we have $\partial_j^{tan}\partial_i^{tan} f = \beta_1+\beta_2+\beta_3$, where

$$\begin{aligned}
\beta_1 &:= \sum_{k,l=1}^{3}\{\delta_{jl}-(\partial_j d)(\partial_l d)\}\{\delta_{ik}-(\partial_i d)(\partial_k d)\}\partial_l\partial_k\tilde{f},\\
\beta_2 &:= -\sum_{k,l=1}^{3}\{\delta_{jl}-(\partial_j d)(\partial_l d)\}(\partial_l\partial_i d)(\partial_k d)\partial_k\tilde{f},\\
\beta_3 &:= -\sum_{k,l=1}^{3}\{\delta_{jl}-(\partial_j d)(\partial_l d)\}(\partial_i d)(\partial_l\partial_k d)\partial_k\tilde{f}.
\end{aligned}$$

From $\partial_k\partial_l\tilde{f}=\partial_l\partial_k\tilde{f}$ it immediately follows that $\alpha_1=\beta_1$. Since $\partial_k\partial_j d=\partial_j\partial_k d$,

$$\begin{aligned}
\alpha_2 &= -(\nabla d\cdot\nabla\tilde{f})\left\{\partial_i\partial_j d-(\partial_i d)\sum_{k=1}^{3}(\partial_k d)(\partial_j\partial_k d)\right\}\\
&= -(\nabla d\cdot\nabla\tilde{f})\left\{\partial_i\partial_j d-(\partial_i d)\partial_j\left(\frac{|\nabla d|^2}{2}\right)\right\}\\
&= -(\nabla d\cdot\nabla\tilde{f})\partial_i\partial_j d.
\end{aligned}$$

Here the last equality follows from $|\nabla d|^2=1$ on N. By the same calculation we have $\beta_2=-(\nabla d\cdot\nabla\tilde{f})\partial_j\partial_i d$. Hence $\alpha_2=\beta_2$ by $\partial_i\partial_j d=\partial_j\partial_i d$. For α_3 and β_3,

$$\begin{aligned}
\alpha_3 &= -\Big[P_\Gamma(\nabla^2 d)\nabla\tilde{f}\Big]_i\partial_j d = [A\nabla_\Gamma f]_i\nu_j,\\
\beta_3 &= -\Big[P_\Gamma(\nabla^2 d)\nabla\tilde{f}\Big]_j\partial_i d = [A\nabla_\Gamma f]_j\nu_i
\end{aligned}$$

by (4.2.2), (4.2.8), (4.2.9), and the definition of the tangential gradient operator. (Note that we calculate values of functions at $y\in\Gamma$.) Therefore, we obtain

$$\partial_i^{tan}\partial_j^{tan} f-\partial_j^{tan}\partial_i^{tan} f=(\alpha_1+\alpha_2+\alpha_3)-(\beta_1+\beta_2+\beta_3)=[A\nabla_\Gamma f]_i\nu_j-[A\nabla_\Gamma f]_j\nu_i,$$

that is, the formula (4.2.11) holds. □

Proof of Lemma 4.2.3. Let v be a general vector field on Γ which may have a nonzero normal component. Since $P_\Gamma \nabla_\Gamma v = \nabla_\Gamma v$ and $(\nabla_\Gamma v)^T P_\Gamma = (\nabla_\Gamma v)^T$ we have

$$2\mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)P_\Gamma) = \mathrm{div}_\Gamma\big((\nabla_\Gamma v)P_\Gamma\big) + \mathrm{div}_\Gamma\big(P_\Gamma(\nabla_\Gamma v)^T\big). \tag{4.A.2}$$

Let us calculate each term on the right-hand side. For $i,j = 1,2,3$ the (i,j)-entry of $(\nabla_\Gamma v)P_\Gamma$ is of the form

$$\big[(\nabla_\Gamma v)P_\Gamma\big]_{ij} = \sum_{k=1}^{3}(\partial_i^{tan} v_k)(\delta_{jk} - \nu_j\nu_k).$$

Thus the j-th component of $\mathrm{div}_\Gamma\big((\nabla_\Gamma v)P_\Gamma\big)$ is

$$\Big[\mathrm{div}_\Gamma\big((\nabla_\Gamma v)P_\Gamma\big)\Big]_j = \sum_{i=1}^{3}\partial_i^{tan}\big[(\nabla_\Gamma v)P_\Gamma\big]_{ij} = \alpha_1 + \alpha_2 + \alpha_3,$$

where

$$\begin{aligned}
\alpha_1 &:= \sum_{i,k=1}^{3}\{(\partial_i^{tan})^2 v_k\}(\delta_{jk} - \nu_j\nu_k),\\
\alpha_2 &:= -\sum_{i,k=1}^{3}(\partial_i^{tan} v_k)(\partial_i^{tan}\nu_j)\nu_k = \sum_{i,k=1}^{3}(\partial_i^{tan} v_k)A_{ij}\nu_k,\\
\alpha_3 &:= -\sum_{i,k=1}^{3}(\partial_i^{tan} v_k)\nu_j(\partial_i^{tan}\nu_k) = \sum_{i,k=1}^{3}(\partial_i^{tan} v_k)\nu_j A_{ik}.
\end{aligned}$$

Here A_{ij} is the (i,j)-entry of the Weingarten map $A = -\nabla_\Gamma \nu$. By the definitions of Δ_Γ and P_Γ we have $\alpha_1 = \big[P_\Gamma(\Delta_\Gamma v)\big]_j$, where Δ_Γ applies each component of the vector field v. Also, since A is symmetric,

$$\alpha_2 = \sum_{i,k=1}^{3} A_{ji}(\partial_i^{tan} v_k)\nu_k = \big[A(\nabla_\Gamma v)\nu\big]_j.$$

Similarly, we have $\alpha_3 = \mathrm{tr}[A\nabla_\Gamma v]\nu_j$. Therefore, the equality

$$\Big[\mathrm{div}_\Gamma\big((\nabla_\Gamma v)P_\Gamma\big)\Big]_j = \big[P_\Gamma(\Delta_\Gamma v)\big]_j + \big[A(\nabla_\Gamma v)\nu\big]_j + \mathrm{tr}[A\nabla_\Gamma v]\nu_j$$

holds for each $j = 1,2,3$, which means that

$$\mathrm{div}_\Gamma\big((\nabla_\Gamma v)P_\Gamma\big) = P_\Gamma(\Delta_\Gamma v) + A(\nabla_\Gamma v)\nu + \mathrm{tr}[A\nabla_\Gamma v]\nu. \tag{4.A.3}$$

Calculations of the second term $\mathrm{div}_\Gamma\big(P_\Gamma(\nabla_\Gamma v)^T\big)$ are more complicated. Since

$$\Big[P_\Gamma(\nabla_\Gamma v)^T\Big]_{ij} = \sum_{k=1}^{3}(\delta_{ik} - \nu_i\nu_k)\partial_j^{tan} v_k,$$

we have $\Bigl[\mathrm{div}_\Gamma\bigl(P_\Gamma(\nabla_\Gamma v)^T\bigr)\Bigr]_j = \beta_1+\beta_2+\beta_3$, where

$$\begin{aligned}
\beta_1 &:= -\sum_{i,k=1}^3 (\partial_i^{tan}\nu_i)\nu_k\partial_j^{tan}v_k = \sum_{i,k=1}^3 A_{ii}\nu_k\partial_j^{tan}v_k,\\
\beta_2 &:= -\sum_{i,k=1}^3 \nu_i(\partial_i^{tan}\nu_k)\partial_j^{tan}v_k = \sum_{i,k=1}^3 \nu_iA_{ik}\partial_j^{tan}v_k,\\
\beta_3 &:= \sum_{i,k=1}^3 (\delta_{ik}-\nu_i\nu_k)\partial_i^{tan}\partial_j^{tan}v_k.
\end{aligned}$$

By the definition of the mean curvature,

$$\beta_1 = \mathrm{tr}[A]\sum_{k=1}^3(\partial_j^{tan}v_k)\nu_k = H\bigl[(\nabla_\Gamma v)\nu\bigr]_j.$$

Since $A_{ik}=A_{ki}$ and $A\nu=0$,

$$\beta_2 = \sum_{i,k=1}^3(\partial_j^{tan}v_k)A_{ki}\nu_i = -\bigl[(\nabla_\Gamma v)A\nu\bigr]_j = 0.$$

For β_3 we have

$$\beta_3 = \sum_{i=1}^3\partial_i^{tan}\partial_j^{tan}v_i - \sum_{k=1}^3\nu_k\{\nu\cdot\nabla_\Gamma(\partial_j^{tan}v_k)\}.$$

The second term on the right-hand side vanishes since $\nu\cdot\nabla_\Gamma(\partial_j^{tan}v_k)=0$ for each j and k. We apply (4.2.11) to the first term to get

$$\begin{aligned}
\beta_3 &= \sum_{i=1}^3\partial_j^{tan}\partial_i^{tan}v_i + \sum_{i=1}^3[A\nabla_\Gamma v_i]_i\nu_j - \sum_{i=1}^3[A\nabla_\Gamma v_i]_j\nu_i\\
&= \partial_j^{tan}(\mathrm{div}_\Gamma v) + \mathrm{tr}[A\nabla_\Gamma v]\nu_j - \bigl[A(\nabla_\Gamma v)\nu\bigr]_j.
\end{aligned}$$

Therefore, it follows that

$$\Bigl[\mathrm{div}_\Gamma\bigl(P_\Gamma(\nabla_\Gamma v)^T\bigr)\Bigr]_j = \partial_j^{tan}(\mathrm{div}_\Gamma v) + \bigl[(HI_3-A)(\nabla_\Gamma v)\nu\bigr]_j + \mathrm{tr}[A\nabla_\Gamma v]\nu_j$$

for each $j=1,2,3$ and thus

$$\mathrm{div}_\Gamma\bigl(P_\Gamma(\nabla_\Gamma v)^T\bigr) = \nabla_\Gamma(\mathrm{div}_\Gamma v) + (HI_3-A)(\nabla_\Gamma v)\nu + \mathrm{tr}[A\nabla_\Gamma v]\nu. \tag{4.A.4}$$

Substituting (4.A.3) and (4.A.4) for (4.A.2) we obtain the formula (4.2.13). □

Proof of Lemma 4.2.6. For $\rho\in[-\varepsilon,\varepsilon]$ let $\Gamma_\rho := \{x\in\mathbb{R}^3 \mid d(x)=\rho\}$ be a level-set surface of Γ. Suppose that the change of variables formula

$$\int_{\Gamma_\rho} f(z)\,d\mathcal{H}^2(z) = \int_\Gamma f(y+\rho\nu(y))J(y,\rho)\,d\mathcal{H}^2(y) \tag{4.A.5}$$

holds for each $\rho \in [-\varepsilon, \varepsilon]$. Then (4.2.20) and (4.2.21) follow from this formula and

$$\int_{\Omega_\varepsilon} f(x)\, dx = \int_{-\varepsilon}^{\varepsilon} \left(\int_{\Gamma_\rho} f(z)\, d\mathcal{H}^2(z) \right) d\rho,$$

which is the well-known co-area formula (see e.g. [11, Theorem 2.9]), and

$$\int_{\partial\Omega_\varepsilon} f(x)\, d\mathcal{H}^2(x) = \int_{\Gamma_\varepsilon} f(z)\, d\mathcal{H}^2(z) + \int_{\Gamma_{-\varepsilon}} f(z)\, d\mathcal{H}^2(z).$$

Let us prove (4.A.5). Since Γ is compact, we may take finitely many open subsets U_k of $\mathbb{R}^2$ and local parametrizations $\mu_k\colon U_k \to \Gamma$ ($k = 1, \dots, N$) such that $\{\mu_k(U_k)\}_{k=1}^N$ is an open covering of Γ. Let $\{\varphi_k\}_{k=1}^N$ be a partition of unity of Γ subordinate to the covering $\{\mu_k(U_k)\}_{k=1}^N$ and for each $\rho \in [-\varepsilon, \varepsilon]$ and $k = 1, \dots, N$ set

$$\mu_k^\rho(s) := \mu_k(s) + \rho\nu(\mu_k(s)), \quad \varphi_k^\rho(\mu_k^\rho(s)) := \varphi_k(\mu_k(s)), \quad s \in U_k.$$

Then $\mu_k^\rho\colon U_k \to \Gamma_\rho$ is a local parametrization of Γ_ρ whose domain is the same as that of μ_k and $\{\mu_k^\rho(U_k)\}_{k=1}^N$ is an open covering of Γ_ρ. Moreover, $\{\varphi_k^\rho\}_{k=1}$ is a partition of unity of Γ_ρ subordinate to the covering $\{\mu_k^\rho(U_k)\}_{k=1}^N$. By these partitions of unity and the definition of integrals over a surface, the proof of (4.A.5) reduces to showing that, for any local parametrization $\mu\colon U \to \Gamma$ with an open subset U of $\mathbb{R}^2$ and $\mu^\rho\colon U \to \Gamma_\rho$ given by $\mu^\rho(s) := \mu(s) + \rho\nu(\mu(s))$, $s \in U$, the formula

$$\sqrt{\det \theta^\rho(s)} = J(\mu(s), \rho)\sqrt{\det \theta(s)}, \quad s \in U \tag{4.A.6}$$

holds. Here θ is a square matrix of order two given by $\theta := \nabla'\mu(\nabla'\mu)^T$, where

$$\nabla'\mu := \begin{pmatrix} \partial_1'\mu_1 & \partial_1'\mu_2 & \partial_1'\mu_3 \\ \partial_2'\mu_1 & \partial_2'\mu_2 & \partial_2'\mu_3 \end{pmatrix} \quad \left(\partial_i' := \frac{\partial}{\partial s_i}\right),$$

and $\theta^\rho := \nabla'\mu^\rho(\nabla'\mu^\rho)^T$. We define square matrices M and M^ρ of order three as

$$M(s) := \begin{pmatrix} \nabla'\mu(s) \\ [\nu(\mu(s))]^T \end{pmatrix}, \quad M^\rho(s) := \begin{pmatrix} \nabla'\mu^\rho(s) \\ [\nu(\mu(s))]^T \end{pmatrix}.$$

Here we see $\nu(\mu(s))$ as a three-dimensional column vector. In the following argument, we sometimes suppress the argument s and abbreviate $\nu(\mu(s))$ to ν. For $i = 1, 2$ the i-th component of $\nabla'\mu(s)\nu(\mu(s)) \in \mathbb{R}^2$ is $\partial_i\mu(s) \cdot \nu(\mu(s)) = 0$ since $\partial_i\mu(s)$ is tangent to Γ at $\mu(s)$. Therefore, $(\nabla'\mu)\nu = 0$ and

$$MM^T = \begin{pmatrix} \nabla'\mu(\nabla'\mu)^T & (\nabla'\mu)\nu \\ [(\nabla'\mu)\nu]^T & |\nu|^2 \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix},$$

which implies $\det \theta = \det(MM^T) = (\det M)^2$. On the other hand, since

$$\mu^\rho(s) = \mu(s) + \rho\nu(\mu(s)) = \mu(s) + \rho\nabla d(\mu(s))$$

by (4.2.2) and thus

$$\nabla'\mu^\rho(s) = \nabla'\mu(s)\{I_3 + \rho\nabla^2 d(\mu(s))\} = \nabla'\mu(s)\{I_3 - \rho A(\mu(s))\}$$

by (4.2.9), we have $\nabla'\mu^\rho(s)\nu(\mu(s)) = 0$ by $\nabla'\mu(s)\nu(\mu(s)) = 0$ and (4.2.7). Hence as in the case of θ and M we have $\det\theta^\rho = (\det M^\rho)^2$. Moreover, by (4.2.7) and the symmetry of the matrix $I_3 - \rho A$,

$$M^\rho = \begin{pmatrix} (\nabla'\mu)(I_3 - \rho A) \\ \nu^T \end{pmatrix} = \begin{pmatrix} \nabla'\mu \\ \nu^T \end{pmatrix}(I_3 - \rho A) = M(I_3 - \rho A).$$

Hence we get

$$\det\theta^\rho = (\det M^\rho)^2 = \{\det M \cdot \det(I_3 - \rho A)\}^2 = \{\det(I_3 - \rho A)\}^2 \det\theta.$$

Finally we observe that the Weingarten map A has the eigenvalues 0, κ_1, and κ_2 and thus

$$\begin{aligned}\det\{I_3 - \rho A(\mu(s))\} &= 1 \cdot \{1 - \rho\kappa_1(\mu(s))\} \cdot \{1 - \rho\kappa_2(\mu(s))\} \\ &= J(\mu(s), \rho) \quad (> 0 \quad \text{for sufficiently small } \rho)\end{aligned}$$

to obtain the formula (4.A.6). □

Now let us return to the moving surface $\Gamma(t)$ and prove Lemmas 4.2.7 and 4.2.8.

Proof of Lemma 4.2.7. As in the proof of Theorem 4.3.1 we use the abbreviations (4.3.13). Let f be a function on S_T and $\tilde{f}$ an arbitrary extension of f to N_T satisfying $\tilde{f}|_{S_T} = f$. For $(x,t) \in Q_{\varepsilon,T}$ we have $f(\pi,t) = \tilde{f}(\pi,t)$ by $\pi = \pi(x,t) \in \Gamma(t)$ and thus

$$\begin{aligned}\nabla\big(f(\pi,t)\big) &= \nabla\pi(x,t)\nabla\tilde{f}(\pi,t), \\ \partial_t\big(f(\pi,t)\big) &= \partial_t\tilde{f}(\pi,t) + (\partial_t\pi(x,t)\cdot\nabla)\tilde{f}(\pi,t).\end{aligned}$$

Hence it is sufficient for (4.2.25) and (4.2.26) to show that

$$\nabla\pi(x,t) = P_\Gamma(\pi,t) + d(x,t)A(\pi,t) + R(d^2), \tag{4.A.7}$$

$$\partial_t\pi(x,t) = V_\Gamma^N(\pi,t)\nu(\pi,t) + d(x,t)\nabla_\Gamma V_\Gamma^N(\pi,t) + R(d), \tag{4.A.8}$$

since

$$\begin{gathered} A\nabla\tilde{f} = AP_\Gamma\nabla\tilde{f} = A\nabla_\Gamma f, \\ \partial_t\tilde{f} + (V_\Gamma^N\nu\cdot\nabla)\tilde{f} = \partial^\circ f, \quad (\nabla_\Gamma V_\Gamma^N\cdot\nabla)\tilde{f} = (\nabla_\Gamma V_\Gamma^N\cdot\nabla_\Gamma)f \end{gathered}$$

on $\Gamma(t)$ by the definition of the tangential gradient, (4.2.8), and (4.2.22) with $v = V_\Gamma^N\nu$. By $\pi(x,t) = x - d(x,t)\nabla d(x,t)$ and (4.2.2) we have

$$\begin{aligned}\nabla\pi(x,t) &= I_3 - \nabla d(x,t)\otimes\nabla d(x,t) - d(x,t)\nabla^2 d(x,t) \\ &= P_\Gamma(\pi,t) - d(x,t)\nabla^2 d(x,t).\end{aligned}$$

Also, we expand $\nabla^2 d$ in powers of d and apply (4.2.9) to obtain

$$\nabla^2 d(x,t) = \nabla^2 d(\pi,t) + R(d) = -A(\pi,t) + R(d).$$

Hence (4.A.7) follows. Similarly, we differentiate $\pi(x,t) = x - d(x,t)\nabla d(x,t)$ with respect to t and apply (4.2.2) and (4.2.4) to get

$$\partial_t\pi(x,t) = V_\Gamma^N(\pi,t)\nu(\pi,t) - d(x,t)\partial_t\nabla d(x,t).$$

Moreover, by $\partial_t\nabla d = \nabla\partial_t d$, (4.2.4), and (4.A.7),

$$\partial_t\nabla d(x,t) = -\nabla\big(V_\Gamma^N(\pi,t)\big) = -\nabla\pi(x,t)\nabla\widetilde{V}_\Gamma^N(\pi,t) = -\nabla_\Gamma V_\Gamma^N(\pi,t) + R(d),$$

where $\widetilde{V}_\Gamma^N$ is an extension of V_Γ^N to N_T with $\widetilde{V}_\Gamma^N|_{S_T} = V_\Gamma^N$. Applying this to the above equality for $\partial_t\pi$ we obtain (4.A.8). □

Proof of Lemma 4.2.8. We use the abbreviations (4.3.13). For $i, j = 1, 2, 3$, let M_{ij} be the (i, j)-entry of a square matrix M of order three. We differentiate both sides of $D_{ij}(x) = S_{ij}(\pi) + d(x,t)S^1_{ij}(\pi) + R(d^2)$ with respect to x_i and apply (4.A.7) to get

$$\partial_i D_{ij}(x) = \partial_i^{tan} S_{ij}(\pi) + S^1_{ij}(\pi)\partial_i d(x,t) + R(d).$$

Therefore, the j-th component of $\operatorname{div} D(x)$ is

$$\begin{aligned}[\operatorname{div} D(x)]_j &= \sum_{i=1}^{3} \partial_i D_{ij}(x) = \sum_{i=1}^{3}\{\partial_i^{tan} S_{ij}(\pi) + S^1_{ij}(\pi)\partial_i d(x,t)\} + R(d(x,t)) \\ &= [\operatorname{div}_\Gamma S(\pi)]_j + \Big[\big(S^1(\pi)\big)^T \nabla d(x,t)\Big]_j + R(d)\end{aligned}$$

and (4.2.27) follows by (4.2.2). $\square$

4.B Comparison of vector Laplacians

The purpose of this appendix is to give a proof of the formula (4.2.15) in Lemma 4.2.4. Main tools for the proof are the Gauss formula (4.2.14) and

$$\Delta_B X = \operatorname{tr}\overline{\nabla}^2 X = \sum_{i=1}^{2}\Big(\overline{\nabla}_i\overline{\nabla}_i X - \overline{\nabla}_{\overline{\nabla}_i e_i} X\Big) \quad \text{on} \quad \Gamma \tag{4.B.1}$$

for any tangential vector field X on Γ, where $\{e_1, e_2\}$ denotes a local orthonormal frame of $T\Gamma$ (i.e. an orthonormal basis of the tangent plane of Γ defined on a relative open subset of Γ) and $\overline{\nabla}_i := \overline{\nabla}_{e_i}$ (for a proof of (4.B.1) see [26, Proposition 34] and [32, Proposition 2.1 in Appendix C]). Hereafter all calculations are carried out on the surface Γ.

We fix coordinates of $\mathbb{R}^3$ and write x_j ($j = 1, 2, 3$) for the j-th component of a point $x \in \mathbb{R}^3$ under this fixed coordinates. Let $X = (X_1, X_2, X_3)$ be a tangential vector field on Γ and $\{e_1, e_2\}$ be a local orthonormal frame of $T\Gamma$. For $i = 1, 2$, by the Gauss formula (4.2.16) and the fact that $\overline{\nabla}_i X$ is tangential we have

$$\overline{\nabla}_i X = (e_i \cdot \nabla_\Gamma)X - (AX \cdot e_i)\nu = P_\Gamma\{(e_i \cdot \nabla_\Gamma)X\}.$$

Here the second equality follows from $P_\Gamma \nu = 0$. Hence

$$\begin{aligned}\overline{\nabla}_i\overline{\nabla}_i X &= P_\Gamma\big[(e_i \cdot \nabla_\Gamma)\{(e_i \cdot \nabla_\Gamma)X - (AX \cdot e_i)\nu\}\big] \\ &= P_\Gamma\big[(e_i \cdot \nabla_\Gamma)\{(e_i \cdot \nabla_\Gamma)X\}\big] - (AX \cdot e_i)P_\Gamma\{(e_i \cdot \nabla_\Gamma)\nu\},\end{aligned}$$

where we used $P_\Gamma \nu = 0$ again in the second equality. By setting $e_i = (e_i^1, e_i^2, e_i^3)$ the j-th component of the vector $(e_i \cdot \nabla_\Gamma)\{(e_i \cdot \nabla_\Gamma)X\}$ ($j = 1, 2, 3$) is of the form

$$\begin{aligned}\sum_{k,l=1}^{3} e_i^k \partial_k^{tan}(e_i^l \partial_l^{tan} X_j) &= \sum_{k,l=1}^{3}\{e_i^k e_i^l \partial_k^{tan}\partial_l^{tan} X_j + e_i^k(\partial_k^{tan} e_i^l)\partial_l^{tan} X_j\} \\ &= \operatorname{tr}\big[(e_i \otimes e_i)\nabla_\Gamma^2 X_j\big] + \{(e_i \cdot \nabla_\Gamma)e_i\} \cdot \nabla_\Gamma X_j.\end{aligned}$$

Also, by the symmetry of the Weingarten map $A = -\nabla_\Gamma \nu$,

$$[(e_i \cdot \nabla_\Gamma)\nu]_j = \sum_{k=1}^{3} e_i^k \partial_k^{tan}\nu_j = -\sum_{k=1}^{3} e_i^k A_{kj} = -[Ae_i]_j.$$

By these equalities and (4.2.8) the j-th component of $\overline{\nabla}_i \overline{\nabla}_i X$ is

$$\left[\overline{\nabla}_i \overline{\nabla}_i X\right]_j = \sum_{k=1}^{3} [P_\Gamma]_{jk} \Big(\mathrm{tr}\big[(e_i \otimes e_i) \nabla_\Gamma^2 X_k\big] + \{(e_i \cdot \nabla_\Gamma) e_i\} \cdot \nabla_\Gamma X_k \Big) + (AX \cdot e_i)[Ae_i]_j. \quad (4.B.2)$$

On the other hand, $\overline{\nabla}_{\overline{\nabla}_i e_i} X$ is of the form

$$\overline{\nabla}_{\overline{\nabla}_i e_i} X = P_\Gamma \Big\{ \Big(\overline{\nabla}_i e_i \cdot \nabla_\Gamma \Big) X \Big\} = P_\Gamma \Big(\big[\{ P_\Gamma (e_i \cdot \nabla_\Gamma) e_i \} \cdot \nabla_\Gamma \big] X \Big)$$

and, since $\{(P_\Gamma F) \cdot \nabla_\Gamma\} G = (F \cdot \nabla_\Gamma) G$ holds for (not necessarily tangential) vector fields F and G on Γ we have

$$\left[\overline{\nabla}_{\overline{\nabla}_i e_i} X\right]_j = \sum_{k=1}^{3} [P_\Gamma]_{jk} \Big(\{(e_i \cdot \nabla_\Gamma) e_i\} \cdot \nabla_\Gamma X_k \Big). \quad (4.B.3)$$

Applying (4.B.2) and (4.B.3) to (4.B.1) we get

$$[\Delta_B X]_j = \sum_{i=1}^{2} \left(\sum_{k=1}^{3} [P_\Gamma]_{jk} \mathrm{tr}\big[(e_i \otimes e_i) \nabla_\Gamma^2 X_k\big] + (AX \cdot e_i)[Ae_i]_j \right).$$

Furthermore, since e_1 and e_2 form an orthonormal basis of the tangent plane of Γ it follows that $\sum_{i=1}^{2} (e_i \otimes e_i) = P_\Gamma$ and thus

$$\sum_{i=1}^{2} \mathrm{tr}\big[(e_i \otimes e_i) \nabla_\Gamma^2 X_k\big] = \mathrm{tr}[P_\Gamma \nabla_\Gamma^2 X_k] = \mathrm{tr}[\nabla_\Gamma^2 X_k] = \Delta_\Gamma X_k$$

for each $k = 1, 2, 3$ by $P_\Gamma \nabla_\Gamma = \nabla_\Gamma$, and

$$\sum_{i=1}^{2} (AX \cdot e_i) A e_i = \sum_{i=1}^{2} A(e_i \otimes e_i) AX = AP_\Gamma AX = A^2 X,$$

by $(AX \cdot e_i) A e_i = (Ae_i \otimes e_i) AX = A(e_i \otimes e_i) AX$ and (4.2.8). Therefore,

$$[\Delta_B X]_j = \sum_{k=1}^{3} [P_\Gamma]_{jk} \Delta_\Gamma X_k + [A^2 X]_j = [P_\Gamma \Delta_\Gamma X]_j + [A^2 X]_j$$

for each $j = 1, 2, 3$, which yields the formula (4.2.15).

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Chapter 5

Navier–Stokes equations in a curved thin domain

5.1 Introduction

We consider the three-dimensional Navier–Stokes equations

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon - \nu\Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon, \quad \operatorname{div} u^\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon \times (0,\infty) \tag{5.1.1}$$

imposed with Navier's slip boundary conditions

$$u^\varepsilon \cdot n_\varepsilon = 0, \quad [\sigma(u^\varepsilon, p^\varepsilon)n_\varepsilon]_{\mathrm{tan}} + \gamma_\varepsilon u^\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon \times (0,\infty) \tag{5.1.2}$$

and initial condition

$$u^\varepsilon|_{t=0} = u_0^\varepsilon, \quad \text{in} \quad \Omega_\varepsilon. \tag{5.1.3}$$

Here Ω_ε is a curved thin domain in $\mathbb{R}^3$ of the form

$$\Omega_\varepsilon := \{y + rn(y) \mid y \in \Gamma,\, r \in (\varepsilon g_0(y), \varepsilon g_1(y))\}, \quad \varepsilon \in (0,1), \tag{5.1.4}$$

where Γ denotes a two-dimensional closed (i.e. compact and without boundary), connected, and oriented surface in $\mathbb{R}^3$ with unit outward normal vector field n, and g_0 and g_1 are functions on Γ such that $g := g_1 - g_0$ is bounded from below by a positive constant. We denote by Γ_ε and n_ε the boundary of Ω_ε and its unit outward normal vector field. The boundary Γ_ε is the union of the inner and outer boundaries Γ_ε^0 and Γ_ε^1 given by

$$\Gamma_\varepsilon^i := \{y + \varepsilon g_i(y)n(y) \mid y \in \Gamma\}, \quad i = 0, 1.$$

Also, $\nu > 0$ is the viscosity coefficient independent of ε and $\gamma_\varepsilon \geq 0$ is the friction coefficient which takes different values on the inner and outer boundaries, i.e.

$$\gamma_\varepsilon := \gamma_\varepsilon^i \quad \text{on} \quad \Gamma_\varepsilon^i,\, i = 0, 1,$$

where γ_ε^0 and γ_ε^1 are nonnegative constants. We further write

$$\sigma(u^\varepsilon, p^\varepsilon) := 2\nu D(u^\varepsilon) - p^\varepsilon I_3, \quad [\sigma(u^\varepsilon, p^\varepsilon)n_\varepsilon]_{\mathrm{tan}} := P_\varepsilon[\sigma(u^\varepsilon, p^\varepsilon)n_\varepsilon]$$

for the stress tensor and the tangential component of the stress vector on Γ_ε, where

$$D(u^\varepsilon) := \frac{\nabla u^\varepsilon + (\nabla u^\varepsilon)^T}{2}, \quad P_\varepsilon := I_3 - n_\varepsilon \otimes n_\varepsilon$$

are the strain rate tensor and the orthogonal projection onto the tangent plane of Γ_ε. Note that $[\sigma(u^\varepsilon,p^\varepsilon)n_\varepsilon]_{\mathrm{tan}} = 2\nu P_\varepsilon D(u^\varepsilon)n_\varepsilon$ does not depend on p^ε and the slip boundary conditions (5.1.2) can be expressed as

$$u^\varepsilon \cdot n_\varepsilon = 0, \quad 2\nu P_\varepsilon D(u^\varepsilon)n_\varepsilon + \gamma_\varepsilon u^\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon \times (0,\infty).$$

In what follows, we mainly refer to these conditions as the slip boundary conditions.

Partial differential equations in thin domains appear in many applications in solid mechanics (thin elastic bodies), fluid mechanics (lubrication, meteorology, ocean dynamics), etc. They have been studied for a long time since the pioneering works [16, 17] by Hale and Raugel on damped wave and reaction-diffusion equations. In the study of the three-dimensional Navier–Stokes equations in thin domains we expect to show the global-in-time existence of a strong solution for large data according to the smallness of the width of thin domains, since a three-dimensional thin domain with very small width can be considered almost two-dimensional. Raugel and Sell [53] first established the global existence of a strong solution to the Navier–Stokes equations with purely periodic or mixed Dirichlet-periodic boundary conditions in the thin product domain $\Omega_\varepsilon = Q_2 \times (0,\varepsilon)$ with a rectangle Q_2 and sufficiently small $\varepsilon > 0$. Later, Temam and Ziane [65] generalized the results in [53] in the case of the thin product domain $\Omega_\varepsilon = \omega \times (0,\varepsilon)$ with a bounded domain ω in $\mathbb{R}^2$ and other boundary conditions which are combinations of the Dirichlet, free, and periodic boundary conditions. We refer to [23–25, 44, 45] and the references cited therein for further generalization and improvement on the results in [53, 65].

The above cited papers deal with thin product domains whose boundaries and limit sets are both flat, but there are various kinds of nonflat thin domains in physical problems (see [54] for examples of nonflat thin domains). A nonflat thin domain was first considered by Temem and Ziane [66], who studied the Navier–Stokes equations with free boundary conditions in a thin spherical shell

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid a < |x| < a + a\varepsilon\}, \quad a > 0$$

to give a mathematical justification of derivation of the primitive equations for the atmosphere and ocean dynamics (see [38–40]). Another generalization of the shape of a thin domain was made by Iftimie, Raugel, and Sell [26], who studied the Navier–Stokes equations in a flat thin domain with a nonflat top boundary

$$\Omega_\varepsilon = \{x = (x', x_3) \in \mathbb{R}^3 \mid x' \in (0,1)^2,\, 0 < x_3 < \varepsilon g(x')\}, \quad g\colon (0,1)^2 \to \mathbb{R}$$

with periodic boundary conditions on the lateral boundaries and Navier's slip boundary conditions on the top and bottom boundaries. Their result was extended by Hoang [20] and Hoang and Sell [21] to a flat thin domain both of whose top and bottom boundaries are not flat (see also [22] for the study of two-phase flows in a flat thin domain with nonflat top and bottom boundaries).

The slip boundary conditions were proposed by Navier [47], which state that the fluid slips on the boundary with velocity proportional to the tangential component of the stress vector. They arise in the study of the atmosphere and ocean dynamics [38–40] and the homogenization of the no-slip boundary condition on a rough boundary [18, 27]. We observe in Remark 5.1.7 that the slip boundary conditions give a "proper" viscous term in the sense of [12, 62] in surface fluid equations derived as the thin width limit of the Navier–Stokes equations in a curved thin domain.

In this chapter, we establish the global existence of a strong solution to the Navier–Stokes equations (5.1.1)–(5.1.3) in the curved thin domain Ω_ε given by (5.1.4) for large data of order

$\varepsilon^{-1/2}$. Our result is basically the same as those in [20, 21, 26], but an additional assumption is required (see Assumption 2 and Remark 5.1.5).

Another subject of this chapter is a singular limit problem for the Navier–Stokes equations (5.1.1)–(5.1.3) when the curved thin domain Ω_ε degenerates into the closed surface Γ as $\varepsilon \to 0$. We are concerned with derivation of limit equations on Γ and comparison of solutions to the bulk and limit equations. There are several works on the asymptotic behavior of eigenvalues of the Laplacian on a curved thin domain around a hypersurface (see e.g. [29, 34, 57]), but a singular limit problem for evolution equations in curved thin domains has not been studied well. Temam and Ziane [66] first considered this problem for the Navier–Stokes equations in the thin spherical shell and proved the convergence of the average in the thin direction of a solution towards a unique solution to the Navier–Stokes equations on a sphere in $\mathbb{R}^3$. Prizzi, Rinaldi, and Rybakowski [51] studied a reaction-diffusion equations in a curved thin domain degenerating into a lower dimensional manifold and compared the dynamics of the original and limit equations (see also [52]). Later, the present author considered the heat equation in a moving thin domain and derived a limit diffusion equation on its limit evolving surface [42]. In the recent work [43], he also formally derived limit equations of the Navier–Stokes equations in a moving thin domain that is a tubular neighborhood of an evolving closed surface. The purpose of this chapter is to give a mathematical justification (and generalization) of the result in [43] in the case of the stationary curved thin domain of the form (5.1.4).

To state our main results let us give several notations and assumptions. Let $\mathbb{P}_\varepsilon$ be the Helmholtz–Leray projection from $L^2(\Omega_\varepsilon)^3$ onto the solenoidal space

$$L^2_\sigma(\Omega_\varepsilon) = \{u \in L^2(\Omega_\varepsilon)^3 \mid \operatorname{div} u = 0 \text{ in } \Omega_\varepsilon,\ u \cdot n_\varepsilon = 0 \text{ on } \Omega_\varepsilon\}.$$

We denote by A_ε the Stokes operator on $L^2_\sigma(\Omega_\varepsilon)$ associated with slip boundary conditions and write $D(A_\varepsilon)$ for its domain (see Section 5.5.2). With these notations the problem (5.1.1)–(5.1.3) is formulated as an abstract evolution equation

$$\partial_t u^\varepsilon + A_\varepsilon u^\varepsilon + \mathbb{P}_\varepsilon(u^\varepsilon \cdot \nabla)u^\varepsilon = \mathbb{P}_\varepsilon f^\varepsilon, \quad u^\varepsilon|_{t=0} = u_0^\varepsilon.$$

We refer to [8, 10, 61, 64] and the references cited therein for the study of this abstract evolution equation. For a function φ on Ω_ε we define its average in the thin direction by

$$M\varphi(y) := \frac{1}{\varepsilon g(y)} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi(y + rn(y))\, dr, \quad y \in \Gamma.$$

Also, by $M_\tau u := PMu$ we denote the averaged tangential component of a vector field u on Ω_ε, where $P := I_3 - n \otimes n$ is the orthogonal projection onto the tangent plane of Γ (see Section 5.6.1). For a vector field v on Γ we define the surface strain rate tensor

$$D_\Gamma(v) := P\left(\frac{\nabla_\Gamma v + (\nabla_\Gamma v)^T}{2}\right)P,$$

where $\nabla_\Gamma := P\nabla$ is the tangential gradient operator on Γ, and set

$$\mathcal{K}(\Gamma) := \{v \in H^1(\Gamma)^3 \mid v \cdot n = 0 \text{ and } D_\Gamma(v) = 0 \text{ on } \Gamma\}. \tag{5.1.5}$$

A vector field $X \in \mathcal{K}(\Gamma)$ satisfies $\overline{\nabla}_Y X \cdot Z + Y \cdot \overline{\nabla}_Z X = 0$ for all tangential vector fields Y and Z on Γ, where $\overline{\nabla}_Y X := P(Y \cdot \nabla_\Gamma)X$ is the covariant derivative of X along Y (see Appendix 5.C). Such a vector field generates a one-parameter group of isometries of Γ and is called a Killing vector field. It is also known that $\mathcal{K}(\Gamma)$ is finite dimensional. For details of Killing vector fields, we refer to [30, 50, 63].

In the proofs of our main results we need the uniform boundedness and coerciveness in ε of the bilinear form corresponding to the Stokes operator A_ε on $V_\varepsilon := L^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ (see Lemma 5.5.4). To establish them we make the following assumptions on the surface and the friction coefficients.

Assumption 1. There exists a constant $c > 0$ such that

$$\gamma_\varepsilon^i \leq c\varepsilon, \quad i = 0, 1. \tag{5.1.6}$$

Assumption 2. Either of the following conditions is satisfied:

(A1) There exists a constant $c > 0$ such that

$$\max_{i=0,1} \gamma_\varepsilon^i \geq c\varepsilon. \tag{5.1.7}$$

(A2) The function space

$$\mathcal{K}_g(\Gamma) := \{v \in \mathcal{K}(\Gamma) \mid v \cdot \nabla_\Gamma g = 0 \text{ on } \Gamma\} \tag{5.1.8}$$

contains only a trivial vector field, i.e. $\mathcal{K}_g(\Gamma) = \{0\}$.

Assumptions 1 and 2 are imposed in Section 5.5 except for Section 5.5.1 and Sections 5.7, 5.8, and 5.10.

Now let us give the main results of this chapter. The first result is the global-in-time existence of a strong solution for large data.

Theorem 5.1.1. *Let Ω_ε be the curved thin domain given by* (5.1.4)*. Suppose that*

- *the closed surface Γ is of class C^5,*
- *$g_0, g_1 \in C^4(\Gamma)$ satisfy $g = g_1 - g_0 \geq c$ on Γ with some constant $c > 0$, and*
- *Assumptions 1 and 2 are satisfied.*

Then there exist constants $\varepsilon_0, c_0 \in (0,1)$ such that the following statement holds: for each $\varepsilon \in (0, \varepsilon_0)$ if given data $u_0^\varepsilon \in V_\varepsilon$ and $f^\varepsilon \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$ satisfy

$$\begin{aligned} &\|A_\varepsilon^{1/2} u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \|\mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))} \\ &\qquad + \|M_\tau u_0^\varepsilon\|^2_{L^2(\Gamma)} + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|^2_{L^\infty(0,T;H^{-1}(\Gamma,T\Gamma))} \leq c_0 \varepsilon^{-1}, \end{aligned} \tag{5.1.9}$$

then there exists a global-in-time strong solution

$$u^\varepsilon \in C([0,\infty); V_\varepsilon) \cap L^2_{loc}([0,\infty); D(A_\varepsilon)) \cap H^1_{loc}([0,\infty); L^2_\sigma(\Omega_\varepsilon))$$

to the Navier–Stokes equations (5.1.1)–(5.1.3)*.*

In (5.1.9) we write $H^{-1}(\Gamma, T\Gamma)$ for the dual space of

$$H^1(\Gamma, T\Gamma) := \{v \in H^1(\Gamma)^3 \mid v \cdot n = 0 \text{ on } \Gamma\}.$$

Note that $V_\varepsilon = D(A_\varepsilon^{1/2})$ and the $L^2(\Omega_\varepsilon)$-norm of $A_\varepsilon^{1/2} u$ for $u \in V_\varepsilon$ is uniformly equivalent in ε to the canonical $H^1(\Omega_\varepsilon)$-norm of u (see Lemma 5.5.5). We also establish several estimates for a strong solution in terms of ε.

Theorem 5.1.2. *Let c_1, c_2, α, and β be positive constants. Then, under the same assumptions as in Theorem 5.1.1, there exists $\varepsilon_1 \in (0,1)$ such that the following statement holds: for $\varepsilon \in (0,\varepsilon_1)$ if $u_0^\varepsilon \in V_\varepsilon$ and $f^\varepsilon \in L^2(0,\infty;L^2(\Omega_\varepsilon)^3)$ satisfy*

$$\begin{aligned} \|A_\varepsilon^{1/2}u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2 &\leq c_1\varepsilon^{-1+\alpha}, \\ \|M_\tau u_0^\varepsilon\|_{L^2(\Gamma)}^2 + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))}^2 &\leq c_2\varepsilon^{-1+\beta}, \end{aligned} \tag{5.1.10}$$

then there exists a global-in-time strong solution u^ε to (5.1.1)–(5.1.3) *satisfying*

$$\begin{aligned} \|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 &\leq c(\varepsilon^{1+\alpha}+\varepsilon^\beta), \\ \int_0^t \|u^\varepsilon(s)\|_{H^1(\Omega_\varepsilon)}^2\,ds &\leq c(\varepsilon^{1+\alpha}+\varepsilon^\beta)(1+t), \end{aligned} \tag{5.1.11}$$

and

$$\begin{aligned} \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon)}^2 &\leq c(\varepsilon^{-1+\alpha}+\varepsilon^{-1+\beta}), \\ \int_0^t \|u^\varepsilon(s)\|_{H^2(\Omega_\varepsilon)}^2\,ds &\leq c(\varepsilon^{-1+\alpha}+\varepsilon^{-1+\beta})(1+t) \end{aligned} \tag{5.1.12}$$

for all $t \geq 0$, where $c>0$ is a constant independent of ε, u^ε, and t.

The proofs of Theorems 5.1.1 and 5.1.2 are given in Section 5.8.

Next we give results on a singular limit problem for the Navier–Stokes equations (5.1.1)–(5.1.3) as the curved thin domain Ω_ε degenerates into the closed surface Γ. We define function spaces of tangential vector fields on Γ

$$\begin{aligned} L^2(\Gamma,T\Gamma) &:= \{v \in L^2(\Gamma)^3 \mid v\cdot n = 0 \text{ on } \Gamma\}, \\ L^2_{g\sigma}(\Gamma,T\Gamma) &:= \{v \in L^2(\Gamma,T\Gamma) \mid \mathrm{div}_\Gamma(gv) = 0 \text{ on } \Gamma\}, \end{aligned}$$

and $V_g := L^2_{g\sigma}(\Gamma,T\Gamma)\cap H^1(\Gamma,T\Gamma)$, where div_Γ is the surface divergence operator on Γ (see Sections 5.2 and 5.9).

Theorem 5.1.3. *For $\varepsilon \in (0,1)$ let $u_0^\varepsilon \in V_\varepsilon$ and $f^\varepsilon \in L^2(0,\infty;L^2(\Omega_\varepsilon)^3)$. Under the same assumptions as in Theorem 5.1.1, suppose further that the following conditions are satisfied:*

(a) There exist $c>0$, $\varepsilon_2 \in (0,1)$, and $\alpha \in (0,1)$ such that

$$\|A_\varepsilon^{1/2}u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2 \leq c\varepsilon^{-1+\alpha}$$

for all $\varepsilon \in (0,\varepsilon_2)$.

(b) There exist $v_0 \in L^2(\Gamma,T\Gamma)$ and $f \in L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))$ such that

$$\begin{aligned} \lim_{\varepsilon\to 0} M_\tau u_0^\varepsilon &= v_0 \quad \textit{weakly in} \quad L^2(\Gamma,T\Gamma), \\ \lim_{\varepsilon\to 0} M_\tau \mathbb{P}_\varepsilon f^\varepsilon &= f \quad \textit{weakly-}\star \textit{ in} \quad L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma)). \end{aligned}$$

(c) For $i=0,1$ there exists $\gamma^i \geq 0$ such that $\lim_{\varepsilon\to 0}\varepsilon^{-1}\gamma_\varepsilon^i = \gamma^i$.

Then there exists $\varepsilon_3 \in (0,1)$ *such that the Navier–Stokes equations* (5.1.1)–(5.1.3) *admit a global strong solution* u^ε *for each* $\varepsilon \in (0,\varepsilon_3)$ *and*

$$\lim_{\varepsilon\to 0} Mu^\varepsilon \cdot n = 0 \quad \textit{strongly in} \quad C([0,\infty);L^2(\Gamma)).$$

Moreover, there exists a vector field

$$v \in C([0,\infty);L^2_{g\sigma}(\Gamma,T\Gamma)) \cap L^2_{loc}([0,\infty);V_g) \cap H^1_{loc}([0,\infty);H^{-1}(\Gamma,T\Gamma))$$

such that, for each $T>0$*,*

$$\begin{aligned} \lim_{\varepsilon\to 0} M_\tau u^\varepsilon &= v && \textit{weakly in} \quad L^2(0,T;H^1(\Gamma,T\Gamma)), \\ \lim_{\varepsilon\to 0} \partial_t M_\tau u^\varepsilon &= \partial_t v && \textit{weakly in} \quad L^2(0,T;H^{-1}(\Gamma,T\Gamma)), \end{aligned}$$

and v *is a unique weak solution to the equations*

$$\begin{aligned} g\Big(\partial_t v + \overline{\nabla}_v v\Big) - 2\nu\left\{P\mathrm{div}_\Gamma[gD_\Gamma(v)] - \frac{1}{g}(\nabla_\Gamma g \otimes \nabla_\Gamma g)v\right\}& \\ + (\gamma^0+\gamma^1)v + g\nabla_\Gamma q = gf \quad \textit{on} \quad \Gamma\times(0,\infty)& \end{aligned} \tag{5.1.13}$$

and

$$\mathrm{div}_\Gamma(gv) = 0 \quad \textit{on} \quad \Gamma\times(0,\infty), \quad v|_{t=0} = v_0 \quad \textit{on} \quad \Gamma \tag{5.1.14}$$

with an associated pressure q*.*

We give the definition of a weak solution to (5.1.13)–(5.1.14) and prove Theorem 5.1.3 in Section 5.10.5 (see also Lemma 5.10.21 for construction of an associated pressure). Here $\overline{\nabla}_v v = P(v\cdot\nabla_\Gamma)v$ is the covariant derivative of v along itself. Note that we do not divide (5.1.13) by g since they correspond to the weighted Helmholtz–Leray decomposition

$$v = v_g + g\nabla_\Gamma q, \quad v \in L^2(\Gamma,T\Gamma),\ v_g \in L^2_{g\sigma}(\Gamma,T\Gamma),\ g\nabla_\Gamma q \in L^2_{g\sigma}(\Gamma,T\Gamma)^\perp,$$

which we derive in Section 5.9.3. We also point out that the weak limit v_0 of $M_\tau u_0^\varepsilon$ actually belongs to $L^2_{g\sigma}(\Gamma,T\Gamma)$, while $M_\tau u_0^\varepsilon$ does not so in general (see Lemma 5.10.22).

If the weak and weak-$\star$ convergence of $M_\tau u_0^\varepsilon$ and $M_\tau\mathbb{P}_\varepsilon f^\varepsilon$ are replaced by the strong convergence, then we get the strong convergence of $M_\tau u^\varepsilon$.

Theorem 5.1.4. *For* $\varepsilon\in(0,1)$ *let* $u_0^\varepsilon \in V_\varepsilon$ *and* $f^\varepsilon \in L^\infty(0,\infty;L^2(\Omega_\varepsilon)^3)$*. Suppose that the assumptions in Theorem 5.1.3 are satisfied with the condition (b) replaced by the following condition:*

(b') There exist $v_0 \in L^2(\Gamma,T\Gamma)$ *and* $f \in L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))$ *such that*

$$\begin{aligned} \lim_{\varepsilon\to 0} M_\tau u_0^\varepsilon &= v_0 && \textit{strongly in} \quad L^2(\Gamma,T\Gamma), \\ \lim_{\varepsilon\to 0} M_\tau\mathbb{P}_\varepsilon f^\varepsilon &= f && \textit{strongly in} \quad L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma)). \end{aligned}$$

Then the statements in Theorem 5.1.3 hold. Moreover, for each $T>0$ *we have*

$$\lim_{\varepsilon\to 0} M_\tau u^\varepsilon = v \quad \textit{strongly in} \quad C([0,T];L^2(\Gamma,T\Gamma)) \cap L^2(0,T;H^1(\Gamma,T\Gamma)).$$

We give an estimate for the difference between $M_\tau u^\varepsilon$ and v (see Theorem 5.10.23) and prove Theorem 5.1.4 in Section 5.10.6. Moreover, we estimate the difference between a strong solution u^ε to (5.1.1)–(5.1.3) and a weak solution v to (5.1.13)–(5.1.14) in Ω_ε (see Theorem 5.10.25). It is worth noting that the normal derivative (with respect to Γ) of u^ε is compared with a surface vector field of the form

$$-W(y)v(y,t) + \frac{1}{g(y)}\{v(y,t)\cdot\nabla_\Gamma g(y)\}n(y), \quad (y,t)\in\Gamma\times(0,T),$$

where $W = -\nabla_\Gamma n$ is the Weingarten map (or shape operator) of Γ (see Theorem 5.10.26). Therefore, the normal derivative of u^ε is not necessarily small even though the curved thin domain Ω_ε and the limit surface Γ are stationary.

Let us explain the idea of the proofs of our main results. In the proof of the global existence of a strong solution (Theorem 5.1.1) we follow the arguments in [20,21] to show that the H^1-norm of a strong solution is bounded uniformly in time by a standard energy method. By the same arguments we also get uniform estimates for a strong solution (Theorem 5.1.2). The main tools for the proof are an extension of a surface vector field to Ω_ε that satisfies the impermeable boundary condition (the first equation in (5.1.2)) given in Section 5.3.3 and average operators in the thin direction defined and investigated in Section 5.6. Using them, the slip boundary conditions, and Sobolev type inequalities on Ω_ε and Γ, we derive a good estimate for the trilinear term

$$\big((u\cdot\nabla)u, A_\varepsilon u\big)_{L^2(\Omega_\varepsilon)}, \quad u\in D(A_\varepsilon)$$

in Section 5.7. A key idea for the proof is to decompose a vector field on Ω_ε into the average part, which is almost two-dimensional, and the residual part, which satisfies the impermeable boundary condition (see Section 5.6.3). Such decomposition enables us to apply a product estimate for a function on Ω_ε and that on Γ to the average part (see Corollary 5.6.19) and a good L^∞-estimate following from the Agmon and Poincaré inequalities to the residual part (see Lemma 5.6.22).

For the proof of the global existence and uniform estimates of a strong solution, we also require the uniform equivalence of the norms

$$c^{-1}\|u\|_{H^1(\Omega_\varepsilon)} \le \|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)} \le c\|u\|_{H^1(\Omega_\varepsilon)}$$

for $u\in V_\varepsilon = D(A_\varepsilon^{1/2})$ with a constant $c>0$ independent of ε (see Lemma 5.5.5). It follows from the uniform boundedness and coerciveness of the bilinear form corresponding to the Stokes problem in Ω_ε with slip boundary conditions, for which the uniform Korn inequalities on Ω_ε established in Section 5.4.1 and Assumptions 1 and 2 are essential (see Lemma 5.5.4). It is more difficult to show the uniform equivalence of the H^2-norms

$$c^{-1}\|u\|_{H^2(\Omega_\varepsilon)} \le \|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \le c\|u\|_{H^2(\Omega_\varepsilon)}$$

for $u\in D(A_\varepsilon)$ (see Lemma 5.5.11). The right-hand inequality follows from a uniform L^2-estimate for the difference between the Stokes and Laplace operators (see Lemma 5.5.8). To prove the left-hand inequality, we derive a uniform a priori estimate for the vector Laplacian with slip boundary conditions, which involves calculations of covariant derivatives on the boundary Γ_ε (see Section 5.5.4).

To prove Theorems 5.1.3 and 5.1.4 on a singular limit problem, we proceed as in [42] to transform a weak formulation for the Navier–Stokes equations (5.1.1)–(5.1.3) into that for the averaged tangential component of a strong solution to (5.1.1)–(5.1.3) with residual terms (see Section 5.10.2). For this purpose, we approximate the bilinear and trilinear forms in the weak formulation for (5.1.1)–(5.1.3) by those in the weak formulation for (5.1.13)–(5.1.14) by using the slip boundary conditions and the average operators (see Section 5.6.4). Moreover, we need to construct an appropriate test function for (5.1.1)–(5.1.3) from a test function for (5.1.13)–(5.1.14), which is a weighted solenoidal vector field on Γ. To this end, we use the impermeable extension of a surface vector field to Ω_ε (see Section 5.3.3) and a uniform H^1-estimate for the gradient part of the Helmholtz–Leray decomposition on Ω_ε (see Lemma 5.5.1).

After transformation of the weak formulation, we derive the energy estimate for the averaged tangential component of a strong solution by using its weak formulation in Section 5.10.3. In derivation of the energy estimate, we cannot substitute the averaged tangential component itself for its weak formulation since it is not weighted solenoidal on Γ in general. To overcome this difficulty, we employ the weighted Helmholtz–Leray projection on Γ (see Section 5.9.3) to replace the averaged tangential component with its weighted solenoidal part. Then we derive the energy estimate for the weighted solenoidal part by substituting it for its weak formulation and apply an estimate for the gradient part of the weighted Helmholtz–Leray decomposition on Γ to obtain the energy estimate for the original averaged tangential component of a strong solution, which enables us to show that the averaged tangential component converges weakly as $\varepsilon \to 0$ and that the limit is a weak solution to the limit equations (5.1.13)–(5.1.14) (see Section 5.10.5). We also derive an estimate for the difference between the averaged tangential component of a strong solution to (5.1.1)–(5.1.3) and a weak solution to (5.1.13)–(5.1.14) by using their weak formulations in Section 5.10.6. Here we again use the weighted Helmholtz–Leray projection on Γ to take the difference of the solutions as a test function in the weak formulations.

Now let us give remarks on Assumption 2 and the limit equations (5.1.13)–(5.1.14).

Remark 5.1.5. In the case of the perfect slip boundary conditions, i.e. the boundary conditions (5.1.2) with $\gamma_\varepsilon = 0$, we need to assume that the condition (A2)

$$\mathcal{K}_g(\Gamma) = \{v \in H^1(\Gamma)^3 \mid v \cdot n = 0,\, D_\Gamma(v) = 0,\, v \cdot \nabla_\Gamma g = 0 \text{ on } \Gamma\} = \{0\}$$

in Assumption 2 is satisfied. This assumption is also made in [20]. On the other hand, the authors of [21, 26] consider the Navier–Stokes equations in

$$\Omega_\varepsilon = \{x = (x', x_3) \in \mathbb{R}^3 \mid x' \in \mathbb{T}^2,\, \varepsilon g_0(x') < x_3 < \varepsilon g_1(x')\}, \quad g_0, g_1 \colon \mathbb{T}^2 \to \mathbb{R}$$

with perfect slip boundary conditions on the top and bottom boundaries without assuming that $\mathcal{K}_g(\mathbb{T}^2) = \{0\}$ (here $\mathbb{T}^2$ is the flat torus). When $\Gamma = \mathbb{T}^2$, we have

$$\mathcal{K}_g(\mathbb{T}^2) = \{(a, 0) \in \mathbb{R}^3 \mid a \in \mathbb{R}^2,\, a \cdot \nabla_2 g = 0 \text{ in } \mathbb{T}^2\},$$

where $g = g_1 - g_0$ and ∇_2 is the gradient operator in $x' \in \mathbb{R}^2$. Assuming that

$$\mathcal{K}_g(\mathbb{T}^2) = \{(a, 0) \in \mathbb{R}^3 \mid a \in \mathbb{R}^2,\, a \cdot \nabla_2 g_0 = a \cdot \nabla_2 g_1 = 0 \text{ in } \mathbb{T}^2\}$$

in [21, 26] (in fact $g_0 = 0$ in [26]), the authors get $\mathcal{K}_g(\mathbb{T}^2) \subset L^2_\sigma(\Omega_\varepsilon)$, i.e. any vector in $\mathcal{K}_g(\mathbb{T}^2)$ satisfies the divergence-free condition and the impermeable boundary condition. Based on this fact, they decompose

$$L^2(\Omega_\varepsilon)^2 = \widehat{L}^2_\sigma(\Omega_\varepsilon) \oplus \mathcal{K}_g(\mathbb{T}^2) \oplus G^2(\Omega_\varepsilon),$$

where (we denote by $\mathcal{K}_g(\mathbb{T}^2)^\perp$ the orthogonal complement of $\mathcal{K}_g(\mathbb{T}^2)$ in $L^2(\Omega_\varepsilon)^3$)

$$\widehat{L}^2_\sigma(\Omega_\varepsilon) = L^2_\sigma(\Omega_\varepsilon) \cap \mathcal{K}_g(\mathbb{T}^2)^\perp, \quad G^2(\Omega_\varepsilon) = \{\nabla q \in L^2(\Omega_\varepsilon) \mid q \in H^1(\Omega_\varepsilon)\},$$

and study the abstract evolution equation

$$\partial_t u^\varepsilon + \widehat{A}_\varepsilon u^\varepsilon + \widehat{\mathbb{P}}_\varepsilon (u^\varepsilon \cdot \nabla) u^\varepsilon = \widehat{\mathbb{P}}_\varepsilon f^\varepsilon, \quad u^\varepsilon|_{t=0} = u_0^\varepsilon \tag{5.1.15}$$

in $\widehat{L}^2_\sigma(\Omega_\varepsilon)$, where $\widehat{\mathbb{P}}_\varepsilon$ is the orthogonal projection from $L^2(\Omega_\varepsilon)^3$ onto $\widehat{L}^2_\sigma(\Omega_\varepsilon)$ and $\widehat{A}_\varepsilon$ is the Stokes operator on $\widehat{L}^2_\sigma(\Omega_\varepsilon)$ associated with perfect slip boundary conditions. The reason why they consider (5.1.15) in $\widehat{L}^2_\sigma(\Omega_\varepsilon)$ is that they prove the uniform coerciveness in ε of the bilinear form corresponding to $\widehat{A}_\varepsilon$ on $\widehat{V}_\varepsilon = \widehat{L}^2_\sigma(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3$ (see [21, Proposition 4.12] and [26, Theorem 2.2]).

In this case, however, we need to be careful about recovery of the original Navier–Stokes equations. By (5.1.15) with a strong solution

$$u^\varepsilon \in C([0,\infty); \widehat{V}_\varepsilon) \cap L^2_{loc}([0,\infty); D(\widehat{A}_\varepsilon)) \cap H^1_{loc}([0,\infty); \widehat{L}^2_\sigma(\Omega_\varepsilon)),$$

we a priori get

$$\partial_t u^\varepsilon - \nu \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon + a = f^\varepsilon$$

in $L^2(\Omega_\varepsilon)^3$ with some function $a = a(t) \in \mathcal{K}_g(\mathbb{T}^2)$, but it vanishes if we assume that $f^\varepsilon(t) \in \mathcal{K}_g(\mathbb{T}^2)^\perp$ for all $t \geq 0$. Indeed, noting that $\partial_t u^\varepsilon, \nabla p^\varepsilon \in \mathcal{K}_g(\mathbb{T}^2)^\perp$ we take the inner product of the above equation with a to get

$$\int_{\Omega_\varepsilon} |a|^2\,dx = \nu \int_{\Omega_\varepsilon} \Delta u^\varepsilon \cdot a\,dx - \int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon \cdot a\,dx.$$

By integration by parts (see (5.5.13)), the divergence-free and perfect slip boundary conditions on u^ε, and $D(a) = 0$ (note that $a \in \mathcal{K}_g(\mathbb{T}^2)$ is independent of x) we have

$$\int_{\Omega_\varepsilon} \Delta u^\varepsilon \cdot a\,dx = -2 \int_{\Omega_\varepsilon} D(u^\varepsilon) : D(a)\,dx = 0. \tag{5.1.16}$$

Moreover, by integration by parts, the impermeable boundary condition on u^ε, and $\operatorname{div} a = 0$ we see that

$$\int_{\Omega_\varepsilon} (u^\varepsilon \cdot \nabla) u^\varepsilon\,dx = -\frac{1}{2} \int_{\Omega_\varepsilon} |u^\varepsilon|^2 \operatorname{div} a\,dx = 0. \tag{5.1.17}$$

By these equalities we obtain $\|a(t)\|^2_{L^2(\Omega_\varepsilon)} = 0$, i.e. $a(t) = 0$ for all $t \geq 0$, and the original Navier–Stokes equations are properly recovered.

These arguments are not applicable to our case. To prove the uniform coerciveness in ε of the bilinear form corresponding to the Stokes problem in the curved thin domain Ω_ε given by (5.1.4) with perfect slip boundary conditions, we need to work on the space

$$\widehat{L}^2_\sigma(\Omega_\varepsilon) := \{u \in L^2_\sigma(\Omega_\varepsilon) \mid (u, \bar{v})_{L^2(\Omega_\varepsilon)} = 0 \text{ for all } v \in \mathcal{K}_g(\Gamma)\},$$

where $\bar{v}$ is the constant extension of $v\colon \Gamma \to \mathbb{R}^3$ in the normal direction of Γ (see Lemmas 5.4.3 and 5.4.5). In this case we have a decomposition

$$L^2(\Omega_\varepsilon) = \widehat{L}^2_\sigma(\Omega_\varepsilon) \oplus [L^2_\sigma(\Omega_\varepsilon) \cap \overline{\mathcal{K}}_g(\Gamma)] \oplus G^2(\Omega_\varepsilon), \quad \overline{\mathcal{K}}_g(\Gamma) := \{\bar{v} \mid v \in \mathcal{K}_g(\Gamma)\}.$$

Here we note that we do not know whether or not $\overline{\mathcal{K}}_g(\Gamma) \subset L^2_\sigma(\Omega_\varepsilon)$. For $\bar{v} \in \overline{\mathcal{K}}_g(\Gamma)$ we have $\operatorname{div} \bar{v} = 0$ in Ω_ε (see Lemma 5.B.1), but for $x = y + \varepsilon g_i(y)n(y) \in \Gamma_\varepsilon^i$ with $y \in \Gamma$, $i = 0, 1$,

$$\bar{v}(x) \cdot n_\varepsilon(x) = (-1)^i \varepsilon \frac{v(y) \cdot \tau_\varepsilon^i(y)}{\sqrt{1 + \varepsilon^2 |\tau_\varepsilon^i(y)|^2}} \quad (\tau_\varepsilon^i(y) := \{I_3 - \varepsilon g_i(y)W(y)\}^{-1} \nabla_\Gamma g_i(y))$$

does not vanish in general just by $v \cdot \nabla_\Gamma g = 0$ on Γ (here W is the Weingarten map of Γ, see Lemma 5.2.10). If we solve the abstract evolution equation (5.1.15) in $\widehat{L}^2_\sigma(\Omega_\varepsilon)$ to obtain a strong solution u^ε, then we have

$$\partial_t u^\varepsilon - \nu \Delta u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon + \bar{v} = f^\varepsilon$$

in $L^2(\Omega_\varepsilon)^3$ with some $\bar{v} = \bar{v}(x,t) \in L^2_\sigma(\Omega_\varepsilon) \cap \overline{\mathcal{K}}_g(\Gamma)$. However, we cannot get $\bar{v} = 0$ even if we assume that $f^\varepsilon \in \overline{\mathcal{K}}_g(\Gamma)^\perp$, since the equality (5.1.16) with a replaced by $\bar{v}$ is not valid in general. Indeed, by (5.2.12) we have

$$D(\bar{v})(x) = \frac{1}{2}\Big[\{I_3 - rW(y)\}^{-1} \nabla_\Gamma v(y) + \{\nabla_\Gamma v(y)\}^T \{I_3 - rW(y)\}^{-1}\Big],$$
$$x = y + rn(y) \in \Omega_\varepsilon,\ y \in \Gamma,\ r \in (\varepsilon g_0(y), \varepsilon g_1(y)),$$

which does not vanish just by $D_\Gamma(v) = 0$ on Γ. (The equality (5.1.17) is still valid for $\bar{v}$ since $\operatorname{div} \bar{v} = 0$ in Ω_ε.) To avoid this difficulty in recovery of the original Navier–Stokes equations we assume $\mathcal{K}_g(\Gamma) = \{0\}$ in the case of the perfect slip boundary conditions. This assumption is also essential for derivation of an estimate for the difference between the Stokes and Laplace operators (see Remark 5.5.9).

Remark 5.1.6. As we mentioned in Remark 5.1.5, the assumption $\mathcal{K}_g(\Gamma) = \{0\}$ is necessary for our results in the case of the perfect slip boundary conditions, i.e. the boundary conditions (5.1.2) with $\gamma_\varepsilon = 0$. A typical example violating this assumption is the thin spherical shell

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid 1 < |x| < 1 + \varepsilon\}, \tag{5.1.18}$$

whose limit surface is the unit sphere S^2 in $\mathbb{R}^3$. In this case $\nabla_\Gamma g = 0$ on S^2 and the restriction on S^2 of $u(x) = e_3 \times x$, $x \in \mathbb{R}^3$ with $e_3 = (0,0,1)$ is a Killing vector field on S^2, and thus $\mathcal{K}_g(S^2) = \mathcal{K}(S^2) \neq \{0\}$. Hence our results do not cover the case of the thin spherical shell with perfect slip boundary conditions.

The Navier–Stokes equations in the thin spherical shell was studied by Temam and Ziane [66]. They imposed the Hodge (or de Rham) boundary conditions

$$u \cdot n_\varepsilon = 0, \quad \operatorname{curl} u \times n_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon,$$

which is called the free boundary conditions in [66]. It is mentioned in [66] that the Hodge boundary conditions are equivalent to the perfect slip boundary conditions, but it is not true for the thin spherical shell given by (5.1.18). Indeed, for $u(x) = e_3 \times x$ we easily see that $D(u) = 0$ and $\operatorname{curl} u = 2e_3$ in $\mathbb{R}^3$. Hence on the inner boundary S^2 with unit outward normal $n_\varepsilon(x) = -x$ we have

$$u \cdot n_\varepsilon = -(e_3 \times x) \cdot x = 0, \quad P_\varepsilon D(u) n_\varepsilon = 0, \quad \operatorname{curl} u \times n_\varepsilon = -2e_3 \times x$$

for $x \in S^2$ and the last vector does not vanish in general. More generally, the Hodge boundary conditions are equivalent to the perfect slip boundary conditions only when a boundary is a part of a plane. For example, on the plane $\{(x',0) \mid x' \in \mathbb{R}^2\}$ these boundary conditions reduce to

$$u_3(x',0) = 0, \quad \partial_3 u_1(x',0) = \partial_3 u_2(x',0) = 0, \quad x' \in \mathbb{R}^2.$$

The difference between the Hodge and perfect slip boundary conditions is due to the curvature of a boundary. It also appears in limit equations on a limit surface of the Navier–Stokes equations in a curved thin domain (see Remark 5.1.7 below).

Remark 5.1.7. If $g \equiv 1$ and $\gamma^0 = \gamma^1 = 0$, the limit equations (5.1.13)–(5.1.14) become

$$\partial_t v + \overline{\nabla}_v v - 2\nu P \mathrm{div}_\Gamma[D_\Gamma(v)] + \nabla_\Gamma q = f, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma \times (0,\infty).$$

In [43, Lemma 2.5], it is shown that

$$2P\mathrm{div}_\Gamma[D_\Gamma(v)] = \Delta_B v + Kv \quad \text{on} \quad \Gamma$$

for a tangential vector field v on Γ satisfying $\mathrm{div}_\Gamma v = 0$ on Γ, where $\Delta_B = -\overline{\nabla}^*\overline{\nabla}$ is the Bochner Laplacian (see Appendix 5.C) and K is the Gaussian curvature of Γ (see Section 5.2.1). Hence the above equations are of the form

$$\partial_t v + \overline{\nabla}_v v - \nu(\Delta_B v + Kv) + \nabla_\Gamma q = f, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad \Gamma \times (0,\infty). \tag{5.1.19}$$

These equations are called the "proper" Navier–Stokes equations on a Riemannian manifold in [12, 62] and were studied by Mitrea and Taylor [41], Nagasawa [46], and Taylor [62]. Note that $Kv = \mathrm{Ric}(v)$ for a tangential vector field v on the embedded surface Γ in $\mathbb{R}^3$, where Ric is the Ricci curvature. Also, the Bochner Laplacian is related to the Hodge Laplacian $\Delta_D = -(d_\Gamma d_\Gamma^* + d_\Gamma^* d_\Gamma)$ with the exterior derivative d_Γ by the Weitzenböck formula $\Delta_D = \Delta_B - \mathrm{Ric}$. For details, see e.g. [30, 50].

On the other hand, Temam and Ziane [66] derived the limit equations

$$\partial_t v + \overline{\nabla}_v v - \nu\Delta_2 v + \nabla_\Gamma q = f, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad S^2 \times (0,\infty)$$

from the Navier–Stokes equations in the thin spherical shell given by (5.1.18). Here $\Delta_2 v = P\Delta\bar{v}$ is the tangential vector Laplacian of a tangential vector field v on S^2, where $\bar{v}$ is the constant extension of v in the normal direction of S^2 (see [66, Appendix]). In terms of our notations given in Section 5.2.1 it is expressed as

$$\Delta_2 v = P\Delta\bar{v} = P\Delta_\Gamma v = \Delta_B v - W^2 v \quad \text{on} \quad S^2$$

by Lemmas 5.2.3 and 5.C.5. Moreover, when $\Gamma = S^2$ we have $W = -P$ and thus $W^2 v = v$ for the tangential vector field v. Thus, the limit equations in [66] become

$$\partial_t v + \overline{\nabla}_v v - \nu(\Delta_B v - v) + \nabla_\Gamma q = f, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad S^2 \times (0,\infty). \tag{5.1.20}$$

Since $K = 1$ for the unit sphere S^2 our limit equations (5.1.19) are formally of the form

$$\partial_t v + \overline{\nabla}_v v - \nu(\Delta_B v + v) + \nabla_\Gamma q = f, \quad \mathrm{div}_\Gamma v = 0 \quad \text{on} \quad S^2 \times (0,\infty).$$

(Note that our results do not cover the case of the thin spherical shell given by (5.1.18) with perfect slip boundary conditions.) The sign of v in the viscous term of this system is opposite to that of (5.1.20), which produces different structures of the limit equations such as the stability of a solution. As we mentioned in Remark 5.1.6, this is due to the difference between the Hodge and perfect slip boundary conditions on a curved boundary.

This chapter is organized as follows. In Section 5.2 we introduce notations for surface quantities and function spaces on a closed surface and give the definition and basic properties of a curved thin domain. We present in Section 5.3 fundamental formulas and inequalities for functions on the surface and thin domain. In Section 5.4 we prove the uniform Korn inequalities on the thin domain and the Korn inequalities on the surface. We define the Stokes operator on the thin domain with slip boundary conditions and show its uniform norm equivalence in Section 5.5. Also, we give a uniform estimate for the gradient part of the Helmholtz–Leray decomposition on the thin domain, which is used in the study of a singular limit problem. In Section 5.6 we define average operators in the thin direction and derive several estimates for the average of functions on the thin domain. The main purpose of Section 5.6 is to give decomposition of a vector field on the thin domain into the average and residual parts and to establish useful estimates for them. We also use the average operators to approximate bilinear and trilinear forms for functions on the thin domains by those for functions on the limit surface. In Section 5.7 we prove a good estimate for the trilinear term, i.e. the L^2-inner product of the inertial term and the Stokes operator. The main ingredients for the proof are the estimates for the Stokes and average operators given in Sections 5.5 and 5.6. Using the estimate for the trilinear term, we establish the global existence and uniform estimates of a strong solution (Theorems 5.1.1 and 5.1.2) in Section 5.8. The last two sections are devoted to the study of a singular limit problem when the curved thin domain degenerates into the closed surface. In Section 5.9 we deal with weighted solenoidal spaces on a closed surface. We give characterization of the annihilator of a weighted solenoidal space and prove the weighted Helmholtz–Leray decomposition on the surface with several estimates for the gradient part. In Section 5.10 we investigate the behavior of the average in the thin direction of a strong solution to the Navier–Stokes equations in the curved thin domain. Our goal is to show the convergence of the average towards a weak solution to the limit equations on the limit surface (Theorems 5.1.3 and 5.1.4). In Appendix 5.A we fix notations on vectors and matrices. We also prove some lemmas by elementary vector calculus. In Appendix 5.B we give the proofs of lemmas in Section 5.2 involving calculations of surface quantities of the surface and the boundary of the thin domain. We introduce the Riemannian connection on a surface and show formulas for the covariant derivative of a tangential vector field in Appendix 5.C.

5.2 Preliminaries

In this section we give notations and formulas on several quantities for a two-dimensional closed surface and a thin domain in $\mathbb{R}^3$. Some lemmas in this section are proved just by direct calculations involving differential geometry of surfaces. We give their proofs in Appendix 5.B to avoid making this section too long.

Throughout this chapter we denote by c a general positive constant independent of the parameter ε. Also, we fix a coordinate system of $\mathbb{R}^3$ and write x_i, $i = 1, 2, 3$ for the i-th component of a point $x \in \mathbb{R}^3$ under the fixed coordinate system. For a vector $a \in \mathbb{R}^3$ we denote by a_i, $i = 1, 2, 3$ the i-th component of a. Sometimes we write a^i or $[a]_i$ instead of a_i. Also, for a matrix $A \in \mathbb{R}^{3\times 3}$ and $i, j = 1, 2, 3$ we denote by A_{ij} or $[A]_{ij}$ the (i, j)-entry of A. Other notations and basic formulas on vectors and matrices are given in Appendix 5.A.

5.2.1 Closed surface

Let Γ be a two-dimensional closed (i.e. compact and without boundary), connected, and oriented surface in $\mathbb{R}^3$. We assume that Γ is of class C^ℓ with $\ell \geq 2$. By n and d we denote the unit outward normal vector field of Γ and the signed distance function from Γ increasing in the direction of n. Also, let κ_1 and κ_2 be the principal curvatures of Γ. By the regularity of Γ we have $n \in C^{\ell-1}(\Gamma)^3$ and $\kappa_1, \kappa_2 \in C^{\ell-2}(\Gamma)$. In particular, κ_1 and κ_2 are bounded on the compact set Γ. Hence we can take a tubular neighborhood $N := \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x,\Gamma) < \delta\}$, $\delta > 0$ of Γ such that for each $x \in N$ there exists a unique point $\pi(x) \in \Gamma$ satisfying

$$x = \pi(x) + d(x)n(\pi(x)), \quad \nabla d(x) = n(\pi(x)). \tag{5.2.1}$$

Moreover, d and π are of class C^ℓ and $C^{\ell-1}$ on $\overline{N}$ (see [15, Section 14.6] for details). By the boundedness of κ_1 and κ_2 we also have

$$c^{-1} \leq 1 - r\kappa_i(y) \leq c \quad \text{for all} \quad y \in \Gamma,\, r \in (-\delta,\delta),\, i = 1,2 \tag{5.2.2}$$

if we take $\delta > 0$ sufficiently small.

Let us define differential operators on the surface Γ. For $y \in \Gamma$ we set

$$P(y) := I_3 - n(y) \otimes n(y), \quad Q(y) := n(y) \otimes n(y).$$

By the definitions and the regularity of Γ we have $P, Q \in C^{\ell-1}(\Gamma)^{3\times 3}$ and

$$I_3 = P + Q, \quad PQ = QP = 0, \quad P^T = P^2 = P, \quad Q^T = Q^2 = Q \quad \text{on} \quad \Gamma.$$

The matrices P and Q are the orthogonal projections onto the tangent plane and the normal direction of Γ. In particular, we have $|P| = |Q| = 1$ on Γ. For $\eta \in C^1(\Gamma)$ we define the tangential gradient and the tangential derivatives of η as

$$\nabla_\Gamma \eta(y) := P(y)\nabla\tilde{\eta}(y), \quad \underline{D}_i\eta(y) := \sum_{j=1}^3 P_{ij}(y)\partial_j\tilde{\eta}(y), \quad y \in \Gamma,\, i = 1,2,3$$

so that $\nabla_\Gamma\eta = (\underline{D}_1\eta, \underline{D}_2\eta, \underline{D}_3\eta)$. Here $\tilde{\eta}$ is a C^1-extension of η to N with $\tilde{\eta}|_\Gamma = \eta$. We easily see that

$$P\nabla_\Gamma\eta = \nabla_\Gamma\eta, \quad n \cdot \nabla_\Gamma\eta = 0 \quad \text{on} \quad \Gamma. \tag{5.2.3}$$

Note that the values of $\nabla_\Gamma\eta$ and $\underline{D}_i\eta$ are independent of the choice of an extension $\tilde{\eta}$ (see e.g. [11, Lemma 2.4]). In particular, the constant extension $\bar{\eta} := \eta \circ \pi$ of η in the normal direction of Γ satisfies

$$\nabla\bar{\eta}(y) = \nabla_\Gamma\eta(y), \quad \partial_i\bar{\eta}(y) = \underline{D}_i\eta(y), \quad y \in \Gamma,\, i = 1,2,3 \tag{5.2.4}$$

since $\nabla\pi(y) = P(y)$ for $y \in \Gamma$ by (5.2.1) and $d(y) = 0$. In what follows, a function $\bar{\eta}$ with an overline always stands for the constant extension of a function η on Γ in the normal direction of Γ. The tangential Hessian matrix of $\eta \in C^2(\Gamma)$ and the Laplace–Beltrami operator are given by

$$\nabla_\Gamma^2\eta := (\underline{D}_i\underline{D}_j\eta)_{i,j}, \quad \Delta_\Gamma\eta := \mathrm{tr}[\nabla_\Gamma^2\eta] = \sum_{i=1}^3 \underline{D}_i^2\eta \quad \text{on} \quad \Gamma.$$

For a (not necessarily tangential) vector field $v = (v_1, v_2, v_3) \in C^1(\Gamma)^3$, we define the tangential gradient matrix and the surface divergence of v by

$$\nabla_\Gamma v = \begin{pmatrix} \underline{D}_1 v_1 & \underline{D}_1 v_2 & \underline{D}_1 v_3 \\ \underline{D}_2 v_1 & \underline{D}_2 v_2 & \underline{D}_2 v_3 \\ \underline{D}_3 v_1 & \underline{D}_3 v_2 & \underline{D}_3 v_3 \end{pmatrix}, \quad \mathrm{div}_\Gamma v = \mathrm{tr}[\nabla_\Gamma v] = \sum_{i=1}^3 \underline{D}_i v_i \quad \text{on} \quad \Gamma.$$

Note that $\nabla_\Gamma v = P\nabla\tilde{v}$ on Γ for any C^1-extension $\tilde{v}$ of v to N with $\tilde{v}|_\Gamma = v$. When $v \in C^2(\Gamma)^3$ we write

$$|\nabla_\Gamma^2 v|^2 := \sum_{i,j,k=1}^3 |\underline{D}_i \underline{D}_j v_k|^2, \quad \Delta_\Gamma v := (\Delta_\Gamma v_1, \Delta_\Gamma v_2, \Delta_\Gamma v_3) \quad \text{on} \quad \Gamma.$$

For a matrix-valued function $A \in C^1(\Gamma)^{3\times 3}$ of the form

$$A = (A_{ij})_{i,j} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

we define the surface divergence of A as a vector field on Γ with j-th component

$$[\mathrm{div}_\Gamma A]_j := \sum_{i=1}^2 \underline{D}_i A_{ij}, \quad j = 1, 2, 3.$$

Next we give surface quantities on Γ. We define the Weingarten map W, (twice) the mean curvature H, and the Gaussian curvature K of Γ by

$$W := -\nabla_\Gamma n, \quad H := \mathrm{tr}[W] = -\mathrm{div}_\Gamma n, \quad K := \kappa_1 \kappa_2 \quad \text{on} \quad \Gamma. \tag{5.2.5}$$

Note that W, H, and K are of class $C^{\ell-2}$ by the smoothness of Γ.

Lemma 5.2.1. *The Weingarten map W is symmetric and satisfies*

$$Wn = 0, \quad PW = WP = W \quad \text{on} \quad \Gamma. \tag{5.2.6}$$

We also have

$$\mathrm{div}_\Gamma P = Hn \quad \text{on} \quad \Gamma. \tag{5.2.7}$$

Proof. For $y \in \Gamma$ we have $W(y) = -\nabla\bar{n}(y) = -\nabla^2 d(y)$ by (5.2.1) and (5.2.4). Hence W is symmetric. Taking the tangential gradient of $|n|^2 = 1$ on Γ we get the first relation of (5.2.6). Also, the second relation immediately follows from the first one. For $j = 1, 2, 3$ the j-th component of $\mathrm{div}_\Gamma P$ is of the form

$$[\mathrm{div}_\Gamma P]_j = \sum_{i=1}^3 \underline{D}_i(\delta_{ij} - n_i n_j) = \sum_{i=1}^3 (W_{ii} n_j + W_{ij} n_i) = Hn_j + [W^T n]_j$$

by (5.2.5) and thus $\mathrm{div}_\Gamma P = Hn + W^T n$ on Γ. Applying $W^T = W$ and $Wn = 0$ to this equality we obtain (5.2.7). $\square$

By (5.2.1) the Weingarten map W has the eigenvalue zero associated with eigenvector n. Note that the other eigenvalues of W are the principal curvatures κ_1 and κ_2 (see e.g. [15, Section 14.6] and [33, Section VII.5]) and thus $H = \kappa_1 + \kappa_2$ on Γ.

When we calculate the derivatives of the constant extension of a function on Γ, the inverse matrix of $I_3 - rW(y)$ for $y \in \Gamma$ and $r \in (-\delta, \delta)$ appears.

Lemma 5.2.2. *The matrix $I_3 - rW(y)$ is invertible and we have*

$$\{I_3 - rW(y)\}^{-1}P(y) = P(y)\{I_3 - rW(y)\}^{-1} \tag{5.2.8}$$

for all $y \in \Gamma$ and $r \in (-\delta, \delta)$. Moreover, there exists a constant $c > 0$ such that

$$c^{-1}|a| \leq \left|\{I_3 - rW(y)\}^k a\right| \leq c|a|, \quad k = \pm 1, \tag{5.2.9}$$

$$\left|I_3 - \{I_3 - rW(y)\}^{-1}\right| \leq c|r| \tag{5.2.10}$$

for all $y \in \Gamma$, $r \in (-\delta, \delta)$, and $a \in \mathbb{R}^3$.

Lemma 5.2.3. *For all $x \in N$ we have*

$$\nabla \pi(x) = \left\{I_3 - d(x)\overline{W}(x)\right\}^{-1} \overline{P}(x). \tag{5.2.11}$$

Therefore, the constant extension $\bar{\eta} = \eta \circ \pi$ of $\eta \in C^1(\Gamma)$ satisfies

$$\nabla \bar{\eta}(x) = \left\{I_3 - d(x)\overline{W}(x)\right\}^{-1} \overline{\nabla_\Gamma \eta}(x), \quad x \in N \tag{5.2.12}$$

and there exists a constant $c > 0$ independent of η such that

$$c^{-1}\left|\overline{\nabla_\Gamma \eta}(x)\right| \leq |\nabla \bar{\eta}(x)| \leq c\left|\overline{\nabla_\Gamma \eta}(x)\right|, \tag{5.2.13}$$

$$\left|\nabla \bar{\eta}(x) - \overline{\nabla_\Gamma \eta}(x)\right| \leq c\left|d(x)\overline{\nabla_\Gamma \eta}(x)\right| \tag{5.2.14}$$

for all $x \in N$. If Γ is of class C^3 and $\eta \in C^2(\Gamma)$, then we have

$$|\nabla^2 \eta(x)| \leq c\left(\left|\overline{\nabla_\Gamma \eta}(x)\right| + \left|\overline{\nabla_\Gamma^2 \eta}(x)\right|\right), \quad x \in N. \tag{5.2.15}$$

Moreover, $\Delta \bar{\eta} = \Delta_\Gamma \eta$ on Γ.

We give the proofs of Lemmas 5.2.2 and 5.2.3 in Appendix 5.B.

Since $\bar{n} = \nabla d$ in N by (5.2.1), we use (5.2.12) and $W = -\nabla_\Gamma n$ on Γ to obtain

$$\nabla \bar{n}(x) = \nabla^2 d(x) = -\left\{I_3 - d(x)\overline{W}(x)\right\}^{-1} \overline{W}(x), \quad x \in N. \tag{5.2.16}$$

If Γ is of class C^3, then $n \in C^2(\Gamma)^3$ and thus

$$|\nabla \bar{n}(x)| \leq c, \quad |\nabla^2 \bar{n}(x)| \leq c, \quad x \in N \tag{5.2.17}$$

with some constant $c > 0$ by (5.2.12) and (5.2.15).

Let us define the weak tangential derivatives of a function on Γ and the Sobolev spaces on Γ. For a vector field $v \in C^1(\Gamma)^3$ we have

$$\int_\Gamma \mathrm{div}_\Gamma v \, d\mathcal{H}^2 = \int_\Gamma \mathrm{div}_\Gamma (Pv) \, d\mathcal{H}^2 + \int_\Gamma \mathrm{div}_\Gamma [(v \cdot n)n] \, d\mathcal{H}^2$$

by the decomposition $v = Pv + (v\cdot n)n$, where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. The first integral on the right-hand side vanishes by the Stokes theorem, since Pv is tangential on the closed surface Γ. Moreover, to the second integral we apply $\mathrm{div}_\Gamma(\xi n) = \nabla_\Gamma\xi\cdot n + \xi\,\mathrm{div}_\Gamma n = -\xi H$ for $\xi \in C^1(\Gamma)$. Then we get

$$\int_\Gamma \mathrm{div}_\Gamma v\,d\mathcal{H}^2 = -\int_\Gamma (v\cdot n)H\,d\mathcal{H}^2$$

for all $v \in C^1(\Gamma)^3$. In particular, for $\eta, \xi \in C^1(\Gamma)$ we set $v = \eta\xi e_i$ in the above formula, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$, to obtain

$$\int_\Gamma (\eta \underline{D}_i\xi + \xi\underline{D}_i\eta)\,d\mathcal{H}^2 = -\int_\Gamma \eta\xi H n_i\,d\mathcal{H}^2, \quad i = 1,2,3. \tag{5.2.18}$$

Based on this identity, for $p \in [1,\infty]$ and $i = 1,2,3$ we say that $\eta \in L^p(\Gamma)$ has the i-th weak tangential derivative if there exists $\eta_i \in L^p(\Gamma)$ such that

$$\int_\Gamma \eta_i\xi\,d\mathcal{H}^2 = -\int_\Gamma \eta(\underline{D}_i\xi + \xi H n_i)\,d\mathcal{H}^2 \tag{5.2.19}$$

for all $\xi \in C^1(\Gamma)$. In this case we write $\underline{D}_i\eta = \eta_i$ and define the Sobolev space

$$W^{1,p}(\Gamma) := \{\eta \in L^p(\Gamma) \mid \underline{D}_i\eta \in L^p(\Gamma) \text{ for all } i = 1,2,3\},$$
$$\|\eta\|_{W^{1,p}(\Gamma)} := \begin{cases} \left(\|\eta\|_{L^p(\Gamma)}^p + \|\nabla_\Gamma\eta\|_{L^p(\Gamma)}^p\right)^{1/p} & \text{if} \quad p \in [1,\infty), \\ \|\eta\|_{L^\infty(\Gamma)} + \|\nabla_\Gamma\eta\|_{L^\infty(\Gamma)} & \text{if} \quad p = \infty. \end{cases}$$

Note that $W^{1,p}(\Gamma)$ is a Banach space. In particular, $H^1(\Gamma) = W^{1,2}(\Gamma)$ is a Hilbert space equipped with inner product $(\eta,\xi)_{H^1(\Gamma)} := (\eta,\xi)_{L^2(\Gamma)} + (\nabla_\Gamma\eta, \nabla_\Gamma\xi)_{L^2(\Gamma)}$. We also define the second order Sobolev space

$$W^{2,p}(\Gamma) := \{\eta \in W^{1,p}(\Gamma) \mid \underline{D}_i\underline{D}_j\eta \in L^p(\Gamma) \text{ for all } i,j = 1,2,3\},$$
$$\|\eta\|_{W^{2,p}(\Gamma)} := \begin{cases} \left(\|\eta\|_{W^{1,p}(\Gamma)}^p + \|\nabla_\Gamma^2\eta\|_{L^p(\Gamma)}^p\right)^{1/p} & \text{if} \quad p \in [1,\infty), \\ \|\eta\|_{W^{1,\infty}(\Gamma)} + \|\nabla_\Gamma^2\eta\|_{L^\infty(\Gamma)} & \text{if} \quad p = \infty. \end{cases}$$

Then $W^{2,p}(\Gamma)$ is again a Banach space and $H^2(\Gamma) = W^{2,2}(\Gamma)$ is a Hilbert space. In what follows, we write $W^{0,p}(\Gamma) = L^p(\Gamma)$ for $p \in [1,\infty]$.

Lemma 5.2.4. *Let $p \in [1,\infty)$ and $m = 0,1,\dots,\ell$. Then $C^\ell(\Gamma)$ is dense in $W^{m,p}(\Gamma)$.*

We prove Lemma 5.2.4 in Appendix 5.B by standard localization and mollification arguments. As in the case of a flat domain, Poincaré's inequality holds on Γ.

Lemma 5.2.5. *Let $p \in [1,\infty)$. There exists a constant $c > 0$ such that*

$$\|\eta\|_{L^p(\Gamma)} \leq c\|\nabla_\Gamma\eta\|_{L^p(\Gamma)} \tag{5.2.20}$$

for all $\eta \in W^{1,p}(\Gamma)$ satisfying $\int_\Gamma \eta\,d\mathcal{H}^2 = 0$.

We refer to [11, Theorem 2.12] for the proof of Lemma 5.2.5. Note that the proof given there applies to a closed, connected, and oriented hypersurface of class C^2.

Let $\mathcal{X}(\Gamma)$ be a function space on Γ like $C^m(\Gamma)$, $L^p(\Gamma)$, and $W^{m,p}(\Gamma)$. We define the space of all tangential vector fields on Γ whose components in $\mathcal{X}(\Gamma)$ by

$$\mathcal{X}(\Gamma, T\Gamma) := \{v \in \mathcal{X}(\Gamma)^3 \mid v \cdot n = 0 \text{ on } \Gamma\}.$$

Note that $W^{m,p}(\Gamma, T\Gamma)$ is a closed subspace of $W^{m,p}(\Gamma)^3$, and thus a Banach space. Moreover, an element of $W^{m,p}(\Gamma, T\Gamma)$ with $p \neq \infty$ can be approximated by smooth tangential vector fields on Γ.

Lemma 5.2.6. *Let $p \in [1, \infty)$ and $m = 0, 1, \dots, \ell - 1$. Then $C^{\ell-1}(\Gamma, T\Gamma)$ is dense in $W^{m,p}(\Gamma, T\Gamma)$ with respect to the norm $\|\cdot\|_{W^{m,p}(\Gamma)}$.*

Proof. Let $v \in W^{m,p}(\Gamma, T\Gamma) \subset W^{m,p}(\Gamma)^3$. By Lemma 5.2.4 we can take a sequence $\{\tilde{v}_k\}_{k=1}^\infty$ in $C^\ell(\Gamma)^3$ that converges to v strongly in $W^{m,p}(\Gamma)^3$. For each $k \in \mathbb{N}$ we set $v_k := P\tilde{v}_k \in C^{\ell-1}(\Gamma, T\Gamma)$ (note that P is of class $C^{\ell-1}$ on Γ). Then since v is tangential on Γ, we have $v - v_k = P(v - \tilde{v}_k)$ on Γ. By this equality, $P \in C^{\ell-1}(\Gamma)^{3\times 3}$, and the strong convergence of $\{\tilde{v}_k\}_{k=1}^\infty$ to v in $W^{m,p}(\Gamma)^3$ we see that

$$\|v - v_k\|_{W^{m,p}(\Gamma)} \leq c\|v - \tilde{v}_k\|_{W^{m,p}(\Gamma)} \to 0 \quad \text{as} \quad k \to \infty.$$

Hence $\{v_k\}_{k=1}^\infty$ converges to v strongly in $W^{m,p}(\Gamma, T\Gamma)$ and the claim is valid. $\square$

Let $H^{-1}(\Gamma)$ be the dual space of $H^1(\Gamma)$ and $\langle\cdot,\cdot\rangle_\Gamma$ be the duality product between $H^{-1}(\Gamma)$ and $H^1(\Gamma)$. We consider $\eta \in L^2(\Gamma)$ as an element of $H^{-1}(\Gamma)$ by setting

$$\langle \eta, \xi\rangle_\Gamma := (\eta, \xi)_{L^2(\Gamma)}, \quad \xi \in H^1(\Gamma) \tag{5.2.21}$$

to get the compact embeddings $H^1(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow H^{-1}(\Gamma)$. For $\eta \in W^{1,\infty}(\Gamma)$, $\xi \in H^{-1}(\Gamma)$, and $\varphi \in H^1(\Gamma)$ we see that

$$|\langle \xi, \eta\varphi\rangle_\Gamma| \leq \|\xi\|_{H^{-1}(\Gamma)}\|\eta\varphi\|_{H^1(\Gamma)} \leq c\|\eta\|_{W^{1,\infty}(\Gamma)}\|\xi\|_{H^{-1}(\Gamma)}\|\varphi\|_{H^1(\Gamma)},$$

where $c > 0$ is independent of η, ξ, and φ. Hence we can define $\eta\xi \in H^{-1}(\Gamma)$ by

$$\langle \eta\xi, \varphi\rangle_\Gamma := \langle \xi, \eta\varphi\rangle_\Gamma, \quad \varphi \in H^1(\Gamma). \tag{5.2.22}$$

Similarly, for $\eta \in L^2(\Gamma)$ we can define $\underline{D}_i\eta \in H^{-1}(\Gamma)$, $i = 1, 2, 3$ by

$$\langle \underline{D}_i\eta, \xi\rangle_\Gamma := -(\eta, \underline{D}_i\xi + \xi H n_i)_{L^2(\Gamma)}, \quad \xi \in H^1(\Gamma) \tag{5.2.23}$$

since n and H are bounded on Γ. Based on this definition we consider the weak tangential gradient $\nabla_\Gamma\eta$ of $\eta \in L^2(\Gamma)$ as an element of $H^{-1}(\Gamma)^3$ satisfying

$$\langle \nabla_\Gamma\eta, v\rangle_\Gamma = -(\eta, \mathrm{div}_\Gamma v + (v \cdot n)H)_{L^2(\Gamma)}, \quad v \in H^1(\Gamma)^3. \tag{5.2.24}$$

Also, the surface divergence of $v \in L^2(\Gamma)^3$ is given by

$$\langle \mathrm{div}_\Gamma v, \eta\rangle_\Gamma = -(v, \nabla_\Gamma\eta + \eta H n)_{L^2(\Gamma)}, \quad \eta \in H^1(\Gamma). \tag{5.2.25}$$

Let $H^{-1}(\Gamma, T\Gamma)$ be the dual of $H^1(\Gamma, T\Gamma)$ and $[\cdot,\cdot]_{T\Gamma}$ the duality product between $H^{-1}(\Gamma, T\Gamma)$ and $H^1(\Gamma, T\Gamma)$. It is homeomorphic to a quotient space of $H^{-1}(\Gamma)^3$.

Lemma 5.2.7. *For $f \in H^{-1}(\Gamma)^3$ we define an equivalence class*

$$[f] := \{\tilde{f} \in H^{-1}(\Gamma)^3 \mid Pf = P\tilde{f} \text{ in } H^{-1}(\Gamma)^3\}.$$

Then the quotient space $\mathcal{Q} := \{[f] \mid f \in H^{-1}(\Gamma)^3\}$ is homeomorphic to $H^{-1}(\Gamma, T\Gamma)$.

Note that $\mathcal{Q}$ is a Banach space equipped with norm $\|[f]\|_{\mathcal{Q}} := \inf_{\tilde{f}\in[f]} \|\tilde{f}\|_{H^{-1}(\Gamma)}$ (see e.g. [56] for details).

Proof. Let $f_1, f_2 \in H^{-1}(\Gamma)^3$. If $Pf_1 = Pf_2$ in $H^{-1}(\Gamma)^3$, then $\langle f_1, v\rangle_\Gamma = \langle f_2, v\rangle_\Gamma$ for all $v \in H^1(\Gamma, T\Gamma)$ by (5.2.22) and $Pv = v$. Hence we can define a linear operator L from $\mathcal{Q}$ to $H^{-1}(\Gamma, T\Gamma)$ by $[L[f], v]_{T\Gamma} := \langle \tilde{f}, v\rangle_\Gamma$ for $[f] \in \mathcal{Q}$ and $v \in H^1(\Gamma, T\Gamma)$, where $\tilde{f}$ is any element of $[f]$. By this definition we also see that

$$\|L[f]\|_{H^{-1}(\Gamma,T\Gamma)} \leq \inf_{\tilde{f}\in[f]} \|\tilde{f}\|_{H^{-1}(\Gamma)} = \|[f]\|_{\mathcal{Q}}.$$

Hence L is bounded. Moreover, if $L[f_1] = L[f_2]$ in $H^{-1}(\Gamma, T\Gamma)$, then

$$\langle Pf_1, v\rangle_\Gamma = \langle f_1, Pv\rangle_\Gamma = [L[f_1], Pv]_{T\Gamma} = [L[f_2], Pv]_{T\Gamma} = \langle f_2, Pv\rangle_\Gamma = \langle Pf_2, v\rangle_\Gamma$$

for all $v \in H^1(\Gamma)^3$ and thus $Pf_1 = Pf_2$ in $H^{-1}(\Gamma)^3$, which means that $[f_1] = [f_2]$ and L is injective. To show its surjectivity, let $F \in H^{-1}(\Gamma, T\Gamma)$. Since $H^1(\Gamma, T\Gamma)$ is a Hilbert space equipped with inner product of $H^1(\Gamma)^3$, by the Riesz representation theorem there exists $v_F \in H^1(\Gamma, T\Gamma)$ such that $[F, v]_{T\Gamma} = (v_F, v)_{H^1(\Gamma)}$ for all $v \in H^1(\Gamma, T\Gamma)$. Then setting $f_F := v_F - \sum_{i=1}^3 \underline{D}_i^2 v_F \in H^{-1}(\Gamma)^3$ we observe by (5.2.21), (5.2.23), and $\sum_{i=1}^3 n_i \underline{D}_i v_F^j = 0$ in $L^2(\Gamma)$ for $j = 1, 2, 3$ that

$$\begin{aligned}
[F, v]_{T\Gamma} &= \sum_{i,j=1}^3 \{(v_F^j, v^j)_{L^2(\Gamma)} + (\underline{D}_i v_F^j, \underline{D}_i v^j)_{L^2(\Gamma)}\} \\
&= \sum_{i,j=1}^3 \langle v_F^j - \underline{D}_i^2 v_F^j, v^j\rangle_\Gamma = \langle f_F, v\rangle_\Gamma = [L[f_F], v]_{T\Gamma}
\end{aligned}$$

for all $v \in H^1(\Gamma, T\Gamma)$, i.e. $F = L[f_F]$ in $H^{-1}(\Gamma, T\Gamma)$. Hence $L\colon \mathcal{Q} \to H^{-1}(\Gamma, T\Gamma)$ is a bounded, injective, and surjective linear operator. Since its inverse is also bounded by the open mapping theorem, $\mathcal{Q}$ is homeomorphic to $H^{-1}(\Gamma, T\Gamma)$. □

In what follows, we identify $L[f] \in H^{-1}(\Gamma, T\Gamma)$ in the proof of Lemma 5.2.7 with equivalence class $[f]$ for $f \in H^{-1}(\Gamma)^3$. We further identify $[f]$ with its representative Pf and write $[Pf, v]_{T\Gamma} = \langle f, v\rangle_\Gamma$ for $v \in H^1(\Gamma, T\Gamma)$. When $Pf = f$ in $H^{-1}(\Gamma)^3$, we take f as a representative of $[f]$ instead of Pf. For example, if $\eta \in L^2(\Gamma)$, then

$$\langle \nabla_\Gamma \eta, v\rangle_\Gamma = -(\eta, \mathrm{div}_\Gamma v + (v \cdot n)H)_{L^2(\Gamma)} = -\bigl(\eta, \mathrm{div}_\Gamma(Pv)\bigr)_{L^2(\Gamma)} = \langle P\nabla_\Gamma \eta, v\rangle_\Gamma$$

for all $v \in H^1(\Gamma)^3$ and thus $P\nabla_\Gamma \eta = \nabla_\Gamma \eta$ in $H^{-1}(\Gamma)^3$. In this case we have

$$[\nabla_\Gamma \eta, v]_{T\Gamma} = -(\eta, \mathrm{div}_\Gamma v)_{L^2(\Gamma)}, \quad \eta \in L^2(\Gamma),\ v \in H^1(\Gamma, T\Gamma). \tag{5.2.26}$$

For $\eta \in W^{1,\infty}(\Gamma)$ and $f \in H^{-1}(\Gamma, T\Gamma)$ we can define $\eta f \in H^{-1}(\Gamma, T\Gamma)$ by

$$[\eta f, v]_{T\Gamma} := [f, \eta v]_{T\Gamma}, \quad v \in H^1(\Gamma, T\Gamma) \tag{5.2.27}$$

since $\eta v \in H^1(\Gamma, T\Gamma)$ and $\|\eta v\|_{H^1(\Gamma)} \leq c\|\eta\|_{W^{1,\infty}(\Gamma)}\|v\|_{H^1(\Gamma)}$. In Section 5.9 we give the characterization of the annihilators in $H^{-1}(\Gamma)^3$ and $H^{-1}(\Gamma, T\Gamma)$ of solenoidal spaces on Γ.

Since Γ is not of class C^∞, the space $C^\infty(\Gamma)$ does not make sense and we can not consider distributions on Γ. To consider the time derivative of functions with values in function spaces on Γ, we introduce the notion of distributions with values in a Banach space (see [37,61,64] for details). For $T>0$ and a Banach space $\mathcal{X}$ we define $\mathcal{D}'(0,T;\mathcal{X})$ as the space of all continuous linear operators from $C_c^\infty(0,T)$, the space of all smooth and compactly supported functions on $(0,T)$, into $\mathcal{X}$. Here we say that $f\colon C_c^\infty(0,T)\to\mathcal{X}$ is continuous if $\{f(\varphi_k)\}_{k=1}^\infty$ converges to $f(\varphi)$ strongly in $\mathcal{X}$ when $\varphi_k,\varphi\in C_c^\infty(0,T)$, $k\in\mathbb{N}$ are supported in the same closed interval $[a,b]\subset(0,T)$ and $\{\partial_t^l\varphi_k\}_{k=1}^\infty$ converges to $\partial_t^l\varphi$ uniformly on $[a,b]$ for all $l\geq 0$. We consider $L^2(0,T;\mathcal{X})\subset\mathcal{D}'(0,T;\mathcal{X})$ by identifying $f\in L^2(0,T;\mathcal{X})$ with

$$\hat{f}(\varphi):=\int_0^T\varphi(t)f(t)\,dt\in\mathcal{X},\quad \varphi\in C_c^\infty(0,T).$$

For $f\in\mathcal{D}'(0,T;\mathcal{X})$ we define the time derivative $\partial_t f\in\mathcal{D}'(0,T;\mathcal{X})$ by $\partial_t f(\varphi):=-f(\partial_t\varphi)$ for $\varphi\in C_c^\infty(0,T)$. When $f\in L^2(0,T;\mathcal{X})$, we have

$$\partial_t f(\varphi)=-f(\partial_t\varphi)=-\int_0^T\partial_t\varphi(t)f(t)\,dt\in\mathcal{X},\quad \varphi\in C_c^\infty(0,T). \tag{5.2.28}$$

If there exists $\xi\in L^2(0,T;\mathcal{X})$ such that

$$\partial_t f(\varphi)=\xi(\varphi),\quad \text{i.e.}\quad -\int_0^T\partial_t\varphi(t)f(t)\,dt=\int_0^T\varphi(t)\xi(t)\,dt\quad(\text{in }\mathcal{X})$$

for all $\varphi\in C_c^\infty(0,T)$, then we write $\partial_t f=\xi\in L^2(0,T;\mathcal{X})$ and define

$$H^1(0,T;\mathcal{X}):=\{f\in L^2(0,T;\mathcal{X})\mid\partial_t f\in L^2(0,T;\mathcal{X})\}.$$

When $q\in L^2(0,T;L^2(\Gamma))$, we can consider the time derivative of $\nabla_\Gamma q$ as an element of $\mathcal{D}'(0,T;H^{-1}(\Gamma,T\Gamma))$. Let us show that the time derivative commutes with the tangential gradient in an appropriate sense.

Lemma 5.2.8. *Let $q\in L^2(0,T;L^2(\Gamma))$. Then*

$$\nabla_\Gamma[\partial_t q(\varphi)]=[\partial_t(\nabla_\Gamma q)](\varphi)\quad\textit{in}\quad H^{-1}(\Gamma,T\Gamma)$$

for all $\varphi\in C_c^\infty(0,T)$.

Proof. For all $v\in H^1(\Gamma,T\Gamma)$ we observe by (5.2.24) and (5.2.28) that

$$\begin{aligned}[\nabla_\Gamma[\partial_t q(\varphi)],v]_{T\Gamma}&=(q(\partial_t\varphi),\mathrm{div}_\Gamma v)_{L^2(\Gamma)}=\int_0^T\partial_t\varphi(t)(q(t),\mathrm{div}_\Gamma v)_{L^2(\Gamma)}\,dt\\&=-\int_0^T\partial_t\varphi(t)[\nabla_\Gamma q(t),v]_{T\Gamma}\,dt=\Big[[\partial_t(\nabla_\Gamma q)](\varphi),v\Big]_{T\Gamma}.\end{aligned}$$

Hence the claim is valid. □

Let $q\in L^2(0,T;L^2(\Gamma))$. Based on Lemma 5.2.8, we consider the tangential gradient of $\partial_t q\in\mathcal{D}'(0,T;L^2(\Gamma))$ as an element of $\mathcal{D}'(0,T;H^{-1}(\Gamma,T\Gamma))$ given by

$$[\nabla_\Gamma(\partial_t q)](\varphi):=\nabla_\Gamma[(\partial_t q)(\varphi)]=[\partial_t(\nabla_\Gamma q)](\varphi)\in H^{-1}(\Gamma,T\Gamma) \tag{5.2.29}$$

for $\varphi\in C_c^\infty(0,T)$. We use this relation in construction of an associated pressure in the limit equations (see Lemma 5.10.21).

5.2.2 Curved thin domain

From now on, we assume that the closed surface Γ is of class C^5 (except for Section 5.9). Let $g_0, g_1 \in C^4(\Gamma)$ such that

$$g(y) := g_1(y) - g_0(y) \geq c \quad \text{for all} \quad y \in \Gamma \tag{5.2.30}$$

with a constant $c > 0$. For $\varepsilon \in (0,1)$ we define a curved thin domain Ω_ε in $\mathbb{R}^3$ as

$$\Omega_\varepsilon := \{y + rn(y) \mid y \in \Gamma,\, \varepsilon g_0(y) < r < \varepsilon g_1(y)\}. \tag{5.2.31}$$

We write Γ_ε for the boundary of Ω_ε. It is the union of the inner boundary Γ_ε^0 and the outer boundary Γ_ε^1 given by

$$\Gamma_\varepsilon^i := \{y + \varepsilon g_i(y) n(y) \mid y \in \Gamma\}, \quad i = 0, 1.$$

Since g_0 and g_1 are bounded on Γ, there exists $\tilde{\varepsilon} \in (0,1)$ such that $\overline{\Omega}_\varepsilon \subset N$ for all $\varepsilon \in (0, \tilde{\varepsilon})$. Replacing g_0 and g_1 by $\tilde{\varepsilon} g_0$ and $\tilde{\varepsilon} g_1$ we may assume $\tilde{\varepsilon} = 1$. Note that the boundary Γ_ε is of class C^4 since Γ is of class C^5, $n \in C^4(\Gamma)^3$, and $g_0, g_1 \in C^4(\Gamma)$. We use this fact in the proof of a uniform a priori estimate for the vector Laplacian (see Section 5.5.4).

Let us give surface quantities on Γ_ε. We define vector fields τ_ε^i and n_ε^i on Γ as

$$\tau_\varepsilon^i(y) := \{I_3 - \varepsilon g_i(y) W(y)\}^{-1} \nabla_\Gamma g_i(y), \tag{5.2.32}$$

$$n_\varepsilon^i(y) := (-1)^{i+1} \frac{n(y) - \varepsilon \tau_\varepsilon^i(y)}{\sqrt{1 + \varepsilon^2 |\tau_\varepsilon^i(y)|^2}}. \tag{5.2.33}$$

for $y \in \Gamma$ and $i = 0, 1$. Note that τ_ε^i is tangential on Γ by $n \cdot \nabla_\Gamma g_i = 0$ and $Wn = 0$. Also, τ_ε^i and n_ε^i are bounded on Γ uniformly in ε along with their first and second order tangential derivatives.

Lemma 5.2.9. *There exists a constant $c > 0$ independent of ε such that*

$$|\tau_\varepsilon^i(y)| \leq c, \quad |\underline{D}_k \tau_\varepsilon^i(y)| \leq c, \quad |\underline{D}_l \underline{D}_k \tau_\varepsilon^i(y)| \leq c, \tag{5.2.34}$$

$$|\tau_\varepsilon^i(y) - \nabla_\Gamma g_i(y)| \leq c\varepsilon, \quad |\nabla_\Gamma \tau_\varepsilon^i(y) - \nabla_\Gamma^2 g_i(y)| \leq c\varepsilon \tag{5.2.35}$$

for all $y \in \Gamma$, $i = 0, 1$, and $k, l = 1, 2, 3$. We also have

$$|n_\varepsilon^i| = 1, \quad |\underline{D}_k n_\varepsilon^i(y)| \leq c, \quad |\underline{D}_l \underline{D}_k n_\varepsilon^i(y)| \leq c, \tag{5.2.36}$$

$$|n_\varepsilon^0(y) + n_\varepsilon^1(y)| \leq c\varepsilon, \quad |\nabla_\Gamma n_\varepsilon^0(y) + \nabla_\Gamma n_\varepsilon^1(y)| \leq c\varepsilon \tag{5.2.37}$$

for all $y \in \Gamma$, $i = 0, 1$, and $k, l = 1, 2, 3$.

Let n_ε be the unit outward normal vector field of Γ_ε. For $i = 0, 1$ the direction of n_ε on Γ_ε^i is the same as that of $(-1)^{i+1}\bar{n}$ since the signed distance function d from Γ increases in the direction of n.

Lemma 5.2.10. *The unit outward normal vector field n_ε of Γ_ε is given by*

$$n_\varepsilon(x) = \bar{n}_\varepsilon^i(x), \quad x \in \Gamma_\varepsilon^i,\, i = 0, 1. \tag{5.2.38}$$

Here $\bar{n}_\varepsilon^i = n_\varepsilon^i \circ \pi$ is the constant extension of the vector field n_ε^i given by (5.2.33).

The proofs of Lemmas 5.2.9 and 5.2.10 are given in Appendix 5.B.

As in the case of the surface Γ, we use n_ε to define the orthogonal projections $P_\varepsilon := I_3 - n_\varepsilon \otimes n_\varepsilon$ and $Q_\varepsilon := n_\varepsilon \otimes n_\varepsilon$ onto the tangent plane and the normal direction of Γ_ε, and the tangential gradient and the tangential derivatives

$$\nabla_{\Gamma_\varepsilon}\varphi := P_\varepsilon \nabla\tilde{\varphi}, \quad \underline{D}_i^\varepsilon \varphi := \sum_{j=1}^3 [P_\varepsilon]_{ij}\partial_j\tilde{\varphi} \quad \text{on} \quad \Gamma_\varepsilon,\ i = 1,2,3$$

for $\varphi \in C^1(\Gamma_\varepsilon)$, where $\tilde{\varphi}$ is an arbitrary C^1-extension of φ to an open neighborhood of Γ_ε with $\tilde{\varphi}|_{\Gamma_\varepsilon} = \varphi$. For $u \in C^1(\Gamma_\varepsilon)^3$ we define the tangential gradient matrix and the surface divergence of u as

$$\nabla_{\Gamma_\varepsilon} u := \begin{pmatrix} \underline{D}_1^\varepsilon u_1 & \underline{D}_1^\varepsilon u_2 & \underline{D}_1^\varepsilon u_3 \\ \underline{D}_2^\varepsilon u_1 & \underline{D}_2^\varepsilon u_2 & \underline{D}_2^\varepsilon u_3 \\ \underline{D}_3^\varepsilon u_1 & \underline{D}_3^\varepsilon u_2 & \underline{D}_3^\varepsilon u_3 \end{pmatrix}, \quad \mathrm{div}_{\Gamma_\varepsilon} u := \mathrm{tr}[\nabla_{\Gamma_\varepsilon} u] = \sum_{i=1}^3 \underline{D}_i^\varepsilon u_i \quad \text{on} \quad \Gamma_\varepsilon.$$

The Weingarten map W_ε and (twice) the mean curvature H_ε of Γ_ε are given by

$$W_\varepsilon := -\nabla_{\Gamma_\varepsilon} n_\varepsilon, \quad H_\varepsilon := -\mathrm{div}_{\Gamma_\varepsilon} n_\varepsilon = \mathrm{tr}[W_\varepsilon] \quad \text{on} \quad \Gamma_\varepsilon.$$

Note that, as in the case of Γ, the matrices P_ε, Q_ε, and W_ε are symmetric and

$$\nabla_{\Gamma_\varepsilon} u = P_\varepsilon \nabla\tilde{u}, \quad P_\varepsilon W_\varepsilon = W_\varepsilon P_\varepsilon = W_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon \tag{5.2.39}$$

for $u \in C^1(\Gamma_\varepsilon)^3$, where $\tilde{u}$ is an arbitrary C^1-extension of u to an open neighborhood of Γ_ε with $\tilde{u}|_{\Gamma_\varepsilon} = u$. We also define the weak tangential derivatives of functions on Γ_ε and the Sobolev spaces $W^{m,p}(\Gamma_\varepsilon)$ for $m = 1,2$ and $p \in [1,\infty)$ as in Section 5.2.1.

Since the unit outward normal n_ε to Γ_ε has the expression (5.2.33), we can compare the surface quantities on Γ_ε with those on Γ.

Lemma 5.2.11. *There exists a constant $c > 0$ independent of ε such that*

$$\left|n_\varepsilon(x) - (-1)^{i+1}\left\{\bar{n}(x) - \varepsilon\overline{\nabla_\Gamma g_i}(x)\right\}\right| \le c\varepsilon^2, \tag{5.2.40}$$

$$\left|P_\varepsilon(x) - \overline{P}(x)\right| \le c\varepsilon, \quad \left|Q_\varepsilon(x) - \overline{Q}(x)\right| \le c\varepsilon, \tag{5.2.41}$$

$$\left|W_\varepsilon(x) - (-1)^{i+1}\overline{W}(x)\right| \le c\varepsilon, \quad \left|H_\varepsilon(x) - (-1)^{i+1}\overline{H}(x)\right| \le c\varepsilon, \tag{5.2.42}$$

$$\left|\underline{D}_j^\varepsilon W_\varepsilon(x) - (-1)^{i+1}\overline{\underline{D}_j W}(x)\right| \le c\varepsilon \tag{5.2.43}$$

for all $x \in \Gamma_\varepsilon^i$, $i = 0,1$, and $j = 1,2,3$.

From Lemma 5.2.11 it immediately follows that W_ε, H_ε, and $\underline{D}_j^\varepsilon W_\varepsilon$, $j = 1,2,3$ are uniformly bounded in ε on Γ_ε (note that $|n_\varepsilon| = |P_\varepsilon| = |Q_\varepsilon| = 1$ on Γ_ε). Moreover, we can compare the surface quantities on the inner and outer boundaries.

Lemma 5.2.12. *There exists a constant $c > 0$ independent of ε such that*

$$|F_\varepsilon(y + \varepsilon g_1(y)n(y)) - F_\varepsilon(y + \varepsilon g_0(y)n(y))| \le c\varepsilon, \tag{5.2.44}$$

$$|G_\varepsilon(y + \varepsilon g_1(y)n(y)) + G_\varepsilon(y + \varepsilon g_0(y)n(y))| \le c\varepsilon \tag{5.2.45}$$

for all $y \in \Gamma$, where $F_\varepsilon = P_\varepsilon, Q_\varepsilon$ and $G_\varepsilon = W_\varepsilon, H_\varepsilon, \underline{D}_j^\varepsilon W_\varepsilon$ with $j = 1,2,3$.

The proofs of Lemmas 5.2.11 and 5.2.12 are given in Appendix 5.B.

Next we give transformation formulas of integrals over Ω_ε and Γ_ε. For functions φ on Ω_ε and η on Γ_ε^i, $i=0,1$ we use the notations

$$\varphi^\sharp(y,r) := \varphi(y+rn(y)), \qquad y\in\Gamma,\ r\in(\varepsilon g_0(y),\varepsilon g_1(y)), \tag{5.2.46}$$

$$\eta_i^\sharp(y) := \eta(y+\varepsilon g_i(y)n(y)), \quad y\in\Gamma. \tag{5.2.47}$$

Let $J=J(y,r)$ be a function given by

$$J(y,r) := \det[I_3 - rW(y)] = \{1-r\kappa_1(y)\}\{1-r\kappa_2(y)\} \tag{5.2.48}$$

for $y\in\Gamma$ and $r\in(-\delta,\delta)$. By (5.2.2) and $\kappa_1,\kappa_2\in C^3(\Gamma)$ we have

$$c^{-1}\le J(y,r)\le c, \quad |\nabla_\Gamma J(y,r)|\le c, \quad \left|\frac{\partial J}{\partial r}(y,r)\right|\le c \tag{5.2.49}$$

for all $y\in\Gamma$ and $r\in(-\delta,\delta)$ (here $\nabla_\Gamma J$ stands for the tangential gradient of J with respect to $y\in\Gamma$). Also, we easily observe that

$$|J(y,r)-1|\le c\varepsilon \quad \text{for all} \quad y\in\Gamma,\ r\in[\varepsilon g_0(y),\varepsilon g_1(y)]. \tag{5.2.50}$$

The function J is the Jacobian appearing in the change of variables formula

$$\int_{\Omega_\varepsilon}\varphi(x)\,dx = \int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}\varphi(y+rn(y))J(y,r)\,dr\,d\mathcal{H}^2(y) \tag{5.2.51}$$

for a function φ on Ω_ε (see e.g. [15, Section 14.6]). The formula (5.2.51) can be seen as a co-area formula. From (5.2.49) and (5.2.51) it immediately follows that

$$c^{-1}\|\varphi\|_{L^p(\Omega_\varepsilon)}^p \le \int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}|\varphi^\sharp(y,r)|^p\,dr\,d\mathcal{H}^2(y)\le c\|\varphi\|_{L^p(\Omega_\varepsilon)}^p \tag{5.2.52}$$

for all $p\in[1,\infty)$ and $\varphi\in L^p(\Omega_\varepsilon)$, where we used the notation (5.2.46). We frequently use this inequality in the sequel.

Lemma 5.2.13. *Let η be a function on Γ and $\bar\eta := \eta\circ\pi$ its constant extension in the normal direction of Γ. Then $\eta\in L^p(\Gamma)$, $p\in[1,\infty)$ if and only if $\bar\eta\in L^p(\Omega_\varepsilon)$. Moreover, there exists a constant $c>0$ independent of ε and η such that*

$$c^{-1}\varepsilon^{1/p}\|\eta\|_{L^p(\Gamma)}\le\|\bar\eta\|_{L^p(\Omega_\varepsilon)}\le c\varepsilon^{1/p}\|\eta\|_{L^p(\Gamma)}. \tag{5.2.53}$$

Also, $\eta\in W^{1,p}(\Gamma)$ if and only if $\bar\eta\in W^{1,p}(\Omega_\varepsilon)$ and we have

$$c^{-1}\varepsilon^{1/p}\|\nabla_\Gamma\eta\|_{L^p(\Gamma)}\le\|\nabla\bar\eta\|_{L^p(\Omega_\varepsilon)}\le c\varepsilon^{1/p}\|\nabla_\Gamma\eta\|_{L^p(\Gamma)} \tag{5.2.54}$$

and therefore

$$c^{-1}\varepsilon^{1/p}\|\eta\|_{W^{1,p}(\Gamma)}\le\|\bar\eta\|_{W^{1,p}(\Omega_\varepsilon)}\le c\varepsilon^{1/p}\|\eta\|_{W^{1,p}(\Gamma)}. \tag{5.2.55}$$

Proof. The change of variables formula (5.2.51) implies that

$$\|\bar\eta\|_{L^p(\Omega_\varepsilon)}^p = \int_\Gamma|\eta(y)|^p\left(\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}J(y,r)\,dr\right)d\mathcal{H}^2(y).$$

Hence the inequality (5.2.53) follows from (5.2.30) and (5.2.49). Similarly, we get (5.2.54) by (5.2.13), (5.2.30), (5.2.49), and (5.2.51). $\square$

Lemma 5.2.14. *For $p \in [1,\infty)$ let $\eta \in W^{2,p}(\Gamma)$. Then $\bar{\eta} := \eta \circ \pi \in W^{2,p}(\Omega_\varepsilon)$ and*

$$\|\bar{\eta}\|_{W^{2,p}(\Omega_\varepsilon)} \leq c\varepsilon^{1/p}\|\eta\|_{W^{2,p}(\Gamma)} \tag{5.2.56}$$

with a constant $c > 0$ independent of ε and η.

Proof. From (5.2.15) and (5.2.53) it follows that

$$\|\nabla^2\bar{\eta}\|_{L^p(\Omega_\varepsilon)} \leq c\left(\left\|\overline{\nabla_\Gamma \eta}\right\|_{L^p(\Omega_\varepsilon)} + \left\|\overline{\nabla_\Gamma^2 \eta}\right\|_{L^p(\Omega_\varepsilon)}\right) \leq c\varepsilon^{1/p}\|\eta\|_{W^{2,p}(\Gamma)}.$$

Combining this inequality with (5.2.55) we obtain (5.2.56). □

We also give a change of variables formula for integrals over Γ_ε.

Lemma 5.2.15. *For $\varphi \in L^1(\Gamma_\varepsilon^i)$, $i = 0,1$ let $\varphi_i^\sharp$ be given by (5.2.47). Then*

$$\int_{\Gamma_\varepsilon^i} \varphi(x)\,d\mathcal{H}^2(x) = \int_\Gamma \varphi_i^\sharp(y)J(y,\varepsilon g_i(y))\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}\,d\mathcal{H}^2(y), \tag{5.2.57}$$

where τ_ε^i is given by (5.2.32). Moreover, if $\varphi \in L^p(\Gamma_\varepsilon^i)$, $p \in [1,\infty)$ then $\varphi_i^\sharp \in L^p(\Gamma)$ and there exists a constant $c > 0$ independent of ε such that

$$c^{-1}\|\varphi\|_{L^p(\Gamma_\varepsilon^i)} \leq \|\varphi_i^\sharp\|_{L^p(\Gamma)} \leq c\|\varphi\|_{L^p(\Gamma_\varepsilon^i)}. \tag{5.2.58}$$

Proof. In Lemma 5.B.2 we show a change of variables formula

$$\int_{\Gamma_h} \varphi(x)\,d\mathcal{H}^2(x) = \int_\Gamma \varphi^\sharp(y)J(y,h(y))\sqrt{1+|\tau_h(y)|^2}\,d\mathcal{H}^2(y)$$

for an integrable function φ on a parametrized surface $\Gamma_h := \{y + h(y)n(y) \mid y \in \Gamma\}$, where $h \in C^1(\Gamma)$ satisfies $|h| < \delta$ on Γ and

$$\varphi^\sharp(y) := \varphi(y + h(y)n(y)), \quad \tau_h(y) := \{I_3 - h(y)W(y)\}^{-1}\nabla_\Gamma h(y), \quad y \in \Gamma.$$

Setting $h = \varepsilon g_i$, $i = 0,1$ in the above formula we obtain (5.2.57). Also, (5.2.58) follows from the formula (5.2.57) and the inequalities (5.2.34) and (5.2.49). □

5.3 Fundamental tools for analysis

5.3.1 Sobolev inequalities

Let us give several Sobolev inequalities on Γ and Ω_ε. First we prove Ladyzhenskaya's inequality on the two-dimensional closed surface Γ.

Lemma 5.3.1. *There exists a constant $c > 0$ such that*

$$\|\eta\|_{L^4(\Gamma)} \leq c\|\eta\|_{L^2(\Gamma)}^{1/2}\|\eta\|_{H^1(\Gamma)}^{1/2} \tag{5.3.1}$$

for all $\eta \in H^1(\Gamma)$.

Proof. Since Γ is compact and without boundary, by a standard localization argument with a partition of unity of Γ it is sufficient to prove

$$\|\eta\|_{L^4(\mu(U))} \leq c\|\eta\|_{L^2(\mu(U))}^{1/2}\|\eta\|_{H^1(\mu(U))}^{1/2} \tag{5.3.2}$$

for a bounded open set U in $\mathbb{R}^2$, a local parametrization $\mu\colon U \to \Gamma$, and $\eta \in H^1(\Gamma)$ compactly supported in $\mu(U)$. For such η, the function $\tilde{\eta} := \eta \circ \mu$ is in $H^1(\mathbb{R}^2)$ and compactly supported in U. Hence Ladyzhenskaya's inequality on $\mathbb{R}^2$ (see [35, Lemma 1 in Chapter 1, Section 1.1]) yields

$$\|\tilde{\eta}\|_{L^4(U)} \leq \sqrt{2}\|\tilde{\eta}\|_{L^2(U)}^{1/2}\|\nabla_s\tilde{\eta}\|_{L^2(U)}^{1/2}, \tag{5.3.3}$$

where ∇_s is the gradient operator in $s \in \mathbb{R}^2$. To deduce (5.3.2) from (5.3.3) we set

$$\nabla_s\mu := \begin{pmatrix} \partial_{s_1}\mu_1 & \partial_{s_1}\mu_2 & \partial_{s_1}\mu_3 \\ \partial_{s_2}\mu_1 & \partial_{s_2}\mu_2 & \partial_{s_2}\mu_3 \end{pmatrix}, \quad \theta := \nabla_s\mu(\nabla_s\mu)^T$$

and recall that integrals over the surface are given by

$$\begin{aligned} \|\eta\|_{L^p(\mu(U))}^p &= \int_U |\tilde{\eta}|^p\sqrt{\det\theta}\,ds, \quad p = 2,4, \\ \|\nabla_\Gamma\eta\|_{L^2(\mu(U))}^2 &= \int_U |(\nabla_\Gamma\eta)\circ\mu|^2\sqrt{\det\theta}\,ds. \end{aligned}$$

Since μ is of class C^5 (note that Γ is of class C^5), the determinant of θ is continuous and does not vanish on U. In particular, it is bounded from above and below by positive constants on the support of $\tilde{\eta}$ since it is a compact subset of U. Hence

$$c^{-1}\|\eta\|_{L^p(\mu(U))} \leq \|\tilde{\eta}\|_{L^p(U)} \leq c\|\eta\|_{L^p(\mu(U))}, \quad p = 2,4 \tag{5.3.4}$$

with a constant $c > 0$. Also, differentiating both sides of $\tilde{\eta}(s) = \eta(\mu(s)) = \bar{\eta}(\mu(s))$ with respect to s_i, $i = 1,2$ and using (5.2.4) (note that $\mu(s) \in \Gamma$) we get

$$\partial_{s_i}\tilde{\eta}(s) = \partial_{s_i}\mu(s)\cdot\nabla\bar{\eta}(\mu(s)) = \partial_{s_i}\mu(s)\cdot\nabla_\Gamma\eta(\mu(s)), \quad s \in U.$$

From this equality and the fact that μ is of class C^5 and the determinant of θ is bounded from below by a positive constant on the support of $\tilde{\eta}$ we deduce that

$$\|\nabla_s\tilde{\eta}\|_{L^2(U)}^2 \leq c\int_U |(\nabla_\Gamma\eta)\circ\mu|^2\,ds \leq c\|\nabla_\Gamma\eta\|_{L^2(\mu(U))}^2.$$

Applying this inequality and (5.3.4) to (5.3.3) we obtain (5.3.2). □

Next we give Poincaré type inequalities on the curved thin domain Ω_ε. By ∂_n we denote the directional derivative in the normal direction of Γ, i.e. for a function φ on Ω_ε and $x \in \Omega_\varepsilon$ we set

$$\partial_n\varphi(x) := (\bar{n}(x)\cdot\nabla)\varphi(x) = \frac{d}{dr}\big(\varphi(y + rn(y))\big)\Big|_{r=d(x)} \quad (y = \pi(x) \in \Gamma). \tag{5.3.5}$$

Note that for the constant extension of a function η on Γ we have

$$\partial_n\bar{\eta}(x) = (\bar{n}(x)\cdot\nabla)\bar{\eta}(x) = 0, \quad x \in \Omega_\varepsilon. \tag{5.3.6}$$

Lemma 5.3.2. *Let $\varphi \in W^{1,p}(\Omega_\varepsilon)$ with $p \in [1,\infty)$. There exists a constant $c > 0$ independent of ε such that*

$$\|\varphi\|_{L^p(\Omega_\varepsilon)} \leq c\left(\varepsilon^{1/p}\|\varphi\|_{L^p(\Gamma_\varepsilon^i)} + \varepsilon\|\partial_n\varphi\|_{L^p(\Omega_\varepsilon)}\right), \quad i = 0,1, \tag{5.3.7}$$

$$\|\varphi\|_{L^p(\Gamma_\varepsilon^i)} \leq c\left(\varepsilon^{-1/p}\|\varphi\|_{L^p(\Omega_\varepsilon)} + \varepsilon^{1-1/p}\|\partial_n\varphi\|_{L^p(\Omega_\varepsilon)}\right), \quad i = 0,1. \tag{5.3.8}$$

Proof. We show (5.3.7) and (5.3.8) for $i = 0$. Their proofs for $i = 1$ are the same. We use the notations (5.2.46) and (5.2.47). Let $y \in \Gamma$ and $r \in (\varepsilon g_0(y), \varepsilon g_1(y))$. Since $\partial\varphi^\sharp/\partial r = (\partial_n\varphi)^\sharp$ by (5.3.5), we have

$$\varphi^\sharp(y,r) = \varphi^\sharp(y,\varepsilon g_0(y)) + \int_{\varepsilon g_0(y)}^{r} (\partial_n\varphi)^\sharp(y,\tilde{r})\,d\tilde{r}. \tag{5.3.9}$$

From (5.3.9) and Hölder's inequality it follows that

$$\begin{aligned}
|\varphi^\sharp(y,r)| &\leq |\varphi_0^\sharp(y)| + \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n\varphi)^\sharp(y,\tilde{r})|\,d\tilde{r} \\
&\leq |\varphi_0^\sharp(y)| + c\varepsilon^{1-1/p}\left(\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n\varphi)^\sharp(y,\tilde{r})|^p\,d\tilde{r}\right)^{1/p}.
\end{aligned}$$

Here $\varphi_0^\sharp(y) = \varphi^\sharp(y,\varepsilon g_0(y))$. Noting that the right-hand side is independent of r, we integrate the p-th power of both sides of the above inequality with respect to r to get

$$\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\varphi^\sharp(y,r)|^p\,dr \leq c\left(\varepsilon|\varphi_0^\sharp(y)|^p + \varepsilon^p\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n\varphi)^\sharp(y,\tilde{r})|^p\,d\tilde{r}\right). \tag{5.3.10}$$

Hence the inequalities (5.2.52) and (5.3.10) yield that

$$\|\varphi\|_{L^p(\Omega_\varepsilon)}^p \leq c\int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\varphi^\sharp(y,r)|^p\,dr\,d\mathcal{H}^2(y) \leq c\left(\varepsilon\|\varphi_0^\sharp\|_{L^p(\Gamma)}^p + \varepsilon^p\|\partial_n\varphi\|_{L^p(\Omega_\varepsilon)}^p\right).$$

Applying (5.2.58) to the first term on the right-hand side we obtain (5.3.7).

Next let us prove (5.3.8). From (5.3.9) we deduce that

$$|\varphi_0^\sharp(y)|^p \leq c\left(\varepsilon^{-1}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\varphi^\sharp(y,r)|^p\,dr + \varepsilon^{p-1}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n\varphi)^\sharp(y,\tilde{r})|^p\,d\tilde{r}\right)$$

as in the proof of (5.3.10). This inequality and (5.2.52) imply that

$$\|\varphi_0^\sharp\|_{L^p(\Gamma)} \leq c\left(\varepsilon^{-1/p}\|\varphi\|_{L^p(\Omega_\varepsilon)} + \varepsilon^{1-1/p}\|\partial_n\varphi\|_{L^p(\Omega_\varepsilon)}\right).$$

Hence we apply (5.2.58) to the left-hand side of the above inequality to get (5.3.8). $\square$

We also show Agmon's inequality on Ω_ε, which gives an estimate for the $L^\infty(\Omega_\varepsilon)$-norm of a function in $H^2(\Omega_\varepsilon)$ with explicit dependence on ε of a bound.

Lemma 5.3.3. *There exists a constant $c > 0$ independent of ε such that*

$$\begin{aligned}
\|\varphi\|_{L^\infty(\Omega_\varepsilon)} &\leq c\varepsilon^{-1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/4}\|\varphi\|_{H^2(\Omega_\varepsilon)}^{1/2} \\
&\qquad\times\left(\|\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\partial_n\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon^2\|\partial_n^2\varphi\|_{L^2(\Omega_\varepsilon)}\right)^{1/4}
\end{aligned} \tag{5.3.11}$$

for all $\varphi \in H^2(\Omega_\varepsilon)$.

Proof. We use the anisotropic Agmon inequality (see [65, Proposition 2.2])

$$\|\Phi\|_{L^\infty(V)} \leq c\|\Phi\|_{L^2(V)}^{1/4} \prod_{i=1}^{3} \left(\|\Phi\|_{L^2(V)} + \|\partial_i\Phi\|_{L^2(V)} + \|\partial_i^2\Phi\|_{L^2(V)}\right)^{1/4} \tag{5.3.12}$$

for $\Phi \in H^2(V)$ with $V = (0,1)^3$. To this end, we localize φ by using a partition of unity on Γ and transform it into a function on V. Since Γ is compact, we can take a finite number of open sets U_k in $\mathbb{R}^2$ and local parametrizations $\mu_k \colon U_k \to \Gamma$, $k = 1, \dots, k_0$ such that $\{\mu_k(U_k)\}_{k=1}^{k_0}$ is an open covering of Γ. Then setting

$$\zeta_k(s) = \mu_k(s') + \varepsilon\{(1-s_3)g_0(\mu_k(s')) + s_3 g_1(\mu_k(s'))\}n(\mu_k(s'))$$

for $s = (s', s_3) \in V_k := U_k \times (0,1)$ we see that $\{\zeta_k(V_k)\}_{k=1}^{k_0}$ is an open covering of Ω_ε. Let $\{\eta_k\}_{k=1}^{k_0}$ be a partition of unity on Γ subordinate to the covering $\{\mu_k(U_k)\}_{k=1}^{k_0}$ and $\bar{\eta}_k := \eta_k \circ \pi$ the constant extension of η_k. Then $\{\bar{\eta}_k\}_{k=1}^{k_0}$ is a partition of unity on Ω_ε subordinate to the covering $\{\zeta_k(V_k)\}_{k=1}^{k_0}$. Hence to prove (5.3.11) it is sufficient to show

$$\begin{aligned}\|\varphi_k\|_{L^\infty(\zeta_k(V_k))} \leq c\varepsilon^{-1/2}&\|\varphi_k\|_{L^2(\zeta_k(V_k))}^{1/4}\|\varphi_k\|_{H^2(\zeta_k(V_k))}^{1/2} \\ &\times \left(\|\varphi_k\|_{L^2(\zeta_k(V_k))} + \varepsilon\|\partial_n\varphi_k\|_{L^2(\zeta_k(V_k))} + \varepsilon^2\|\partial_n^2\varphi_k\|_{L^2(\zeta_k(V_k))}\right)^{1/4}\end{aligned} \tag{5.3.13}$$

for each $k = 1, \dots, k_0$, where $\varphi_k := \bar{\eta}_k\varphi$ (note that $\partial_n\varphi_k = \bar{\eta}_k\partial_n\varphi$ by $\partial_n\bar{\eta}_k = 0$). Let us prove (5.3.13). Hereafter we suppress the index k. By taking the open subset U of $\mathbb{R}^2$ small and scaling it, we may assume $U = (0,1)^2$ and $V = (0,1)^3$. Since Γ is of class C^5, the local parametrization μ is of class C^5 and thus the mapping

$$\zeta(s) = \mu(s') + \varepsilon\{(1-s_3)\bar{g}_0(\mu(s')) + s_3\bar{g}_1(\mu(s'))\}\bar{n}(\mu(s')), \quad s = (s', s_3) \in V$$

is of class C^4 by $g_0, g_1 \in C^4(\Gamma)$ and $n \in C^4(\Gamma)^3$ (here $\bar{g}_i = g_i \circ \pi$ and $\bar{n} = n \circ \pi$). We differentiate $\zeta(s)$ and apply (5.2.4) and (5.2.16) with $y = \mu(s') \in \Gamma$ to get

$$\begin{aligned}\partial_{s_i}\zeta(s) &= \left[I_3 - \varepsilon\big\{(1-s_3)g_0(\mu(s')) + s_3 g_1(\mu(s'))\big\}W(\mu(s'))\right]\partial_{s_i}\mu(s') \\ &\qquad + \varepsilon\partial_{s_i}\mu(s') \cdot \{(1-s_3)\nabla_\Gamma g_0(\mu(s')) + s_3\nabla_\Gamma g_1(\mu(s'))\}n(\mu(s')), \\ \partial_{s_3}\zeta(s) &= \varepsilon g(\mu(s'))n(\mu(s'))\end{aligned} \tag{5.3.14}$$

for $s = (s', s_3) \in V$ and $i = 1, 2$. From these formulas it follows that

$$\det\nabla_s\zeta(s) = \varepsilon g(\mu(s'))J(\mu(s'), h_\varepsilon(s))\sqrt{\det\theta(s')}, \quad s = (s', s_3) \in V, \tag{5.3.15}$$

where $\nabla_s\zeta$ is the gradient matrix of ζ in s and

$$\begin{gathered} h_\varepsilon(s) := \varepsilon\{(1-s_3)g_0(\mu(s')) + s_3 g_1(\mu(s'))\}, \\ \theta(s') := \nabla_{s'}\mu(s')\nabla_{s'}\mu(s')^T, \quad \nabla_{s'}\mu := \begin{pmatrix} \partial_{s_1}\mu_1 & \partial_{s_1}\mu_2 & \partial_{s_1}\mu_3 \\ \partial_{s_2}\mu_1 & \partial_{s_2}\mu_2 & \partial_{s_2}\mu_3 \end{pmatrix}. \end{gathered}$$

(We give detailed calculations for (5.3.15) in Appendix 5.B.) Let $\Phi(s) := \varphi(\zeta(s))$ for $s \in V$. Since φ is localized by the constant extension of a cut-off function on Γ, the function Φ is supported in $\mathcal{K} \times (0,1)$ with some compact subset $\mathcal{K}$ of U. Then the determinant of θ is bounded from below by a positive constant on $\mathcal{K}$ since it is continuous and does not vanish on

U (note that the local parametrization μ is of class C^5). By this fact, (5.2.30), and (5.2.49) we have

$$\det \nabla_s \zeta(s) \geq c\varepsilon, \quad s \in \mathcal{K} \times (0,1) \tag{5.3.16}$$

with a constant $c > 0$. Since $\Phi = \varphi \circ \zeta$ is supported in $\mathcal{K} \times (0,1)$ and ζ is bounded on $\mathcal{K} \times (0,1)$ along with its first and second order derivatives (note that ζ is of class C^4 on V and depends linearly on s_3), we observe by the change of variables formula

$$\int_{\zeta(V)} \varphi(x)\,dx = \int_V \Phi(s) \det \nabla_s \zeta(s)\,ds, \tag{5.3.17}$$

the inequality (5.3.16), and $\varphi \in H^2(\Omega_\varepsilon)$ that $\Phi \in H^2(V)$ and

$$\|\Phi\|_{L^\infty(V)} = \|\varphi\|_{L^\infty(\zeta(V))}, \quad \|\Phi\|_{L^2(V)} \leq c\varepsilon^{-1/2}\|\varphi\|_{L^2(\zeta(V))}. \tag{5.3.18}$$

Moreover, by the chain rule of differentiation we have

$$\begin{aligned}
&\partial_{s_i}\Phi(s) = \partial_{s_i}\zeta(s) \cdot \nabla\varphi(\zeta(s)), \\
&\partial_{s_i}^2\Phi(s) = \partial_{s_i}^2\zeta(s) \cdot \nabla\varphi(\zeta(s)) + \partial_{s_i}\zeta(s) \cdot \nabla^2\varphi(\zeta(s))\partial_{s_i}\zeta(s), \\
&\partial_{s_3}\Phi(s) = \varepsilon g(\mu(s'))\partial_n\varphi(\zeta(s)), \quad \partial_{s_3}^2\Phi(s) = \varepsilon^2 g(\mu(s'))^2\partial_n^2\varphi(\zeta(s))
\end{aligned}$$

for $s = (s', s_3) \in V$ and $i = 1,2$, where the last two equalities follow from (5.3.14) and $\partial_n\varphi = (\bar{n} \cdot \nabla)\varphi$. To the above equalities we apply the boundedness of g on Γ and that of the first and second order derivatives of ζ on $\mathcal{K} \times (0,1)$ to get

$$\begin{gathered}
|\partial_{s_i}\Phi(s)| \leq c|\nabla\varphi(\zeta(s))|, \quad |\partial_{s_i}^2\Phi(s)| \leq c(|\nabla\varphi(\zeta(s))| + |\nabla^2\varphi(\zeta(s))|), \quad i = 1,2, \\
|\partial_{s_3}^k\Phi(s)| \leq c\varepsilon^k|\partial_n^k\varphi(\zeta(s))|, \quad k = 1,2
\end{gathered}$$

for $s \in \mathcal{K} \times (0,1)$. Noting that $\Phi = \varphi \circ \zeta$ is supported in $\mathcal{K} \times (0,1)$, we deduce from the above inequalities, (5.3.16), and (5.3.17) that

$$\|\partial_{s_i}^k\Phi\|_{L^2(V)} \leq c\varepsilon^{-1/2}\|\varphi\|_{H^k(\zeta(V))}, \quad \|\partial_{s_3}^k\Phi\|_{L^2(V)} \leq c\varepsilon^{k-1/2}\|\partial_n^k\varphi\|_{L^2(\zeta(V))} \tag{5.3.19}$$

for $i,k = 1,2$. Finally, applying the anisotropic Agmon inequality (5.3.12) to $\Phi \in H^2(V)$ and using (5.3.18) and (5.3.19) we obtain (5.3.13). $\square$

5.3.2 Consequences of the boundary conditions

In this subsection we derive several properties from the boundary conditions

$$u \cdot n_\varepsilon = 0, \tag{5.3.20}$$

$$2\nu P_\varepsilon D(u)n_\varepsilon + \gamma_\varepsilon u = 0, \tag{5.3.21}$$

where $D(u) =: (\nabla u)_S = \{\nabla u + (\nabla u)^T\}/2$ is the strain rate tensor. First we consider vector fields satisfying the impermeable boundary condition (5.3.20).

Lemma 5.3.4. *For $i = 0,1$ let $u \in C(\Gamma_\varepsilon^i)^3$ satisfy* (5.3.20) *on Γ_ε^i. Then*

$$u \cdot \bar{n} = \varepsilon u \cdot \bar{\tau}_\varepsilon^i, \quad |u \cdot \bar{n}| \leq c\varepsilon|u| \quad \text{on} \quad \Gamma_\varepsilon^i, \tag{5.3.22}$$

where τ_ε^i is given by (5.2.32) *and $c > 0$ is a constant independent of ε and u.*

Proof. The first equality of (5.3.22) is an immediate consequence of (5.2.33), (5.2.38), and (5.3.20) on Γ_ε^i. This equality and the first inequality of (5.2.34) implies the second inequality of (5.3.22). □

As a consequence of Lemma 5.3.4 we derive Poincaré's inequalities for the normal component (with respect to Γ) of a vector field on Ω_ε.

Lemma 5.3.5. *Let $p \in [1,\infty)$. There exists $c>0$ independent of ε such that*

$$\|u\cdot\bar{n}\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{W^{1,p}(\Omega_\varepsilon)} \tag{5.3.23}$$

for all $u \in W^{1,p}(\Omega_\varepsilon)^3$ satisfying (5.3.20) on Γ_ε^0 or on Γ_ε^1. We also have

$$\left\|\overline{P}\nabla(u\cdot\bar{n})\right\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{W^{2,p}(\Omega_\varepsilon)} \tag{5.3.24}$$

for all $u \in W^{2,p}(\Omega_\varepsilon)^3$ satisfying (5.3.20) on Γ_ε^0 or on Γ_ε^1.

Proof. Let $u \in W^{1,p}(\Omega_\varepsilon)^3$. We may assume that u satisfies (5.3.20) on Γ_ε^0 without loss of generality. By (5.3.6) and (5.3.7) with $i=0$,

$$\|u\cdot\bar{n}\|_{L^p(\Omega_\varepsilon)} \leq c\left(\varepsilon^{1/p}\|u\cdot\bar{n}\|_{L^p(\Gamma_\varepsilon^0)} + \varepsilon\|\partial_n u\|_{L^p(\Omega_\varepsilon)}\right). \tag{5.3.25}$$

Moreover, we apply the second inequality of (5.3.22) and then use (5.3.8) with $i=0$ to the first term on the right-hand side of (5.3.25) to get

$$\|u\cdot\bar{n}\|_{L^p(\Gamma_\varepsilon^0)} \leq c\varepsilon\|u\|_{L^p(\Gamma_\varepsilon^0)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{1,p}(\Omega_\varepsilon)}. \tag{5.3.26}$$

Combining (5.3.25) and (5.3.26) we obtain (5.3.23).

Next suppose that $u \in W^{2,p}(\Omega_\varepsilon)^3$ satisfies (5.3.20) on Γ_ε^0. Noting that

$$\left|\partial_n\left[\overline{P}\nabla(u\cdot\bar{n})\right]\right| \leq c(|\nabla u| + |\nabla^2 u|) \quad \text{in} \quad \Omega_\varepsilon$$

by (5.2.17) and (5.3.6), we apply (5.3.7) with $i=0$ to get

$$\left\|\overline{P}\nabla(u\cdot\bar{n})\right\|_{L^p(\Omega_\varepsilon)} \leq c\left(\varepsilon^{1/p}\left\|\overline{P}\nabla(u\cdot\bar{n})\right\|_{L^p(\Gamma_\varepsilon^0)} + \varepsilon\|u\|_{W^{2,p}(\Omega_\varepsilon)}\right) \tag{5.3.27}$$

Since the tangential gradient on Γ_ε depends only on the value of a function on Γ_ε, we see by (5.2.39) and the first equality of (5.3.22) that

$$\begin{aligned}\overline{P}\nabla(u\cdot\bar{n}) &= \nabla_{\Gamma_\varepsilon}(u\cdot\bar{n}) + \left(\overline{P}-P_\varepsilon\right)\nabla(u\cdot\bar{n}) = \varepsilon\nabla_{\Gamma_\varepsilon}(u\cdot\bar{\tau}_\varepsilon^0) + \left(\overline{P}-P_\varepsilon\right)\nabla(u\cdot\bar{n}) \\ &= \varepsilon P_\varepsilon\nabla(u\cdot\bar{\tau}_\varepsilon^0) + \left(\overline{P}-P_\varepsilon\right)\nabla(u\cdot\bar{n})\end{aligned}$$

on Γ_ε^0. By this formula, (5.2.13), (5.2.17), (5.2.34), and (5.2.41) we have

$$\left|\overline{P}\nabla(u\cdot\bar{n})\right| \leq c\varepsilon(|u| + |\nabla u|) \quad \text{on} \quad \Gamma_\varepsilon^0.$$

From this inequality and (5.3.8) it follows that

$$\left\|\overline{P}\nabla(u\cdot\bar{n})\right\|_{L^p(\Gamma_\varepsilon^0)} \leq c\varepsilon\left(\|u\|_{L^p(\Gamma_\varepsilon^0)} + \|\nabla u\|_{L^p(\Gamma_\varepsilon^0)}\right) \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}.$$

Applying this inequality to the right-hand side of (5.3.27) we obtain (5.3.24). □

The next lemma gives an expression for the normal component of the directional derivatives on Γ_ε for vector fields satisfying the impermeable boundary condition in terms of the Weingarten map of Γ_ε.

Lemma 5.3.6. *Let $i = 0, 1$. For $u_1, u_2 \in C^1(\overline{\Omega}_\varepsilon)^3$ satisfying* (5.3.20) *on Γ_ε^i we have*

$$(u_1 \cdot \nabla)u_2 \cdot n_\varepsilon = W_\varepsilon u_1 \cdot u_2 = u_1 \cdot W_\varepsilon u_2 \quad \text{on} \quad \Gamma_\varepsilon^i. \tag{5.3.28}$$

Proof. The equality $u_1 \cdot n_\varepsilon = 0$ on Γ_ε^i means that u_1 is tangential on Γ_ε^i. Hence

$$(u_1 \cdot \nabla)u_2 \cdot n_\varepsilon = (u_1 \cdot \nabla_{\Gamma_\varepsilon})u_2 \cdot n_\varepsilon = u_1 \cdot \nabla_{\Gamma_\varepsilon}(u_2 \cdot n_\varepsilon) - u_2 \cdot (u_1 \cdot \nabla_{\Gamma_\varepsilon})n_\varepsilon$$

on Γ_ε^i. The first term on the right-hand side vanishes by $u_2 \cdot n_\varepsilon = 0$ on Γ_ε^i (note that the tangential gradient on Γ_ε^i depends only on the values of a function on Γ_ε^i). Also, by $-\nabla_{\Gamma_\varepsilon} n_\varepsilon = W_\varepsilon = W_\varepsilon^T$ we have

$$(u_1 \cdot \nabla_{\Gamma_\varepsilon})n_\varepsilon = -W_\varepsilon^T u_1 = -W_\varepsilon u_1 \quad \text{on} \quad \Gamma_\varepsilon^i.$$

Combining the above two equalities we obtain (5.3.28). □

Note that $(u_1 \cdot \nabla)u_2 \cdot n_\varepsilon$ can be expressed without using the derivatives of u_1 and u_2 by (5.3.28). We use this fact in the analysis of boundary integrals, see Lemma 5.4.1.

Next we prove formulas for vector fields satisfying the slip boundary conditions (5.3.20)–(5.3.21).

Lemma 5.3.7. *Let $i = 0, 1$. For $u \in C^2(\overline{\Omega}_\varepsilon)^3$ satisfying* (5.3.20)–(5.3.21) *on Γ_ε^i we have*

$$P_\varepsilon(n_\varepsilon \cdot \nabla)u = -W_\varepsilon u - \frac{\gamma_\varepsilon}{\nu}u \quad \text{on} \quad \Gamma_\varepsilon^i, \tag{5.3.29}$$

$$n_\varepsilon \times \operatorname{curl} u = -n_\varepsilon \times \left\{n_\varepsilon \times \left(2W_\varepsilon u + \frac{\gamma_\varepsilon}{\nu}u\right)\right\} \quad \text{on} \quad \Gamma_\varepsilon^i. \tag{5.3.30}$$

Proof. Applying the tangential gradient operator $\nabla_{\Gamma_\varepsilon}$ to $u \cdot n_\varepsilon = 0$ on Γ_ε^i we have

$$(\nabla_{\Gamma_\varepsilon} u)n_\varepsilon = -(\nabla_{\Gamma_\varepsilon} n_\varepsilon)u = W_\varepsilon u \quad \text{on} \quad \Gamma_\varepsilon^i. \tag{5.3.31}$$

From this equality, (5.3.21), and

$$2P_\varepsilon D(u)n_\varepsilon = P_\varepsilon\{(\nabla u)n_\varepsilon + (\nabla u)^T n_\varepsilon\} = (\nabla_{\Gamma_\varepsilon} u)n_\varepsilon + P_\varepsilon(n_\varepsilon \cdot \nabla)u$$

by (5.2.39) we deduce that

$$P_\varepsilon(n_\varepsilon \cdot \nabla)u = -(\nabla_{\Gamma_\varepsilon} u)n_\varepsilon - \frac{\gamma_\varepsilon}{\nu}u = -W_\varepsilon u - \frac{\gamma_\varepsilon}{\nu}u \quad \text{on} \quad \Gamma_\varepsilon^i.$$

Hence (5.3.29) holds. To prove (5.3.30) we observe that the vector field $n_\varepsilon \times \operatorname{curl} u$ is tangential on Γ_ε^i. By this fact, (5.2.39), (5.3.29), and (5.3.31) we have

$$\begin{aligned} n_\varepsilon \times \operatorname{curl} u = P_\varepsilon(n_\varepsilon \times \operatorname{curl} u) &= P_\varepsilon\{(\nabla u)n_\varepsilon - (\nabla u)^T n_\varepsilon\} \\ &= (\nabla_{\Gamma_\varepsilon} u)n_\varepsilon - P_\varepsilon(n_\varepsilon \cdot \nabla)u = 2W_\varepsilon u + \frac{\gamma_\varepsilon}{\nu}u \end{aligned}$$

on Γ_ε^i. The equality (5.3.30) follows from this equality and the identity

$$a \times (a \times b) = (a \cdot b)a - |a|^2 b, \quad a, b \in \mathbb{R}^3$$

with $a = n_\varepsilon$ and $b = 2W_\varepsilon u + \nu^{-1}\gamma_\varepsilon u$ since $n_\varepsilon \cdot u = 0$, $n_\varepsilon \cdot W_\varepsilon u = 0$, and $|n_\varepsilon|^2 = 1$ on Γ_ε^i. □

Let us derive an estimate for the $L^p(\Omega_\varepsilon)$-norm of the tangential component (with respect to Γ) of the stress vector $D(u)\bar{n}$. It is used in the study of a singular limit problem for (5.1.1)–(5.1.3) as ε tends to zero.

Lemma 5.3.8. *Suppose that Assumption 1 is satisfied, i.e. the inequality* (5.1.6) *holds and let $p \in [1,\infty)$. Then there exists $c > 0$ independent of ε such that*

$$\left\|\overline{P}D(u)\bar{n}\right\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{W^{2,p}(\Omega_\varepsilon)} \tag{5.3.32}$$

for all $u \in W^{2,p}(\Omega_\varepsilon)^3$ satisfying (5.3.21) *on Γ_ε^0 or on Γ_ε^1.*

Proof. We proceed as in the proof of Lemma 5.3.5. Let $u \in W^{2,p}(\Omega_\varepsilon)^3$ satisfy (5.3.21) on Γ_ε^i for $i = 0$ or $i = 1$. By (5.3.6) and the boundedness of n and P we have

$$\left|\partial_n\left[\overline{P}D(u)\bar{n}\right]\right| \leq c|\nabla^2 u| \quad \text{in} \quad \Omega_\varepsilon.$$

Hence we use (5.3.7) to get

$$\left\|\overline{P}D(u)\bar{n}\right\|_{L^p(\Omega_\varepsilon)} \leq c\left(\varepsilon^{1/p}\left\|\overline{P}D(u)\bar{n}\right\|_{L^p(\Gamma_\varepsilon^i)} + \varepsilon\|u\|_{W^{2,p}(\Omega_\varepsilon)}\right). \tag{5.3.33}$$

Moreover, since u satisfies (5.3.21) on Γ_ε^i, we have

$$\begin{aligned}\overline{P}D(u)\bar{n} &= (-1)^{i+1}P_\varepsilon D(u)n_\varepsilon + P_\varepsilon D(u)\{\bar{n} - (-1)^{i+1}n_\varepsilon\} + \left(\overline{P} - P_\varepsilon\right)D(u)\bar{n} \\ &= (-1)^i\frac{\gamma_\varepsilon}{2\nu}u + P_\varepsilon D(u)\{\bar{n} - (-1)^{i+1}n_\varepsilon\} + \left(\overline{P} - P_\varepsilon\right)D(u)\bar{n}\end{aligned}$$

on Γ_ε^i. Applying (5.1.6), (5.2.40), and (5.2.41) to the last line of this equality and noting that P_ε is bounded uniformly in ε we deduce that

$$\left|\overline{P}D(u)\bar{n}\right| \leq c\varepsilon(|u| + |\nabla u|) \quad \text{on} \quad \Gamma_\varepsilon^i.$$

By this inequality and (5.3.8) we get

$$\left\|\overline{P}D(u)\bar{n}\right\|_{L^p(\Gamma_\varepsilon^i)} \leq c\varepsilon\left(\|u\|_{L^p(\Gamma_\varepsilon^i)} + \|\nabla u\|_{L^p(\Gamma_\varepsilon^i)}\right) \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}.$$

We apply this inequality to (5.3.33) to conclude that (5.3.32) is valid. □

Finally, we compare the tangential component (with respect to Γ) of the normal derivative of a vector field u on Ω_ε with $-\overline{W}u$.

Lemma 5.3.9. *Under Assumption 1, there exists $c > 0$ independent of ε such that*

$$\left\|\overline{P}\partial_n u + \overline{W}u\right\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{W^{2,p}(\Omega_\varepsilon)} \tag{5.3.34}$$

for all $u \in W^{2,p}(\Omega_\varepsilon)^3$ with $p \in [1,\infty)$ satisfying the slip boundary conditions (5.3.20)–(5.3.21) *on Γ_ε^0 or on Γ_ε^1. Here $\partial_n u$ is the normal derivative of u given by* (5.3.5).

Proof. For $i = 0$ or $i = 1$ let $u \in W^{2,p}(\Omega_\varepsilon)^3$ satisfy (5.3.20)–(5.3.21) on Γ_ε^i. By (5.3.6) and the boundedness of n, P, and W on Γ we have

$$\left|\partial_n\left[\overline{P}\partial_n u + \overline{W}u\right]\right| \leq c(|\nabla u| + |\nabla^2 u|) \quad \text{in} \quad \Omega_\varepsilon.$$

We apply (5.3.7) and the above inequality to get

$$\left\|\overline{P}\partial_n u+\overline{W}u\right\|_{L^p(\Omega_\varepsilon)} \le c\left(\varepsilon^{1/p}\left\|\overline{P}\partial_n u+\overline{W}u\right\|_{L^p(\Gamma_\varepsilon^i)}+\varepsilon\|u\|_{W^{2,p}(\Omega_\varepsilon)}\right). \tag{5.3.35}$$

Moreover, since $\partial_n u=(\bar{n}\cdot\nabla)u$ and u satisfies (5.3.20)–(5.3.21) on Γ_ε^i,

$$\begin{aligned}\overline{P}\partial_n u &= (-1)^{i+1}P_\varepsilon(n_\varepsilon\cdot\nabla)u+P_\varepsilon[\{\bar{n}-(-1)^{i+1}n_\varepsilon\}\cdot\nabla]u+\left(\overline{P}-P_\varepsilon\right)(\bar{n}\cdot\nabla)u \\ &= (-1)^{i+1}\left(-W_\varepsilon u-\frac{\gamma_\varepsilon}{\nu}u\right)+P_\varepsilon[\{\bar{n}-(-1)^{i+1}n_\varepsilon\}\cdot\nabla]u+\left(\overline{P}-P_\varepsilon\right)(\bar{n}\cdot\nabla)u\end{aligned}$$

on Γ_ε^i by (5.3.29). Hence by (5.1.6) and (5.2.40)–(5.2.42) we get

$$\left|\overline{P}\partial_n u+\overline{W}u\right| \le \left|\overline{W}u-(-1)^{i+1}W_\varepsilon u\right|+c\varepsilon(|u|+|\nabla u|)\le c\varepsilon(|u|+|\nabla u|)$$

on Γ_ε^i, which together with (5.3.8) implies that

$$\left\|\overline{P}\partial_n u+\overline{W}u\right\|_{L^p(\Gamma_\varepsilon^i)} \le c\varepsilon\left(\|u\|_{L^p(\Gamma_\varepsilon^i)}+\|\nabla u\|_{L^p(\Gamma_\varepsilon^i)}\right)\le c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}.$$

Applying this inequality to (5.3.35) we obtain (5.3.34). □

5.3.3 Impermeable extension of surface vector fields

In the analysis of integrals over Ω_ε involving a vector field on Γ it is convenient to consider its extension to Ω_ε satisfying the impermeable boundary condition on Γ_ε. Let τ_ε^0 and τ_ε^1 be the vector fields on Γ given by (5.2.32). We define a vector field Ψ_ε on N by

$$\Psi_\varepsilon(x):=\frac{1}{\bar{g}(x)}\left\{\left(d(x)-\varepsilon\bar{g}_0(x)\right)\bar{\tau}_\varepsilon^1(x)+\left(\varepsilon\bar{g}_1(x)-d(x)\right)\bar{\tau}_\varepsilon^0(x)\right\},\quad x\in N. \tag{5.3.36}$$

By definition, $\Psi_\varepsilon=\varepsilon\bar{\tau}_\varepsilon^i$ on Γ_ε^i, $i=0,1$. Let us give several estimates for Ψ_ε.

Lemma 5.3.10. *There exists a constant $c>0$ independent of ε such that*

$$|\Psi_\varepsilon|\le c\varepsilon,\quad |\nabla\Psi_\varepsilon|\le c,\quad |\nabla^2\Psi_\varepsilon|\le c\quad \text{in}\quad \Omega_\varepsilon. \tag{5.3.37}$$

Moreover, we have

$$\left|\overline{P}\nabla\Psi_\varepsilon\right|\le c\varepsilon,\quad \left|\partial_n\Psi_\varepsilon-\frac{1}{\bar{g}}\overline{\nabla_\Gamma g}\right|\le c\varepsilon\quad \text{in}\quad \Omega_\varepsilon. \tag{5.3.38}$$

Proof. Applying (5.2.34) and

$$0\le d(x)-\varepsilon\bar{g}_0(x)\le\varepsilon\bar{g}(x),\quad 0\le\varepsilon\bar{g}_1(x)-d(x)\le\varepsilon\bar{g}(x),\quad x\in\Omega_\varepsilon \tag{5.3.39}$$

to (5.3.36) we get the first inequality of (5.3.37). Also, by $\nabla d=\bar{n}$ in N we have

$$\nabla\Psi_\varepsilon=\frac{1}{\bar{g}}\{\bar{n}\otimes(\bar{\tau}_\varepsilon^1-\bar{\tau}_\varepsilon^0)+F_\varepsilon\}\quad \text{in}\quad N, \tag{5.3.40}$$

where F_ε is a 3×3 matrix-valued function on N is given by

$$F_\varepsilon:=-\nabla\bar{g}\otimes\Psi_\varepsilon+\varepsilon(\nabla\bar{g}_1\otimes\bar{\tau}_\varepsilon^0-\nabla\bar{g}_0\otimes\bar{\tau}_\varepsilon^1)+(d-\varepsilon\bar{g}_0)\nabla\bar{\tau}_\varepsilon^1+(\varepsilon\bar{g}_1-d)\nabla\bar{\tau}_\varepsilon^0.$$

By (5.2.30), (5.2.34), and $|n| = 1$ on Γ we see that the first term on the right-hand side of (5.3.40) is bounded on N uniformly in ε. Moreover, from (5.2.13), (5.2.30), (5.2.34), the first inequality of (5.3.37), (5.3.39), and $g_0, g_1 \in C^4(\Gamma)$ we deduce that

$$|F_\varepsilon| \leq c\varepsilon \quad \text{in} \quad \Omega_\varepsilon. \tag{5.3.41}$$

Hence the second inequality of (5.3.37) follows. Similarly, differentiating both sides of (5.3.40) and using (5.2.13), (5.2.15), (5.2.34), the first and second inequalities of (5.3.37), and $g_0, g_1 \in C^4(\Gamma)$ we can derive the last inequality of (5.3.37).

Let us prove (5.3.38). First note that

$$\begin{aligned} &P[n \otimes (\tau_\varepsilon^1 - \tau_\varepsilon^0)] = (Pn) \otimes (\tau_\varepsilon^1 - \tau_\varepsilon^0) = 0, \\ &[(\tau_\varepsilon^1 - \tau_\varepsilon^0) \otimes n]n = |n|^2(\tau_\varepsilon^1 - \tau_\varepsilon^0) = \tau_\varepsilon^1 - \tau_\varepsilon^0 \end{aligned}$$

on Γ. These equalities and (5.3.5) imply that

$$\overline{P}\nabla\Psi_\varepsilon = \frac{1}{\bar{g}}\overline{P}F_\varepsilon, \quad \partial_n\Psi_\varepsilon = (\nabla\Psi_\varepsilon)^T\bar{n} = \frac{1}{\bar{g}}(\tau_\varepsilon^1 - \tau_\varepsilon^0) + F_\varepsilon^T\bar{n} \quad \text{in} \quad N.$$

Hence we see by (5.2.30), (5.3.41), and $|P| = 1$ on Γ that

$$\left|\overline{P}\nabla\Psi_\varepsilon\right| \leq c\left|\overline{P}F_\varepsilon\right| \leq c|F_\varepsilon| \leq c\varepsilon \quad \text{in} \quad \Omega_\varepsilon.$$

Also, applying (5.2.30), (5.2.35), and (5.3.41) to the equality for $\partial_n\Psi_\varepsilon$ we obtain

$$\left|\partial_n\Psi_\varepsilon - \frac{1}{\bar{g}}\overline{\nabla_\Gamma g}\right| \leq \frac{1}{\bar{g}}\sum_{i=0,1}\left|\bar{\tau}_\varepsilon^i - \overline{\nabla_\Gamma g_i}\right| + |F_\varepsilon| \leq c\varepsilon \quad \text{in} \quad \Omega_\varepsilon.$$

Hence (5.3.38) is valid. □

For a tangential vector field v on Γ (i.e. $v \cdot n = 0$ on Γ) we define

$$E_\varepsilon v(x) := \bar{v}(x) + \{\bar{v}(x) \cdot \Psi_\varepsilon(x)\}\bar{n}(x), \quad x \in N, \tag{5.3.42}$$

where $\bar{v}$ and $\bar{n}$ are the constant extensions of v and n. By the definition of Ψ_ε we easily see that $E_\varepsilon v$ satisfies the impermeable boundary condition on Γ_ε.

Lemma 5.3.11. *For all $v \in C(\Gamma, T\Gamma)$ we have $E_\varepsilon v \cdot n_\varepsilon = 0$ on Γ_ε.*

Proof. For $i = 0, 1$ we observe by (5.2.33), (5.2.38), and $v \cdot n = 0$ on Γ that

$$\bar{v} \cdot n_\varepsilon = (-1)^i\frac{\varepsilon\bar{v} \cdot \bar{\tau}_\varepsilon^i}{\sqrt{1 + \varepsilon^2|\bar{\tau}_\varepsilon^i|^2}}, \quad \bar{n} \cdot n_\varepsilon = \frac{(-1)^{i+1}}{\sqrt{1 + \varepsilon^2|\bar{\tau}_\varepsilon^i|^2}} \quad \text{on} \quad \Gamma_\varepsilon^i.$$

From these equalities and $\Psi_\varepsilon = \varepsilon\bar{\tau}_\varepsilon^i$ on Γ_ε^i by (5.3.36) we get $E_\varepsilon v \cdot n_\varepsilon = 0$ on Γ_ε^i. □

Also, it is easy to show that $E_\varepsilon v \in W^{m,p}(\Omega_\varepsilon)$ for $v \in W^{m,p}(\Gamma, T\Gamma)$.

Lemma 5.3.12. *There exists a constant $c > 0$ independent of ε such that*

$$\|E_\varepsilon v\|_{W^{m,p}(\Omega_\varepsilon)} \leq c\varepsilon^{1/2}\|v\|_{W^{m,p}(\Gamma)} \tag{5.3.43}$$

for all $v \in W^{m,p}(\Gamma, T\Gamma)$ with $p \in [1, \infty)$ and $m = 0, 1, 2$.

Proof. By (5.2.13), (5.2.15), and (5.3.37) we have

$$|E_\varepsilon v| \leq c|\bar{v}|, \quad |\nabla E_\varepsilon v| \leq c\left(|\bar{v}| + \left|\overline{\nabla_\Gamma v}\right|\right), \quad |\nabla^2 E_\varepsilon v| \leq c\left(|\bar{v}| + \left|\overline{\nabla_\Gamma v}\right| + \left|\overline{\nabla_\Gamma^2 v}\right|\right)$$

in Ω_ε. These inequalities, (5.2.53), (5.2.55), and (5.2.56) imply (5.3.43). □

If $\widetilde{\Omega}_\varepsilon$ is a flat thin domain of the form

$$\widetilde{\Omega}_\varepsilon = \{x = (x', x_3) \in \mathbb{R}^3 \mid x' \in \omega,\, \varepsilon\tilde{g}_0(x') < x_3 < \varepsilon\tilde{g}_1(x')\},$$

where ω is a domain in $\mathbb{R}^2$ and $\tilde{g}_0$ and $\tilde{g}_1$ are functions on ω, then we have

$$\operatorname{div}(E_\varepsilon v)(x) = \frac{1}{\tilde{g}(x')}\operatorname{div}(\tilde{g}v)(x'), \quad x = (x', x_3) \in \widetilde{\Omega}_\varepsilon \quad (\tilde{g} := \tilde{g}_1 - \tilde{g}_0)$$

for $v\colon \omega \to \mathbb{R}^2$ (see [21, Lemma 4.24] and [26, Remark 3.1]). This is not the case for the curved thin domain Ω_ε given by (5.1.4) because the principal curvatures of the surface Γ do not vanish in general. However, we can show that the difference between $\operatorname{div}(E_\varepsilon v)$ and $g^{-1}\operatorname{div}_\Gamma(gv)$ is of order ε in Ω_ε.

Lemma 5.3.13. *There exists a constant $c > 0$ independent of ε such that*

$$\left|\nabla E_\varepsilon v - \left\{\overline{\nabla_\Gamma v} + \frac{1}{\bar{g}}\left(\bar{v}\cdot\overline{\nabla_\Gamma g}\right)\overline{Q}\right\}\right| \leq c\varepsilon\left(|\bar{v}| + \left|\overline{\nabla_\Gamma v}\right|\right) \quad \text{in} \quad \Omega_\varepsilon \tag{5.3.44}$$

for all $v \in C^1(\Gamma, T\Gamma)$. Moreover, we have

$$\left|\operatorname{div}(E_\varepsilon v) - \frac{1}{\bar{g}}\overline{\operatorname{div}_\Gamma(gv)}\right| \leq c\varepsilon\left(|\bar{v}| + \left|\overline{\nabla_\Gamma v}\right|\right) \quad \text{in} \quad \Omega_\varepsilon. \tag{5.3.45}$$

Proof. From (5.3.42), $Q = n\otimes n$, and $(n\otimes\nabla_\Gamma g)v = (v\cdot\nabla_\Gamma g)n$ it follows that

$$\begin{gathered}\nabla E_\varepsilon v = \nabla\bar{v} + [(\nabla\bar{v})\Psi_\varepsilon + (\nabla\Psi_\varepsilon)\bar{v}]\otimes\bar{n} + (\bar{v}\cdot\Psi_\varepsilon)\nabla\bar{n}, \\ \left(\bar{v}\cdot\overline{\nabla_\Gamma g}\right)\overline{Q} = \left[\left(\bar{n}\otimes\overline{\nabla_\Gamma g}\right)\bar{v}\right]\otimes\bar{n}\end{gathered}$$

in N. Hence

$$\begin{aligned}\left|\nabla E_\varepsilon v - \left\{\overline{\nabla_\Gamma v} + \frac{1}{\bar{g}}\left(\bar{v}\cdot\overline{\nabla_\Gamma g}\right)\overline{Q}\right\}\right| &\leq \left|\nabla\bar{v} - \overline{\nabla_\Gamma v}\right| + |\{(\nabla\bar{v})\Psi_\varepsilon\}\otimes\bar{n}| + |(\bar{v}\cdot\Psi_\varepsilon)\nabla\bar{n}| \\ &\quad + \left|\left[\left(\nabla\Psi_\varepsilon - \frac{1}{\bar{g}}\bar{n}\otimes\overline{\nabla_\Gamma g}\right)\bar{v}\right]\otimes\bar{n}\right|.\end{aligned} \tag{5.3.46}$$

Since $\nabla\Psi_\varepsilon = \overline{P}\nabla\Psi_\varepsilon + \overline{Q}\nabla\Psi_\varepsilon = \overline{P}\nabla\Psi_\varepsilon + \bar{n}\otimes\partial_n\Psi_\varepsilon$, by (5.3.38) we get

$$\left|\nabla\Psi_\varepsilon - \frac{1}{\bar{g}}\bar{n}\otimes\overline{\nabla_\Gamma g}\right| \leq \left|\overline{P}\nabla\Psi_\varepsilon\right| + \left|\bar{n}\otimes\left(\partial_n\Psi_\varepsilon - \frac{1}{\bar{g}}\overline{\nabla_\Gamma g}\right)\right| \leq c\varepsilon \quad \text{in} \quad \Omega_\varepsilon.$$

Applying this inequality, (5.2.13), (5.2.14), (5.2.17), (5.3.37), and $|d| \leq c\varepsilon$ in Ω_ε to the right-hand side of (5.3.46) we obtain (5.3.44). Also, since $\operatorname{tr}[Q] = n\cdot n = 1$,

$$\operatorname{tr}\left[\nabla_\Gamma v + \frac{1}{g}(v\cdot\nabla_\Gamma g)Q\right] = \operatorname{div}_\Gamma v + \frac{1}{g}(v\cdot\nabla_\Gamma g) = \frac{1}{g}\operatorname{div}_\Gamma(gv) \quad \text{on} \quad \Gamma.$$

Hence the inequality (5.3.45) follows from (5.3.44). □

As a consequence of Lemma 5.3.13 we have the L^p-estimate for $\mathrm{div}(E_\varepsilon v)$ on Ω_ε.

Lemma 5.3.14. *Let $p \in [1,\infty)$. There exists $c > 0$ independent of ε such that*

$$\|\mathrm{div}(E_\varepsilon v)\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon^{1/p} \left(\|\mathrm{div}_\Gamma(gv)\|_{L^p(\Gamma)} + \varepsilon\|v\|_{W^{1,p}(\Gamma)}\right) \tag{5.3.47}$$

for all $v \in W^{1,p}(\Gamma, T\Gamma)$. In particular, if v satisfies $\mathrm{div}_\Gamma(gv) = 0$ on Γ, then

$$\|\mathrm{div}(E_\varepsilon v)\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon^{1+1/p}\|v\|_{W^{1,p}(\Gamma)}. \tag{5.3.48}$$

Proof. By (5.2.30) and (5.3.45) we have

$$|\mathrm{div}(E_\varepsilon v)| \leq c\left\{\left|\overline{\mathrm{div}_\Gamma(gv)}\right| + \varepsilon\left(|\bar{v}| + \left|\overline{\nabla_\Gamma v}\right|\right)\right\} \quad \text{in} \quad \Omega_\varepsilon.$$

The inequality (5.3.47) follows from this inequality and (5.2.53). □

5.4 Korn inequalities on a thin domain and a surface

In this section we establish the Korn inequalities on Ω_ε and Γ, which play a fundamental role in the study of the Stokes operator on Ω_ε associated with slip boundary conditions and the corresponding limit operator on Γ.

5.4.1 Uniform Korn inequalities on a thin domain

For the proof of the global existence of a strong solution to (5.1.1)–(5.1.3), it is essential that the bilinear form corresponding to the Stokes operator on Ω_ε with slip boundary conditions is uniformly coercive in ε (see Section 5.5.2). To show the uniform coerciveness of the bilinear form, let us prove the uniform Korn inequalities on Ω_ε. First we give an estimate for the L^2-norm of the gradient matrix of a vector field on Ω_ε.

Lemma 5.4.1. *There exists a constant $c_{K,1} > 0$ independent of ε such that*

$$\|\nabla u\|^2_{L^2(\Omega_\varepsilon)} \leq 4\|D(u)\|^2_{L^2(\Omega_\varepsilon)} + c_{K,1}\|u\|^2_{L^2(\Omega_\varepsilon)} \tag{5.4.1}$$

for all $\varepsilon \in (0,1)$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying (5.3.20) on Γ_ε.

Let us prove an auxiliary density result.

Lemma 5.4.2. *Let $u \in H^1(\Omega_\varepsilon)^3$ satisfy (5.3.20) on Γ_ε. Then there exists a sequence $\{u_k\}_{k=1}^\infty$ in $C^2(\overline{\Omega}_\varepsilon)^3$ such that u_k satisfies (5.3.20) on Γ_ε for each $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} \|u - u_k\|_{H^1(\Omega_\varepsilon)} = 0.$$

Proof. We follow the idea of the proof of [8, Theorem IV.4.7], but here it is not necessary to localize a vector field on Ω_ε. For $x \in N$ we define

$$\tilde{n}(x) := \frac{1}{\varepsilon\bar{g}(x)}\left\{\left(d(x) - \varepsilon\bar{g}_0(x)\right)\bar{n}^1_\varepsilon(x) + \left(\varepsilon\bar{g}_1(x) - d(x)\right)\bar{n}^0_\varepsilon(x)\right\},$$

where n^0_ε and n^1_ε are given by (5.2.33) and $\bar{\eta} = \eta \circ \pi$ denotes the constant extension of a function η on Γ. Then $\tilde{n} \in C^2(N)$ by the regularity of Γ, g_0, and g_1. Moreover, $\tilde{n} = n_\varepsilon$ on Γ_ε by Lemma 5.2.10. Hence if $u \in H^1(\Omega_\varepsilon)^3$ satisfies (5.3.20) on Γ_ε, then we have $u \cdot \tilde{n} \in H^1_0(\Omega_\varepsilon)$

and $w := u - (u \cdot \tilde{n})\tilde{n} \in H^1(\Omega_\varepsilon)^3$. Since Γ_ε is of class C^1, there exist sequences $\{\varphi_k\}_{k=1}^\infty$ in $C_c^\infty(\Omega_\varepsilon)$ and $\{w_k\}_{k=1}^\infty$ in $C^\infty(\overline{\Omega}_\varepsilon)^3$ such that

$$\lim_{k\to\infty} \|u \cdot \tilde{n} - \varphi_k\|_{H^1(\Omega_\varepsilon)} = \lim_{k\to\infty} \|w - w_k\|_{H^1(\Omega_\varepsilon)} = 0.$$

Here $C_c^\infty(\Omega_\varepsilon)$ is the space of all smooth and compactly supported functions on Ω_ε. Therefore, setting $u_k := \varphi_k\tilde{n} + w_k - (w_k \cdot \tilde{n})\tilde{n} \in C^2(\overline{\Omega}_\varepsilon)$ we see that

$$u_k \cdot n_\varepsilon = u_k \cdot \tilde{n} = \varphi_k = 0 \quad \text{on} \quad \Gamma_\varepsilon$$

for each $k \in \mathbb{N}$ and (note that $u = (u \cdot \tilde{n})\tilde{n} + w$ and $w \cdot \tilde{n} = 0$ in Ω_ε)

$$\begin{aligned}\|u - u_k\|_{H^1(\Omega_\varepsilon)} &= \|(u \cdot \tilde{n} - \varphi_k)\tilde{n} + (w - w_k) - \{(w - w_k) \cdot \tilde{n}\}\tilde{n}\|_{H^1(\Omega_\varepsilon)} \\ &\leq c\left(\|u \cdot \tilde{n} - \varphi_k\|_{H^1(\Omega_\varepsilon)} + \|w - w_k\|_{H^1(\Omega_\varepsilon)}\right) \to 0\end{aligned}$$

as $k \to \infty$. $\square$

Proof of Lemma 5.4.1. By Lemma 5.4.2 it is sufficient to show (5.4.1) for all $u \in C^2(\overline{\Omega}_\varepsilon)^3$ satisfying (5.3.20) on Γ_ε. Since $2|D(u)|^2 = |\nabla u|^2 + \nabla u : (\nabla u)^T$,

$$2\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 = \|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 + \int_{\Omega_\varepsilon} \nabla u : (\nabla u)^T\, dx.$$

To the second them on the right-hand side we apply integration by parts twice (note that u is of class C^2 on $\overline{\Omega}_\varepsilon$) and then use (5.3.20) on Γ_ε to get

$$\int_{\Omega_\varepsilon} \nabla u : (\nabla u)^T\, dx = \int_{\Omega_\varepsilon} (\operatorname{div} u)^2\, dx + \int_{\Gamma_\varepsilon} (u \cdot \nabla)u \cdot n_\varepsilon\, d\mathcal{H}^2.$$

Here the first them on the right-hand side is nonnegative. Therefore,

$$\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \leq 2\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 + \left|\int_{\Gamma_\varepsilon} (u \cdot \nabla)u \cdot n_\varepsilon\, d\mathcal{H}^2\right|. \tag{5.4.2}$$

Let us estimate the integral over Γ_ε in (5.4.2). Since u satisfies (5.3.20) on Γ_ε, we can apply (5.3.28) to the integrand of the boundary integral to get

$$\int_{\Gamma_\varepsilon} (u \cdot \nabla)u \cdot n_\varepsilon\, d\mathcal{H}^2 = \int_{\Gamma_\varepsilon} u \cdot W_\varepsilon u\, d\mathcal{H}^2 = \sum_{i=0,1} \int_{\Gamma_\varepsilon^i} u \cdot W_\varepsilon u\, d\mathcal{H}^2. \tag{5.4.3}$$

To estimate the right-hand side we set

$$\begin{aligned}F_i(y) &:= \sqrt{1 + \varepsilon^2|\tau_\varepsilon^i(y)|^2}\, W_{\varepsilon,i}^\sharp(y), \quad i = 0, 1, \\ F(y, r) &:= \frac{1}{\varepsilon g(y)}\left\{\big(r - \varepsilon g_0(y)\big)F_1(y) - \big(\varepsilon g_1(y) - r\big)F_0(y)\right\}, \\ \varphi(y, r) &:= u^\sharp(y, r) \cdot F(y, r)u^\sharp(y, r)J(y, r)\end{aligned} \tag{5.4.4}$$

for $y \in \Gamma$ and $r \in [\varepsilon g_0(y), \varepsilon g_1(y)]$. Here and in what follows we use the notations (5.2.46) and (5.2.47), and we sometimes suppress the arguments y and r. By (5.4.4) we observe that

$$[u \cdot W_\varepsilon u]_i^\sharp(y)\sqrt{1 + \varepsilon^2|\tau_\varepsilon^i(y)|^2}J(y, \varepsilon g_i(y)) = (-1)^{i+1}\varphi(y, \varepsilon g_i(y)), \quad y \in \Gamma,\, i = 0, 1.$$

From this relation and (5.2.57) we deduce that

$$\begin{aligned}\sum_{i=0,1}\int_{\Gamma_\varepsilon^i}[u\cdot W_\varepsilon u](x)\,d\mathcal{H}^2(x)&=\int_\Gamma\{\varphi(y,\varepsilon g_1(y))-\varphi(y,\varepsilon g_0(y))\}\,d\mathcal{H}^2(y)\\&=\int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}\frac{\partial\varphi}{\partial r}(y,r)\,dr\,d\mathcal{H}^2(y).\end{aligned}\tag{5.4.5}$$

To estimate the integrand on the last line, we use (5.2.49) to get

$$\left|\frac{\partial\varphi}{\partial r}\right|\leq c\left\{\left(|F|+\left|\frac{\partial F}{\partial r}\right|\right)|u^\sharp|^2+|F||u^\sharp||(\nabla u)^\sharp|\right\}.\tag{5.4.6}$$

By (5.2.34) and the uniform boundedness in ε of W_ε on Γ_ε we observe that F_0 and F_1 are bounded on Γ uniformly in ε. Thus we have

$$|F(y,r)|\leq\frac{c}{\varepsilon g(y)}\big\{\big(r-\varepsilon g_0(y)\big)+\big(\varepsilon g_1(y)-r\big)\big\}=c\tag{5.4.7}$$

for $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$. Also, by $\partial F/\partial r=(\varepsilon g)^{-1}(F_1+F_0)$ and (5.4.4),

$$\left|\frac{\partial F}{\partial r}\right|\leq c\varepsilon^{-1}\left(|W_{\varepsilon,1}^\sharp+W_{\varepsilon,0}^\sharp|+\sum_{i=0,1}\left(\sqrt{1+\varepsilon^2|\tau_\varepsilon^i|^2}-1\right)|W_{\varepsilon,i}^\sharp|\right).\tag{5.4.8}$$

By the mean value theorem for the function $\sqrt{1+s}$, $s\geq 0$ and (5.2.34) we have

$$(0\leq)\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}-1\leq\frac{\varepsilon^2}{2}|\tau_\varepsilon^i(y)|^2\leq c\varepsilon^2,\quad y\in\Gamma.\tag{5.4.9}$$

We apply this inequality, (5.2.45) with $G_\varepsilon=W_\varepsilon$, and the uniform boundedness in ε of W_ε to the right-hand side of (5.4.8) to obtain

$$\left|\frac{\partial F}{\partial r}(y,r)\right|\leq c\quad\text{for all}\quad y\in\Gamma,\,r\in[\varepsilon g_0(y),\varepsilon g_1(y)].\tag{5.4.10}$$

From (5.4.6), (5.4.7), and (5.4.10) we deduce that

$$\left|\frac{\partial\varphi}{\partial r}(y,r)\right|\leq c\Big(|u^\sharp(y,r)|^2+\Big[|u^\sharp||(\nabla u)^\sharp|\Big](y,r)\Big)\tag{5.4.11}$$

for all $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$, where $c>0$ is a constant independent of ε. Applying (5.4.5) and (5.4.11) to (5.4.3) and using (5.2.52) and Holdör's inequality we see that

$$\begin{aligned}\left|\int_{\Gamma_\varepsilon}(u\cdot\nabla)u\cdot n_\varepsilon\,d\mathcal{H}^2\right|&\leq c\int_\Gamma\int_{\varepsilon g_0}^{\varepsilon g_1}\Big(|u^\sharp|^2+|u^\sharp||(\nabla u)^\sharp|\Big)\,dr\,d\mathcal{H}^2\\&\leq c\left(\|u\|_{L^2(\Omega_\varepsilon)}^2+\|u\|_{L^2(\Omega_\varepsilon)}\|\nabla u\|_{L^2(\Omega_\varepsilon)}\right).\end{aligned}\tag{5.4.12}$$

By (5.4.2), (5.4.12), and Young's inequality we obtain

$$\begin{aligned}\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2&\leq 2\|D(u)\|_{L^2(\Omega_\varepsilon)}^2+c\left(\|u\|_{L^2(\Omega_\varepsilon)}^2+\|u\|_{L^2(\Omega_\varepsilon)}\|\nabla u\|_{L^2(\Omega_\varepsilon)}\right)\\&\leq 2\|D(u)\|_{L^2(\Omega_\varepsilon)}^2+c\|u\|_{L^2(\Omega_\varepsilon)}^2+\frac{1}{2}\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2.\end{aligned}$$

Hence (5.4.1) follows. □

Next we show a uniform H^1-estimate for a vector field u on Ω_ε with $u \cdot n_\varepsilon = 0$ on Γ_ε by the L^2-norm of the strain rate tensor on Ω_ε. To this end, we need to impose another condition on u (see (5.4.14) and Remark 5.4.6).

Lemma 5.4.3. *For given $\alpha > 0$ and $\beta \in [0,1)$ there exist constants*

$$\varepsilon_K = \varepsilon_K(\alpha,\beta) \in (0,1), \quad c_{K,2} = c_{K,2}(\alpha,\beta) > 0$$

independent of ε such that

$$\|u\|^2_{L^2(\Omega_\varepsilon)} \leq \alpha \|\nabla u\|^2_{L^2(\Omega_\varepsilon)} + c_{K,2}\|D(u)\|^2_{L^2(\Omega_\varepsilon)} \tag{5.4.13}$$

for all $\varepsilon \in (0,\varepsilon_K)$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying (5.3.20) *on Γ_ε and*

$$\left|(u,\bar{v})_{L^2(\Omega_\varepsilon)}\right| \leq \beta \|u\|_{L^2(\Omega_\varepsilon)}\|\bar{v}\|_{L^2(\Omega_\varepsilon)} \quad \text{for all} \quad v \in \mathcal{K}_g(\Gamma). \tag{5.4.14}$$

Here $\mathcal{K}_g(\Gamma)$ is the function space given by (5.1.8) *and $\bar{v} = v \circ \pi$.*

To prove Lemma 5.4.3 we use a change of variables formula to transform integrals over Ω_ε into those over the domain Ω_1 with fixed width (note that we assume $\overline{\Omega}_1 \subset N$ by scaling g_0 and g_1). Define a mapping $\Phi_\varepsilon \colon \Omega_1 \to \Omega_\varepsilon$ by

$$\Phi_\varepsilon(X) := \pi(X) + \varepsilon d(X)\bar{n}(X), \quad X \in \Omega_1. \tag{5.4.15}$$

We easily see that Φ_ε is bijective and its inverse $\Phi_\varepsilon^{-1} \colon \Omega_\varepsilon \to \Omega_1$ is given by

$$\Phi_\varepsilon^{-1}(x) := \pi(x) + \varepsilon^{-1} d(x)\bar{n}(x), \quad x \in \Omega_\varepsilon.$$

Also, by (5.2.6), (5.2.11), and (5.2.16) we have

$$\nabla\Phi_\varepsilon = \left(I_3 - d\overline{W}\right)^{-1}\left(I_3 - \varepsilon d\overline{W}\right)\overline{P} + \varepsilon\overline{Q} \quad \text{on} \quad \Omega_1.$$

Hence taking an orthonormal basis of $\mathbb{R}^3$ that consists of n and the other eigenvectors of W we get

$$\det \nabla\Phi_\varepsilon(X) = \varepsilon J(\pi(X), d(X))^{-1} J(\pi(X), \varepsilon d(X)), \quad X \in \Omega_1$$

and the change of variables formula

$$\int_{\Omega_\varepsilon} \varphi(x)\,dx = \varepsilon \int_{\Omega_1} \varphi(\Phi_\varepsilon(X)) J(\pi(X), d(X))^{-1} J(\pi(X), \varepsilon d(X))\,dX \tag{5.4.16}$$

for a function φ on Ω_ε. In particular, by (5.2.49) we have

$$c\varepsilon^{-1}\|\varphi\|^2_{L^2(\Omega_\varepsilon)} \leq \|\varphi \circ \Phi_\varepsilon\|^2_{L^2(\Omega_1)} \leq c'\varepsilon^{-1}\|\varphi\|^2_{L^2(\Omega_\varepsilon)} \tag{5.4.17}$$

for $\varphi \in L^2(\Omega_\varepsilon)$, where c and c' are positive constants independent of ε. By (5.4.17) and direct calculations of matrices we can show the following auxiliary inequalities for the proof of Lemma 5.4.3 (for a proof, see Appendix 5.A).

Lemma 5.4.4. *For $u \in H^1(\Omega_\varepsilon)^3$ let $U := u \circ \Phi_\varepsilon$. Then*

$$\varepsilon^{-1}\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \geq c\left(\left\|\overline{P}\nabla U\right\|_{L^2(\Omega_1)}^2 + \varepsilon^{-2}\|\partial_n U\|_{L^2(\Omega_1)}^2\right) \tag{5.4.18}$$

with a constant $c > 0$ independent of ε and u, where $\partial_n U = (\bar{n}\cdot\nabla)U$ is the normal derivative (with respect to Γ) of U. We also have

$$\varepsilon^{-1}\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 \geq c\left(\left\|\overline{P}F_\varepsilon(U)_S\overline{P}\right\|_{L^2(\Omega_1)}^2 + \varepsilon^{-2}\|\partial_n(U\cdot\bar{n})\|_{L^2(\Omega_1)}^2\right), \tag{5.4.19}$$

where $F_\varepsilon(U)_S = \{F_\varepsilon(U) + F_\varepsilon(U)^T\}/2$ is the symmetric part of the matrix

$$F_\varepsilon(U) := \left(I_3 - \varepsilon d\overline{W}\right)^{-1}\left(I_3 - d\overline{W}\right)\nabla U. \tag{5.4.20}$$

Proof of Lemma 5.4.3. Following the idea of the proof of [21, Lemma 4.14] we prove (5.4.13) by contradiction. Assume to the contrary that there exist a sequence $\{\varepsilon_k\}_{k=1}^\infty$ of positive numbers with $\lim_{k\to\infty}\varepsilon_k = 0$ and vector fields $u_k \in H^1(\Omega_{\varepsilon_k})^3$ satisfying (5.3.20) on Γ_{ε_k}, (5.4.14), and

$$\|u_k\|_{L^2(\Omega_{\varepsilon_k})}^2 > \alpha\|\nabla u_k\|_{L^2(\Omega_{\varepsilon_k})}^2 + k\|D(u_k)\|_{L^2(\Omega_{\varepsilon_k})}^2, \quad k\in\mathbb{N}. \tag{5.4.21}$$

For each $k \in \mathbb{N}$ let $U_k := u_k \circ \Phi_{\varepsilon_k} \in H^1(\Omega_1)^3$. Dividing both sides of (5.4.21) by ε_k and using (5.4.17), (5.4.18), and (5.4.19) we get

$$\begin{aligned}\|U_k\|_{L^2(\Omega_1)}^2 > c\alpha&\left(\left\|\overline{P}\nabla U_k\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n U_k\|_{L^2(\Omega_1)}^2\right)\\ &+ ck\left(\left\|\overline{P}F_{\varepsilon_k}(U_k)_S\overline{P}\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n(U_k\cdot\bar{n})\|_{L^2(\Omega_1)}^2\right),\end{aligned}$$

where $F_{\varepsilon_k}(U_k)_S$ is the symmetric part of the matrix $F_{\varepsilon_k}(U_k)$ given by (5.4.20). By this inequality we have $U_k \neq 0$ and thus we may assume

$$\|U_k\|_{L^2(\Omega_1)} = 1, \quad k\in\mathbb{N} \tag{5.4.22}$$

by replacing U_k with $U_k/\|U_k\|_{L^2(\Omega_1)}$. Then we get

$$\left\|\overline{P}\nabla U_k\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n U_k\|_{L^2(\Omega_1)}^2 < c\alpha^{-1}, \tag{5.4.23}$$

$$\left\|\overline{P}F_{\varepsilon_k}(U_k)_S\overline{P}\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n(U_k\cdot\bar{n})\|_{L^2(\Omega_1)}^2 < ck^{-1}. \tag{5.4.24}$$

From (5.4.22), (5.4.23), and

$$|\nabla U_k|^2 = \left|\overline{P}\nabla U_k\right|^2 + \left|\overline{Q}\nabla U_k\right|^2, \quad \left|\overline{Q}\nabla U_k\right| = |\bar{n}\otimes\partial_n U_k| = |\partial_n U_k|$$

it follows that $\{U_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega_1)^3$. Hence there exists a subsequence of $\{U_k\}_{k=1}^\infty$, which we refer to as $\{U_k\}_{k=1}^\infty$ again, that converges to some $U \in H^1(\Omega_1)^3$ weakly in $H^1(\Omega_1)^3$. By the compact embedding $H^1(\Omega_1) \hookrightarrow L^2(\Omega_1)$ we also see that $\{U_k\}_{k=1}^\infty$ converges to U strongly in $L^2(\Omega_1)^3$ and thus

$$\|U\|_{L^2(\Omega_1)} = \lim_{k\to\infty}\|U_k\|_{L^2(\Omega_1)} = 1 \tag{5.4.25}$$

by (5.4.22). Our goal is to show $U = 0$ on Ω_1, which contradicts with (5.4.25). Since

$$\lim_{k\to\infty}\|\partial_n U_k\|_{L^2(\Omega_1)} = 0 \tag{5.4.26}$$

by (5.4.23) and $\{U_k\}_{k=1}^\infty$ converges to U weakly in $H^1(\Omega_1)^3$, it follows that $\partial_n U = 0$ on Ω_1, i.e. U is independent of the normal direction of Γ. Hence setting

$$V(y) := U(y + g_0(y)n(y)), \quad y \in \Gamma$$

we can consider U as the constant extension of V, i.e. $U = \overline{V}$ on Ω_1. Moreover, from (5.3.8) with $\varepsilon = 1$ and $\partial_n \overline{V} = 0$ on Ω_1 we deduce that

$$\left\|U_k - \overline{V}\right\|_{L^2(\Gamma_1)} \leq c\left(\left\|U_k - \overline{V}\right\|_{L^2(\Omega_1)} + \|\partial_n U_k\|_{L^2(\Omega_1)}\right).$$

Thus, by the strong convergence of $\{U_k\}_{k=1}^\infty$ to $\overline{V} = U$ in $L^2(\Omega_1)^3$ and (5.4.26),

$$\lim_{k\to\infty} \left\|U_k - \overline{V}\right\|_{L^2(\Gamma_1)} = 0. \tag{5.4.27}$$

Now let us prove that $V \in \mathcal{K}_g(\Gamma)$. Since $\overline{V} = U \in H^1(\Omega_\varepsilon)^3$, we have $V \in H^1(\Gamma)^3$ (see Lemma 5.2.13). To show that V is tangential on Γ we recall that the vector field u_k satisfies (5.3.20) on Γ_{ε_k}. Hence we can use (5.3.22) to get

$$|u_k \cdot \bar{n}| \leq c\varepsilon_k |u_k| \quad \text{on} \quad \Gamma_{\varepsilon_k}, \quad \text{i.e.} \quad |U_k \cdot \bar{n}| \leq c\varepsilon_k |U_k| \quad \text{on} \quad \Gamma_1.$$

From this inequality and (5.3.8) with $\varepsilon = 1$ it follows that

$$\|U_k \cdot \bar{n}\|_{L^2(\Gamma_1)} \leq c\varepsilon_k \|U_k\|_{L^2(\Gamma_1)} \leq c\varepsilon_k \|U_k\|_{H^1(\Omega_1)} \to 0 \quad \text{as} \quad k \to \infty \tag{5.4.28}$$

since $\{U_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega_1)^3$. Combining this with (5.4.27) we get $\overline{V} \cdot \bar{n} = 0$ on Γ_1, which implies that $V \cdot n = 0$ on Γ, i.e. V is tangential on Γ.

Next we show that $D_\Gamma(V) = 0$ on Γ. To this end we observe that

$$F_{\varepsilon_k}(U_k) = \left(I_3 - \varepsilon_k d\overline{W}\right)^{-1}\left(I_3 - d\overline{W}\right)\nabla U_k$$

converges weakly to $\overline{\nabla_\Gamma V}$ in $L^2(\Omega_1)^{3\times 3}$. Indeed, by (5.2.10) and $|d| \leq c$ on Ω_1,

$$\left|I_3 - \left\{I_3 - \varepsilon_k d(X)\overline{W}(X)\right\}^{-1}\right| \leq c\varepsilon_k |d(X)| \leq c\varepsilon_k \to 0 \quad \text{as} \quad k \to \infty$$

for all $X \in \Omega_1$, and thus $(I_3 - \varepsilon_k d\overline{W})^{-1}$ converges uniformly to I_3 on Ω_1. By this fact, the weak convergence of $\{U_k\}_{k=1}^\infty$ to $U = \overline{V}$ in $H^1(\Omega_1)^3$, and (5.2.12) we get

$$\lim_{k\to\infty} F_{\varepsilon_k}(U_k) = \left(I_3 - d\overline{W}\right)\nabla\overline{V} = \overline{\nabla_\Gamma V} \quad \text{weakly in} \quad L^2(\Omega_1)^{3\times 3}.$$

and thus

$$\lim_{k\to\infty} \overline{P} F_{\varepsilon_k}(U_k)_S \overline{P} = \overline{P}\left(\overline{\nabla_\Gamma V}\right)_S \overline{P} = \overline{D_\Gamma(V)} \quad \text{weakly in} \quad L^2(\Omega_1)^{3\times 3}.$$

From this fact and

$$\lim_{k\to\infty} \left\|\overline{P} F_{\varepsilon_k}(U_k)_S \overline{P}\right\|_{L^2(\Omega_1)} = 0$$

by (5.4.24) it follows that $\overline{D_\Gamma(V)} = 0$ on Ω_1, i.e. $D_\Gamma(V) = 0$ on Γ.

It remains to prove $V\cdot\nabla_\Gamma g=0$ on Γ. In what follows, we use the notations (5.2.46) and (5.2.47) (with $\varepsilon=1$). Since u_k satisfies (5.3.20) on Γ_{ε_k}, we see by (5.3.22) that

$$u_k\cdot\bar\tau^i_{\varepsilon_k}=\varepsilon_k^{-1}u_k\cdot\bar n \quad\text{on}\quad \Gamma^i_{\varepsilon_k},\ i=0,1.$$

By this equality we get $U_k\cdot\bar\tau^i_{\varepsilon_k}=\varepsilon_k^{-1}U_k\cdot\bar n$ on Γ^i_1, $i=0,1$, or equivalently,

$$U^\sharp_{k,i}(y)\cdot\tau^i_{\varepsilon_k}(y)=\varepsilon_k^{-1}U^\sharp_{k,i}(y)\cdot n(y),\quad y\in\Gamma,\ i=0,1.$$

Hence

$$\|U^\sharp_{k,1}\cdot\tau^1_{\varepsilon_k}-U^\sharp_{k,0}\cdot\tau^0_{\varepsilon_k}\|_{L^2(\Gamma)}\leq\varepsilon_k^{-1}\|U^\sharp_{k,1}\cdot n-U^\sharp_{k,0}\cdot n\|_{L^2(\Gamma)}. \tag{5.4.29}$$

Moreover, since $\bar n^\sharp(y,r)=n(y)$ for $y\in\Gamma$ and $r\in(g_0(y),g_1(y))$, we have

$$(U^\sharp_{k,1}\cdot n)(y)-(U^\sharp_{k,0}\cdot n)(y)=\int_{g_0(y)}^{g_1(y)}\frac{\partial}{\partial r}\Big((U_k\cdot\bar n)^\sharp(y,r)\Big)\,dr=\int_{g_0(y)}^{g_1(y)}[\partial_n(U_k\cdot\bar n)]^\sharp(y,r)\,dr.$$

Hence by Hölder's inequality, (5.2.52), and (5.4.24),

$$\begin{aligned}\|U^\sharp_{k,1}\cdot n-U^\sharp_{k,0}\cdot n\|^2_{L^2(\Gamma)}&=\int_\Gamma\left(\int_{g_0(y)}^{g_1(y)}[\partial_n(U_k\cdot\bar n)]^\sharp(y,r)\,dr\right)^2d\mathcal{H}^2(y)\\&\leq c\|\partial_n(U_k\cdot\bar n)\|^2_{L^2(\Omega_1)}\leq c\varepsilon_k^2k^{-1}.\end{aligned}$$

Applying this inequality to the right-hand side of (5.4.29) we get

$$\|U^\sharp_{k,1}\cdot\tau^1_{\varepsilon_k}-U^\sharp_{k,0}\cdot\tau^0_{\varepsilon_k}\|_{L^2(\Gamma)}\leq ck^{-1/2}\to0\quad\text{as}\quad k\to\infty. \tag{5.4.30}$$

Also, by (5.2.34), (5.2.35), and (5.2.58),

$$\begin{aligned}\|U^\sharp_{k,i}\cdot\tau^i_{\varepsilon_k}-V\cdot\nabla_\Gamma g_i\|_{L^2(\Gamma)}&\leq\|(U^\sharp_{k,i}-V)\cdot\tau^i_{\varepsilon_k}\|_{L^2(\Gamma)}+\|V\cdot(\tau^i_{\varepsilon_k}-\nabla_\Gamma g_i)\|_{L^2(\Gamma)}\\&\leq c\left(\|U^\sharp_{k,i}-V\|_{L^2(\Gamma)}+\varepsilon_k\|V\|_{L^2(\Gamma)}\right)\\&\leq c\left(\left\|U_k-\overline{V}\right\|_{L^2(\Gamma^i_1)}+\varepsilon_k\|V\|_{L^2(\Gamma)}\right).\end{aligned}$$

Since the right-hand side tends to zero as $k\to\infty$ by (5.4.27), we get

$$\lim_{k\to\infty}\|U^\sharp_{k,i}\cdot\tau^i_{\varepsilon_k}-V\cdot\nabla_\Gamma g_i\|_{L^2(\Gamma)}=0,\quad i=0,1.$$

Combining this equality with (5.4.30) we obtain

$$\|V\cdot\nabla_\Gamma g_1-V\cdot\nabla_\Gamma g_0\|_{L^2(\Gamma)}=0,$$

i.e. $V\cdot\nabla_\Gamma g=0$ on Γ. Therefore, the vector field V is in $\mathcal{K}_g(\Gamma)$.

Finally, we recall that $u_k\in H^1(\Omega_{\varepsilon_k})^3$ satisfies (5.4.14) and thus

$$\left|\left(u_k,\overline{V}\right)_{L^2(\Omega_{\varepsilon_k})}\right|\leq\beta\|u_k\|_{L^2(\Omega_{\varepsilon_k})}\left\|\overline{V}\right\|_{L^2(\Omega_{\varepsilon_k})} \tag{5.4.31}$$

with $\beta \in [0,1)$ by $V \in \mathcal{K}_g(\Gamma)$. Let us express this inequality in terms of U_k and send $k \to \infty$. Noting that $\overline{V}(\Phi_{\varepsilon_k}(X)) = \overline{V}(X)$ for $X \in \Omega_1$ by $\pi(\Phi_{\varepsilon_k}(X)) = \pi(X)$, we use the change of variables formula (5.4.16) to get

$$\left(u_k, \overline{V}\right)_{L^2(\Omega_{\varepsilon_k})} = \varepsilon_k \int_{\Omega_1} U_k \overline{V} \varphi_k \, dX,$$

where $\varphi_k(X) := J(\pi(X), d(X))^{-1} J(\pi(X), \varepsilon_k d(X))$ for $X \in \Omega_1$. Here U_k converges strongly to $U = \overline{V}$ in $L^2(\Omega_1)^3$ as $k \to \infty$. Moreover, for all $X \in \Omega_1$ we see by (5.2.49) and (5.2.50) that

$$|\varphi_k(X) - J(\pi(X), d(X))^{-1}| \leq c\varepsilon_k \to 0 \quad \text{as} \quad k \to \infty$$

and thus φ_k converges uniformly to $J(\pi(\cdot), d(\cdot))^{-1}$ on Ω_1. Hence we obtain

$$\lim_{k\to\infty} \varepsilon_k^{-1} \left(u_k, \overline{V}\right)_{L^2(\Omega_{\varepsilon_k})} = \lim_{k\to\infty} \int_{\Omega_1} U_k \overline{V} \varphi_k \, dX = \int_{\Omega_1} \left|\overline{V}\right|^2 J(\pi(\cdot), d(\cdot))^{-1} \, dX.$$

By the change of variables formula (5.2.51) the last term is of the form

$$\int_\Gamma \int_{g_0(y)}^{g_1(y)} |V(y)|^2 J(y,r)^{-1} J(y,r) \, dr \, d\mathcal{H}^2(y) = \int_\Gamma g(y)|V(y)|^2 \, d\mathcal{H}^2(y).$$

Therefore,

$$\lim_{k\to\infty} \varepsilon_k^{-1} \left(u_k, \overline{V}\right)_{L^2(\Omega_{\varepsilon_k})} = \|g^{1/2} V\|_{L^2(\Gamma)}^2. \tag{5.4.32}$$

By the same arguments we also have

$$\lim_{k\to\infty} \varepsilon_k^{-1} \|u_k\|_{L^2(\Omega_{\varepsilon_k})}^2 = \lim_{k\to\infty} \varepsilon_k^{-1} \left\|\overline{V}\right\|_{L^2(\Omega_{\varepsilon_k})}^2 = \|g^{1/2} V\|_{L^2(\Gamma)}^2. \tag{5.4.33}$$

Now we divide both sides of (5.4.31) by ε_k, send $k \to \infty$, and use (5.4.32) and (5.4.33). Then we obtain

$$\|g^{1/2} V\|_{L^2(\Gamma)}^2 \leq \beta \|g^{1/2} V\|_{L^2(\Gamma)}^2.$$

Since $\beta < 1$, we observe by this inequality and (5.2.30) that $V = 0$ on Γ. This shows $U = \overline{V} = 0$ in Ω_1, which contradicts with (5.4.25). Hence (5.4.13) is valid. □

Lemma 5.4.5. *For given $\beta \in [0,1)$ there exist $\varepsilon_\beta \in (0,1)$ and $c_\beta > 0$ such that*

$$\|u\|_{H^1(\Omega_\varepsilon)}^2 \leq c_\beta \|D(u)\|_{L^2(\Omega_\varepsilon)}^2 \tag{5.4.34}$$

for all $\varepsilon \in (0, \varepsilon_\beta)$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying (5.3.20) *on Γ_ε and* (5.4.14).

Proof. Let $c_{K,1} > 0$ be the constant given in Lemma 5.4.1. Also, let $\varepsilon_K \in (0,1)$ and $c_{K,2} > 0$ be the constants given in Lemma 5.4.3 with $\alpha := 1/2c_{K,1}$. For $\varepsilon \in (0, \varepsilon_K)$ let $u \in H^1(\Omega_\varepsilon)^3$ satisfy (5.3.20) on Γ_ε and (5.4.14). By (5.4.1) and (5.4.13) we have

$$\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \leq (4 + c_{K,1}c_{K,2})\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 + c_{K,1}\alpha \|\nabla u\|_{L^2(\Omega_\varepsilon)}^2.$$

Since $\alpha = 1/2c_{K,1}$, the above inequality implies that

$$\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \leq c_{\beta,1} \|D(u)\|_{L^2(\Omega_\varepsilon)}^2, \quad c_{\beta,1} := 2(4 + c_{K,1}c_{K,2}). \tag{5.4.35}$$

From this inequality and (5.4.13) we further deduce that

$$\|u\|_{L^2(\Omega_\varepsilon)}^2 \leq c_{\beta,2} \|D(u)\|_{L^2(\Omega_\varepsilon)}^2, \quad c_{\beta,2} := 2(2c_{K,1}^{-1} + c_{K,2}). \tag{5.4.36}$$

By (5.4.35) and (5.4.36) we get (5.4.34) with $\varepsilon_\beta := \varepsilon_K$ and $c_\beta := c_{\beta,1} + c_{\beta,2}$. □

Remark 5.4.6. The uniform Korn inequality (5.4.34) was proved by Lewicka and Müller under a slightly more general assumptions on the curved thin domain Ω_ε, see [36, Theorem 2.2]. Here we gave another proof of the same inequality.

Lewicka and Müller also proved that (5.4.34) is not valid if we drop the condition (5.4.14). In the proof, they used a nontrivial Killing vector field in $\mathcal{K}_g(\Gamma)$ to construct a vector field $v_\varepsilon \in H^1(\Omega_\varepsilon)^3$ satisfying $v_\varepsilon \cdot n_\varepsilon = 0$ on Γ_ε but

$$\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \geq c\varepsilon, \quad \|D(v_\varepsilon)\|_{L^2(\Omega_\varepsilon)}^2 \leq c\varepsilon^3.$$

See [36, Section 4] for details.

5.4.2 Korn inequalities on a surface

In this subsection we prove the Korn inequalities on Γ, which are used in the study of a singular limit problem of the Navier–Stokes equations (5.1.1)–(5.1.3) as Ω_ε degenerates into Γ.

Lemma 5.4.7. *There exists a constant $c > 0$ such that*

$$\|\nabla_\Gamma v\|_{L^2(\Gamma)}^2 \leq c\left(\|D_\Gamma(v)\|_{L^2(\Gamma)}^2 + \|v\|_{L^2(\Gamma)}^2\right) \tag{5.4.37}$$

for all $v \in H^1(\Gamma, T\Gamma)$. Here $D_\Gamma(v)$ is the surface strain rate tensor given by

$$D_\Gamma(v) := P(\nabla_\Gamma v)_S P, \quad (\nabla_\Gamma v)_S = \frac{\nabla_\Gamma v + (\nabla_\Gamma v)^T}{2}. \tag{5.4.38}$$

Proof. For sufficiently small $\varepsilon > 0$ let $N_\varepsilon := \{x \in \mathbb{R}^3 \mid -\varepsilon < d(x) < \varepsilon\}$ be the tubular neighborhood of Γ such that $\overline{N_\varepsilon} \subset N$. As in the proof of Lemma 5.4.1 we can show that

$$\|\nabla u\|_{L^2(N_\varepsilon)}^2 \leq 4\|D(u)\|_{L^2(N_\varepsilon)}^2 + c\|u\|_{L^2(N_\varepsilon)}^2 \tag{5.4.39}$$

for all $u \in H^1(N_\varepsilon)^3$ satisfying $u \cdot n_{N_\varepsilon} = 0$ on ∂N_ε, where $c > 0$ is a constant independent of ε and n_{N_ε} is the unit outward normal vector field of the boundary ∂N_ε. Let $\bar{v} = v \circ \pi$ be the constant extension of $v \in H^1(\Gamma, T\Gamma)$. By Lemma 5.2.13 we see that $\bar{v} \in H^1(N_\varepsilon)^3$. Moreover, since N_ε is the tubular neighborhood of Γ, the unit outward normal n_{N_ε} is given by

$$n_{N_\varepsilon}(x) = \begin{cases} \bar{n}(x) & \text{if} \quad d(x) = \varepsilon, \\ -\bar{n}(x) & \text{if} \quad d(x) = -\varepsilon \end{cases}$$

for $x \in \partial N_\varepsilon$. By this fact and $v \cdot n = 0$ on Γ we see that $\bar{v} \cdot n_{N_\varepsilon} = 0$ on ∂N_ε. Hence we can apply (5.4.39) to $\bar{v}$ to get

$$\|\nabla \bar{v}\|_{L^2(N_\varepsilon)}^2 \leq 4\|D(\bar{v})\|_{L^2(N_\varepsilon)}^2 + c\|\bar{v}\|_{L^2(N_\varepsilon)}^2. \tag{5.4.40}$$

Let us derive (5.4.37) from (5.4.40). By (5.2.53) and (5.2.54) with Ω_ε replaced by N_ε,

$$\|\nabla \bar{v}\|_{L^2(N_\varepsilon)}^2 \geq c\varepsilon\|\nabla_\Gamma v\|_{L^2(\Gamma)}^2, \quad \|\bar{v}\|_{L^2(N_\varepsilon)}^2 \leq c\varepsilon\|v\|_{L^2(\Gamma)}^2. \tag{5.4.41}$$

To estimate the $L^2(N_\varepsilon)$-norm of $D(\bar{v})$ we see by (5.2.14) and $|d| \leq \varepsilon$ in N_ε that

$$\left|D(\bar{v}) - \left(\overline{\nabla_\Gamma v}\right)_S\right| \leq \left|\nabla \bar{v} - \overline{\nabla_\Gamma v}\right| \leq c\varepsilon \left|\overline{\nabla_\Gamma v}\right| \quad \text{in} \quad N_\varepsilon. \tag{5.4.42}$$

Moreover, by $I_3 = P + Q$, $\nabla_\Gamma v = P\nabla_\Gamma v$,

$$(\nabla_\Gamma v)Q = \{(\nabla_\Gamma v)n\} \otimes n = \{\nabla_\Gamma(v \cdot n) - (\nabla_\Gamma n)v\} \otimes n = (Wv) \otimes n \quad \text{on} \quad \Gamma,$$

which follows from $v \cdot n = 0$ and $-\nabla_\Gamma n = W$, and (5.2.6) we have

$$\nabla_\Gamma v = P(\nabla_\Gamma v)P + P(\nabla_\Gamma v)Q = P(\nabla_\Gamma v)P + (Wv) \otimes n \quad \text{on} \quad \Gamma.$$

Hence $(\nabla_\Gamma v)_S = D_\Gamma(v) + [(Wv) \otimes n]_S$ on Γ. From this equality, (5.4.42), and

$$|[(Wv) \otimes n]_S| \leq |(Wv) \otimes n| = |Wv||n| \leq c|v| \quad \text{on} \quad \Gamma$$

by $|n| = 1$ and the boundedness of W we deduce that

$$|D(\bar{v})| \leq \left|\left(\overline{\nabla_\Gamma v}\right)_S\right| + \left|D(\bar{v}) - \left(\overline{\nabla_\Gamma v}\right)_S\right| \leq \left|\overline{D_\Gamma(v)}\right| + c\left(|\bar{v}| + \varepsilon\left|\overline{\nabla_\Gamma v}\right|\right)$$

in N_ε. This inequality and (5.2.53) show that

$$\|D(\bar{v})\|_{L^2(N_\varepsilon)}^2 \leq c\varepsilon\left(\|D_\Gamma(v)\|_{L^2(\Gamma)}^2 + \|v\|_{L^2(\Gamma)}^2 + \varepsilon^2\|\nabla_\Gamma v\|_{L^2(\Gamma)}^2\right). \tag{5.4.43}$$

Now we apply (5.4.41) and (5.4.43) to (5.4.40) and then divide both sides by ε to get

$$\|\nabla_\Gamma v\|_{L^2(\Gamma)}^2 \leq c_1\left(\|D_\Gamma(v)\|_{L^2(\Gamma)}^2 + \|v\|_{L^2(\Gamma)}^2 + \varepsilon^2\|\nabla_\Gamma v\|_{L^2(\Gamma)}^2\right)$$

with some constant $c_1 > 0$ independent of ε. We take $\varepsilon > 0$ so small that $c_1\varepsilon^2 \leq 1/2$ in the above inequality to obtain

$$\|\nabla_\Gamma v\|_{L^2(\Gamma)}^2 \leq c_1\left(\|D_\Gamma(v)\|_{L^2(\Gamma)}^2 + \|v\|_{L^2(\Gamma)}^2\right) + \frac{1}{2}\|\nabla_\Gamma v\|_{L^2(\Gamma)}^2.$$

Hence the inequality (5.4.37) follows. □

Lemma 5.4.8. *For given $\beta \in [0,1)$, there exists $c_\beta > 0$ such that*

$$\|v\|_{H^1(\Gamma)}^2 \leq c_\beta\left(\|D_\Gamma(v)\|_{L^2(\Gamma)}^2 + \|v \cdot \nabla_\Gamma g\|_{L^2(\Gamma)}^2\right) \tag{5.4.44}$$

for all $v \in H^1(\Gamma, T\Gamma)$ satisfying

$$\left|(v,w)_{L^2(\Gamma)}\right| \leq \beta\|v\|_{L^2(\Gamma)}\|w\|_{L^2(\Gamma)} \quad \textit{for all} \quad w \in \mathcal{K}_g(\Gamma), \tag{5.4.45}$$

where $\mathcal{K}_g(\Gamma)$ is the function space given by (5.1.8)*.*

Proof. We prove the inequality

$$\|v\|_{L^2(\Gamma)}^2 \leq c\left(\|D_\Gamma(v)\|_{L^2(\Gamma)}^2 + \|v \cdot \nabla_\Gamma g\|_{L^2(\Gamma)}^2\right). \tag{5.4.46}$$

Assume to the contrary that for each $k \in \mathbb{N}$ there exists $v_k \in H^1(\Gamma, T\Gamma)$ such that

$$\|v_k\|_{L^2(\Gamma)}^2 > k\left(\|D_\Gamma(v_k)\|_{L^2(\Gamma)}^2 + \|v_k \cdot \nabla_\Gamma g\|_{L^2(\Gamma)}^2\right)$$

and v_k satisfies (5.4.45). Since $v_k \neq 0$, we may assume $\|v_k\|_{L^2(\Gamma)} = 1$ by replacing v_k with $v_k/\|v_k\|_{L^2(\Gamma)}$. Then by the above inequality we have

$$\|D_\Gamma(v_k)\|_{L^2(\Gamma)}^2 + \|v_k \cdot \nabla_\Gamma g\|_{L^2(\Gamma)}^2 < \frac{1}{k}. \tag{5.4.47}$$

By $\|v_k\|_{L^2(\Gamma)} = 1$, (5.4.37), and (5.4.47), the sequence $\{v_k\}_{k=1}^\infty$ is bounded in $H^1(\Gamma, T\Gamma)$. Hence, using the compact embedding $H^1(\Gamma, T\Gamma) \hookrightarrow L^2(\Gamma, T\Gamma)$, we can take a subsequence of $\{v_k\}_{k=1}^\infty$, which we denote by $\{v_k\}_{k=1}^\infty$ again, that converges to some $v \in H^1(\Gamma, T\Gamma)$ weakly in $H^1(\Gamma, T\Gamma)$ and strongly in $L^2(\Gamma, T\Gamma)$, and thus

$$\|v\|_{L^2(\Gamma)} = \lim_{k\to\infty} \|v_k\|_{L^2(\Gamma)} = 1. \tag{5.4.48}$$

Moreover, since $\{D_\Gamma(v_k)\}_{k=1}^\infty$ and $\{v_k \cdot \nabla_\Gamma g\}_{k=1}^\infty$ converge to $D_\Gamma(v)$ and $v \cdot \nabla_\Gamma g$ weakly in $L^2(\Gamma)^{3\times 3}$ and $L^2(\Gamma)$, we send $k \to \infty$ in (5.4.47) to get

$$D_\Gamma(v) = 0, \quad v \cdot \nabla_\Gamma g = 0 \quad \text{on} \quad \Gamma, \quad \text{i.e.} \quad v \in \mathcal{K}_g(\Gamma).$$

Now we recall that v_k satisfies (5.4.45). Hence by $\|v_k\|_{L^2(\Gamma)} = 1$ and (5.4.48) we have

$$\left|(v_k, v)_{L^2(\Gamma)}\right| \leq \beta \|v_k\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} = \beta \quad \text{for all} \quad k \in \mathbb{N}.$$

We send $k \to \infty$ in the above equality and then use the strong convergence of $\{v_k\}_{k=1}^\infty$ to v in $L^2(\Gamma, T\Gamma)$ and (5.4.48). Then we have $1 \leq \beta$, which contradicts with $\beta \in [0,1)$. Hence (5.4.46) holds and we get (5.4.44) by combining (5.4.37) and (5.4.46). $\square$

5.5 Stokes operator with slip boundary conditions

In this section we prove inequalities for the Helmholtz–Leray projection on Ω_ε and the Stokes operator associated with slip boundary conditions. We use the inequality (5.5.1) given in Section 5.5.1 for the study of a singular limit problem and inequalities and formulas in other subsections for the proof of the global existence of a strong solution.

5.5.1 Helmholtz–Leray projection on a thin domain

Let $L^2_\sigma(\Omega_\varepsilon)$ be the closure of the space $C^\infty_{c,\sigma}(\Omega_\varepsilon) := \{u \in C^\infty_c(\Omega_\varepsilon)^3 \mid \operatorname{div} u = 0 \text{ in } \Omega_\varepsilon\}$ in $L^2(\Omega_\varepsilon)^3$. It is well-known (see e.g. [8, 14, 64]) that $L^2_\sigma(\Omega_\varepsilon)$ is characterized by

$$L^2_\sigma(\Omega_\varepsilon) = \{u \in L^2(\Omega_\varepsilon)^3 \mid \operatorname{div} u = 0 \text{ in } \Omega_\varepsilon,\ u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}$$

and the Helmholtz–Leray decomposition $L^2(\Omega_\varepsilon)^3 = L^2_\sigma(\Omega_\varepsilon) \oplus L^2_\sigma(\Omega_\varepsilon)^\perp$ holds with

$$L^2_\sigma(\Omega_\varepsilon)^\perp = \{\nabla p \in L^2(\Omega_\varepsilon)^3 \mid p \in H^1(\Omega_\varepsilon)\}.$$

Let $\mathbb{P}_\varepsilon$ be the Helmholtz–Leray projection from $L^2(\Omega_\varepsilon)^3$ onto $L^2_\sigma(\Omega_\varepsilon)$. It is given by $\mathbb{P}_\varepsilon u = u - \nabla\varphi$ for $u \in L^2(\Omega_\varepsilon)^3$, where $\varphi \in H^1(\Omega_\varepsilon)$ is a weak solution to the Neumann problem of Poisson's equation

$$\Delta\varphi = \operatorname{div} u \quad \text{in} \quad \Omega_\varepsilon, \quad \frac{\partial\varphi}{\partial n_\varepsilon} = u \cdot n_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon.$$

Moreover, by the elliptic regularity theorem (see [13, 15]) we have $\mathbb{P}_\varepsilon u \in H^1(\Omega_\varepsilon)^3$ when $u \in H^1(\Omega_\varepsilon)^3$. Our aim is to give a uniform estimate for the $H^1(\Omega_\varepsilon)$-norm of the difference $u - \mathbb{P}_\varepsilon u$ for $u \in H^1(\Omega_\varepsilon)^3$ satisfying $u \cdot n_\varepsilon = 0$ on Γ_ε.

Lemma 5.5.1. *There exist constants $\varepsilon_\sigma \in (0,1)$ and $c > 0$ such that*

$$\|u - \mathbb{P}_\varepsilon u\|_{H^1(\Omega_\varepsilon)} \leq c\|\operatorname{div} u\|_{L^2(\Omega_\varepsilon)} \tag{5.5.1}$$

for all $\varepsilon \in (0, \varepsilon_\sigma)$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying $u \cdot n_\varepsilon = 0$ on Γ_ε.

To prove Lemma 5.5.1 we first derive the uniform Poincaré inequality for $\varphi \in H^1(\Omega_\varepsilon)$ whose integral over Ω_ε vanishes.

Lemma 5.5.2. *There exist constants $\varepsilon_\sigma \in (0,1)$ and $c > 0$ such that*

$$\|\varphi\|_{L^2(\Omega_\varepsilon)} \leq c\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)} \tag{5.5.2}$$

for all $\varepsilon \in (0,\varepsilon_\sigma)$ and $\varphi \in H^1(\Omega_\varepsilon)$ satisfying $\int_{\Omega_\varepsilon} \varphi\,dx = 0$.

Proof. As in the proof of Lemma 5.4.4, we prove (5.5.2) by contradiction. Assume to the contrary that there exist a sequence $\{\varepsilon_k\}_{k=1}^\infty$ of positive numbers that converges to zero and $\varphi_k \in H^1(\Omega_{\varepsilon_k})$ such that

$$\|\varphi_k\|_{L^2(\Omega_{\varepsilon_k})}^2 > k\|\nabla\varphi_k\|_{L^2(\Omega_{\varepsilon_k})}^2, \quad \int_{\Omega_{\varepsilon_k}} \varphi_k\,dx = 0, \quad k \in \mathbb{N}. \tag{5.5.3}$$

For each $k \in \mathbb{N}$ let Φ_{ε_k} be the bijection from Ω_1 onto Ω_{ε_k} given by (5.4.15) and define $\xi_k := \varphi_k \circ \Phi_{\varepsilon_k} \in H^1(\Omega_1)$. We divide both sides of the first inequality of (5.5.3) by ε_k and apply (5.4.17) and (5.4.18) to $u = (\varphi_k, 0, 0)$ and $U = (\xi_k, 0, 0)$ to get

$$\|\xi_k\|_{L^2(\Omega_1)}^2 > ck\left(\left\|\overline{P}\nabla\xi_k\right\|_{L^2(\Omega_1)}^2 + \varepsilon_k^{-2}\|\partial_n\xi_k\|_{L^2(\Omega_1)}^2\right),$$

where $\partial_n\xi_k$ is the normal derivative of ξ_k given by (5.3.5). Since $\|\xi_k\|_{L^2(\Omega_1)} > 0$, we may assume that $\|\xi_k\|_{L^2(\Omega_1)} = 1$ by replacing ξ_k with $\xi_k/\|\xi_k\|_{L^2(\Omega_1)}$. Then by the above inequality we get

$$\left\|\overline{P}\nabla\xi_k\right\|_{L^2(\Omega_1)}^2 < ck^{-1}, \quad \|\partial_n\xi_k\|_{L^2(\Omega_1)}^2 < c\varepsilon_k^2 k^{-1}. \tag{5.5.4}$$

By these inequalities, $\|\xi_k\|_{L^2(\Omega_1)} = 1$, and

$$|\nabla\xi_k|^2 = \left|\overline{P}\nabla\xi_k\right|^2 + \left|\overline{Q}\nabla\xi_k\right|^2, \quad \left|\overline{Q}\nabla\xi_k\right| = |\bar{n} \otimes \partial_n\xi_k| = |\partial_n\xi_k|,$$

we observe that $\{\xi_k\}_{k=1}^\infty$ is bounded in $H^1(\Omega_1)$. From this fact and the compact embedding $H^1(\Omega_1) \hookrightarrow L^2(\Omega_1)$ we deduce that there exists a subsequence of $\{\xi_k\}_{k=1}^\infty$, which we denote by $\{\xi_k\}_{k=1}^\infty$ again, that converges to some $\xi \in H^1(\Omega_1)$ strongly in $L^2(\Omega_1)$ and weakly in $H^1(\Omega_1)$. Hence

$$\|\xi\|_{L^2(\Omega_1)} = \lim_{k\to\infty} \|\xi_k\|_{L^2(\Omega_1)} = 1. \tag{5.5.5}$$

By the weak convergence of $\{\xi_k\}_{k=1}^\infty$ to ξ in $H^1(\Omega_1)$ and (5.5.4) we have $\overline{P}\nabla\xi = 0$ and $\partial_n\xi = 0$ in Ω_1. For $y \in \Gamma$ let $\eta(y) := \xi(y + g_0(y)n(y))$. Then by $\partial_n\xi = 0$ we see that $\xi = \bar{\eta}$ is the constant extension of η, and thus $\eta \in H^1(\Gamma)$ by $\xi \in H^1(\Omega_1)$ and Lemma 5.2.13. Moreover, by $\overline{P}\nabla\xi = 0$, (5.2.3), (5.2.8), and (5.2.12) we have

$$\overline{P}\left(I_3 - d\overline{W}\right)^{-1}\overline{\nabla_\Gamma\eta} = \left(I_3 - d\overline{W}\right)^{-1}\overline{\nabla_\Gamma\eta} = 0 \quad \text{in} \quad \Omega_1,$$

which implies that $\overline{\nabla_\Gamma\eta} = 0$ in Ω_1, i.e. $\nabla_\Gamma\eta = 0$ on Γ. Hence setting

$$\hat{\eta} := \eta - \frac{1}{|\Gamma|}\int_\Gamma \eta\,d\mathcal{H}^2 \quad \text{on} \quad \Gamma,$$

where $|\Gamma|$ is the area of Γ, we can apply Poincaré's inequality (5.2.20) to $\hat{\eta}$ to get

$$\|\hat{\eta}\|_{L^2(\Gamma)} \leq c\|\nabla_\Gamma \hat{\eta}\|_{L^2(\Gamma)} = c\|\nabla_\Gamma \eta\|_{L^2(\Gamma)} = 0$$

and thus $\hat{\eta} = 0$ on Γ, i.e. $\eta = |\Gamma|^{-1}\int_\Gamma \eta\, d\mathcal{H}^2$ is constant. Now we return to the second equality of (5.5.3) and use the change of variables formula (5.4.16) to get

$$\int_{\Omega_1} \xi_k(X)J(\pi(X),d(X))^{-1}J(\pi(X),\varepsilon_k d(X))\,dX = 0.$$

Let $k \to \infty$ in this equality. Then since $J(\pi(\cdot),\varepsilon_k d(\cdot))$ converges to one uniformly in Ω_1 as $k \to \infty$ by (5.2.50) and $\{\xi_k\}_{k=1}^\infty$ converges to $\xi = \bar{\eta}$ strongly in $L^2(\Omega_1)$,

$$\int_{\Omega_1} \eta(\pi(X))J(\pi(X),d(X))^{-1}\,dX = 0.$$

Moreover, by the change of variables formula (5.2.51) and the fact that η is constant on Γ, we obtain

$$\eta\int_\Gamma\int_{g_0(y)}^{g_1(y)} J(y,r)^{-1}J(y,r)\,dr\,d\mathcal{H}^2(y) = \eta\int_\Gamma g(y)\,d\mathcal{H}^2(y) = 0,$$

which together with (5.2.30) yields that $\eta = 0$. Hence

$$\|\xi\|_{L^2(\Omega_1)} = \|\bar{\eta}\|_{L^2(\Omega_1)} = 0,$$

which contradicts with (5.5.5). Therefore, the uniform inequality (5.5.2) is valid. □

Next we consider the Neumann problem of Poisson's equation

$$\Delta\varphi = -\xi \quad \text{in} \quad \Omega_\varepsilon, \quad \frac{\partial\varphi}{\partial n_\varepsilon} = 0 \quad \text{on} \quad \Gamma_\varepsilon \tag{5.5.6}$$

for $\xi \in H^{-1}(\Omega_\varepsilon)$ satisfying $\langle\xi,1\rangle_{\Omega_\varepsilon} = 0$, where $\langle\cdot,\cdot\rangle_{\Omega_\varepsilon}$ denotes the duality product between $H^{-1}(\Omega_\varepsilon)$ and $H^1(\Omega_\varepsilon)$. By the Lax–Milgram theory there exists a unique weak solution $\varphi \in H^1(\Omega_\varepsilon)$ satisfying

$$(\nabla\varphi,\nabla\zeta)_{L^2(\Omega_\varepsilon)} = \langle\xi,\zeta\rangle_{\Omega_\varepsilon} \quad \text{for all} \quad \zeta \in H^1(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon}\varphi\,dx = 0. \tag{5.5.7}$$

Moreover, by the elliptic regularity theorem, if $\xi \in L^2(\Omega_\varepsilon)$ then we have $\varphi \in H^2(\Omega_\varepsilon)$ and there exists a constant $c_\varepsilon > 0$ depending on ε such that

$$\|\varphi\|_{H^2(\Omega_\varepsilon)} \leq c_\varepsilon\|\xi\|_{L^2(\Omega_\varepsilon)}. \tag{5.5.8}$$

In this case, the equation (5.5.6) is satisfied in the strong sense. Let us show that we can take a constant c_ε in (5.5.8) independently of ε.

Lemma 5.5.3. *Let ε_σ be the constant given in Lemma 5.5.2 and $\varepsilon \in (0,\varepsilon_\sigma)$. Suppose that $\xi \in L^2(\Omega_\varepsilon)$ satisfies $\int_{\Omega_\varepsilon}\xi\,dx = 0$. Then there exists a constant $c > 0$ independent of ε and ξ such that*

$$\|\varphi\|_{H^2(\Omega_\varepsilon)} \leq c\|\xi\|_{L^2(\Omega_\varepsilon)} \tag{5.5.9}$$

for a unique solution $\varphi \in H^2(\Omega_\varepsilon)$ to the problem (5.5.6).

Proof. Setting $\zeta = \varphi$ in (5.5.7) and using (5.5.2) we immediately get

$$\|\varphi\|_{H^1(\Omega_\varepsilon)} \leq \|\xi\|_{H^{-1}(\Omega_\varepsilon)} \leq \|\xi\|_{L^2(\Omega_\varepsilon)}.$$

Hence it is sufficient to show that

$$\|\nabla^2\varphi\|_{L^2(\Omega_\varepsilon)} \leq c\left(\|\Delta\varphi\|_{L^2(\Omega_\varepsilon)} + \|\varphi\|_{H^1(\Omega_\varepsilon)}\right) = c\left(\|\xi\|_{L^2(\Omega_\varepsilon)} + \|\varphi\|_{H^1(\Omega_\varepsilon)}\right) \tag{5.5.10}$$

with some constant $c > 0$ independent of ε (note that $\Delta\varphi = \xi$ a.e. in Ω_ε).

Since $C_c^\infty(\Omega_\varepsilon)$ is dense in $L^2(\Omega_\varepsilon)$, we can take a sequence $\{\xi_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\Omega_\varepsilon)$ that converges strongly to ξ in $L^2(\Omega_\varepsilon)$. For each $k \in \mathbb{N}$ let

$$\tilde{\xi}_k(x) := \xi_k(x) - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \xi_k(y)\,dy, \quad x \in \Omega_\varepsilon$$

and $\varphi_k \in H^1(\Omega_\varepsilon)$ be a unique weak solution to (5.5.6) with ξ replaced by $\tilde{\xi}_k$ (note that $\langle\tilde{\xi}_k, 1\rangle_{\Omega_\varepsilon} = (\tilde{\xi}_k, 1)_{L^2(\Omega_\varepsilon)} = 0$). Since $\tilde{\xi}_k \in C^\infty(\overline{\Omega}_\varepsilon)$ and Γ_ε is of class C^4, the elliptic regularity theorem yields that $\varphi_k \in H^3(\Omega_\varepsilon)$. Moreover, by the strong convergence of $\{\xi_k\}_{k=1}^\infty$ to ξ in $L^2(\Omega_\varepsilon)$ and

$$\lim_{k\to\infty}\int_{\Omega_\varepsilon} \xi_k\,dx = \lim_{k\to\infty}(\xi_k, 1)_{L^2(\Omega_\varepsilon)} = (\xi, 1)_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \xi\,dx = 0$$

we observe that

$$\|\xi - \tilde{\xi}_k\|_{L^2(\Omega_\varepsilon)} \leq \|\xi - \xi_k\|_{L^2(\Omega_\varepsilon)} + \frac{1}{|\Omega_\varepsilon|^{1/2}}\left|\int_{\Omega_\varepsilon} \xi_k\,dx\right| \to 0 \quad \text{as} \quad k \to \infty.$$

Since $\varphi - \varphi_k$ is a unique solution to (5.5.6) for the source term $\xi - \tilde{\xi}_k$, by (5.5.8) and the above convergence we obtain

$$\|\varphi - \varphi_k\|_{H^2(\Omega_\varepsilon)} \leq c_\varepsilon\|\xi - \tilde{\xi}_k\|_{L^2(\Omega_\varepsilon)} \to 0 \quad \text{as} \quad k \to \infty.$$

(Note that the constant c_ε does not depend on k.) Hence we can derive (5.5.10) by showing the same inequality for φ_k and sending $k \to \infty$.

From now on, we fix and suppress the subscript k. Hence we suppose that φ is in $H^3(\Omega_\varepsilon)$ and satisfies (5.5.6) in the strong sense. In particular, we have

$$\nabla\varphi \cdot n_\varepsilon = \frac{\partial\varphi}{\partial n_\varepsilon} = 0 \quad \text{on} \quad \Gamma_\varepsilon. \tag{5.5.11}$$

By the regularity of φ we can carry out integration by parts twice to get

$$\begin{aligned}\|\nabla^2\varphi\|_{L^2(\Omega_\varepsilon)}^2 &= \|\Delta\varphi\|_{L^2(\Omega_\varepsilon)}^2 + \int_{\Gamma_\varepsilon}\{(\nabla\varphi\cdot\nabla)\nabla\varphi\cdot n_\varepsilon - (\nabla\varphi\cdot n_\varepsilon)\Delta\varphi\}\,d\mathcal{H}^2 \\ &= \|\Delta\varphi\|_{L^2(\Omega_\varepsilon)}^2 + \int_{\Gamma_\varepsilon}(\nabla\varphi\cdot\nabla)\nabla\varphi\cdot n_\varepsilon\,d\mathcal{H}^2.\end{aligned} \tag{5.5.12}$$

Here we used (5.5.11) in the second equality. Moreover, based on (5.5.11) we can show as in the proof of Lemma 5.4.1 (see (5.4.12)) that

$$\left|\int_{\Gamma_\varepsilon}(\nabla\varphi\cdot\nabla)\nabla\varphi\cdot n_\varepsilon\,d\mathcal{H}^2\right| \leq c\left(\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}\|\nabla^2\varphi\|_{L^2(\Omega_\varepsilon)}\right).$$

Applying this inequality to (5.5.12) and using Young's inequality we obtain

$$\|\nabla^2\varphi\|_{L^2(\Omega_\varepsilon)}^2 \leq \|\Delta\varphi\|_{L^2(\Omega_\varepsilon)}^2 + c\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{2}\|\nabla^2\varphi\|_{L^2(\Omega_\varepsilon)},$$

which yields (5.5.10). Hence the inequality (5.5.9) follows. $\square$

Now let us derive the uniform estimate (5.5.1) for the difference $u - \mathbb{P}_\varepsilon u$.

Proof of Lemma 5.5.1. Let ε_σ be the constant given in Lemma 5.5.2 and $\varepsilon \in (0, \varepsilon_\sigma)$. Suppose that $u \in H^1(\Omega_\varepsilon)^3$ satisfies $u \cdot n_\varepsilon = 0$ on Γ_ε. Then for $\xi := -\operatorname{div} u \in L^2(\Omega_\varepsilon)$ the divergence theorem implies that

$$\langle \xi, 1 \rangle_{\Omega_\varepsilon} = \int_{\Omega_\varepsilon} \xi\, dx = -\int_{\Gamma_\varepsilon} u \cdot n_\varepsilon\, d\mathcal{H}^2 = 0.$$

The Helmholtz–Leray projection of u is given by $\mathbb{P}_\varepsilon u = u - \nabla\varphi$, where $\varphi \in H^1(\Omega_\varepsilon)$ is a unique weak solution to (5.5.6) with $\xi = -\operatorname{div} u$. Since $\xi \in L^2(\Omega_\varepsilon)$, Lemma 5.5.3 yields

$$\|u - \mathbb{P}_\varepsilon u\|_{H^1(\Omega_\varepsilon)} = \|\nabla\varphi\|_{H^1(\Omega_\varepsilon)} \leq c\|\xi\|_{L^2(\Omega_\varepsilon)} = c\|\operatorname{div} u\|_{L^2(\Omega_\varepsilon)}$$

with a constant $c > 0$ independent of ε. Hence (5.5.1) is valid. □

5.5.2 Definition and basic properties of the Stokes operator

Let us define the Stokes operator associated with slip boundary conditions and give its basic properties. For $u_1 \in H^2(\Omega_\varepsilon)^3$ and $u_2 \in H^1(\Omega_\varepsilon)^3$ integration by parts yields

$$\int_{\Omega_\varepsilon} \{\Delta u_1 + \nabla(\operatorname{div} u_1)\} \cdot u_2\, dx = -2\int_{\Omega_\varepsilon} D(u_1) : D(u_2)\, dx + 2\int_{\Gamma_\varepsilon} [D(u_1)n_\varepsilon] \cdot u_2\, d\mathcal{H}^2. \tag{5.5.13}$$

Hence if u_1 and u_2 satisfy $\operatorname{div} u_1 = 0$ in Ω_ε and

$$u_1 \cdot n_\varepsilon = 0, \quad 2\nu P_\varepsilon D(u_1)n_\varepsilon + \gamma_\varepsilon u_1 = 0, \quad u_2 \cdot n_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon,$$

then from the above identity we have

$$\nu \int_{\Omega_\varepsilon} \Delta u_1 \cdot u_2\, dx = -2\nu \int_{\Omega_\varepsilon} D(u_1) : D(u_2)\, dx - \sum_{i=0,1} \gamma_\varepsilon^i \int_{\Gamma_\varepsilon^i} u_1 \cdot u_2\, d\mathcal{H}^2.$$

Based on this observation we define a bilinear form

$$a_\varepsilon(u_1, u_2) := 2\nu \int_{\Omega_\varepsilon} D(u_1) : D(u_2)\, dx + \sum_{i=0,1} \gamma_\varepsilon^i \int_{\Gamma_\varepsilon^i} u_1 \cdot u_2\, d\mathcal{H}^2 \tag{5.5.14}$$

for $u_1, u_2 \in H^1(\Omega_\varepsilon)^3$. By definition, a_ε is symmetric on $H^1(\Omega_\varepsilon)^3$.

Lemma 5.5.4. *Under Assumptions 1 and 2, there exist $\varepsilon_0 \in (0,1)$ and $c > 0$ such that*

$$c^{-1}\|u\|_{H^1(\Omega_\varepsilon)}^2 \leq a_\varepsilon(u,u) \leq c\|u\|_{H^1(\Omega_\varepsilon)}^2 \tag{5.5.15}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying (5.3.20) *on Γ_ε.*

Proof. Let $u \in H^1(\Omega_\varepsilon)^3$. By (5.1.6) in Assumption 1 and (5.3.8) we have

$$\gamma_\varepsilon^i \|u\|_{L^2(\Gamma_\varepsilon^i)}^2 \leq c\varepsilon\left(\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon\|\partial_n u\|_{L^2(\Omega_\varepsilon)}^2\right) \leq c\|u\|_{H^1(\Omega_\varepsilon)}^2$$

for $i = 0, 1$. Combining this with

$$\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 = \frac{1}{2}\left(\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 + \int_{\Omega_\varepsilon} \nabla u : (\nabla u)^T\, dx\right) \leq \|\nabla u\|_{L^2(\Omega_\varepsilon)}^2$$

by Hölder's inequality we get the right-hand inequality of (5.5.15).

Let us prove the left-hand inequality of (5.5.15). First we assume that the condition (A1) in Assumption 2 is satisfied. Without loss of generality, we may assume that $\gamma_\varepsilon^0 = \max_{i=0,1}\gamma_\varepsilon^i$. For $u \in H^1(\Omega_\varepsilon)^3$ we use (5.3.7) with $i = 0$ to get

$$\|u\|_{L^2(\Omega_\varepsilon)}^2 \leq c\left(\varepsilon\|u\|_{L^2(\Gamma_\varepsilon^0)}^2 + \varepsilon^2\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2\right).$$

To the right-hand side we apply (5.1.7) and (5.4.1). Then

$$\begin{aligned}\|u\|_{L^2(\Omega_\varepsilon)}^2 &\leq c\left(\gamma_\varepsilon^0\|u\|_{L^2(\Gamma_\varepsilon^0)}^2 + \varepsilon^2\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon^2\|u\|_{L^2(\Omega_\varepsilon)}^2\right)\\ &\leq c_1 a_\varepsilon(u,u) + c_2\varepsilon^2\|u\|_{L^2(\Omega_\varepsilon)}^2\end{aligned}$$

with positive constants c_1 and c_2 independent of ε. We set $\varepsilon_1 := 1/\sqrt{2c_2}$ and take $\varepsilon \in (0,\varepsilon_1)$ in the above inequality to get

$$\|u\|_{L^2(\Omega_\varepsilon)}^2 \leq 2c_1 a_\varepsilon(u,u)$$

Moreover, from this inequality and (5.4.1) it follows that

$$\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \leq c a_\varepsilon(u,u).$$

By the above two inequalities we see that the left-hand inequality of (5.5.15) is valid.

Next we suppose that the condition (A2) in Assumption 2 holds. In this case the condition (5.4.14) is automatically satisfied for any $\beta \in [0,1)$. We fix $\beta \in [0,1)$ and apply Lemma 5.4.5 to obtain

$$\|u\|_{H^1(\Omega_\varepsilon)}^2 \leq c_\beta\|D(u)\|_{L^2(\Omega_\varepsilon)}^2 \leq c_\beta a_\varepsilon(u,u)$$

for all $\varepsilon \in (0,\varepsilon_\beta)$ and $u \in H^1(\Omega_\varepsilon)^3$ satisfying (5.3.20) on Γ_ε, where $\varepsilon_\beta \in (0,1)$ and $c_\beta > 0$ are the constants given in Lemma 5.4.5. Hence the left-hand inequality of (5.5.15) holds and we conclude that the lemma is valid with $\varepsilon_0 := \min\{\varepsilon_1,\varepsilon_\beta\}$. □

Throughout the rest of this section (except for Lemmas 5.5.6 and 5.5.7) we suppose that Assumptions 1 and 2 are satisfied and fix the number $\varepsilon_0 \in (0,1)$ given in Lemma 5.5.4. For $\varepsilon \in (0,\varepsilon_0)$ we define a function space

$$V_\varepsilon := L_\sigma^2(\Omega_\varepsilon) \cap H^1(\Omega_\varepsilon)^3 = \{u \in H^1(\Omega_\varepsilon)^3 \mid \operatorname{div} u = 0 \text{ in } \Omega_\varepsilon,\ u\cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}.$$

By Lemma 5.5.4 the bilinear form a_ε restricted to $V_\varepsilon \times V_\varepsilon$ is continuous, coercive, and symmetric on the Hilbert space V_ε (equipped with $H^1(\Omega_\varepsilon)$-inner product). Hence the Lax–Milgram theorem yields that there exists a bounded linear operator A_ε from V_ε into its dual space V_ε' such that

$${}_{V_\varepsilon'}\langle A_\varepsilon u_1, u_2\rangle_{V_\varepsilon} = a_\varepsilon(u_1,u_2), \quad u_1,u_2 \in V_\varepsilon,$$

where ${}_{V_\varepsilon'}\langle\cdot,\cdot\rangle_{V_\varepsilon}$ stands for the duality product between V_ε' and V_ε (see e.g. [8]). We consider A_ε as an unbounded operator on $L_\sigma^2(\Omega_\varepsilon)$ with its domain

$$D(A_\varepsilon) = \{u \in V_\varepsilon \mid A_\varepsilon u \in L_\sigma^2(\Omega_\varepsilon)\}.$$

Then from the Lax–Milgram theory it follows that A_ε is a positive self-adjoint operator on $L^2_\sigma(\Omega_\varepsilon)$. Moreover, by a regularity result on the Stokes problem with slip boundary conditions (see [5, 60]) we know that

$$\begin{aligned} D(A_\varepsilon) &= \{u \in V_\varepsilon \cap H^2(\Omega_\varepsilon)^3 \mid 2\nu P_\varepsilon D(u)n_\varepsilon + \gamma_\varepsilon u = 0 \text{ on } \Gamma_\varepsilon\}, \\ A_\varepsilon u &= -\nu \mathbb{P}_\epsilon \Delta u \quad \text{for} \quad u \in D(A_\varepsilon). \end{aligned}$$

Here $\mathbb{P}_\varepsilon$ is the Helmholtz–Leray projection (see Section 5.5.1). Note that $D(A_\varepsilon^{1/2}) = V_\varepsilon$ and

$$(A_\varepsilon u_1, u_2)_{L^2(\Omega_\varepsilon)} = (A_\varepsilon^{1/2} u_1, A_\varepsilon^{1/2} u_2)_{L^2(\Omega_\varepsilon)} \tag{5.5.16}$$

for all $u_1 \in D(A_\varepsilon)$ and $u_2 \in V_\varepsilon$. Moreover, the identity

$$\|A_\varepsilon^{1/2} u\|^2_{L^2(\Omega_\varepsilon)} = a_\varepsilon(u,u) = 2\nu \|D(u)\|^2_{L^2(\Omega_\varepsilon)} + \gamma^0_\varepsilon \|u\|^2_{L^2(\Gamma^0_\varepsilon)} + \gamma^1_\varepsilon \|u\|^2_{L^2(\Gamma^1_\varepsilon)} \tag{5.5.17}$$

holds for all $u \in V_\varepsilon$.

Lemma 5.5.5. *There exists a constant $c > 0$ independent of ε such that*

$$c^{-1}\|u\|_{H^1(\Omega_\varepsilon)} \le \|A_\varepsilon^{1/2} u\|_{L^2(\Omega_\varepsilon)} \le c\|u\|_{H^1(\Omega_\varepsilon)} \tag{5.5.18}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $u \in V_\varepsilon$. Moreover, if $u \in D(A_\varepsilon)$, then we have

$$\|A_\varepsilon^{1/2} u\|_{L^2(\Omega_\varepsilon)} \le c\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}. \tag{5.5.19}$$

Proof. The inequality (5.5.18) is an immediate consequence of (5.5.15) and (5.5.17). To prove (5.5.19) we see by Hölder's inequality that

$$\|A_\varepsilon^{1/2} u\|^2_{L^2(\Omega_\varepsilon)} = (u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)} \le \|u\|_{L^2(\Omega_\varepsilon)} \|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}$$

for $u \in D(A_\varepsilon)$. Applying $\|u\|_{L^2(\Omega_\varepsilon)} \le c\|A_\varepsilon^{1/2} u\|_{L^2(\Omega_\varepsilon)}$ by (5.5.18) to the right-hand side we obtain (5.5.19). □

5.5.3 Uniform regularity estimates for the Stokes operator

In this subsection we estimate the difference between the Stokes and Laplace operators and show the uniform equivalence of the norms $\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}$ and $\|u\|_{H^2(\Omega_\varepsilon)}$ for $u \in D(A_\varepsilon)$.

First we give an integration by parts formula for the curl of a vector field on Ω_ε. For $x \in N$ we set

$$\begin{aligned} \tilde{n}_1(x) &:= \frac{1}{\varepsilon \bar{g}(x)} \bigl\{ \bigl(d(x) - \varepsilon \bar{g}_0(x)\bigr) \bar{n}^1_\varepsilon(x) - \bigl(\varepsilon \bar{g}_1(x) - d(x)\bigr) \bar{n}^0_\varepsilon(x) \bigr\}, \\ \tilde{n}_2(x) &:= \frac{1}{\varepsilon \bar{g}(x)} \left\{ \bigl(d(x) - \varepsilon \bar{g}_0(x)\bigr) \frac{\gamma^1_\varepsilon}{\nu} \bar{n}^1_\varepsilon(x) + \bigl(\varepsilon \bar{g}_1(x) - d(x)\bigr) \frac{\gamma^0_\varepsilon}{\nu} \bar{n}^0_\varepsilon(x) \right\}, \\ \widetilde{W}(x) &:= \frac{1}{\varepsilon \bar{g}(x)} \bigl\{ \bigl(d(x) - \varepsilon \bar{g}_0(x)\bigr) W^1_\varepsilon(x) - \bigl(\varepsilon \bar{g}_1(x) - d(x)\bigr) W^0_\varepsilon(x) \bigr\}. \end{aligned}$$

Where n^i_ε, $i = 0, 1$ is given by (5.2.33) and

$$W^i_\varepsilon(x) := -\{I_3 - \bar{n}^i_\varepsilon(x) \otimes \bar{n}^i_\varepsilon(x)\} \nabla \bar{n}^i_\varepsilon(x), \quad x \in N,\ i = 0, 1.$$

By definition it immediately follows that

$$\tilde{n}_1 = (-1)^{i+1} n_\varepsilon, \quad \tilde{n}_2 = \frac{\gamma_\varepsilon}{\nu} n_\varepsilon, \quad \widetilde{W} = (-1)^{i+1} W_\varepsilon \quad \text{on} \quad \Gamma^i_\varepsilon,\ i = 0, 1. \tag{5.5.20}$$

Lemma 5.5.6. *For a vector field $u\colon \Omega_\varepsilon \to \mathbb{R}^3$ define $G(u) := G_1(u) + G_2(u)$, where*

$$G_1(u) := 2\tilde{n}_1 \times \widetilde{W}u, \quad G_2(u) := \tilde{n}_2 \times u \quad \text{in} \quad \Omega_\varepsilon. \tag{5.5.21}$$

Then, under Assumption 1 that (5.1.6) *is valid, there exists $c > 0$ independent of ε such that*

$$|G(u)| \leq c|u|, \quad |\nabla G(u)| \leq c(|u| + |\nabla u|) \quad \text{in} \quad \Omega_\varepsilon. \tag{5.5.22}$$

Proof. By (5.2.13), (5.2.15), (5.2.36), and $g_0, g_1 \in C^4(\Gamma)$ we see that

$$|\partial_x^\alpha \bar{n}_\varepsilon^i(x)| \leq c, \quad |\partial_x^\alpha \bar{g}_i(x)| \leq c, \quad x \in N,\ i = 0, 1,\ |\alpha| = 0, 1, 2, \tag{5.5.23}$$

where $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$ with $\alpha_i \geq 0$, $i = 1, 2, 3$ and $c > 0$ is a constant independent of ε. From this it also follows that

$$|W_\varepsilon^i(x)| \leq c, \quad |\partial_k W_\varepsilon^i(x)| \leq c, \quad x \in N,\ i = 0, 1,\ k = 1, 2, 3. \tag{5.5.24}$$

By (5.1.6), (5.3.39), (5.5.23), and (5.5.24) we have

$$|\tilde{n}_1| \leq c, \quad |\tilde{n}_2| \leq c\varepsilon, \quad \left|\widetilde{W}\right| \leq c \quad \text{in} \quad \Omega_\varepsilon. \tag{5.5.25}$$

Thus the first inequality of (5.5.22) is valid. To prove the second inequality of (5.5.22) we need to estimate the first order derivatives of $\tilde{n}_1$, $\tilde{n}_2$, and $\widetilde{W}$. As in the proof of Lemma 5.3.10, by direct calculations with $\nabla d = \bar{n}$ in N and the inequalities (5.2.30), (5.3.39), (5.5.23), and (5.5.24) we observe that

$$\nabla \tilde{n}_1 = \frac{1}{\varepsilon \bar{g}} \bar{n} \otimes (\bar{n}_\varepsilon^0 + \bar{n}_\varepsilon^1) + f_1, \quad \nabla \tilde{n}_2 = \frac{1}{\varepsilon \bar{g}} \bar{n} \otimes \left(\frac{\gamma_\varepsilon^1}{\nu}\bar{n}_\varepsilon^1 - \frac{\gamma_\varepsilon^0}{\nu}\bar{n}_\varepsilon^0\right) + f_2,$$
$$\partial_k \widetilde{W} = \frac{1}{\varepsilon \bar{g}} \bar{n}_k (W_\varepsilon^0 + W_\varepsilon^1) + F_k, \quad k = 1, 2, 3$$

in Ω_ε, where f_1, f_2, and F_k are bounded on Ω_ε uniformly in ε. We apply (5.2.37) to $\nabla \tilde{n}_1$, (5.1.6) and (5.2.36) to $\nabla \tilde{n}_2$, and use (5.2.30) to show that

$$|\nabla \tilde{n}_1| \leq \frac{1}{\varepsilon \bar{g}} |\bar{n}_\varepsilon^0 + \bar{n}_\varepsilon^1| + |f_1| \leq c, \quad |\nabla \tilde{n}_2| \leq \frac{c(\gamma_\varepsilon^0 + \gamma_\varepsilon^1)}{\varepsilon \bar{g}} + |f_2| \leq c \tag{5.5.26}$$

in Ω_ε. To estimate the first order derivatives of $\widetilde{W}$ we see by (5.2.12) that

$$\begin{aligned} W_\varepsilon^0 + W_\varepsilon^1 = \{(\bar{n}_\varepsilon^0 + \bar{n}_\varepsilon^1) \otimes \bar{n}_\varepsilon^0 - \bar{n}_\varepsilon^1 \otimes (\bar{n}_\varepsilon^0 + \bar{n}_\varepsilon^1)\}\left(I_3 - d\overline{W}\right)^{-1}\overline{\nabla_\Gamma n_\varepsilon^0} \\ - (I_3 - \bar{n}_\varepsilon^1 \otimes \bar{n}_\varepsilon^1)\left(I_3 - d\overline{W}\right)^{-1}\left(\overline{\nabla_\Gamma n_\varepsilon^0} + \overline{\nabla_\Gamma n_\varepsilon^1}\right) \end{aligned}$$

in N. Hence we get $|W_\varepsilon^0 + W_\varepsilon^1| \leq c\varepsilon$ in N by (5.2.9), (5.2.36), and (5.2.37) and

$$\left|\partial_k \widetilde{W}\right| \leq \frac{1}{\varepsilon \bar{g}} |W_\varepsilon^0 + W_\varepsilon^1| + |F_k| \leq c \quad \text{in} \quad \Omega_\varepsilon. \tag{5.5.27}$$

Applying (5.5.25), (5.5.26), and (5.5.27) to the gradients of G_1 and G_2 given by (5.5.21) we obtain the second inequality of (5.5.22). $\square$

Lemma 5.5.7. *We have the integration by parts formula*

$$\int_{\Omega_\varepsilon} \operatorname{curl}\operatorname{curl} u \cdot \Phi\,dx = -\int_{\Omega_\varepsilon} \operatorname{curl} G(u) \cdot \Phi\,dx + \int_{\Omega_\varepsilon} \{\operatorname{curl} u + G(u)\} \cdot \operatorname{curl}\Phi\,dx \tag{5.5.28}$$

for all $u \in H^2(\Omega_\varepsilon)^3$ *satisfying the slip boundary conditions* (5.3.20)–(5.3.21) *on* Γ_ε *and* $\Phi \in L^2(\Omega_\varepsilon)^3$ *with* $\operatorname{curl}\Phi \in L^2(\Omega_\varepsilon)^3$, *where* $G(u)$ *is given in Lemma 5.5.6.*

Proof. By standard cut-off, dilatation, and mollification arguments, we can show as in the proof of [64, Chapter 1, Theorem 1.1] that for $\Phi \in L^2(\Omega_\varepsilon)^3$ with $\operatorname{curl}\Phi \in L^2(\Omega_\varepsilon)^3$ there exists a sequence $\{\Phi_k\}_{k=1}^\infty$ of vector fields in $C^\infty(\overline{\Omega}_\varepsilon)^3$ such that

$$\lim_{k\to\infty} \|\Phi - \Phi_k\|_{L^2(\Omega_\varepsilon)} = \lim_{k\to\infty} \|\operatorname{curl}\Phi - \operatorname{curl}\Phi_k\|_{L^2(\Omega_\varepsilon)} = 0.$$

Thus, by a density argument, it is sufficient to prove (5.5.28) for all $\Phi \in C^\infty(\overline{\Omega}_\varepsilon)^3$.

Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy (5.3.20) and (5.3.21) on Γ_ε and $\Phi \in C^\infty(\overline{\Omega}_\varepsilon)^3$. Then

$$\int_{\Omega_\varepsilon} \operatorname{curl}\operatorname{curl} u \cdot \Phi\,dx = \int_{\Gamma_\varepsilon} (n_\varepsilon \times \operatorname{curl} u) \cdot \Phi\,d\mathcal{H}^2 + \int_{\Omega_\varepsilon} \operatorname{curl} u \cdot \operatorname{curl}\Phi\,dx \tag{5.5.29}$$

by integration by parts. To the boundary integral we apply

$$\begin{aligned} n_\varepsilon \times \operatorname{curl} u &= -n_\varepsilon \times \left\{ n_\varepsilon \times \left(2W_\varepsilon u + \frac{\gamma_\varepsilon}{\nu} u \right) \right\} \\ &= -n_\varepsilon \times \left(2\tilde{n}_1 \times \widetilde{W} u + \tilde{n}_2 \times u \right) = -n_\varepsilon \times G(u) \end{aligned}$$

on Γ_ε by (5.3.30), (5.5.20), and (5.5.21), and then use integration by parts to get

$$\begin{aligned} \int_{\Gamma_\varepsilon} (n_\varepsilon \times \operatorname{curl} u) \cdot \Phi\,d\mathcal{H}^2 &= -\int_{\Gamma_\varepsilon} \{n_\varepsilon \times G(u)\} \cdot \Phi\,d\mathcal{H}^2 \\ &= \int_{\Omega_\varepsilon} \{G(u) \cdot \operatorname{curl}\Phi - \operatorname{curl} G(u) \cdot \Phi\}\,dx. \end{aligned}$$

Substituting this for (5.5.29) we obtain (5.5.28) □

Using the formula (5.5.28), we derive an estimate for the difference between the Stokes and Laplace operators as in [19, Theorem 2.1].

Lemma 5.5.8. *There exists a constant* $c > 0$ *independent of* ε *such that*

$$\|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)} \tag{5.5.30}$$

for all $\varepsilon \in (0, \varepsilon_0)$ *and* $u \in D(A_\varepsilon)$.

Note that the $H^1(\Omega_\varepsilon)$-norm of u appears in the right-hand side of (5.5.30) instead of its $H^2(\Omega_\varepsilon)$-norm.

Proof. Let $u \in D(A_\varepsilon)$. Since $A_\varepsilon u = -\nu\mathbb{P}_\varepsilon\Delta u \in L^2_\sigma(\Omega_\varepsilon)$, there exists $q \in H^1(\Omega_\varepsilon)$ such that $A_\varepsilon u + \nu\Delta u = \nabla q$. Then by $A_\varepsilon u \in L^2_\sigma(\Omega_\varepsilon)$ and $\nabla q \in L^2_\sigma(\Omega_\varepsilon)^\perp$ we get

$$\|A_\varepsilon u + \nu\Delta u\|^2_{L^2(\Omega_\varepsilon)} = (A_\varepsilon u, \nabla q)_{L^2(\Omega_\varepsilon)} + (\nu\Delta u, \nabla q)_{L^2(\Omega_\varepsilon)} = (\nu\Delta u, \nabla q)_{L^2(\Omega_\varepsilon)}.$$

Moreover, noting that

$$\operatorname{curl}\nabla q = 0, \quad \Delta u = -\operatorname{curl}\operatorname{curl} u \quad \text{in} \quad \Omega_\varepsilon$$

by $\operatorname{div} u = 0$, we apply (5.5.28) with $\Phi = -\nabla q$ to the last term to get

$$\begin{aligned}(\nu\Delta u, \nabla q)_{L^2(\Omega_\varepsilon)} &= \nu(\operatorname{curl}\operatorname{curl} u, -\nabla q)_{L^2(\Omega_\varepsilon)} \\ &= \nu(\operatorname{curl} G(u), \nabla q)_{L^2(\Omega_\varepsilon)} \\ &= \nu(\operatorname{curl} G(u), A_\varepsilon u + \nu\Delta u)_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

where $G(u)$ is given in Lemma 5.5.6. From this equality and (5.5.22) we deduce that

$$\begin{aligned}\|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)}^2 &\leq \nu\|\operatorname{curl} G(u)\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} \\ &\leq c\|u\|_{H^1(\Omega_\varepsilon)}\|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)}\end{aligned}$$

with some constant $c > 0$ independent of ε. Thus (5.5.30) follows. □

Remark 5.5.9. In the proof of Lemma 5.5.8, Assumption 2 is essential since it enables us to consider the Stokes operator A_ε on the usual solenoidal space $L^2_\sigma(\Omega_\varepsilon)$ and thus the curl of $A_\varepsilon u + \nu\Delta u = \nabla q$ vanishes. If we drop Assumption 2 and consider the Stokes operator $\widehat{A}_\varepsilon$ on

$$\widehat{L}^2_\sigma(\Omega_\varepsilon) = L^2_\sigma(\Omega_\varepsilon) \cap \overline{\mathcal{K}}_g(\Gamma)^\perp, \quad \overline{\mathcal{K}}_g(\Gamma) = \{\bar{v} \mid v \in \mathcal{K}_g(\Gamma)\},$$

where $\mathcal{K}_g(\Gamma)$ is given by (5.1.8) and $\bar{v} = v \circ \pi$ is the constant extension of v, then

$$\widehat{A}_\varepsilon u + \nu\Delta u = \nabla q + \bar{v}, \quad q \in H^1(\Omega_\varepsilon),\, \bar{v} \in L^2_\sigma(\Omega_\varepsilon) \cap \overline{K}_g(\Gamma)$$

for $u \in D(\widehat{A}_\varepsilon)$. In this case, however, we cannot prove a similar inequality to (5.5.30) since the curl of $\bar{v}$ does not vanish in general. This difficulty does not occur in the proof of [19, Theorem 2.1] since in that case $\mathcal{K}_g(\Gamma)$ reduces to

$$\mathcal{K}_g(\mathbb{T}^2) = \{(a, 0) \in \mathbb{R}^3 \mid a \in \mathbb{R}^2,\, a \cdot \nabla_2 g = 0 \text{ in } \mathbb{T}^2\}$$

and the curl of the constant $(a, 0) \in \mathcal{K}_g(\mathbb{T}^2)$ automatically vanishes.

Next we show that for $u \in D(A_\varepsilon)$ the norm $\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}$ is bounded from above and below by the canonical $H^2(\Omega_\varepsilon)$-norm of u with constants independent of ε.

Lemma 5.5.10. *There exists a constant $c > 0$ independent of ε such that*

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq c\left(\|\Delta u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right) \tag{5.5.31}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $u \in H^2(\Omega_\varepsilon)^3$ satisfying (5.3.20) *and* (5.3.21) *on Γ_ε.*

The proof of Lemma 5.5.10 is similar to that of Lemma 5.5.3, but we need to carry out calculations a lot and use some formulas for the Riemannian connection on Γ_ε. We give it in the next subsection.

Using Lemmas 5.5.8 and 5.5.10 we prove the uniform equivalence of the $L^2(\Omega_\varepsilon)$-norm of $A_\varepsilon u$ and the $H^2(\Omega_\varepsilon)$-norm of $u \in D(A_\varepsilon)$.

Lemma 5.5.11. *There exists a constant $c > 0$ independent of ε such that*

$$c^{-1}\|u\|_{H^2(\Omega_\varepsilon)} \leq \|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^2(\Omega_\varepsilon)} \tag{5.5.32}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $u \in D(A_\varepsilon)$.

Proof. By (5.5.30) and (5.5.31) we have

$$\begin{aligned}
\|u\|_{H^2(\Omega_\varepsilon)} &\leq c\left(\|\Delta u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right) \\
&\leq c\left(\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} + \|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right) \\
&\leq c\left(\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} + \|u\|_{H^1(\Omega_\varepsilon)}\right).
\end{aligned}$$

Applying (5.5.18) and (5.5.19) to the second term on the right-hand side we obtain the left-hand inequality of (5.5.32). Also, from (5.5.30) and $\|u\|_{H^1(\Omega_\varepsilon)} \leq \|u\|_{H^2(\Omega_\varepsilon)}$ we deduce that

$$\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \leq \|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} + \|\nu\Delta u\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^2(\Omega_\varepsilon)}.$$

Hence the right-hand inequality of (5.5.32) holds. □

As a consequence of Lemmas 5.5.5 and 5.5.11, we obtain an interpolation inequality for a vector field in $D(A_\varepsilon)$.

Lemma 5.5.12. *There exists a constant $c > 0$ independent of ε such that*

$$\|u\|_{H^1(\Omega_\varepsilon)} \leq c\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2} \tag{5.5.33}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $u \in D(A_\varepsilon)$.

Proof. Let $u \in D(A_\varepsilon)$. From (5.5.16) and (5.5.18) it follows that

$$\|u\|_{H^1(\Omega_\varepsilon)}^2 \leq c\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}^2 = c(A_\varepsilon u, u)_{L^2(\Omega_\varepsilon)} \leq c\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}\|u\|_{L^2(\Omega_\varepsilon)}.$$

Applying (5.5.32) to the right-hand side of this inequality we get

$$\|u\|_{H^1(\Omega_\varepsilon)}^2 \leq c\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}.$$

Hence (5.5.33) is valid. □

5.5.4 Uniform a priori estimate for the vector Laplacian

The purpose of this subsection is to give the proof of Lemma 5.5.10. First we give an approximation result of a vector field in $H^2(\Omega_\varepsilon)^3$ satisfying the slip boundary conditions. Let us consider the equilibrium equations of linear elasticity with slip boundary conditions

$$\begin{cases} \Delta u + \nabla(\operatorname{div} u) = f & \text{in} \quad \Omega_\varepsilon, \\ u \cdot n_\varepsilon = 0, \quad 2\nu P_\varepsilon D(u)n_\varepsilon + \gamma_\varepsilon u = 0 & \text{on} \quad \Gamma_\varepsilon, \end{cases} \tag{5.5.34}$$

where $f\colon \Omega_\varepsilon \to \mathbb{R}^3$ is an external force (and we artificially impose the slip boundary conditions). By the identity (5.5.13) and the slip boundary conditions we observe that the bilinear form corresponding to the problem (5.5.34) is a_ε given by (5.5.14). Thus, under Assumptions 1 and 2, Lemma 5.5.4 and the Lax–Milgram theory show that for each $f \in H^{-1}(\Omega_\varepsilon)^3$ there exists a unique weak solution $u \in H^1(\Omega_\varepsilon)^3$ satisfying $u \cdot n_\varepsilon = 0$ on Γ_ε and

$$a_\varepsilon(u, \Phi) = \langle f, \Phi\rangle_{\Omega_\varepsilon} \quad \text{for all} \quad \Phi \in H^1(\Omega_\varepsilon)^3,$$

where $\langle\cdot,\cdot\rangle_{\Omega_\varepsilon}$ stands for the duality product between $H^{-1}(\Omega_\varepsilon)$ and $H^1(\Omega_\varepsilon)$.

Lemma 5.5.13. *Let $u \in H^1(\Omega_\varepsilon)^3$ be a unique weak solution to the problem* (5.5.34)*. Assume that $f \in L^2(\Omega_\varepsilon)^3$. Then $u \in H^2(\Omega_\varepsilon)^3$ and it satisfies* (5.5.34) *a.e. in Ω_ε and on Γ_ε. Moreover, there exists a constant $c_\varepsilon > 0$ depending on ε such that*

$$\|u\|_{H^2(\Omega_\varepsilon)} \leq c_\varepsilon \|f\|_{L^2(\Omega_\varepsilon)}. \tag{5.5.35}$$

If in addition $f \in H^1(\Omega_\varepsilon)^3$, then $u \in H^3(\Omega_\varepsilon)^3$.

Proof. The inequality (5.5.35) and the H^2-regularity of u are proved by a standard localization argument and a method of the difference quotient. Here we omit their proofs since they are the same as those of [5, Theorem 1.2] and [60, Theorem 2], which establish the H^2-regularity of a weak solution to the Stokes problem with slip boundary conditions.

Also, the H^3-regularity of u is proved by induction and a localization argument as in the case of a general second order elliptic equation. For details, see [13, Theorem 5 in Section 6.3]. (Note that the C^4-regularity of the boundary Γ_ε is required for the H^3-regularity of u, see Section 5.2.2 and the proofs of [5, Theorem 1.2] and [60, Theorem 2].) □

Based on Lemma 5.5.13 we show that a vector field in $H^2(\Omega_\varepsilon)^3$ is approximated by those in $H^3(\Omega_\varepsilon)^3$ under the slip boundary conditions.

Lemma 5.5.14. *Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy the slip boundary conditions* (5.3.20)–(5.3.21) *on Γ_ε. Then there exists a sequence $\{u_k\}_{k=1}^\infty$ in $H^3(\Omega_\varepsilon)^3$ such that u_k satisfies* (5.3.20)–(5.3.21) *on Γ_ε for each $k \in \mathbb{N}$ and $\lim_{k\to\infty} \|u - u_k\|_{H^2(\Omega_\varepsilon)} = 0$.*

Proof. Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy (5.3.20)–(5.3.21) on Γ_ε and $f := \Delta u + \nabla(\operatorname{div} u)$. Since $f \in L^2(\Omega_\varepsilon)^3$, there exists a sequence $\{f_k\}_{k=1}^\infty$ in $C_c^\infty(\Omega_\varepsilon)^3$ that converges to f strongly in $L^2(\Omega_\varepsilon)^3$. For each $k \in \mathbb{N}$ let u_k be a unique weak solution to the problem (5.5.34) with external force f_k. Then since $f_k \in C_c^\infty(\Omega_\varepsilon)^3$, we see by Lemma 5.5.13 that $u_k \in H^3(\Omega_\varepsilon)^3$ and it satisfies (5.3.20)–(5.3.21) on Γ_ε. Moreover, by (5.5.35) and the fact that $u - u_k$ is a unique weak solution to (5.5.34) with external force $f - f_k$, we have

$$\|u - u_k\|_{H^2(\Omega_\varepsilon)} \leq c_\varepsilon \|f - f_k\|_{L^2(\Omega_\varepsilon)}.$$

Letting $k \to \infty$ in this inequality and using the strong convergence of $\{f_k\}_{k=1}^\infty$ to f in $L^2(\Omega_\varepsilon)^3$ we obtain $\lim_{k\to\infty} \|u - u_k\|_{H^2(\Omega_\varepsilon)} = 0$ (note that the constant c_ε does not depend on k). □

Now let us show Lemma 5.5.10. As in Section 5.2.1 we denote by

$$H^m(\Gamma_\varepsilon, T\Gamma_\varepsilon) := \{u \in H^m(\Gamma_\varepsilon)^3 \mid u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon\}, \quad m = 0, 1, 2$$

the space of all tangential vector fields on Γ_ε of class H^m (here we write $H^0 = L^2$). For $u \in H^1(\Gamma_\varepsilon, T\Gamma_\varepsilon)$ and $v \in L^2(\Gamma_\varepsilon, T\Gamma_\varepsilon)$ we define the covariant derivative

$$\overline{\nabla}_v^\varepsilon u := P_\varepsilon(v \cdot \nabla)\tilde{u} = P_\varepsilon(v \cdot \nabla_{\Gamma_\varepsilon})u \quad \text{on} \quad \Gamma_\varepsilon,$$

where $\tilde{u}$ is any H^1-extension of u to an open neighborhood of Γ_ε with $\tilde{u}|_{\Gamma_\varepsilon} = u$. We use the formulas for the covariant derivatives given in Appendix 5.C.

Proof of Lemma 5.5.10. Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy (5.3.20)–(5.3.21) on Γ_ε. Since

$$\|u\|_{H^2(\Omega_\varepsilon)}^2 = \|u\|_{H^1(\Omega_\varepsilon)}^2 + \|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}^2,$$

it is sufficient for (5.5.31) to show that

$$\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}^2 \leq c\left(\|\Delta u\|_{L^2(\Omega_\varepsilon)}^2 + \|u\|_{H^1(\Omega_\varepsilon)}^2\right). \tag{5.5.36}$$

Moreover, by Lemma 5.5.14 we may assume that $u \in H^3(\Omega_\varepsilon)^3$, and thus its trace on Γ_ε is in $H^2(\Gamma_\varepsilon, T\Gamma_\varepsilon)$ (note that u satisfies $u \cdot n_\varepsilon = 0$ on Γ_ε). For such u we can carry out integration by parts twice to get

$$\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}^2 = \|\Delta u\|_{L^2(\Omega_\varepsilon)}^2 + \int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\}\, d\mathcal{H}^2. \tag{5.5.37}$$

Here $(n_\varepsilon \cdot \nabla)\nabla u$ denotes a 3×3 matrix whose (i,j)-entry is given by

$$[(n_\varepsilon \cdot \nabla)\nabla u]_{ij} := (n_\varepsilon \cdot \nabla)\partial_i u_j, \quad i,j = 1,2,3.$$

Let us estimate the boundary integral in (5.5.37). Since u satisfies the slip boundary conditions (5.3.20)–(5.3.21) on Γ_ε, we see by Lemma 5.3.7 that

$$(n_\varepsilon \cdot \nabla)u = -W_\varepsilon u - \tilde{\gamma}_\varepsilon u + \xi_\varepsilon n_\varepsilon \quad \text{on} \quad \Gamma_\varepsilon, \tag{5.5.38}$$

where $\tilde{\gamma}_\varepsilon := \gamma_\varepsilon/\nu$ and $\xi_\varepsilon := (n_\varepsilon \cdot \nabla)u \cdot n_\varepsilon = \nabla u : Q_\varepsilon$ (note that u and $W_\varepsilon u$ are tangential on Γ_ε). The first step is to show that

$$\int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\}\, d\mathcal{H}^2 = \sum_{k=1}^4 \int_{\Gamma_\varepsilon} \varphi_k \, d\mathcal{H}^2, \tag{5.5.39}$$

where

$$\begin{aligned}
\varphi_1 &:= -2\{\nabla_{\Gamma_\varepsilon} W_\varepsilon \cdot u + (\nabla u)W_\varepsilon + \tilde{\gamma}_\varepsilon \nabla u\} : P_\varepsilon(\nabla u)P_\varepsilon, \\
\varphi_2 &:= W_\varepsilon \nabla u : (\nabla u)P_\varepsilon \\
&\qquad - 2(u \cdot \mathrm{div}_{\Gamma_\varepsilon} W_\varepsilon + 2\nabla u : W_\varepsilon)(\nabla u : Q_\varepsilon) + H_\varepsilon(\nabla u : Q_\varepsilon)^2, \\
\varphi_3 &:= -(W_\varepsilon^3 u - H_\varepsilon W_\varepsilon^2 u) \cdot u, \\
\varphi_4 &:= -\tilde{\gamma}_\varepsilon(2W_\varepsilon^2 u - 2H_\varepsilon W_\varepsilon u - \tilde{\gamma}_\varepsilon H_\varepsilon u) \cdot u.
\end{aligned} \tag{5.5.40}$$

In (5.5.40) we used the notation $\nabla_{\Gamma_\varepsilon} W_\varepsilon \cdot u$ for the 3×3 matrix given by

$$[\nabla_{\Gamma_\varepsilon} W_\varepsilon \cdot u]_{ij} := \sum_{k=1}^3 (\underline{D}_i^\varepsilon [W_\varepsilon]_{jk}) u_k, \quad i,j = 1,2,3. \tag{5.5.41}$$

Using a partition of unity of Γ_ε we may assume that $u|_{\Gamma_\varepsilon}$ is compactly supported in a relatively open subset U of Γ_ε on which we can take a local orthonormal frame $\{\tau_1, \tau_2\}$ (see Appendix 5.C). Since $\{\tau_1, \tau_2, n_\varepsilon\}$ is an orthonormal basis of $\mathbb{R}^3$,

$$\nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\} = (\nabla u)^T : [\{(n_\varepsilon \cdot \nabla)\nabla u\}^T - \Delta u \otimes n_\varepsilon] = \sum_{i=1}^3 \eta_i \tag{5.5.42}$$

on U, where

$$\eta_i := (\nabla u)^T \tau_i \cdot [\{(n_\varepsilon \cdot \nabla)\nabla u\}^T \tau_i - (\Delta u \otimes n_\varepsilon)\tau_i], \quad i = 1,2, \tag{5.5.43}$$

$$\eta_3 := (\nabla u)^T n_\varepsilon \cdot [\{(n_\varepsilon \cdot \nabla)\nabla u\}^T n_\varepsilon - (\Delta u \otimes n_\varepsilon)n_\varepsilon]. \tag{5.5.44}$$

In what follows, we carry out calculations on U. By (5.C.2) and $\tau_i\cdot n_\varepsilon=0$,

$$\begin{aligned}(\nabla u)^T\tau_i &= (\tau_i\cdot\nabla)u=\overline{\nabla}_i^\varepsilon u+(W_\varepsilon u\cdot\tau_i)n_\varepsilon,\\ (\Delta u\otimes n_\varepsilon)\tau_i &= (\tau_i\cdot n_\varepsilon)\Delta u=0,\end{aligned} \tag{5.5.45}$$

where $\overline{\nabla}_i^\varepsilon:=\overline{\nabla}_{\tau_i}^\varepsilon$, $i=1,2$. Writing τ_i^j and n_ε^j, $j=1,2,3$ for the j-th component of τ_i and n_ε, we see that the j-th component of $\{(n_\varepsilon\cdot\nabla)\nabla u\}^T\tau_i$ is of the form

$$\begin{aligned}\sum_{k,l=1}^3 n_\varepsilon^k(\partial_k\partial_l u_j)\tau_i^l &= \sum_{k=1}^3 n_\varepsilon^k(\tau_i\cdot\nabla)(\partial_k u_j)=\sum_{k=1}^3 n_\varepsilon^k(\tau_i\cdot\nabla_{\Gamma_\varepsilon})(\partial_k u_j)\\ &= \sum_{k=1}^3\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})(n_\varepsilon^k\partial_k u_j)-(\tau_i\cdot\nabla_{\Gamma_\varepsilon}n_\varepsilon^k)\partial_k u_j\}\\ &= (\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(n_\varepsilon\cdot\nabla)u_j\}-\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})n_\varepsilon\cdot\nabla\}u_j.\end{aligned}$$

(Note that τ_i is tangential on Γ_ε and the tangential derivatives depend only on the values of functions on Γ_ε). Therefore,

$$\{(n_\varepsilon\cdot\nabla)\nabla u\}^T\tau_i=(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(n_\varepsilon\cdot\nabla)u\}-\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})n_\varepsilon\cdot\nabla\}u.$$

By (5.5.38), (5.C.2), $W_\varepsilon^T=W_\varepsilon$, and

$$(\tau_i\cdot\nabla_{\Gamma_\varepsilon})n_\varepsilon=(\nabla_{\Gamma_\varepsilon}n_\varepsilon)^T\tau_i=-W_\varepsilon^T\tau_i=-W_\varepsilon\tau_i, \tag{5.5.46}$$

we further observe that

$$\begin{aligned}\{(n_\varepsilon\cdot\nabla)\nabla u\}^T\tau_i=-\overline{\nabla}_i^\varepsilon(W_\varepsilon u)-\tilde\gamma_\varepsilon\overline{\nabla}_i^\varepsilon u+\overline{\nabla}_{W_\varepsilon\tau_i}^\varepsilon u-\xi_\varepsilon W_\varepsilon\tau_i&\\ +\{(-\tilde\gamma_\varepsilon W_\varepsilon u+\nabla_{\Gamma_\varepsilon}\xi_\varepsilon)\cdot\tau_i\}n_\varepsilon.&\end{aligned} \tag{5.5.47}$$

Note that the first four terms on the right-hand side of (5.5.47) are tangential on Γ_ε. From (5.5.43), (5.5.45), and (5.5.47) we deduce that

$$\eta_i=-\overline{\nabla}_i^\varepsilon u\cdot\left\{\overline{\nabla}_i^\varepsilon(W_\varepsilon u)+\tilde\gamma_\varepsilon\overline{\nabla}_i^\varepsilon u-\overline{\nabla}_{W_\varepsilon\tau_i}^\varepsilon u+\xi_\varepsilon W_\varepsilon\tau_i\right\}+(W_\varepsilon u\cdot\tau_i)\{(-\tilde\gamma_\varepsilon W_\varepsilon u+\nabla_{\Gamma_\varepsilon}\xi_\varepsilon)\cdot\tau_i\}$$

for $i=1,2$. Hence by (5.C.9)–(5.C.13) and the fact that $\{\tau_1,\tau_2\}$ is an orthonormal basis of the tangent plane of Γ_ε we obtain

$$\begin{aligned}\eta_1+\eta_2=-\{\nabla_{\Gamma_\varepsilon}(W_\varepsilon u)+\tilde\gamma_\varepsilon\nabla_{\Gamma_\varepsilon}u-W_\varepsilon\nabla_{\Gamma_\varepsilon}u\}:(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon&\\ -\xi_\varepsilon(\nabla_{\Gamma_\varepsilon}u:W_\varepsilon)+W_\varepsilon u\cdot(-\tilde\gamma_\varepsilon W_\varepsilon u+\nabla_{\Gamma_\varepsilon}\xi_\varepsilon).&\end{aligned} \tag{5.5.48}$$

To calculate η_3 we observe that the j-th component of $\{(n_\varepsilon\cdot\nabla)\nabla u\}^T n_\varepsilon$ is given by

$$\begin{aligned}\sum_{k,l=1}^3 n_\varepsilon^k(\partial_k\partial_l u_j)n_\varepsilon^l &= \mathrm{tr}[Q_\varepsilon\nabla^2 u_j]=\mathrm{tr}[\nabla^2 u_j]-\mathrm{tr}[P_\varepsilon\nabla^2 u_j]\\ &= \Delta u_j-\sum_{i=1,2}P_\varepsilon(\nabla^2 u_j)\tau_i\cdot\tau_i-P_\varepsilon(\nabla^2 u)n_\varepsilon\cdot n_\varepsilon\\ &= \Delta u_j-\sum_{i=1,2}(\tau_i\cdot\nabla)\nabla u_j\cdot\tau_i\end{aligned}$$

for $j=1,2,3$. In the last equality we used $P_\varepsilon^T=P_\varepsilon$, $P_\varepsilon\tau_i=\tau_i$, and $P_\varepsilon n_\varepsilon=0$. For the second term on the last line we further see that

$$(\tau_i\cdot\nabla)\nabla u_j\cdot\tau_i=(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\nabla u_j\cdot\tau_i=(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(\tau_i\cdot\nabla)u_j\}-\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\tau_i\cdot\nabla\}u_j.$$

From these equalities and $(\Delta u\otimes n_\varepsilon)n_\varepsilon=(n_\varepsilon\cdot n_\varepsilon)\Delta u=\Delta u$ it follows that

$$\begin{aligned}\{(n_\varepsilon\cdot\nabla)\nabla u\}^T n_\varepsilon-(\Delta u\otimes n_\varepsilon)n_\varepsilon&\\ =-\sum_{i=1,2}[(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(\tau_i\cdot\nabla)u\}-\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\tau_i\cdot\nabla\}u].&\end{aligned}\tag{5.5.49}$$

By (5.5.38), (5.5.46), and (5.C.2) we have

$$\begin{aligned}(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\{(\tau_i\cdot\nabla)u\}&=(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\left\{\overline{\nabla}_i^\varepsilon u+(W_\varepsilon u\cdot\tau_i)n_\varepsilon\right\}\\&=\overline{\nabla}_i^\varepsilon\overline{\nabla}_i^\varepsilon u-(W_\varepsilon u\cdot\tau_i)W_\varepsilon\tau_i+\left\{W_\varepsilon\overline{\nabla}_i^\varepsilon u\cdot\tau_i+\tau_i\cdot\nabla_{\Gamma_\varepsilon}(W_\varepsilon u\cdot\tau_i)\right\}n_\varepsilon\end{aligned}$$

and

$$\begin{aligned}\{(\tau_i\cdot\nabla_{\Gamma_\varepsilon})\tau_i\cdot\nabla\}u&=\left[\left\{\overline{\nabla}_i^\varepsilon\tau_i+(W_\varepsilon\tau_i\cdot\tau_i)n_\varepsilon\right\}\cdot\nabla\right]u\\&=\overline{\nabla}_{\overline{\nabla}_i^\varepsilon\tau_i}^\varepsilon u-(W_\varepsilon\tau_i\cdot\tau_i)(W_\varepsilon u+\tilde{\gamma}_\varepsilon u)+\left(W_\varepsilon u\cdot\overline{\nabla}_i^\varepsilon\tau_i+\xi_\varepsilon W_\varepsilon\tau_i\cdot\tau_i\right)n_\varepsilon.\end{aligned}$$

We substitute these expressions for (5.5.49) and use

$$\sum_{i=1,2}(W_\varepsilon u\cdot\tau_i)W_\varepsilon\tau_i=\sum_{i=1,2}W_\varepsilon(\tau_i\otimes\tau_i)W_\varepsilon u=W_\varepsilon P_\varepsilon W_\varepsilon u=W_\varepsilon^2 u$$

by $P_\varepsilon=\sum_{i=1,2}\tau_i\otimes\tau_i$ and $P_\varepsilon W_\varepsilon=W_\varepsilon$,

$$\sum_{i=1,2}\left\{\tau_i\cdot\nabla_{\Gamma_\varepsilon}(W_\varepsilon u\cdot\tau_i)-W_\varepsilon u\cdot\overline{\nabla}_i^\varepsilon\tau_i\right\}=\sum_{i=1,2}\overline{\nabla}_i^\varepsilon(W_\varepsilon u)\cdot\tau_i=\mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u)$$

by (5.C.5) and (5.C.9), and the formulas (5.C.8) and (5.C.10) to deduce that

$$\begin{aligned}\{(n_\varepsilon\cdot\nabla)\nabla u\}^T n_\varepsilon-(\Delta u\otimes n_\varepsilon)n_\varepsilon=-\sum_{i=1,2}\left(\overline{\nabla}_i^\varepsilon\overline{\nabla}_i^\varepsilon u-\overline{\nabla}_{\overline{\nabla}_i^\varepsilon\tau_i}^\varepsilon u\right)+W_\varepsilon^2u-H_\varepsilon W_\varepsilon u-\tilde{\gamma}_\varepsilon H_\varepsilon u&\\-\{\nabla_{\Gamma_\varepsilon}u:W_\varepsilon+\mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u)-\xi_\varepsilon H_\varepsilon\}n_\varepsilon.&\end{aligned}\tag{5.5.50}$$

Hence we take the inner product of (5.5.38) and (5.5.50) to obtain

$$\begin{aligned}\eta_3=\sum_{i=1,2}\left(\overline{\nabla}_i^\varepsilon\overline{\nabla}_i^\varepsilon u-\overline{\nabla}_{\overline{\nabla}_i^\varepsilon\tau_i}^\varepsilon u\right)\cdot(W_\varepsilon u+\tilde{\gamma}_\varepsilon u)-(W_\varepsilon^2u-H_\varepsilon W_\varepsilon u-\tilde{\gamma}_\varepsilon H_\varepsilon u)\cdot(W_\varepsilon u+\tilde{\gamma}_\varepsilon u)&\\-\xi_\varepsilon\{\nabla_{\Gamma_\varepsilon}u:W_\varepsilon+\mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u)-\xi_\varepsilon H_\varepsilon\}.&\end{aligned}\tag{5.5.51}$$

Now we observe by (5.2.39) and direct calculations that

$$\nabla_{\Gamma_\varepsilon}u:(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon=P_\varepsilon(\nabla u):P_\varepsilon(\nabla u)P_\varepsilon=\nabla u:P_\varepsilon^TP_\varepsilon(\nabla u)P_\varepsilon.$$

Since $P_\varepsilon^T=P_\varepsilon^2=P_\varepsilon$, the above equality implies that

$$\nabla_{\Gamma_\varepsilon}u:(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon=\nabla u:P_\varepsilon(\nabla u)P_\varepsilon.\tag{5.5.52}$$

By the same calculations and (5.2.39) we have

$$\begin{aligned}\nabla_{\Gamma_\varepsilon}(W_\varepsilon u):(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon &= \{\nabla_{\Gamma_\varepsilon}W_\varepsilon\cdot u+(\nabla_{\Gamma_\varepsilon}u)W_\varepsilon\}:(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon \\ &= \{\nabla_{\Gamma_\varepsilon}W_\varepsilon\cdot u+(\nabla u)W_\varepsilon\}:P_\varepsilon(\nabla u)P_\varepsilon,\end{aligned} \tag{5.5.53}$$

where the matrix $\nabla_{\Gamma_\varepsilon}W_\varepsilon\cdot u$ is given by (5.5.41), and

$$W_\varepsilon(\nabla_{\Gamma_\varepsilon}u):(\nabla_{\Gamma_\varepsilon}u)P_\varepsilon = W_\varepsilon(\nabla u):(\nabla u)P_\varepsilon, \quad \nabla_{\Gamma_\varepsilon}u:W_\varepsilon=\nabla u:W_\varepsilon. \tag{5.5.54}$$

Also, it is easy to see that

$$\begin{aligned}W_\varepsilon u\cdot\nabla_{\Gamma_\varepsilon}\xi_\varepsilon &= \mathrm{div}_{\Gamma_\varepsilon}(\xi_\varepsilon W_\varepsilon u)-\xi_\varepsilon\mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u), \\ \mathrm{div}_{\Gamma_\varepsilon}(W_\varepsilon u) &= u\cdot\mathrm{div}_{\Gamma_\varepsilon}W_\varepsilon+\nabla_{\Gamma_\varepsilon}u:W_\varepsilon = u\cdot\mathrm{div}_{\Gamma_\varepsilon}W_\varepsilon+\nabla u:W_\varepsilon.\end{aligned} \tag{5.5.55}$$

Hence we deduce from (5.5.42), (5.5.48), (5.5.51)–(5.5.55), and $W_\varepsilon^T=W_\varepsilon$ that

$$\begin{aligned}&\int_{\Gamma_\varepsilon}\nabla u:\{(n_\varepsilon\cdot\nabla)\nabla u-n_\varepsilon\otimes\Delta u\}\,d\mathcal{H}^2 \\ &\qquad = \sum_{i=1,2}\int_{\Gamma_\varepsilon}\left(\overline{\nabla}_i^\varepsilon\overline{\nabla}_i^\varepsilon u-\overline{\nabla}^\varepsilon_{\overline{\nabla}_i^\varepsilon\tau_i}u\right)\cdot(W_\varepsilon u+\tilde{\gamma}_\varepsilon u)\,d\mathcal{H}^2 \\ &\qquad\qquad +\int_{\Gamma_\varepsilon}\left(\frac{1}{2}\varphi_1+\sum_{k=2}^4\varphi_k\right)d\mathcal{H}^2+\int_{\Gamma_\varepsilon}\mathrm{div}_{\Gamma_\varepsilon}(\xi_\varepsilon W_\varepsilon u)\,d\mathcal{H}^2,\end{aligned}$$

where $\varphi_1,\dots,\varphi_4$ are given by (5.5.40). The last integral on the right-hand side vanishes by the Stokes theorem since $\xi_\varepsilon W_\varepsilon u$ is tangential on Γ_ε. Moreover, applying (5.C.15) to the first term and then use (5.C.12), (5.5.52), and (5.5.53) we observe that

$$\sum_{i=1,2}\int_{\Gamma_\varepsilon}\left(\overline{\nabla}_i^\varepsilon\overline{\nabla}_i^\varepsilon u-\overline{\nabla}^\varepsilon_{\overline{\nabla}_i^\varepsilon\tau_i}u\right)\cdot(W_\varepsilon u+\tilde{\gamma}_\varepsilon u)\,d\mathcal{H}^2=\frac{1}{2}\int_{\Gamma_\varepsilon}\varphi_1\,d\mathcal{H}^2.$$

Hence the equality (5.5.39) follows.

The second step is to show that

$$\left|\int_{\Gamma_\varepsilon}\varphi_k\,d\mathcal{H}^2\right|\leq c\left(\|u\|_{H^1(\Omega_\varepsilon)}^2+\|u\|_{H^1(\Omega_\varepsilon)}\|\nabla^2u\|_{L^2(\Omega_\varepsilon)}\right), \quad k=1,2, \tag{5.5.56}$$

$$\left|\int_{\Gamma_\varepsilon}\varphi_k\,d\mathcal{H}^2\right|\leq c\|u\|_{H^1(\Omega_\varepsilon)}^2, \quad k=3,4 \tag{5.5.57}$$

with a constant $c>0$ independent of ε. The estimate (5.5.57) for $k=4$ is an easy consequence of (5.1.6), (5.3.8), and the uniform boundedness in ε of W_ε and H_ε on Γ_ε:

$$\left|\int_{\Gamma_\varepsilon}\varphi_4\,d\mathcal{H}^2\right|\leq c\varepsilon\|u\|_{L^2(\Gamma_\varepsilon)}^2\leq c\varepsilon(\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2+\varepsilon\|\partial_nu\|_{L^2(\Omega_\varepsilon)}^2)\leq c\|u\|_{H^1(\Omega_\varepsilon)}^2.$$

Let us prove the estimate (5.5.56) for $k=1$. We proceed as in the proof of Lemma 5.4.1. In what follows, we use the notations (5.2.46) and (5.2.47) and sometimes suppress the arguments y and r. For $y\in\Gamma$, $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$, and $j,k,l=1,2,3$ we set

$$\begin{aligned}F(y,r) &:= \frac{1}{\varepsilon g(y)}\big\{\big(r-\varepsilon g_0(y)\big)W_{\varepsilon,1}^\sharp(y)-\big(\varepsilon g_1(y)-r\big)W_{\varepsilon,0}^\sharp(y)\big\}, \\ G_{jk}^l(y,r) &:= \frac{1}{\varepsilon g(y)}\big\{\big(r-\varepsilon g_0(y)\big)\big(\underline{D}_j^\varepsilon[W_\varepsilon]_{kl}\big)_1^\sharp(y)-\big(\varepsilon g_1(y)-r\big)\big(\underline{D}_j^\varepsilon[W_\varepsilon]_{kl}\big)_0^\sharp(y)\big\}, \\ \tilde{\gamma}(y,r) &:= \frac{1}{\varepsilon g(y)}\big\{\big(r-\varepsilon g_0(y)\big)\tilde{\gamma}_\varepsilon^1-\big(\varepsilon g_1(y)-r\big)\tilde{\gamma}_\varepsilon^0\big\},\end{aligned}$$

where $\tilde{\gamma}_\varepsilon^i := \gamma_\varepsilon^i/\nu$, $i=0,1$. Then we have

$$\begin{aligned}[][\nabla_{\Gamma_\varepsilon}W_\varepsilon\cdot u+(\nabla u)W_\varepsilon+\tilde{\gamma}_\varepsilon\nabla u]_i^\sharp(y)\\ &=(-1)^{i+1}[G\cdot u^\sharp+(\nabla u)^\sharp F+\tilde{\gamma}(\nabla u)^\sharp](y,\varepsilon g_i(y)),\quad y\in\Gamma,\ i=0,1,\end{aligned}\tag{5.5.58}$$

where $G\cdot u^\sharp$ denotes a 3×3 matrix whose (j,k)-entry is given by

$$[G\cdot u^\sharp]_{jk}:=\sum_{l=1}^3 G_{jk}^l u_l^\sharp,\quad j,k=1,2,3.$$

Moreover, by (5.1.6), (5.2.45) for W_ε and $\underline{D}_j^\varepsilon W_\varepsilon$ with $j=1,2,3$,

$$|r-\varepsilon g_i(y)|\leq \varepsilon g(y)\leq c\varepsilon,\quad y\in\Gamma,\ r\in[\varepsilon g_0(y),\varepsilon g_1(y)],\ i=0,1,\tag{5.5.59}$$

and the uniform boundedness in ε of W_ε and $\underline{D}_j^\varepsilon W_\varepsilon$ on Γ_ε we have

$$|\eta(y,r)|+\left|\frac{\partial\eta}{\partial r}(y,r)\right|\leq c,\quad \eta=F,G_{jk}^l,\tilde{\gamma}\tag{5.5.60}$$

for all $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$ with a constant $c>0$ independent of ε. We also define matrix-valued functions R and S by

$$R(y,r):=\frac{1}{\varepsilon g(y)}\bigl\{\bigl(r-\varepsilon g_0(y)\bigr)P_{\varepsilon,1}^\sharp(y)+\bigl(\varepsilon g_1(y)-r\bigr)P_{\varepsilon,0}^\sharp(y)\bigr\}$$

for $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$, and

$$\begin{aligned}S_i(y)&:=\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}\,P_{\varepsilon,i}^\sharp(y),\quad i=0,1,\\ S(y,r)&:=\frac{1}{\varepsilon g(y)}\bigl\{\bigl(r-\varepsilon g_0(y)\bigr)S_1(y)+\bigl(\varepsilon g_1(y)-r\bigr)S_0(y)\bigr\}.\end{aligned}$$

Then we easily observe that

$$\sqrt{1+\varepsilon^2|\tau_\varepsilon^i(y)|^2}\,[P_\varepsilon(\nabla u)P_\varepsilon]_i^\sharp(y)=[R(\nabla u)^\sharp S](y,\varepsilon g_i(y)),\quad y\in\Gamma,\ i=0,1.\tag{5.5.61}$$

Moreover, from (5.2.44) for P_ε, (5.4.9), and (5.5.59) we deduce that

$$|\eta(y,r)|+\left|\frac{\partial\eta}{\partial r}(y,r)\right|\leq c,\quad \eta=R,S\tag{5.5.62}$$

for all $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$ with a constant $c>0$ independent of ε. Now let us define a function $\Phi_1=\Phi_1(y,r)$ for $y\in\Gamma$ and $r\in[\varepsilon g_0(y),\varepsilon g_1(y)]$ by

$$\Phi_1(y,r):=-2[\{G\cdot u^\sharp+(\nabla u)^\sharp F+\tilde{\gamma}(\nabla u)^\sharp\}:R(\nabla u)S](y,r)J(y,r).$$

Then by the change of variables formula (5.2.57) and the equalities (5.5.58) and (5.5.61) we observe that

$$\begin{aligned}\int_{\Gamma_\varepsilon}\varphi_1(x)\,d\mathcal{H}^2(x)&=\sum_{i=0,1}\int_{\Gamma_\varepsilon^i}\varphi_1(x)\,d\mathcal{H}^2(x)=\int_\Gamma\{\Phi_1(y,\varepsilon g_1(y))-\Phi_1(y,\varepsilon g_0(y))\}\,d\mathcal{H}^2(y)\\ &=\int_\Gamma\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)}\frac{\partial\Phi_1}{\partial r}(y,r)\,dr\,d\mathcal{H}^2(y).\end{aligned}$$

Furthermore, the inequalities (5.2.49), (5.5.60), and (5.5.62) imply that

$$\left|\frac{\partial \Phi_1}{\partial r}\right| \leq c\{|u^\sharp|^2 + |(\nabla u)^\sharp|^2 + (|u^\sharp| + |(\nabla u)^\sharp|)|(\nabla^2 u)^\sharp|\}$$

with some constant $c > 0$ independent of ε (here we also used Young's inequality). From the above relations, (5.2.52), and Hölder's inequality it follows that

$$\begin{aligned}\left|\int_{\Gamma_\varepsilon} \varphi_1 \, d\mathcal{H}^2\right| &\leq c\int_\Gamma \int_{\varepsilon g_0}^{\varepsilon g_1} \{|u^\sharp|^2 + |(\nabla u)^\sharp|^2 + (|u^\sharp| + |(\nabla u)^\sharp|)|(\nabla^2 u)^\sharp|\} \, dr \, d\mathcal{H}^2 \\ &\leq c\left(\|u\|_{H^1(\Omega_\varepsilon)}^2 + \|u\|_{H^1(\Omega_\varepsilon)}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}\right).\end{aligned}$$

(Note that $\|u\|_{L^2(\Omega_\varepsilon)} \leq \|u\|_{H^1(\Omega_\varepsilon)}$.) Thus, the inequality (5.5.56) for $k = 1$ holds. By the same arguments we can show (5.5.56) for $k = 2$ and (5.5.57) for $k = 3$.

Finally, from (5.5.39), (5.5.56), and (5.5.57) we deduce that

$$\left|\int_{\Gamma_\varepsilon} \nabla u : \{(n_\varepsilon \cdot \nabla)\nabla u - n_\varepsilon \otimes \Delta u\} \, d\mathcal{H}^2\right| \leq c\left(\|u\|_{H^1(\Omega_\varepsilon)}^2 + \|u\|_{H^1(\Omega_\varepsilon)}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}\right).$$

We apply this inequality to (5.5.37) and then use Young's inequality to obtain

$$\begin{aligned}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}^2 &\leq \|\Delta u\|_{L^2(\Omega_\varepsilon)}^2 + c\left(\|u\|_{H^1(\Omega_\varepsilon)}^2 + \|u\|_{H^1(\Omega_\varepsilon)}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}\right) \\ &\leq \|\Delta u\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{2}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)}^2 + c\|u\|_{H^1(\Omega_\varepsilon)}^2,\end{aligned}$$

which yields (5.5.36). Hence the inequality (5.5.31) is valid. □

5.6 Average operators in the thin direction

5.6.1 Definition and basic inequalities of the average operators

In this section we investigate average operators in the thin direction and establish several inequalities related to them, which are useful in the analysis of the Navier–Stokes equations in the curved thin domain Ω_ε. Throughout this section we assume $\varepsilon \in (0,1)$.

Definition 5.6.1. We define the average operator M in the normal direction of Γ as

$$M\varphi(y) := \frac{1}{\varepsilon g(y)} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi(y + rn(y)) \, dr, \quad y \in \Gamma \tag{5.6.1}$$

for a function φ on Ω_ε. The operator M is also applied to a vector field $u\colon \Omega_\varepsilon \to \mathbb{R}^3$ and we define the averaged tangential component $M_\tau u$ of u by

$$M_\tau u(y) := P(y)Mu(y) = \frac{1}{\varepsilon g(y)} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} P(y)u(y + rn(y)) \, dr, \quad y \in \Gamma. \tag{5.6.2}$$

For the sake of simplicity, we denote the tangential and normal components (with respect to the surface Γ) of a vector field $u\colon \Omega_\varepsilon \to \mathbb{R}^3$ by

$$u_\tau(x) := \overline{P}(x)u(x), \quad u_n(x) := \{u(x) \cdot \bar{n}(x)\}\bar{n}(x), \quad x \in \Omega_\varepsilon, \tag{5.6.3}$$

where $\overline{P}$ and $\bar{n}$ are the constant extensions of P and n in the normal direction of Γ, so that $u = u_\tau + u_n$ and $u_\tau \cdot u_n = 0$ (note that u_n is a vector field). Also, we use the notations (5.2.46) and (5.2.47) and sometimes suppress the arguments of functions. For example, we write

$$M\varphi = \frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1} \varphi^\sharp\,dr, \quad M_\tau u = \frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1} u_\tau^\sharp\,dr.$$

Let us derive basic inequalities for the average operators M and M_τ. First note that for a vector field u on Ω_ε we have $M_\tau u = Mu_\tau$ by (5.6.2) and (5.6.3). Hence the following inequalities for M are also valid for M_τ.

Lemma 5.6.2. *Let $p \in [1,\infty)$. There exists $c > 0$ independent of ε such that*

$$\|M\varphi\|_{L^p(\Gamma)} \le c\varepsilon^{-1/p}\|\varphi\|_{L^p(\Omega_\varepsilon)}, \tag{5.6.4}$$

$$\left\|\overline{M\varphi}\right\|_{L^p(\Omega_\varepsilon)} \le c\|\varphi\|_{L^p(\Omega_\varepsilon)} \tag{5.6.5}$$

for all $\varphi \in L^p(\Omega_\varepsilon)$. Here $\overline{M\varphi} := (M\varphi)\circ\pi$ is the constant extension of $M\varphi$.

Proof. By Hölder's inequality and (5.2.30),

$$|M\varphi(y)|^p = \left|\frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi^\sharp(y,r)\,dr\right|^p \le c\varepsilon^{-1}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\varphi^\sharp(y,r)|^p\,dr$$

for all $y \in \Gamma$. Integrating both sides of the above inequality over Γ and using (5.2.52) we obtain (5.6.4). The inequality (5.6.5) follows from (5.2.53) and (5.6.4). $\square$

Lemma 5.6.3. *Let $p \in [1,\infty)$. There exists $c > 0$ independent of ε such that*

$$\left\|\varphi - \overline{M\varphi}\right\|_{L^p(\Omega_\varepsilon)} \le c\varepsilon\|\partial_n\varphi\|_{L^p(\Omega_\varepsilon)}, \tag{5.6.6}$$

$$\left\|\varphi - \overline{M\varphi}\right\|_{L^p(\Gamma_\varepsilon^i)} \le c\varepsilon^{1-1/p}\|\partial_n\varphi\|_{L^p(\Omega_\varepsilon)}, \quad i = 0,1 \tag{5.6.7}$$

for all $\varphi \in W^{1,p}(\Omega_\varepsilon)$, where $\partial_n\varphi$ is the normal derivative of φ given by (5.3.5).

Proof. For $y \in \Gamma$ and $r \in (\varepsilon g_0(y), \varepsilon g_1(y))$ we have

$$\varphi^\sharp(y,r) - M\varphi(y) = \frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \{\varphi^\sharp(y,r) - \varphi^\sharp(y,r_1)\}\,dr_1. \tag{5.6.8}$$

Since $\partial\varphi^\sharp/\partial r = (\partial_n\varphi)^\sharp$ by (5.2.46) and (5.3.5),

$$|\varphi^\sharp(y,r) - \varphi^\sharp(y,r_1)| = \left|\int_{r_1}^{r} \frac{\partial}{\partial r_2}\big(\varphi^\sharp(y,r_2)\big)\,dr_2\right| \le \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n\varphi)^\sharp(y,r_2)|\,dr_2.$$

Noting that the right-hand side is independent of r_1, we apply the above inequality to the right-hand side of (5.6.8) to get

$$|\varphi^\sharp(y,r) - M\varphi(y)| \le \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n\varphi)^\sharp(y,r_2)|\,dr_2 \tag{5.6.9}$$

for all $y \in \Gamma$ and $r \in (\varepsilon g_0(y), \varepsilon g_1(y))$. We use Hölder's inequality to (5.6.9) to get

$$|\varphi^\sharp(y,r) - M\varphi(y)| \leq c\varepsilon^{1-1/p} \left(\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n \varphi)^\sharp(y,r_2)|^p \, dr_2 \right)^{1/p}.$$

for all $y \in \Gamma$ and $r \in (\varepsilon g_0(y), \varepsilon g_1(y))$. Note that the right-hand side is independent of r. The above inequality and (5.2.52) imply that

$$\begin{aligned}
\left\| \varphi - \overline{M\varphi} \right\|_{L^p(\Omega_\varepsilon)}^p &\leq c \int_\Gamma \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |\varphi^\sharp(y,r) - M\varphi(y)|^p \, dr \, d\mathcal{H}^2(y) \\
&\leq c \int_\Gamma \varepsilon g(y) \left(c\varepsilon^{p-1} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} |(\partial_n \varphi)^\sharp(y,r_2)|^p \, dr_2 \right) d\mathcal{H}^2(y) \\
&\leq c\varepsilon^p \|\partial_n \varphi\|_{L^p(\Omega_\varepsilon)}^p.
\end{aligned}$$

Hence (5.6.6) holds. We also have (5.6.7) by applying (5.3.8) to $\varphi - \overline{M\varphi}$ and using (5.3.6) and (5.6.6). □

Lemma 5.6.4. *Let $p \in [1,\infty)$. There exists $c > 0$ independent of ε such that*

$$\|Mu \cdot n\|_{L^p(\Gamma)} \leq c\varepsilon^{1-1/p} \|u\|_{W^{1,p}(\Omega_\varepsilon)} \tag{5.6.10}$$

for all $u \in W^{1,p}(\Omega_\varepsilon)^3$ satisfying $u \cdot n_\varepsilon = 0$ on Γ_ε^0 or on Γ_ε^1.

Proof. Applying (5.6.4) to $Mu \cdot n = M(u \cdot \bar{n})$ and using (5.3.23) we obtain (5.6.10). □

Unlike the case of a flat thin domain, the constant extension of the average operator on $L^2(\Omega_\varepsilon)$ is not symmetric because the Jacobian $J(y,r)$ appears in the change of variables formula (5.2.51). However, its skew-symmetric part is small of order ε.

Lemma 5.6.5. *There exists a constant $c > 0$ independent of ε such that*

$$\left| \left(\overline{M\varphi}, \xi \right)_{L^2(\Omega_\varepsilon)} - \left(\varphi, \overline{M\xi} \right)_{L^2(\Omega_\varepsilon)} \right| \leq c\varepsilon \|\varphi\|_{L^2(\Omega_\varepsilon)} \|\xi\|_{L^2(\Omega_\varepsilon)} \tag{5.6.11}$$

for all $\varphi, \xi \in L^2(\Omega_\varepsilon)$.

Proof. By (5.2.51) and (5.6.1) we have

$$\begin{aligned}
\left(\overline{M\varphi}, \xi \right)_{L^2(\Omega_\varepsilon)} &= \int_\Gamma M\varphi \left(\int_{\varepsilon g_0}^{\varepsilon g_1} \xi^\sharp J \, dr \right) d\mathcal{H}^2 \\
&= \varepsilon (M\varphi, gM\xi)_{L^2(\Gamma)} + \int_\Gamma M\varphi \left(\int_{\varepsilon g_0}^{\varepsilon g_1} \xi^\sharp (J-1) \, dr \right) d\mathcal{H}^2
\end{aligned}$$

and

$$\left(\varphi, \overline{M\xi} \right)_{L^2(\Omega_\varepsilon)} = \varepsilon (gM\varphi, M\xi)_{L^2(\Gamma)} + \int_\Gamma \left(\int_{\varepsilon g_0}^{\varepsilon g_1} \varphi^\sharp (J-1) \, dr \right) M\xi \, d\mathcal{H}^2.$$

Since $(M\varphi, gM\xi)_{L^2(\Gamma)} = (gM\varphi, M\xi)_{L^2(\Gamma)}$, we see by (5.2.50) that

$$\begin{aligned}
&\left| \left(\overline{M\varphi}, \xi \right)_{L^2(\Omega_\varepsilon)} - \left(\varphi, \overline{M\xi} \right)_{L^2(\Omega_\varepsilon)} \right| \\
&\qquad \leq c\varepsilon \left\{ \int_\Gamma |M\varphi| \left(\int_{\varepsilon g_0}^{\varepsilon g_1} |\xi^\sharp| \, dr \right) d\mathcal{H}^2 + \int_\Gamma \left(\int_{\varepsilon g_0}^{\varepsilon g_1} |\varphi^\sharp| \, dr \right) |M\xi| \, d\mathcal{H}^2 \right\}.
\end{aligned} \tag{5.6.12}$$

Moreover, applying Hölder's inequality twice and using (5.2.52) and (5.6.4) we get

$$\begin{aligned}\int_\Gamma |M\varphi| \left(\int_{\varepsilon g_0}^{\varepsilon g_1} |\xi^\sharp|\,dr\right) d\mathcal{H}^2 &\leq \|M\varphi\|_{L^2(\Gamma)} \left\{\int_\Gamma \varepsilon g \left(\int_{\varepsilon g_0}^{\varepsilon g_1} |\xi^\sharp|^2\,dr\right) d\mathcal{H}^2\right\}^{1/2} \\ &\leq c\|\varphi\|_{L^2(\Omega_\varepsilon)}\|\xi\|_{L^2(\Omega_\varepsilon)}\end{aligned}$$

and a similar inequality for the second term on the right-hand side of (5.6.12). Hence (5.6.11) follows. □

Now let us consider the time derivative of the average operator.

Lemma 5.6.6. *Let $\varphi \in H^1(0,T;L^2(\Omega_\varepsilon))$, $T>0$. Then*

$$\partial_t M\varphi = M(\partial_t\varphi) \in L^2(0,T;L^2(\Gamma))$$

and there exists a constant $c>0$ independent of ε and φ such that

$$\|\partial_t M\varphi\|_{L^2(0,T;L^2(\Gamma))} \leq c\varepsilon^{-1/2}\|\partial_t\varphi\|_{L^2(0,T;L^2(\Omega_\varepsilon))}. \tag{5.6.13}$$

Proof. First note that $M(\partial_t\varphi) \in L^2(0,T;L^2(\Gamma))$ by $\partial_t\varphi \in L^2(0,T;L^2(\Omega_\varepsilon))$ and (5.6.4). The relation $\partial_t M\varphi = M(\partial_t\varphi)$ is formally trivial since g_0, g_1, and the surface quantities on Γ are independent of time. To prove it rigorously we show that

$$\int_0^T \partial_t\xi(t)M\varphi(t)\,dt = -\int_0^T \xi(t)[M(\partial_t\varphi)](t)\,dt \quad \text{in} \quad L^2(\Gamma)$$

for all $\xi \in C_c^\infty(0,T)$. Since $L^2(\Gamma)$ is a Hilbert space, this is equivalent to

$$\int_0^T \partial_t\xi(t)(M\varphi(t),\eta)_{L^2(\Gamma)}\,dt = -\int_0^T \xi(t)([M(\partial_t\varphi)](t),\eta)_{L^2(\Gamma)}\,dt \tag{5.6.14}$$

for all $\xi \in C_c^\infty(0,T)$ and $\eta \in L^2(\Gamma)$. We define a function $\tilde\eta$ on Ω_ε by

$$\tilde\eta(x) := \frac{\eta(\pi(x))}{\varepsilon g(\pi(x))J(\pi(x),d(x))}, \quad x \in \Omega_\varepsilon.$$

Then by (5.2.30), (5.2.49), and (5.2.53) we see that $\tilde\eta \in L^2(\Omega_\varepsilon)$. Also, by the change of variables formula (5.2.51) and the definition (5.6.1) of M we have

$$(\varphi(t),\tilde\eta)_{L^2(\Omega_\varepsilon)} = \int_\Gamma \left(\frac{1}{\varepsilon g}\int_{\varepsilon g_0}^{\varepsilon g_1} \varphi^\sharp(t)\,dr\right)\eta\,d\mathcal{H}^2 = (M\varphi(t),\eta)_{L^2(\Gamma)} \tag{5.6.15}$$

for all $t \in (0,T)$. Hence

$$\int_0^T \partial_t\xi(t)(M\varphi(t),\eta)_{L^2(\Gamma)}\,dt = \int_0^T \partial_t\xi(t)(\varphi(t),\tilde\eta)_{L^2(\Omega_\varepsilon)}\,dt. \tag{5.6.16}$$

Moreover, since $\varphi \in H^1(0,T;L^2(\Omega_\varepsilon))$ and $\tilde\eta \in L^2(\Omega_\varepsilon)$ is independent of time,

$$\int_0^T \partial_t\xi(t)(\varphi(t),\tilde\eta)_{L^2(\Omega_\varepsilon)}\,dt = -\int_0^T \xi(t)(\partial_t\varphi(t),\tilde\eta)_{L^2(\Omega_\varepsilon)}\,dt.$$

Applying this equality to the right-hand side of (5.6.16) and using (5.6.15) with $\varphi(t)$ replaced by $\partial_t\varphi(t)$ we obtain (5.6.14). Hence the relation $\partial_t M\varphi = M(\partial_t\varphi)$ is valid and the inequality (5.6.13) follows from (5.6.4). □

5.6.2 Tangential derivatives of the average operators

Let us give several formulas and inequalities for the tangential derivatives of the average operators.

Lemma 5.6.7. *For $\varphi \in C^1(\Omega_\varepsilon)$ we have*

$$\nabla_\Gamma M\varphi = M(B\nabla\varphi) + M\big((\partial_n\varphi)\psi_\varepsilon\big) \quad \text{on} \quad \Gamma, \tag{5.6.17}$$

where the matrix-valued function B and the vector field ψ_ε are given by

$$\begin{aligned} B(x) &:= \big\{I_3 - d(x)\overline{W}(x)\big\}\,\overline{P}(x), \\ \psi_\varepsilon(x) &:= \frac{1}{\bar{g}(x)}\,\big\{\big(d(x)-\varepsilon\bar{g}_0(x)\big)\overline{\nabla_\Gamma g_1}(x) + \big(\varepsilon\bar{g}_1(x)-d(x)\big)\overline{\nabla_\Gamma g_0}(x)\big\} \end{aligned} \tag{5.6.18}$$

for $x \in N$. Here the functions on the right-hand sides of (5.6.18) *except for $d(x)$ are the constant extensions of the functions on Γ.*

Proof. The constant extension of $M\varphi$ is given by

$$\overline{M\varphi}(x) = \frac{1}{\varepsilon\bar{g}(x)}\int_{\varepsilon\bar{g}_0(x)}^{\varepsilon\bar{g}_1(x)} \varphi(\pi(x)+r\bar{n}(x))\,dr, \quad x \in N.$$

We differentiate both sides of this equality with respect to $x \in N$ and set $x = y \in \Gamma$. Then by (5.2.4), (5.2.6), (5.2.11), and (5.2.16) with $d(y) = 0$ we get

$$\nabla_\Gamma M\varphi(y) = \frac{I(y)}{\varepsilon g(y)} + \frac{1}{\varepsilon g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \{I_3 - rW(y)\}P(y)(\nabla\varphi)^\sharp(y,r)\,dr \tag{5.6.19}$$

for $y \in \Gamma$. Here and hereafter we use the notations (5.2.46) and (5.2.47) and set

$$I(y) := -\frac{\nabla_\Gamma g(y)}{g(y)}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \varphi^\sharp(y,r)\,dr + \varepsilon\varphi_1^\sharp(y)\nabla_\Gamma g_1(y) - \varepsilon\varphi_0^\sharp(y)\nabla_\Gamma g_0(y).$$

To the right-hand side we apply

$$\varepsilon\varphi_1^\sharp(y)\nabla_\Gamma g_1(y) - \varepsilon\varphi_0^\sharp(y)\nabla_\Gamma g_0(y) = \Big[(\varphi\psi_\varepsilon)^\sharp(y,r)\Big]_{r=\varepsilon g_0(y)}^{\varepsilon g_1(y)} = \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \frac{\partial}{\partial r}\big((\varphi\psi_\varepsilon)^\sharp(y,r)\big)\,dr$$

and the equalities $\partial\varphi^\sharp/\partial r = (\partial_n\varphi)^\sharp$ and $\partial\psi_\varepsilon^\sharp/\partial r = \nabla_\Gamma g/g$ by

$$\psi_\varepsilon^\sharp(y,r) = \frac{1}{g(y)}\big\{\big(r-\varepsilon g_0(y)\big)\nabla_\Gamma g_1(y) + \big(\varepsilon g_1(y)-r\big)\nabla_\Gamma g_0(y)\big\}.$$

Then we have

$$I(y) = \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \big((\partial_n\varphi)\psi_\varepsilon\big)^\sharp(y,r)\,dr = \varepsilon g(y)\big[M\big((\partial_n\varphi)\psi_\varepsilon\big)\big](y)$$

for all $y \in \Gamma$. Applying this and $\{I_3 - rW(y)\}P(y) = B^\sharp(y,r)$ to the right-hand side of (5.6.19) we obtain (5.6.17). □

Remark 5.6.8. By (5.3.39) and (5.6.18) there exists $c>0$ independent of ε such that

$$|B| \leq c, \quad |\psi_\varepsilon| \leq c\varepsilon \quad \text{in} \quad \Omega_\varepsilon. \tag{5.6.20}$$

Also, from (5.2.13), (5.2.30), and $\nabla d = \bar{n}$ in N it follows that

$$|\nabla B| \leq c, \quad |\nabla \psi_\varepsilon| \leq c, \quad \left|\nabla\psi_\varepsilon - \frac{1}{\bar{g}}\bar{n}\otimes\overline{\nabla_\Gamma g}\right| \leq c\varepsilon \quad \text{in} \quad \Omega_\varepsilon. \tag{5.6.21}$$

Lemma 5.6.9. *Let $m=1,2$ and $p\in[1,\infty)$. For $\varphi\in W^{m,p}(\Omega_\varepsilon)$ we have*

$$\|M\varphi\|_{W^{m,p}(\Gamma)} \leq c\varepsilon^{-1/p}\|\varphi\|_{W^{m,p}(\Omega_\varepsilon)}, \tag{5.6.22}$$

$$\left\|\overline{M\varphi}\right\|_{W^{m,p}(\Omega_\varepsilon)} \leq c\|\varphi\|_{W^{m,p}(\Omega_\varepsilon)} \tag{5.6.23}$$

with some constant $c>0$ independent of ε and φ.

Proof. Let $\varphi\in W^{1,p}(\Omega_\varepsilon)$. From (5.6.4) and (5.6.17) it follows that

$$\begin{aligned}\|\nabla_\Gamma M\varphi\|_{L^p(\Gamma)} &\leq c\left(\|M(B\nabla\varphi)\|_{L^p(\Gamma)} + \left\|M\big((\partial_n\varphi)\psi_\varepsilon\big)\right\|_{L^p(\Gamma)}\right)\\ &\leq c\varepsilon^{-1/p}\left(\|B\nabla\varphi\|_{L^p(\Omega_\varepsilon)} + \|(\partial_n\varphi)\psi_\varepsilon\|_{L^p(\Omega_\varepsilon)}\right).\end{aligned}$$

Here B and ψ_ε are bounded on Ω_ε uniformly in ε (see Remark 5.6.8). Hence

$$\|\nabla_\Gamma M\varphi\|_{L^p(\Gamma)} \leq c\varepsilon^{-1/p}\|\nabla\varphi\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon^{-1/p}\|\varphi\|_{W^{1,p}(\Omega_\varepsilon)} \tag{5.6.24}$$

Combining (5.6.24) with (5.6.4) we obtain (5.6.22) with $m=1$. When $\varphi\in W^{2,p}(\Omega_\varepsilon)$, we apply (5.6.24) with φ replaced by $B\nabla\varphi$ and $(\partial_n\varphi)\psi_\varepsilon$. Then by (5.6.20)–(5.6.21) we see that

$$\|\nabla_\Gamma M(B\nabla\varphi)\|_{L^p(\Gamma)} + \left\|\nabla_\Gamma M\big((\partial_n\varphi)\psi_\varepsilon\big)\right\|_{L^p(\Gamma)} \leq c\varepsilon^{-1/p}\|\varphi\|_{W^{2,p}(\Omega_\varepsilon)}.$$

Therefore, applying ∇_Γ to (5.6.17) and using the above inequality we get

$$\|\nabla_\Gamma^2 M\varphi\|_{L^p(\Gamma)} \leq c\varepsilon^{-1/p}\|\varphi\|_{W^{2,p}(\Omega_\varepsilon)}$$

and (5.6.22) with $m=2$ follows from this inequality, (5.6.4), and (5.6.24). The inequality (5.6.23) is an immediate consequence of (5.2.55), (5.2.56), and (5.6.22). ☐

Lemma 5.6.10. *There exists a constant $c>0$ independent of ε such that*

$$\left\|\overline{P}\nabla\varphi - \overline{\nabla_\Gamma M\varphi}\right\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\|\varphi\|_{W^{2,p}(\Omega_\varepsilon)} \tag{5.6.25}$$

$$\left\|\overline{P}\nabla\varphi - \overline{\nabla_\Gamma M\varphi}\right\|_{L^p(\Gamma_\varepsilon^i)} \leq c\varepsilon^{1-1/p}\|\varphi\|_{W^{2,p}(\Omega_\varepsilon)}, \quad i=0,1 \tag{5.6.26}$$

for all $\varphi\in W^{2,p}(\Omega_\varepsilon)$ with $p\in[1,\infty)$.

Proof. By (5.6.17) and (5.6.18) we have $\overline{P}\nabla\varphi - \overline{\nabla_\Gamma M\varphi} = u+\bar{v}$ in Ω_ε, where

$$\begin{aligned} u(x) &:= \overline{P}(x)\nabla\varphi(x) - \left[M\Big(\overline{P}\nabla\varphi\Big)\right](\pi(x)), \quad x\in\Omega_\varepsilon,\\ v(y) &:= \left[M\Big(d\overline{W}\nabla\varphi\Big)\right](y) - \left[M\big((\partial_n\varphi)\psi_\varepsilon\big)\right](y), \quad y\in\Gamma.\end{aligned}$$

We apply (5.6.6) to u and use (5.3.6) to get

$$\|u\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\left\|\partial_n\Big(\overline{P}\nabla\varphi\Big)\right\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\|\varphi\|_{W^{2,p}(\Omega_\varepsilon)}. \tag{5.6.27}$$

Also, from (5.6.5) and

$$\left|d\overline{W}\nabla\varphi\right| \leq c\varepsilon|\nabla\varphi|, \quad |(\partial_n\varphi)\psi_\varepsilon| \leq c\varepsilon|\nabla\varphi| \quad \text{in} \quad \Omega_\varepsilon$$

by (5.6.20) and $|d| \leq c\varepsilon$ in Ω_ε it follows that

$$\|\bar{v}\|_{L^p(\Omega_\varepsilon)} \leq c\left(\left\|d\overline{W}\nabla\varphi\right\|_{L^p(\Omega_\varepsilon)} + \|(\partial_n\varphi)\psi_\varepsilon\|_{L^p(\Omega_\varepsilon)}\right) \leq c\varepsilon\|\nabla\varphi\|_{L^p(\Omega_\varepsilon)}. \tag{5.6.28}$$

Combining (5.6.27) and (5.6.28) we obtain

$$\left\|\overline{P}\nabla\varphi - \overline{\nabla_\Gamma M\varphi}\right\|_{L^p(\Omega_\varepsilon)} \leq \|u\|_{L^p(\Omega_\varepsilon)} + \|\bar{v}\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\|\varphi\|_{W^{2,p}(\Omega_\varepsilon)}.$$

Hence (5.6.25) holds. Also, (5.6.26) follows from (5.3.6), (5.3.8), and (5.6.25). □

Lemma 5.6.11. *There exists a constant $c > 0$ independent of ε such that*

$$\|Mu\cdot n\|_{W^{1,p}(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)} \tag{5.6.29}$$

for all $u \in W^{2,p}(\Omega_\varepsilon)^3$, $p \in [1,\infty)$ satisfying $u\cdot n_\varepsilon = 0$ on Γ_ε^0 or on Γ_ε^1.

Proof. The estimate for the $L^p(\Gamma)$-norm of $Mu\cdot n$ is given in Lemma 5.6.4. Let us consider the tangential gradient of $Mu\cdot n = M(u\cdot\bar{n})$. By (5.6.17),

$$\nabla_\Gamma(Mu\cdot n) = \nabla_\Gamma M(u\cdot\bar{n}) = M\big(B\nabla(u\cdot\bar{n})\big) + M(\partial_n(u\cdot\bar{n})\psi_\varepsilon) \quad \text{on} \quad \Gamma.$$

To the right-hand side we apply (5.6.4) and

$$|B\nabla(u\cdot\bar{n})| \leq c\left|\overline{P}\nabla(u\cdot\bar{n})\right|, \quad |\partial_n(u\cdot\bar{n})\psi_\varepsilon| \leq c\varepsilon|\nabla u| \quad \text{in} \quad \Omega_\varepsilon$$

by (5.2.9), (5.3.6), and (5.6.20) to get

$$\begin{aligned}\|\nabla_\Gamma(Mu\cdot n)\|_{L^p(\Gamma)} &\leq c\left(\left\|M\big(B\nabla(u\cdot\bar{n})\big)\right\|_{L^p(\Gamma)} + \|M(\partial_n(u\cdot\bar{n})\psi_\varepsilon)\|_{L^p(\Gamma)}\right)\\ &\leq c\varepsilon^{-1/p}\left(\|B\nabla(u\cdot\bar{n})\|_{L^p(\Omega_\varepsilon)} + \|\partial_n(u\cdot\bar{n})\psi_\varepsilon\|_{L^p(\Omega_\varepsilon)}\right)\\ &\leq c\left(\varepsilon^{-1/p}\left\|\overline{P}\nabla(u\cdot\bar{n})\right\|_{L^p(\Omega_\varepsilon)} + \varepsilon^{1-1/p}\|\nabla u\|_{L^p(\Omega_\varepsilon)}\right).\end{aligned}$$

Applying (5.3.24) to the first term on the last line we obtain

$$\|\nabla_\Gamma(Mu\cdot n)\|_{L^p(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}.$$

Hence (5.6.29) follows. □

Next we establish estimates for the weighted surface divergence of the average of a solenoidal vector field on Ω_ε. They are useful for the proof of the global existence of a strong solution as well as the study of a singular limit problem for (5.1.1)–(5.1.3).

Lemma 5.6.12. *Let $p \in [1,\infty)$. There exists $c > 0$ independent of ε such that*

$$\|\mathrm{div}_\Gamma(gMu)\|_{L^p(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{1,p}(\Omega_\varepsilon)} \tag{5.6.30}$$

for all $u \in W^{1,p}(\Omega_\varepsilon)^3$ satisfying $\mathrm{div}\, u = 0$ in Ω_ε and $u\cdot n_\varepsilon = 0$ on Γ_ε. If in addition $u \in W^{2,p}(\Omega_\varepsilon)^3$, then we have

$$\|\mathrm{div}_\Gamma(gMu)\|_{W^{1,p}(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}. \tag{5.6.31}$$

Proof. As in the proofs of the previous lemmas we use the notations (5.2.46) and (5.2.47). Let $u \in W^{1,p}(\Omega_\varepsilon)^3$ satisfy $\operatorname{div} u = 0$ in Ω_ε and $u \cdot n_\varepsilon = 0$ on Γ_ε. First we show that

$$\operatorname{div}_\Gamma(gMu) = \sum_{j=1}^{4} \eta_j \quad \text{on} \quad \Gamma, \tag{5.6.32}$$

where $\eta_1, \dots, \eta_4$ are functions on Γ defined by

$$\begin{aligned} \eta_1 &:= \sum_{i=0,1} (-1)^i (u_i^\sharp - Mu) \cdot \tau_\varepsilon^i, \quad \eta_2 := \sum_{i=0,1} (-1)^i Mu \cdot (\tau_\varepsilon^i - \nabla_\Gamma g_i), \\ \eta_3 &:= -gM\left(d \operatorname{tr}\left[\overline{W}\nabla u\right]\right), \quad \eta_4 := gM(\partial_n u \cdot \psi_\varepsilon) \end{aligned} \tag{5.6.33}$$

with τ_ε^i, $i = 0, 1$ given by (5.2.32). By (5.6.17) and (5.6.18) we have

$$\begin{aligned} g \operatorname{div}_\Gamma(Mu) &= g \operatorname{tr}[\nabla_\Gamma Mu] = gM(\operatorname{tr}[B\nabla u]) + gM(\operatorname{tr}[\psi_\varepsilon \otimes \partial_n u]) \\ &= gM\left(\operatorname{tr}\left[\overline{P}\nabla u\right]\right) + \eta_3 + \eta_4 \end{aligned}$$

on Γ. Moreover, since $\operatorname{div} u = 0$ in Ω_ε and $(\bar{n} \otimes \bar{n})\nabla u = \bar{n} \otimes \partial_n u$,

$$\operatorname{tr}\left[\overline{P}\nabla u\right] = \operatorname{div} u - \operatorname{tr}[(\bar{n} \otimes \bar{n})\nabla u] = -\bar{n} \cdot \partial_n u \quad \text{in} \quad \Omega_\varepsilon.$$

By these equalities and $\operatorname{div}_\Gamma(gMu) = \nabla_\Gamma g \cdot Mu + g \operatorname{div}_\Gamma(Mu)$ we get

$$\operatorname{div}_\Gamma(gMu) = \nabla_\Gamma g \cdot Mu - gM(\partial_n u \cdot \bar{n}) + \eta_3 + \eta_4 \quad \text{on} \quad \Gamma. \tag{5.6.34}$$

Let us calculate the second term on the right-hand side. Since

$$(\partial_n u \cdot \bar{n})^\sharp(y, r) = \frac{\partial}{\partial r}\Big((u \cdot \bar{n})^\sharp(y, r)\Big), \quad y \in \Gamma,\ r \in (\varepsilon g_0(y), \varepsilon g_1(y))$$

by $\partial_n \bar{n} = 0$ in Ω_ε, we have

$$\begin{aligned} g(y)[M(\partial_n u \cdot \bar{n})](y) &= \frac{1}{\varepsilon}\int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} \frac{\partial}{\partial r}\Big((u \cdot \bar{n})^\sharp(y, r)\Big)\,dr \\ &= \frac{1}{\varepsilon}\{(u \cdot \bar{n})^\sharp(y, \varepsilon g_1(y)) - (u \cdot \bar{n})^\sharp(y, \varepsilon g_0(y))\} \end{aligned}$$

for $y \in \Gamma$. Moreover, since $u \cdot n_\varepsilon = 0$ on Γ_ε, we use (5.3.22) to get

$$(u \cdot \bar{n})^\sharp(y, \varepsilon g_i(y)) = \varepsilon(u \cdot \bar{\tau}_\varepsilon^i)^\sharp(y, \varepsilon g_i(y)) = \varepsilon(u_i^\sharp \cdot \tau_\varepsilon^i)(y), \quad y \in \Gamma.$$

Hence

$$gM(\partial_n u \cdot \bar{n}) = u_1^\sharp \cdot \tau_\varepsilon^1 - u_0^\sharp \cdot \tau_\varepsilon^0 = \nabla_\Gamma g \cdot Mu - \eta_1 - \eta_2 \quad \text{on} \quad \Gamma. \tag{5.6.35}$$

Combining (5.6.34) and (5.6.35) we obtain (5.6.32). Let us estimate the L^p-norms of the functions $\eta_1, \dots, \eta_4$ on Γ. By (5.2.34), (5.2.58), and (5.6.7),

$$\|\eta_1\|_{L^p(\Gamma)} \leq c \sum_{i=0,1} \left\|u - \overline{Mu}\right\|_{L^p(\Gamma_\varepsilon^i)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{1,p}(\Omega_\varepsilon)}. \tag{5.6.36}$$

The first inequality of (5.2.35) and (5.6.4) imply that

$$\|\eta_2\|_{L^p(\Gamma)} \leq c\varepsilon\|Mu\|_{L^p(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{L^p(\Omega_\varepsilon)}. \tag{5.6.37}$$

To η_3 and η_4 we apply (5.6.4) and

$$\left|d\,\mathrm{tr}\Big[\overline{W}\nabla u\Big]\right| \leq c\varepsilon|\nabla u|, \quad |\partial_n u\cdot\psi_\varepsilon| \leq c\varepsilon|\nabla u| \quad \text{in} \quad \Omega_\varepsilon$$

by (5.6.20), $|d| \leq c\varepsilon$ in Ω_ε, and the boundedness of W to get

$$\begin{aligned} \|\eta_3\|_{L^p(\Gamma)} &\leq c\varepsilon^{-1/p}\left\|d\,\mathrm{tr}\Big[\overline{W}\nabla u\Big]\right\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon^{1-1/p}\|\nabla u\|_{L^p(\Omega_\varepsilon)}, \\ \|\eta_4\|_{L^p(\Gamma)} &\leq c\varepsilon^{-1/p}\|\partial_n u\cdot\psi_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon^{1-1/p}\|\nabla u\|_{L^p(\Omega_\varepsilon)}. \end{aligned} \tag{5.6.38}$$

Applying (5.6.36), (5.6.37), and (5.6.38) to (5.6.32) we obtain (5.6.30).

Now we suppose $u \in W^{2,p}(\Omega_\varepsilon)^3$ and estimate the L^p-norms of $\nabla_\Gamma\eta_1,\dots,\nabla_\Gamma\eta_4$ on Γ. By (5.2.47) and

$$\nabla_\Gamma\big(y+\varepsilon g_i(y)n(y)\big) = P(y)+\varepsilon\{\nabla_\Gamma g_i(y)\otimes n(y)-g_i(y)W(y)\}, \quad y\in\Gamma$$

we have $\nabla_\Gamma u_i^\sharp = (P+\varepsilon G_i)(\nabla u)_i^\sharp$ on Γ, where

$$G_i(y) := \nabla_\Gamma g_i(y)\otimes n(y) - g_i(y)W(y), \quad y\in\Gamma,\ i=0,1.$$

Hence

$$\begin{aligned} \nabla_\Gamma\eta_1 &= \sum_{i=0,1}(-1)^i\{(\nabla_\Gamma u_i^\sharp-\nabla_\Gamma Mu)\tau_\varepsilon^i+(\nabla_\Gamma\tau_\varepsilon^i)(u_i^\sharp-Mu)\} \\ &= \sum_{i=0,1}(-1)^i\Big[\big\{\big(P(\nabla u)_i^\sharp-\nabla_\Gamma Mu\big)+\varepsilon G_i(\nabla u)_i^\sharp\big\}\tau_\varepsilon^i+(\nabla_\Gamma\tau_\varepsilon^i)(u_i^\sharp-Mu)\Big] \end{aligned}$$

on Γ. Since G_0 and G_1 are bounded on Γ, we see by (5.2.34) that

$$|\nabla_\Gamma\eta_1| \leq c\sum_{i=0,1}(|P(\nabla u)_i^\sharp-\nabla_\Gamma Mu|+\varepsilon|(\nabla u)_i^\sharp|+|u_i^\sharp-Mu|) \quad \text{on} \quad \Gamma.$$

From this inequality and (5.2.58) we deduce that

$$\begin{aligned} \|\nabla_\Gamma\eta_1\|_{L^p(\Gamma)} &\leq c\sum_{i=0,1}\Big(\|P(\nabla u)_i^\sharp-\nabla_\Gamma Mu\|_{L^p(\Gamma)}+\varepsilon\|(\nabla u)_i^\sharp\|_{L^p(\Gamma)}+\|u_i^\sharp-Mu\|_{L^p(\Gamma)}\Big) \\ &\leq c\sum_{i=0,1}\Big(\left\|\overline{P}\nabla u-\overline{\nabla_\Gamma Mu}\right\|_{L^p(\Gamma_\varepsilon^i)}+\varepsilon\|\nabla u\|_{L^p(\Gamma_\varepsilon^i)}+\left\|u-\overline{Mu}\right\|_{L^p(\Gamma_\varepsilon^i)}\Big). \end{aligned}$$

To the right-hand side we apply (5.3.8), (5.6.7), and (5.6.26) to obtain

$$\|\nabla_\Gamma\eta_1\|_{L^p(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}. \tag{5.6.39}$$

Next we estimate the L^p-norm of $\nabla_\Gamma\eta_2$ on Γ. Since

$$\nabla_\Gamma\eta_2 = \sum_{i=0,1}(-1)^i\{(\nabla_\Gamma Mu)(\tau_\varepsilon^i-\nabla_\Gamma g_i)+(\nabla_\Gamma\tau_\varepsilon^i-\nabla_\Gamma^2 g_i)Mu\} \quad \text{on} \quad \Gamma,$$

it follows from (5.2.35) that $|\nabla_\Gamma \eta_2| \leq c\varepsilon(|Mu| + |\nabla_\Gamma Mu|)$ on Γ. By this inequality and (5.6.22) with $m = 1$ we get

$$\|\nabla_\Gamma \eta_2\|_{L^p(\Gamma)} \leq c\varepsilon \|Mu\|_{W^{1,p}(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{1,p}(\Omega_\varepsilon)}. \tag{5.6.40}$$

Let us consider the tangential gradient of $\eta_3 = -gM(d\phi)$, where $\phi := \mathrm{tr}[\overline{W}\nabla u]$. By (5.6.17) we see that

$$\nabla_\Gamma \eta_3 = -M(d\phi)\nabla_\Gamma g - g\bigl\{M(\phi B\nabla d) + M(dB\nabla\phi) + M\bigl(\partial_n(d\phi)\psi_\varepsilon\bigr)\bigr\}$$

on Γ. Moreover, since $B\nabla d = 0$ in Ω_ε by $\nabla d = \bar{n}$ in Ω_ε and $Pn = 0$ on Γ, the second term on the right-hand side vanishes. These facts and (5.6.4) imply that

$$\begin{aligned} \|\nabla_\Gamma \eta_3\|_{L^p(\Gamma)} &\leq c\left(\|M(d\phi)\|_{L^p(\Gamma)} + \|M(dB\nabla\phi)\|_{L^p(\Gamma)} + \bigl\|M\bigl(\partial_n(d\phi)\psi_\varepsilon\bigr)\bigr\|_{L^p(\Gamma)}\right) \\ &\leq c\varepsilon^{-1/p}\left(\|d\phi\|_{L^p(\Omega_\varepsilon)} + \|dB\nabla\phi\|_{L^p(\Omega_\varepsilon)} + \|\partial_n(d\phi)\psi_\varepsilon\|_{L^p(\Omega_\varepsilon)}\right). \end{aligned}$$

To the second line we apply

$$|d\phi| \leq c\varepsilon|\nabla u|, \quad |dB\nabla\phi| \leq c\varepsilon(|\nabla u| + |\nabla^2 u|), \quad |\partial_n(d\phi)\psi_\varepsilon| \leq c\varepsilon(|\nabla u| + |\nabla^2 u|)$$

in Ω_ε by (5.6.20), $|d| \leq c\varepsilon$ in Ω_ε, and $\partial_n d = \bar{n}\cdot\nabla d = |\bar{n}|^2 = 1$ in N to obtain

$$\|\nabla_\Gamma \eta_3\|_{L^p(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}. \tag{5.6.41}$$

Let us estimate the $L^p(\Gamma)$-norm of $\nabla_\Gamma \eta_4$. Setting $\xi := \partial_n u \cdot \psi_\varepsilon$ we have

$$\nabla_\Gamma \eta_4 = (M\xi)\nabla_\Gamma g + g\bigl\{M(B\nabla\xi) + M\bigl((\partial_n\xi)\psi_\varepsilon\bigr)\bigr\} \quad \text{on} \quad \Gamma$$

by (5.6.17). From this equality and (5.6.4) we deduce that

$$\begin{aligned} \|\nabla_\Gamma \eta_4\|_{L^p(\Gamma)} &\leq c\left(\|M\xi\|_{L^p(\Gamma)} + \|M(B\nabla\xi)\|_{L^p(\Gamma)} + \bigl\|M\bigl((\partial_n\xi)\psi_\varepsilon\bigr)\bigr\|_{L^p(\Gamma)}\right) \\ &\leq c\varepsilon^{-1/p}\left(\|\xi\|_{L^p(\Omega_\varepsilon)} + \|B\nabla\xi\|_{L^p(\Omega_\varepsilon)} + \|(\partial_n\xi)\psi_\varepsilon\|_{L^p(\Omega_\varepsilon)}\right). \end{aligned} \tag{5.6.42}$$

We apply (5.6.20) and (5.6.21) to $\xi = \partial_n u \cdot \psi_\varepsilon$ and $(\partial_n\xi)\psi_\varepsilon$ to obtain

$$\begin{aligned} \|\xi\|_{L^p(\Omega_\varepsilon)} &\leq c\varepsilon\|\nabla u\|_{L^p(\Omega_\varepsilon)}, \\ \|(\partial_n\xi)\psi_\varepsilon\|_{L^p(\Omega_\varepsilon)} &\leq c\varepsilon\left(\|\nabla u\|_{L^p(\Omega_\varepsilon)} + \|\nabla^2 u\|_{L^p(\Omega_\varepsilon)}\right). \end{aligned} \tag{5.6.43}$$

Moreover, by (5.6.21) and $P(n\otimes\nabla_\Gamma g) = (Pn)\otimes\nabla_\Gamma g = 0$ on Γ we see that

$$\bigl|\overline{P}\nabla\psi_\varepsilon\bigr| = \left|\overline{P}\left(\nabla\psi_\varepsilon - \frac{1}{\bar{g}}\bar{n}\otimes\overline{\nabla_\Gamma g}\right)\right| \leq c\varepsilon \quad \text{in} \quad \Omega_\varepsilon.$$

Using this inequality and (5.6.20) to $B\nabla\xi = B\{\nabla(\partial_n u)\}\psi_\varepsilon + B(\nabla\psi_\varepsilon)\partial_n u$ we get

$$\|B\nabla\xi\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\left(\|\nabla u\|_{L^p(\Omega_\varepsilon)} + \|\nabla^2 u\|_{L^p(\Omega_\varepsilon)}\right). \tag{5.6.44}$$

From (5.6.42), (5.6.43), and (5.6.44) it follows that

$$\|\nabla_\Gamma \eta_4\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}. \tag{5.6.45}$$

Finally, from (5.6.32), (5.6.39), (5.6.40), (5.6.41), and (5.6.45) we deduce that

$$\bigl\|\nabla_\Gamma\bigl(\mathrm{div}_\Gamma(gMu)\bigr)\bigr\|_{L^p(\Gamma)} \leq \sum_{j=1}^{4}\|\nabla_\Gamma \eta_j\|_{L^p(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}$$

and conclude that (5.6.31) is valid. □

Lemma 5.6.13. *There exists a constant $c > 0$ independent of ε such that*

$$\|\mathrm{div}_\Gamma(gM_\tau u)\|_{L^p(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{1,p}(\Omega_\varepsilon)} \tag{5.6.46}$$

for all $u \in W^{1,p}(\Omega_\varepsilon)^3$, $p \in [1,\infty)$ satisfying $\mathrm{div}\, u = 0$ *in Ω_ε and $u \cdot n_\varepsilon = 0$ on Γ_ε. If in addition $u \in W^{2,p}(\Omega_\varepsilon)^3$, then we have*

$$\|\mathrm{div}_\Gamma(gM_\tau u)\|_{W^{1,p}(\Gamma)} \leq c\varepsilon^{1-1/p}\|u\|_{W^{2,p}(\Omega_\varepsilon)}. \tag{5.6.47}$$

Proof. Let $\varphi := g(Mu \cdot n)$. Then

$$\mathrm{div}_\Gamma(gM_\tau u) = \mathrm{div}_\Gamma(gMu) - \mathrm{div}_\Gamma(\varphi n) = \mathrm{div}_\Gamma(gMu) + \varphi H$$

by $\nabla_\Gamma \varphi \cdot n = 0$ and $\mathrm{div}_\Gamma n = -H$. Hence

$$\begin{aligned} |\mathrm{div}_\Gamma(gM_\tau u)| &\leq c(|\mathrm{div}_\Gamma(gMu)| + |Mu \cdot n|), \\ \big|\nabla_\Gamma\big(\mathrm{div}_\Gamma(gM_\tau u)\big)\big| &\leq c\left(\big|\nabla_\Gamma\big(\mathrm{div}_\Gamma(gMu)\big)\big| + |Mu \cdot n| + |\nabla_\Gamma(Mu \cdot n)|\right) \end{aligned}$$

on Γ. These inequalities, (5.6.10), and (5.6.29)–(5.6.31) imply (5.6.46) and (5.6.47). □

As a consequence of Lemmas 5.6.10 and 5.6.12, we get a relation for the normal derivative (with respect to Γ) of a vector field on Ω_ε and its averaged tangential component.

Lemma 5.6.14. *Let $p \in [1,\infty)$. There exists $c > 0$ independent of ε such that*

$$\left\|\partial_n u \cdot \bar{n} - \frac{1}{\bar{g}}\overline{M_\tau u} \cdot \overline{\nabla_\Gamma g}\right\|_{L^p(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{W^{2,p}(\Omega_\varepsilon)} \tag{5.6.48}$$

for all $u \in W^{2,p}(\Omega_\varepsilon)^3$ satisfying $\mathrm{div}\, u = 0$ *in Ω_ε and $u \cdot n_\varepsilon = 0$ on Γ_ε. Here $\partial_n u$ is the normal derivative of u given by* (5.3.5).

Proof. Since $Q = n \otimes n$, $I_3 = P + Q$ on Γ and $\partial_n u = (\bar{n} \cdot \nabla)u = (\nabla u)^T \bar{n}$ in Ω_ε,

$$\partial_n u \cdot \bar{n} = \mathrm{tr}[\bar{n} \otimes \partial_n u] = \mathrm{tr}\Big[\overline{Q}\nabla u\Big] = \mathrm{div}\, u - \mathrm{tr}\Big[\overline{P}\nabla u\Big] \quad \text{in} \quad \Omega_\varepsilon.$$

Also, since $\nabla_\Gamma g$ is tangential on Γ,

$$\frac{1}{g}M_\tau u \cdot \nabla_\Gamma g = \frac{1}{g}Mu \cdot \nabla_\Gamma g = \frac{1}{g}\mathrm{div}_\Gamma(gMu) - \mathrm{div}_\Gamma(Mu) = \frac{1}{g}\mathrm{div}_\Gamma(gMu) - \mathrm{tr}[\nabla_\Gamma Mu]$$

on Γ. From these equalities, $\mathrm{div}\, u = 0$ in Ω_ε, and (5.2.30) we deduce that

$$\left\|\partial_n u \cdot \bar{n} - \frac{1}{\bar{g}}\overline{M_\tau u} \cdot \overline{\nabla_\Gamma g}\right\|_{L^p(\Omega_\varepsilon)} \leq \left\|\overline{P}\nabla u - \overline{\nabla_\Gamma Mu}\right\|_{L^p(\Omega_\varepsilon)} + c\left\|\overline{\mathrm{div}_\Gamma(gMu)}\right\|_{L^p(\Omega_\varepsilon)}.$$

Applying (5.2.53), (5.6.25), and (5.6.30) to the right-hand side we obtain (5.6.48). □

When $u \in L^2(\Omega_\varepsilon)^3$, we can consider the weighted surface divergence of $M_\tau u$ as an element of $H^{-1}(\Gamma)$. In particular, if $u \in L^2_\sigma(\Omega_\varepsilon)$ then we have an estimate for its $H^{-1}(\Gamma)$-norm similar to (5.6.46).

Lemma 5.6.15. *There exists a constant $c > 0$ in dependent of ε such that*

$$\|\mathrm{div}_\Gamma(gM_\tau u)\|_{H^{-1}(\Gamma)} \leq c\varepsilon^{1/2}\|u\|_{L^2(\Omega_\varepsilon)} \tag{5.6.49}$$

for all $u \in L^2_\sigma(\Omega_\varepsilon)$.

Proof. We use the notation (5.2.46) and suppress the arguments of functions. Let η be an arbitrary function in $H^1(\Gamma)$. By (5.2.25) we have

$$\langle \mathrm{div}_\Gamma(gM_\tau u), \eta\rangle_\Gamma = -\int_\Gamma gM_\tau u \cdot \nabla_\Gamma \eta \, d\mathcal{H}^2 - \int_\Gamma g(M_\tau u \cdot n)\eta H \, d\mathcal{H}^2.$$

The second term on the right-hand side vanishes by $M_\tau u \cdot n = 0$ on Γ. To estimate the first term, we observe by $M_\tau u \cdot \nabla_\Gamma \eta = Mu \cdot \nabla_\Gamma \eta$ and (5.6.1) that

$$\int_\Gamma gM_\tau u \cdot \nabla_\Gamma \eta \, d\mathcal{H}^2 = \varepsilon^{-1} \int_\Gamma \int_{\varepsilon g_0}^{\varepsilon g_1} u^\sharp \cdot \nabla_\Gamma \eta \, dr \, d\mathcal{H}^2.$$

From this equality and the change of variables formula (5.2.51) it follows that

$$\left| \varepsilon^{-1} \int_{\Omega_\varepsilon} u \cdot \nabla \bar{\eta} \, dx - \int_\Gamma gM_\tau u \cdot \nabla_\Gamma \eta \, d\mathcal{H}^2 \right| \leq \varepsilon^{-1}(I_1 + I_2),$$

where $\bar{\eta} = \eta \circ \pi$ is the constant extension of η and

$$I_1 := \left| \int_{\Omega_\varepsilon} u \cdot \left(\nabla \bar{\eta} - \overline{\nabla_\Gamma \eta} \right) dx \right|, \quad I_2 := \left| \int_\Gamma \int_{\varepsilon g_0}^{\varepsilon g_1} (u^\sharp \cdot \nabla_\Gamma \eta)(J-1) \, dr \, d\mathcal{H}^2 \right|.$$

To I_1 we apply (5.2.14) with $|d| \leq c\varepsilon$ in Ω_ε, Hölder's inequality, and (5.2.53) to get

$$I_1 \leq c\varepsilon \|u\|_{L^2(\Omega_\varepsilon)} \left\| \overline{\nabla_\Gamma \eta} \right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{3/2} \|u\|_{L^2(\Omega_\varepsilon)} \|\nabla_\Gamma \eta\|_{L^2(\Gamma)}.$$

We also have the same inequality for I_2 by (5.2.50), (5.2.52), and (5.2.53). Hence

$$\left| \varepsilon^{-1} \int_{\Omega_\varepsilon} u \cdot \nabla \bar{\eta} \, dx - \int_\Gamma gM_\tau u \cdot \nabla_\Gamma \eta \, d\mathcal{H}^2 \right| \leq \varepsilon^{-1}(I_1 + I_2) \leq c\varepsilon^{1/2} \|u\|_{L^2(\Omega_\varepsilon)} \|\nabla_\Gamma \eta\|_{L^2(\Gamma)}.$$

Here $\int_{\Omega_\varepsilon} u \cdot \nabla \bar{\eta} \, dx = 0$ by $u \in L^2_\sigma(\Omega_\varepsilon)$ and $\nabla \bar{\eta} \in L^2_\sigma(\Omega_\varepsilon)^\perp$. Therefore,

$$|\langle \mathrm{div}_\Gamma(gM_\tau u), \eta\rangle_\Gamma| = \left| \int_\Gamma gM_\tau u \cdot \nabla_\Gamma \eta \, d\mathcal{H}^2 \right| \leq c\varepsilon^{1/2} \|u\|_{L^2(\Omega_\varepsilon)} \|\nabla_\Gamma \eta\|_{L^2(\Gamma)}.$$

Since this inequality holds for all $\eta \in H^1(\Gamma)$, the inequality (5.6.49) is valid. □

5.6.3 Decomposition of vector fields into the average and residual parts

In the study of the Navier–Stokes equations in a thin domain it is convenient to decompose a three-dimensional vector field into an almost two-dimensional vector field and its residual term and analyze them separately. Moreover, we can derive a good L^∞-estimate for the residual term if it satisfies the impermeable boundary condition. To give such a decomposition we use the impermeable extension operator E_ε given by (5.3.42) in Section 5.3.3.

Definition 5.6.16. For a vector field u on Ω_ε we define

$$u^a(x) := E_\varepsilon M_\tau u(x) = \overline{M_\tau u}(x) + \left\{ \overline{M_\tau u}(x) \cdot \Psi_\varepsilon(x) \right\} \bar{n}(x), \quad x \in N, \tag{5.6.50}$$

where Ψ_ε is the vector field given by (5.3.36) and $M_\tau u$ is the averaged tangential component of u given by (5.6.2). Also, we denote by $u^r := u - u^a$ the residual part of u.

From Lemmas 5.3.12, 5.6.2, and 5.6.9 we observe that if $u \in H^m(\Omega_\varepsilon)^3$, $m = 0, 1, 2$, then u^a and u^r are in the same space (here we write $H^0 = L^2$).

Lemma 5.6.17. *For $u \in H^m(\Omega_\varepsilon)^3$ with $m = 0, 1, 2$ we have $u^a, u^r \in H^m(\Omega_\varepsilon)^3$ and*

$$\|u^a\|_{H^m(\Omega_\varepsilon)} \le c\|u\|_{H^m(\Omega_\varepsilon)}, \quad \|u^r\|_{H^m(\Omega_\varepsilon)} \le c\|u\|_{H^m(\Omega_\varepsilon)} \tag{5.6.51}$$

with a constant $c > 0$ independent of ε and u.

Since the average part u^a can be seen as almost two-dimensional, we expect to have a good L^2-estimate for the product of u^a and a function on Ω_ε. Indeed, we can apply the following product estimate of functions on Γ and Ω_ε to u^a.

Lemma 5.6.18. *There exists a constant $c > 0$ independent of ε such that*

$$\|\bar{\eta}\varphi\|_{L^2(\Omega_\varepsilon)} \le c\|\eta\|_{L^2(\Gamma)}^{1/2}\|\eta\|_{H^1(\Gamma)}^{1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}^{1/2} \tag{5.6.52}$$

for all $\eta \in H^1(\Gamma)$ and $\varphi \in H^1(\Omega_\varepsilon)$, where $\bar{\eta} := \eta \circ \pi$ is the constant extension of η.

Proof. Throughout the proof, we use the notation (5.2.46) and suppress the arguments of functions. By (5.2.52) and (5.6.1) we have

$$\|\bar{\eta}\varphi\|_{L^2(\Omega_\varepsilon)}^2 \le c\int_\Gamma |\eta|^2 \left(\int_{\varepsilon g_0}^{\varepsilon g_1} |\varphi^\sharp|^2\,dr\right) d\mathcal{H}^2 = c\varepsilon \int_\Gamma g|\eta|^2 M(|\varphi|^2)\,d\mathcal{H}^2.$$

Noting that g is bounded on Γ, we apply Hölder's inequality to the last integral to get

$$\|\bar{\eta}\varphi\|_{L^2(\Omega_\varepsilon)}^2 \le c\varepsilon\|\eta\|_{L^4(\Gamma)}^2\|M(|\varphi|^2)\|_{L^2(\Gamma)}. \tag{5.6.53}$$

Here the $L^4(\Gamma)$-norm of η is estimated by Ladyzhenskaya's inequality (5.3.1). To estimate the $L^2(\Gamma)$-norm of $M(|\varphi|^2)$ we see that $M(|\varphi|^2) \in W^{1,1}(\Gamma)$ for $\varphi \in H^1(\Omega_\varepsilon)$. Indeed, by (5.2.30) and (5.2.52) we have

$$\|M(|\varphi|^2)\|_{L^1(\Gamma)} = \int_\Gamma \frac{1}{\varepsilon g}\left(\int_{\varepsilon g_0}^{\varepsilon g_1} |\varphi^\sharp|^2\,dr\right) d\mathcal{H}^2 \le c\varepsilon^{-1}\|\varphi\|_{L^2(\Omega_\varepsilon)}^2. \tag{5.6.54}$$

Also, by (5.6.17), (5.6.20), and $\nabla(|\varphi|^2) = 2\varphi\nabla\varphi$ we have

$$|\nabla_\Gamma M(|\varphi|^2)| \le |M\big(B\nabla(|\varphi|^2)\big)| + |M\big(\partial_n(|\varphi|^2)\psi_\varepsilon\big)| \le cM(|\varphi\nabla\varphi|) \quad \text{on} \quad \Gamma.$$

Hence from (5.6.4) and Hölder's inequality we deduce that

$$\begin{aligned} \|\nabla_\Gamma M(|\varphi|^2)\|_{L^1(\Gamma)} &\le c\|M(|\varphi\nabla\varphi|)\|_{L^1(\Gamma)} \le c\varepsilon^{-1}\|\varphi\nabla\varphi\|_{L^1(\Omega_\varepsilon)} \\ &\le c\varepsilon^{-1}\|\varphi\|_{L^2(\Omega_\varepsilon)}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{5.6.55}$$

Now we observe that the Sobolev embedding $W^{1,1}(\Gamma) \hookrightarrow L^2(\Gamma)$ is valid since $\Gamma \subset \mathbb{R}^3$ is a two-dimensional compact surface without boundary (see e.g. [3, Theorem 2.20]). By this fact, (5.6.54), (5.6.55), and $\|\varphi\|_{L^2(\Omega_\varepsilon)} \le \|\varphi\|_{H^1(\Omega_\varepsilon)}$ we obtain

$$\|M(|\varphi|^2)\|_{L^2(\Gamma)} \le c\|M(|\varphi|^2)\|_{W^{1,1}(\Gamma)} \le c\varepsilon^{-1}\|\varphi\|_{L^2(\Omega_\varepsilon)}\|\varphi\|_{H^1(\Omega_\varepsilon)}.$$

Finally, we apply the above inequality and (5.3.1) to (5.6.53) to get

$$\|\bar{\eta}\varphi\|_{L^2(\Omega_\varepsilon)}^2 \le c\|\eta\|_{L^2(\Gamma)}\|\eta\|_{H^1(\Gamma)}\|\varphi\|_{L^2(\Omega_\varepsilon)}\|\varphi\|_{H^1(\Omega_\varepsilon)},$$

which shows (5.6.52). $\square$

Corollary 5.6.19. *For $\varphi \in H^1(\Omega_\varepsilon)$, $u \in H^1(\Omega_\varepsilon)^3$, and u^a given by (5.6.50) we have*

$$\big\| |u^a|\,\varphi \big\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}^{1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}^{1/2} \tag{5.6.56}$$

with a constant $c > 0$ independent of ε, φ, and u. If in addition $u \in H^2(\Omega_\varepsilon)^3$, then

$$\big\| |\nabla u^a|\,\varphi \big\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}^{1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}. \tag{5.6.57}$$

Proof. By (5.3.37) and $|M_\tau u| = |PMu| \leq |Mu|$ on Γ we have

$$|u^a| \leq (1+|\Psi_\varepsilon|)\left|\overline{M_\tau u}\right| \leq c\left|\overline{Mu}\right| \quad \text{in} \quad \Omega_\varepsilon.$$

We apply this inequality and (5.6.52) with $\eta = |Mu|$ to the L^2-norm of $|u^a|\,\varphi$ on Ω_ε and then use (5.6.4) and (5.6.22) to obtain (5.6.56).

Let us prove (5.6.57). We differentiate both sides of (5.6.50) to get

$$\nabla u^a = \nabla\Big(\overline{M_\tau u}\Big) + \Big[\Big\{\nabla\Big(\overline{M_\tau u}\Big)\Big\}\,\Psi_\varepsilon + (\nabla\Psi_\varepsilon)\overline{M_\tau u}\Big]\otimes\bar{n} + \Big(\overline{M_\tau u}\cdot\Psi_\varepsilon\Big)\nabla\bar{n}$$

in Ω_ε. To the right-hand side we apply (5.2.13), (5.2.17), and (5.3.37) to get

$$|\nabla u^a| \leq c\left(\left|\overline{M_\tau u}\right| + \left|\overline{\nabla_\Gamma M_\tau u}\right|\right) \leq c\left(\left|\overline{Mu}\right| + \left|\overline{\nabla_\Gamma Mu}\right|\right) \quad \text{in} \quad \Omega_\varepsilon.$$

Here the second inequality follows from $M_\tau u = PMu$ and $P \in C^4(\Gamma)^{3\times 3}$. Applying the above inequality and (5.6.52) with $\eta = |Mu|$ and $\eta = |\nabla_\Gamma Mu|$ to the L^2-norm of $|\nabla u^a|\,\varphi$ on Ω_ε and then using (5.6.4) and (5.6.22) we obtain (5.6.57). □

Next we show Poincaré type inequalities for the residual part $u^r = u - u^a$. For any vector field u on Ω_ε we have $u^a\cdot n_\varepsilon = 0$ on Γ_ε by Lemma 5.3.11. Hence $u^r\cdot n_\varepsilon = 0$ on Γ_ε if u itself satisfies the same impermeable boundary condition, which is essential for derivation of Poincaré type inequalities.

Lemma 5.6.20. *Let $u \in H^1(\Omega_\varepsilon)^3$ satisfy $u\cdot n_\varepsilon = 0$ on Γ_ε. Then there exists a constant $c > 0$ independent of ε and u such that*

$$\|u^r\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|\partial_n u^r\|_{L^2(\Omega_\varepsilon)} \tag{5.6.58}$$

for $u^r = u - u^a$, where $\partial_n u^r$ is the normal derivative of u^r given by (5.3.5).

Proof. We use the notation (5.6.3) for the tangential and normal components (with respect to Γ) of a vector field on Ω_ε. Since $u_\tau^r = u_\tau - \overline{M_\tau u}$ in Ω_ε by (5.6.50) and $M_\tau u = Mu_\tau$ on Γ, we use (5.6.6) to get

$$\|u_\tau^r\|_{L^2(\Omega_\varepsilon)} = \left\|u_\tau - \overline{Mu_\tau}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|\partial_n u_\tau\|_{L^2(\Omega_\varepsilon)} = c\varepsilon\|\partial_n u_\tau^r\|_{L^2(\Omega_\varepsilon)}.$$

Here the last equality follows from (5.3.6) with $\eta = M_\tau u$. From this inequality and

$$\|\partial_n u_\tau^r\|_{L^2(\Omega_\varepsilon)} = \left\|\overline{P}(\partial_n u^r)\right\|_{L^2(\Omega_\varepsilon)} \leq \|\partial_n u^r\|_{L^2(\Omega_\varepsilon)}$$

by $u_\tau^r = \overline{P}u^r$ and $\partial_n\overline{P} = 0$ in Ω_ε we deduce that

$$\|u_\tau^r\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|\partial_n u^r\|_{L^2(\Omega_\varepsilon)}. \tag{5.6.59}$$

To estimate the $L^2(\Omega_\varepsilon)$-norm of u_n^r, we see that u^r satisfies $u^r\cdot n_\varepsilon = 0$ on Γ_ε since u and u^a satisfy the same boundary condition. Hence we can apply (5.3.23) to get

$$\|u_n^r\|_{L^2(\Omega_\varepsilon)} = \|u^r\cdot\bar{n}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|\partial_n u^r\|_{L^2(\Omega_\varepsilon)}.$$

Combining (5.6.59) and the above inequality we obtain (5.6.58). □

Lemma 5.6.21. *Suppose that Assumption 1 is satisfied, i.e. the inequality* (5.1.6) *is valid. Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy $\operatorname{div} u = 0$ in Ω_ε and the slip boundary conditions* (5.3.20)–(5.3.21) *on Γ_ε. Then $u^r = u - u^a$ satisfies*

$$\|\nabla u^r\|_{L^2(\Omega_\varepsilon)} \leq c\left(\varepsilon\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}\right). \tag{5.6.60}$$

Here $c > 0$ is a constant independent of ε and u.

Proof. As in the proof of Lemma 5.6.20, we use the notation (5.6.3). Based on the equalities $u^r = u^r_\tau + u^r_n$ and $I_3 = \overline{P} + \overline{Q}$ we split the gradient matrix of u^r into

$$\nabla u^r = \overline{P}\nabla u^r_\tau + \overline{Q}\nabla u^r_\tau + \overline{P}\nabla u^r_n + \overline{Q}\nabla u^r_n \quad \text{in} \quad \Omega_\varepsilon \tag{5.6.61}$$

and estimate the $L^2(\Omega_\varepsilon)$-norm of each term on the right-hand side.

First we derive an estimate for $\overline{P}\nabla u^r_\tau$. We differentiate $u^r_\tau = u_\tau - \overline{Mu_\tau}$ and use (5.2.8), (5.2.12), and $P\nabla_\Gamma = \nabla_\Gamma$ to get

$$\overline{P}\nabla u^r_\tau = \left(\overline{P}\nabla u_\tau - \overline{\nabla_\Gamma Mu_\tau}\right) - \left\{I_3 - \left(I_3 - d\overline{W}\right)^{-1}\right\}\overline{\nabla_\Gamma Mu_\tau} \quad \text{in} \quad \Omega_\varepsilon.$$

Hence by (5.2.10) with $|d| \leq c\varepsilon$ in Ω_ε, (5.2.53), (5.6.22), and (5.6.25) we see that

$$\begin{aligned} \left\|\overline{P}\nabla u^r_\tau\right\|_{L^2(\Omega_\varepsilon)} &\leq \left\|\overline{P}\nabla u_\tau - \overline{\nabla_\Gamma Mu_\tau}\right\|_{L^2(\Omega_\varepsilon)} + c\varepsilon\left\|\overline{\nabla_\Gamma Mu_\tau}\right\|_{L^2(\Omega_\varepsilon)} \\ &\leq c\varepsilon\|u_\tau\|_{H^2(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{H^2(\Omega_\varepsilon)}. \end{aligned} \tag{5.6.62}$$

Next we deal with $\overline{Q}\nabla u^r_\tau$. By $u^r_\tau = \overline{P}u - \overline{M_\tau u}$ and (5.3.6),

$$\left|\overline{Q}\nabla u^r_\tau\right| = |\bar{n} \otimes \partial_n u^r_\tau| = |\partial_n u^r_\tau| = \left|\overline{P}\partial_n u\right| \quad \text{in} \quad \Omega_\varepsilon.$$

Since we suppose that Assumption 1 is satisfied and that u satisfies (5.3.20)–(5.3.21) on Γ_ε, we can use (5.3.34) to get

$$\begin{aligned} \left\|\overline{Q}\nabla u^r_\tau\right\|_{L^2(\Omega_\varepsilon)} = \left\|\overline{P}\partial_n u\right\|_{L^2(\Omega_\varepsilon)} &\leq \left\|\overline{P}\partial_n u + \overline{W}u\right\|_{L^2(\Omega_\varepsilon)} + \left\|\overline{W}u\right\|_{L^2(\Omega_\varepsilon)} \\ &\leq c\left(\varepsilon\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}\right). \end{aligned} \tag{5.6.63}$$

Let us estimate the $L^2(\Omega_\varepsilon)$-norm of $\overline{P}\nabla u^r_n$. Since $u^r_n = (u^r \cdot \bar{n})\bar{n}$ we have

$$\overline{P}\nabla u^r_n = \left[\overline{P}\nabla(u^r \cdot \bar{n})\right] \otimes \bar{n} - (u^r \cdot \bar{n})\overline{P}\nabla\bar{n} \quad \text{in} \quad \Omega_\varepsilon.$$

By this formula, (5.2.13), (5.2.17), and $|n| = |P| = 1$ on Γ,

$$\left\|\overline{P}\nabla u^r_n\right\|_{L^2(\Omega_\varepsilon)} \leq c\left(\left\|\overline{P}\nabla(u^r \cdot \bar{n})\right\|_{L^2(\Omega_\varepsilon)} + \|u^r\|_{L^2(\Omega_\varepsilon)}\right).$$

Here u^r satisfies $u^r \cdot n_\varepsilon = 0$ on Γ_ε since u satisfies the same boundary condition. Thus, we can apply (5.3.24) and (5.6.58) to the above inequality to get

$$\left\|\overline{P}\nabla u^r_n\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{H^2(\Omega_\varepsilon)}. \tag{5.6.64}$$

Now let us consider $\overline{Q}\nabla u^r_n = \bar{n} \otimes \partial_n u^r_n$. Since

$$u^r_n = (u^r \cdot \bar{n})\bar{n} = \left(u \cdot \bar{n} - \overline{M_\tau u} \cdot \Psi_\varepsilon\right)\bar{n}, \quad \partial_n u^r_n = \left(\partial_n u \cdot \bar{n} - \overline{M_\tau u} \cdot \partial_n\Psi_\varepsilon\right)\bar{n}$$

by (5.3.6) and (5.6.50), we have

$$\begin{aligned}|\overline{Q}\nabla u_n^r| &= |\partial_n u_n^r| = \left|\partial_n u\cdot\bar{n} - \overline{M_\tau u}\cdot\partial_n\Psi_\varepsilon\right| \\ &\leq \left|\partial_n u\cdot\bar{n} - \frac{1}{\bar{g}}\overline{M_\tau u}\cdot\overline{\nabla_\Gamma g}\right| + \left|\overline{M_\tau u}\right|\left|\partial_n\Psi_\varepsilon - \frac{1}{\bar{g}}\overline{\nabla_\Gamma g}\right|\end{aligned}$$

in Ω_ε. By this inequality, $|M_\tau u|\leq|Mu|$ on Γ, (5.3.38), (5.6.5), and (5.6.48) we see that

$$\left\|\overline{Q}\nabla u_n^r\right\|_{L^2(\Omega_\varepsilon)} \leq \left\|\partial_n u\cdot\bar{n} - \frac{1}{\bar{g}}\overline{M_\tau u}\cdot\overline{\nabla_\Gamma g}\right\|_{L^2(\Omega_\varepsilon)} + c\varepsilon\left\|\overline{Mu}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|u\|_{H^2(\Omega_\varepsilon)}. \quad (5.6.65)$$

Here we used the fact that u satisfies $\operatorname{div} u = 0$ in Ω_ε and (5.3.20) on Γ_ε to apply (5.6.48). Finally, applying (5.6.62)–(5.6.65) to the right-hand side of (5.6.61) we obtain (5.6.60). $\square$

As a consequence of Lemmas 5.6.20 and 5.6.21, we obtain a good L^∞-estimate for the residual term u^r on Ω_ε.

Lemma 5.6.22. *Under Assumption 1, let $u\in H^2(\Omega_\varepsilon)^3$ satisfy $\operatorname{div} u = 0$ in Ω_ε and (5.3.20)–(5.3.21) on Γ_ε. Then there exists a constant $c>0$ independent of ε and u such that*

$$\|u^r\|_{L^\infty(\Omega_\varepsilon)} \leq c\left(\varepsilon^{1/2}\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}\right) \quad (5.6.66)$$

for $u^r = u - u^a$, where u^a is given by (5.6.50).

Proof. Since $u^r\in H^2(\Omega_\varepsilon)^3$, we can use Agmon's inequality (5.3.11) to get

$$\begin{aligned}\|u^r\|_{L^\infty(\Omega_\varepsilon)} &\leq c\varepsilon^{-1/2}\|u^r\|_{L^2(\Omega_\varepsilon)}^{1/4}\|u^r\|_{H^2(\Omega_\varepsilon)}^{1/2} \\ &\qquad\times\left(\|u^r\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\partial_n u^r\|_{L^2(\Omega_\varepsilon)} + \varepsilon^2\|\partial_n^2 u^r\|_{L^2(\Omega_\varepsilon)}\right)^{1/4}.\end{aligned}$$

To the right-hand side of this inequality we apply

$$\begin{gathered}\|\partial_n^2 u^r\|_{L^2(\Omega_\varepsilon)} \leq c\|u^r\|_{H^2(\Omega_\varepsilon)} \leq c\|u\|_{H^2(\Omega_\varepsilon)}, \\ \|u^r\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|\partial_n u^r\|_{L^2(\Omega_\varepsilon)} \leq c\left(\varepsilon^2\|u\|_{H^2(\Omega_\varepsilon)} + \varepsilon\|u\|_{L^2(\Omega_\varepsilon)}\right)\end{gathered}$$

by (5.6.51), (5.6.58), and (5.6.60) to get

$$\begin{aligned}\|u^r\|_{L^\infty(\Omega_\varepsilon)} &\leq c\varepsilon^{-1/2}\left(\varepsilon^2\|u\|_{H^2(\Omega_\varepsilon)} + \varepsilon\|u\|_{L^2(\Omega_\varepsilon)}\right)^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2} \\ &= c\left(\varepsilon\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}\right)^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}.\end{aligned}$$

Using $(a+b)^{1/2}\leq a^{1/2}+b^{1/2}$ for $a,b\geq 0$ to this inequality we obtain (5.6.66). $\square$

Finally, let us estimate the $L^2(\Omega_\varepsilon)$-norm of $u\otimes u$ and $(u\cdot\nabla)u$ by using the product estimate for the average part and the L^∞-estimate for the residual part.

Lemma 5.6.23. *Under Assumption 1, let $u\in H^2(\Omega_\varepsilon)^3$ satisfy $\operatorname{div} u = 0$ in Ω_ε and (5.3.20)–(5.3.21) on Γ_ε. Then there exists a constant $c>0$ independent of ε and u such that*

$$\begin{aligned}\|u\otimes u\|_{L^2(\Omega_\varepsilon)} &\leq c\Big(\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)} \\ &\qquad+\varepsilon^{1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{3/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}\Big) \quad (5.6.67)\end{aligned}$$

and

$$\|(u\cdot\nabla)u\|_{L^2(\Omega_\varepsilon)} \leq c\left(\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}. \quad (5.6.68)$$

Proof. Let u^a be the average part of u given by (5.6.50) and $u^r = u - u^a$ the residual part. Since $u \in H^2(\Omega_\varepsilon)^3$ satisfies $\operatorname{div} u = 0$ in Ω_ε and (5.3.20)–(5.3.21) on Γ_ε we have

$$\begin{aligned}
&\|u^a \otimes u\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}, \\
&\|u^r \otimes u\|_{L^2(\Omega_\varepsilon)} \leq \|u^r\|_{L^\infty(\Omega_\varepsilon)}\|u\|_{L^2(\Omega_\varepsilon)} \leq c\left(\varepsilon^{1/2}\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}\right)\|u\|_{L^2(\Omega_\varepsilon)}
\end{aligned}$$

by (5.6.56) and (5.6.66). Applying these inequalities to the right-hand side of

$$\|u \otimes u\|_{L^2(\Omega_\varepsilon)} \leq \|u^a \otimes u\|_{L^2(\Omega_\varepsilon)} + \|u^r \otimes u\|_{L^2(\Omega_\varepsilon)}$$

we obtain (5.6.67). To prove (5.6.68) we observe by (5.5.33) and (5.6.56) that

$$\|(u^a \cdot \nabla)u\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2} \leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}.$$

Also, by (5.5.33) and (5.6.66) we get

$$\begin{aligned}
\|(u^r \cdot \nabla)u\|_{L^2(\Omega_\varepsilon)} &\leq c\|u^r\|_{L^\infty(\Omega_\varepsilon)}\|\nabla u\|_{L^2(\Omega_\varepsilon)} \\
&\leq c\left(\varepsilon^{1/2}\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}\right)\|u\|_{H^1(\Omega_\varepsilon)} \\
&= c\left(\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2} + \varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}\right) \\
&\leq c\left(\|u\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}.
\end{aligned}$$

We use these two estimates to the right-hand side of

$$\|(u \cdot \nabla)u\|_{L^2(\Omega_\varepsilon)} \leq \|(u^a \cdot \nabla)u\|_{L^2(\Omega_\varepsilon)} + \|(u^r \cdot \nabla)u\|_{L^2(\Omega_\varepsilon)}$$

and note that $1 \leq \varepsilon^{-1/2}$ by $\varepsilon < 1$ to obtain (5.6.68). □

5.6.4 Average of bilinear and trilinear forms

We consider approximation of bilinear and trilinear forms for functions on Ω_ε by those for functions on Γ and the average operators. The results in this subsection are fundamental for the study of a singular limit problem for the Navier–Stokes equations (5.1.1)–(5.1.3). Throughout this subsection, we denote by $\bar{\eta} = \eta \circ \pi$ the constant extension of a function η on Γ in the normal direction of Γ. We also use the notations (5.2.46) and (5.2.47) and suppress the arguments of functions.

First we consider the L^2-inner products on Ω_ε and Γ_ε^i, $i = 0, 1$.

Lemma 5.6.24. *There exists a constant $c > 0$ independent of ε such that*

$$\left|\int_{\Omega_\varepsilon} \varphi\bar{\eta}\,dx - \varepsilon\int_\Gamma g(M\varphi)\eta\,d\mathcal{H}^2\right| \leq c\varepsilon^{3/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)} \tag{5.6.69}$$

for all $\varphi \in L^2(\Omega_\varepsilon)$ and $\eta \in L^2(\Gamma)$.

Proof. By the change of variables formula (5.2.51) and the definition (5.6.1) of the average operator,

$$\int_{\Omega_\varepsilon} \varphi\bar{\eta}\,dx - \varepsilon\int_\Gamma g(M\varphi)\eta\,d\mathcal{H}^2 = \int_\Gamma\int_{\varepsilon g_0}^{\varepsilon g_1} \varphi^\sharp\eta(J-1)\,dr\,d\mathcal{H}^2.$$

We apply (5.2.50), (5.2.52), Hölder's inequality, and (5.2.53) to the right-hand side to get

$$\left|\int_\Gamma\int_{\varepsilon g_0}^{\varepsilon g_1} \varphi^\sharp\eta(J-1)\,dr\,d\mathcal{H}^2\right| \leq c\varepsilon\|\varphi\|_{L^2(\Omega_\varepsilon)}\|\bar{\eta}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{3/2}\|\varphi\|_{L^2(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}.$$

Hence we obtain (5.6.69). □

Lemma 5.6.25. *There exists a constant $c > 0$ independent of ε such that*

$$\left|\int_{\Gamma_\varepsilon^i} \varphi\bar{\eta}\, d\mathcal{H}^2 - \int_\Gamma (M\varphi)\eta\, d\mathcal{H}^2\right| \leq \varepsilon^{1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}, \quad i = 0,1 \tag{5.6.70}$$

for all $\varphi \in H^1(\Omega_\varepsilon)$ and $\eta \in L^2(\Gamma)$.

Proof. Using the change of variables formula (5.2.57) we have

$$\int_{\Gamma_\varepsilon^i} \varphi\bar{\eta}\, d\mathcal{H}^2 - \int_\Gamma (M\varphi)\eta\, d\mathcal{H}^2 = I_1 + I_2, \tag{5.6.71}$$

where

$$I_1 := \int_\Gamma \varphi_i^\sharp\eta\left(J_i^\sharp\sqrt{1+\varepsilon^2|\tau_\varepsilon^i|^2} - 1\right) d\mathcal{H}^2, \quad I_2 := \int_\Gamma (\varphi_i^\sharp - M\varphi)\eta\, d\mathcal{H}^2.$$

Here we use the notation (5.2.47) and $J_i^\sharp(y) := J(y, \varepsilon g_i(y))$ for $y \in \Gamma$. By (5.2.34), (5.2.50), and (5.4.9) we observe that

$$\left|J_i^\sharp\sqrt{1+\varepsilon^2|\tau_\varepsilon^i|^2} - 1\right| \leq |J_i^\sharp - 1|\sqrt{1+\varepsilon^2|\tau_\varepsilon^i|^2} + \left(\sqrt{1+\varepsilon^2|\tau_\varepsilon^i|^2} - 1\right) \leq c\varepsilon$$

on Γ. From this inequality, (5.2.58), and (5.3.8) it follows that

$$|I_1| \leq c\varepsilon\|\varphi_i^\sharp\|_{L^2(\Gamma)}\|\eta\|_{L^2(\Gamma)} \leq c\varepsilon\|\varphi\|_{L^2(\Gamma_\varepsilon^i)}\|\eta\|_{L^2(\Gamma)} \leq c\varepsilon^{1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}.$$

Also, by (5.2.58) and (5.6.7) we have

$$|I_2| \leq \|\varphi_i^\sharp - M\varphi\|_{L^2(\Gamma)}\|\eta\|_{L^2(\Gamma)} \leq c\left\|\varphi - \overline{M\varphi}\right\|_{L^2(\Gamma_\varepsilon^i)}\|\eta\|_{L^2(\Gamma)} \leq c\varepsilon^{1/2}\|\varphi\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}.$$

Applying these two estimates to (5.6.71) we obtain (5.6.70). □

Next we deal with bilinear forms including the strain rate tensor

$$D(u) = (\nabla u)_S = \frac{\nabla u + (\nabla u)^T}{2}$$

for a vector field u on Ω_ε.

Lemma 5.6.26. *There exists a constant $c > 0$ independent of ε such that*

$$\left|\int_{\Omega_\varepsilon} D(u) : \overline{A}\, dx - \varepsilon\int_\Gamma gD_\Gamma(M_\tau u) : A\, d\mathcal{H}^2\right| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|A\|_{L^2(\Gamma)} \tag{5.6.72}$$

for all $u \in H^1(\Omega_\varepsilon)^3$ and $A \in L^2(\Gamma)^{3\times 3}$ satisfying

$$u \cdot n_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad PA = AP = A \quad \text{on} \quad \Gamma.$$

Here $D_\Gamma(M_\tau u)$ is the surface strain rate tensor given by (5.4.38).

Proof. Since $PA = AP = A$ and $P^T = P$ on Γ, we have

$$PM\bigl(D(u)\bigr)P : A = M\bigl(D(u)\bigr) : A \quad \text{on} \quad \Gamma.$$

From this equality, (5.6.69), and $\|D(u)\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)}$ it follows that

$$\left|\int_{\Omega_\varepsilon} D(u) : \overline{A}\,dx - \varepsilon\int_\Gamma gPM\bigl(D(u)\bigr)P : A\,d\mathcal{H}^2\right| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|A\|_{L^2(\Gamma)}. \tag{5.6.73}$$

By (5.6.17), (5.6.18), (5.6.20), and $|d| \leq c\varepsilon$ in Ω_ε we have

$$|\nabla_\Gamma Mu - PM(\nabla u)| \leq \left|M\Bigl(d\overline{W}\nabla u\Bigr)\right| + |M(\psi_\varepsilon \otimes \partial_n u)| \leq c\varepsilon M(|\nabla u|) \quad \text{on} \quad \Gamma. \tag{5.6.74}$$

Noting that $P\nabla_\Gamma Mu = \nabla_\Gamma Mu$ and P is bounded on Γ, we deduce from (5.6.74) that

$$\bigl|D_\Gamma(Mu) - PM\bigl(D(u)\bigr)P\bigr| \leq c|\{\nabla_\Gamma Mu - PM(\nabla u)\}P| \leq c\varepsilon M(|\nabla u|) \quad \text{on} \quad \Gamma.$$

By this inequality, the boundedness of g on Γ, and (5.6.4) we see that

$$\begin{aligned}\left|\int_\Gamma gD_\Gamma(Mu) : A\,d\mathcal{H}^2 - \int_\Gamma gPM\bigl(D(u)\bigr)P : A\,d\mathcal{H}^2\right| &\leq c\varepsilon\|M(|\nabla u|)\|_{L^2(\Gamma)}\|A\|_{L^2(\Gamma)} \\ &\leq c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|A\|_{L^2(\Gamma)}.\end{aligned} \tag{5.6.75}$$

Moreover, by the decomposition $Mu = (Mu\cdot n)n + M_\tau u$ and $-\nabla_\Gamma n = W$ we get

$$\nabla_\Gamma Mu = \nabla_\Gamma(Mu\cdot n)\otimes n - (Mu\cdot n)W + \nabla_\Gamma M_\tau u \quad \text{on} \quad \Gamma.$$

Since $(a\otimes n)P = a\otimes(Pn) = 0$ for any $a \in \mathbb{R}^3$ by $P^T = P$ and $Pn = 0$,

$$P(\nabla_\Gamma Mu)P - P(\nabla_\Gamma M_\tau u)P = -(Mu\cdot n)PWP = -(Mu\cdot n)W \quad \text{on} \quad \Gamma,$$

where we used (5.2.6) in the last equality. By this equality and the boundedness of W on Γ,

$$|D_\Gamma(Mu) - D_\Gamma(M_\tau u)| \leq |P(\nabla_\Gamma Mu)P - P(\nabla_\Gamma M_\tau u)P| \leq c|Mu\cdot n| \quad \text{on} \quad \Gamma.$$

Hence by (5.6.10) we have

$$\begin{aligned}\left|\int_\Gamma gD_\Gamma(Mu) : A\,d\mathcal{H}^2 - \int_\Gamma gD_\Gamma(M_\tau u) : A\,d\mathcal{H}^2\right| &\leq c\|Mu\cdot n\|_{L^2(\Gamma)}\|A\|_{L^2(\Gamma)} \\ &\leq c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|A\|_{L^2(\Gamma)}.\end{aligned}$$

Combining this inequality, (5.6.73), and (5.6.75) we obtain (5.6.72). ☐

Lemma 5.6.27. *There exists a constant $c > 0$ independent of ε such that*

$$\left|\int_{\Omega_\varepsilon}\Bigl(D(u) : \overline{Q}\Bigr)\bar{\eta}\,dx - \varepsilon\int_\Gamma (M_\tau u\cdot\nabla_\Gamma g)\eta\,d\mathcal{H}^2\right| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)} \tag{5.6.76}$$

for all $u \in H^1(\Omega_\varepsilon)^3$ satisfying $u\cdot n_\varepsilon = 0$ on Γ_ε and $\eta \in L^2(\Gamma)$.

Proof. First note that, by the change of variables formula (5.2.51), the inequalities (5.2.50), (5.2.52), and (5.2.53), and $|Q| = |n| = 1$ we have

$$\left|\int_{\Omega_\varepsilon}\Big(D(u):\overline{Q}\Big)\bar{\eta}\,dx - \int_\Gamma\left(\int_{\varepsilon g_0}^{\varepsilon g_1} D(u)^\sharp : Q\,dr\right)\eta\,d\mathcal{H}^2\right| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}. \quad (5.6.77)$$

Let us calculate the integral of $D(u)^\sharp(y,r) : Q(y)$ with respect to r. Since $Q = n\otimes n$ is symmetric, we have

$$\begin{aligned} D(u)(y+rn(y)) : Q(y) &= \nabla u(y+rn(y)) : n(y)\otimes n(y) = [(n(y)\cdot\nabla)u](y+rn(y))\cdot n(y) \\ &= \frac{\partial}{\partial r}\Big(u(y+rn(y))\Big)\cdot n(y) \end{aligned}$$

for all $y\in\Gamma$ and $r\in(\varepsilon g_0(y),\varepsilon g_1(y))$, i.e. $D(u)^\sharp : Q = (\partial u^\sharp/\partial r)\cdot n$. Hence

$$\int_{\varepsilon g_0}^{\varepsilon g_1} D(u)^\sharp : Q\,dr = \left(\int_{\varepsilon g_0}^{\varepsilon g_1}\frac{\partial u^\sharp}{\partial r}\,dr\right)\cdot n = u_1^\sharp\cdot n - u_0^\sharp\cdot n \quad \text{on} \quad \Gamma. \quad (5.6.78)$$

Here and hereafter we use the notation (5.2.47). Since u satisfies $u\cdot n_\varepsilon = 0$ on Γ_ε,

$$(-1)^{i+1}u_i^\sharp\cdot n = u_i^\sharp\cdot\{(-1)^{i+1}(n-\varepsilon\nabla_\Gamma g_i) - n_{\varepsilon,i}^\sharp\} + \varepsilon(-1)^{i+1}u_i^\sharp\cdot\nabla_\Gamma g_i$$

on Γ for $i = 0,1$. From this equality and $g = g_1 - g_0$ it follows that

$$\begin{aligned} u_1^\sharp\cdot n - u_0^\sharp\cdot n &= \sum_{i=0,1}(-1)^{i+1}u_i^\sharp\cdot n \\ &= \sum_{i=0,1}u_i^\sharp\cdot\{(-1)^{i+1}(n-\varepsilon\nabla_\Gamma g_i) - n_{\varepsilon,i}^\sharp\} \\ &\qquad + \varepsilon\sum_{i=0,1}(-1)^{i+1}(u_i^\sharp - Mu)\cdot\nabla_\Gamma g_i + \varepsilon Mu\cdot\nabla_\Gamma g \end{aligned}$$

on Γ. Applying (5.2.40) to the second line we get

$$|(u_1^\sharp\cdot n - u_0^\sharp\cdot n) - \varepsilon Mu\cdot\nabla_\Gamma g| \leq c\varepsilon\sum_{i=0,1}(\varepsilon|u_i^\sharp| + |u_i^\sharp - Mu|) \quad \text{on} \quad \Gamma. \quad (5.6.79)$$

Combining (5.6.78) and (5.6.79) and using Hölder's inequality we see that

$$\begin{aligned} &\left|\int_\Gamma\left(\int_{\varepsilon g_0}^{\varepsilon g_1} D(u)^\sharp : Q\,dr\right)\eta\,d\mathcal{H}^2 - \varepsilon\int_\Gamma(Mu\cdot\nabla_\Gamma g)\eta\,d\mathcal{H}^2\right| \\ &\qquad\qquad \leq c\varepsilon\sum_{i=0,1}\Big(\varepsilon\|u_i^\sharp\|_{L^2(\Gamma)} + \|u_i^\sharp - Mu\|_{L^2(\Gamma)}\Big)\|\eta\|_{L^2(\Gamma)}. \end{aligned}$$

Moreover, by (5.2.58), (5.3.8), and (5.6.7),

$$\varepsilon\|u_i^\sharp\|_{L^2(\Gamma)} + \|u_i^\sharp - Mu\|_{L^2(\Gamma)} \leq c\left(\varepsilon\|u\|_{L^2(\Gamma_\varepsilon^i)} + \left\|u - \overline{Mu}\right\|_{L^2(\Gamma_\varepsilon^i)}\right) \leq c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}.$$

From the above two estimates we deduce that

$$\begin{aligned} &\left|\int_\Gamma\left(\int_{\varepsilon g_0}^{\varepsilon g_1} D(u)^\sharp : Q\,dr\right)\eta\,d\mathcal{H}^2 - \varepsilon\int_\Gamma(Mu\cdot\nabla_\Gamma g)\eta\,d\mathcal{H}^2\right| \\ &\qquad\qquad \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}. \quad (5.6.80) \end{aligned}$$

Finally, we combine (5.6.77) and (5.6.80) and note that $Mu\cdot\nabla_\Gamma g = M_\tau u\cdot\nabla_\Gamma g$ by the fact that $\nabla_\Gamma g$ is tangential on Γ to obtain the inequality (5.6.76). $\square$

Lemma 5.6.28. *Suppose that Assumption 1 is satisfied, i.e. the inequality* (5.1.6) *holds. Let* $u^\varepsilon \in H^2(\Omega_\varepsilon)^3$ *satisfy* (5.3.21) *on* Γ_ε^0 *or on* Γ_ε^1 *and* $v \in L^2(\Gamma, T\Gamma)$*. Then we have*

$$\left|\int_{\Omega_\varepsilon} D(u) : \bar{v} \otimes \bar{n}\, dx\right| \leq c\varepsilon^{3/2} \|u\|_{H^2(\Omega_\varepsilon)} \|v\|_{L^2(\Gamma)}, \tag{5.6.81}$$

where $c > 0$ *is a constant independent of* ε*,* u*, and* v*.*

Proof. Since v is tangential on Γ,

$$D(u) : \bar{v} \otimes \bar{n} = \mathrm{tr}[D(u)^T(\bar{v} \otimes \bar{n})] = (D(u)^T\bar{v}) \cdot \bar{n} = \bar{v} \cdot (D(u)\bar{n}) = \bar{v} \cdot \overline{P} D(u)\bar{n}$$

in Ω_ε. Hence by (5.2.53) and (5.3.32) we see that

$$\left|\int_{\Omega_\varepsilon} D(u) : \bar{v} \otimes \bar{n}\, dx\right| \leq c \left\|\overline{P} D(u)\bar{n}\right\|_{L^2(\Omega_\varepsilon)} \|\bar{v}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{3/2} \|u\|_{H^2(\Omega_\varepsilon)} \|v\|_{L^2(\Gamma)}.$$

Here we used Assumption 1 and the condition on u to apply (5.3.32). □

Now let us derive estimates for trilinear forms. The main tools for the estimates are the product estimate (5.6.52) for functions on Γ and Ω_ε and the L^∞-estimate (5.6.66) for the residual part of a vector field on Ω_ε.

Lemma 5.6.29. *Let* $u_1 \in H^2(\Omega_\varepsilon)^3$*,* $u_2 \in H^1(\Omega_\varepsilon)^3$*, and* $A \in L^2(\Gamma)^{3\times 3}$*. Under Assumption 1, suppose further that* u_1 *satisfies* $\mathrm{div}\, u_1 = 0$ *in* Ω_ε *and* (5.3.20)–(5.3.21) *on* Γ_ε *and that* A *satisfies* $PA = AP = A$ *on* Γ*. Then*

$$\left|\int_{\Omega_\varepsilon} u_1 \otimes u_2 : \overline{A}\, dx - \varepsilon \int_\Gamma g(M_\tau u_1) \otimes (M_\tau u_2) : A\, d\mathcal{H}^2\right| \leq cR_\varepsilon(u_1, u_2) \|A\|_{L^2(\Gamma)}, \tag{5.6.82}$$

where $c > 0$ *is a constant independent of* ε*,* u_1*,* u_2*, and* A *and*

$$\begin{aligned} R_\varepsilon(u_1, u_2) &:= \varepsilon \|u_1\|_{H^1(\Omega_\varepsilon)} \|u_2\|_{H^1(\Omega_\varepsilon)} \\ &\quad + \left(\varepsilon \|u_1\|_{H^2(\Omega_\varepsilon)} + \varepsilon^{1/2} \|u_1\|_{L^2(\Omega_\varepsilon)}^{1/2} \|u_1\|_{H^2(\Omega_\varepsilon)}^{1/2}\right) \|u_2\|_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{5.6.83}$$

Proof. We use the notations (5.6.3) for the tangential and normal components (with respect to Γ) of a vector field on Ω_ε. Since $PA = AP = A$ and $P^T = P$ on Γ,

$$u_1 \otimes u_2 : \overline{A} = \overline{P}(u_1 \otimes u_2)\overline{P} : \overline{A} = u_{1,\tau} \otimes u_{2,\tau} : \overline{A} \quad \text{in} \quad \Omega_\varepsilon.$$

Using this equality we decompose the difference

$$\int_{\Omega_\varepsilon} u_1 \otimes u_2 : \overline{A}\, dx - \varepsilon \int_\Gamma g(M_\tau u_1) \otimes (M_\tau u_2) : A\, d\mathcal{H}^2 = I_1 + I_2 \tag{5.6.84}$$

into

$$\begin{aligned} I_1 &:= \int_{\Omega_\varepsilon} u_{1,\tau} \otimes u_{2,\tau} : \overline{A}\, dx - \int_{\Omega_\varepsilon} \left(\overline{M_\tau u_1}\right) \otimes u_{2,\tau} : \overline{A}\, dx, \\ I_2 &:= \int_{\Omega_\varepsilon} \left(\overline{M_\tau u_1}\right) \otimes u_{2,\tau} : \overline{A}\, dx - \varepsilon \int_\Gamma g(M_\tau u_1) \otimes (M_\tau u_2) : A\, d\mathcal{H}^2. \end{aligned}$$

Let u_1^a be the average part of u_1 given by (5.6.50) and $u_1^r := u_1 - u_1^a$. Since

$$u_{1,\tau} - \overline{M_\tau u_1} = \overline{P}u_1 - \overline{P}u_1^a = \overline{P}u_1^r, \quad u_{2,\tau} = \overline{P}u_2 \quad \text{in} \quad \Omega_\varepsilon, \quad |P| = 1 \quad \text{on} \quad \Gamma,$$

where the first equality follows from (5.6.50), we have

$$|I_1| = \left|\int_{\Omega_\varepsilon} \left(\overline{P}u_1^r\right) \otimes u_{2,\tau} : \overline{A}\,dx\right| \leq c\|u_1^r\|_{L^\infty(\Omega_\varepsilon)}\|u_2\|_{L^2(\Omega_\varepsilon)}\left\|\overline{A}\right\|_{L^2(\Omega_\varepsilon)}.$$

We apply (5.2.53) and (5.6.66) to the right-hand side to obtain

$$|I_1| \leq c\left(\varepsilon\|u_1\|_{H^2(\Omega_\varepsilon)} + \varepsilon^{1/2}\|u_1\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u_1\|_{H^2(\Omega_\varepsilon)}^{1/2}\right)\|u_2\|_{L^2(\Omega_\varepsilon)}\|A\|_{L^2(\Gamma)}. \tag{5.6.85}$$

Here we used Assumption 1 and the conditions on u_1 to apply (5.6.66).

Let us estimate I_2. Noting that $M_\tau u_2 = Mu_{2,\tau}$ on Γ, we use the change of variables formula (5.2.51) and the inequalities (5.2.50), (5.2.52), and

$$|M_\tau u_1| = |PMu_1| \leq |Mu_1| \quad \text{on} \quad \Gamma, \quad |u_{2,\tau}| \leq |u_2| \quad \text{in} \quad \Omega_\varepsilon$$

to deduce that

$$\begin{aligned} |I_2| = \left|\int_\Gamma (M_\tau u_1) \otimes \left(\int_{\varepsilon g_0}^{\varepsilon g_1} u_{2,\tau}^\sharp (J-1)\,dr\right) : A\,d\mathcal{H}^2\right| &\leq c\varepsilon\int_{\Omega_\varepsilon} \left|\overline{M_\tau u_1}\right| |u_{2,\tau}| \left|\overline{A}\right|\,dx \\ &\leq c\varepsilon\left\|\,\left|\overline{Mu_1}\right|\,|u_2|\,\right\|_{L^2(\Omega_\varepsilon)}\left\|\overline{A}\right\|_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

Moreover, from (5.6.4), (5.6.22), and (5.6.52) it follows that

$$\begin{aligned} \left\|\,\left|\overline{Mu_1}\right|\,|u_2|\,\right\|_{L^2(\Omega_\varepsilon)} &\leq c\|Mu_1\|_{L^2(\Gamma)}^{1/2}\|Mu_1\|_{H^1(\Gamma)}^{1/2}\|u_2\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u_2\|_{H^1(\Omega_\varepsilon)}^{1/2} \\ &\leq c\varepsilon^{-1/2}\|u_1\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u_1\|_{H^1(\Omega_\varepsilon)}^{1/2}\|u_2\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u_2\|_{H^1(\Omega_\varepsilon)}^{1/2}. \end{aligned}$$

We apply this inequality and (5.2.53) to the above estimate for I_2 to get

$$\begin{aligned} |I_2| &\leq c\varepsilon\|u_1\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u_1\|_{H^1(\Omega_\varepsilon)}^{1/2}\|u_2\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u_2\|_{H^1(\Omega_\varepsilon)}^{1/2}\|A\|_{L^2(\Gamma)}. \\ &\leq c\varepsilon\|u_1\|_{H^1(\Omega_\varepsilon)}\|u_2\|_{H^1(\Omega_\varepsilon)}\|A\|_{L^2(\Gamma)}. \end{aligned} \tag{5.6.86}$$

By (5.6.84), (5.6.85), and (5.6.86) we conclude that the inequality (5.6.82) is valid with $R_\varepsilon(u_1, u_2)$ given by (5.6.83). □

Lemma 5.6.30. *Let $u_1, u_2 \in H^1(\Omega_\varepsilon)^3$ and $v \in H^1(\Gamma)^3$. Suppose that u_2 satisfies $u_2 \cdot n_\varepsilon = 0$ on Γ_ε^0 or on Γ_ε^1. Then there exists a constant $c > 0$ independent of ε, u_1, u_2, and v such that*

$$\left|\int_{\Omega_\varepsilon} u_1 \otimes u_2 : \bar{v} \otimes \bar{n}\,dx\right| \leq c\varepsilon\|u_1\|_{H^1(\Omega_\varepsilon)}\|u_2\|_{H^1(\Omega_\varepsilon)}\|v\|_{H^1(\Gamma)}. \tag{5.6.87}$$

Proof. Since $u_1 \otimes u_2 : \bar{v} \otimes \bar{n} = (u_1 \cdot \bar{v})(u_2 \cdot \bar{n})$, we use (5.3.23) and (5.6.52) to get

$$\begin{aligned} \left|\int_{\Omega_\varepsilon} u_1 \otimes u_2 : \bar{v} \otimes \bar{n}\,dx\right| &\leq \|u_1 \cdot \bar{v}\|_{L^2(\Omega_\varepsilon)}\|u_2 \cdot \bar{n}\|_{L^2(\Omega_\varepsilon)} \\ &\leq c\varepsilon\|u_1\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u_1\|_{H^1(\Omega_\varepsilon)}^{1/2}\|v\|_{L^2(\Gamma)}^{1/2}\|v\|_{H^1(\Gamma)}^{1/2}\|u_2\|_{H^1(\Omega_\varepsilon)} \\ &\leq c\varepsilon\|u_1\|_{H^1(\Omega_\varepsilon)}\|u_2\|_{H^1(\Omega_\varepsilon)}\|v\|_{H^1(\Gamma)}. \end{aligned}$$

Thus, the inequality (5.6.87) holds. □

5.7 Estimate for the trilinear term

The purpose of this section is to give an estimate for the trilinear term, which is essential for our proof of the global existence of a strong solution. Throughout this section we impose Assumptions 1 and 2 and let ε_0 be the positive constant given in Lemma 5.5.4.

Lemma 5.7.1. *For any $\alpha > 0$ there exist $c_\alpha^1, c_\alpha^2 > 0$ independent of ε such that*

$$\begin{aligned}\left|\left((u\cdot\nabla)u, A_\varepsilon u\right)_{L^2(\Omega_\varepsilon)}\right| &\leq \left(\alpha + c_\alpha^1\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}^2 \\ &\quad + c_\alpha^2\left(\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^4 + \varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2\right)\end{aligned} \tag{5.7.1}$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $u \in D(A_\varepsilon)$. (In fact, c_α^1 does not depend on α.)

The main tools for the proof of Lemma 5.7.1 are the estimates given in Sections 5.5.3 and 5.6.3. We also use the following inequality for the tangential component (with respect to Γ) of the curl of the average part u^a (for a proof, see Appendix 5.A).

Lemma 5.7.2. *For $u \in C^1(\Omega_\varepsilon)^3$ let $u^a := E_\varepsilon M_\tau u$ be given by* (5.6.50). *Then*

$$\left|\overline{P}\operatorname{curl} u^a\right| \leq c\left(\left|\overline{Mu}\right| + \varepsilon\left|\overline{\nabla_\Gamma Mu}\right|\right) \quad \text{in} \quad \Omega_\varepsilon, \tag{5.7.2}$$

where $c > 0$ is a constant independent of ε and u.

Proof of Lemma 5.7.1. For $u \in D(A_\varepsilon)$ let u^a be given by (5.6.50), $u^r := u - u^a$, and $\omega := \operatorname{curl} u$. Since $(u\cdot\nabla)u = \omega\times u + \nabla(|u|^2)/2$ and $(\nabla(|u|^2), A_\varepsilon u)_{L^2(\Omega_\varepsilon)} = 0$ by $A_\varepsilon u \in L^2_\sigma(\Omega_\varepsilon)$ and $\nabla(|u|^2) \in L^2_\sigma(\Omega_\varepsilon)^\perp$, we have

$$\left((u\cdot\nabla)u, A_\varepsilon u\right)_{L^2(\Omega_\varepsilon)} = (\omega\times u, A_\varepsilon u)_{L^2(\Omega_\varepsilon)} = I_1 + I_2 + I_3,$$

where

$$\begin{gathered} I_1 := (\omega\times u^r, A_\varepsilon u)_{L^2(\Omega_\varepsilon)}, \\ I_2 := (\omega\times u^a, A_\varepsilon u + \nu\Delta u)_{L^2(\Omega_\varepsilon)}, \quad I_3 := (\omega\times u^a, -\nu\Delta u)_{L^2(\Omega_\varepsilon)}. \end{gathered}$$

Let us estimate I_1, I_2, and I_3 separately. By (5.5.32) and (5.6.66),

$$\begin{aligned} |I_1| &\leq \|u^r\|_{L^\infty(\Omega_\varepsilon)}\|\omega\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)} \\ &\leq c\left(\varepsilon^{1/2}\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}\right)\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)} \\ &= c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^2 + c\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^{3/2}. \end{aligned}$$

To the second term we apply Young's inequality $ab \leq \alpha a^{4/3} + c_\alpha b^4$ to get

$$|I_1| \leq \left(\alpha + c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}^2 + c_\alpha\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^4. \tag{5.7.3}$$

(Note that the constant c in the above inequality does not depend on α.)

Next we deal with I_2. By (5.6.56) we have

$$\begin{aligned} \|\omega\times u^a\|_{L^2(\Omega_\varepsilon)} &\leq c\varepsilon^{-1/2}\|\omega\|_{L^2(\Omega_\varepsilon)}^{1/2}\|\omega\|_{H^1(\Omega_\varepsilon)}^{1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}^{1/2} \\ &\leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}. \end{aligned} \tag{5.7.4}$$

From this inequality, (5.5.30), and (5.5.33) it follows that

$$\begin{aligned}
|I_2| &\leq \|\omega \times u^a\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon u + \nu\Delta u\|_{L^2(\Omega_\varepsilon)} \\
&\leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}^2\|u\|_{H^2(\Omega_\varepsilon)}^{1/2} \\
&\leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}.
\end{aligned}$$

Applying Young's inequality $ab \leq \alpha a^2 + c_\alpha b^2$ to the last line we further get

$$|I_2| \leq \alpha\|u\|_{H^2(\Omega_\varepsilon)}^2 + c_\alpha\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2. \tag{5.7.5}$$

The estimate for I_3 is more complicated. Setting $\Phi := \omega \times u^a$ we have

$$I_3 = -\nu(\Delta u, \Phi)_{L^2(\Omega_\varepsilon)} = \nu(\operatorname{curl}\omega, \Phi)_{L^2(\Omega_\varepsilon)}$$

by $\operatorname{div} u = 0$ in Ω_ε. Here Φ is in $H^1(\Omega_\varepsilon)^3$ since $\omega \in H^1(\Omega_\varepsilon)^3$, $u^a \in H^2(\Omega_\varepsilon)^3$, and the Sobolev embeddings $H^1(\Omega_\varepsilon) \hookrightarrow L^4(\Omega_\varepsilon)$ and $H^2(\Omega_\varepsilon) \hookrightarrow L^\infty(\Omega_\varepsilon)$ hold (see [1]). Hence we can apply the identity (5.5.28) to get

$$I_3 = -\nu(\operatorname{curl} G(u), \Phi)_{L^2(\Omega_\varepsilon)} + \nu(\omega + G(u), \operatorname{curl}\Phi)_{L^2(\Omega_\varepsilon)} = J_1 + J_2 + J_3,$$

where $G(u)$ is given in Lemma 5.5.6 and

$$\begin{gathered}
J_1 := -\nu(\operatorname{curl} G(u), \Phi)_{L^2(\Omega_\varepsilon)}, \\
J_2 := \nu(G(u), \operatorname{curl}\Phi)_{L^2(\Omega_\varepsilon)}, \quad J_3 := \nu(\omega, \operatorname{curl}\Phi)_{L^2(\Omega_\varepsilon)}.
\end{gathered}$$

From (5.7.4) and (5.5.22) we deduce that

$$|J_1| \leq c\|\nabla G(u)\|_{L^2(\Omega_\varepsilon)}\|\Phi\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}^2\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}.$$

Then using (5.5.33) and Young's inequality $ab \leq \alpha a^2 + c_\alpha b^2$ we get

$$\begin{aligned}
|J_1| &\leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)} \\
&\leq \alpha\|u\|_{H^2(\Omega_\varepsilon)}^2 + c_\alpha\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2.
\end{aligned} \tag{5.7.6}$$

Let us estimate J_2. By (5.5.22),

$$|\operatorname{curl}\Phi| \leq c(|\nabla\omega||u^a| + |\omega||\nabla u^a|) \leq c(|u^a||\nabla^2 u| + |\nabla u^a||\nabla u|) \quad \text{in} \quad \Omega_\varepsilon,$$

and Hölder's inequality we have

$$\begin{aligned}
|J_2| &\leq c\int_{\Omega_\varepsilon} |u|(|u^a||\nabla^2 u| + |\nabla u^a||\nabla u|)\,dx \\
&\leq c\left(\|\,|u^a|\,|u|\,\|_{L^2(\Omega_\varepsilon)}\|\nabla^2 u\|_{L^2(\Omega_\varepsilon)} + \|\,|\nabla u^a|\,|u|\,\|_{L^2(\Omega_\varepsilon)}\|\nabla u\|_{L^2(\Omega_\varepsilon)}\right).
\end{aligned}$$

To the last line we apply (5.6.56) and (5.6.57) to obtain

$$\begin{aligned}
|J_2| &\leq c\varepsilon^{-1/2}\left(\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)} + \|u\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}^2\|u\|_{H^2(\Omega_\varepsilon)}^{1/2}\right) \\
&\leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)},
\end{aligned}$$

where the second inequality follows from (5.5.33). Hence Young's inequality yields

$$|J_2| \leq \alpha \|u\|^2_{H^2(\Omega_\varepsilon)} + c_\alpha \varepsilon^{-1} \|u\|^2_{L^2(\Omega_\varepsilon)} \|u\|^2_{H^1(\Omega_\varepsilon)}. \tag{5.7.7}$$

To derive an estimate for $J_3 = \nu(\omega, \operatorname{curl}\Phi)_{L^2(\Omega_\varepsilon)}$ we observe that

$$\operatorname{curl}\Phi = (u^a \cdot \nabla)\omega - (\omega \cdot \nabla)u^a + (\operatorname{div} u^a)\omega - (\operatorname{div}\omega)u^a.$$

Moreover, $\operatorname{div}\omega = \operatorname{div}\operatorname{curl} u = 0$ and

$$\int_{\Omega_\varepsilon} \omega \cdot (u^a \cdot \nabla)\omega \, dx = -\frac{1}{2}\int_{\Omega_\varepsilon} (\operatorname{div} u^a)|\omega|^2 \, dx$$

by integration by parts and $u^a \cdot n_\varepsilon = 0$ on Γ_ε (see Section 5.6.3). Therefore,

$$\begin{aligned} J_3 &= \nu(\omega, (u^a \cdot \nabla)\omega - (\omega \cdot \nabla)u^a + (\operatorname{div} u^a)\omega)_{L^2(\Omega_\varepsilon)} \\ &= \frac{\nu}{2}(\operatorname{div} u^a, |\omega|^2)_{L^2(\Omega_\varepsilon)} - \nu(\omega, (\omega \cdot \nabla)u^a)_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{5.7.8}$$

Noting that $u^a = E_\varepsilon M_\tau u$ is given by (5.6.50), we split the first term into

$$\begin{aligned} (\operatorname{div} u^a, |\omega|^2)_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \frac{1}{\bar{g}}\left(\overline{\operatorname{div}_\Gamma(gM_\tau u)}\right)|\omega|^2 \, dx \\ + \int_{\Omega_\varepsilon}\left(\operatorname{div}(E_\varepsilon M_\tau u) - \frac{1}{\bar{g}}\overline{\operatorname{div}_\Gamma(gM_\tau u)}\right)|\omega|^2 \, dx. \end{aligned}$$

Applying (5.2.30), (5.3.45), and Hölder's inequality to the right-hand side we have

$$|(\operatorname{div} u^a, |\omega|^2)_{L^2(\Omega_\varepsilon)}| \leq c(K_1 + \varepsilon K_2 + \varepsilon K_3)\|\omega\|_{L^2(\Omega_\varepsilon)},$$

where

$$\begin{gathered} K_1 := \left\| \left|\overline{\operatorname{div}_\Gamma(gM_\tau u)}\right| |\omega| \right\|_{L^2(\Omega_\varepsilon)}, \\ K_2 := \left\| \left|\overline{M_\tau u}\right| |\omega| \right\|_{L^2(\Omega_\varepsilon)}, \quad K_3 := \left\| \left|\overline{\nabla_\Gamma M_\tau u}\right| |\omega| \right\|_{L^2(\Omega_\varepsilon)}. \end{gathered}$$

To K_1 we apply (5.6.52) and then use (5.6.46) and (5.6.47) to get

$$\begin{aligned} K_1 &\leq c\|\operatorname{div}_\Gamma(gM_\tau u)\|^{1/2}_{L^2(\Gamma)}\|\operatorname{div}_\Gamma(gM_\tau u)\|^{1/2}_{H^1(\Gamma)}\|\omega\|^{1/2}_{L^2(\Omega_\varepsilon)}\|\omega\|^{1/2}_{H^1(\Omega_\varepsilon)} \\ &\leq c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)} \leq c\varepsilon^{1/2}\|u\|^2_{H^2(\Omega_\varepsilon)}. \end{aligned}$$

Similarly, applying (5.6.52) to K_2 and K_3 and noting that

$$\begin{aligned} \|M_\tau u\|_{H^k(\Gamma)} &\leq c\|Mu\|_{H^k(\Gamma)} \leq c\varepsilon^{-1/2}\|u\|_{H^k(\Omega_\varepsilon)}, \\ \|\nabla_\Gamma M_\tau u\|_{H^k(\Gamma)} &\leq c\|Mu\|_{H^{k+1}(\Gamma)} \leq c\varepsilon^{-1/2}\|u\|_{H^{k+1}(\Omega_\varepsilon)} \end{aligned}$$

for $k = 0, 1$ (with $H^0 = L^2$) by $M_\tau u = PMu$ on Γ, $P \in C^4(\Gamma)^{3\times 3}$, (5.6.4), and (5.6.22) we obtain

$$\begin{aligned} K_2 &\leq c\varepsilon^{-1/2}\|u\|^{1/2}_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|^{1/2}_{H^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1/2}\|u\|^2_{H^2(\Omega_\varepsilon)}, \\ K_3 &\leq c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1/2}\|u\|^2_{H^2(\Omega_\varepsilon)}. \end{aligned}$$

From these inequalities and $\|\omega\|_{L^2(\Omega_\varepsilon)} \leq c\|u\|_{H^1(\Omega_\varepsilon)}$ we deduce that

$$|(\operatorname{div} u^a, |\omega|^2)_{L^2(\Omega_\varepsilon)}| \leq c(K_1 + \varepsilon K_2 + \varepsilon K_3)\|\omega\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|^2_{H^2(\Omega_\varepsilon)}. \quad (5.7.9)$$

Let us estimate $(\omega, (\omega\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}$. Using $\omega = \operatorname{curl} u^r + \operatorname{curl} u^a$ we have

$$(\omega, (\omega\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)} = (\omega, (\operatorname{curl} u^r\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)} + (\omega, (\operatorname{curl} u^a\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}.$$

The first term on the right-hand side is bounded by

$$|(\omega, (\operatorname{curl} u^r\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}| \leq c\|\nabla u^r\|_{L^2(\Omega_\varepsilon)}\|\,|\nabla u^a|\,|\omega|\,\|_{L^2(\Omega_\varepsilon)}.$$

To the right-hand side we apply (5.6.60) and

$$\begin{aligned}\|\,|\nabla u^a|\,|\omega|\,\|_{L^2(\Omega_\varepsilon)} &\leq c\varepsilon^{-1/2}\|\omega\|^{1/2}_{L^2(\Omega_\varepsilon)}\|\omega\|^{1/2}_{H^1(\Omega_\varepsilon)}\|u\|^{1/2}_{H^1(\Omega_\varepsilon)}\|u\|^{1/2}_{H^2(\Omega_\varepsilon)} \\ &\leq c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}\end{aligned} \quad (5.7.10)$$

by (5.6.57). Then we get

$$\begin{aligned}&|(\omega, (\operatorname{curl} u^r\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}| \\ &\qquad \leq c\left(\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|^2_{H^2(\Omega_\varepsilon)} + \varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}\right).\end{aligned} \quad (5.7.11)$$

Also, we decompose $(\omega, (\operatorname{curl} u^a\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}$ into the sum of

$$L_1 := \left(\omega, \left((\overline{P}\operatorname{curl} u^a)\cdot\nabla\right)u^a\right)_{L^2(\Omega_\varepsilon)}, \quad L_2 := (\omega, (\operatorname{curl} u^a\cdot\bar{n})\partial_n u^a)_{L^2(\Omega_\varepsilon)}.$$

To L_1 we apply (5.7.2) and Hölder's inequality to get

$$\begin{aligned}|L_1| &\leq c\int_{\Omega_\varepsilon}|\omega|\left(\left|\overline{Mu}\right| + \varepsilon\left|\overline{\nabla_\Gamma Mu}\right|\right)|\nabla u^a|\,dx \\ &\leq c\|\,|\nabla u^a|\,|\omega|\,\|_{L^2(\Omega_\varepsilon)}\left(\left\|\overline{Mu}\right\|_{L^2(\Omega_\varepsilon)} + \varepsilon\left\|\overline{\nabla_\Gamma Mu}\right\|_{L^2(\Omega_\varepsilon)}\right).\end{aligned}$$

Hence from (5.6.5), (5.6.23), (5.7.10), and $\|u\|_{H^1(\Omega_\varepsilon)} \leq \|u\|_{H^2(\Omega_\varepsilon)}$ it follows that

$$\begin{aligned}|L_1| &\leq c\varepsilon^{-1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}\left(\|u\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|u\|_{H^1(\Omega_\varepsilon)}\right) \\ &\leq c\left(\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|^2_{H^2(\Omega_\varepsilon)} + \varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}\right).\end{aligned}$$

To estimate L_2 we see by the definition (5.6.50) of u^a, (5.3.6), and (5.3.37) that

$$|\partial_n u^a| = \left|\overline{M_\tau u}\cdot\partial_n\Psi_\varepsilon\right| \leq c\left|\overline{M_\tau u}\right| = c\left|\overline{PMu}\right| \leq c\left|\overline{Mu}\right| \quad \text{in} \quad \Omega_\varepsilon.$$

We apply this inequality and $|\operatorname{curl} u^a\cdot\bar{n}| \leq c|\nabla u^a|$ to L_2 and use (5.6.5) and (5.7.10). Then we have

$$\begin{aligned}|L_2| &\leq c\int_{\Omega_\varepsilon}|\omega||\nabla u^a|\left|\overline{Mu}\right|\,dx \leq c\|\,|\nabla u^a|\,|\omega|\,\|_{L^2(\Omega_\varepsilon)}\left\|\overline{Mu}\right\|_{L^2(\Omega_\varepsilon)} \\ &\leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}.\end{aligned}$$

Applying the above inequalities to $(\omega,(\operatorname{curl} u^a\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}=L_1+L_2$ we obtain

$$\begin{aligned}&|(\omega,(\operatorname{curl} u^a\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}|\\ &\qquad\leq c\left(\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^2+\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}\right).\end{aligned}$$

From this inequality and (5.7.11) we deduce that

$$\begin{aligned}|(\omega,(\omega\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}|&\leq|(\omega,(\operatorname{curl} u^r\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}|+|(\omega,(\operatorname{curl} u^a\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}|\\ &\leq c\left(\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}^2+\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}\right).\end{aligned}$$

Since Young's inequality implies that

$$\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|u\|_{H^1(\Omega_\varepsilon)}\|u\|_{H^2(\Omega_\varepsilon)}\leq\alpha\|u\|_{H^2(\Omega_\varepsilon)}^2+c_\alpha\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2,$$

we further get

$$|(\omega,(\operatorname{curl} u^a\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}|\leq\left(\alpha+c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}^2+c_\alpha\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2.$$

We apply this inequality and (5.7.9) to (5.7.8) to show that

$$\begin{aligned}|J_3|&\leq c\left(|(\operatorname{div} u^a,|\omega|^2)_{L^2(\Omega_\varepsilon)}|+|(\omega,(\operatorname{curl} u^a\cdot\nabla)u^a)_{L^2(\Omega_\varepsilon)}|\right)\\ &\leq c\left(\alpha+\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}^2+c_\alpha\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2.\end{aligned}\tag{5.7.12}$$

Since $I_3=J_1+J_2+J_3$, we see by (5.7.6), (5.7.7), and (5.7.12) that

$$|I_3|\leq c\left(\alpha+\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}^2+c_\alpha\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2$$

and this inequality combined with (5.7.3) and (5.7.5) yields

$$\begin{aligned}\left|\left((u\cdot\nabla)u,A_\varepsilon u\right)_{L^2(\Omega_\varepsilon)}\right|&\leq|I_1|+|I_2|+|I_3|\\ &\leq\left(c_1\alpha+c_2\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}\right)\|u\|_{H^2(\Omega_\varepsilon)}^2\\ &\qquad+c_\alpha\left(\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^4+\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|u\|_{H^1(\Omega_\varepsilon)}^2\right)\end{aligned}$$

with positive constants c_1, c_2, and c_α independent of ε. Replacing $c_1\alpha$ by α in the above inequality we obtain (5.7.1). □

Finally, we fix α and write (5.7.1) in terms of the Stokes operator A_ε.

Corollary 5.7.3. *There exist $d_1,d_2>0$ independent of ε such that*

$$\begin{aligned}\left|\left((u\cdot\nabla)u,A_\varepsilon u\right)_{L^2(\Omega_\varepsilon)}\right|&\leq\left(\frac{1}{4}+d_1\varepsilon^{1/2}\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}\right)\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}^2\\ &\qquad+d_2\left(\|u\|_{L^2(\Omega_\varepsilon)}^2\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}^4+\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}^2\right)\end{aligned}\tag{5.7.13}$$

for all $\varepsilon\in(0,\varepsilon_0)$ and $u\in D(A_\varepsilon)$.

Proof. Applying (5.5.18) and (5.5.32) to the right-hand side of (5.7.1) we get

$$\begin{aligned}\left|\left((u\cdot\nabla)u,A_\varepsilon u\right)_{L^2(\Omega_\varepsilon)}\right|&\leq\left(c\alpha+d_\alpha^1\varepsilon^{1/2}\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}\right)\|A_\varepsilon u\|_{L^2(\Omega_\varepsilon)}^2\\ &\qquad+d_\alpha^2\left(\|u\|_{L^2(\Omega_\varepsilon)}^2\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}^4+\varepsilon^{-1}\|u\|_{L^2(\Omega_\varepsilon)}^2\|A_\varepsilon^{1/2}u\|_{L^2(\Omega_\varepsilon)}^2\right)\end{aligned}$$

with positive constants c, d_α^1, and d_α^2 independent of ε. We take $\alpha=1/4c$ in the above inequality to obtain (5.7.13). □

5.8 Global existence and uniform estimates of a strong solution

Based on the estimate (5.7.13) for the trilinear term and the inequalities given in the previous sections we prove Theorems 5.1.1 and 5.1.2. As in Section 5.7 we suppose that Assumptions 1 and 2 are satisfied and let ε_0 be the positive constant given in Lemma 5.5.4.

First we recall the well-known result on the local-in-time existence of a strong solution (see e.g. [8, 10, 61, 64]).

Theorem 5.8.1. *For given $u_0^\varepsilon \in V_\varepsilon$ and $f^\varepsilon \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$ there exist $T_0 > 0$ depending on Ω_ε, ν, u_0^ε, and f^ε and a strong solution u^ε to the Navier–Stokes equations* (5.1.1)–(5.1.3) *on $[0, T_0)$ satisfying*

$$u^\varepsilon \in C([0,T]; V_\varepsilon) \cap L^2(0,T; D(A_\varepsilon)) \cap H^1(0,T; L^2_\sigma(\Omega_\varepsilon)) \quad \text{for all} \quad T \in (0, T_0).$$

If u^ε is maximally defined on the time interval $[0, T_{\max})$ and $T_{\max}$ is finite, then

$$\lim_{t\to T_{\max}^-} \|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} = \infty.$$

To establish the global-in-time existence of a strong solution u^ε we show that the $L^2(\Omega_\varepsilon)$-norm of $A_\varepsilon^{1/2}u^\varepsilon(t)$ is bounded uniformly in t. We argue by a standard energy method and use the uniform Gronwall inequality (see [59, Lemma D.3]).

Lemma 5.8.2 (Uniform Gronwall inequality)**.** *Let z, ξ, and ζ be nonnegative functions in $L^1_{loc}([0,T);\mathbb{R})$, $T \in (0,\infty]$. Suppose that $z \in C(0,T;\mathbb{R})$ and*

$$\frac{dz}{dt}(t) \le \xi(t)z(t) + \zeta(t) \quad \text{for a.a.} \quad t \in (0,T).$$

Then $z \in L^\infty_{loc}(0,T;\mathbb{R})$ and

$$z(t_2) \le \left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} z(s)\,ds + \int_{t_1}^{t_2}\zeta(s)\,ds\right)\exp\left(\int_{t_1}^{t_2}\xi(s)\,ds\right)$$

for all $t_1, t_2 \in (0,T)$ with $t_1 < t_2$.

We also use an estimate for the duality product between a vector field on Ω_ε and the constant extension of a tangential vector field on Γ.

Lemma 5.8.3. *There exists a constant $c > 0$ independent of ε such that*

$$\left|(\bar{v}, u)_{L^2(\Omega_\varepsilon)}\right| \le c\varepsilon^{1/2}\|v\|_{H^{-1}(\Gamma,T\Gamma)}\|u\|_{H^1(\Omega_\varepsilon)} \tag{5.8.1}$$

for all $v \in L^2(\Gamma, T\Gamma)$ and $u \in H^1(\Omega_\varepsilon)^3$, where $\bar{v} := v \circ \pi$ is the constant extension of v in the normal direction of Γ.

Proof. For the sake of simplicity, we use the notations (5.2.46) and (5.2.47) and suppress the arguments of functions. By the change of variables formula (5.2.51),

$$(\bar{v}, u)_{L^2(\Omega_\varepsilon)} = \int_\Gamma v\left(\int_{\varepsilon g_0}^{\varepsilon g_1} u^\sharp J\,dr\right)d\mathcal{H}^2 = (v,\eta)_{L^2(\Gamma)}, \tag{5.8.2}$$

where the vector field $\eta\colon\Gamma\to\mathbb{R}^3$ is given by

$$\eta(y) := \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} u^\sharp(y,r)J(y,r)\,dr, \quad y\in\Gamma.$$

Let us estimate the $H^1(\Gamma)$-norm of η. By (5.2.49), Hölder's inequality, and (5.2.52),

$$\|\eta\|_{L^2(\Gamma)}^2 \le \int_\Gamma \varepsilon g\left(\int_{\varepsilon g_0}^{\varepsilon g_1} |u^\sharp|^2\,dr\right) d\mathcal{H}^2 \le c\varepsilon\|u\|_{L^2(\Omega_\varepsilon)}^2. \tag{5.8.3}$$

Moreover, by the same calculations as in the proof of Lemma 5.6.7 we see that

$$\nabla_\Gamma\eta = \int_{\varepsilon g_0}^{\varepsilon g_1} \left\{\frac{\partial}{\partial r}\Big(J\psi_\varepsilon^\sharp\otimes u^\sharp\Big) + J(B\nabla u)^\sharp + \nabla_\Gamma J\otimes u^\sharp\right\} dr$$

on Γ, where B and ψ_ε are given by (5.6.18). To the right-hand side we apply the inequalities (5.2.49), (5.6.20), and (5.6.21) to obtain

$$|\nabla_\Gamma\eta| \le c\int_{\varepsilon g_0}^{\varepsilon g_1} (|u^\sharp| + |(\nabla u)^\sharp|)\,dr \quad \text{on} \quad \Gamma.$$

Hence Hölder's inequality and (5.2.52) imply that

$$\|\nabla_\Gamma\eta\|_{L^2(\Gamma)}^2 \le c\int_\Gamma \varepsilon g\left(\int_{\varepsilon g_0}^{\varepsilon g_1} (|u^\sharp|^2 + |(\nabla u)^\sharp|^2)\,dr\right) d\mathcal{H}^2 \le c\varepsilon\|u\|_{H^1(\Omega_\varepsilon)}^2. \tag{5.8.4}$$

By (5.8.3) and (5.8.4) we get $\eta\in H^1(\Gamma)^3$ and thus $P\eta\in H^1(\Gamma,T\Gamma)$. Moreover,

$$\|P\eta\|_{H^1(\Gamma)} \le c\|\eta\|_{H^1(\Gamma)} \le c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}. \tag{5.8.5}$$

Now we observe that

$$(\bar v, u)_{L^2(\Omega_\varepsilon)} = (v,\eta)_{L^2(\Gamma)} = (v, P\eta)_{L^2(\Gamma)} = [v, P\eta]_{T\Gamma}$$

by (5.8.2), the fact that v is tangential on Γ, and $P\eta\in H^1(\Gamma,T\Gamma)$. Hence

$$\left|(\bar v, u)_{L^2(\Omega_\varepsilon)}\right| = \left|[v,P\eta]_{T\Gamma}\right| \le \|v\|_{H^{-1}(\Gamma,T\Gamma)}\|P\eta\|_{H^1(\Gamma)}$$

and we obtain (5.8.1) by applying (5.8.5) to the right-hand side of this inequality. □

Now we are ready to establish the global-in-time existence of a strong solution to the Navier–Stokes equations (5.1.1)–(5.1.3).

Proof of Theorem 5.1.1. We follow the idea of the proofs of [20, Theorem 7.4] and [21, Theorem 3.1]. In what follows, we use the notation (5.6.3) for the tangential and normal components (with respect to Γ) of a vector field on Ω_ε. We also write c for a general positive constant independent of ε, c_0, and $T_{\max}$.

Let $u_0^\varepsilon\in V_\varepsilon$ and $f\in L^\infty(0,\infty;L^2(\Omega_\varepsilon)^3)$ satisfy (5.1.9) with

$$c_0 := \min\left\{1, \frac{d_3^2}{4}, \frac{d_3^2}{4d_4}\right\}, \quad d_3 := \frac{1}{4d_1}, \tag{5.8.6}$$

where d_1 is the positive constant given in Corollary 5.7.3 and d_4 is a positive constant given later. Noting that $M_\tau u_0^\varepsilon = Mu_{0,\tau}^\varepsilon$ on Γ and

$$u_0^\varepsilon = u_{0,n}^\varepsilon + \left(u_{0,\tau}^\varepsilon - \overline{Mu_{0,\tau}^\varepsilon}\right) + \overline{M_\tau u_0^\varepsilon}, \quad |u_{0,n}^\varepsilon| = |u_0^\varepsilon \cdot \bar{n}| \quad \text{in} \quad \Omega_\varepsilon,$$

we apply (5.2.53), (5.3.23), and (5.6.6) to the right-hand side of

$$\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c\left(\|u_0^\varepsilon \cdot \bar{n}\|_{L^2(\Omega_\varepsilon)}^2 + \left\|u_{0,\tau}^\varepsilon - \overline{Mu_{0,\tau}^\varepsilon}\right\|_{L^2(\Omega_\varepsilon)}^2 + \left\|\overline{M_\tau u_0^\varepsilon}\right\|_{L^2(\Omega_\varepsilon)}^2\right)$$

and then use $\|u_{0,\tau}^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq c\|u_0^\varepsilon\|_{H^1(\Omega_\varepsilon)}$ and (5.5.18) to get

$$\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c\left(\varepsilon^2\|A_\varepsilon^{1/2}u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon\|M_\tau u_0^\varepsilon\|_{L^2(\Gamma)}^2\right). \tag{5.8.7}$$

Hence from (5.1.9) and $c_0 \leq 1$ it follows that

$$\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq cc_0 \leq c. \tag{5.8.8}$$

Let u^ε be a strong solution to the Navier–Stokes equations (5.1.1)–(5.1.3) defined on the maximal time interval $[0, T_{\max})$. First we derive estimates for

$$\|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2, \quad \int_t^{\min\{t+1,T_{\max}\}} \|A_\varepsilon^{1/2}u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds, \quad t \in [0, T_{\max})$$

with explicit dependence of constants on ε. Taking the $L^2(\Omega_\varepsilon)$-inner product of

$$\partial_t u^\varepsilon + A_\varepsilon u^\varepsilon + \mathbb{P}_\varepsilon(u^\varepsilon \cdot \nabla)u^\varepsilon = \mathbb{P}_\varepsilon f^\varepsilon \quad \text{on} \quad (0, T_{\max}) \tag{5.8.9}$$

with u^ε and using

$$(\mathbb{P}_\varepsilon(u^\varepsilon \cdot \nabla)u^\varepsilon, u^\varepsilon)_{L^2(\Omega_\varepsilon)} = \big((u^\varepsilon \cdot \nabla)u^\varepsilon, u^\varepsilon\big)_{L^2(\Omega_\varepsilon)} = 0$$

by integration by parts, $\operatorname{div} u^\varepsilon = 0$ in Ω_ε, and $u^\varepsilon \cdot n_\varepsilon = 0$ on Γ_ε we get

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 = (\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon)_{L^2(\Omega_\varepsilon)} \quad \text{on} \quad (0, T_{\max}). \tag{5.8.10}$$

We split the right-hand side into the sum of

$$I_1 := \left(\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon - \overline{M_\tau u^\varepsilon}\right)_{L^2(\Omega_\varepsilon)}, \quad I_2 := \left(\mathbb{P}_\varepsilon f^\varepsilon, \overline{M_\tau u^\varepsilon}\right)_{L^2(\Omega_\varepsilon)}.$$

Applying (5.3.23) and (5.6.6) to $u^\varepsilon - \overline{M_\tau u^\varepsilon} = u_n^\varepsilon + (u_\tau^\varepsilon - \overline{Mu_\tau^\varepsilon})$ we get

$$\left\|u^\varepsilon - \overline{M_\tau u^\varepsilon}\right\|_{L^2(\Omega_\varepsilon)} \leq \|u^\varepsilon \cdot \bar{n}\|_{L^2(\Omega_\varepsilon)} + \left\|u_\tau^\varepsilon - \overline{Mu_\tau^\varepsilon}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}$$

(Note that $\|u_\tau^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq c\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}$.) This inequality and (5.5.18) show that

$$|I_1| \leq \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}\left\|u^\varepsilon - \overline{M_\tau u^\varepsilon}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

To I_2 we use (5.6.11) and (5.8.1) to get

$$\begin{aligned}|I_2| &\leq \left|\left(\overline{M_\tau\mathbb{P}_\varepsilon f^\varepsilon}, u^\varepsilon\right)_{L^2(\Omega_\varepsilon)}\right| + \left|\left(\mathbb{P}_\varepsilon f^\varepsilon, \overline{M_\tau u^\varepsilon}\right)_{L^2(\Omega_\varepsilon)} - \left(\overline{M_\tau\mathbb{P}_\varepsilon f^\varepsilon}, u^\varepsilon\right)_{L^2(\Omega_\varepsilon)}\right| \\ &\leq c\left(\varepsilon^{1/2}\|M_\tau\mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma,T\Gamma)}\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon\|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}\right).\end{aligned}$$

Applying these estimates to $(\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon)_{L^2(\Omega_\varepsilon)} = I_1 + I_2$ and using the inequality

$$\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \|u^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq c\|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)} \tag{5.8.11}$$

by (5.5.18) and Young's inequality we obtain

$$|(\mathbb{P}_\varepsilon f^\varepsilon, u^\varepsilon)_{L^2(\Omega_\varepsilon)}| \leq \frac{1}{2}\|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + c\left(\varepsilon^2\|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon\|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma, T\Gamma)}^2\right).$$

From this inequality and (5.8.10) we deduce that

$$\frac{d}{dt}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c\left(\varepsilon^2\|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon\|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma, T\Gamma)}^2\right) \tag{5.8.12}$$

on $(0, T_{\max})$. By (5.8.11) we further get

$$\frac{d}{dt}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{a_1}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c\left(\varepsilon^2\|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon\|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma, T\Gamma)}^2\right) \tag{5.8.13}$$

on $(0, T_{\max})$ with some constant $a_1 > 0$ independent of ε, c_0, and $T_{\max}$. For each $t \in (0, T_{\max})$ we multiply both sides of (5.8.13) at $s \in (0, t)$ by $e^{(s-t)/a_1}$ and integrate them over $(0, t)$. Then we have

$$\begin{aligned}\|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 &\leq e^{-t/a_1}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \\ &\quad + ca_1(1 - e^{-t/a_1})\left(\varepsilon^2\|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2 + \varepsilon\|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;H^{-1}(\Gamma, T\Gamma))}^2\right).\end{aligned} \tag{5.8.14}$$

Also, integrating (5.8.12) over (t, t_*) with $t_* := \min\{t+1, T_{\max}\}$ we deduce that

$$\begin{aligned}\int_t^{t_*}\|A_\varepsilon^{1/2} u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds &\leq \|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \\ &\quad + c\left(\varepsilon^2\|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2 + \varepsilon\|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;H^{-1}(\Gamma, T\Gamma))}^2\right).\end{aligned} \tag{5.8.15}$$

Hence we apply (5.1.9) and (5.8.8) to the right-hand sides of (5.8.14) and (5.8.15) to get

$$\|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 + \int_t^{t_*}\|A_\varepsilon^{1/2} u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \leq cc_0 \quad \text{for all} \quad t \in [0, T_{\max}). \tag{5.8.16}$$

Next we prove the uniform boundedness of $\|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}$ in $t \in [0, T_{\max})$. Our goal is to show that

$$\varepsilon^{1/2}\|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} < d_3 = \frac{1}{4d_1} \quad \text{for all} \quad t \in [0, T_{\max}). \tag{5.8.17}$$

If (5.8.17) is valid, then Theorem 5.8.1 implies that $T_{\max} = \infty$, i.e. the strong solution u^ε exists on the whole time interval $[0, \infty)$. First note that (5.8.17) is valid at $t = 0$ by (5.1.9) and (5.8.6). Let us prove (5.8.17) for all $t \in (0, T_{\max})$ by contradiction. Assume to the contrary that there exists $T \in (0, T_{\max})$ such that

$$\varepsilon^{1/2}\|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} < d_3 \quad \text{for all} \quad t \in [0, T), \tag{5.8.18}$$

$$\varepsilon^{1/2}\|A_\varepsilon^{1/2} u^\varepsilon(T)\|_{L^2(\Omega_\varepsilon)} = d_3. \tag{5.8.19}$$

(Note that $u^\varepsilon \in C([0,T_{\max});V_\varepsilon)$ and thus $\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}$ is continuous in t.) We consider (5.8.9) on $(0,T]$ and take its $L^2(\Omega_\varepsilon)$-inner product with $A_\varepsilon u^\varepsilon$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 &+ \|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \\ &\leq \left|\left((u^\varepsilon\cdot\nabla)u^\varepsilon, A_\varepsilon u^\varepsilon\right)_{L^2(\Omega_\varepsilon)}\right| + |(\mathbb{P}_\varepsilon f^\varepsilon, A_\varepsilon u^\varepsilon)_{L^2(\Omega_\varepsilon)}|. \qquad (5.8.20)\end{aligned}$$

on $(0,T]$. To the first term on the right-hand side we apply (5.7.13) and (5.8.18)–(5.8.19). Then noting that $d_3 = 1/4d_1$ we see that

$$\begin{aligned}\left|\left((u^\varepsilon\cdot\nabla)u^\varepsilon, A_\varepsilon u^\varepsilon\right)_{L^2(\Omega_\varepsilon)}\right| &\leq \frac{1}{2}\|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \\ &\quad + d_2\left(\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^4 + \varepsilon^{-1}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\right)\end{aligned}$$

on $(0,T]$. Also, Young's inequality yields that

$$|(\mathbb{P}_\varepsilon f^\varepsilon, A_\varepsilon u^\varepsilon)_{L^2(\Omega_\varepsilon)}| \leq \frac{1}{4}\|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2.$$

Applying these inequalities to the right-hand side of (5.8.20) we obtain

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{2}\|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq \xi\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \zeta \quad \text{on} \quad (0,T], \qquad (5.8.21)$$

where the functions ξ and ζ are given by

$$\begin{aligned}\xi(t) &:= 2d_2\|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2, \\ \zeta(t) &:= 2\left(d_2\varepsilon^{-1}\|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2\right)\end{aligned} \qquad (5.8.22)$$

for $t\in(0,T]$. By (5.8.16), (5.8.18), and (5.8.19) we see that

$$\xi \leq cc_0\varepsilon^{-1}, \quad \zeta \leq c\left(c_0\varepsilon^{-1}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\right) \quad \text{on} \quad (0,T].$$

Applying these inequalities to (5.8.21) we have

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{2}\|A_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c\left(c_0\varepsilon^{-1}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\right) \qquad (5.8.23)$$

on $(0,T]$. From (5.5.19) and (5.8.23) we further deduce that

$$\frac{d}{dt}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{a_2}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c\left(c_0\varepsilon^{-1}\|A_\varepsilon^{1/2}u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\right)$$

with a constant $a_2>0$ independent of ε, c_0, and $T_{\max}$. For $t\in(0,T]$ we multiply both sides of the above inequality at $s\in(0,t)$ by $e^{(s-t)/a_2}$ and integrate them over $(0,t)$ to get

$$\begin{aligned}\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \leq e^{-t/a_2}\|A_\varepsilon^{1/2}u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 &+ cc_0\varepsilon^{-1}\int_0^t e^{(s-t)/a_2}\|A_\varepsilon^{1/2}u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \\ &+ ca_2(1-e^{-t/a_2})\|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2. \qquad (5.8.24)\end{aligned}$$

When $t \leq T_* := \min\{1, T\}$, we apply (5.1.9) and (5.8.16) to the right-hand side of (5.8.24) and use $c_0 \leq 1$ by (5.8.6) to deduce that

$$\|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \leq cc_0(1+c_0)\varepsilon^{-1} \leq cc_0\varepsilon^{-1} \quad \text{for all} \quad t \in (0, T_*]. \tag{5.8.25}$$

Now we suppose that $T \geq 1$ and estimate $\|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}$ for $t \in [1, T]$. Since

$$\frac{d}{dt}\|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq \xi \|A_\varepsilon^{1/2} u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \zeta \quad \text{on} \quad (0, T]$$

by (5.8.21), we can use Lemma 5.8.2 with $z(t) = \|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2$ to obtain

$$\|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \leq \left(\int_{t-1}^t \|A_\varepsilon^{1/2} u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds + \int_{t-1}^t \zeta(s)\,ds\right) \exp\left(\int_{t-1}^t \xi(s)\,ds\right) \tag{5.8.26}$$

for all $t \in [1, T]$. Moreover, the functions ξ and ζ given by (5.8.22) satisfy

$$\begin{aligned}
&\int_{t-1}^t \xi(s)\,ds \leq cc_0 \int_{t-1}^t \|A_\varepsilon^{1/2} u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \leq c, \\
&\int_{t-1}^t \zeta(s)\,ds \leq c\left(c_0\varepsilon^{-1}\int_{t-1}^t \|A_\varepsilon^{1/2} u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2\right) \leq cc_0\varepsilon^{-1}
\end{aligned}$$

by (5.1.9), (5.8.16), and $c_0 \leq 1$. Using these inequalities and (5.8.16) to (5.8.26) we have

$$\|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \leq cc_0\varepsilon^{-1} \quad \text{for all} \quad t \in [1, T]. \tag{5.8.27}$$

Now we combine (5.8.25) and (5.8.27) to observe that

$$\|A_\varepsilon^{1/2} u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \leq d_4 c_0 \varepsilon^{-1} \quad \text{for all} \quad t \in (0, T]$$

with some constant $d_4 > 0$ independent of ε, c_0, and T. Hence if we define c_0 by (5.8.6), then by setting $t = T$ in the above inequality we get

$$\|A_\varepsilon^{1/2} u^\varepsilon(T)\|_{L^2(\Omega_\varepsilon)}^2 \leq \frac{d_3^2\varepsilon^{-1}}{4}, \quad \text{i.e.} \quad \varepsilon^{1/2}\|A_\varepsilon^{1/2} u^\varepsilon(T)\|_{L^2(\Omega_\varepsilon)} \leq \frac{d_3}{2} < d_3,$$

which contradicts with (5.8.19). Therefore, the inequality (5.8.17) is valid and we conclude by Theorem 5.8.1 that $T_{\max} = \infty$, i.e. the strong solution u^ε to (5.1.1)–(5.1.3) exists on the whole time interval $[0, \infty)$. □

Using the inequalities given in the proof of Theorem 5.1.1, we can also show the uniform estimates (5.1.11) and (5.1.12) for a strong solution.

Proof of Theorem 5.1.2. Let $\varepsilon_0, c_0 \in (0, 1)$ be the constants given in Theorem 5.1.1. Since α and β are positive, we can take $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$c_1\varepsilon^\alpha \leq c_0, \quad c_2\varepsilon^\beta \leq c_0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_1).$$

Hence, for $\varepsilon \in (0, \varepsilon_1)$ if the initial velocity u_0^ε and the external force f^ε satisfy (5.1.10), then they also satisfy (5.1.9) and Theorem 5.1.1 implies that there exists a global strong solution u^ε to (5.1.1)–(5.1.3).

Let us derive the estimates (5.1.11) and (5.1.12) for the strong solution u^ε. Hereafter we denote by c a general positive constant independent of ε. First note that

$$\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \leq c(\varepsilon^{1+\alpha}+\varepsilon^\beta) \tag{5.8.28}$$

by (5.1.10) and (5.8.7). We apply this inequality and (5.1.10) to (5.8.14) to get

$$\|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \leq c(\varepsilon^{1+\alpha}+\varepsilon^\beta) \quad \text{for all} \quad t \geq 0. \tag{5.8.29}$$

Also, integrating (5.8.12) over $[0,t]$ and using (5.1.10) and (5.8.28) we have

$$\int_0^t \|A_\varepsilon^{1/2}u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \leq c(\varepsilon^{1+\alpha}+\varepsilon^\beta)(1+t) \quad \text{for all} \quad t \geq 0. \tag{5.8.30}$$

Combining (5.8.29) and (5.8.30) with (5.5.18) we obtain (5.1.11).

Next let us prove (5.1.12). We use (5.1.10) and (5.8.28) to (5.8.15) to deduce that

$$\int_t^{t+1} \|A_\varepsilon^{1/2}u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \leq c(\varepsilon^{1+\alpha}+\varepsilon^\beta) \quad \text{for all} \quad t \geq 0. \tag{5.8.31}$$

(Note that $t_* = \min\{t+1, T_{\max}\} = t+1$ in (5.8.15) since $T_{\max} = \infty$.) Since (5.8.17) and (5.8.29) are valid for all $t \geq 0$, as in the proof of Theorem 5.1.1 we can show that (5.8.24) holds for all $t \geq 0$. When $t \in [0,1]$, we apply (5.1.10) and (5.8.30) to the right-hand side of (5.8.24) to obtain

$$\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \leq c(\varepsilon^{-1+\alpha}+\varepsilon^{-1+\beta}) \quad \text{for all} \quad t \in [0,1]. \tag{5.8.32}$$

Let $t \geq 1$. In (5.8.26) the functions ξ and ζ given by (5.8.22) satisfy

$$\begin{aligned}
\int_{t-1}^t \xi(s)\,ds &\leq c\int_{t-1}^t \|A_\varepsilon^{1/2}u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \leq c, \\
\int_{t-1}^t \zeta(s)\,ds &\leq c\left(\varepsilon^{-1}\int_{t-1}^t \|A_\varepsilon^{1/2}u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2\right) \\
&\leq c(\varepsilon^{-1+\alpha}+\varepsilon^{-1+\beta})
\end{aligned}$$

by (5.1.10), (5.8.16), and (5.8.31). Applying these inequalities and (5.8.31) to (5.8.26) we get

$$\|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \leq c(\varepsilon^{-1+\alpha}+\varepsilon^{-1+\beta}) \quad \text{for all} \quad t \geq 1. \tag{5.8.33}$$

By (5.5.18), (5.8.32), and (5.8.33) we obtain the first inequality of (5.1.12). To derive the second inequality we observe that (5.8.23) is valid for all $s \geq 0$ since (5.8.16) and (5.8.17) hold on the whole time interval $[0,\infty)$. Hence for each $t \geq 0$ we integrate (5.8.23) over $[0,t]$ and use (5.1.10) and (5.8.30) to get

$$\int_0^t \|A_\varepsilon u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \leq c(\varepsilon^{-1+\alpha}+\varepsilon^{-1+\beta})(1+t).$$

This inequality combined with (5.5.32) yields the second inequality of (5.1.12). □

For the study of a singular limit problem in Section 5.10 we need estimates for the tensor product and the time derivative of a strong solution with $\beta = 1$. Let us derive them by using the inequalities given in Lemma 5.6.23.

Theorem 5.8.4. *Let c_1 and c_2 be positive constants, $\alpha \in (0,1]$, and $\beta = 1$. Under the same assumptions as in Theorem 5.1.1, let ε_1 be the constant given in Theorem 5.1.2. For $\varepsilon \in (0,\varepsilon_1)$ suppose that $u_0^\varepsilon \in V_\varepsilon$ and $f^\varepsilon \in L^2(0,\infty;L^2(\Omega_\varepsilon)^3)$ satisfy (5.1.10). Then there exists a global strong solution u^ε to (5.1.1)–(5.1.3) such that*

$$\begin{aligned} \|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 &\le c\varepsilon, \quad \int_0^t \|u^\varepsilon(s)\|_{H^1(\Omega_\varepsilon)}^2\,ds \le c\varepsilon(1+t), \\ \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon)}^2 &\le c\varepsilon^{-1+\alpha}, \quad \int_0^t \|u^\varepsilon(s)\|_{H^2(\Omega_\varepsilon)}^2\,ds \le c\varepsilon^{-1+\alpha}(1+t) \end{aligned} \tag{5.8.34}$$

for all $t \ge 0$, and

$$\int_0^t \|[u^\varepsilon \otimes u^\varepsilon](s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \le c\varepsilon(1+t), \tag{5.8.35}$$

$$\int_0^t \|\partial_t u^\varepsilon(s)\|_{L^2(\Omega_\varepsilon)}^2\,ds \le c\varepsilon^{-1+\alpha}(1+t) \tag{5.8.36}$$

for all $t \ge 0$, where $c > 0$ is a constant independent of ε, u^ε, and t.

Proof. A global strong solution u^ε exists by Theorem 5.1.2. Also, we have (5.8.34) by (5.1.11) and (5.1.12) since $\beta = 1$ and $\varepsilon \le \varepsilon^\alpha$ by $\alpha \le 1$ and $\varepsilon < 1$.

Let us derive (5.8.35). Hereafter we suppress the argument s of integrands. Noting that $u^\varepsilon \in L^2_{loc}([0,\infty);D(A_\varepsilon))$ satisfies the conditions in Lemma 5.6.23, we use (5.6.67) to get

$$\begin{aligned} \int_0^t \|u^\varepsilon \otimes u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\,ds \le c\Bigg(&\varepsilon^{-1}\int_0^t \|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \|u\|_{H^1(\Omega_\varepsilon)}^2\,ds \\ &+\varepsilon\int_0^t \|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}^2\,ds + \int_0^t \|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^3\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}\,ds\Bigg) \end{aligned} \tag{5.8.37}$$

for all $t \ge 0$. By (5.8.34) we have

$$\int_0^t \|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}^2\,ds \le c\varepsilon^2(1+t), \tag{5.8.38}$$

$$\int_0^t \|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}^2\,ds \le c\varepsilon^\alpha(1+t). \tag{5.8.39}$$

Also, Hölder's inequality and (5.8.34) imply that

$$\int_0^t \|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^3\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}\,ds \le c\varepsilon^{3/2}t^{1/2}\left(\int_0^t \|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}^2\,ds\right)^{1/2} \le c\varepsilon^{1+\alpha/2}(1+t). \tag{5.8.40}$$

Applying (5.8.38)–(5.8.40) to (5.8.37) and noting that $\varepsilon^\alpha, \varepsilon^{\alpha/2} \le 1$ we obtain (5.8.35).

Next we prove (5.8.36). We take the $L^2(\Omega_\varepsilon)$-inner product of $\partial_t u^\varepsilon$ with

$$\partial_t u^\varepsilon + A_\varepsilon u^\varepsilon + \mathbb{P}_\varepsilon[(u^\varepsilon\cdot\nabla)u^\varepsilon] = \mathbb{P}_\varepsilon f^\varepsilon \quad \text{on} \quad (0,\infty)$$

and then integrate it over $(0,t)$ with $t > 0$ and use Young's inequality to get

$$\begin{aligned} &\int_0^t \|\partial_t u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\,ds + \|A_\varepsilon^{1/2}u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \\ &\qquad \le \|A_\varepsilon^{1/2}u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + c\int_0^t\left(\|(u^\varepsilon\cdot\nabla)u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\right)ds. \end{aligned} \tag{5.8.41}$$

Let us estimate the integral of $(u^\varepsilon \cdot \nabla)u^\varepsilon$. By (5.6.68) we have

$$\int_0^t \|(u^\varepsilon \cdot \nabla)u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\,ds \leq c\int_0^t \left(\varepsilon^{-1}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}^2\right)\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}^2\,ds.$$

To the right-hand side we apply (5.8.39) and

$$\int_0^r \|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}^2\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}^2\,ds \leq c\varepsilon^{-2+2\alpha}(1+t) \tag{5.8.42}$$

by (5.8.34), and then use $\varepsilon^\alpha \leq 1$ by $\varepsilon < 1$ to deduce that

$$\int_0^t \|(u^\varepsilon \cdot \nabla)u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\,ds \leq c\varepsilon^{-1+\alpha}(1+t).$$

Applying this inequality and (5.1.10) to (5.8.41) we obtain (5.8.36). □

5.9 Weighted solenoidal spaces on a surface

In the study of a singular limit problem for the Navier–Stokes equations (5.1.1)–(5.1.3) we deal with a weighted solenoidal space of the form

$$H^1_{g\sigma}(\Gamma, T\Gamma) = \{v \in H^1(\Gamma, T\Gamma) \mid \mathrm{div}_\Gamma(gv) = 0 \text{ on } \Gamma\}.$$

The purpose of this section is to establish several properties of weighted solenoidal spaces on a surface. Throughout this section we assume that Γ is a two-dimensional closed, connected, and oriented surface in $\mathbb{R}^3$ of class C^2. We use the notations for the surface quantities on Γ given in Section 5.2.1.

5.9.1 Nečas inequality on a surface

Let $q \in L^2(\Gamma)$. We consider q and its weak tangential gradient as elements in $H^{-1}(\Gamma)$ and $H^{-1}(\Gamma, T\Gamma)$ given by (5.2.21) and (5.2.26). By these equalities we easily get

$$\|q\|_{H^{-1}(\Gamma)} + \|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)} \leq c\|q\|_{L^2(\Gamma)}.$$

For bounded Lipschitz domains in $\mathbb{R}^m$, $m \in \mathbb{N}$ the inverse inequality is also valid and known as the Nečas inequality (see [48, Chapter 3, Lemma 7.1]). Let us show the Nečas inequality on the surface Γ.

Lemma 5.9.1. *There exists a constant $c > 0$ such that*

$$\|q\|_{L^2(\Gamma)} \leq c\left(\|q\|_{H^{-1}(\Gamma)} + \|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)}\right) \tag{5.9.1}$$

for all $q \in L^2(\Gamma)$.

In the proof of Lemma 5.9.1 we use the Načas inequality on the whole space

$$\|\tilde{q}\|_{L^2(\mathbb{R}^2)} \leq c\left(\|\tilde{q}\|_{H^{-1}(\mathbb{R}^2)} + \|\nabla_s\tilde{q}\|_{H^{-1}(\mathbb{R}^2)}\right) \tag{5.9.2}$$

for $\tilde{q} \in L^2(\mathbb{R}^2)$, where $H^{-1}(\mathbb{R}^2)$ is the dual space of $H^1(\mathbb{R}^2)$ (via the $L^2(\mathbb{R}^2)$-inner product). Also, $\tilde{q} \in H^{-1}(\mathbb{R}^2)$ and $\nabla_s\tilde{q} \in H^{-1}(\mathbb{R}^2)^2$ are given by

$$\langle\tilde{q},\xi\rangle_{\mathbb{R}^2} := (\tilde{q},\xi)_{L^2(\mathbb{R}^2)}, \quad \langle\nabla_s\tilde{q},\varphi\rangle_{\mathbb{R}^2} := -(q, \partial_{s_1}\varphi_1 + \partial_{s_2}\varphi_2)_{L^2(\mathbb{R}^2)}$$

for $\xi \in H^1(\mathbb{R}^2)$ and $\varphi = (\varphi_1,\varphi_2) \in H^1(\mathbb{R}^2)^2$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$ is the duality product between $H^{-1}(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$. The inequality (5.9.2) follows from the characterization of the L^2-Sobolev spaces on $\mathbb{R}^2$ by the Fourier transform. See the proof of [48, Chapter 3, Lemma 7.1] for details.

Proof. First we note that it is sufficient to show (5.9.1) when q is compactly supported in a relatively open subset of Γ on which we can take a local coordinate system. To see this, let $q \in L^2(\Gamma)$ and $\eta \in C^2(\Gamma)$. For $\xi \in H^1(\Gamma)$ we have

$$|\langle \eta q, \xi\rangle_\Gamma| = |\langle q, \eta\xi\rangle_\Gamma| \leq \|q\|_{H^{-1}(\Gamma)}\|\eta\xi\|_{H^1(\Gamma)} \leq c\|\eta\|_{W^{1,\infty}(\Gamma)}\|q\|_{H^{-1}(\Gamma)}\|\xi\|_{H^1(\Gamma)},$$

where $c > 0$ is a constant independent of q, η, and ξ. Also,

$$\begin{aligned}[\nabla_\Gamma(\eta q), v]_{T\Gamma} &= -(\eta q, \mathrm{div}_\Gamma v)_{L^2(\Gamma)} = -\big(q, \mathrm{div}_\Gamma(\eta v)\big)_{L^2(\Gamma)} + (q, \nabla_\Gamma\eta\cdot v)_{L^2(\Gamma)} \\ &= [\nabla_\Gamma q, \eta v]_{T\Gamma} + \langle q, \nabla_\Gamma \eta \cdot v\rangle_\Gamma\end{aligned}$$

for all $v \in H^1(\Gamma, T\Gamma)$ by (5.2.26) (note that $\eta v \in H^1(\Gamma, T\Gamma)$) and thus

$$\begin{aligned}|[\nabla_\Gamma(\eta q), v]_{T\Gamma}| &\leq \|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)}\|\eta v\|_{H^1(\Gamma)} + \|q\|_{H^{-1}(\Gamma)}\|\nabla_\Gamma\eta\cdot v\|_{H^1(\Gamma)} \\ &\leq c\|\eta\|_{W^{2,\infty}(\Gamma)}\left(\|q\|_{H^{-1}(\Gamma)} + \|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)}\right)\|v\|_{H^1(\Gamma)}.\end{aligned}$$

From the above inequalities it follows that

$$\begin{aligned}\|\eta q\|_{H^{-1}(\Gamma)} &\leq c\|\eta\|_{W^{1,\infty}(\Gamma)}\|q\|_{H^{-1}(\Gamma)}, \\ \|\nabla_\Gamma(\eta q)\|_{H^{-1}(\Gamma,T\Gamma)} &\leq c\|\eta\|_{W^{2,\infty}(\Gamma)}\left(\|q\|_{H^{-1}(\Gamma)} + \|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)}\right).\end{aligned}$$

By these inequalities we can get (5.9.1) for q if we localize it by a partition of unity of Γ consisting of functions in $C^2(\Gamma)$ (we can take such functions since Γ is of class C^2) and prove (5.9.1) for each localized function.

From now on, we assume that $q \in L^2(\Gamma)$ is compactly supported in a relatively open subset $\mu(U)$ of Γ, where U is an open subset of $\mathbb{R}^2$ and $\mu\colon U \to \mathbb{R}^2$ is a local parametrization of Γ. (Note that μ is of class C^2 on U since Γ is of class C^2.) Let $\tilde{q} := q\circ\mu$ on U and $\mathcal{K}$ be the support of $\tilde{q}$, which is a compact subset of U. The Riemannian metric $\theta = (\theta_{ij})_{i,j}$ of Γ is locally defined by

$$\theta_{ij}(s) := \partial_{s_i}\mu(s)\cdot\partial_{s_j}\mu(s), \quad s \in U,\ i,j = 1,2.$$

We write $\theta^{-1} = (\theta^{ij})_{i,j}$ for the inverse matrix of θ. Since μ is of class C^2, there exists a constant $c > 0$ such that

$$|\partial_s^\alpha \mu(s)| \leq c, \quad s \in \mathcal{K},\ |\alpha| = 1,2, \tag{5.9.3}$$

where $\partial_s^\alpha = \partial_{s_1}^{\alpha_1}\partial_{s_2}^{\alpha_2}$ for $\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}^2$ with $\alpha_i \geq 0$, $i = 1,2$. By this inequality and the formula $\partial_{s_i}\theta^{-1} = -\theta^{-1}(\partial_{s_i}\theta)\theta^{-1}$ we also have

$$|\theta(s)| \leq c, \quad |\theta^{-1}(s)| \leq c, \quad |\partial_{s_i}\theta(s)| \leq c, \quad |\partial_{s_i}\theta^{-1}(s)| \leq c, \quad s \in \mathcal{K},\ i = 1,2. \tag{5.9.4}$$

Moreover, since the determinant of θ is continuous and does not vanish on U, there exists a constant $c > 0$ such that

$$c^{-1} \leq \det\theta(s) \leq c, \quad s \in \mathcal{K}. \tag{5.9.5}$$

We extend $\tilde{q}$ to $\mathbb{R}^2$ by setting zero outside of U. Then by (5.9.5) we get

$$\|q\|_{L^2(\Gamma)}^2 = \int_{\mathcal{K}} |\tilde{q}|^2\sqrt{\det\theta}\,ds \leq c\|\tilde{q}\|_{L^2(\mathcal{K})}^2 = c\|\tilde{q}\|_{L^2(\mathbb{R}^2)}^2$$

and we apply the inequality (5.9.2) to the last term to obtain

$$\|q\|_{L^2(\Gamma)} \leq c\left(\|\tilde{q}\|_{H^{-1}(\mathbb{R}^2)} + \|\nabla_s\tilde{q}\|_{H^{-1}(\mathbb{R}^2)}\right). \tag{5.9.6}$$

Let us estimate the right-hand side of (5.9.6) by that of (5.9.1). To this end, we first consider the duality product $\langle\tilde{q},\xi\rangle_{\mathbb{R}^2} = (\tilde{q},\xi)_{L^2(\mathbb{R}^2)}$ for an arbitrary $\xi \in H^1(\mathbb{R}^2)$. Since $\tilde{q}$ is supported in $\mathcal{K}$, we may assume that ξ is also supported in $\mathcal{K}$. We define a function η on $\mu(\mathcal{K}) \subset \Gamma$ by

$$\eta(\mu(s)) := \frac{\xi(s)}{\sqrt{\det\theta(s)}}, \quad s \in \mathcal{K} \tag{5.9.7}$$

and extend it to Γ by setting zero outside of $\mu(\mathcal{K})$. Then we have

$$\langle\tilde{q},\xi\rangle_{\mathbb{R}^2} = (\tilde{q},\xi)_{L^2(\mathbb{R}^2)} = \int_{\mathcal{K}} \tilde{q}\left(\frac{\xi}{\sqrt{\det\theta}}\right)\sqrt{\det\theta}\,ds = \int_{\mu(\mathcal{K})} q\eta\,d\mathcal{H}^2. \tag{5.9.8}$$

Let us show $\eta \in H^1(\Gamma)$. Since η is supported in $\mu(\mathcal{K})$, we see by (5.9.5) that

$$\|\eta\|_{L^2(\Gamma)}^2 = \int_{\mathcal{K}} |\eta\circ\mu|^2\sqrt{\det\theta}\,ds \leq c\|\xi\|_{L^2(\mathcal{K})}^2 = c\|\xi\|_{L^2(\mathbb{R}^2)}^2. \tag{5.9.9}$$

To estimate the $L^2(\Gamma)$-norm of the tangential gradient of η, we differentiate both sides of (5.9.7) with respect to s_i, $i = 1, 2$ and use (5.9.4) and (5.9.5) to get

$$|\partial_{s_i}(\eta\circ\mu)(s)| \leq c(|\xi(s)| + |\partial_{s_i}\xi(s)|), \quad s \in \mathcal{K}.$$

Since the norm of $(\nabla_\Gamma\eta)\circ\mu = \sum_{i,j=1}^2 \theta^{ij}\partial_{s_i}(\eta\circ\mu)\partial_{s_j}\mu$ is given by

$$|(\nabla_\Gamma\eta)\circ\mu|^2 = \sum_{i,j=1}^2 \theta^{ij}\partial_{s_i}(\eta\circ\mu)\partial_{s_j}(\eta\circ\mu) \quad \text{in} \quad \mathcal{K},$$

we apply the above inequality and (5.9.4) to the right-hand side to get

$$|(\nabla_\Gamma\eta)\circ\mu|^2 \leq c(|\xi|^2 + |\nabla_s\xi|^2) \quad \text{in} \quad \mathcal{K},$$

where ∇_s is the gradient operator in $s \in \mathbb{R}^2$. Noting that η and ξ are supported in $\mu(\mathcal{K})$ and $\mathcal{K}$, respectively, we use this inequality and (5.9.5) to obtain

$$\|\nabla_\Gamma\eta\|_{L^2(\Gamma)}^2 = \int_{\mathcal{K}} |(\nabla_\Gamma\eta)\circ\mu|^2\sqrt{\det\theta}\,ds \leq c\|\xi\|_{H^1(\mathcal{K})}^2 = c\|\xi\|_{H^1(\mathbb{R}^2)}^2.$$

By this inequality and (5.9.9) we have $\eta \in H^1(\Gamma)$ and $\|\eta\|_{H^1(\Gamma)} \leq c\|\xi\|_{H^1(\mathbb{R}^2)}$. Hence

$$\langle\tilde{q},\xi\rangle_{\mathbb{R}^2} = \int_{\mu(\mathcal{K})} q\eta\,d\mathcal{H}^2 = (q,\eta)_{L^2(\Gamma)} = \langle q,\eta\rangle_\Gamma$$

by (5.9.8) and the above inequality implies that

$$\left|\langle\tilde{q},\xi\rangle_{\mathbb{R}^2}\right| = \left|\langle q,\eta\rangle_\Gamma\right| \leq \|q\|_{H^{-1}(\Gamma)}\|\eta\|_{H^1(\Gamma)} \leq c\|q\|_{H^{-1}(\Gamma)}\|\xi\|_{H^1(\mathbb{R}^2)}$$

for all $\xi \in H^1(\mathbb{R}^2)$. Therefore,

$$\|\tilde{q}\|_{H^{-1}(\mathbb{R}^2)} \leq c\|q\|_{H^{-1}(\Gamma)}. \tag{5.9.10}$$

Next let $\varphi = (\varphi_1, \varphi_2) \in H^1(\mathbb{R}^2)^2$ and we consider the duality product

$$\langle \nabla_s \tilde{q}, \varphi \rangle_{\mathbb{R}^2} = -(\tilde{q}, \partial_{s_1}\varphi_1 + \partial_{s_2}\varphi_2)_{L^2(\mathbb{R}^2)}.$$

We may assume again that φ is supported in $\mathcal{K}$ since $\tilde{q}$ is so. To express this product in terms of an integral over Γ, we recall that the surface divergence of a tangential vector field $X(\mu(s)) = \sum_{i=1}^2 X_i(s)\partial_{s_i}\mu(s)$, $s \in \mathcal{K}$ is given by

$$\mathrm{div}_\Gamma X(\mu(s)) = \frac{1}{\sqrt{\det\theta(s)}} \sum_{i=1}^2 \frac{\partial}{\partial s_i}\Big(X_i(s)\sqrt{\det\theta(s)}\Big), \quad s \in \mathcal{K}.$$

Based on this formula, we define a tangential vector field Φ on $\mu(\mathcal{K})$ by

$$\Phi(\mu(s)) := \frac{1}{\sqrt{\det\theta(s)}} \sum_{i=1}^2 \varphi_i(s)\partial_{s_i}\mu(s), \quad s \in \mathcal{K} \tag{5.9.11}$$

and extend it to Γ by setting zero outside of $\mu(\mathcal{K})$. Then we have

$$\mathrm{div}_\Gamma \Phi(\mu(s)) = \frac{\partial_{s_1}\varphi_1(s) + \partial_{s_2}\varphi_2(s)}{\sqrt{\det\theta(s)}}, \quad s \in \mathcal{K}$$

and thus we get (note that $\tilde{q} = q \circ \mu$ and φ are supported in $\mathcal{K}$)

$$\begin{aligned}\langle \nabla_s \tilde{q}, \varphi \rangle_{\mathbb{R}^2} &= -(\tilde{q}, \partial_{s_1}\varphi_1 + \partial_{s_2}\varphi_2)_{L^2(\mathbb{R}^2)} \\ &= -\int_{\mathcal{K}} \tilde{q}\left(\frac{\partial_{s_1}\varphi_1 + \partial_{s_2}\varphi_2}{\sqrt{\det\theta}}\right)\sqrt{\det\theta}\,ds = -\int_{\mu(\mathcal{K})} q\,\mathrm{div}_\Gamma\Phi\,d\mathcal{H}^2.\end{aligned} \tag{5.9.12}$$

Let us estimate the $H^1(\Gamma)$-norm of Φ. By (5.9.4), (5.9.5), and (5.9.11) we have

$$|\Phi \circ \mu|^2 = \frac{1}{\det\theta} \sum_{i,j=1}^2 \theta_{ij}\varphi_i\varphi_j \leq c|\varphi|^2 \quad \text{in} \quad \mathcal{K}.$$

Since Φ is supported in $\mu(\mathcal{K})$, the above inequality and (5.9.5) imply

$$\|\Phi\|_{L^2(\Gamma)}^2 = \int_{\mathcal{K}} |\Phi \circ \mu|^2 \sqrt{\det\theta}\,ds \leq c\|\varphi\|_{L^2(\mathcal{K})}^2 = c\|\varphi\|_{L^2(\mathbb{R}^2)}^2, \tag{5.9.13}$$

where the last equality follows from the fact that φ is supported in $\mathcal{K}$. To estimate the tangential derivatives of Φ, let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and $\Phi_k := \Phi \cdot e_k$ for $k = 1, 2, 3$. Then Φ_k is supported in $\mu(\mathcal{K})$ and

$$\Phi_k(\mu(s)) = \frac{1}{\sqrt{\det\theta(s)}} \sum_{j=1}^2 \varphi_j(s)\partial_{s_j}\mu(s) \cdot e_k, \quad s \in \mathcal{K}.$$

We differentiate both sides with respect to s_i and use (5.9.3), (5.9.4), and (5.9.5) to get

$$|\partial_{s_i}(\Phi_k \circ \mu)(s)| \leq c\left(|\varphi(s)| + |\nabla_s\varphi(s)|\right), \quad s \in \mathcal{K},\ i = 1, 2.$$

From this inequality and (5.9.4) we deduce that

$$|(\nabla_\Gamma \Phi_k) \circ \mu|^2 = \sum_{i,j=1}^2 \theta^{ij}\partial_{s_i}(\Phi_k \circ \mu)\partial_{s_j}(\Phi_k \circ \mu) \leq c(|\varphi|^2 + |\nabla_s\varphi|^2)$$

in $\mathcal{K}$ for $k = 1, 2, 3$, and thus $|(\nabla_\Gamma \Phi) \circ \mu|^2 \leq c(|\varphi|^2 + |\nabla_s \varphi|^2)$ in $\mathcal{K}$. Noting that Φ and φ are supported in $\mu(\mathcal{K})$ and $\mathcal{K}$, respectively, we use this inequality and (5.9.5) to observe that

$$\|\nabla_\Gamma \Phi\|_{L^2(\Gamma)}^2 = \int_{\mathcal{K}} |(\nabla_\Gamma \Phi) \circ \mu|^2 \sqrt{\det \theta}\, ds \leq c\|\varphi\|_{H^1(\mathcal{K})}^2 = c\|\varphi\|_{H^1(\mathbb{R}^2)}^2.$$

From this inequality and (5.9.13) it follows that $\Phi \in H^1(\Gamma, T\Gamma)$ and

$$\|\Phi\|_{H^1(\Gamma)} \leq c\|\varphi\|_{H^1(\mathbb{R}^2)}. \tag{5.9.14}$$

Now we return to (5.9.12) and use (5.2.26) to get

$$\langle \nabla_s \tilde{q}, \varphi \rangle_{\mathbb{R}^2} = -\int_{\mu(\mathcal{K})} q \operatorname{div}_\Gamma \Phi \, d\mathcal{H}^2 = -(q, \operatorname{div}_\Gamma \Phi)_{L^2(\Gamma)} = [\nabla_\Gamma q, \Phi]_{T\Gamma}.$$

Hence by (5.9.14) we obtain

$$\left|\langle \nabla_s \tilde{q}, \varphi \rangle_{\mathbb{R}^2}\right| = \left|[\nabla_\Gamma q, \Phi]_{T\Gamma}\right| \leq \|\nabla_\Gamma q\|_{H^{-1}(\Gamma, T\Gamma)} \|\Phi\|_{H^1(\Gamma)} \leq c\|\nabla_\Gamma q\|_{H^{-1}(\Gamma, T\Gamma)} \|\varphi\|_{H^1(\mathbb{R}^2)}$$

for all $\varphi \in H^1(\mathbb{R}^2)^2$, which implies that

$$\|\nabla_s \tilde{q}\|_{H^{-1}(\mathbb{R}^2)} \leq c\|\nabla_\Gamma q\|_{H^{-1}(\Gamma, T\Gamma)}.$$

Finally, we apply this inequality and (5.9.10) to (5.9.6) to conclude that the inequality (5.9.1) is valid. $\square$

Next we prove Poincaré's inequality for $q \in L^2(\Gamma)$ based on the Nečas inequality (5.9.1). We first show that the tangential gradient of q vanishes in $H^{-1}(\Gamma, T\Gamma)$ if and only if q is constant on Γ.

Lemma 5.9.2. *Let $q \in L^2(\Gamma)$. Then*

$$\nabla_\Gamma q = 0 \quad \text{in} \quad H^{-1}(\Gamma, T\Gamma)$$

if and only if q is constant on Γ.

Proof. Suppose first that q is constant on Γ. Then for all $v \in H^1(\Gamma, T\Gamma)$ we have

$$[\nabla_\Gamma q, v]_{T\Gamma} = -q \int_\Gamma \operatorname{div}_\Gamma v \, d\mathcal{H}^2 = 0$$

by (5.2.26) and the Stokes theorem. Hence $\nabla_\Gamma q = 0$ in $H^{-1}(\Gamma, T\Gamma)$.

Conversely, assume that $\nabla_\Gamma q = 0$ in $H^{-1}(\Gamma, T\Gamma)$. We first prove $q \in H^1(\Gamma)$. For $\eta \in C^1(\Gamma)$ and $i = 1, 2, 3$ we set $v := \eta P e_i$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Then $v \in H^1(\Gamma, T\Gamma)$ since P is of class C^1. Moreover,

$$\operatorname{div}_\Gamma v = \nabla_\Gamma \eta \cdot P e_i + \eta(\operatorname{div}_\Gamma P \cdot e_i) = \underline{D}_i \eta + \eta H n_i \quad \text{on} \quad \Gamma$$

by $P^T = P$, (5.2.3), and (5.2.7). From this equality we deduce that

$$0 = [\nabla_\Gamma q, v]_{T\Gamma} = -(q, \operatorname{div}_\Gamma v)_{L^2(\Gamma)} = -(q, \underline{D}_i \eta + \eta H n_i)_{L^2(\Gamma)}$$

for all $\eta \in C^1(\Gamma)$. Hence by the definition of the weak tangential derivative in $L^2(\Gamma)$ (see (5.2.19)) we get $\underline{D}_i q = 0$ in $L^2(\Gamma)$ for $i = 1,2,3$, which shows that $q \in H^1(\Gamma)$ and $\nabla_\Gamma q = 0$ in $L^2(\Gamma)^3$. Now we set

$$\hat{q} := q - \frac{1}{|\Gamma|}\int_\Gamma q\,d\mathcal{H}^2 \quad \text{on} \quad \Gamma,$$

where $|\Gamma|$ is the area of Γ. Then we have $\hat{q} \in H^1(\Gamma)$ and $\int_\Gamma \hat{q}\,d\mathcal{H}^2 = 0$. Hence we can apply Poincaré's inequality (5.2.20) to $\hat{q}$ to get

$$\|\hat{q}\|_{L^2(\Gamma)} \leq c\|\nabla_\Gamma \hat{q}\|_{L^2(\Gamma)} = c\|\nabla_\Gamma q\|_{L^2(\Gamma)} = 0,$$

i.e. $\hat{q} = 0$ on Γ, which implies that $q = |\Gamma|^{-1}\int_\Gamma q\,d\mathcal{H}^2$ is constant on Γ. □

Next we estimate $q \in L^2(\Gamma)$ in $H^{-1}(\Gamma)$ by its tangential gradient.

Lemma 5.9.3. *There exists a constant $c > 0$ such that*

$$\|q\|_{H^{-1}(\Gamma)} \leq c\left(\left|\int_\Gamma q\,d\mathcal{H}^2\right| + \|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)}\right) \tag{5.9.15}$$

for all $q \in L^2(\Gamma)$.

Proof. We prove (5.9.15) by contradiction. Assume to the contrary that for each $k \in \mathbb{N}$ there exists $q_k \in L^2(\Gamma)$ such that

$$\|q_k\|_{H^{-1}(\Gamma)} > k\left(\left|\int_\Gamma q_k\,d\mathcal{H}^2\right| + \|\nabla_\Gamma q_k\|_{H^{-1}(\Gamma,T\Gamma)}\right).$$

Replacing q_k with $q_k/\|q_k\|_{H^{-1}(\Gamma)}$ we may assume that

$$\|q_k\|_{H^{-1}(\Gamma)} = 1, \quad \left|\int_\Gamma q_k\,d\mathcal{H}^2\right| + \|\nabla_\Gamma q_k\|_{H^{-1}(\Gamma,T\Gamma)} < k^{-1}. \tag{5.9.16}$$

From the second inequality it follows that

$$\lim_{k\to\infty}\int_\Gamma q_k\,d\mathcal{H}^2 = \lim_{k\to\infty}(q_k,1)_{L^2(\Gamma)} = 0, \quad \lim_{k\to\infty}\|\nabla_\Gamma q_k\|_{H^{-1}(\Gamma,T\Gamma)} = 0. \tag{5.9.17}$$

By (5.9.1) and (5.9.16) the sequence $\{q_k\}_{k=1}^\infty$ is bounded in $L^2(\Gamma)$. Hence there exists a subsequence of $\{q_k\}_{k=1}^\infty$, which is referred to as $\{q_k\}_{k=1}^\infty$ again, that converges to some q weakly in $L^2(\Gamma)$. Since the embedding $L^2(\Gamma) \hookrightarrow H^{-1}(\Gamma)$ is compact, by taking a subsequence we may assume that $\{q_k\}_{k=1}^\infty$ converges to q strongly in $H^{-1}(\Gamma)$. Hence by the first equality of (5.9.16) we have

$$\|q\|_{H^{-1}(\Gamma)} = \lim_{k\to\infty}\|q_k\|_{H^{-1}(\Gamma)} = 1. \tag{5.9.18}$$

By (5.2.26) and the weak convergence of $\{q_k\}_{k=1}^\infty$ to q in $L^2(\Gamma)$ we see that $\{\nabla_\Gamma q_k\}_{k=1}^\infty$ converges to $\nabla_\Gamma q$ weakly in $H^{-1}(\Gamma,T\Gamma)$. Hence by (5.9.17) we get

$$\int_\Gamma q\,d\mathcal{H}^2 = (q,1)_{L^2(\Gamma)} = 0, \quad \|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)} = 0.$$

By these equalities and Lemma 5.9.2 we find that $q = 0$ on Γ and thus $\|q\|_{H^{-1}(\Gamma)} = 0$, which contradicts with (5.9.18). Hence (5.9.15) is valid. □

Combining (5.9.1) and (5.9.15) we obtain Poincaré's inequality for $q \in L^2(\Gamma)$.

Lemma 5.9.4. *There exists a constant $c > 0$ such that*

$$\|q\|_{L^2(\Gamma)} \leq c\left(\left|\int_\Gamma q\,d\mathcal{H}^2\right| + \|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)}\right) \tag{5.9.19}$$

for all $q \in L^2(\Gamma)$.

5.9.2 Characterization of the annihilator of a weighted solenoidal space

Let $g \in C^1(\Gamma)$ satisfy $g \geq c$ on Γ with some constant $c > 0$. We define a weighted solenoidal space of tangential vector fields

$$H^1_{g\sigma}(\Gamma, T\Gamma) := \{v \in H^1(\Gamma, T\Gamma) \mid \mathrm{div}_\Gamma(gv) = 0 \text{ on } \Gamma\},$$

If $q \in L^2(\Gamma)$, then (5.2.26) and (5.2.27) imply that

$$[g\nabla_\Gamma q, v]_{T\Gamma} = -\big(q, \mathrm{div}_\Gamma(gv)\big)_{L^2(\Gamma)} = 0 \quad \text{for all} \quad v \in H^1_{g\sigma}(\Gamma, T\Gamma).$$

Let us prove the converse of this statement for an element of $H^{-1}(\Gamma, T\Gamma)$, which is a weighted version of de Rham's theorem.

Theorem 5.9.5. *Suppose that $f \in H^{-1}(\Gamma, T\Gamma)$ satisfies*

$$[f, v]_{T\Gamma} = 0 \quad \textit{for all} \quad v \in H^1_{g\sigma}(\Gamma, T\Gamma).$$

Then there exists a unique $q \in L^2(\Gamma)$ such that

$$f = g\nabla_\Gamma q \quad \textit{in} \quad H^{-1}(\Gamma, T\Gamma), \qquad \int_\Gamma q\,d\mathcal{H}^2 = 0.$$

Moreover, there exists a constant $c > 0$ independent of f such that

$$\|q\|_{L^2(\Gamma)} \leq c\|f\|_{H^{-1}(\Gamma,T\Gamma)}. \tag{5.9.20}$$

We give auxiliary lemmas for Theorem 5.9.5.

Lemma 5.9.6. *There exists $c > 0$ such that*

$$c^{-1}\|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)} \leq \|g\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)} \leq c\|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)} \tag{5.9.21}$$

for all $q \in L^2(\Gamma)$.

Proof. Since $g \in C^1(\Gamma)$ is bounded from below by a positive constant, we have

$$\begin{aligned} |[\nabla_\Gamma q, v]_{T\Gamma}| = \left|[g\nabla_\Gamma q, g^{-1}v]_{T\Gamma}\right| &\leq \|g\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)}\|g^{-1}v\|_{H^1(\Gamma)} \\ &\leq c\|g\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)}\|v\|_{H^1(\Gamma)} \end{aligned}$$

for all $v \in H^1(\Gamma, T\Gamma)$. Hence the left-hand inequality of (5.9.21) holds. Similarly, we can show the right-hand inequality of (5.9.21). □

Lemma 5.9.7. *The subspace*

$$\mathcal{X} := \{g\nabla_\Gamma q \in H^{-1}(\Gamma, T\Gamma) \mid q \in L^2(\Gamma)\} \tag{5.9.22}$$

is closed in $H^{-1}(\Gamma, T\Gamma)$.

Proof. Let $\{q_k\}_{k=1}^\infty$ be a sequence in $L^2(\Gamma)$ such that $\{g\nabla_\Gamma q_k\}_{k=1}^\infty$ converges to some f strongly in $H^{-1}(\Gamma, T\Gamma)$. For each $k \in \mathbb{N}$ we subtract the average of q_k over Γ from q_k to assume $\int_\Gamma q_k \, d\mathcal{H}^2 = 0$ without changing $g\nabla_\Gamma q_k$ (see Lemma 5.9.2). Then by (5.9.19) and (5.9.21) we see that

$$\|q_k - q_l\|_{L^2(\Gamma)} \leq c\|g\nabla_\Gamma q_k - g\nabla_\Gamma q_l\|_{H^{-1}(\Gamma,T\Gamma)} \to 0 \quad \text{as} \quad k,l \to \infty.$$

Hence $\{q_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(\Gamma)$ and converges to some q strongly in $L^2(\Gamma)$. Then by (5.2.26) and (5.2.27) we easily see that

$$\|g\nabla_\Gamma q - g\nabla_\Gamma q_k\|_{H^{-1}(\Gamma,T\Gamma)} \leq c\|q - q_k\|_{L^2(\Gamma)} \to 0 \quad \text{as} \quad k \to \infty.$$

Since $\{g\nabla_\Gamma q_k\}_{k=1}^\infty$ converges to f strongly in $H^{-1}(\Gamma, T\Gamma)$, we conclude by the above convergence that $f = g\nabla_\Gamma q \in \mathcal{X}$. Therefore, $\mathcal{X}$ is closed in $H^{-1}(\Gamma, T\Gamma)$. $\square$

To prove Theorem 5.9.5 we use basic results of functional analysis. Let $\mathcal{B}$ be a Banach space. For a subset $\mathcal{X}$ of $\mathcal{B}$ we define the annihilator of $\mathcal{X}$ by

$$\mathcal{X}^\perp := \{f \in \mathcal{B}' \mid {}_{\mathcal{B}'}\langle f, v\rangle_{\mathcal{B}} = 0 \text{ for all } v \in \mathcal{X}\},$$

where $\mathcal{B}'$ is the dual of $\mathcal{B}$ and ${}_{\mathcal{B}'}\langle \cdot, \cdot\rangle_{\mathcal{B}}$ is the duality product between $\mathcal{B}'$ and $\mathcal{B}$.

Lemma 5.9.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of $\mathcal{B}$. If $\mathcal{X} \subset \mathcal{Y}$ in $\mathcal{B}$, then $\mathcal{Y}^\perp \subset \mathcal{X}^\perp$ in $\mathcal{B}'$.*

Lemma 5.9.9. *If $\mathcal{B}$ is reflexive and $\mathcal{X}$ is a closed subspace of $\mathcal{B}$, then $(\mathcal{X}^\perp)^\perp = \mathcal{X}$.*

Lemma 5.9.8 is an immediate consequence of the definition of the annihilator. Also, Lemma 5.9.9 follows from the Hahn–Banach theorem, see e.g. [56, Theorem 4.7].

Proof of Theorem 5.9.5. Since $H^1(\Gamma, T\Gamma)$ is a Hilbert space, its dual $H^{-1}(\Gamma, T\Gamma)$ is also a Hilbert space and thus reflexive. Let $\mathcal{X}$ be the subspace of $H^{-1}(\Gamma, T\Gamma)$ given by (5.9.22) and $v \in \mathcal{X}^\perp \subset H^1(\Gamma, T\Gamma)$. Then for all $q \in L^2(\Gamma)$ we have

$$0 = [g\nabla_\Gamma q, v]_{T\Gamma} = -\big(q, \operatorname{div}_\Gamma(gv)\big)_{L^2(\Gamma)}$$

by $g\nabla_\Gamma q \in \mathcal{X}$, (5.2.26), and (5.2.27) and thus $\operatorname{div}_\Gamma(gv) = 0$ on Γ, i.e. $v \in H^1_{g\sigma}(\Gamma, T\Gamma)$. Hence $\mathcal{X}^\perp \subset H^1_{g\sigma}(\Gamma, T\Gamma)$ in $H^1(\Gamma, T\Gamma)$ and by Lemma 5.9.8 we have

$$H^1_{g\sigma}(\Gamma, T\Gamma)^\perp = \{f \in H^{-1}(\Gamma, T\Gamma) \mid [f, v]_{T\Gamma} = 0 \text{ for all } v \in H^1_{g\sigma}(\Gamma, T\Gamma)\} \subset (\mathcal{X}^\perp)^\perp.$$

Since $\mathcal{X}$ is closed in $H^{-1}(\Gamma, T\Gamma)$ by Lemma 5.9.7, we have $(\mathcal{X}^\perp)^\perp = \mathcal{X}$ by Lemma 5.9.9. Hence $H^1_{g\sigma}(\Gamma, T\Gamma)^\perp \subset \mathcal{X}$, i.e. for $f \in H^1_{g\sigma}(\Gamma, T\Gamma)^\perp$ there exists $q \in L^2(\Gamma)$ such that $f = g\nabla_\Gamma q$ in $H^{-1}(\Gamma, T\Gamma)$. Moreover, subtracting the average of q over Γ from q we may assume $\int_\Gamma q \, d\mathcal{H}^2 = 0$ without changing $g\nabla_\Gamma q$ (see Lemma 5.9.2). Therefore, the existence part of the theorem is valid. To prove the uniqueness, suppose that $q_1, q_2 \in L^2(\Gamma)$ satisfy

$$g\nabla_\Gamma q_1 = g\nabla_\Gamma q_2 \quad \text{in} \quad H^{-1}(\Gamma, T\Gamma), \qquad \int_\Gamma q_1 \, d\mathcal{H}^2 = \int_\Gamma q_2 \, d\mathcal{H}^2 = 0.$$

Then $\nabla_\Gamma(q_1 - q_2) = 0$ in $H^{-1}(\Gamma, T\Gamma)$ by (5.9.21) and thus $q_1 - q_2$ is constant on Γ by Lemma 5.9.2. Since $\int_\Gamma (q_1 - q_2) \, d\mathcal{H}^2 = 0$, the constant $q_1 - q_2$ is equal to zero, i.e. $q_1 = q_2$ on Γ. Hence the uniqueness is also valid. Finally, the estimate (5.9.20) follows from (5.9.19) with $\int_\Gamma q \, d\mathcal{H}^2 = 0$ and (5.9.21). $\square$

5.9.3 Weighted Helmholtz–Leray decomposition of tangential vector fields

The aim of this subsection is to establish the weighted Helmholtz–Leray decomposition of $L^2(\Gamma, T\Gamma)$ and give estimates for the gradient part of the decomposition. For $v \in L^2(\Gamma)^3$ we consider $\mathrm{div}_\Gamma(gv)$ as an element of $H^{-1}(\Gamma)$ given by (5.2.25). We define a subspace of $L^2(\Gamma, T\Gamma)$ by

$$L^2_{g\sigma}(\Gamma, T\Gamma) := \{v \in L^2(\Gamma, T\Gamma) \mid \mathrm{div}_\Gamma(gv) = 0 \text{ in } H^{-1}(\Gamma)\}.$$

By (5.2.25) and $g \in C^1(\Gamma)$ we easily deduce that

$$\|\mathrm{div}_\Gamma(gv)\|_{H^{-1}(\Gamma)} \leq c\|v\|_{L^2(\Gamma)} \quad \text{for all} \quad v \in L^2(\Gamma)^3.$$

Hence $L^2_{g\sigma}(\Gamma, T\Gamma)$ is closed in $L^2(\Gamma, T\Gamma)$ (note that $L^2(\Gamma, T\Gamma)$ is closed in $L^2(\Gamma)$). Let $L^2_{g\sigma}(\Gamma, T\Gamma)^\perp$ be the orthogonal complement of $L^2_{g\sigma}(\Gamma, T\Gamma)$ in $L^2(\Gamma, T\Gamma)$.

Lemma 5.9.10. *The orthogonal complement of $L^2_{g\sigma}(\Gamma, T\Gamma)$ is of the form*

$$L^2_{g\sigma}(\Gamma, T\Gamma)^\perp = \{g\nabla_\Gamma q \in L^2(\Gamma, T\Gamma) \mid q \in H^1(\Gamma)\}.$$

Proof. Let $\mathcal{X} := \{g\nabla_\Gamma q \in L^2(\Gamma, T\Gamma) \mid q \in H^1(\Gamma)\}$. By (5.2.25) we have

$$(v, g\nabla_\Gamma q)_{L^2(\Gamma)} = (gv, \nabla_\Gamma q)_{L^2(\Gamma)} = -\langle \mathrm{div}_\Gamma(gv), q\rangle_\Gamma = 0$$

for all $v \in L^2_{g\sigma}(\Gamma, T\Gamma)$ and $q \in H^1(\Gamma)$, which shows $\mathcal{X} \subset L^2_{g\sigma}(\Gamma, T\Gamma)^\perp$. Conversely, let $f \in L^2_{g\sigma}(\Gamma, T\Gamma)^\perp$. We consider $f = Pf$ in $H^{-1}(\Gamma, T\Gamma)$ (see Section 5.2.1) to get

$$[f, v]_{T\Gamma} = (f, v)_{L^2(\Gamma)} = 0 \quad \text{for all} \quad v \in H^1_{g\sigma}(\Gamma, T\Gamma) \subset L^2_{g\sigma}(\Gamma, T\Gamma).$$

Hence by Theorem 5.9.5 there exists $q \in L^2(\Gamma)$ such that $f = g\nabla_\Gamma q$ in $H^{-1}(\Gamma, T\Gamma)$. To prove $q \in H^1(\Gamma)$ let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and $v := g^{-1}\eta Pe_i$ for $\eta \in C^1(\Gamma)$ and $i = 1, 2, 3$. Then $v \in H^1(\Gamma, T\Gamma)$ since P and g are of class C^1 and $g \geq c > 0$ on Γ. Moreover,

$$\mathrm{div}_\Gamma(gv) = \nabla_\Gamma\eta \cdot Pe_i + \eta(\mathrm{div}_\Gamma P \cdot e_i) = \underline{D}_i\eta + \eta Hn_i \quad \text{on} \quad \Gamma$$

by the symmetry of P, (5.2.3), and (5.2.7). Hence we get

$$\begin{aligned} -(q, \underline{D}_i\eta + \eta Hn_i)_{L^2(\Gamma)} &= -\big(q, \mathrm{div}_\Gamma(gv)\big)_{L^2(\Gamma)} = [g\nabla_\Gamma q, v]_{T\Gamma} \\ &= [f, v]_{T\Gamma} = (f, v)_{L^2(\Gamma)} = (g^{-1}f_i, \eta)_{L^2(\Gamma)} \end{aligned}$$

for all $\eta \in C^1(\Gamma)$, where f_i is the i-th component of f (note that f is tangential on Γ). From this equality and the definition of the weak tangential derivative in $L^2(\Gamma)$ (see (5.2.19)) it follows that $\underline{D}_i q = g^{-1}f_i \in L^2(\Gamma)$ for $i = 1, 2, 3$. Hence we get $q \in H^1(\Gamma)$ and $f = g\nabla_\Gamma q \in \mathcal{X}$, and the inclusion $L^2_{g\sigma}(\Gamma, T\Gamma)^\perp \subset \mathcal{X}$ holds. □

For $q_1, q_2 \in H^1(\Gamma)$ we have $g\nabla_\Gamma q_1 = g\nabla_\Gamma q_2$ on Γ if and only if $q_1 - q_2$ is constant on Γ by (5.2.30) and Lemma 5.9.2. By this fact and Lemma 5.9.10 we obtain the weighted Helmholtz–Leray decomposition of tangential vector fields on Γ.

Theorem 5.9.11. *For each $v \in L^2(\Gamma, T\Gamma)$ we have the orthogonal decomposition*

$$v = v_g + g\nabla_\Gamma q, \quad v_g \in L^2_{g\sigma}(\Gamma, T\Gamma),\ g\nabla_\Gamma q \in L^2_{g\sigma}(\Gamma, T\Gamma)^\perp.$$

Here $q \in H^1(\Gamma)$ is uniquely determined up to a constant.

Note that here we derived the weighted Helmholtz–Leray decomposition without introducing the notion of differential forms. When $g \equiv 1$ it is also proved in the recent work [55] without calculus of differential forms, where the solenoidal part is further decomposed into the curl of some function and a harmonic field whose surface divergence and curl vanish.

Next we consider approximation of surface solenoidal vector fields. In general, for $v \in L^2(\Gamma, T\Gamma)$ satisfying $\mathrm{div}_\Gamma v = 0$ in $H^{-1}(\Gamma)$ and $\eta \in C^1(\Gamma)$, the surface divergence $\mathrm{div}_\Gamma(\eta v) = \nabla_\Gamma \eta \cdot v$ does not vanish in $H^{-1}(\Gamma)$. Hence standard localization and mollification arguments with a partition of unity of Γ do not work on approximation of surface solenoidal vector fields by smooth ones. Instead, we use a solution to Poisson's equation on Γ.

Lemma 5.9.12. *For each $\eta \in H^{-1}(\Gamma)$ satisfying $\langle \eta, 1\rangle_\Gamma = 0$ there exists a unique weak solution $q \in H^1(\Gamma)$ to Poisson's equation*

$$\Delta_\Gamma q = -\eta \quad \text{on} \quad \Gamma, \qquad \int_\Gamma q \, d\mathcal{H}^2 = 0 \tag{5.9.23}$$

in the sense that

$$(\nabla_\Gamma q, \nabla_\Gamma \xi)_{L^2(\Gamma)} = \langle \eta, \xi\rangle_\Gamma \quad \text{for all} \quad \xi \in H^1(\Gamma). \tag{5.9.24}$$

Moreover, there exists a constant $c > 0$ such that

$$\|q\|_{H^1(\Gamma)} \leq c\|\eta\|_{H^{-1}(\Gamma)}. \tag{5.9.25}$$

If in addition $\eta \in L^2(\Gamma)$, then $q \in H^2(\Gamma)$ and

$$\|q\|_{H^2(\Gamma)} \leq c\|\eta\|_{L^2(\Gamma)}. \tag{5.9.26}$$

The existence and uniqueness of a weak solution to (5.9.23) and the estimate (5.9.25) follow from Poincaré's inequality (5.2.20) and the Lax–Milgram theorem. Also, the H^2-regularity and (5.9.26) are proved by a standard localization argument and the elliptic regularity theorem. For details, see [11, Theorems 3.1 and 3.3].

Lemma 5.9.13. *The space $H^1_{g\sigma}(\Gamma, T\Gamma)$ is dense in $L^2_{g\sigma}(\Gamma, T\Gamma)$.*

Proof. Let $v \in L^2_{g\sigma}(\Gamma, T\Gamma)$. By Lemma 5.2.6 we can take a sequence $\{\tilde{v}_k\}_{k=1}^\infty$ in $C^1(\Gamma, T\Gamma)$ that converges to v strongly in $L^2(\Gamma, T\Gamma)$. For each $k \in \mathbb{N}$ we have

$$\|\mathrm{div}_\Gamma(g\tilde{v}_k)\|_{H^{-1}(\Gamma)} = \|\mathrm{div}_\Gamma[g(\tilde{v}_k - v)]\|_{H^{-1}(\Gamma)} \leq c\|\tilde{v}_k - v\|_{L^2(\Gamma)} \tag{5.9.27}$$

by $\mathrm{div}_\Gamma(gv) = 0$ in $H^{-1}(\Gamma)$ and (5.2.25). We consider Poisson's equation (5.9.23) with source term $\eta_k := -\mathrm{div}_\Gamma(g\tilde{v}_k) \in L^2(\Gamma)$. By Lemma 5.9.12 there exists a unique solution $q_k \in H^2(\Gamma)$ to (5.9.23). Moreover, by (5.9.25) and (5.9.27) we have

$$\|q_k\|_{H^1(\Gamma)} \leq c\|\eta_k\|_{H^{-1}(\Gamma)} = c\|\mathrm{div}_\Gamma(g\tilde{v}_k)\|_{H^{-1}(\Gamma)} \leq c\|v - \tilde{v}_k\|_{L^2(\Gamma)}.$$

Hence $v_k := \tilde{v}_k - g^{-1}\nabla_\Gamma q_k \in H^1_{g\sigma}(\Gamma, T\Gamma)$ and

$$\|v - v_k\|_{L^2(\Gamma)} \leq \|v - \tilde{v}_k\|_{L^2(\Gamma)} + c\|q_k\|_{H^1(\Gamma)} \leq c\|v - \tilde{v}_k\|_{L^2(\Gamma)} \to 0$$

as $k \to \infty$ by $g \geq c > 0$ on Γ and the strong convergence of $\{\tilde{v}_k\}_{k=1}^\infty$ to v in $L^2(\Gamma, T\Gamma)$, which shows that $H^1_{g\sigma}(\Gamma, T\Gamma)$ is dense in $L^2_{g\sigma}(\Gamma, T\Gamma)$. $\square$

Let $\mathbb{P}_g$ be the orthogonal projection from $L^2(\Gamma, T\Gamma)$ onto $L^2_{g\sigma}(\Gamma, T\Gamma)$. We call it the weighted Helmholtz–Leray projection. In the study of a singular limit problem for (5.1.1)–(5.1.3) we need to estimate the difference $v - \mathbb{P}_g v$ for $v \in L^2(\Gamma, T\Gamma)$.

Lemma 5.9.14. *There exists a constant $c > 0$ such that*

$$\|v - \mathbb{P}_g v\|_{L^2(\Gamma)} \leq c\|\mathrm{div}_\Gamma(gv)\|_{H^{-1}(\Gamma)} \tag{5.9.28}$$

for all $v \in L^2(\Gamma, T\Gamma)$. If in addition $v \in H^1(\Gamma, T\Gamma)$, then $\mathbb{P}_g v \in H^1_{g\sigma}(\Gamma, T\Gamma)$ and

$$\|v - \mathbb{P}_g v\|_{H^1(\Gamma)} \leq c\|\mathrm{div}_\Gamma(gv)\|_{L^2(\Gamma)}. \tag{5.9.29}$$

Proof. Let $v \in L^2(\Gamma, T\Gamma)$ and $\eta := -\mathrm{div}_\Gamma(gv) \in H^{-1}(\Gamma)$. Since v is tangential on Γ, we see by (5.2.25) that

$$\langle \eta, 1\rangle_\Gamma = (gv, Hn)_{L^2(\Gamma)} = \int_\Gamma g(v \cdot n)H\, d\mathcal{H}^2 = 0.$$

Hence by Lemma 5.9.12 there exists a unique weak solution $q \in H^1(\Gamma)$ to (5.9.23) with $\eta = -\mathrm{div}_\Gamma(gv)$. Then since $v - g^{-1}\nabla_\Gamma q \in L^2_{g\sigma}(\Gamma, T\Gamma)$, we have $\mathbb{P}_g v = v - g^{-1}\nabla_\Gamma q$ by the uniqueness of the weighted Helmholtz–Leray decomposition. Moreover, since

$$\|q\|_{H^1(\Gamma)} \leq c\|\eta\|_{H^{-1}(\Gamma)} = c\|\mathrm{div}_\Gamma(gv)\|_{H^{-1}(\Gamma)}$$

by (5.9.25) and $g \in C^1(\Gamma)$ is bounded from below by a positive constant, we obtain

$$\|v - \mathbb{P}_g v\|_{L^2(\Gamma)} = \|g^{-1}\nabla_\Gamma q\|_{L^2(\Gamma)} \leq c\|q\|_{H^1(\Gamma)} \leq c\|\mathrm{div}_\Gamma(gv)\|_{H^{-1}(\Gamma)}.$$

Hence the inequality (5.9.28) holds.

Next suppose that $v \in H^1(\Gamma, T\Gamma)$. Then since $\eta = -\mathrm{div}_\Gamma(gv) \in L^2(\Gamma)$ Lemma 5.9.12 implies that that $q \in H^2(\Gamma)$ and thus $\mathbb{P}_g v = v - g^{-1}\nabla_\Gamma q \in H^1_{g\sigma}(\Gamma, T\Gamma)$. Moreover, since

$$\|q\|_{H^2(\Gamma)} \leq c\|\eta\|_{L^2(\Gamma)} = c\|\mathrm{div}_\Gamma(gv)\|_{L^2(\Gamma)}$$

by (5.9.26) and $g \in C^1(\Gamma)$ is bounded from below by a positive constant,

$$\|v - \mathbb{P}_g v\|_{H^1(\Gamma)} = \|g^{-1}\nabla_\Gamma q\|_{H^1(\Gamma)} \leq c\|q\|_{H^2(\Gamma)} \leq c\|\mathrm{div}_\Gamma(gv)\|_{L^2(\Gamma)},$$

i.e. the inequality (5.9.29) is valid. □

Corollary 5.9.15. *There exists a constant $c > 0$ such that*

$$\|\mathbb{P}_g v\|_{H^k(\Gamma)} \leq c\|v\|_{H^k(\Gamma)} \tag{5.9.30}$$

for all $v \in H^k(\Gamma, T\Gamma)$, $k = 0, 1$ (note that $H^0 = L^2$).

Proof. If $k = 0$, then (5.9.30) holds with $c = 1$ since $\mathbb{P}_g$ is the orthogonal projection from $L^2(\Gamma, T\Gamma)$ onto $L^2_{g\sigma}(\Gamma, T\Gamma)$. Moreover, the inequality (5.9.30) for $k = 1$ follows from (5.9.29) and $\|\mathrm{div}_\Gamma(gv)\|_{L^2(\Gamma)} \leq c\|v\|_{H^1(\Gamma)}$. □

Next we consider the time derivative of $v - \mathbb{P}_g v$. We derive an estimate for the time derivative of a weak solution to Poisson's equation (5.9.23).

Lemma 5.9.16. *Let $T>0$. Suppose that $\eta\in H^1(0,T;H^{-1}(\Gamma))$ satisfies*

$$\langle \eta(t),1\rangle_\Gamma = 0 \quad \textit{for all} \quad t\in[0,T].$$

For each $t\in[0,T]$ let $q(t)\in H^1(\Gamma)$ be a unique weak solution to (5.9.23) with source term $\eta(t)$. Then $q\in H^1(0,T;H^1(\Gamma))$ and there exists a constant $c>0$ such that

$$\|\partial_t q\|_{L^2(0,T;H^1(\Gamma))} \leq c\|\partial_t\eta\|_{L^2(0,T;H^{-1}(\Gamma))}. \tag{5.9.31}$$

Moreover, for a.a. $t\in(0,T)$ the time derivative $\partial_t q(t)\in H^1(\Gamma)$ is a unique weak solution to (5.9.23) with source term $\partial_t\eta(t)$.

Note that, when $\eta\in H^1(0,T;H^{-1}(\Gamma))$, $\eta(t)\in H^{-1}(\Gamma)$ is well-defined for each $t\in[0,T]$ since $H^1(0,T;H^{-1}(\Gamma))$ is continuously embedded into $C([0,T];H^{-1}(\Gamma))$.

Proof. First note that $q\in L^2(0,T;H^1(\Gamma))$ by (5.9.25) and $\eta\in L^2(0,T;H^{-1}(\Gamma))$. Let us prove $\partial_t q\in L^2(0,T;H^1(\Gamma))$ by means of the difference quotient. Fix $\delta\in(0,T/2)$ and $h\in\mathbb{R}\setminus\{0\}$ with $|h|<\delta/2$. For $t\in(\delta,T-\delta)$ we define

$$D_hq(t) := \frac{q(t+h)-q(t)}{h}\in H^1(\Gamma), \quad D_h\eta(t) := \frac{\eta(t+h)-\eta(t)}{h}\in H^{-1}(\Gamma).$$

Note that these definitions make sense since $t+h\in(\delta/2,T-\delta/2)$. Moreover,

$$\int_\Gamma D_hq(t)\,d\mathcal{H}^2 = 0, \quad \langle D_h\eta(t),1\rangle_\Gamma = 0, \quad t\in(\delta,T-\delta)$$

since $q(t)$ and $\eta(t)$ satisfy the same equalities for all $t\in[0,T]$. For $\xi\in H^1(\Gamma)$ we subtract (5.9.24) for $q(t)$ from that for $q(t+h)$ and divide both sides by h to get

$$(\nabla_\Gamma D_hq(t),\nabla_\Gamma\xi)_{L^2(\Gamma)} = \langle D_h\eta(t),\xi\rangle_\Gamma. \tag{5.9.32}$$

Since this equality holds for all $\xi\in H^1(\Gamma)$, the function $D_hq(t)$ is a unique weak solution to (5.9.23) with source term $D_h\eta(t)$. Hence by (5.9.25) we have

$$\|D_hq(t)\|_{H^1(\Gamma)} \leq c\|D_h\eta(t)\|_{H^{-1}(\Gamma)}, \quad t\in(\delta,T-\delta).$$

Note that the constant $c>0$ in this inequality does not depend on t, δ, and h. From this inequality it immediately follows that

$$\|D_hq\|_{L^2(\delta,T-\delta;H^1(\Gamma))} \leq c\|D_h\eta\|_{L^2(\delta,T-\delta;H^{-1}(\Gamma))}.$$

Moreover, since $\eta\in H^1(0,T;H^{-1}(\Gamma))$, we have

$$\|D_h\eta\|_{L^2(\delta,T-\delta;H^{-1}(\Gamma))} \leq c\|\partial_t\eta\|_{L^2(0,T;H^{-1}(\Gamma))}$$

with a constant $c>0$ independent of h and δ (see [13, Section 5.8, Theorem 3 (i)]). Combining the above two estimates we obtain

$$\|D_hq\|_{L^2(\delta,T-\delta;H^1(\Gamma))} \leq c\|\partial_t\eta\|_{L^2(0,T;H^{-1}(\Gamma))}$$

for all $h\in\mathbb{R}\setminus\{0\}$ with $|h|<\delta/2$. Since the right-hand side of this inequality is independent of h, it follows that $\partial_tq\in L^2(\delta,T-\delta;H^1(\Gamma))$ and

$$\|\partial_tq\|_{L^2(\delta,T-\delta;H^1(\Gamma))} \leq c\|\partial_t\eta\|_{L^2(0,T;H^{-1}(\Gamma))}$$

for all $\delta \in (0,T/2)$ (see [13, Section 5.8, Theorem 3 (ii)]). In particular, we have $\partial_t q(t) \in H^1(\Gamma)$ for a.a. $t \in (0,T)$. Since the right-hand side of the above inequality is independent of δ, the monotone convergence theorem yields

$$\|\partial_t q\|_{L^2(0,T;H^1(\Gamma))} = \lim_{\delta\to 0} \|\partial_t q\|_{L^2(\delta,T-\delta;H^1(\Gamma))} \leq c\|\partial_t \eta\|_{L^2(0,T;H^{-1}(\Gamma))}$$

and thus $\partial_t q \in L^2(0,T;H^1(\Gamma))$ and (5.9.31) is valid.

Next we show that $\partial_t q(t)$ is a unique weak solution to (5.9.23) with $\partial_t \eta(t)$ for a.a. $t \in (0,T)$. Let $\xi \in H^1(\Gamma)$ and $\varphi \in C_c^\infty(0,T)$. Suppose that φ is supported in $(\delta, T-\delta)$ with $\delta \in (0,T/2)$. We extend φ to $\mathbb{R}$ by setting zero outside of $(0,T)$. For $h \in \mathbb{R} \setminus \{0\}$, $|h| < \delta/2$ we multiply both sides of (5.9.32) by $\varphi(t)$, integrate them over $(\delta, T-\delta)$, and make the change of a variable

$$\int_\delta^{T-\delta} \psi(t+h)\varphi(t)\,dt = \int_{\delta+h}^{T-\delta+h} \psi(s)\varphi(s-h)\,ds$$

for $\psi(t) = (\nabla_\Gamma q(t), \nabla_\Gamma \xi)_{L^2(\Gamma)}$, $\langle \eta(t), \xi\rangle_\Gamma$ to get

$$-\int_0^T (\nabla_\Gamma q(t), \nabla_\Gamma \xi)_{L^2(\Gamma)} D_{-h}\varphi(t)\,dt = -\int_0^T \langle \eta(t), \xi\rangle_\Gamma D_{-h}\varphi(t)\,dt,$$

where $D_{-h}\varphi(t) := \{\varphi(t-h)-\varphi(t)\}/(-h)$ (note that φ is supported in $(\delta, T-\delta)$). Let $h \to 0$ in this equality. Then since $D_{-h}\varphi$ converges to $\partial_t\varphi$ uniformly on $(0,T)$,

$$-\int_0^T (\nabla_\Gamma q(t), \nabla_\Gamma \xi)_{L^2(\Gamma)} \partial_t\varphi(t)\,dt = -\int_0^T \langle \eta(t), \xi\rangle_\Gamma \partial_t\varphi(t)\,dt$$

for all $\varphi \in C_c^\infty(0,T)$. By this equality, $q \in H^1(0,T;H^1(\Gamma))$, and $\eta \in H^1(0,T;H^{-1}(\Gamma))$,

$$([\nabla_\Gamma(\partial_t q)](t), \nabla_\Gamma \xi)_{L^2(\Gamma)} = \langle \partial_t \eta(t), \xi\rangle_\Gamma$$

for all $\xi \in H^1(\Gamma)$ and a.a. $t \in (0,T)$. Here we note that $\partial_t(\nabla_\Gamma q) = \nabla_\Gamma(\partial_t q)$ a.e. on $\Gamma \times (0,T)$ by $q \in H^1(0,T;H^1(\Gamma))$. In the same way we can show $\int_\Gamma \partial_t q(t)\,d\mathcal{H}^2 = 0$ for a.a. $t \in (0,T)$ since $q(t)$ satisfies the same equality for all $t \in [0,T]$. Hence $\partial_t q(t)$ is a unique weak solution to (5.9.23) with $\partial_t \eta(t)$ for a.a. $t \in (0,T)$. $\Box$

Based on Lemma 5.9.16 we give an estimate for the time derivative of $v - \mathbb{P}_g v$.

Lemma 5.9.17. *Let $v \in H^1(0,T;L^2(\Gamma,T\Gamma))$, $T > 0$. Then*

$$\mathbb{P}_g v \in H^1(0,T;L^2_{g\sigma}(\Gamma,T\Gamma))$$

and there exists a constant $c > 0$ such that

$$\|\partial_t v - \partial_t \mathbb{P}_g v\|_{L^2(0,T;L^2(\Gamma))} \leq c\|\mathrm{div}_\Gamma(g\partial_t v)\|_{L^2(0,T;H^{-1}(\Gamma))}. \tag{5.9.33}$$

Proof. Let $\eta := -\mathrm{div}_\Gamma(gv)$. Since $v \in H^1(0,T;L^2(\Gamma,T\Gamma))$, we have

$$\eta \in H^1(0,T;H^{-1}(\Gamma)), \quad \partial_t\eta = -\mathrm{div}_\Gamma(g\partial_t v) \in L^2(0,T;H^{-1}(\Gamma)).$$

Here the second relation is due to the fact that g and P are independent of time (note that P appears in the definition of the tangential derivatives). For each $t \in [0,T]$ let $q(t) \in H^1(\Gamma)$

be a unique weak solution to (5.9.23) with $\eta(t) = -\mathrm{div}_\Gamma[gv(t)]$, which satisfies $\langle \eta(t), 1\rangle_\Gamma = 0$ as in the proof of Lemma 5.9.14. Then Lemma 5.9.16 implies that $q \in H^1(0,T;H^1(\Gamma))$ and

$$\|\partial_t q\|_{L^2(0,T;H^1(\Gamma))} \leq c\|\partial_t \eta\|_{L^2(0,T;H^{-1}(\Gamma))} = c\|\mathrm{div}_\Gamma(g\partial_t v)\|_{L^2(0,T;H^{-1}(\Gamma))}.$$

Moreover, for a.a. $t \in (0,T)$ the time derivative $\partial_t q(t) \in H^1(\Gamma)$ is a unique weak solution to (5.9.23) with $\partial_t \eta(t) = -\mathrm{div}_\Gamma[g\partial_t v(t)]$. By these facts and

$$\mathbb{P}_g v = v - g^{-1}\nabla_\Gamma q, \quad \partial_t \mathbb{P}_g v = \partial_t v - g^{-1}\nabla_\Gamma(\partial_t q) \quad \text{a.e. on} \quad \Gamma \times (0,T)$$

we observe that $\mathbb{P}_g v \in H^1(0,T;L^2_{g\sigma}(\Gamma,T\Gamma))$ and

$$\begin{aligned}\|\partial_t v - \partial_t \mathbb{P}_g v\|_{L^2(0,T;L^2(\Gamma))} = \|g^{-1}\nabla_\Gamma(\partial_t q)\|_{L^2(0,T;L^2(\Gamma))} &\leq c\|\partial_t q\|_{L^2(0,T;H^1(\Gamma))} \\ &\leq c\|\mathrm{div}_\Gamma(g\partial_t v)\|_{L^2(0,T;H^{-1}(\Gamma))},\end{aligned}$$

where we also used the inequality $g \geq c > 0$ on Γ. Hence the lemma is valid. □

5.9.4 Solenoidal spaces of general vector fields

In this subsection we briefly investigate solenoidal spaces of general (not necessarily tangential) vector fields on Γ. Although the results of this subsection are not used in the sequel, we believe that they are useful for the future study of surface fluid equations including fluid equations on an evolving surface (see e.g. [28, 31, 32, 43]). For the sake of simplicity, we only consider the case $g \equiv 1$ and give a remark on the case of general g at the end of this subsection.

Let $q \in L^2(\Gamma)$. By (5.2.21) and (5.2.24) we have

$$\langle \nabla_\Gamma q + qHn, v\rangle_\Gamma = -(q, \mathrm{div}_\Gamma v)_{L^2(\Gamma)} \tag{5.9.34}$$

for all $v \in H^1(\Gamma)^3$. Hence $\langle \nabla_\Gamma q + qHn, v\rangle_\Gamma = 0$ for all v in the solenoidal space

$$H^1_\sigma(\Gamma) := \{v \in H^1(\Gamma)^3 \mid \mathrm{div}_\Gamma v = 0 \text{ on } \Gamma\}.$$

Our goal is to prove that each element of the annihilator of $H^1_\sigma(\Gamma)$ is of the form $\nabla_\Gamma q + qHn$. To this end, we give two properties of a functional of this form.

Lemma 5.9.18. *Let $q \in L^2(\Gamma)$. Then*

$$\nabla_\Gamma q + qHn = 0 \quad \textit{in} \quad H^{-1}(\Gamma)^3$$

if and only if $q = 0$ on Γ.

Proof. We first note that for all $q \in L^2(\Gamma)$ we have

$$\|\nabla_\Gamma q\|_{H^{-1}(\Gamma,T\Gamma)} \leq \|\nabla_\Gamma q + qHn\|_{H^{-1}(\Gamma)} \tag{5.9.35}$$

since $[\nabla_\Gamma q, v]_{T\Gamma} = \langle \nabla_\Gamma q + qHn, v\rangle_\Gamma$ for all $v \in H^1(\Gamma, T\Gamma)$ (see Section 5.2.1).

Suppose that $\nabla_\Gamma q + qHn = 0$ in $H^{-1}(\Gamma)^3$. Then $\nabla_\Gamma q = 0$ in $H^{-1}(\Gamma, T\Gamma)$ by (5.9.35) and thus q is constant on Γ by Lemma 5.9.2. To prove $q = 0$, we set $v := \xi n$ in (5.9.34) for an arbitrary $\xi \in H^1(\Gamma)$ (note that n is of class C^1 on Γ) to get

$$0 = \langle \nabla_\Gamma q + qHn, v\rangle_\Gamma = q\int_\Gamma \xi H \, d\mathcal{H}^2.$$

Since $H \in C(\Gamma) \subset L^2(\Gamma)$ and $H^1(\Gamma)$ is dense in $L^2(\Gamma)$ (see Lemma 5.2.4), we observe by the above equality and a density argument that

$$q \int_\Gamma H^2 \, d\mathcal{H}^2 = 0.$$

Moreover, for the compact surface Γ in $\mathbb{R}^3$ it is known (see e.g. (16.32) in [15]) that

$$\frac{1}{4} \int_\Gamma H^2 \, d\mathcal{H}^2 \geq 4\pi.$$

(Note that in our definition H is not divided by the dimension of Γ.) Hence $q = 0$.

Conversely, if $q = 0$ on Γ, then $\nabla_\Gamma q + qHn = 0$ in $H^{-1}(\Gamma)^3$ by (5.9.34). □

Lemma 5.9.19. *There exists a constant $c > 0$ such that*

$$\|q\|_{L^2(\Gamma)} \leq c \|\nabla_\Gamma q + qHn\|_{H^{-1}(\Gamma)} \tag{5.9.36}$$

for all $q \in L^2(\Gamma)$.

Proof. By the Nečas inequality (5.9.1) and (5.9.35) it is sufficient to show that

$$\|q\|_{H^{-1}(\Gamma)} \leq c \|\nabla_\Gamma q + qHn\|_{H^{-1}(\Gamma)} \tag{5.9.37}$$

for all $q \in L^2(\Gamma)$. Assume to the contrary that there exists $q_k \in L^2(\Gamma)$ such that

$$\|q_k\|_{H^{-1}(\Gamma)} > k \|\nabla_\Gamma q_k + q_k Hn\|_{H^{-1}(\Gamma)} \tag{5.9.38}$$

for each $k \in \mathbb{N}$. Since $\|q_k\|_{H^{-1}(\Gamma)} \neq 0$, we may assume that $\|q_k\|_{H^{-1}(\Gamma)} = 1$ for all $k \in \mathbb{N}$ by replacing q_k with $q_k / \|q_k\|_{H^{-1}(\Gamma)}$. Then we observe by (5.9.1), (5.9.35), and (5.9.38) that $\{q_k\}_{k=1}^\infty$ is bounded in $L^2(\Gamma)$. By this fact and the compact embedding $L^2(\Gamma) \hookrightarrow H^{-1}(\Gamma)$ we can take a subsequence of $\{q_k\}_{k=1}^\infty$, which we denote by $\{q_k\}_{k=1}^\infty$ again, that converges to some $q \in L^2(\Gamma)$ weakly in $L^2(\Gamma)$ and strongly in $H^{-1}(\Gamma)$. Moreover, the weak convergence of $\{q_k\}_{k=1}^\infty$ to q in $L^2(\Gamma)$ and (5.9.34) imply that

$$\lim_{k\to\infty} (\nabla_\Gamma q_k + q_k Hn) = \nabla_\Gamma q + qHn \quad \text{weakly in} \quad H^{-1}(\Gamma)^3.$$

By this fact, (5.9.38), and $\|q_k\|_{H^{-1}(\Gamma)} = 1$ we have

$$\|\nabla_\Gamma q + qHn\|_{H^{-1}(\Gamma)} \leq \liminf_{k\to\infty} \|\nabla_\Gamma q_k + q_k Hn\|_{H^{-1}(\Gamma)} = 0.$$

Hence $\nabla_\Gamma q + qHn = 0$ in $H^{-1}(\Gamma)$ and $q = 0$ on Γ by Lemma 5.9.18. However, the strong convergence of $\{q_k\}_{k=1}^\infty$ to q in $H^{-1}(\Gamma)$ implies that

$$\|q\|_{H^{-1}(\Gamma)} = \lim_{k\to\infty} \|q_k\|_{H^{-1}(\Gamma)} = 1, \tag{5.9.39}$$

which contradicts with $q = 0$. Therefore, the inequality (5.9.37) is valid. □

Now we prove de Rham's theorem for the annihilator of $H^1_\sigma(\Gamma)$.

Theorem 5.9.20. *Suppose that $f \in H^{-1}(\Gamma)^3$ satisfies*

$$\langle f, v \rangle_\Gamma = 0 \quad \text{for all} \quad v \in H^1_\sigma(\Gamma).$$

Then there exists a unique $q \in L^2(\Gamma)$ such that

$$f = \nabla_\Gamma q + qHn \quad \text{in} \quad H^{-1}(\Gamma)^3, \quad \|q\|_{L^2(\Gamma)} \leq c \|f\|_{H^{-1}(\Gamma)}$$

with a constant $c > 0$ independent of f.

Proof. Using (5.9.36) we can show as in the proof of Lemma 5.9.7 that the subspace

$$\mathcal{X} := \{\nabla_\Gamma q + qHn \in H^{-1}(\Gamma)^3 \mid q \in L^2(\Gamma)\}$$

is closed in $H^{-1}(\Gamma)^3$. Moreover, by (5.9.34) we easily observe that $\mathcal{X}^\perp \subset H^1_\sigma(\Gamma)$ in $H^1(\Gamma)^3$. Noting that the dual $H^{-1}(\Gamma)^3$ of the Hilbert space $H^1(\Gamma)^3$ is reflexive, we use Lemmas 5.9.8 and 5.9.9 to obtain

$$H^1_\sigma(\Gamma)^\perp = \{f \in H^{-1}(\Gamma)^3 \mid \langle f, v\rangle_\Gamma = 0 \text{ for all } v \in H^1_\sigma(\Gamma)\} \subset (\mathcal{X}^\perp)^\perp = \mathcal{X}$$

in $H^{-1}(\Gamma)^3$. Hence the existence part of the theorem is valid. Also, the uniqueness and the estimate immediately follow from Lemma 5.9.19. □

Next we derive the Helmholtz–Leray decomposition of general vector fields on Γ. We define a subspace of $L^2(\Gamma)^3$ by

$$L^2_\sigma(\Gamma) := \{v \in L^2(\Gamma)^3 \mid \mathrm{div}_\Gamma v = 0 \text{ in } H^{-1}(\Gamma)\}.$$

By (5.2.25) we have $\|\mathrm{div}_\Gamma v\|_{H^{-1}(\Gamma)} \leq c\|v\|_{L^2(\Gamma)}$ for all $v \in L^2(\Gamma)^3$. Hence $L^2_\sigma(\Gamma)$ is closed in $L^2(\Gamma)^3$. Let us give the characterization of the orthogonal complement of $L^2_\sigma(\Gamma)$ in $L^2(\Gamma)^3$.

Lemma 5.9.21. *The orthogonal complement of $L^2_\sigma(\Gamma)$ in $L^2(\Gamma)^3$ is of the form*

$$L^2_\sigma(\Gamma)^\perp = \{\nabla_\Gamma q + qHn \in L^2(\Gamma)^3 \mid q \in H^1(\Gamma)\}.$$

Proof. The proof is similar to that of Lemma 5.9.10. By (5.2.25) we immediately get $\nabla_\Gamma q + qHn \in L^2_\sigma(\Gamma)^\perp$ for all $q \in H^1(\Gamma)$. Conversely, let $f \in L^2_\sigma(\Gamma)^\perp$. Then we have $\langle f, v\rangle_\Gamma = 0$ for all $v \in H^1_\sigma(\Gamma) \subset L^2_\sigma(\Gamma)$ by (5.2.21). Hence by Theorem 5.9.20 there exists a unique $q \in L^2(\Gamma)$ such that $f = \nabla_\Gamma q + qHn$ in $H^{-1}(\Gamma)^3$. To prove $q \in H^1(\Gamma)$ let $v := \eta e_i$ for $\eta \in C^1(\Gamma)$ and $i = 1, 2, 3$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Then since $v \in H^1(\Gamma)^3$ and $\mathrm{div}_\Gamma v = \underline{D}_i \eta$, we have

$$\begin{aligned} -(q, \underline{D}_i\eta)_{L^2(\Gamma)} &= -(q, \mathrm{div}_\Gamma v)_{L^2} = \langle \nabla_\Gamma q + qHn, v\rangle_\Gamma \\ &= \langle f, v\rangle_\Gamma = (f, v)_{L^2(\Gamma)} = (f_i, \eta)_{L^2(\Gamma)}, \end{aligned}$$

where f_i is the i-th component of f. From this equality we deduce that

$$-(q, \underline{D}_i\eta + \eta H n_i)_{L^2(\Gamma)} = (f_i - qHn_i, \eta)_{L^2(\Gamma)} \quad \text{for all} \quad \eta \in C^1(\Gamma).$$

This means that $\underline{D}_i q = f_i - qHn_i \in L^2(\Gamma)$ by the definition of the weak tangential derivative in $L^2(\Gamma)$ (see (5.2.19)). Hence we obtain $q \in H^1(\Gamma)$ and $f = \nabla_\Gamma q + qHn$ in $L^2(\Gamma)^3$. □

The result of Lemma 5.9.21 was given in [31, Lemma 2.7] (see also [32, Theorem 1.1]). Here we gave another proof of it. By Lemmas 5.9.18 and 5.9.21 we obtain the Helmholtz–Leray decomposition of vector fields in $L^2(\Gamma)^3$ with uniqueness of the gradient part.

Theorem 5.9.22. *For each $v \in L^2(\Gamma)^3$ we have the orthogonal decomposition*

$$v = v_\sigma + \nabla_\Gamma q + qHn, \quad v_\sigma \in L^2_\sigma(\Gamma),\ \nabla_\Gamma q + qHn \in L^2_\sigma(\Gamma)^\perp.$$

Here $q \in H^1(\Gamma)$ is uniquely determined.

The Helmholtz–Leray decomposition in Theorem 5.9.22 was already stated in [32] without an explicit formulation (see a remark after [32, Theorem 1.1]).

Remark 5.9.23. Theorem 5.9.22 applied to a tangential vector field on Γ does not imply the tangential Helmholtz–Leray decomposition (with $g \equiv 1$) given in Theorem 5.9.11 in general. To see this, suppose that Γ is strictly convex and thus the mean curvature H of Γ does not vanish on the whole surface. Let $v \in L^2(\Gamma, T\Gamma)$ be a tangential vector field on Γ such that $\mathrm{div}_\Gamma v \neq 0$ in $H^{-1}(\Gamma)$. By Theorem 5.9.22 we get the orthogonal decomposition

$$v = v_\sigma + \nabla_\Gamma q + qHn, \quad v_\sigma \in L^2_\sigma(\Gamma),\ \nabla_\Gamma q + qHn \in L^2_\sigma(\Gamma)^\perp$$

with a unique $q \in H^1(\Gamma)$. Since v is tangential on Γ,

$$0 = v_\sigma \cdot n + qH, \quad \text{i.e.} \quad v_\sigma \cdot n = -qH \quad \text{on} \quad \Gamma.$$

Moreover, $q \neq 0$ in $H^1(\Gamma)$ by $\mathrm{div}_\Gamma v \neq 0$ in $H^{-1}(\Gamma)$. By this property and the fact that H does not vanish on the whole surface Γ by the strict convexity of Γ we see that $v_\sigma \cdot n = -qH \neq 0$ in $L^2(\Gamma)$. Hence the solenoidal part v_σ of v given by Theorem 5.9.22 is not tangential on Γ, while the solenoidal part v_g (with $g \equiv 1$) of the same v given by Theorem 5.9.11 must be tangential on Γ.

The vector field $\nabla_\Gamma q + qHn$ appears in the interface equations of two-phase flows [4, 6, 49] as well as the Navier–Stokes equations on an evolving surface [28, 31]. By (5.2.7) we observe that the surface divergence of qP is of this form:

$$\mathrm{div}_\Gamma(qP) = P\nabla_\Gamma q + q\,\mathrm{div}_\Gamma P = \nabla_\Gamma q + qHn.$$

The tensor qP is a part of the Boussinesq–Scriven surface stress tensor [2, 7, 58]

$$S_\Gamma = \{q + (\lambda_s - \mu_s)\mathrm{div}_\Gamma v\}P + 2\mu_s D_\Gamma(v).$$

Here q is the surface tension, λ_s is the surface dilatational viscosity, μ_s is the surface shear viscosity, v is the total velocity of surface flow, and $D_\Gamma(v)$ is the surface strain rate tensor given by (5.4.38).

Finally, we give a remark on the case of general g. Let $g \in C^1(\Gamma)$ be bounded from below by a positive constant. We define weighted solenoidal spaces

$$\begin{aligned} L^2_{g\sigma}(\Gamma) &:= \{v \in L^2(\Gamma)^3 \mid \mathrm{div}_\Gamma(gv) = 0 \text{ in } H^{-1}(\Gamma)\}, \\ H^1_{g\sigma}(\Gamma) &:= \{v \in H^1(\Gamma)^3 \mid \mathrm{div}_\Gamma(gv) = 0 \text{ on } \Gamma\}. \end{aligned}$$

By (5.2.22), (5.2.24), and (5.2.25) we have

$$\begin{aligned} \langle g(\nabla_\Gamma q + qHn), v\rangle_\Gamma &= -\big(q, \mathrm{div}_\Gamma(gv)\big)_{L^2(\Gamma)}, & q \in L^2(\Gamma),\ v \in H^1(\Gamma)^3, \\ \langle \mathrm{div}_\Gamma(gw), \eta\rangle_\Gamma &= -\big(w, g(\nabla_\Gamma \eta + \eta Hn)\big)_{L^2(\Gamma)}, & w \in L^2(\Gamma)^3,\ \eta \in H^1(\Gamma). \end{aligned}$$

Using these formulas and applying Theorem 5.9.20 or Lemma 5.9.21 to $g^{-1}f$ for f in $H^1_{g\sigma}(\Gamma)^\perp$ or $L^2_{g\sigma}(\Gamma)^\perp$, we can show the following weighted version of the main results in this subsection.

Theorem 5.9.24. *Suppose that $f \in H^{-1}(\Gamma)^3$ satisfies*

$$\langle f, v\rangle_\Gamma = 0 \quad \textit{for all} \quad v \in H^1_{g\sigma}(\Gamma).$$

Then there exists a unique $q \in L^2(\Gamma)$ such that

$$f = g(\nabla_\Gamma q + qHn) \quad \textit{in} \quad H^{-1}(\Gamma)^3, \quad \|q\|_{L^2(\Gamma)} \leq c\|f\|_{H^{-1}(\Gamma)}$$

with a constant $c > 0$ independent of f.

Theorem 5.9.25. *For each $v \in L^2(\Gamma)^3$ we have the orthogonal decomposition*

$$v = v_g + g(\nabla_\Gamma q + qHn), \quad v_g \in L^2_{g\sigma}(\Gamma),\ g(\nabla_\Gamma q + qHn) \in L^2_{g\sigma}(\Gamma)^\perp.$$

Here $q \in H^1(\Gamma)$ is uniquely determined.

5.10 Singular limit problem on degeneration of a thin domain

In this section we study a singular limit problem for the Navier–Stokes equations (5.1.1)–(5.1.3) as the curved thin domain Ω_ε degenerates into the closed surface Γ. Our goal is to derive the limit system on Γ of (5.1.1)–(5.1.3) and compare it with the bulk system.

Throughout this section we impose Assumptions 1 and 2 and let ε_1 and ε_σ be the positive constants given in Theorem 5.1.2 and Lemma 5.5.1, respectively. We assume that $\varepsilon \in (0, \varepsilon_1')$ with $\varepsilon_1' := \min\{\varepsilon_1, \varepsilon_\sigma\}$ and the assumptions in Theorem 5.8.4 are satisfied. Also, we denote by $\bar{\eta} = \eta \circ \pi$ the constant extension of a function η on Γ to the normal direction of Γ.

5.10.1 Weak formulation for the bulk system

Our ansatz is a weak formulation for (5.1.1)–(5.1.3) satisfied by a strong solution. Let

$$u^\varepsilon \in C([0,\infty); V_\varepsilon) \cap L^2_{loc}([0,\infty); D(A_\varepsilon)) \cap H^1_{loc}([0,\infty); L^2_\sigma(\Omega_\varepsilon))$$

be the global strong solution to (5.1.1)–(5.1.3) given by Theorem 5.8.4. It satisfies

$$\int_0^T \{(\partial_t u^\varepsilon, \varphi)_{L^2(\Omega_\varepsilon)} + a_\varepsilon(u^\varepsilon, \varphi) + b_\varepsilon(u^\varepsilon, u^\varepsilon, \varphi)\}\, dt = \int_0^T (\mathbb{P}_\varepsilon f^\varepsilon, \varphi)_{L^2(\Omega_\varepsilon)}\, dt \tag{5.10.1}$$

for all $T > 0$ and $\varphi \in L^2(0,T; V_\varepsilon)$, and $u^\varepsilon|_{t=0} = u_0^\varepsilon$ in V_ε. Here a_ε is the bilinear form given by (5.5.14), i.e.

$$a_\varepsilon(u_1, u_2) = 2\nu \int_{\Omega_\varepsilon} D(u_1) : D(u_2)\, dx + \sum_{i=0,1} \gamma_\varepsilon^i \int_{\Gamma_\varepsilon^i} u_1 \cdot u_2\, d\mathcal{H}^2$$

for $u_1, u_2 \in H^1(\Omega_\varepsilon)^3$ and b_ε is a trilinear form defined by

$$b_\varepsilon(u_1, u_2, u_3) := -\int_{\Omega_\varepsilon} u_1 \otimes u_2 : \nabla u_3\, dx \tag{5.10.2}$$

for $u_1, u_2, u_3 \in H^1(\Omega_\varepsilon)^3$. Note that, if $u_1 \in V_\varepsilon$, then we have

$$\int_{\Omega_\varepsilon} (u_1 \cdot \nabla) u_2 \cdot u_3\, dx = b(u_1, u_2, u_3)$$

by the integration by parts formula

$$\int_{\Omega_\varepsilon} (u_1 \cdot \nabla) u_2 \cdot u_3\, dx = \int_{\Gamma_\varepsilon} (u_1 \cdot n_\varepsilon)(u_2 \cdot u_3)\, d\mathcal{H}^2 - \int_{\Omega_\varepsilon} \{(\operatorname{div} u_1)(u_2 \cdot u_3) + u_1 \otimes u_2 : \nabla u_3\}\, dx$$

and the conditions $\operatorname{div} u_1 = 0$ in Ω_ε and $u_1 \cdot n_\varepsilon = 0$ on Γ_ε.

Our goal is to derive the limit of the weak formulation (5.10.1) as well as to show the convergence of the average of the strong solution u^ε as $\varepsilon \to 0$.

5.10.2 Average of the weak formulation

The first step is to derive a weak formulation satisfied by the averaged tangential component of the strong solution u^ε, in which we take a test function from the weighted solenoidal space

$$V_g := H^1_{g\sigma}(\Gamma, T\Gamma) = \{v \in H^1(\Gamma, T\Gamma) \mid \operatorname{div}_\Gamma(gv) = 0 \text{ on } \Gamma\}.$$

Since the constant extension of a vector field in V_g is not in V_ε, we need to construct an appropriate test function in V_ε from a weighted solenoidal vector field on Γ. To this end, we use the impermeable extension operator E_ε given by (5.3.42) and the Helmholtz–Leray projection $\mathbb{P}_\varepsilon$ onto $L^2_\sigma(\Omega_\varepsilon)$.

Lemma 5.10.1. *For $\eta \in V_g$ let $\eta_\varepsilon := \mathbb{P}_\varepsilon E_\varepsilon \eta$. Then $\eta_\varepsilon \in V_\varepsilon$ and*

$$\|\eta_\varepsilon - \bar{\eta}\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla \eta_\varepsilon - \overline{F(\eta)}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{3/2}\|\eta\|_{H^1(\Gamma)}, \tag{5.10.3}$$

$$\|\eta_\varepsilon - \bar{\eta}\|_{L^2(\Gamma_\varepsilon)} \leq c\varepsilon\|\eta\|_{H^1(\Gamma)}, \tag{5.10.4}$$

where $c > 0$ is a constant independent of ε and η, and

$$F(\eta) := \nabla_\Gamma \eta + \frac{1}{g}(\eta \cdot \nabla_\Gamma g)Q \quad \text{on} \quad \Gamma. \tag{5.10.5}$$

Proof. Since $\eta \in H^1(\Gamma)^3$, we have $E_\varepsilon \eta \in H^1(\Omega_\varepsilon)^3$ by Lemma 5.3.12 and thus $\eta_\varepsilon \in V_\varepsilon$. Let us derive the estimates (5.10.3) and (5.10.4). By the definition (5.3.42) of the extension $E_\varepsilon \eta$ and the inequalities (5.2.53) and (5.3.37) we have

$$\|E_\varepsilon \eta - \bar{\eta}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon\|\bar{\eta}\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{3/2}\|\eta\|_{L^2(\Gamma)}. \tag{5.10.6}$$

Also, from (5.2.53) and (5.3.44) it follows that

$$\left\|\nabla E_\varepsilon \eta - \overline{F(\eta)}\right\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon \left(\|\bar{\eta}\|_{L^2(\Omega_\varepsilon)} + \left\|\overline{\nabla_\Gamma \eta}\right\|_{L^2(\Omega_\varepsilon)}\right) \leq c\varepsilon^{3/2}\|\eta\|_{H^1(\Gamma)}. \tag{5.10.7}$$

Since $E_\varepsilon \eta \in H^1(\Omega_\varepsilon)^3$ satisfies $E_\varepsilon \eta \cdot n_\varepsilon = 0$ on Γ_ε (see Lemma 5.3.11), we can apply (5.5.1) to $u = E_\varepsilon \eta$ and $\mathbb{P}_\varepsilon u = \eta_\varepsilon$ to get

$$\|\eta_\varepsilon - E_\varepsilon \eta\|_{H^1(\Omega_\varepsilon)} \leq c\|\mathrm{div}(E_\varepsilon \eta)\|_{L^2(\Omega_\varepsilon)}.$$

Moreover, noting that $\eta \in V_g$ satisfies $\mathrm{div}_\Gamma(gv) = 0$ on Γ, we use (5.3.48) to the right-hand side of the above inequality to observe that

$$\|\eta_\varepsilon - E_\varepsilon \eta\|_{H^1(\Omega_\varepsilon)} \leq c\varepsilon^{3/2}\|\eta\|_{H^1(\Gamma)}. \tag{5.10.8}$$

By (5.10.6)–(5.10.8) we obtain (5.10.3). To prove (5.10.4) we use (5.3.8), (5.10.3), and $\partial_n \bar{\eta} = 0$ in Ω_ε to get

$$\begin{aligned}
\|\eta_\varepsilon - \bar{\eta}\|_{L^2(\Gamma_\varepsilon)} &\leq c\left(\varepsilon^{-1/2}\|\eta_\varepsilon - \bar{\eta}\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{1/2}\|\partial_n \eta_\varepsilon - \partial_n \bar{\eta}\|_{L^2(\Omega_\varepsilon)}\right) \\
&\leq c\left(\varepsilon\|\eta\|_{H^1(\Gamma)} + \varepsilon^{1/2}\|\eta_\varepsilon\|_{H^1(\Omega_\varepsilon)}\right).
\end{aligned}$$

(Recall that $\partial_n = \bar{n} \cdot \nabla$ is the directional derivative in the normal direction of Γ). Moreover, from (5.3.43) and (5.10.8) we deduce that

$$\|\eta_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq \|E_\varepsilon \eta\|_{H^1(\Omega_\varepsilon)} + \|\eta_\varepsilon - E_\varepsilon \eta\|_{H^1(\Omega_\varepsilon)} \leq c\varepsilon^{1/2}\|\eta\|_{H^1(\Gamma)}.$$

Therefore, the inequality (5.10.4) follows. □

Next we approximate the bilinear and trilinear forms a_ε and b_ε by bilinear and trilinear forms for tangential vector fields on Γ. Let γ^0 and γ^1 be nonnegative constants. For $v_1, v_2 \in H^1(\Gamma, T\Gamma)$ we define

$$\begin{aligned}
a_g(v_1, v_2) := 2\nu \int_\Gamma \left\{ g D_\Gamma(v_1) : D_\Gamma(v_2) + \frac{1}{g}(v_1 \cdot \nabla_\Gamma g)(v_2 \cdot \nabla_\Gamma g) \right\} d\mathcal{H}^2 \\
+ (\gamma^0 + \gamma^1) \int_\Gamma v_1 \cdot v_2 \, d\mathcal{H}^2,
\end{aligned} \tag{5.10.9}$$

where $D_\Gamma(v_1)$ is the surface strain rate tensor given by (5.4.38). Also, we set

$$b_g(v_1,v_2,v_3) := -\int_\Gamma g(v_1\otimes v_2) : \nabla_\Gamma v_3\,d\mathcal{H}^2 \tag{5.10.10}$$

for $v_1,v_2,v_3\in H^1(\Gamma,T\Gamma)$. Let us give their basic properties.

Lemma 5.10.2. *There exists a constant $c>0$ such that*

$$\|\nabla_\Gamma v\|^2_{L^2(\Gamma)} \le c\left\{a_g(v,v)+\|v\|^2_{L^2(\Gamma)}\right\} \tag{5.10.11}$$

for all $v\in H^1(\Gamma,T\Gamma)$.

Proof. By (5.2.30) and the Korn inequality (5.4.37) we have

$$\begin{aligned}\|\nabla_\Gamma v\|^2_{L^2(\Gamma)} &\le c\left(\|D_\Gamma(v)\|^2_{L^2(\Gamma)}+\|v\|^2_{L^2(\Gamma)}\right) \le c\left(2\nu\|g^{1/2}D_\Gamma(v)\|^2_{L^2(\Gamma)}+\|v\|^2_{L^2(\Gamma)}\right)\\ &\le c\left\{a_g(v,v)+\|v\|^2_{L^2(\Gamma)}\right\}\end{aligned}$$

for all $v\in H^1(\Gamma,T\Gamma)$. Hence (5.10.11) holds. $\square$

Lemma 5.10.3. *There exists a constant $c>0$ such that*

$$|b_g(v_1,v_2,v_3)| \le c\|v_1\|^{1/2}_{L^2(\Gamma)}\|v_1\|^{1/2}_{H^1(\Gamma)}\|v_2\|^{1/2}_{L^2(\Gamma)}\|v_2\|^{1/2}_{H^1(\Gamma)}\|v_3\|_{H^1(\Gamma)} \tag{5.10.12}$$

for all $v_1,v_2,v_3\in H^1(\Gamma,T\Gamma)$. Moreover,

$$b_g(v_1,v_2,v_3) = -b_g(v_1,v_3,v_2), \quad b_g(v_1,v_2,v_2)=0 \tag{5.10.13}$$

for all $v_1\in V_g$ and $v_2,v_3\in H^1(\Gamma,T\Gamma)$.

Proof. The inequality (5.10.12) follows from Hölder's inequality and Ladyzhenskaya's inequality (5.3.1). Let $v_1\in V_g$ and $v_2,v_3\in H^1(\Gamma,T\Gamma)$. For $a\in\mathbb{R}^3$ and $i=1,2,3$ we denote by a^i the i-th component of a. Since

$$\begin{aligned}g(v_1\otimes v_2):\nabla_\Gamma v_3 &= \sum_{i,j=1}^3 g v_1^i v_2^j \underline{D}_i v_3^j = \sum_{i,j=1}^3 \{\underline{D}_i(g v_1^i v_2^j v_3^j) - v_2^j v_3^j \underline{D}_i(g v_1^i) - g v_1^i v_3^j \underline{D}_i v_2^j\}\\ &= \mathrm{div}_\Gamma[g(v_2\cdot v_3)v_1] - (v_2\cdot v_3)\mathrm{div}_\Gamma(g v_1) - v_1\otimes v_3 : \nabla_\Gamma v_2\end{aligned}$$

on Γ and $\mathrm{div}_\Gamma(gv_1)=0$ by $v_1\in V_g$, we have

$$b_g(v_1,v_2,v_3) = \int_\Gamma \mathrm{div}_\Gamma[g(v_2\cdot v_3)v_1]\,d\mathcal{H}^2 - b_g(v_1,v_3,v_2).$$

Here the first term on the right-hand side vanishes by the Stokes theorem, since $g(v_2\cdot v_3)v_1$ is tangential on the closed surface Γ. Hence the first equality (5.10.13) follows. We also get the second equality by setting $v_2=v_3$ in the first one. $\square$

Now let us approximate the bilinear and trilinear forms a_ε and b_ε by a_g and b_g.

Lemma 5.10.4. *Let $u \in H^2(\Omega_\varepsilon)^3$ satisfy the slip boundary conditions* (5.3.20)–(5.3.21) *on Γ_ε. Also, let $\eta \in V_g$ and $\eta_\varepsilon := \mathbb{P}_\varepsilon E_\varepsilon \eta$. Then*

$$|a_\varepsilon(u,\eta_\varepsilon) - \varepsilon a_g(M_\tau u,\eta)| \leq cR_\varepsilon^a(u)\|\eta\|_{H^1(\Gamma)}, \tag{5.10.14}$$

where $c>0$ is a constant independent of ε, u, and η, and

$$R_\varepsilon^a(u) := \varepsilon^{3/2}\|u\|_{H^2(\Omega_\varepsilon)} + \varepsilon^{1/2}\|u\|_{L^2(\Omega_\varepsilon)}\sum_{i=0,1}|\varepsilon^{-1}\gamma_\varepsilon^i - \gamma^i|. \tag{5.10.15}$$

Proof. Let $F(\eta)$ be the matrix given by (5.10.5) and

$$J_1 := \bigl(D(u), D(\eta_\varepsilon)\bigr)_{L^2(\Omega_\varepsilon)} - \Bigl(D(u), \overline{F(\eta)}\Bigr)_{L^2(\Omega_\varepsilon)},$$
$$J_2 := \Bigl(D(u), \overline{F(\eta)}\Bigr)_{L^2(\Omega_\varepsilon)} - \varepsilon\Bigl\{\bigl(gD_\Gamma(M_\tau u), D_\Gamma(\eta)\bigr)_{L^2(\Gamma)} + (M_\tau u\cdot\nabla_\Gamma g, g^{-1}\eta\cdot\nabla_\Gamma g)_{L^2(\Gamma)}\Bigr\}.$$

We also define

$$K_1 := \sum_{i=0,1}\gamma_\varepsilon^i\{(u,\eta_\varepsilon)_{L^2(\Gamma_\varepsilon^i)} - (u,\bar\eta)_{L^2(\Gamma_\varepsilon^i)}\},$$
$$K_2 := \sum_{i=0,1}\gamma_\varepsilon^i\{(u,\bar\eta)_{L^2(\Gamma_\varepsilon^i)} - (M_\tau u,\eta)_{L^2(\Gamma)}\}, \quad K_3 := \sum_{i=0,1}(\gamma_\varepsilon^i - \varepsilon\gamma^i)(M_\tau u,\eta)_{L^2(\Gamma)}$$

so that

$$a_\varepsilon(u,\eta_\varepsilon) - \varepsilon a_g(M_\tau u,\eta) = 2\nu(J_1+J_2) + K_1 + K_2 + K_3.$$

Let us estimate each term on the right-hand side. Since $D(u)$ is symmetric,

$$D(u) : D(\eta_\varepsilon) = D(u) : \nabla\eta_\varepsilon \quad \text{in} \quad \Omega_\varepsilon.$$

By this equality and (5.10.3) we have

$$|J_1| \leq \|D(u)\|_{L^2(\Omega_\varepsilon)}\Bigl\|\nabla\eta_\varepsilon - \overline{F(\eta)}\Bigr\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{H^1(\Gamma)}. \tag{5.10.16}$$

Next we deal with J_2. Since $I_3 = P + Q$, $\nabla_\Gamma\eta = P\nabla_\Gamma\eta$, and $\eta\cdot n = 0$,

$$\nabla_\Gamma\eta = (\nabla_\Gamma\eta)P + (\nabla_\Gamma\eta)Q = P(\nabla_\Gamma\eta)P + \{(\nabla_\Gamma\eta)n\}\otimes n,$$
$$(\nabla_\Gamma\eta)n = \nabla_\Gamma(\eta\cdot n) - (\nabla_\Gamma n)\eta = W\eta$$

on Γ. From these relations we deduce that $F(\eta) = A + v\otimes n + \xi Q$ on Γ, where

$$A := P(\nabla_\Gamma\eta)P, \quad v := W\eta, \quad \xi := g^{-1}\eta\cdot\nabla_\Gamma g.$$

Moreover, by the symmetry of $D_\Gamma(M_\tau u)$ we see that

$$D_\Gamma(M_\tau u) : D_\Gamma(\eta) = D_\Gamma(M_\tau u) : P(\nabla_\Gamma\eta)P = D_\Gamma(M_\tau u) : A$$

on Γ. Hence we have $J_2 = J_2^1 + J_2^2 + J_2^3$, where

$$J_2^1 := \Bigl(D(u), \overline{A}\Bigr)_{L^2(\Omega_\varepsilon)} - \varepsilon(gD_\Gamma(M_\tau u), A)_{L^2(\Gamma)},$$
$$J_2^2 := \Bigl(D(u), \overline{\xi Q}\Bigr)_{L^2(\Omega_\varepsilon)} - \varepsilon(M_\tau u\cdot\nabla_\Gamma g, \xi)_{L^2(\Gamma)}, \quad J_2^3 := (D(u), \bar v\otimes\bar n)_{L^2(\Omega_\varepsilon)}.$$

Since u and A satisfies the conditions in Lemma 5.6.26, we can apply (5.6.72) to J_2^1 to get

$$|J_2^1| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|A\|_{L^2(\Gamma)} \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{H^1(\Gamma)}.$$

Also, since u satisfies $u \cdot n_\varepsilon = 0$ on Γ_ε, we use (5.6.76) to deduce that

$$|J_2^2| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\xi\|_{L^2(\Gamma)} \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}.$$

To estimate J_2^3, we note that $v = W\eta \in H^1(\Gamma, T\Gamma)$ and u satisfies the boundary condition (5.3.21) on Γ_ε. Hence the inequality (5.6.81) implies that

$$|J_2^3| \leq c\varepsilon^{3/2}\|u\|_{H^2(\Omega_\varepsilon)}\|v\|_{L^2(\Gamma)} \leq c\varepsilon^{3/2}\|u\|_{H^2(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}.$$

From the above three estimates it follows that

$$|J_2| \leq |J_2^1| + |J_2^2| + |J_2^3| \leq c\varepsilon^{3/2}\|u\|_{H^2(\Omega_\varepsilon)}\|\eta\|_{H^1(\Gamma)}. \tag{5.10.17}$$

Now let us estimate K_1, K_2, and K_3. To K_1 we apply (5.1.6), (5.3.8), and (5.10.4) to get

$$|K_1| \leq c\varepsilon\|u\|_{L^2(\Gamma_\varepsilon)}\|\eta_\varepsilon - \bar{\eta}\|_{L^2(\Gamma_\varepsilon)} \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{H^1(\Gamma)}. \tag{5.10.18}$$

Also, since η is tangential on Γ, we have $M_\tau u \cdot \eta = Mu \cdot \eta$ on Γ and thus

$$|K_2| \leq c\varepsilon \sum_{i=0,1} \left|(u, \bar{\eta})_{L^2(\Gamma_\varepsilon^i)} - (Mu, \eta)_{L^2(\Gamma)}\right| \leq c\varepsilon^{3/2}\|u\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)} \tag{5.10.19}$$

by (5.1.6) and (5.6.70). To K_3 we just use (5.6.4) to obtain

$$\begin{aligned} |K_3| &\leq c\varepsilon^{-1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)} \sum_{i=0,1} |\gamma_\varepsilon^i - \varepsilon\gamma^i| \\ &= c\varepsilon^{1/2}\|u\|_{L^2(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)} \sum_{i=0,1} |\varepsilon^{-1}\gamma_\varepsilon^i - \gamma^i|. \end{aligned} \tag{5.10.20}$$

Finally, we deduce from (5.10.16)–(5.10.20) that

$$|a_\varepsilon(u, \eta_\varepsilon) - \varepsilon a_g(M_\tau u, \eta)| \leq c(|J_1| + |J_2| + |K_1| + |K_2| + |K_3|) \leq cR_\varepsilon^a(u)\|\eta\|_{H^1(\Gamma)},$$

where $R_\varepsilon^a(u)$ is given by (5.10.15). Hence (5.10.14) is valid. $\square$

Lemma 5.10.5. *Let $u_1 \in H^2(\Omega_\varepsilon)^3$, $u_2 \in H^1(\Omega_\varepsilon)^3$, $\eta \in V_g$, and $\eta_\varepsilon := \mathbb{P}_\varepsilon E_\varepsilon \eta$. Suppose that u_1 satisfies $\operatorname{div} u_1 = 0$ in Ω_ε and (5.3.20)–(5.3.21) on Γ_ε and that u_2 satisfies (5.3.20) on Γ_ε^0 or on Γ_ε^1. Then*

$$|b_\varepsilon(u_1, u_2, \eta_\varepsilon) - \varepsilon b_g(M_\tau u_1, M_\tau u_2, \eta)| \leq cR_\varepsilon^b(u_1, u_2)\|\eta\|_{H^1(\Gamma)}. \tag{5.10.21}$$

Here $c > 0$ is a constant independent of ε, u_1, u_2, and η, and

$$\begin{aligned} R_\varepsilon^b(u_1, u_2) := \varepsilon^{3/2}\|u_1 \otimes u_2\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|u_1\|_{H^1(\Omega_\varepsilon)}\|u_2\|_{H^1(\Omega_\varepsilon)} \\ + \left(\varepsilon\|u_1\|_{H^2(\Omega_\varepsilon)} + \varepsilon^{1/2}\|u_1\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u_1\|_{H^2(\Omega_\varepsilon)}^{1/2}\right)\|u_2\|_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{5.10.22}$$

Proof. Let $F(\eta)$ be the matrix given by (5.10.5). By (5.10.3) we see that

$$\begin{aligned}\left|b_\varepsilon(u_1,u_2,\eta_\varepsilon)-\left(u_1\otimes u_2,\overline{F(\eta)}\right)_{L^2(\Omega_\varepsilon)}\right| &\leq \|u_1\otimes u_2\|_{L^2(\Omega_\varepsilon)}\left\|\nabla\eta_\varepsilon-\overline{F(\eta)}\right\|_{L^2(\Omega_\varepsilon)} \\ &\leq c\varepsilon^{3/2}\|u_1\otimes u_2\|_{L^2(\Omega_\varepsilon)}\|\eta\|_{H^1(\Gamma)}.\end{aligned} \tag{5.10.23}$$

Since η is tangential on Γ, we have

$$\nabla_\Gamma\eta=(\nabla_\Gamma\eta)P+\{(\nabla_\Gamma\eta)n\}\otimes n=P(\nabla_\Gamma\eta)P+(W\eta)\otimes n \quad \text{on} \quad \Gamma$$

as in the proof of Lemma 5.10.4. Based on this formula, we decompose

$$F(\eta)=A+v\otimes n,\quad A:=P(\nabla_\Gamma\eta)P,\quad v:=W\eta+g^{-1}(\eta\cdot\nabla_\Gamma g)n \quad \text{on} \quad \Gamma.$$

Then since u_1 and A satisfies the conditions in Lemma 5.6.29, we can use (5.6.82) to get

$$\begin{aligned}\left|\left(u_1\otimes u_2,\overline{A}\right)_{L^2(\Omega_\varepsilon)}-\varepsilon(g(M_\tau u_1)\otimes(M_\tau u_2),A)_{L^2(\Gamma)}\right| &\leq cR_\varepsilon(u_1,u_2)\|A\|_{L^2(\Gamma)} \\ &\leq cR_\varepsilon(u_1,u_2)\|\eta\|_{H^1(\Gamma)},\end{aligned} \tag{5.10.24}$$

where $R_\varepsilon(u_1,u_2)$ is given by (5.6.83). Also, since $v\in H^1(\Gamma)^3$ and u_2 satisfies (5.3.20) on Γ_ε^0 or on Γ_ε^1, the inequality (5.6.87) yields that

$$\begin{aligned}|(u_1\otimes u_2,\bar{v}\otimes\bar{n})_{L^2(\Omega_\varepsilon)}| &\leq c\varepsilon\|u_1\|_{H^1(\Omega_\varepsilon)}\|u_2\|_{H^1(\Omega_\varepsilon)}\|v\|_{H^1(\Gamma)} \\ &\leq c\varepsilon\|u_1\|_{H^1(\Omega_\varepsilon)}\|u_2\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{H^1(\Gamma)}.\end{aligned} \tag{5.10.25}$$

Noting that $F(\eta)=A+v\otimes n$ on Γ and $R_\varepsilon(u_1,u_2)$ is of the form (5.6.83), we combine (5.10.23), (5.10.24), and (5.10.25) to obtain

$$\left|b_\varepsilon(u_1,u_2,\eta_\varepsilon)-\varepsilon(g(M_\tau u_1)\otimes(M_\tau u_2),A)_{L^2(\Gamma)}\right|\leq cR_\varepsilon^b(u_1,u_2)\|\eta\|_{H^1(\Gamma)}, \tag{5.10.26}$$

where $R_\varepsilon^b(u_1,u_2)$ is given by (5.10.22). Finally, we observe that

$$(M_\tau u_1)\otimes(M_\tau u_2):A=(M_\tau u_1)\otimes(M_\tau u_2):\nabla_\Gamma\eta \quad \text{on} \quad \Gamma$$

by $A=P(\nabla_\Gamma\eta)P$ and the fact that $M_\tau u_1$ and $M_\tau u_2$ are tangential on Γ. Therefore,

$$(g(M_\tau u_1)\otimes(M_\tau u_2),A)_{L^2(\Gamma)}=b_g(M_\tau u_1,M_\tau u_2,\eta)$$

and the inequality (5.10.21) follows from (5.10.26). □

Now let us derive a weak formulation for the averaged tangential component of u^ε from (5.10.1).

Lemma 5.10.6. *Suppose that the assumptions in Theorem 5.8.4 hold. For $\varepsilon\in(0,\varepsilon_1')$ let u^ε be the global strong solution to* (5.1.1)–(5.1.3) *given by Theorem 5.8.4. Then*

$$M_\tau u^\varepsilon\in C([0,\infty);H^1(\Gamma,T\Gamma))\cap H^1_{loc}([0,\infty);L^2(\Gamma,T\Gamma))$$

and for all $\eta\in L^2(0,T;V_g)$, $T>0$ we have

$$\begin{aligned}\int_0^T\{(g\partial_t M_\tau u^\varepsilon,\eta)_{L^2(\Gamma)}+a_g(M_\tau u,\eta)+b_g(M_\tau u,M_\tau u,\eta)\}\,dt \\ =\int_0^T(gM_\tau\mathbb{P}_\varepsilon f^\varepsilon,\eta)_{L^2(\Gamma)}\,dt+R_\varepsilon^1(\eta).\end{aligned} \tag{5.10.27}$$

Here the residual term $R_\varepsilon^1(\eta)$ *satisfies*

$$|R_\varepsilon^1(\eta)| \leq c\left(\varepsilon^{\alpha/4}+\sum_{i=0,1}|\varepsilon^{-1}\gamma_\varepsilon^i-\gamma^i|\right)(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \tag{5.10.28}$$

with a constant $c>0$ *independent of* ε, u^ε, η, *and* T.

Proof. The space-time regularity of $M_\tau u^\varepsilon$ follows from that of u^ε and Lemmas 5.6.2, 5.6.6, and 5.6.9. Let us show that $M_\tau u^\varepsilon$ satisfies (5.10.27). For $\eta \in L^2(0,T;V_g)$ we set $\eta_\varepsilon := \mathbb{P}_\varepsilon E_\varepsilon \eta$. Then $\eta_\varepsilon \in L^2(0,T;V_\varepsilon)$ by Lemma 5.10.1 and thus we can substitute it for φ in (5.10.1):

$$\int_0^T \{(\partial_t u^\varepsilon,\eta_\varepsilon)_{L^2(\Omega_\varepsilon)}+a_\varepsilon(u^\varepsilon,\eta_\varepsilon)+b_\varepsilon(u^\varepsilon,u^\varepsilon,\eta_\varepsilon)\}\,dt = \int_0^T (\mathbb{P}_\varepsilon f^\varepsilon,\eta_\varepsilon)_{L^2(\Omega_\varepsilon)}\,dt.$$

We divide both sides of this equality by ε and replace each term by the corresponding term of (5.10.27). Then we get (5.10.27) with $R_\varepsilon^1(\eta) := \varepsilon^{-1}(I_1+I_2+I_3+I_4)$, where

$$\begin{aligned}
I_1 &:= \int_0^T (\partial_t u^\varepsilon,\eta_\varepsilon)_{L^2(\Omega_\varepsilon)}\,dt - \varepsilon\int_0^T (g\partial_t M_\tau u^\varepsilon,\eta)_{L^2(\Gamma)}\,dt,\\
I_2 &:= \int_0^T a_\varepsilon(u^\varepsilon,\eta_\varepsilon)\,dt - \varepsilon\int_0^T a_g(M_\tau u,\eta)\,dt,\\
I_3 &:= \int_0^T b_\varepsilon(u^\varepsilon,u^\varepsilon,\eta_\varepsilon)\,dt - \varepsilon\int_0^T b_g(M_\tau u^\varepsilon,M_\tau u^\varepsilon,\eta)\,dt,\\
I_4 &:= \int_0^T (\mathbb{P}_\varepsilon f^\varepsilon,\eta_\varepsilon)_{L^2(\Omega_\varepsilon)}\,dt - \varepsilon\int_0^T (gM_\tau\mathbb{P}_\varepsilon f^\varepsilon,\eta)_{L^2(\Gamma)}\,dt.
\end{aligned}$$

Let us estimate these differences. First note that

$$(g\partial_t M_\tau u^\varepsilon,\eta)_{L^2(\Gamma)} = (gM_\tau(\partial_t u^\varepsilon),\eta)_{L^2(\Gamma)} = (gM(\partial_t u^\varepsilon),\eta)_{L^2(\Gamma)}$$

by Lemma 5.6.6 and the fact that η is tangential on Γ. Thus, by (5.6.69) and (5.10.4),

$$\begin{aligned}
&|(\partial_t u^\varepsilon,\eta_\varepsilon)_{L^2(\Omega_\varepsilon)} - \varepsilon(g\partial_t M_\tau u^\varepsilon,\eta)_{L^2(\Gamma)}|\\
&\quad \leq |(\partial_t u^\varepsilon,\bar\eta)_{L^2(\Omega_\varepsilon)} - \varepsilon(gM(\partial_t u^\varepsilon),\eta)_{L^2(\Gamma)}| + \|\partial_t u^\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\eta_\varepsilon-\bar\eta\|_{L^2(\Omega_\varepsilon)}\\
&\quad \leq c\varepsilon^{3/2}\|\partial_t u^\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\eta\|_{L^2(\Gamma)}.
\end{aligned}$$

From this inequality, Hölder's inequality, and (5.8.36) it follows that

$$\begin{aligned}
|I_1| &\leq c\varepsilon^{3/2}\|\partial_t u^\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\|\eta\|_{L^2(0,T;L^2(\Gamma))}\\
&\leq c\varepsilon^{1+\alpha/2}(1+T)^{1/2}\|\eta\|_{L^2(0,T;L^2(\Gamma))}.
\end{aligned} \tag{5.10.29}$$

In the same way, we apply (5.6.69) and (5.10.4) to I_4 and then use (5.1.10) to get

$$|I_4| \leq c\varepsilon^{1+\alpha/2}T^{1/2}\|\eta\|_{L^2(0,T;L^2(\Gamma))}. \tag{5.10.30}$$

Next we deal with I_2. By (5.10.14) we see that

$$|I_2| \leq c\left(\int_0^T R_\varepsilon^a(u^\varepsilon)^2\,dt\right)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))},$$

where $R_\varepsilon^a(u^\varepsilon)$ is given by (5.10.15). Moreover, by (5.8.34) we have

$$\begin{aligned}\int_0^T R_\varepsilon^a(u^\varepsilon)^2\,dt &\leq c\left(\varepsilon^3\int_0^T \|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}^2\,dt+\varepsilon\gamma(\varepsilon)^2\int_0^T\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\,dt\right)\\ &\leq c\varepsilon^2\{\varepsilon^\alpha+\gamma(\varepsilon)^2\}(1+T),\end{aligned}$$

where $\gamma(\varepsilon):=\sum_{i=0,1}|\varepsilon^{-1}\gamma_\varepsilon^i-\gamma^i|$. Therefore,

$$|I_2|\leq c\varepsilon\{\varepsilon^{\alpha/2}+\gamma(\varepsilon)\}(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))}. \tag{5.10.31}$$

Now let us estimate I_3. By (5.10.21) we have

$$|I_3|\leq c\left(\int_0^T R_\varepsilon^b(u,u)^2\,dt\right)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))},$$

where $R_\varepsilon^b(u,u)$ is given by (5.10.22). To estimate the right-hand side, we see that

$$\int_0^T\|u\|_{H^1(\Omega_\varepsilon)}^4\,dt\leq\left(\sup_{t\in(0,T)}\|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon)}^2\right)\int_0^T\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}^2\,dt\leq c\varepsilon^\alpha(1+T)$$

by (5.8.34). Using this inequality, (5.8.35), (5.8.39), and (5.8.40) we deduce that

$$\begin{aligned}\int_0^T R_\varepsilon^b(u,u)^2\,dt\leq c\Bigg(&\varepsilon^3\int_0^T\|u^\varepsilon\otimes u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\,dt+\varepsilon^2\int_0^T\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}^4\,dt\\ &+\varepsilon^2\int_0^T\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}^2\,dt+\varepsilon\int_0^T\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^3\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}\,dt\Bigg)\\ \leq c\varepsilon^2&(\varepsilon^2+\varepsilon^\alpha+\varepsilon^{\alpha/2})(1+T)\leq c\varepsilon^{2+\alpha/2}(1+T).\end{aligned}$$

Hence we obtain

$$|I_3|\leq c\varepsilon^{1+\alpha/4}(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))}. \tag{5.10.32}$$

Finally, by (5.10.29)–(5.10.32) we observe that

$$|R_\varepsilon^1(\eta)|\leq\varepsilon^{-1}\sum_{j=1}^4|I_j|\leq c\{\varepsilon^{\alpha/4}+\varepsilon^{\alpha/2}+\gamma(\varepsilon)\}(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))}$$

and thus (5.10.28) holds by $\gamma(\varepsilon)=\sum_{i=0,1}|\varepsilon^{-1}\gamma_\varepsilon^i-\gamma^i|$ and $\varepsilon^{\alpha/2}\leq\varepsilon^{\alpha/4}$. □

Remark 5.10.7. By $u\in L_{loc}^2([0,\infty);D(A_\varepsilon))$ and Lemma 5.6.9 we also have

$$M_\tau u^\varepsilon\in L_{loc}^2([0,\infty);H^2(\Gamma,T\Gamma)).$$

We do not use this property in the sequel.

5.10.3 Energy estimate for the average of a solution

Next we prove the energy estimate for the averaged tangential component of the strong solution u^ε. We would like to take the averaged tangential component $M_\tau u^\varepsilon$ as a test function of the weak formulation (5.10.27), but it is not allowed since $M_\tau u^\varepsilon$ is not in V_g, i.e. the surface divergence of $gM_\tau u^\varepsilon$ does not vanish in general. To overcome this difficulty, we use the weighted Helmholtz–Leray projection $\mathbb{P}_g$ onto $L^2_{g\sigma}(\Gamma, T\Gamma)$ given in Section 5.9.3 and replace $M_\tau u^\varepsilon$ in (5.10.27) with $\mathbb{P}_g M_\tau u^\varepsilon$.

Lemma 5.10.8. *Let $u \in L^2_\sigma(\Omega_\varepsilon)$. Then $\mathbb{P}_g M_\tau u \in L^2_{g\sigma}(\Gamma, T\Gamma)$ and*

$$\|M_\tau u - \mathbb{P}_g M_\tau u\|_{L^2(\Gamma)} \leq c\varepsilon^{1/2}\|u\|_{L^2(\Omega_\varepsilon)}, \tag{5.10.33}$$

where $c > 0$ is a constant independent of ε. Also, if $u \in V_\varepsilon$, then $\mathbb{P}_g M_\tau u \in V_g$ and

$$\|M_\tau u - \mathbb{P}_g M_\tau u\|_{H^1(\Gamma)} \leq c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}. \tag{5.10.34}$$

Proof. Let $u \in L^2_\sigma(\Omega_\varepsilon)$. Since $M_\tau u \in L^2(\Gamma, T\Gamma)$ by Lemma 5.6.2, we have $\mathbb{P}_g M_\tau u \in L^2_{g\sigma}(\Gamma, T\Gamma)$. Moreover, from (5.6.49) and (5.9.28) it follows that

$$\|M_\tau u - \mathbb{P}_g M_\tau u\|_{L^2(\Gamma)} \leq c\|\mathrm{div}_\Gamma(gM_\tau u)\|_{H^{-1}(\Gamma)} \leq c\varepsilon^{1/2}\|u\|_{L^2(\Omega_\varepsilon)}.$$

Hence (5.10.33) is valid. If $u \in V_\varepsilon$, then $M_\tau u \in H^1(\Gamma, T\Gamma)$ by Lemma 5.6.9 and thus Lemma 5.9.14 shows that $\mathbb{P}_g M_\tau u \in V_g$. Also, we deduce from (5.6.46) and (5.9.29) that

$$\|M_\tau u - \mathbb{P}_g M_\tau u\|_{H^1(\Gamma)} \leq c\|\mathrm{div}_\Gamma(gM_\tau u)\|_{L^2(\Gamma)} \leq c\varepsilon^{1/2}\|u\|_{H^1(\Omega_\varepsilon)}.$$

Thus, the inequality (5.10.34) holds. □

Lemma 5.10.9. *For $T > 0$, let $u \in H^1(0, T; L^2_\sigma(\Omega_\varepsilon))$. Then*

$$\mathbb{P}_g M_\tau u \in H^1(0, T; L^2_{g\sigma}(\Gamma, T\Gamma))$$

and there exists a constant $c > 0$ independent of ε and u such that

$$\|\partial_t M_\tau u - \partial_t \mathbb{P}_g M_\tau u\|_{L^2(0,T;L^2(\Gamma))} \leq c\varepsilon^{1/2}\|\partial_t u\|_{L^2(0,T;L^2(\Omega_\varepsilon))}. \tag{5.10.35}$$

Proof. By the space-time regularity of u and Lemmas 5.6.2 and 5.6.6 we see that $M_\tau u$ is in $H^1(0, T; L^2(\Gamma, T\Gamma))$. Hence $\mathbb{P}_g M_\tau u \in H^1(0, T; L^2_{g\sigma}(\Gamma, T\Gamma))$ and

$$\|\partial_t M_\tau u - \partial_t \mathbb{P}_g M_\tau u\|_{L^2(0,T;L^2(\Gamma))} \leq c\|\mathrm{div}_\Gamma(g\partial_t M_\tau u)\|_{L^2(0,T;H^{-1}(\Gamma))}$$

by Lemma 5.9.17. Since $\partial_t M_\tau u = M_\tau(\partial_t u)$ in $L^2(0, T; L^2(\Gamma, T\Gamma))$ by Lemma 5.6.6, we further observe by $\partial_t u \in L^2(0, T; L^2_\sigma(\Omega_\varepsilon))$ and (5.6.49) that

$$\|\mathrm{div}_\Gamma(g\partial_t M_\tau u)\|_{L^2(0,T;H^{-1}(\Gamma))} = \|\mathrm{div}_\Gamma[gM_\tau(\partial_t u)]\|_{L^2(0,T;H^{-1}(\Gamma))} \leq c\varepsilon^{1/2}\|\partial_t u\|_{L^2(0,T;L^2(\Omega_\varepsilon))}.$$

Combining the above two inequalities we obtain (5.10.35). □

Using Lemmas 5.10.8 and 5.10.9 we derive a weak formulation for $\mathbb{P}_g M_\tau u$.

Lemma 5.10.10. *Let u^ε be as in Lemma 5.10.6. Then*

$$v^\varepsilon := \mathbb{P}_g M_\tau u^\varepsilon \in C([0,\infty);V_g) \cap H^1_{loc}([0,\infty), L^2_{g\sigma}(\Gamma, T\Gamma))$$

and there exists a constant $c > 0$ independent of ε and u^ε such that

$$\|M_\tau u^\varepsilon(t) - v^\varepsilon(t)\|^2_{L^2(\Gamma)} \leq c\varepsilon^2, \quad \int_0^t \|M_\tau u^\varepsilon(s) - v^\varepsilon(s)\|^2_{H^1(\Gamma)}\,ds \leq c\varepsilon^2(1+t) \tag{5.10.36}$$

for all $t \geq 0$. Moreover, for all $\eta \in L^2(0,T;V_g)$, $T > 0$ we have

$$\begin{aligned}\int_0^T \{(g\partial_t v^\varepsilon, \eta)_{L^2(\Gamma)} + a_g(v^\varepsilon,\eta) + b_g(v^\varepsilon, v^\varepsilon, \eta)\}\,dt& \\ = \int_0^T (gM_\tau \mathbb{P}_\varepsilon f^\varepsilon, \eta)_{L^2(\Gamma)}\,dt + R^1_\varepsilon(\eta) + R^2_\varepsilon(\eta),&\end{aligned} \tag{5.10.37}$$

where $R^1_\varepsilon(\eta)$ is given in Lemma 5.10.6 and $R^2_\varepsilon(\eta)$ satisfies

$$|R^2_\varepsilon(\eta)| \leq c\varepsilon^{\alpha/2}(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \tag{5.10.38}$$

with a constant $c > 0$ independent of ε, v^ε, η, and T.

Proof. By Lemmas 5.10.6, 5.10.8, and 5.10.9 we obtain the space-time regularity of v^ε. Also, the inequalities (5.10.36) immediately follow from (5.8.34), (5.10.33), and (5.10.34). For $\eta \in L^2(0,T;V_g)$ let

$$\begin{aligned}I_1 &:= \int_0^T (g\partial_t M_\tau u^\varepsilon, \eta)_{L^2(\Gamma)}\,dt - \int_0^T (g\partial_t v^\varepsilon, \eta)_{L^2(\Gamma)}\,dt, \\ I_2 &:= \int_0^T a_g(M_\tau u^\varepsilon, \eta)\,dt - \int_0^T a_g(v^\varepsilon,\eta)\,dt, \\ I_3 &:= \int_0^T b_g(M_\tau u^\varepsilon, M_\tau u^\varepsilon, \eta)\,dt - \int_0^T b_g(v^\varepsilon, v^\varepsilon, \eta)\,dt.\end{aligned}$$

Then by (5.10.27) we obtain (5.10.37) with $R^2_\varepsilon(\eta) := I_1 + I_2 + I_3$. Let us estimate I_1, I_2, and I_3. By (5.8.36) and (5.10.35) we have

$$\begin{aligned}|I_1| &\leq c\|\partial_t M_\tau u^\varepsilon - \partial_t v^\varepsilon\|_{L^2(0,T;L^2(\Gamma))}\|\eta\|_{L^2(0,T;L^2(\Gamma))} \\ &\leq c\varepsilon^{1/2}\|\partial_t u^\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\|\eta\|_{L^2(0,T;L^2(\Gamma))} \\ &\leq c(1+T)^{1/2}\varepsilon^{\alpha/2}\|\eta\|_{L^2(0,T;L^2(\Gamma))}.\end{aligned} \tag{5.10.39}$$

Also, we use (5.2.30) and (5.10.36) to get

$$|I_2| \leq c\|M_\tau u^\varepsilon - v^\varepsilon\|_{L^2(0,T;H^1(\Gamma))}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \leq c(1+T)^{1/2}\varepsilon\|\eta\|_{L^2(0,T;H^1(\Gamma))}. \tag{5.10.40}$$

Now let us consider I_3. Using Hölder's inequality twice we have

$$\begin{aligned}&|b_g(M_\tau u^\varepsilon, M_\tau u^\varepsilon, \eta) - b_g(v^\varepsilon, v^\varepsilon, \eta)| \\ &\qquad\leq \|M_\tau u^\varepsilon - v^\varepsilon\|_{L^4(\Gamma)}\left(\|M_\tau u^\varepsilon\|_{L^4(\Gamma)} + \|v^\varepsilon\|_{L^4(\Gamma)}\right)\|\nabla_\Gamma \eta\|_{L^2(\Gamma)}.\end{aligned}$$

Moreover, by (5.3.1), (5.6.4), (5.6.22), (5.9.30), (5.10.33), and (5.10.34) we observe that

$$\begin{aligned}\|w\|_{L^4(\Gamma)} &\leq c\varepsilon^{-1/2}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}^{1/2}, \quad w = M_\tau u^\varepsilon, v^\varepsilon, \\ \|M_\tau u^\varepsilon - v^\varepsilon\|_{L^4(\Gamma)} &\leq c\varepsilon^{1/2}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^{1/2}\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}^{1/2}.\end{aligned}$$

Hence

$$|b_g(M_\tau u^\varepsilon, M_\tau u^\varepsilon, \eta) - b_g(v^\varepsilon, v^\varepsilon, \eta)| \leq c\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}\|\eta\|_{H^1(\Gamma)}$$

and we use (5.8.38) to get

$$\begin{aligned}|I_3| &\leq c\left(\int_0^T \|u^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}^2\,dt\right)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \\ &\leq c\varepsilon(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))}.\end{aligned} \tag{5.10.41}$$

By (5.10.39), (5.10.40), (5.10.41), and $\varepsilon \leq \varepsilon^{\alpha/2}$ we obtain (5.10.38). □

Based on (5.10.37) we prove the energy estimate for $v^\varepsilon = \mathbb{P}_g M_\tau u^\varepsilon$.

Lemma 5.10.11. *Let u^ε be as in Lemma 5.10.6 and $v^\varepsilon = \mathbb{P}_g M_\tau u^\varepsilon$. Then*

$$\max_{t\in[0,T]}\|v^\varepsilon(t)\|_{L^2(\Gamma)}^2 + \int_0^T \|\nabla_\Gamma v^\varepsilon(t)\|_{L^2(\Gamma)}^2\,dt \leq c_T \tag{5.10.42}$$

for all $T \geq 0$, where $c_T > 0$ is a constant depending on T and independent of ε.

Proof. For $t \in [0,T]$ let $1_{[0,t]}\colon \mathbb{R} \to \mathbb{R}$ be the characteristic function of the time interval $[0,t]$. Since $v^\varepsilon \in C([0,\infty);V_g)$, we can take $\eta := 1_{[0,t]}v^\varepsilon$ as a test function in (5.10.37). Then using (5.10.13) we obtain

$$\begin{aligned}&\int_0^t \{(g\partial_s v^\varepsilon, v^\varepsilon)_{L^2(\Gamma)} + a_g(v^\varepsilon, v^\varepsilon)\}\,ds \\ &\qquad = \int_0^t (gM_\tau\mathbb{P}_\varepsilon f^\varepsilon, v^\varepsilon)_{L^2(\Gamma)}\,ds + R_\varepsilon^1(v^\varepsilon) + R_\varepsilon^2(v^\varepsilon),\end{aligned} \tag{5.10.43}$$

where $R_\varepsilon^1(v^\varepsilon)$ and $R_\varepsilon^2(v^\varepsilon)$ satisfy (5.10.28) and (5.10.38), respectively. Since g is nonnegative (see (5.2.30)) and independent of time,

$$\begin{aligned}\int_0^t (g\partial_s v^\varepsilon, v^\varepsilon)_{L^2(\Gamma)}\,ds &= \frac{1}{2}\int_0^t \frac{d}{ds}\|g^{1/2}v^\varepsilon\|_{L^2(\Gamma)}^2\,ds \\ &= \frac{1}{2}\|g^{1/2}v^\varepsilon(t)\|_{L^2(\Gamma)}^2 - \frac{1}{2}\|g^{1/2}v^\varepsilon(0)\|_{L^2(\Gamma)}^2.\end{aligned} \tag{5.10.44}$$

Also, we see by (5.10.11) that

$$\int_0^t \|\nabla_\Gamma v^\varepsilon\|_{L^2(\Gamma)}^2\,ds \leq c\int_0^t \left\{a_g(v^\varepsilon, v^\varepsilon) + \|v^\varepsilon\|_{L^2(\Gamma)}^2\right\}\,ds. \tag{5.10.45}$$

For the right-hand side of (5.10.43), we consider $M_\tau\mathbb{P}_\varepsilon f^\varepsilon = PM_\tau\mathbb{P}_\varepsilon f^\varepsilon$ as an element of $H^{-1}(\Gamma,T\Gamma)$ (see Section 5.2.1). Then we have

$$\begin{aligned}\int_0^t(gM_\tau\mathbb{P}_\varepsilon f^\varepsilon,v^\varepsilon)_{L^2(\Gamma)}\,ds &= \int_0^t[M_\tau\mathbb{P}_\varepsilon f^\varepsilon,gv^\varepsilon]_{T\Gamma}\,ds \\ &\leq c\int_0^t\|M_\tau\mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma,T\Gamma)}\|gv^\varepsilon\|_{H^1(\Gamma)}\,ds \\ &\leq c\int_0^t\|M_\tau\mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma,T\Gamma)}\|v^\varepsilon\|_{H^1(\Gamma)}\,ds.\end{aligned} \tag{5.10.46}$$

To estimate the residual terms, we note that $\varepsilon^{-1}\gamma_\varepsilon^0$ and $\varepsilon^{-1}\gamma_\varepsilon^1$ are bounded by (5.1.6). Hence we see by (5.10.28) and (5.10.38) (with T replaced by t) that

$$|R_\varepsilon^1(v^\varepsilon)|+|R_\varepsilon^2(v^\varepsilon)|\leq c(1+t)^{1/2}\left(\int_0^t\|v^\varepsilon\|_{H^1(\Gamma)}^2\,ds\right)^{1/2}. \tag{5.10.47}$$

Now we deduce from (5.10.43)–(5.10.47) that

$$\begin{aligned}&\|g^{1/2}v^\varepsilon(t)\|_{L^2(\Gamma)}^2+\int_0^t\|\nabla_\Gamma v^\varepsilon\|_{L^2(\Gamma)}^2\,ds \\ &\qquad\leq c\left\{\|g^{1/2}v^\varepsilon(0)\|_{L^2(\Gamma)}^2+\int_0^t\left(\|v^\varepsilon\|_{L^2(\Gamma)}^2+\|M_\tau\mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma,T\Gamma)}\|v^\varepsilon\|_{H^1(\Gamma)}\right)ds\right\} \\ &\qquad\qquad+c(1+t)^{1/2}\left(\int_0^t\|v^\varepsilon\|_{H^1(\Gamma)}^2\,ds\right)^{1/2}.\end{aligned}$$

Noting that $\|v^\varepsilon\|_{H^1(\Gamma)}^2=\|v^\varepsilon\|_{L^2(\Gamma)}^2+\|\nabla_\Gamma v^\varepsilon\|_{L^2(\Gamma)}^2$, we apply Young's inequality to the last two terms of the above inequality to get

$$\begin{aligned}&\|g^{1/2}v^\varepsilon(t)\|_{L^2(\Gamma)}^2+\int_0^t\|\nabla_\Gamma v^\varepsilon\|_{L^2(\Gamma)}^2\,ds \\ &\qquad\leq c\left\{\|g^{1/2}v^\varepsilon(0)\|_{L^2(\Gamma)}^2+\int_0^t\left(\|v^\varepsilon\|_{L^2(\Gamma)}^2+\|M_\tau\mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma,T\Gamma)}^2\right)ds+1+t\right\} \\ &\qquad\qquad+\frac{1}{2}\int_0^t\|\nabla_\Gamma v^\varepsilon\|_{L^2(\Gamma)}^2\,ds.\end{aligned}$$

Then we make the integral of $\|\nabla_\Gamma v^\varepsilon\|_{L^2(\Gamma)}^2$ on the right-hand side absorbed into the left-hand side and use the inequalities (5.1.10) with $\beta=1$, (5.2.30), and

$$\|g^{1/2}v^\varepsilon(0)\|_{L^2(\Gamma)}\leq c\|v^\varepsilon(0)\|_{L^2(\Gamma)}\leq c\|M_\tau u^\varepsilon(0)\|_{L^2(\Gamma)}=c\|M_\tau u_0^\varepsilon\|_{L^2(\Gamma)}$$

by (5.9.30) with $k=0$ (note that $v^\varepsilon=\mathbb{P}_gM_\tau u^\varepsilon$) and $u^\varepsilon(0)=u_0^\varepsilon$ in V_ε to obtain

$$\|v^\varepsilon(t)\|_{L^2(\Gamma)}^2+\int_0^t\|\nabla_\Gamma v^\varepsilon\|_{L^2(\Gamma)}^2\,ds\leq c\left(1+t+\int_0^t\|v^\varepsilon\|_{L^2(\Gamma)}^2\,ds\right) \tag{5.10.48}$$

for all $t\in[0,T]$. From this inequality we deduce that

$$\|v^\varepsilon(t)\|_{L^2(\Gamma)}^2+1\leq c\left\{1+\int_0^t\left(\|v^\varepsilon\|_{L^2(\Gamma)}^2+1\right)ds\right\},\quad t\in[0,T]$$

and thus Gronwall's inequality yields

$$\|v^\varepsilon(t)\|_{L^2(\Gamma)}^2 + 1 \leq ce^{ct} \leq ce^{cT}, \quad t \in [0,T].$$

Applying this inequality to (5.10.48) with $t = T$ we also get

$$\int_0^T \|\nabla_\Gamma v^\varepsilon\|_{L^2(\Gamma)}^2 \, dt \leq c(1 + T + e^{cT}).$$

Hence we conclude that (5.10.42) holds with $c_T := c(1 + T + e^{cT})$, where $c > 0$ is a constant independent of ε and T. □

As a consequence of (5.10.36) and (5.10.42) we obtain the energy estimate for $M_\tau u^\varepsilon$.

Corollary 5.10.12. *Let u^ε be as in Lemma 5.10.6. Then*

$$\max_{t\in[0,T]} \|M_\tau u^\varepsilon(t)\|_{L^2(\Gamma)}^2 + \int_0^T \|\nabla_\Gamma M_\tau u^\varepsilon(t)\|_{L^2(\Gamma)}^2 \, dt \leq c_T \tag{5.10.49}$$

for all $T \geq 0$, where $c_T > 0$ is a constant depending on T and independent of ε.

5.10.4 Estimate for the time derivative of the average

By the energy estimate (5.10.49) we observe that (a subsequence of) $M_\tau u^\varepsilon$ converges weakly in an appropriate function space. However, to show the convergence of the trilinear term in (5.10.27) we also require the strong convergence of $M_\tau u^\varepsilon$, which is proved by the Aubin–Lions lemma. For this purpose, let us estimate the time derivative of $M_\tau u^\varepsilon$. We first construct an appropriate test function.

Lemma 5.10.13. *For each $w \in H^1(\Gamma, T\Gamma)$ there exist $\eta \in V_g$ and $q \in H^2(\Gamma)$ such that $w = g\eta + g\nabla_\Gamma q$ and*

$$\|\eta\|_{H^1(\Gamma)} \leq c\|w\|_{H^1(\Gamma)}, \tag{5.10.50}$$

where $c > 0$ is a constant independent of w.

Proof. Let $w \in H^1(\Gamma, T\Gamma)$ and $\xi := -\mathrm{div}_\Gamma w \in L^2(\Gamma)$. Since w is tangential on the closed surface Γ, the integral of ξ over Γ vanishes by the Stokes theorem. Also,

$$\|q\|_{L^2(\Gamma)} \leq c\|\nabla_\Gamma q\|_{L^2(\Gamma)} \leq c\|g^{1/2}\nabla_\Gamma q\|_{L^2(\Gamma)} \tag{5.10.51}$$

for all $q \in H^1(\Gamma)$ with $\int_\Gamma q \, d\mathcal{H}^2 = 0$ by Poincaré's inequality (5.2.20) and (5.2.30). Hence the Lax–Milgram theorem shows that the problem

$$\mathrm{div}_\Gamma(g\nabla_\Gamma q) = -\xi \quad \text{on} \quad \Gamma, \qquad \int_\Gamma q \, d\mathcal{H}^2 = 0$$

admits a unique weak solution $q \in H^1(\Gamma)$ in the sense that

$$(g\nabla_\Gamma q, \nabla_\Gamma \varphi)_{L^2(\Gamma)} = (\xi, \varphi)_{L^2(\Gamma)} \quad \text{for all} \quad \varphi \in H^1(\Gamma). \tag{5.10.52}$$

From this equality with $\varphi = q$ and (5.10.51) we deduce that

$$\|q\|_{H^1(\Gamma)} \leq c\|\xi\|_{L^2(\Gamma)} = c\|\mathrm{div}_\Gamma w\|_{L^2(\Gamma)} \leq c\|w\|_{H^1(\Gamma)}. \tag{5.10.53}$$

Moreover, replacing φ by $g^{-1}\varphi$ in (5.10.52) we get

$$(\nabla_\Gamma q, \nabla_\Gamma \varphi)_{L^2(\Gamma)} = (g^{-1}(\xi + \nabla_\Gamma g \cdot \nabla_\Gamma q), \varphi)_{L^2(\Gamma)} \quad \text{for all} \quad \varphi \in H^1(\Gamma),$$

which combined with (5.2.20) shows that q is a unique weak solution to

$$\Delta_\Gamma \psi = -g^{-1}(\xi + \nabla_\Gamma g \cdot \nabla_\Gamma q) \in L^2(\Gamma), \quad \int_\Gamma \psi \, d\mathcal{H}^2 = 0.$$

(Note that the integral of the source term over Γ vanishes by (5.10.52).) Hence Lemma 5.9.12 and the inequalities (5.2.30) and (5.10.53) imply that $q \in H^2(\Gamma)$ and

$$\|q\|_{H^2(\Gamma)} \leq c\|g^{-1}(\xi + \nabla_\Gamma g \cdot \nabla_\Gamma q)\|_{L^2(\Gamma)} \leq c\|w\|_{H^1(\Gamma)}. \tag{5.10.54}$$

Now let $\eta := g^{-1}w - \nabla_\Gamma q$ on Γ. Then by $q \in H^2(\Gamma)$ and $\mathrm{div}_\Gamma(g\nabla_\Gamma q) = \mathrm{div}_\Gamma w$ on Γ we have $\eta \in V_g$. Moreover, from (5.2.30) and (5.10.54) it follows that

$$\|\eta\|_{H^1(\Gamma)} \leq c\left(\|w\|_{H^1(\Gamma)} + \|\nabla_\Gamma q\|_{H^1(\Gamma)}\right) \leq c\|w\|_{H^1(\Gamma)}.$$

Hence we obtain $w = g\eta + g\nabla_\Gamma q$ and (5.10.50). $\square$

As in the previous section, we estimate the time derivative of v^ε and then derive an estimate for the time derivative of $M_\tau u^\varepsilon$ by using a difference estimate.

Lemma 5.10.14. *Let u^ε be as in Lemma 5.10.6 and $v^\varepsilon = \mathbb{P}_g M_\tau u^\varepsilon$. Then*

$$\|\partial_t v^\varepsilon\|_{L^2(0,T;H^{-1}(\Gamma,T\Gamma))} \leq c_T \tag{5.10.55}$$

for all $T > 0$, where $c_T > 0$ is a constant depending on T and independent of ε.

Proof. Let $w \in L^2(0,T;H^1(\Gamma,T\Gamma))$. By Lemma 5.10.13 we can take $\eta \in L^2(0,T;V_g)$ and $q \in L^2(0,T;H^2(\Gamma))$ such that $w = g\eta + g\nabla_\Gamma q$. Since $\partial_t v^\varepsilon(t) \in L^2_{g\sigma}(\Gamma,T\Gamma)$ and $g\nabla_\Gamma q(t) \in L^2_{g\sigma}(\Gamma,T\Gamma)^\perp$ for a.a. $t \in (0,T)$ by Lemmas 5.9.10 and 5.10.10,

$$\int_0^T (\partial_t v^\varepsilon, g\nabla_\Gamma q)_{L^2(\Gamma)} \, dt = 0.$$

By this equality and $g\eta = w - g\nabla_\Gamma q$ we have

$$\int_0^T (g\partial_t v^\varepsilon, \eta)_{L^2(\Gamma)} \, dt = \int_0^T (\partial_t v^\varepsilon, g\eta)_{L^2(\Gamma)} \, dt = \int_0^T (\partial_t v^\varepsilon, w)_{L^2(\Gamma)} \, dt.$$

We substitute $\eta = g^{-1}w - \nabla_\Gamma q$ for (5.10.37) and use this equality. Then

$$\begin{aligned}\int_0^T (\partial_t v^\varepsilon, w)_{L^2(\Gamma)} \, dt = -\int_0^T a_g(v^\varepsilon, \eta) \, dt - \int_0^T b_g(v^\varepsilon, v^\varepsilon, \eta) \, dt \\ + \int_0^T (gM_\tau \mathbb{P}_\varepsilon f^\varepsilon, \eta)_{L^2(\Gamma)} \, dt + R^1_\varepsilon(\eta) + R^2_\varepsilon(\eta),\end{aligned} \tag{5.10.56}$$

where $R^1_\varepsilon(\eta)$ and $R^2_\varepsilon(\eta)$ are given in Lemmas 5.10.6 and 5.10.10. To the first term on the right-hand side we apply (5.2.30), (5.10.42), and (5.10.50) to get

$$\left|\int_0^T a_g(v^\varepsilon, \eta) \, dt\right| \leq c\|v^\varepsilon\|_{L^2(0,T;H^1(\Gamma))}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \leq c_T\|w\|_{L^2(0,T;H^1(\Gamma))}.$$

Here and in what follows we denote by c_T a general positive constant depending on T and independent of ε. Also, by (5.10.12), (5.10.42), and (5.10.50) we see that

$$\begin{aligned}\left|\int_0^T b_g(v^\varepsilon, v^\varepsilon, \eta)\,dt\right| &\le c\int_0^T \|v^\varepsilon\|_{L^2(\Gamma)}\|v^\varepsilon\|_{H^1(\Gamma)}\|\eta\|_{H^1(\Gamma)}\,dt \\ &\le c\|v^\varepsilon\|_{L^\infty(0,T;L^2(\Gamma))}\|v^\varepsilon\|_{L^2(0,T;H^1(\Gamma))}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \\ &\le c_T\|w\|_{L^2(0,T;H^1(\Gamma))}.\end{aligned}$$

For the other terms we proceed as in the proof of Lemma 5.10.11 (see (5.10.46) and (5.10.47)) and use (5.1.10) with $\beta = 1$ and (5.10.50). Then we get

$$\begin{aligned}\left|\int_0^T (gM_\tau\mathbb{P}_\varepsilon f^\varepsilon, \eta)_{L^2(\Gamma)}\,dt\right| &\le c\int_0^T \|M_\tau\mathbb{P}_\varepsilon f^\varepsilon\|_{H^{-1}(\Gamma,T\Gamma)}\|\eta\|_{H^1(\Gamma)}\,dt \\ &\le cT^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \le cT^{1/2}\|w\|_{L^2(0,T;H^1(\Gamma))}\end{aligned}$$

and

$$|R_\varepsilon^1(\eta)| + |R_\varepsilon^2(\eta)| \le c(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \le c(1+T)^{1/2}\|w\|_{L^2(0,T;H^1(\Gamma))}.$$

Applying these inequalities to the right-hand side of (5.10.56) we obtain

$$\left|\int_0^T (\partial_t v^\varepsilon, w)_{L^2(\Gamma)}\,dt\right| \le c_T\|w\|_{L^2(0,T;H^1(\Gamma))}$$

for all $w \in L^2(0,T;H^1(\Gamma,T\Gamma))$ (note that w is not necessarily a weighted solenoidal vector field). Hence (5.10.55) holds. □

Corollary 5.10.15. *Let u^ε be as in Lemma 5.10.6. Then*

$$\|\partial_t M_\tau u^\varepsilon\|_{L^2(0,T;H^{-1}(\Gamma,T\Gamma))} \le c_T \tag{5.10.57}$$

for all $T > 0$, where $c_T > 0$ is a constant depending on T and independent of ε.

Proof. Let $v^\varepsilon = \mathbb{P}_g M_\tau u^\varepsilon$. Noting that $\|v\|_{H^{-1}(\Gamma,T\Gamma)} \le \|v\|_{L^2(\Gamma)}$ for $v \in L^2(\Gamma,T\Gamma)$, we see by (5.8.36) and (5.10.35) that

$$\begin{aligned}\|\partial_t M_\tau u^\varepsilon - \partial_t v^\varepsilon\|_{L^2(0,T;H^{-1}(\Gamma,T\Gamma))} &\le \varepsilon^{1/2}\|\partial_t u^\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \\ &\le c\varepsilon^{\alpha/2}(1+T)^{1/2} \le c(1+T)^{1/2}.\end{aligned}$$

Combining this inequality and (5.10.55) we obtain (5.10.57). □

Remark 5.10.16. In construction of a weak solution to the Navier–Stokes equations, we usually estimate the time derivative of an approximate solution in the dual of a solenoidal space. However, in Lemma 5.10.14 we proved an estimate for $\partial_t v^\varepsilon$ in $H^{-1}(\Gamma,T\Gamma)$, not in the dual V_g' of V_g. This is because we multiply $\partial_t v^\varepsilon$ by g in (5.10.37). When $f \in V_g'$, we can not define a functional $gf\colon v \mapsto {}_{V_g'}\langle f, gv\rangle_{V_g}$ for $v \in V_g$ since gv is not in V_g in general (here ${}_{V_g'}\langle\cdot,\cdot\rangle_{V_g}$ stands for the duality product between V_g' and V_g). Since this issue does not occur for a functional in $H^{-1}(\Gamma,T\Gamma)$ (see (5.2.27)), we consider $\partial_t v^\varepsilon$ and $\partial_t M_\tau u^\varepsilon$ in this space.

5.10.5 Weak convergence of the average and characterization of the limit

The goal of this subsection is to prove Theorem 5.1.3. We proceed as in the case of the two-dimensional Navier–Stokes equations (see e.g. [8, 10, 61, 64]). First we give a definition of a weak solution to the limit equations (5.1.13)–(5.1.14) based on (5.10.27).

Definition 5.10.17. Let $T > 0$, $v_0 \in L^2_{g\sigma}(\Gamma, T\Gamma)$, and $f \in L^2(0, T; H^{-1}(\Gamma, T\Gamma))$. We say that a vector field

$$v \in L^\infty(0, T; L^2_{g\sigma}(\Gamma, T\Gamma)) \cap L^2(0, T; V_g) \quad \text{with} \quad \partial_t v \in L^1(0, T; H^{-1}(\Gamma, T\Gamma))$$

is a weak solution to the equations (5.1.13)–(5.1.14) on $[0, T)$ if it satisfies

$$\int_0^T \{[g\partial_t v, \eta]_{T\Gamma} + a_g(v, \eta) + b_g(v, v, \eta)\}\, dt = \int_0^T [gf, \eta]_{T\Gamma}\, dt \tag{5.10.58}$$

for all $\eta \in C_c(0, T; V_g)$ and $v|_{t=0} = v_0$ in $H^{-1}(\Gamma, T\Gamma)$.

Definition 5.10.18. Let $v_0 \in L^2_{g\sigma}(\Gamma, T\Gamma)$ and $f \in L^2_{loc}([0, \infty); H^{-1}(\Gamma, T\Gamma))$. We say that v is a weak solution to (5.1.13)–(5.1.14) on $[0, \infty)$ if it is a weak solution to (5.1.13)–(5.1.14) on $[0, T)$ for all $T > 0$.

For $T > 0$, a weak solution to (5.1.13)–(5.1.14) on $[0, T)$ is continuous on $[0, T]$ with values in $H^{-1}(\Gamma, T\Gamma)$ and thus the initial condition makes sense. In fact, it becomes a continuous function with values in $L^2(\Gamma, T\Gamma)$.

Lemma 5.10.19. *Let $T > 0$ and $f \in L^2(0, T; H^{-1}(\Gamma, T\Gamma))$. Suppose that*

$$v \in L^\infty(0, T; L^2_{g\sigma}(\Gamma, T\Gamma)) \cap L^2(0, T; V_g) \quad \textit{with} \quad \partial_t v \in L^1(0, T; H^{-1}(\Gamma, T\Gamma))$$

satisfies (5.10.58) *for all $\eta \in C_c(0, T; V_g)$. Then*

$$v \in C([0, T]; L^2_{g\sigma}(\Gamma, T\Gamma)), \quad \partial_t v \in L^2(0, T; H^{-1}(\Gamma, T\Gamma)),$$

and (5.10.58) *is valid for all $\eta \in L^2(0, T; V_g)$.*

Note that here the initial condition $v|_{t=0} = v_0$ in $H^{-1}(\Gamma, T\Gamma)$ is not imposed.

Proof. We estimate $\partial_t v$ as in the proof of Lemma 5.10.14, where we used the fact that $\partial_t v^\varepsilon(t) \in L^2_{g\sigma}(\Gamma, T\Gamma)$ for a.a. $t \in (0, T)$. This is not valid for v, but we have

$$[\partial_t v(t), g\nabla_\Gamma q]_{T\Gamma} = 0 \quad \text{for all } q \in H^2(\Gamma) \text{ and a.a. } t \in (0, T). \tag{5.10.59}$$

Indeed, for all $\xi \in C_c^\infty(0, T)$ we have

$$\begin{aligned} \int_0^T \xi(t)[\partial_t v(t), g\nabla_\Gamma q]_{T\Gamma}\, dt &= -\int_0^T \partial_t \xi(t)[v(t), g\nabla_\Gamma q]_{T\Gamma}\, dt \\ &= -\int_0^T \partial_t \xi(t)(v(t), g\nabla_\Gamma q)_{L^2(\Gamma)}\, dt = 0 \end{aligned}$$

by $v(t) \in L^2_{g\sigma}(\Gamma, T\Gamma)$ for a.a. $t \in (0, T)$ and $g\nabla_\Gamma q \in L^2_{g\sigma}(\Gamma, T\Gamma)^\perp$ (see Lemma 5.9.10). Hence (5.10.59) is valid. Now let $w \in C_c(0, T; H^1(\Gamma, T\Gamma))$. By Lemma 5.10.13 we can take $\eta \in C_c(0, T; V_g)$ and $q \in C_c(0, T; H^2(\Gamma))$ such that $w = g\eta + g\nabla_\Gamma q$. Moreover,

$$\int_0^T [\partial_t v, w]_{T\Gamma}\, dt = \int_0^T \big([\partial_t v, g\eta]_{T\Gamma} + [\partial_t v, g\nabla_\Gamma q]_{T\Gamma}\big)\, dt = \int_0^T [g\partial_t v, \eta]_{T\Gamma}\, dt$$

by (5.10.59). We substitute η for (5.10.58). Then using the above equality, (5.10.50), and

$$\left|\int_0^T b_g(v,v,\eta)\,dt\right| \leq c\|v\|_{L^\infty(0,T;L^2(\Gamma))}\|v\|_{L^2(0,T;H^1(\Gamma))}\|\eta\|_{L^2(0,T;H^1(\Gamma))} \tag{5.10.60}$$

by (5.10.12) we calculate as in the proof of Lemma 5.10.14 to get

$$\left|\int_0^T [\partial_t v, w]_{T\Gamma}\,dt\right| \leq c\|w\|_{L^2(0,T;H^1(\Gamma))} \quad \text{for all} \quad w \in C_c(0,T;H^1(\Gamma,T\Gamma)).$$

Since $C_c(0,T;H^1(\Gamma,T\Gamma))$ is dense in $L^2(0,T;H^1(\Gamma,T\Gamma))$, this inequality implies

$$\partial_t v \in L^2(0,T;H^{-1}(\Gamma,T\Gamma)). \tag{5.10.61}$$

Combining this property with

$$v \in L^2(0,T;V_g) \subset L^2(0,T;H^1(\Gamma,T\Gamma))$$

we apply the interpolation result of Lions–Magenes [37, Chapter 1, Theorem 3.1] (see also [64, Chapter III, Lemma 1.2]) to v to get

$$v \in C([0,T];L^2(\Gamma,T\Gamma)).$$

Moreover, since $v \in L^\infty(0,T;L^2_{g\sigma}(\Gamma,T\Gamma))$, the vector field $v(t)$ is in $L^2_{g\sigma}(\Gamma,T\Gamma)$ for a.a. $t \in (0,T)$ and, in particular, for all t in a dense subset of $[0,T]$. Thus, by the continuity of $v(t)$ on $[0,T]$ in $L^2(\Gamma,T\Gamma)$ and the fact that $L^2_{g\sigma}(\Gamma,T\Gamma)$ is closed in $L^2(\Gamma,T\Gamma)$, we observe that $v(t) \in L^2_{g\sigma}(\Gamma,T\Gamma)$ for all $t \in [0,T]$ and

$$v \in C([0,T];L^2_{g\sigma}(\Gamma,T\Gamma)).$$

Finally, since $C_c(0,T;V_g)$ is dense in $L^2(0,T;V_g)$ and both sides of (5.10.58) are linear and continuous for $\eta \in L^2(0,T;V_g)$ by (5.10.60) and (5.10.61), the equality (5.10.58) is also valid for all $\eta \in L^2(0,T;V_g)$. □

By Lemma 5.10.19 the value of a weak solution to (5.1.13)–(5.1.14) at $t=0$ is well-defined as a vector field on Γ. Hence we can consider the initial condition $v|_{t=0} = v_0$ as an equality for vector fields on Γ. Let us show the uniqueness of a weak solution to (5.1.13)–(5.1.14) and the existence of an associated pressure.

Lemma 5.10.20. *For given $v_0 \in L^2_{g\sigma}(\Gamma,T\Gamma)$ and $f \in L^2(0,T;H^{-1}(\Gamma,T\Gamma))$, $T>0$ there exists at most one weak solution to* (5.1.13)–(5.1.14) *on* $[0,T)$.

Proof. Let v_1 and v_2 be weak solutions to (5.1.13)–(5.1.14) and $w := v_1 - v_2$. Then

$$w \in C([0,T];L^2(\Gamma,T\Gamma)), \quad \partial_t w \in L^2(0,T;H^{-1}(\Gamma,T\Gamma)) \tag{5.10.62}$$

by Lemma 5.10.19 and $w|_{t=0} = 0$ on Γ. Moreover, subtracting the weak formulation (5.10.58) for v_2 from that for v_1 we get

$$\int_0^T \{[g\partial_t w,\eta]_{T\Gamma} + a_g(w,\eta) + b_g(w,v_1,\eta) + b_g(v_2,w,\eta)\}\,ds = 0 \tag{5.10.63}$$

for all $\eta \in L^2(0,T;V_g)$. For each $t\in[0,T]$ let $\eta(s) := 1_{[0,t]}(s)w(s)$, $s\in[0,T]$, where $1_{[0,t]}$ is the characteristic function of the time interval $[0,t]$. Since $\eta\in L^2(0,T;V_g)$ we can substitute it for (5.10.63). Then we use

$$\begin{aligned}\int_0^t [g\partial_s w, w]_{T\Gamma}\,ds &= \frac{1}{2}\int_0^t \frac{d}{ds}\|g^{1/2}w\|_{L^2(\Gamma)}^2\,ds \\ &= \frac{1}{2}\|g^{1/2}w(t)\|_{L^2(\Gamma)}^2 - \frac{1}{2}\|g^{1/2}w(0)\|_{L^2(\Gamma)}^2 \\ &\geq c\left(\|w(t)\|_{L^2(\Gamma)}^2 - \|w(0)\|_{L^2(\Gamma)}^2\right)\end{aligned} \tag{5.10.64}$$

by (5.10.62) and the fact that g is bounded on Γ from above and below by positive constants (see (5.2.30)), the inequality (5.10.11), and

$$|b_g(w,v_1,w)| = |b_g(w,w,v_1)| \leq c\|w\|_{L^2(\Gamma)}\|w\|_{H^1(\Gamma)}\|v_1\|_{H^1(\Gamma)}$$

and $b_g(v_2,w,w)=0$ by (5.10.12) and (5.10.13) to obtain

$$\begin{aligned}&\|w(t)\|_{L^2(\Gamma)}^2 + \int_0^t \|\nabla_\Gamma w\|_{L^2(\Gamma)}^2\,ds \\ &\qquad \leq c\left\{\|w(0)\|_{L^2(\Gamma)}^2 + \int_0^t \left(\|w\|_{L^2(\Gamma)}^2 + \|w\|_{L^2(\Gamma)}\|w\|_{H^1(\Gamma)}\|v_1\|_{H^1(\Gamma)}\right)ds\right\}.\end{aligned}$$

We further apply Young's inequality to the last term to get

$$\begin{aligned}&\|w(t)\|_{L^2(\Gamma)}^2 + \int_0^t \|\nabla_\Gamma w\|_{L^2(\Gamma)}^2\,ds \\ &\qquad \leq c\left\{\|w(0)\|_{L^2(\Gamma)}^2 + \int_0^t \left(1+\|v_1\|_{H^1(\Gamma)}^2\right)\|w\|_{L^2(\Gamma)}^2\,ds\right\} + \frac{1}{2}\int_0^t \|\nabla_\Gamma w\|_{L^2(\Gamma)}^2\,ds.\end{aligned}$$

Then we subtract the half of the integral of $\|\nabla_\Gamma w\|_{L^2(\Gamma)}^2$ from both sides and use $w|_{t=0}=0$ on Γ to find that

$$\|w(t)\|_{L^2(\Gamma)}^2 \leq c\int_0^t \left(1+\|v_1\|_{H^1(\Gamma)}^2\right)\|w\|_{L^2(\Gamma)}^2\,ds \quad \text{for all} \quad t\in[0,T].$$

(Here we omit the integral of $\|\nabla_\Gamma w\|_{L^2(\Gamma)}^2$ on the left-hand side.) Since $1+\|v_1\|_{H^1(\Gamma)}^2$ is integrable on $(0,T)$, we can use Gronwall's inequality to the above inequality to get $\|w(t)\|_{L^2(\Gamma)}^2 = 0$ for all $t\in[0,T]$. Hence $v_1=v_2$. □

Lemma 5.10.21. *Let v be a weak solution to* (5.1.13)–(5.1.14) *on* $[0,T)$, $T>0$. *Then there exists a unique $\hat{q}\in C([0,T];L^2(\Gamma))$ such that $\int_\Gamma \hat{q}(t)\,d\mathcal{H}^2 = 0$ for all $t\in[0,T]$ and*

$$\begin{aligned}g\left(\partial_t v + \overline{\nabla}_v v\right) - 2\nu\left\{P\operatorname{div}_\Gamma[gD_\Gamma(v)] - \frac{1}{g}(\nabla_\Gamma g\otimes\nabla_\Gamma g)v\right\} \\ + (\gamma^0+\gamma^1)v + g\nabla_\Gamma q = gf \quad \text{in} \quad \mathcal{D}'(0,T;H^{-1}(\Gamma,T\Gamma))\end{aligned} \tag{5.10.65}$$

with $q := \partial_t\hat{q} \in \mathcal{D}'(0,T;L^2(\Gamma))$ (see Section 5.2.1).

Here $\overline{\nabla}_v v = P(v\cdot\nabla_\Gamma)v$ is the covariant derivative of v along itself (see Appendix 5.C). Also, recall that we identity $H^{-1}(\Gamma,T\Gamma)$ with quotient space

$$\mathcal{Q} = \{[f] \mid f\in H^{-1}(\Gamma)^3\}, \quad [f] = \{\tilde{f}\in H^{-1}(\Gamma)^3 \mid Pf = P\tilde{f} \text{ in } H^{-1}(\Gamma)^3\}$$

and take Pf (or f when $Pf=f$ in $H^{-1}(\Gamma)^3$) as a representative of the equivalence class $[f]$ to write $[Pf,v]_{T\Gamma} = \langle f,v\rangle_\Gamma$ for $v\in H^1(\Gamma,T\Gamma)$ (see Section 5.2.1).

Proof. Let $A \in L^2(\Gamma)^{3\times 3}$ and $\eta \in H^1(\Gamma, T\Gamma)$. If $A^T = A$ and $PA = AP = A$ on Γ, then by (5.2.23) and $An = APn = 0$ we see that

$$\begin{aligned} \big(gA, D_\Gamma(\eta)\big)_{L^2(\Gamma)} &= (gA, \nabla_\Gamma \eta)_{L^2(\Gamma)} = \sum_{i,j=1}^{3} (gA_{ij}, \underline{D}_i \eta_j)_{L^2(\Gamma)} \\ &= -\sum_{i,j=1}^{3} \big\{\langle \underline{D}_i(gA_{ij}), \eta_j\rangle_\Gamma + (gA_{ij}Hn_i, \eta_j)_{L^2(\Gamma)}\big\} \\ &= -\big\{\langle \mathrm{div}_\Gamma(gA), \eta\rangle_\Gamma + (gHAn, \eta)_{L^2(\Gamma)}\big\} = -[P\mathrm{div}_\Gamma(gA), \eta]_{T\Gamma}. \end{aligned}$$

Also, for $v \in V_g$ and $\eta \in H^1(\Gamma, T\Gamma)$ we use (5.10.13) and $v \otimes \eta : \nabla_\Gamma v = (v \cdot \nabla_\Gamma)v \cdot \eta$ to get

$$b_g(v, v, \eta) = -b_g(v, \eta, v) = \big(g(v \cdot \nabla_\Gamma)v, \eta\big)_{L^2(\Gamma)} = (gP(v \cdot \nabla_\Gamma)v, \eta)_{L^2(\Gamma)} = \Big(g\overline{\nabla}_v v, \eta\Big)_{L^2(\Gamma)}.$$

Let v be a weak solution to (5.1.13)–(5.1.14) on $[0, T)$. We apply the above equalities with $A = D_\Gamma(v)$ and $(v \cdot \nabla_\Gamma g)(\eta \cdot \nabla_\Gamma g) = (\nabla_\Gamma g \otimes \nabla_\Gamma g)v \cdot \eta$ to (5.10.58) to obtain

$$\int_0^T \big([g\partial_t v, \eta]_{T\Gamma} + [A_g v, \eta]_{T\Gamma} + [B_g(v, v), \eta]_{T\Gamma}\big)\, ds = \int_0^T [gf, \eta]_{T\Gamma}\, ds \tag{5.10.66}$$

for all $\eta \in L^2(0, T; V_g)$ (see Lemma 5.10.19), where $B_g(v, v) := g\overline{\nabla}_v v$ and

$$A_g v := -2\nu \left\{P\mathrm{div}_\Gamma[gD_\Gamma(v)] - \frac{1}{g}(\nabla_\Gamma g \otimes \nabla_\Gamma g)v\right\} + (\gamma^0 + \gamma^1)v. \tag{5.10.67}$$

Since $A_g v, B_g(v, v), f \in L^2(0, T; H^{-1}(\Gamma, T\Gamma))$ by Definition 5.10.17, the functions

$$\widehat{A}_g v(t) := \int_0^t A_g v(s)\, ds, \quad \widehat{B}_g(v, v)(t) := \int_0^t B_g(v(s), v(s))\, ds,$$
$$\hat{f}(t) := \int_0^t f(s)\, ds, \quad t \in [0, T]$$

are continuous with values in $H^{-1}(\Gamma, T\Gamma)$. For each $t \in [0, T]$ and $\xi \in V_g$ we take a test function $\eta(s) := 1_{[0,t]}(s)\xi$, $s \in [0, T]$ in (5.10.66), where $1_{[0,t]} \colon \mathbb{R} \to \mathbb{R}$ is the characteristic function of $[0, t]$. Then since ξ is independent of time,

$$v \in H^1(0, T; H^{-1}(\Gamma, T\Gamma)) \subset C([0, T]; H^{-1}(\Gamma, T\Gamma)),$$

and $v|_{t=0} = v_0$ in $H^{-1}(\Gamma, T\Gamma)$, we have $[F(t), \xi]_{T\Gamma} = 0$ for all $\xi \in V_g$, where

$$F := gv - gv_0 + \widehat{A}_g v + \widehat{B}_g(v, v) - g\hat{f} \in C([0, T]; H^{-1}(\Gamma, T\Gamma)).$$

Hence by Theorem 5.9.5 there exists a unique $\hat{q}(t) \in L^2(\Gamma)$ such that

$$F(t) = -g\nabla_\Gamma \hat{q}(t) \quad \text{in} \quad H^{-1}(\Gamma, T\Gamma), \qquad \int_\Gamma \hat{q}(t)\, d\mathcal{H}^2 = 0$$

for all $t \in [0, T]$. Moreover, by (5.9.20) and $F \in C([0, T]; H^{-1}(\Gamma, T\Gamma))$ we see that

$$\hat{q} \in C([0, T]; L^2(\Gamma)) \subset L^2(0, T; L^2(\Gamma))$$

and thus $q := \partial_t\hat{q} \in \mathcal{D}'(0,T;L^2(\Gamma))$ is well-defined. Now we have

$$0 = -\int_0^T \partial_t\varphi(t)\{F(t) + g\nabla_\Gamma\hat{q}(t)\}\,dt = \int_0^T \varphi(t)\{\partial_t F(t) + g\partial_t(\nabla_\Gamma\hat{q})(t)\}\,dt$$

in $H^{-1}(\Gamma,T\Gamma)$ for all $\varphi \in C_c^\infty(0,T)$, which means that

$$\partial_t F(t) + g\partial_t(\nabla_\Gamma\hat{q})(t) = 0 \quad \text{in} \quad \mathcal{D}'(0,T;H^{-1}(\Gamma,T\Gamma)).$$

Moreover, $\partial_t(\nabla_\Gamma\hat{q}) = \nabla_\Gamma q$ by (5.2.29) and

$$\partial_t F = g\partial_t v + A_g v + B_g(v,v) - gf = g\Big(\partial_t v + \overline{\nabla}_v v\Big) + A_g v - gf$$

with $A_g v$ given by (5.10.67). Hence (5.10.65) is valid. □

Before starting the proof of Theorem 5.1.3, we give an auxiliary statement on the weak limit of the averaged tangential component of a vector field in $L^2_\sigma(\Omega_\varepsilon)$ as $\varepsilon \to 0$.

Lemma 5.10.22. *For $\varepsilon \in (0,1)$ let $u^\varepsilon \in L^2_\sigma(\Omega_\varepsilon)$. Also, let $v \in L^2(\Gamma,T\Gamma)$. Suppose that $M_\tau u^\varepsilon$ converges to v weakly in $L^2(\Gamma,T\Gamma)$ as $\varepsilon \to 0$ and*

$$\|u^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{-1+\alpha} \quad \textit{for sufficiently small} \quad \varepsilon \in (0,1)$$

with some $c, \alpha > 0$. Then $v \in L^2_{g\sigma}(\Gamma,T\Gamma)$.

Proof. By (5.2.25) and the weak convergence of $\{M_\tau u^\varepsilon\}_\varepsilon$ to v in $L^2(\Gamma,T\Gamma)$ we see that $\mathrm{div}_\Gamma(gM_\tau u^\varepsilon)$ converges to $\mathrm{div}_\Gamma(gv)$ weakly in $H^{-1}(\Gamma)$ as $\varepsilon \to 0$. Moreover, by (5.6.49) and the assumption on the $L^2(\Omega_\varepsilon)$-norm of u^ε we have

$$\|\mathrm{div}_\Gamma(gM_\tau u^\varepsilon)\|_{H^{-1}(\Gamma)} \leq c\varepsilon^{1/2}\|u^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c\varepsilon^{\alpha/2}$$

for sufficiently small $\varepsilon \in (0,1)$ with $\alpha > 0$. Hence

$$\|\mathrm{div}_\Gamma(gv)\|_{H^{-1}(\Gamma)} \leq \liminf_{\varepsilon\to 0}\|\mathrm{div}_\Gamma(gM_\tau u^\varepsilon)\|_{H^{-1}(\Gamma)} = 0$$

and $\mathrm{div}_\Gamma(gv) = 0$ in $H^{-1}(\Gamma)$, which means that $v \in L^2_{g\sigma}(\Gamma,T\Gamma)$. □

Now we are ready to prove Theorem 5.1.3.

Proof of Theorem 5.1.3. For $\varepsilon \in (0,1)$ suppose that the initial velocity u_0^ε and the external force f^ε satisfy the assumptions of Theorem 5.1.3. Then $\{M_\tau u_0^\varepsilon\}_\varepsilon$ and $\{M_\tau \mathbb{P}_\varepsilon f^\varepsilon\}_\varepsilon$ are bounded in $L^2(\Gamma,T\Gamma)$ and $L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))$, respectively, by the condition (b). By this fact and the condition (a) we see that the inequalities (5.1.10) hold with $\beta = 1$ for $\varepsilon \in (0,\varepsilon_2)$. Hence Theorem 5.8.4 implies that there exists a global strong solution u^ε to (5.1.1)–(5.1.3) satisfying (5.8.34)–(5.8.36) for each $\varepsilon \in (0,\varepsilon_3)$ with $\varepsilon_3 := \min\{\varepsilon_1', \varepsilon_2\}$, where $\varepsilon_1' = \min\{\varepsilon_1, \varepsilon_\sigma\}$ with ε_1 and ε_σ given in Theorem 5.1.2 and Lemma 5.5.1. Moreover, by (5.6.10) and (5.8.34) we have

$$\sup_{t\in[0,\infty)}\|Mu^\varepsilon(t)\cdot n\|_{L^2(\Gamma)} \leq c\varepsilon^{1/2}\sup_{t\in[0,\infty)}\|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon)} \leq c\varepsilon^{\alpha/2} \to 0$$

as $\varepsilon \to 0$. Hence $\{Mu^\varepsilon \cdot n\}_\varepsilon$ converges to zero strongly in $C([0,\infty);L^2(\Gamma))$.

Now let us consider the averaged tangential component $M_\tau u^\varepsilon$. First note that the weak limit v_0 of $\{M_\tau u_0^\varepsilon\}_\varepsilon$ is in $L^2_{g\sigma}(\Gamma,T\Gamma)$ by the condition (a), the inequality (5.5.18), and Lemma 5.10.22. Since all results in the previous subsections apply to u^ε, for fixed $T > 0$ we see by (5.10.49) and (5.10.57) that

- $\{M_\tau u^\varepsilon\}_\varepsilon$ is bounded in $L^\infty(0,T;L^2(\Gamma,T\Gamma))\cap L^2(0,T;H^1(\Gamma,T\Gamma))$,
- $\{\partial_t M_\tau u^\varepsilon\}_\varepsilon$ is bounded in $L^2(0,T;H^{-1}(\Gamma,T\Gamma))$.

Hence there exist $\varepsilon_k\in(0,\varepsilon_3)$, $k\in\mathbb{N}$ and

$$v\in L^\infty(0,T;L^2(\Gamma,T\Gamma))\cap L^2(0,T;H^1(\Gamma,T\Gamma))\quad\text{with}\quad \partial_t v\in L^2(0,T;H^{-1}(\Gamma,T\Gamma))$$

such that $\lim_{k\to\infty}\varepsilon_k=0$ and

$$\begin{aligned}
\lim_{k\to\infty} M_\tau u^{\varepsilon_k} &= v &&\text{weakly-}\star\text{ in} && L^\infty(0,T;L^2(\Gamma,T\Gamma)),\\
\lim_{k\to\infty} M_\tau u^{\varepsilon_k} &= v &&\text{weakly in} && L^2(0,T;H^1(\Gamma,T\Gamma)),\\
\lim_{k\to\infty} \partial_t M_\tau u^{\varepsilon_k} &= \partial_t v &&\text{weakly in} && L^2(0,T;H^{-1}(\Gamma,T\Gamma)).
\end{aligned}\tag{5.10.68}$$

Moreover, by the Aubin–Lions lemma (see e.g. [8, Theorem II.5.16]) there exists a subsequence of $\{M_\tau u^{\varepsilon_k}\}_{k=1}^\infty$, which we denote by $\{M_\tau u^{\varepsilon_k}\}_{k=1}^\infty$ again, such that

$$\lim_{k\to\infty} M_\tau u^{\varepsilon_k}=v\quad\text{strongly in}\quad L^2(0,T;L^2(\Gamma,T\Gamma)).\tag{5.10.69}$$

Then $v(t)\in L^2_{g\sigma}(\Gamma,T\Gamma)$ for a.a. $t\in(0,T)$ by (5.8.34) and Lemma 5.10.22. Hence

$$v\in L^\infty(0,T;L^2_{g\sigma}(\Gamma,T\Gamma))\cap L^2(0,T;V_g).$$

Let us show that v satisfies (5.10.58) for all $\eta\in C_c(0,T;V_g)$. In what follows, we write c for a general positive constant that may depend on v and η but is independent of ε_k and u^{ε_k}. We consider the weak formulation (5.10.27) for $M_\tau u^{\varepsilon_k}$:

$$\begin{aligned}
\int_0^T \{[g\partial_t M_\tau u^{\varepsilon_k},\eta]_{T\Gamma}+a_g(M_\tau u^{\varepsilon_k},\eta)+b_g(M_\tau u^{\varepsilon_k},M_\tau u^{\varepsilon_k},\eta)\}\,dt&\\
=\int_0^T [gM_\tau\mathbb{P}_{\varepsilon_k}f^{\varepsilon_k},\eta]_{T\Gamma}\,dt+R^1_{\varepsilon_k}(\eta).&
\end{aligned}\tag{5.10.70}$$

Here $\partial_t M_\tau u^{\varepsilon_k}$ and $M_\tau\mathbb{P}_{\varepsilon_k}f^{\varepsilon_k}$ are considered in $H^{-1}(\Gamma,T\Gamma)$ (see Section 5.2.1). Let $k\to\infty$ in this equality. Noting that $[gF,w]_{T\Gamma}=[F,gw]_{T\Gamma}$ for $F\in H^{-1}(\Gamma,T\Gamma)$ and $w\in H^1(\Gamma,T\Gamma)$ (see Section 5.2.1), we deduce from the assumption (b) and (5.10.68) that

$$\begin{aligned}
\lim_{k\to\infty}\int_0^T [g\partial_t M_\tau u^{\varepsilon_k},\eta]_{T\Gamma}\,dt &= \int_0^T [g\partial_t v,\eta]_{T\Gamma}\,dt,\\
\lim_{k\to\infty}\int_0^T a_g(M_\tau u^{\varepsilon_k},\eta)\,dt &= \int_0^T a_g(v,\eta)\,dt,\\
\lim_{k\to\infty}\int_0^T [gM_\tau\mathbb{P}_{\varepsilon_k}f^{\varepsilon_k},\eta]_{T\Gamma}\,dt &= \int_0^T [gf,\eta]_{T\Gamma}\,dt.
\end{aligned}\tag{5.10.71}$$

Also, by (5.10.28), the assumption (c), and $\alpha>0$ we have

$$|R^1_{\varepsilon_k}(\eta)|\le c\left(\varepsilon_k^{\alpha/4}+\sum_{i=0,1}|\varepsilon_k^{-1}\gamma^i_{\varepsilon_k}-\gamma^i|\right)(1+T)^{1/2}\|\eta\|_{L^2(0,T;H^1(\Gamma))}\to 0\tag{5.10.72}$$

as $k \to \infty$. To show the convergence of the trilinear term, we set

$$\begin{aligned}I_1^k &:= \int_0^T b_g(M_\tau u^{\varepsilon_k}, M_\tau u^{\varepsilon_k}, \eta)\,dt - \int_0^T b_g(v, M_\tau u^{\varepsilon_k}, \eta)\,dt, \\ I_2^k &:= \int_0^T b_g(v, M_\tau u^{\varepsilon_k}, \eta)\,dt - \int_0^T b_g(v, v, \eta)\,dt.\end{aligned}$$

Since $\|\eta(t)\|_{H^1(\Gamma)}$ is bounded on $[0,T]$ by $\eta \in C_c(0,T;V_g)$, we see by (5.10.12) that

$$\begin{aligned}|I_1^k| &\leq c\int_0^T \|M_\tau u^{\varepsilon_k} - v\|_{L^2(\Gamma)}^{1/2}\|M_\tau u^{\varepsilon_k} - v\|_{H^1(\Gamma)}^{1/2}\|M_\tau u^{\varepsilon_k}\|_{H^1(\Gamma)}\|\eta\|_{H^1(\Gamma)}\,dt \\ &\leq c\|M_\tau u^{\varepsilon_k} - v\|_{L^2(0,T;L^2(\Gamma))}^{1/2}\|M_\tau u^{\varepsilon_k} - v\|_{L^2(0,T;H^1(\Gamma))}^{1/2}\|M_\tau u^{\varepsilon_k}\|_{L^2(0,T;H^1(\Gamma))}.\end{aligned}$$

Applying (5.10.49) and (5.10.69) to the last line we obtain

$$|I_1^k| \leq c\|M_\tau u^{\varepsilon_k} - v\|_{L^2(0,T;L^2(\Gamma))}^{1/2} \to 0 \quad \text{as} \quad k \to \infty. \tag{5.10.73}$$

For I_2^k, we consider the linear functional

$$\Phi(\xi) := \int_0^T b_g(v, \xi, \eta)\,dt, \quad \xi \in L^2(0,T;H^1(\Gamma,T\Gamma)).$$

By (5.10.12) and the boundedness of $\|\eta(t)\|_{H^1(\Gamma)}$ on $[0,T]$ we get

$$|\Phi(\xi)| \leq c\|\eta\|_{L^\infty(0,T;H^1(\Gamma))}\|v\|_{L^2(0,T;H^1(\Gamma))}\|\xi\|_{L^2(0,T;H^1(\Gamma))}$$

for all $\xi \in L^2(0,T;H^1(\Gamma,T\Gamma))$. Hence Φ is bounded on $L^2(0,T;H^1(\Gamma,T\Gamma))$ and the weak convergence (5.10.68) in $L^2(0,T;H^1(\Gamma,T\Gamma))$ implies that

$$\lim_{k\to\infty} I_2^k = \lim_{k\to\infty}\{\Phi(M_\tau u^{\varepsilon_k}) - \Phi(v)\} = 0.$$

Combining this equality with (5.10.73) we obtain

$$\lim_{k\to\infty}\int_0^T b_g(M_\tau u^{\varepsilon_k}, M_\tau u^{\varepsilon_k}, \eta)\,dt = \int_0^T b_g(v, v, \eta)\,dt. \tag{5.10.74}$$

We send $k \to \infty$ in (5.10.70) and apply (5.10.71), (5.10.72), and (5.10.74) to show that v satisfies (5.10.58) for all $\eta \in C_c(0,T;V_g)$. Moreover, by Lemma 5.10.19 we see that

$$v \in C([0,T], L^2_{g\sigma}(\Gamma,T\Gamma)), \quad \partial_t v \in L^2(0,T;H^{-1}(\Gamma,T\Gamma))$$

and (5.10.58) is valid for all $\eta \in L^2(0,T;V_g)$.

To show that v is a weak solution to (5.1.13)–(5.1.14) on $[0,T)$ we also need to verify the initial condition. Let $\xi \in V_g$ and φ be a smooth function on $[0,T]$ such that $\varphi(0) = 1$ and $\varphi(T) = 0$. We define $\eta \in L^2(0,T;V_g)$ by $\eta(t) := \varphi(t)\xi$ for $t \in [0,T]$ and substitute it for (5.10.58) and (5.10.70). Then we carry out integration by parts for $\partial_t v$ and $\partial_t M_\tau u^\varepsilon$ and use $\varphi(0) = 1$ and $\varphi(T) = 0$ to get

$$(gv(0), \xi)_{L^2(\Gamma)} = J_\infty, \quad (gM_\tau u_0^{\varepsilon_k}, \xi)_{L^2(\Gamma)} = J_k, \tag{5.10.75}$$

where

$$J_\infty := -\int_0^T \partial_t\varphi(gv,\xi)_{L^2(\Gamma)}\,dt + \int_0^T \{a_g(v,\eta)+b_g(v,v,\eta)\}\,dt - \int_0^T [gf,\eta]_{T\Gamma}\,dt$$

and

$$J_k := -\int_0^T \partial_t\varphi(gM_\tau u^{\varepsilon_k},\xi)_{L^2(\Gamma)}\,dt + \int_0^T \{a_g(M_\tau u^{\varepsilon_k},\eta)+b_g(M_\tau u^{\varepsilon_k},M_\tau u^{\varepsilon_k},\eta)\}\,dt \\ - \int_0^T [gM_\tau \mathbb{P}_{\varepsilon_k} f^{\varepsilon_k},\eta]_{T\Gamma}\,dt - R^1_{\varepsilon_k}(\eta).$$

We send $k\to\infty$ in the second equality of (5.10.75). Then the left-hand side converges to $(gv_0,\xi)_{L^2(\Gamma)}$ by the assumption (b). Also, we use (5.10.71), (5.10.72), (5.10.74), and

$$\lim_{k\to\infty}\int_0^T \partial_t\varphi(gM_\tau u^{\varepsilon_k},\xi)_{L^2(\Gamma)}\,dt = \int_0^T \partial_t\varphi(gv,\xi)_{L^2(\Gamma)}\,dt$$

by the strong convergence (5.10.69) to find that $\lim_{k\to\infty} J_k = J_\infty$ (note that in the proof of (5.10.74) we only used the boundedness of $\|\eta(t)\|_{H^1(\Gamma)}$ on $[0,T]$). Hence

$$(gv(0),\xi)_{L^2(\Gamma)} = J_\infty = (gv_0,\xi)_{L^2(\Gamma)} \quad \text{for all} \quad \xi\in V_g.$$

Since V_g is dense in $L^2_{g\sigma}(\Gamma,T\Gamma)$ (see Lemma 5.9.13), the above equality is also valid for all $\xi\in L^2_{g\sigma}(\Gamma,T\Gamma)$. Thus, setting $\xi := v(0)-v_0$ we get

$$(g\{v(0)-v_0\},v(0)-v_0)_{L^2(\Gamma)} = \|g^{1/2}\{v(0)-v_0\}\|^2_{L^2(\Gamma)} = 0,$$

which combined with (5.2.30) shows $v|_{t=0}=v_0$ on Γ. Therefore, v is a unique weak solution to (5.1.13)–(5.1.14) on $[0,T)$ (here the uniqueness follows from Lemma 5.10.20).

Let us prove the convergence of the full sequence

$$\begin{aligned} \lim_{\varepsilon\to0} M_\tau u^\varepsilon &= v && \text{weakly in} && L^2(0,T;H^1(\Gamma,T\Gamma)),\\ \lim_{\varepsilon\to0} \partial_t M_\tau u^\varepsilon &= \partial_t v && \text{weakly in} && L^2(0,T;H^{-1}(\Gamma,T\Gamma)). \end{aligned} \tag{5.10.76}$$

By the boundedness of $\{M_\tau u^\varepsilon\}_\varepsilon$ and $\{\partial_t M_\tau u^\varepsilon\}_\varepsilon$ (see (5.10.49) and (5.10.57)) and the uniqueness of a weak solution to (5.1.13)–(5.1.14) (see Lemma 5.10.20) we can show as above that for any sequence $\{\varepsilon_l\}_{l=1}^\infty$ of positive numbers convergent to zero there exists its subsequence $\{\varepsilon_k\}_{k=1}^\infty$ such that $\{M_\tau u^{\varepsilon_k}\}_{k=1}^\infty$ converges to v in the sense of (5.10.68) and (5.10.69). This proves (5.10.76).

Since the strong solution u^ε to (5.1.1)–(5.1.3) exists globally in time for $\varepsilon\in(0,\varepsilon_3)$, by the above arguments we get a unique weak solution

$$v_T \in C([0,T];L^2_{g\sigma}(\Gamma,T\Gamma))\cap L^2(0,T;V_g)\cap H^1(0,T;H^{-1}(\Gamma,T\Gamma))$$

to (5.1.13)–(5.1.14) on $[0,T)$ satisfying (5.10.76) for all $T>0$. Moreover, if $T<T'$ then $v_T = v_{T'}$ on $[0,T]$ by the uniqueness of a weak solution. Hence we can define

$$v\in C([0,\infty);L^2_{g\sigma}(\Gamma,T\Gamma))\cap L^2_{loc}([0,\infty);V_g)\cap H^1_{loc}([0,\infty);H^{-1}(\Gamma,T\Gamma))$$

by $v:=v_T$ on $[0,T]$ for each $T>0$, which is a unique weak solution to (5.1.13)–(5.1.14) on $[0,\infty)$ and satisfies (5.10.76) for all $T>0$. $\square$

As a consequence of Theorem 5.1.3, we obtain the existence of a weak solution to (5.1.13)–(5.1.14) for the initial velocity v_0 and the external force f given by the weak and weak-$\star$ limit of $M_\tau u_0^\varepsilon$ and $M_\tau \mathbb{P}_\varepsilon f^\varepsilon$, respectively. For general $v_0 \in L^2_{g\sigma}(\Gamma, T\Gamma)$ and $f \in L^2_{loc}([0,\infty); H^{-1}(\Gamma, T\Gamma))$ we can construct a global weak solution to (5.1.13)–(5.1.14) by the Galerkin method as in the case of the Navier–Stokes equations in a two-dimensional bounded domain (see e.g. [8, 10, 64]). Here we just give the outline of construction of a weak solution by the Galerkin method.

Countable basis of V_g. If Assumption 2 and the condition (c) in Theorem 5.1.3 are satisfied, we have $\mathcal{K}_g(\Gamma) = \{0\}$ or $\max_{i=0,1}\gamma^i > 0$. By this fact and the Korn inequalities (5.4.37) and (5.4.44) the bilinear form a_g given by (5.10.9) is coercive (and bounded) on V_g and thus induces a linear homeomorphism A_g from V_g onto V_g' by the Lax–Milgram theorem. We consider A_g as an unbounded operator on $L^2_{g\sigma}(\Gamma, T\Gamma)$ equipped with inner product

$$(v_1, v_2)_{L^2_g(\Gamma)} := (g^{1/2}v_1, g^{1/2}v_2)_{L^2(\Gamma)}, \quad v_1, v_2 \in L^2_{g\sigma}(\Gamma, T\Gamma),$$

which is equivalent to the canonical $L^2(\Gamma)$-inner product by (5.2.30). Then we can show as in the case of the Stokes operator on a bounded domain (see [8, Theorem IV.5.5]) that there exists a sequence $\{w_k\}_{k=1}^\infty$ of eigenvectors of A_g that is an orthonormal basis of $L^2_{g\sigma}(\Gamma, T\Gamma)$ equipped with inner product $(\cdot,\cdot)_{L^2_g(\Gamma)}$ as well as an orthogonal basis of V_g equipped with inner product $a_g(\cdot,\cdot)$. In particular,

$$(gw_i, w_j)_{L^2(\Gamma)} = (w_i, w_j)_{L^2_g(\Gamma)} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{5.10.77}$$

Note that, even if $\mathcal{K}_g(\Gamma) \neq \{0\}$ and $\gamma^0 = \gamma^1 = 0$, we can take such a sequence by replacing a_g with $\tilde{a}_g(v_1, v_2) := a_g(v_1, v_2) + (v_1, v_2)_{L^2(\Gamma)}$, which is coercive and bounded on V_g (see Lemma 5.10.2).

Approximate problem. For $k \in \mathbb{N}$ we seek for an approximate solution

$$v_k(y,t) = \sum_{i=1}^k \xi_i(t) w_i(y), \quad y \in \Gamma,\ t \in [0,T] \quad (T > 0)$$

that satisfies

$$(g\partial_t v_k(t), \eta_k)_{L^2(\Gamma)} + a_g(v_k(t), \eta_k) + b_g(v_k(t), v_k(t), \eta_k) = [gf(t), \eta_k]_{T\Gamma} \tag{5.10.78}$$

for all $\eta_k \in V_g^k$ and $t \in (0,T)$ (with initial condition), where V_g^k is the linear span of $\{w_i\}_{i=1}^k$. (In fact, we need to approximate $f(t)$ by a continuous function.) This problem is equivalent to a system of ordinary differential equations of the form

$$\sum_{i=1}^k (gw_i, w_j)_{L^2(\Gamma)} \frac{d\xi_i}{dt}(t) = \mathcal{P}_j(\xi(t)) + [gf(t), w_j]_{T\Gamma}, \quad j = 1, \dots, k$$

with polynomials $\mathcal{P}_1, \dots, \mathcal{P}_k$ of $\xi = (\xi_1, \dots, \xi_k)$. Applying (5.10.77) to the left-hand side we see that this system reduces to $d\xi_j/dt = \mathcal{P}_j(\xi) + [gf, w_j]_{T\Gamma}$, which we can solve locally by the Cauchy–Lipschitz theorem. Using (5.10.77) we can also derive the energy estimate for v_k and show its global existence.

Estimate for the time derivative of the approximate solution. As in Lemma 5.10.14 (see also Remark 5.10.16), we estimate the time derivative of v_k in $H^{-1}(\Gamma, T\Gamma)$. To this end, we take $w \in H^1(\Gamma, T\Gamma)$ and apply Lemma 5.10.13 to get $w = g\eta + g\nabla_\Gamma q$ with $\eta \in V_g$ satisfying (5.10.50) and $q \in H^2(\Gamma)$. Since $\{w_k\}_{k=1}^\infty$ is an orthogonal basis of V_g equipped with inner product $a_g(\cdot,\cdot)$, which is equivalent to the canonical $H^1(\Gamma)$-inner product,

$$\eta = \sum_{i=1}^\infty a_g(\eta, \tilde{w}_i)\tilde{w}_i \quad \text{in} \quad V_g, \quad \tilde{w}_i := \frac{w_i}{a_g(w_i, w_i)^{1/2}}.$$

Then we set $\eta_k := \sum_{i=1}^k a_g(\eta, \tilde{w}_k)\tilde{w}_k \in V_g^k$ to get

$$\|\eta_k\|_{H^1(\Gamma)} \leq c\|\eta\|_{H^1(\Gamma)} \leq c\|w\|_{H^1(\Gamma)}$$

by (5.10.50) and

$$\begin{aligned}(g\partial_t v_k, \eta_k)_{L^2(\Gamma)} &= (g\partial_t v_k, \eta)_{L^2(\Gamma)} = (\partial_t v_k, g\eta)_{L^2(\Gamma)} \\ &= (\partial_t v_k, w - g\nabla_\Gamma q)_{L^2(\Gamma)} = (\partial_t v_k, w)_{L^2(\Gamma)},\end{aligned}$$

where we used (5.10.77) in the first equality and $\partial_t v_k \in L^2_{g\sigma}(\Gamma, T\Gamma)$ and $g\nabla_\Gamma q \in L^2_{g\sigma}(\Gamma, T\Gamma)^\perp$ in the last equality (note that here we take the canonical $L^2(\Gamma)$-inner product). Hence, substituting η_k for (5.10.78) and using these relations, we can show as in the proof of Lemma 5.10.14 that

$$\|\partial_t v_k(t)\|_{H^{-1}(\Gamma,T\Gamma)} \leq c\left\{\left(1 + \|v_k(t)\|_{L^2(\Gamma)}\right)\|v_k(t)\|_{H^1(\Gamma)} + \|f(t)\|_{H^{-1}(\Gamma,T\Gamma)}\right\}$$

for all $t \in [0,T]$. By this inequality and the energy estimate for v_k we obtain the boundedness of $\{\partial_t v_k\}_{k=1}^\infty$ in $L^2(0,T; H^{-1}(\Gamma, T\Gamma))$ and we can prove the convergence of $\{v_k\}_{k=1}^\infty$ to a weak solution to (5.1.13)–(5.1.14) as in the proof of Theorem 5.1.3.

5.10.6 Strong convergence of the average and error estimates

In this subsection we show the strong convergence of the averaged tangential component of a strong solution to the Navier–Stokes equations (5.1.1)–(5.1.3) towards a weak solution to the limit equations (5.1.13)–(5.1.14). We also estimate the difference between a strong solution to (5.1.1)–(5.1.3) and a weak solution to (5.1.13)–(5.1.14).

Theorem 5.10.23. *Suppose that the assumptions in Theorem 5.8.4 are satisfied. For $\varepsilon \in (0, \varepsilon_1)$ let u^ε be the global strong solution to* (5.1.1)–(5.1.3) *given by Theorem 5.8.4. Also, let $v_0 \in L^2_{g\sigma}(\Gamma, T\Gamma)$, $f \in L^\infty(0,\infty; H^{-1}(\Gamma, T\Gamma))$, and v be a weak solution to* (5.1.13)–(5.1.14) *on* $[0,\infty)$. *Then for all $T > 0$ we have*

$$\begin{aligned}\max_{t\in[0,T]} \|M_\tau u^\varepsilon(t) - v(t)\|^2_{L^2(\Gamma)} &+ \int_0^T \|\nabla_\Gamma M_\tau u^\varepsilon(t) - \nabla_\Gamma v(t)\|^2_{L^2(\Gamma)}\,dt \\ &\leq c_T\left\{\delta(\varepsilon)^2 + \|M_\tau u_0^\varepsilon - v_0\|^2_{L^2(\Gamma)} + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon - f\|^2_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))}\right\}, \quad (5.10.79)\end{aligned}$$

where $c_T > 0$ is a constant depending only on T and

$$\delta(\varepsilon) := \varepsilon^{\alpha/4} + \sum_{i=0,1} |\varepsilon^{-1}\gamma_\varepsilon^i - \gamma^i|. \quad (5.10.80)$$

As in Section 5.10.3, we first compare the auxiliary vector field $v^\varepsilon = \mathbb{P}_g M_\tau u^\varepsilon$ with a weak solution to (5.1.13)–(5.1.14) and then derive (5.10.79) by using the estimate for the difference between v^ε and $M_\tau u^\varepsilon$.

Lemma 5.10.24. *Under the same assumptions as in Theorem 5.10.23, we have*

$$\max_{t\in[0,T]}\|v^\varepsilon(t)-v(t)\|_{L^2(\Gamma)}^2+\int_0^T\|\nabla_\Gamma v^\varepsilon(t)-\nabla_\Gamma v(t)\|_{L^2(\Gamma)}^2\,dt \\ \leq c_T\left\{\delta(\varepsilon)^2+\|v^\varepsilon(0)-v_0\|_{L^2(\Gamma)}^2+\|M_\tau\mathbb{P}_\varepsilon f^\varepsilon-f\|_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))}^2\right\} \tag{5.10.81}$$

for all $T>0$, where $v^\varepsilon=\mathbb{P}_gM_\tau u^\varepsilon$ is given in Lemma 5.10.10, $c_T>0$ is a constant depending only on T, and $\delta(\varepsilon)$ is given by (5.10.80).

Proof. For the sake of simplicity, we set

$$w^\varepsilon:=v^\varepsilon-v,\quad w_0^\varepsilon:=v^\varepsilon(0)-v_0=\mathbb{P}_gM_\tau u_0^\varepsilon-v_0,\quad F^\varepsilon:=M_\tau\mathbb{P}_\varepsilon f^\varepsilon-f.$$

Let $T>0$. We subtract both sides of (5.10.58) from those of (5.10.37) to get

$$\int_0^T\{[g\partial_sw^\varepsilon,\eta]_{T\Gamma}+a_g(w^\varepsilon,\eta)+b_g(w^\varepsilon,v^\varepsilon,\eta)+b_g(v,w^\varepsilon,\eta)\}\,ds \\ =\int_0^T[gF^\varepsilon,\eta]_{T\Gamma}\,ds+R_\varepsilon^1(\eta)+R_\varepsilon^2(\eta)$$

for all $\eta\in L^2(0,T;V_g)$ (see also Lemma 5.10.19), where $R_\varepsilon^1(\eta)$ and $R_\varepsilon^2(\eta)$ are given in Lemmas 5.10.6 and 5.10.10. For each $t\in[0,T]$, let $1_{[0,t]}\colon\mathbb{R}\to\mathbb{R}$ be the characteristic function of $[0,t]$. We substitute $\eta=1_{[0,t]}w^\varepsilon$ for the above equality and calculate as in the proofs of Lemmas 5.10.11 and 5.10.20 by using (5.10.11)–(5.10.13), (5.10.28), (5.10.38), (5.10.64), and Young's inequality. Then we get

$$\|w^\varepsilon(t)\|_{L^2(\Gamma)}^2+\int_0^t\|\nabla_\Gamma w^\varepsilon\|_{L^2(\Gamma)}^2\,ds \\ \leq c\left\{\|w_0^\varepsilon\|_{L^2(\Gamma)}^2+\int_0^t\left(1+\|v^\varepsilon\|_{H^1(\Gamma)}^2\right)\|w^\varepsilon\|_{L^2(\Gamma)}^2\,ds\right. \\ \left.+\int_0^t\|F^\varepsilon\|_{H^{-1}(\Gamma,T\Gamma)}^2\,ds+\delta(\varepsilon)^2(1+t)\right\} \tag{5.10.82}$$

for all $t\in[0,T]$. Here $\delta(\varepsilon)$ given by (5.10.80) comes from (5.10.28) and (5.10.38) (note that $\varepsilon^{\alpha/2}\leq\varepsilon^{\alpha/4}$). This inequality implies that

$$\xi(t)\leq c\left\{\xi(0)+\int_0^t\left(\varphi(s)\xi(s)+\|F^\varepsilon(s)\|_{H^{-1}(\Gamma,T\Gamma)}^2\right)ds\right\}\quad\text{for all}\quad t\in[0,T],$$

where $\xi(t):=\delta(\varepsilon)^2+\|w^\varepsilon(t)\|_{L^2(\Gamma)}^2$ and $\varphi(t):=1+\|v^\varepsilon(t)\|_{H^1(\Gamma)}^2$. Hence by Gronwall's inequality we have

$$\xi(t)\leq c\left(\xi(0)+\int_0^t\|F^\varepsilon(s)\|_{H^{-1}(\Gamma,T\Gamma)}^2\,ds\right)\exp\left(\int_0^t\varphi(s)\,ds\right),\quad t\in[0,T].$$

From this inequality and the estimate (5.10.42) for $\|v^\varepsilon\|_{H^1(\Gamma)}^2$ we deduce that

$$\|w^\varepsilon(t)\|_{L^2(\Gamma)}^2\leq c_T\left\{\delta(\varepsilon)^2+\|w_0^\varepsilon\|_{L^2(\Gamma)}^2+\|F^\varepsilon\|_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))}^2\right\}$$

for all $t\in[0,T]$, where $c_T>0$ is a constant depending only on T. Applying this inequality and (5.10.42) to (5.10.82) we also get the same estimate for the time integral of $\|\nabla_\Gamma w^\varepsilon\|_{L^2(\Gamma)}^2$. Therefore, the inequality (5.10.81) is valid. □

Proof of Theorem 5.10.23. Let $v^\varepsilon = \mathbb{P}_g M_\tau u^\varepsilon$. Since $v_0 \in L^2_{g\sigma}(\Gamma, T\Gamma)$ and $\mathbb{P}_g$ is the orthogonal projection from $L^2(\Gamma, T\Gamma)$ onto $L^2_{g\sigma}(\Gamma, T\Gamma)$,

$$\|v^\varepsilon(0) - v_0\|_{L^2(\Gamma)} = \|\mathbb{P}_g(M_\tau u_0^\varepsilon - v_0)\|_{L^2(\Gamma)} \le \|M_\tau u_0^\varepsilon - v_0\|_{L^2(\Gamma)}.$$

By this inequality, (5.10.36), and (5.10.81) we obtain (5.10.79) (note that $\varepsilon^2 \le \delta(\varepsilon)^2$). $\square$

Now we see that Theorem 5.1.4 is an immediate consequence of Theorem 5.10.23.

Proof of Theorem 5.1.4. The condition (b') implies the condition (b) in Theorem 5.1.3. Thus, the statements in Theorem 5.1.3 are valid. Also, for each $T > 0$ the right-hand side of (5.10.79) converges to zero as $\varepsilon \to 0$ by (5.10.80), $\alpha > 0$, and the conditions (b') and (c). Hence $M_\tau u^\varepsilon$ converges to v strongly in $C([0,T]; L^2(\Gamma, T\Gamma))$ and $L^2(0,T; H^1(\Gamma, T\Gamma))$ as $\varepsilon \to 0$. $\square$

Next let us estimate the difference between a strong solution to (5.1.1)–(5.1.3) and a weak solution to (5.1.13)–(5.1.14) in Ω_ε. Recall that we denote by $\bar\eta = \eta \circ \pi$ the constant extension of a function η on Γ in the normal direction of Γ.

Theorem 5.10.25. *Under the same assumptions as in Theorem 5.10.23, we have*

$$\begin{aligned}\max_{t\in[0,T]} \|u^\varepsilon(t) - \bar v(t)\|^2_{L^2(\Omega_\varepsilon)} &+ \int_0^T \left\|\overline{P}\nabla u^\varepsilon(t) - \overline{\nabla_\Gamma v}(t)\right\|^2_{L^2(\Omega_\varepsilon)} dt \\ &\le c_T\varepsilon\left\{\delta(\varepsilon)^2 + \|M_\tau u_0^\varepsilon - v_0\|^2_{L^2(\Gamma)} + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon - f\|^2_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))}\right\} \end{aligned}\tag{5.10.83}$$

for all $T > 0$, where $c_T > 0$ is a constant depending only on T. In particular,

$$\lim_{\varepsilon\to0} \varepsilon^{-1}\|u^\varepsilon - \bar v\|^2_{C([0,T];L^2(\Omega_\varepsilon))} = \lim_{\varepsilon\to0} \varepsilon^{-1}\left\|\overline{P}\nabla u^\varepsilon - \overline{\nabla_\Gamma v}\right\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} = 0$$

for all $T > 0$ provided that $\lim_{\varepsilon\to0} \varepsilon^{-1}\gamma_\varepsilon^i = \gamma^i$ for $i = 0, 1$ and

$$\lim_{\varepsilon\to0} \|M_\tau u_0^\varepsilon - v_0\|^2_{L^2(\Gamma)} = \lim_{\varepsilon\to0} \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon - f\|^2_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))} = 0.$$

Proof. For the sake of simplicity, we denote by I_ε the right-hand side of (5.10.79). Also, we fix $T > 0$ and suppress $t \in [0,T]$. Let us estimate $u^\varepsilon - \bar v$. Since

$$u^\varepsilon - \bar v = \left(u^\varepsilon - \overline{Mu^\varepsilon}\right) + \left(\overline{Mu^\varepsilon}\cdot\bar n\right)\bar n + \left(\overline{M_\tau u^\varepsilon} - \bar v\right) \quad \text{in} \quad \Omega_\varepsilon,$$

we apply (5.2.53), (5.6.6), and (5.6.10) to the right-hand side to get

$$\|u^\varepsilon - \bar v\|^2_{L^2(\Omega_\varepsilon)} \le c\varepsilon\left(\varepsilon\|u^\varepsilon\|^2_{H^1(\Omega_\varepsilon)} + \|M_\tau u^\varepsilon - v\|^2_{L^2(\Gamma)}\right).$$

Hence we see by (5.8.34), (5.10.79), and (5.10.80) that

$$\|u^\varepsilon(t) - \bar v(t)\|^2_{L^2(\Omega_\varepsilon)} \le c\varepsilon(\varepsilon^\alpha + I_\varepsilon) \le c\varepsilon I_\varepsilon \quad \text{for all} \quad t \in [0,T]. \tag{5.10.84}$$

Next we consider the second term on the left-hand side of (5.10.83). Let

$$J_1 := \left\|\overline{P}\nabla u^\varepsilon - \overline{\nabla_\Gamma M u^\varepsilon}\right\|_{L^2(\Omega_\varepsilon)}, \quad J_2 := \left\|\overline{\nabla_\Gamma[(Mu^\varepsilon\cdot n)n]}\right\|_{L^2(\Omega_\varepsilon)},$$
$$J_3 := \left\|\overline{\nabla_\Gamma M_\tau u^\varepsilon} - \overline{\nabla_\Gamma v}\right\|_{L^2(\Omega_\varepsilon)}.$$

By (5.2.53) and (5.6.25) we have

$$J_1 \leq c\varepsilon\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}, \quad J_3 \leq c\varepsilon^{1/2}\|\nabla_\Gamma M_\tau u^\varepsilon - \nabla_\Gamma v\|_{L^2(\Gamma)}.$$

Also, the inequalities (5.2.53) and (5.6.29) imply that

$$J_2 \leq c\varepsilon^{1/2}\|Mu^\varepsilon \cdot n\|_{H^1(\Gamma)} \leq c\varepsilon\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}.$$

From these inequalities we deduce that

$$\left\|\overline{P}\nabla u^\varepsilon - \overline{\nabla_\Gamma v}\right\|_{L^2(\Omega_\varepsilon)} \leq J_1 + J_2 + J_3 \leq c\varepsilon^{1/2}\left(\varepsilon^{1/2}\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)} + \|\nabla_\Gamma M_\tau u^\varepsilon - \nabla_\Gamma v\|_{L^2(\Gamma)}\right).$$

Thus, by (5.8.34) and (5.10.79) we have

$$\begin{aligned}\int_0^T \left\|\overline{P}\nabla u^\varepsilon - \overline{\nabla_\Gamma v}\right\|_{L^2(\Omega_\varepsilon)}^2 dt &\leq c\varepsilon\int_0^T \left(\varepsilon\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}^2 + \|\nabla_\Gamma M_\tau u^\varepsilon - \nabla_\Gamma v\|_{L^2(\Gamma)}^2\right) dt \\ &\leq c\varepsilon\{\varepsilon^\alpha(1+T) + I_\varepsilon\} \leq c\varepsilon(1+T)I_\varepsilon.\end{aligned}$$

Combining this inequality and (5.10.84) we obtain (5.10.83). □

We can also compare the normal derivative (with respect to Γ) of a strong solution to (5.1.1)–(5.1.3) with a weak solution to (5.1.13)–(5.1.14).

Theorem 5.10.26. *Under the same assumptions as in Theorem 5.10.23, we have*

$$\begin{aligned}\int_0^T &\left(\left\|\overline{P}\partial_n u^\varepsilon(t) + \overline{Wv}(t)\right\|_{L^2(\Omega_\varepsilon)}^2 + \left\|\partial_n u^\varepsilon(t)\cdot\bar{n} - \frac{1}{\bar{g}}\bar{v}(t)\cdot\overline{\nabla_\Gamma g}\right\|_{L^2(\Omega_\varepsilon)}^2\right) dt \\ &\leq c_T\varepsilon\left\{\delta(\varepsilon)^2 + \|M_\tau u_0^\varepsilon - v_0\|_{L^2(\Gamma)}^2 + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon - f\|_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))}^2\right\} \quad (5.10.85)\end{aligned}$$

for all $T>0$, where $\partial_n u = (\bar{n}\cdot\nabla)u$ is the normal derivative of u given by (5.3.5) and $c_T>0$ is a constant depending only on T. Hence, setting

$$V := -Wv + \frac{1}{g}(v\cdot\nabla_\Gamma g)n \quad \text{on} \quad \Gamma\times(0,\infty)$$

we have (note that Wv is tangential on Γ)

$$\lim_{\varepsilon\to0}\varepsilon^{-1}\left\|\partial_n u^\varepsilon - \overline{V}\right\|_{L^2(0,T;L^2(\Omega_\varepsilon))}^2 = 0$$

for all $T>0$ provided that $\lim_{\varepsilon\to0}\varepsilon^{-1}\gamma_\varepsilon^i = \gamma^i$ for $i=0,1$ and

$$\lim_{\varepsilon\to0}\|M_\tau u_0^\varepsilon - v_0\|_{L^2(\Gamma)}^2 = \lim_{\varepsilon\to0}\|M_\tau\mathbb{P}_\varepsilon f^\varepsilon - f\|_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))}^2 = 0.$$

Proof. We fix $T>0$ and suppress $t\in[0,T]$. Let

$$\begin{aligned}J_1 &:= \left\|\overline{P}\partial_n u^\varepsilon + \overline{W}u^\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \quad J_2 := \left\|\overline{WMu^\varepsilon} - \overline{W}u^\varepsilon\right\|_{L^2(\Omega_\varepsilon)}, \\ J_3 &:= \left\|\overline{Wv} - \overline{WMu^\varepsilon}\right\|_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

We apply (5.3.34) to J_1 and (5.6.6) to J_2 to get

$$J_1 \leq c\varepsilon\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)}, \quad J_2 \leq c\varepsilon\|u^\varepsilon\|_{H^1(\Omega_\varepsilon)}.$$

Also, noting that $WMu^\varepsilon = WM_\tau u^\varepsilon$ on Γ by (5.2.6), we use (5.2.53) to get

$$J_3 \le c\varepsilon^{1/2}\|Wv - WM_\tau u^\varepsilon\|_{L^2(\Gamma)} \le c\varepsilon^{1/2}\|v - M_\tau u^\varepsilon\|_{L^2(\Gamma)}.$$

From these inequalities we deduce that

$$\left\|\overline{P}\partial_n u^\varepsilon + \overline{Wv}\right\|_{L^2(\Omega_\varepsilon)} \le J_1 + J_2 + J_3 \le c\varepsilon^{1/2}\left(\varepsilon^{1/2}\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)} + \|M_\tau u^\varepsilon - v\|_{L^2(\Gamma)}\right).$$

We also observe by (5.2.30), (5.2.53), and (5.6.48) that

$$\begin{aligned}
&\left\|\partial_n u^\varepsilon \cdot \bar{n} - \frac{1}{\bar{g}}\bar{v}\cdot\overline{\nabla_\Gamma g}\right\|_{L^2(\Omega_\varepsilon)} \\
&\quad \le \left\|\partial_n u^\varepsilon \cdot \bar{n} - \frac{1}{\bar{g}}\overline{M_\tau u^\varepsilon}\cdot\overline{\nabla_\Gamma g}\right\|_{L^2(\Omega_\varepsilon)} + \left\|\frac{1}{\bar{g}}\left(\overline{M_\tau u^\varepsilon} - \bar{v}\right)\cdot\overline{\nabla_\Gamma g}\right\|_{L^2(\Omega_\varepsilon)} \\
&\quad \le c\varepsilon^{1/2}\left(\varepsilon^{1/2}\|u^\varepsilon\|_{H^2(\Omega_\varepsilon)} + \|M_\tau u^\varepsilon - v\|_{L^2(\Gamma)}\right).
\end{aligned}$$

Hence, as in the proof of Theorem 5.10.25, we integrate the square of the above inequalities over $(0,T)$ and then use (5.8.34) and (5.10.79) to obtain (5.10.85). □

Remark 5.10.27. In the estimate (5.10.85) the Weingarten map W represents the curvatures of the limit surface Γ. On the other hand, the functions g_0 and g_1 with $g = g_1 - g_0$ are used to define the inner and outer boundaries of the curved thin domain Ω_ε. Therefore, roughly speaking, the tangential component (with respect to Γ) of the normal derivative $\partial_n u^\varepsilon$ of the bulk velocity depends only on the shape of Γ, while the geometry of the boundaries of Ω_ε affects only the normal component of $\partial_n u^\varepsilon$.

5.A Notations and basic formulas on vectors and matrices

We give notations and basic formulas on vectors and matrices, and use them to prove Lemmas 5.4.4 and 5.7.2.

For a matrix $A \in \mathbb{R}^{3\times 3}$ we denote by A^T and $A_S := (A + A^T)/2$ the transpose and the symmetric part of A, respectively. We define the tensor product of vectors $a \in \mathbb{R}^l$ and $b \in \mathbb{R}^m$ with $l, m \in \mathbb{N}$ as

$$a \otimes b := \begin{pmatrix} a_1b_1 & \cdots & a_1b_m \\ \vdots & & \vdots \\ a_lb_1 & \cdots & a_lb_m \end{pmatrix}, \quad a = (a_1,\dots,a_l),\, b = (b_1,\dots,b_m).$$

Also, for a vector field $u = (u_1, u_2, u_3)$ on an open subset of $\mathbb{R}^3$ we write

$$\nabla u := \begin{pmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 \\ \partial_3 u_1 & \partial_3 u_2 & \partial_3 u_3 \end{pmatrix}, \quad |\nabla^2 u|^2 := \sum_{i,j,k=1}^3 |\partial_i\partial_j u_k|^2 \quad \left(\partial_i := \frac{\partial}{\partial x_i}\right).$$

We define the inner product of matrices $A, B \in \mathbb{R}^{3\times 3}$ and the norm of A as

$$A : B := \mathrm{tr}[A^T B] = \sum_{i=1}^3 Ae_i \cdot Be_i, \quad |A| := \sqrt{A : A},$$

where $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Note that $A : B$ does not depend on a choice of an orthonormal basis of $\mathbb{R}^3$. In particular, taking the standard basis of $\mathbb{R}^3$ we see that

$$A : B = \sum_{i,j=1}^{3} A_{ij}B_{ij} = B : A = A^T : B^T, \quad AB : C = A : CB^T = B : A^T C$$

for $A, B, C \in \mathbb{R}^{3\times 3}$. Also, for $a, b \in \mathbb{R}^3$ we have $|a \otimes b| = |a||b|$.

Lemma 5.A.1. *Let $n_0 \in \mathbb{R}^3$ and $A_1 \in \mathbb{R}^{3\times 3}$ satisfy $|n_0| = 1$ and $A_1 b \cdot n_0 = 0$ for all $b \in \mathbb{R}^3$. Then for a matrix $B := A_1 + n_0 \otimes a$ with $a \in \mathbb{R}^3$ we have*

$$|B|^2 = |A_1|^2 + |a|^2. \tag{5.A.1}$$

Also, let $\tau_1, \tau_2 \in \mathbb{R}^3$ and $A_2 \in \mathbb{R}^{3\times 3}$ satisfy

$$n_0 \cdot \tau_1 = n_0 \cdot \tau_2 = 0, \quad A_2 n_0 = 0, \quad A_2 b \cdot n_0 = 0 \quad \textit{for all} \quad b \in \mathbb{R}^3.$$

Then for a matrix $C := A_2 + \tau_1 \otimes n_0 + n_0 \otimes \tau_2 + c n_0 \otimes n_0$ with $c \in \mathbb{R}$ we have

$$|C|^2 = |A_2|^2 + |\tau_1|^2 + |\tau_2|^2 + |c|^2. \tag{5.A.2}$$

Proof. We take an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ with $e_3 = n_0$. Then since $(n_0 \otimes a)b = (a \cdot b)n_0$, $A_1 b \cdot n_0 = 0$ for any $b \in \mathbb{R}^3$ and $|n_0| = 1$,

$$\begin{aligned} |Be_i|^2 &= |A_1 e_i + (a \cdot e_i)n_0|^2 = |A_1 e_i|^2 + (a \cdot e_i)^2, \quad i = 1, 2, \\ |Bn_0|^2 &= |A_1 n_0 + (a \cdot n_0)n_0|^2 = |A_1 n_0|^2 + (a \cdot n_0)^2. \end{aligned}$$

Applying these equalities to $|B|^2 = |Be_1|^2 + |Be_2|^2 + |Bn_0|^2$ we get

$$|B|^2 = (|A_1 e_1|^2 + |A_1 e_2|^2 + |A_1 n_0|^2) + \{(a \cdot e_1)^2 + (a \cdot e_2)^2 + (a \cdot n_0)^2\} = |A_1|^2 + |a|^2.$$

Thus (5.A.1) is valid. Next we prove (5.A.2). Since $e_i \cdot n_0 = 0$ and $A_2 e_i \cdot n_0 = 0$,

$$|Ce_i|^2 = |A_2 e_i + (\tau_2 \cdot e_i)n_0|^2 = |A_2 e_i|^2 + (\tau_2 \cdot e_i)^2, \quad i = 1, 2.$$

Also, by $A_2 n_0 = 0$, $|n_0| = 1$, and $\tau_1 \cdot n_0 = 0$,

$$|Cn_0|^2 = |\tau_1 + cn_0|^2 = |\tau_1|^2 + |c|^2.$$

Hence $|C|^2 = |Ce_1|^2 + |Ce_2|^2 + |Cn_0|^2$ is of the form

$$|C|^2 = \sum_{i=1,2} \bigl(|A_2 e_i|^2 + (\tau_2 \cdot e_i)^2\bigr) + |\tau_1|^2 + |c|^2.$$

Here the first two terms on the right-hand side are equal to $|A_2|^2$ and $|\tau_2|^2$ since $A_2 n_0 = 0$ and $\tau_2 \cdot n_0 = 0$. Hence (5.A.2) follows. $\square$

Based on the formulas (5.A.1) and (5.A.2) we prove Lemma 5.4.4.

Proof of Lemma 5.4.4. Let $u \in H^1(\Omega_\varepsilon)^3$ and $U := u \circ \Phi_\varepsilon$, where $\Phi_\varepsilon \colon \Omega_1 \to \Omega_\varepsilon$ is given by (5.4.15). The inverse map of Φ_ε is of the form

$$\Phi_\varepsilon^{-1}(x) := \pi(x) + \varepsilon^{-1} d(x)\bar{n}(x), \quad x \in \Omega_\varepsilon.$$

Then by (5.2.6), (5.2.11), (5.2.16), and $\nabla d = \bar{n}$ we have

$$\nabla\Phi_\varepsilon^{-1}(x) = \left\{I_3 - d(x)\overline{W}(x)\right\}^{-1}\left\{I_3 - \varepsilon^{-1}d(x)\overline{W}(x)\right\}\overline{P}(x) + \varepsilon^{-1}\overline{Q}(x)$$

for $x \in \Omega_\varepsilon$. We set $x = \Phi_\varepsilon(X)$ with $X \in \Omega_1$ in this equality and use

$$d(\Phi_\varepsilon(X)) = \varepsilon d(X), \quad \pi(\Phi_\varepsilon(X)) = \pi(X)$$

to get $\nabla\Phi_\varepsilon^{-1}(\Phi_\varepsilon(X)) = \Lambda_\varepsilon(X)\overline{P}(X) + \varepsilon^{-1}\overline{Q}(X)$, where

$$\Lambda_\varepsilon(X) := \left\{I_3 - \varepsilon d(X)\overline{W}(X)\right\}^{-1}\left\{I_3 - d(X)\overline{W}(X)\right\}, \quad X \in \Omega_1.$$

From this formula and

$$\nabla u(\Phi_\varepsilon(X)) = \nabla\Phi_\varepsilon^{-1}(\Phi_\varepsilon(X))\nabla U(X), \quad \overline{Q}(X)\nabla U(X) = \bar{n}(X) \otimes \partial_n U(X)$$

it follows that

$$\nabla u(\Phi_\varepsilon(X)) = \Lambda_\varepsilon(X)\overline{P}(X)\nabla U(X) + \varepsilon^{-1}\bar{n}(X) \otimes \partial_n U(X), \quad X \in \Omega_1.$$

Hereafter we refer to this equality as

$$(\nabla u) \circ \Phi_\varepsilon = \Lambda_\varepsilon \overline{P}\nabla U + \varepsilon^{-1}\bar{n} \otimes \partial_n U \quad \text{in} \quad \Omega_1. \tag{5.A.3}$$

Since $\Lambda_\varepsilon\overline{P} = \overline{P}\Lambda_\varepsilon$ by (5.2.6) and (5.2.8), we have $(\Lambda_\varepsilon\overline{P}\nabla U)b \cdot \bar{n} = 0$ for any $b \in \mathbb{R}^3$. Hence we can use (5.A.1) with $n_0 = \bar{n}$, $A_1 = \Lambda_\varepsilon\overline{P}\nabla U$, and $a = \varepsilon^{-1}\partial_n U$ to get

$$|(\nabla u) \circ \Phi_\varepsilon|^2 = \left|\Lambda_\varepsilon\overline{P}\nabla U\right|^2 + \varepsilon^{-2}|\partial_n U|^2 \geq c\left(\left|\overline{P}\nabla U\right|^2 + \varepsilon^{-2}|\partial_n U|^2\right) \quad \text{in} \quad \Omega_1,$$

where the second inequality follows from (5.2.9). By this inequality and (5.4.17),

$$\varepsilon^{-1}\|\nabla u\|_{L^2(\Omega_\varepsilon)}^2 \geq c\|(\nabla u) \circ \Phi_\varepsilon\|_{L^2(\Omega_1)}^2 \geq c\left(\left\|\overline{P}\nabla U\right\|_{L^2(\Omega_1)}^2 + \varepsilon^{-2}\|\partial_n U\|_{L^2(\Omega_1)}^2\right).$$

Hence (5.4.18) is valid. To prove (5.4.19) we observe by $I_3 = P + Q$ and (5.3.6) that

$$\begin{aligned}
\overline{P}\nabla U &= \overline{P}(\nabla U)\overline{P} + \overline{P}(\nabla U)\overline{Q} = \overline{P}(\nabla U)\overline{P} + \left[\overline{P}(\nabla U)\bar{n}\right] \otimes \bar{n}, \\
\partial_n U &= \partial_n\left[\overline{P}U + (U \cdot \bar{n})\bar{n}\right] = \overline{P}\partial_n U + \{\partial_n(U \cdot \bar{n})\}\bar{n}.
\end{aligned}$$

We apply these equalities to (5.A.3) and use $\Lambda_\varepsilon\overline{P} = \overline{P}\Lambda_\varepsilon$ by (5.2.6) and (5.2.8) to get

$$\begin{aligned}
(\nabla u) \circ \Phi_\varepsilon = \overline{P}\Lambda_\varepsilon(\nabla U)\overline{P} &+ \left[\overline{P}\Lambda_\varepsilon(\nabla U)\bar{n}\right] \otimes \bar{n} \\
&+ \varepsilon^{-1}\bar{n} \otimes \left(\overline{P}\partial_n U\right) + \varepsilon^{-1}\{\partial_n(U \cdot \bar{n})\}\bar{n} \otimes \bar{n} \quad \text{in} \quad \Omega_1.
\end{aligned}$$

Hence $D(u) \circ \Phi_\varepsilon = \{(\nabla u) \circ \Phi_\varepsilon + (\nabla u)^T \circ \Phi_\varepsilon\}/2$ is of the form

$$D(u) \circ \Phi_\varepsilon = A_2 + \tau \otimes \bar{n} + \bar{n} \otimes \tau + \varepsilon^{-1}\{\partial_n(U \cdot \bar{n})\}\bar{n} \otimes \bar{n} \quad \text{in} \quad \Omega_1,$$

where $F_\varepsilon(U) := \Lambda_\varepsilon \nabla U$ is the matrix (5.4.20) and

$$A_2 := \overline{P} F_\varepsilon(U)_S \overline{P}, \quad \tau := \frac{1}{2}\overline{P}\{\Lambda_\varepsilon(\nabla U)\bar{n} + \varepsilon^{-1}\partial_n U\}.$$

(Note that $F_\varepsilon(U)_S$ is the symmetric part of $F_\varepsilon(U)$.) By the above definitions it is obvious that $\tau \cdot \bar{n} = 0$, $A_2\bar{n} = 0$, and $A_2 b \cdot \bar{n} = 0$ for all $b \in \mathbb{R}^3$. Therefore, we can apply (5.A.2) to $C = D(u) \circ \Phi_\varepsilon$ to obtain

$$|D(u) \circ \Phi_\varepsilon|^2 = |A_2|^2 + 2|\tau|^2 + \varepsilon^{-2}|\partial_n(U \cdot \bar{n})|^2 \geq |A_2|^2 + \varepsilon^{-2}|\partial_n(U \cdot \bar{n})|^2$$

in Ω_1. From this inequality and (5.4.17) we deduce that

$$\varepsilon^{-1}\|D(u)\|^2_{L^2(\Omega_\varepsilon)} \geq c\|D(u) \circ \Phi_\varepsilon\|^2_{L^2(\Omega_1)} \geq c\left(\|A_2\|^2_{L^2(\Omega_1)} + \varepsilon^{-2}\|\partial_n(U \cdot \bar{n})\|^2_{L^2(\Omega_1)}\right).$$

Hence (5.4.19) is valid (note that $A_2 = \overline{P} F_\varepsilon(U)_S \overline{P}$). □

Next we give a formula on the curl of a vector field and show Lemma 5.7.2.

Lemma 5.A.2. *Let U be an open set in $\mathbb{R}^3$ and E_1, E_2, and E_3 vector fields on U such that $\{E_1(x), E_2(x), E_3(x)\}$ is an orthonormal basis of $\mathbb{R}^3$ for each $x \in \mathbb{R}^3$ and*

$$E_1 \times E_2 = E_3, \quad E_2 \times E_3 = E_1, \quad E_3 \times E_1 = E_2 \quad \text{in} \quad U.$$

Then for $u \in C^1(U)^3$ we have

$$\begin{aligned} \operatorname{curl} u = \{&(E_2 \cdot \nabla)u \cdot E_3 - (E_3 \cdot \nabla)u \cdot E_2\}E_1 \\ &+ \{(E_3 \cdot \nabla)u \cdot E_1 - (E_1 \cdot \nabla)u \cdot E_3\}E_2 \\ &+ \{(E_1 \cdot \nabla)u \cdot E_2 - (E_2 \cdot \nabla)u \cdot E_1\}E_3 \quad \text{in} \quad U. \end{aligned} \tag{5.A.4}$$

Proof. By the assumption, $\operatorname{curl} u = \sum_{i=1}^3(\operatorname{curl} u \cdot E_i)E_i$. Since $E_1 = E_2 \times E_3$,

$$\begin{aligned} \operatorname{curl} u \cdot E_1 &= \operatorname{curl} u \cdot (E_2 \times E_3) = E_2 \cdot (E_3 \times \operatorname{curl} u) \\ &= E_2 \cdot \{(\nabla u)E_3 - (\nabla u)^T E_3\} \\ &= (\nabla u)^T E_2 \cdot E_3 - (\nabla u)^T E_3 \cdot E_2 \\ &= (E_2 \cdot \nabla)u \cdot E_3 - (E_3 \cdot \nabla)u \cdot E_2. \end{aligned}$$

Calculating $\operatorname{curl} u \cdot E_i$, $i = 2, 3$ in the same way we observe that (5.A.4) holds. □

Proof of Lemma 5.7.2. Let $u \in C^1(\Omega_\varepsilon)^3$ and $u^a := E_\varepsilon M_\tau u$ be given by (5.6.50). Since the surface Γ is compact, we can take finite relative open sets O_k of Γ and pairs of tangential vector fields $\{\tau_1^k, \tau_2^k\}$ on O_k, $k = 1, \dots, k_0$ such that $\Gamma = \bigcup_{k=1}^{k_0} O_k$, the triplet $\{\tau_1^k, \tau_2^k, n\}$ forms an orthonormal basis of $\mathbb{R}^3$ on O_k, and

$$\tau_1^k \times \tau_2^k = n, \quad \tau_2^k \times n = \tau_1^k, \quad n \times \tau_1^k = \tau_2^k \quad \text{on} \quad O_k$$

for each $k = 1, \dots, k_0$. Then since $\Omega_\varepsilon = \bigcup_{k=1}^{k_0} U_k$, where

$$U_k := \{y + rn(y) \mid y \in O_k,\, r \in (\varepsilon g_0(y), \varepsilon g_1(y))\}, \quad k = 1, \dots, k_0,$$

it is sufficient to show (5.7.2) in U_k for each $k = 1, \dots, k_0$. From now on, we fix and suppress k. We use (5.A.4) to u^a with $E_1 := \bar{\tau}_1$, $E_2 := \bar{\tau}_2$, and $E_3 := \bar{n}$. Then since $P\tau_i = \tau_i$, $i = 1, 2$ and $Pn = 0$,

$$\overline{P}\operatorname{curl} u^a = \{(\bar{\tau}_2 \cdot \nabla)u^a \cdot \bar{n} - (\bar{n} \cdot \nabla)u^a \cdot \bar{\tau}_2\}\bar{\tau}_1 + \{(\bar{n} \cdot \nabla)u^a \cdot \bar{\tau}_1 - (\bar{\tau}_1 \cdot \nabla)u^a \cdot \bar{n}\}\bar{\tau}_2 \quad \text{in} \quad U.$$

From this equality, $(\bar{n} \cdot \nabla)u^a = \partial_n u^a$, and $|\tau_1| = |\tau_2| = |n| = 1$ we deduce that

$$\left|\overline{P}\operatorname{curl} u^a\right| \leq c\left(|\partial_n u^a| + |(\bar{\tau}_1 \cdot \nabla)u^a \cdot \bar{n}| + |(\bar{\tau}_2 \cdot \nabla)u^a \cdot \bar{n}|\right) \quad \text{in} \quad U. \tag{5.A.5}$$

Let us estimate each term on the right-hand side. By (5.3.6), (5.3.37), and (5.6.50),

$$|\partial_n u^a| = \left|\overline{M_\tau u} \cdot \partial_n \Psi_\varepsilon\right| \leq c\left|\overline{M_\tau u}\right| = c\left|\overline{PMu}\right| \leq c\left|\overline{Mu}\right| \quad \text{in} \quad U. \tag{5.A.6}$$

To estimate the other terms we set

$$u_\tau^a := \overline{P}u^a = \overline{M_\tau u}, \quad u_n^a := (u^a \cdot \bar{n})\bar{n} = \left(\overline{M_\tau u} \cdot \Psi_\varepsilon\right)\bar{n}$$

so that $u^a = u_\tau^a + u_n^a$. Let $i = 1, 2$. Since $u_\tau^a \cdot \bar{n} = 0$ in U, we have

$$(\bar{\tau}_i \cdot \nabla)u_\tau^a \cdot \bar{n} = (\bar{\tau}_i \cdot \nabla)(u_\tau^a \cdot \bar{n}) - u_\tau^a \cdot (\bar{\tau}_i \cdot \nabla)\bar{n} = -u_\tau^a \cdot (\bar{\tau}_i \cdot \nabla)\bar{n}.$$

Hence by (5.2.17) and $|\tau_i| = 1$ we get

$$|(\bar{\tau}_i \cdot \nabla)u_\tau^a \cdot \bar{n}| \leq c|u_\tau^a| \leq c\left|\overline{Mu}\right| \quad \text{in} \quad U,\ i = 1, 2. \tag{5.A.7}$$

Also, by $\tau_i = P\tau_i$, $P = P^T$, and $|\tau_i| = 1$ we see that

$$|(\bar{\tau}_i \cdot \nabla)u_n^a| = \left|(\nabla u_n^a)^T \overline{P}\bar{\tau}_i\right| = \left|\left[\overline{P}(\nabla u_n^a)\right]^T \bar{\tau}_i\right| \leq \left|\overline{P}(\nabla u_n^a)\right| \quad \text{in} \quad U.$$

Moreover, by $u_n^a = (\overline{M_\tau u} \cdot \Psi_\varepsilon)\bar{n}$ we have

$$\overline{P}(\nabla u_n^a) = \left[\left\{\overline{P}\nabla\left(\overline{M_\tau u}\right)\right\}\Psi_\varepsilon + \left(\overline{P}\nabla\Psi_\varepsilon\right)\overline{M_\tau u}\right] \otimes \bar{n} + \left(\overline{M_\tau u} \cdot \Psi_\varepsilon\right)\overline{P}\nabla\bar{n}$$

and thus the inequalities (5.2.13), (5.2.17), and (5.3.37) imply that

$$\left|\overline{P}(\nabla u_n^a)\right| \leq c\varepsilon\left(\left|\overline{M_\tau u}\right| + \left|\overline{\nabla_\Gamma M_\tau u}\right|\right) \leq c\varepsilon\left(\left|\overline{Mu}\right| + \left|\overline{\nabla_\Gamma Mu}\right|\right) \quad \text{in} \quad U.$$

Here the last inequality follows from $M_\tau u = PMu$ and $P \in C^4(\Gamma)^{3\times 3}$. Therefore,

$$|(\bar{\tau}_i \cdot \nabla)u_n^a \cdot \bar{n}| \leq c\varepsilon\left(\left|\overline{Mu}\right| + \left|\overline{\nabla_\Gamma Mu}\right|\right) \quad \text{in} \quad U,\ i = 1, 2. \tag{5.A.8}$$

Noting that $u^a = u_\tau^a + u_n^a$, we apply (5.A.6), (5.A.7), and (5.A.8) to the right-hand side of (5.A.5) to conclude that the inequality (5.7.2) is valid in U. $\square$

5.B Calculations involving differential geometry of surfaces

The purpose of this appendix is to give the proofs of the lemmas in Section 5.2 and related results, which involve calculations of the surface quantities on Γ, Γ_ε^0, and Γ_ε^1. We also show the formula (5.3.15) in the proof of Lemma 5.3.3.

First we assume that the closed surface Γ is of class C^ℓ with $\ell \geq 2$ and prove the lemmas given in Section 5.2.1 and a few results on Γ.

Proof of Lemma 5.2.2. Since W has the eigenvalues zero, κ_1, and κ_2,

$$\det[I_3 - rW(y)] = \{1 - r\kappa_1(y)\}\{1 - r\kappa_2(y)\} > 0, \quad y \in \Gamma,\, r \in (-\delta, \delta)$$

by (5.2.2). Hence $I_3 - rW(y)$ is invertible. Also, the equality (5.2.8) follows from (5.2.6).

Let us prove (5.2.9) and (5.2.10). For the sake of simplicity, we fix and suppress the argument $y \in \Gamma$. Since W is real and symmetric by Lemma 5.2.1 and has the eigenvalues κ_1, κ_2, and zero with $Wn = 0$, we can take an orthonormal basis $\{\tau_1, \tau_2, n\}$ of $\mathbb{R}^3$ such that $W\tau_i = \kappa_i \tau_i$, $i = 1, 2$. Then we have

$$(I_3 - rW)\tau_i = (1 - r\kappa_i)\tau_i, \quad (I_3 - rW)n = n$$

for $r \in (-\delta, \delta)$ and $i = 1, 2$, and thus

$$(I_3 - rW)^{-1}\tau_i = (1 - r\kappa_i)^{-1}\tau_i, \quad (I_3 - rW)^{-1}n = n. \tag{5.B.1}$$

Since $\{\tau_1, \tau_2, n\}$ is an orthonormal basis of $\mathbb{R}^3$, these formulas imply that

$$\begin{aligned}(I_3 - rW)^k a &= \sum_{i=1,2} (a \cdot \tau_i)(I_3 - rW)^k \tau_i + (a \cdot n)(I_3 - rW)^k n \\ &= \sum_{i=1,2} (a \cdot \tau_i)(1 - r\kappa_i)^k \tau_i + (a \cdot n)n\end{aligned}$$

for all $a \in \mathbb{R}^3$ and $k = \pm 1$. Hence

$$\left|(I_3 - rW)^k a\right|^2 = \sum_{i=1,2} (a \cdot \tau_i)^2 (1 - r\kappa_i)^{2k} + (a \cdot n)^2$$

and (5.2.9) follows from (5.2.2) and $|a|^2 = (a \cdot \tau_1)^2 + (a \cdot \tau_2)^2 + (a \cdot n)^2$. Also, from (5.B.1), $|\tau_1| = |\tau_2| = 1$, and $|1 - (1 - r\kappa_i)^{-1}| \leq |r\kappa_i(1 - r\kappa_i)^{-1}| \leq c|r|$ by (5.2.2) we deduce that

$$\left|I_3 - (I_3 - rW)^{-1}\right|^2 = \sum_{i=1,2} |1 - (1 - r\kappa_i)^{-1}|^2 \leq c|r|^2.$$

Hence (5.2.10) is valid. □

Proof of Lemma 5.2.3. By (5.2.1) and $n(\pi(x)) = \bar{n}(\pi(x))$ we have

$$\pi(x) = x - d(x)\bar{n}(\pi(x)), \quad x \in N.$$

We differentiate both sides in x and then use $\nabla d = \bar{n}$ and

$$-\nabla \bar{n}(\pi(x)) = -\overline{\nabla_\Gamma n}(\pi(x)) = \overline{W}(\pi(x)) = \overline{W}(x)$$

by (5.2.4) (note that $\pi(x) \in \Gamma$) to get

$$\nabla \pi(x) = I_3 - \bar{n}(x) \otimes \bar{n}(x) - d(x)\nabla\pi(x)\nabla\bar{n}(\pi(x)) = \overline{P}(x) + d(x)\nabla\pi(x)\overline{W}(x)$$

for $x \in N$, which implies that

$$\pi(x)\left\{I_3 - d(x)\overline{W}(x)\right\} = \overline{P}(x), \quad x \in N.$$

Since $I_3 - d(x)\overline{W}(x)$ is invertible for $x \in N$ by Lemma 5.2.2, the equality (5.2.11) follows from this equality and (5.2.8). Also, by (5.2.11) and $P\nabla_\Gamma\eta = \nabla_\Gamma\eta$ on Γ we get (5.2.12). The inequalities (5.2.13) and (5.2.14) follows from (5.2.9), (5.2.10), and (5.2.12).

Now suppose that Γ is of class C^3 and let us prove (5.2.15). For $\eta \in C^2(\Gamma)$ and $i = 1, 2, 3$ we differentiate both sides of (5.2.12) with respect to x_i to get

$$\partial_i \nabla\bar{\eta} = \left\{\partial_i\Big(I_3 - d\overline{W}\Big)^{-1}\right\}\overline{\nabla_\Gamma\eta} + \Big(I_3 - d\overline{W}\Big)^{-1}\partial_i\Big(\overline{\nabla_\Gamma\eta}\Big) \quad \text{in} \quad N. \tag{5.B.2}$$

To estimate the right-hand side we differentiate both sides of

$$\left\{I_3 - d(x)\overline{W}(x)\right\}^{-1}\left\{I_3 - d(x)\overline{W}(x)\right\} = I_3, \quad x \in N$$

with respect to x_i and use $\nabla d = \bar{n}$ to get

$$\partial_i\Big(I_3 - d\overline{W}\Big)^{-1} = \Big(I_3 - d\overline{W}\Big)^{-1}\Big(\bar{n}_i\overline{W} + d\partial_i\overline{W}\Big)\Big(I_3 - d\overline{W}\Big)^{-1} \quad \text{in} \quad N. \tag{5.B.3}$$

Here the right-hand side is bounded on N by (5.2.9), (5.2.12), and the C^1-regularity of W on Γ (note that Γ is of class C^3). Using this fact and (5.2.9) to (5.B.2) we obtain

$$|\partial_i\nabla\bar{\eta}| \le c\left(\left|\overline{\nabla_\Gamma\eta}\right| + \left|\overline{\nabla_\Gamma^2\eta}\right|\right) \quad \text{in} \quad N,\ i = 1, 2, 3,$$

which shows (5.2.15). Moreover, by (5.2.4), (5.B.2), (5.B.3), and $d = 0$ on Γ,

$$\partial_i\nabla\bar{\eta} = n_i W\nabla_\Gamma\eta + \underline{D}_i(\nabla_\Gamma\eta), \quad \text{i.e.} \quad \partial_i\partial_j\bar{\eta} = n_i\sum_{k=1}^3 W_{jk}\underline{D}_k\eta + \underline{D}_i\underline{D}_j\eta$$

on Γ for $i, j = 1, 2, 3$, which implies that

$$\Delta\bar{\eta} = \sum_{i=1}^3 \partial_i^2\bar{\eta} = \sum_{i,k=1}^3 (n_i W_{ik}\underline{D}_k\eta + \underline{D}_i^2\eta) = W^T n \cdot \nabla_\Gamma\eta + \Delta_\Gamma\eta.$$

Since $W^T n = Wn = 0$ by Lemma 5.2.1, we obtain $\Delta\bar{\eta} = \Delta_\Gamma\eta$ on Γ. □

Using the orthonormal frame $\{\tau_1, \tau_2, n\}$ given in the proof of Lemma 5.2.2 and the formula (5.2.12), we can express the divergence of the constant extension of a surface vector field in terms of the surface quantities on Γ.

Lemma 5.B.1. *For $v \in C^1(\Gamma)^3$ let $\bar{v} = v \circ \pi$ be its constant extension. Then*

$$\operatorname{div}\bar{v} = \frac{1}{(1 - d\bar{\kappa}_1)(1 - d\bar{\kappa}_2)}\left\{\Big(1 - d\overline{H}\Big)\overline{\operatorname{div}_\Gamma v} + d\Big(\overline{\nabla_\Gamma v} : \overline{W}\Big)\right\} \quad \text{in} \quad N. \tag{5.B.4}$$

Moreover, $\operatorname{div}\bar{v} = 0$ *in N provided that $D_\Gamma(v) = 0$ on Γ, where $D_\Gamma(v)$ is the surface strain rate tensor given by* (5.4.38).

Proof. Let $x = y + rn(y) \in N$ with $y = \pi(x) \in \Gamma$ and $r = d(x) \in (-\delta, \delta)$. In what follows, we suppress the argument y for functions on Γ. As in the proof of Lemma 5.2.2, we take an orthonormal basis $\{\tau_1,\tau_2,n\}$ of $\mathbb{R}^3$ such that $W\tau_i=\kappa_i\tau_i$ for $i=1,2$. Then by (5.2.12), (5.B.1), and the symmetry of W we have

$$\begin{aligned}\operatorname{div}\bar{v}(x)&=\operatorname{tr}[\nabla\bar{v}(x)]=\operatorname{tr}[(I_3-rW)^{-1}\nabla_\Gamma v]\\&=\sum_{i=1,2}(I_3-rW)^{-1}(\nabla_\Gamma v)\tau_i\cdot\tau_i+(I_3-rW)^{-1}(\nabla_\Gamma v)n\cdot n\\&=\sum_{i=1,2}(\nabla_\Gamma v)\tau_i\cdot(1-r\kappa_i)^{-1}\tau_i+(\nabla_\Gamma v)n\cdot n.\end{aligned}$$

Here $(\nabla_\Gamma v)n\cdot n=P(\nabla_\Gamma v)n\cdot n=0$. Hence

$$\begin{aligned}\operatorname{div}\bar{v}(x)&=\frac{(1-r\kappa_2)(\nabla_\Gamma v)\tau_1\cdot\tau_1+(1-r\kappa_1)(\nabla_\Gamma v)\tau_2\cdot\tau_2}{(1-r\kappa_1)(1-r\kappa_2)}\\&=\frac{\{1-r(\kappa_1+\kappa_2)\}I_1+rI_2}{(1-r\kappa_1)(1-r\kappa_2)},\end{aligned}\tag{5.B.5}$$

where

$$I_1:=\sum_{i=1,2}(\nabla_\Gamma v)\tau_i\cdot\tau_i,\quad I_2:=\sum_{i=1,2}(\nabla_\Gamma v)\tau_i\cdot\kappa_i\tau_i.$$

Since $(\nabla_\Gamma v)n\cdot n=0$ and $\{\tau_1,\tau_2,n\}$ is an orthonormal basis of $\mathbb{R}^3$,

$$I_1=\sum_{i=1,2}(\nabla_\Gamma v)\tau_i\cdot\tau_i+(\nabla_\Gamma v)n\cdot n=\operatorname{tr}[\nabla_\Gamma v]=\operatorname{div}_\Gamma v.\tag{5.B.6}$$

Also, since $W\tau_i=\kappa_i\tau_i$ and $Wn=0$,

$$I_2=\sum_{i=1,2}(\nabla_\Gamma v)\tau_i\cdot W\tau_i+(\nabla_\Gamma v)n\cdot Wn=\nabla_\Gamma v:W.$$

We apply these equalities, $\kappa_1+\kappa_2=H$, and $r=d(x)$ to (5.B.5) to get (5.B.4).

Now let v satisfy $D_\Gamma(v)=0$ on Γ. Since $\{\tau_1,\tau_2,n\}$ is an orthonormal basis of $\mathbb{R}^3$, τ_1 and τ_2 are tangent to Γ (at y). Hence $P\tau_i=\tau_i$ for $i=1,2$ and

$$0=\operatorname{tr}[D_\Gamma(v)]=\sum_{i=1,2}P(\nabla_\Gamma v)P\tau_i\cdot\tau_i+P(\nabla_\Gamma v)Pn\cdot n=\sum_{i=1,2}(\nabla_\Gamma v)\tau_i\cdot\tau_i=\operatorname{div}_\Gamma v$$

by $P^T=P$, $Pn=0$, and (5.B.6). Moreover, by (5.2.6) and the symmetry of W,

$$\nabla_\Gamma v:W=P(\nabla_\Gamma v)P:W=D_\Gamma(v):W=0.$$

Applying these equalities to (5.B.4) we observe that $\operatorname{div}\bar{v}=0$ in N. □

Let us prove the density result on the Sobolev space on Γ.

Proof of Lemma 5.2.4. Here we only show the density of $C^\ell(\Gamma)$ in $W^{m,p}(\Gamma)$ in the case $\ell=m=2$ and $p\in[1,\infty)$. The assertion in other cases are proved similarly.

Let $\eta\in W^{2,p}(\Gamma)$. By a standard localization argument with a partition of unity of Γ (note that Γ is compact), we may assume that there exist an open set U in $\mathbb{R}^2$ and a local parametrization $\mu\colon U\to\Gamma$ such that η is supported in $\mu(\mathcal{K})$, where $\mathcal{K}$ is a compact subset of U. Since Γ is of class C^2, the local parametrization μ is of class C^2 on U and thus there exists a constant $c>0$ such that

$$|\partial_{s_i}\mu(s)|\leq c, \quad |\partial_{s_i}\partial_{s_j}\mu(s)|\leq c, \quad s\in\mathcal{K},\, i,j=1,2. \tag{5.B.7}$$

We denote by $\theta=(\theta_{ij})_{i,j}$ the Riemannian metric of Γ. It is locally given by

$$\theta_{ij}(s):=\partial_{s_i}\mu(s)\cdot\partial_{s_j}\mu(s), \quad s\in U,\, i,j=1,2.$$

Since the determinant of θ is continuous and does not vanish on U, it is bounded from above and below by a positive constant on $\mathcal{K}$:

$$c^{-1}\leq \det\theta(s)\leq c, \quad s\in\mathcal{K}. \tag{5.B.8}$$

Let $\theta^{-1}=(\theta^{ij})_{i,j}$ be the inverse matrix of θ. Since θ_{ij} and its first order derivatives are bounded on $\mathcal{K}$ by (5.B.7) and $\partial_{s_k}\theta^{-1}=-\theta^{-1}(\partial_{s_k}\theta)\theta^{-1}$, we have

$$|\theta^{ij}(s)|\leq c, \quad |\partial_{s_k}\theta^{ij}(s)|\leq c, \quad s\in\mathcal{K},\, i,j,k=1,2. \tag{5.B.9}$$

Moreover, there exists $c>0$ such that

$$c^{-1}|a|^2\leq\theta^{-1}(s)a\cdot a\leq c|a|^2, \quad s\in\mathcal{K},\, a\in\mathbb{R}^2. \tag{5.B.10}$$

To see this, let $X(s,a):=\sum_{i,j=1}^2\theta^{ij}(s)a_i\partial_{s_j}\mu(s)$ for $s\in U$ and $a=(a_1,a_2)\in\mathbb{R}^2$. Since $\partial_{s_1}\mu(s)$ and $\partial_{s_2}\mu(s)$ are linearly independent, $X(s,a)$ vanishes if and only if

$$\sum_{i=1,2}\theta^{ij}(s)a_i=\sum_{i=1,2}\theta^{ji}(s)a_i=0 \quad \text{for} \quad j=1,2, \quad \text{i.e.} \quad \theta^{-1}(s)a=0.$$

(Note that θ^{-1} is symmetric since θ is so.) By this fact we observe that $X(s,a)=0$ if and only if $a=0$ and the function

$$|X(s,a)|^2=\sum_{i,j=1}^2\theta^{ij}(s)a_ia_j=\theta^{-1}(s)a\cdot a$$

is continuous for $(s,a)\in U\times\mathbb{R}^2$ and does not vanish for $a\neq 0$. In particular, it is bounded from above and below by positive constants on the compact set $\mathcal{K}\times S^1$, where S^1 is the unit circle in $\mathbb{R}^2$. Hence (5.B.10) follows.

For $s\in U$ let $\tilde\eta(s):=\eta(\mu(s))$. We show that there exists $c>0$ such that

$$c^{-1}\|\tilde\eta\|_{W^{2,p}(U)}\leq\|\eta\|_{W^{2,p}(\mu(U))}\leq c\|\tilde\eta\|_{W^{2,p}(U)}. \tag{5.B.11}$$

By the definition of an integral over a surface,

$$\int_{\mu(U)}|\eta(y)|^p\,d\mathcal{H}^2(y)=\int_U|\tilde\eta(s)|^p\sqrt{\det\theta(s)}\,ds.$$

Noting that η is supported in $\mu(\mathcal{K})$, we use (5.B.8) to the this equality to get

$$c^{-1}\|\tilde\eta\|_{L^p(U)}\leq\|\eta\|_{L^p(\mu(U))}\leq c\|\tilde\eta\|_{L^p(U)}. \tag{5.B.12}$$

Next we compare the first order derivatives of η and $\tilde{\eta}$. Since

$$\begin{aligned}\nabla_\Gamma\eta(\mu(s)) &= \sum_{i,j=1}^2 \theta^{ij}(s)\partial_{s_i}\tilde{\eta}(s)\partial_{s_j}\mu(s),\\ |\nabla_\Gamma\eta(\mu(s))|^2 &= \sum_{i,j=1}^2 \theta^{ij}(s)\partial_{s_i}\tilde{\eta}(s)\partial_{s_j}\tilde{\eta}(s) = \theta^{-1}(s)\nabla_s\tilde{\eta}(s)\cdot\nabla_s\tilde{\eta}(s)\end{aligned} \tag{5.B.13}$$

for $s \in U$, where ∇_s is the gradient operator in $s \in \mathbb{R}^2$, by (5.B.10) we have

$$c^{-1}|\nabla_s\tilde{\eta}(s)| \leq |\nabla_\Gamma\eta(\mu(s))| \leq c|\nabla_s\tilde{\eta}(s)|, \quad s \in \mathcal{K}.$$

We apply this inequality and (5.B.8) to

$$\int_{\mu(U)} |\nabla_\Gamma\eta(y)|^p \, d\mathcal{H}^2(y) = \int_U |\nabla_\Gamma\eta(\mu(s))|^p \sqrt{\det\theta(s)}\, ds$$

(note that η is supported in $\mu(\mathcal{K})$) to obtain

$$c^{-1}\|\nabla_s\tilde{\eta}\|_{L^p(U)} \leq \|\nabla_\Gamma\eta\|_{L^p(\mu(U))} \leq c\|\nabla_s\tilde{\eta}\|_{L^p(U)}. \tag{5.B.14}$$

Let us consider the second order derivatives of η and $\tilde{\eta}$. For $s \in U$ and $k = 1,2,3$ we set $\tilde{\eta}_k(s) := \underline{D}_k\eta(\mu(s))$. Then by the right-hand inequality of (5.B.14) we see that

$$\|\nabla_\Gamma\underline{D}_k\eta\|_{L^p(\mu(U))} \leq c\|\nabla_s\tilde{\eta}_k\|_{L^p(U)}.$$

Moreover, since $\underline{D}_k\eta = \nabla_\Gamma\eta\cdot e_k$, where $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbb{R}^3$,

$$\tilde{\eta}_k(s) = \nabla_\Gamma\eta(\mu(s))\cdot e_k = \sum_{i,j=1}^2 \theta^{ij}(s)\partial_{s_i}\tilde{\eta}(s)\partial_{s_j}\mu(s)\cdot e_k, \quad s \in U$$

by (5.B.13). Thus, applying ∇_s to both sides and using (5.B.7) and (5.B.9) we get

$$|\nabla_s\tilde{\eta}_k(s)| \leq c(|\nabla_s\tilde{\eta}(s)| + |\nabla_s^2\tilde{\eta}(s)|), \quad s \in \mathcal{K}.$$

Since η is supported in $\mu(\mathcal{K})$, we find by the above inequalities that

$$\|\nabla_\Gamma\underline{D}_k\eta\|_{L^p(\mu(U))} \leq c\|\nabla_s\tilde{\eta}_k\|_{L^p(U)} \leq c\|\nabla_s\tilde{\eta}\|_{W^{1,p}(U)}$$

for $k = 1,2,3$. Therefore,

$$\|\nabla_\Gamma^2\eta\|_{L^p(\mu(U))} \leq c\|\tilde{\eta}\|_{W^{2,p}(U)}. \tag{5.B.15}$$

To estimate the $L^p(U)$-norms of the second order derivatives of $\tilde{\eta}$, we take the inner product of the first equality of (5.B.13) with $\partial_{s_k}\mu(s)$, $k = 1,2,3$ to get

$$\nabla_\Gamma\eta(\mu(s))\cdot\partial_{s_k}\mu(s) = \sum_{i,j=1}^2 \theta^{ij}(s)\theta_{jk}(s)\partial_{s_i}\tilde{\eta}(s) = \partial_{s_k}\tilde{\eta}(s), \quad s \in U.$$

Since $\mu(s) \in \Gamma$, we have $\nabla_\Gamma\eta(\mu(s)) = \overline{\nabla_\Gamma\eta}(\mu(s))$, where the right-hand side stands for the constant extension of $\nabla_\Gamma\eta$ in the normal direction of Γ, and thus

$$\begin{aligned}\partial_{s_l}\Big(\nabla_\Gamma\eta(\mu(s))\Big) &= \partial_{s_l}\Big(\overline{\nabla_\Gamma\eta}(\mu(s))\Big) = \Big[\nabla\Big(\overline{\nabla_\Gamma\eta}\Big)\Big](\mu(s))\partial_{s_l}\mu(s)\\ &= \nabla_\Gamma^2\eta(\mu(s))\partial_{s_l}\mu(s), \quad l = 1,2,3\end{aligned}$$

by (5.2.4). Hence for $s \in U$ and $k, l = 1, 2, 3$ we have

$$\begin{aligned}\partial_{s_l}\partial_{s_k}\tilde{\eta}(s) &= \partial_{s_l}\Big(\nabla_\Gamma\eta(\mu(s))\cdot\partial_{s_k}\mu(s)\Big)\\ &= \nabla_\Gamma^2\eta(\mu(s))\partial_{s_l}\mu(s)\cdot\partial_{s_k}\mu(s) + \nabla_\Gamma\eta(\mu(s))\cdot\partial_{s_l}\partial_{s_k}\mu(s)\end{aligned}$$

and by applying (5.B.7) to the last line we deduce that

$$|\nabla_s^2\tilde{\eta}(s)| \le c(|\nabla_\Gamma\eta(\mu(s))| + |\nabla_\Gamma^2\eta(\mu(s))|), \quad s \in \mathcal{K}.$$

Noting that η is supported in $\mu(\mathcal{K})$ and

$$\int_{\mu(U)} |\nabla_\Gamma^k\eta(y)|^p\,d\mathcal{H}^2(y) = \int_U |\nabla_\Gamma^k\eta(\mu(s))|^p\sqrt{\det\theta(s)}\,ds, \quad k = 1, 2,$$

we use the above inequality and (5.B.8) to the $L^p(U)$-norm of $\nabla_s^2\tilde{\eta}$ to get

$$\|\nabla_s^2\tilde{\eta}\|_{L^p(U)} \le c\|\eta\|_{W^{2,p}(\mu(U))}. \tag{5.B.16}$$

By (5.B.12), (5.B.14), (5.B.15), and (5.B.16) we obtain (5.B.11).

Now we observe by $\eta \in W^{2,p}(\Gamma)$ and the left-hand inequality of (5.B.11) that $\tilde{\eta}$ is in $W^{2,p}(U)$. Since it is also supported in the compact subset $\mathcal{K}$ of U, by a standard mollification argument (see e.g. [1, Lemma 3.16]) we can take a sequence $\{\tilde{\eta}_k\}_{k=1}^\infty$ in $C^\infty(U)$ that converges to $\tilde{\eta}$ strongly in $W^{2,p}(U)$. Hence the right-hand inequality of (5.B.11) yields that

$$\|\eta - \eta_k\|_{W^{2,p}(\mu(U))} \le c\|\tilde{\eta} - \tilde{\eta}_k\|_{W^{2,p}(U)} \to 0 \quad \text{as} \quad k \to \infty,$$

where $\eta_k(y) := \tilde{\eta}_k(\mu^{-1}(y))$ for $k \in \mathbb{N}$ and $y \in \Gamma$. Since the local parametrization $\mu\colon U \to \Gamma$ is of class C^2, we have $\eta_k \in C^2(\mu(U))$ for each $k \in \mathbb{N}$. Hence $C^2(\Gamma)$ is dense in $W^{2,p}(\Gamma)$ (recall that we localized η by using a partition of unity of Γ). □

Next we give a change of variables formula for an integral over a parametrized surface. Let $h \in C^1(\Gamma)$ satisfy $|h| < \delta$ on Γ. We define a parametrized surface

$$\Gamma_h := \{y + h(y)n(y) \mid y \in \Gamma\} \subset \mathbb{R}^3. \tag{5.B.17}$$

Note that $\Gamma_h \subset N$ by $|h| < \delta$ on Γ (see Section 5.2.1). For $y \in \Gamma$ we set

$$\tau_h(y) := \{I_3 - h(y)W(y)\}^{-1}\nabla_\Gamma h(y), \quad n_h(y) := \frac{n(y) - \tau_h(y)}{\sqrt{1 + |\tau_h(y)|^2}}. \tag{5.B.18}$$

Here the vector field τ_h is tangential on Γ. We assume that the orientation of Γ_h is the same as that of Γ. Then as in the proof of Lemma 5.2.10 below we can show that the constant extension $\bar{n}_h := n_h \circ \pi$ gives the unit outward normal vector field of Γ_h.

For $\varphi \in C^1(\Gamma_h)$ we define the tangential gradient $\nabla_{\Gamma_h}\varphi$ as

$$\nabla_{\Gamma_h}\varphi(x) := \{I_3 - \bar{n}_h(x) \otimes \bar{n}_h(x)\}\nabla\tilde{\varphi}(x), \quad x \in \Gamma_h,$$

where $\tilde{\varphi}$ is an arbitrary extension of φ to N satisfying $\tilde{\varphi}|_{\Gamma_h} = \varphi$. Let us give a change of variables formula for integrals over Γ_h.

Lemma 5.B.2. *Suppose that Γ is of class C^2 and $h \in C^1(\Gamma)$ satisfies $|h| < \delta$ on Γ. Let Γ_h be the parametrized surface given by* (5.B.17)*. For $\varphi \in L^1(\Gamma_h)$ we have*

$$\int_{\Gamma_h} \varphi(x)\,d\mathcal{H}^2(x) = \int_{\Gamma} \varphi^\sharp(y) J(y,h(y))\sqrt{1+|\tau_h(y)|^2}\,d\mathcal{H}^2(y), \tag{5.B.19}$$

where $\varphi^\sharp(y) := \varphi(y+h(y)n(y))$ for $y \in \Gamma$ and J and τ_h are given by (5.2.48) *and* (5.B.18)*.*

Before starting to prove Lemma 5.B.2 we give a remark on a partition of unity of Γ_h. Since the surface Γ is compact, we can take finite open subsets U_k of $\mathbb{R}^2$ and local parametrizations $\mu^k\colon U_k \to \Gamma$, $k = 1,\dots,k_0$ such that $\{\mu^k(U_k)\}_{k=1}^{k_0}$ is an open covering of Γ. Let $\{\eta^k\}_{k=1}^{k_0}$ be a partition of unity of Γ subordinate to the covering $\{\mu^k(U_k)\}_{k=1}^{k_0}$. Then setting

$$\mu_h^k(s) := \mu_k(s) + h(\mu_k(s))n(\mu_k(s)), \quad \eta_h^k(\mu_h^k(s)) := \eta^k(\mu^k(s)), \quad s \in U_k$$

we observe that $\mu_h^k\colon U_k \to \Gamma_h$, $k = 1,\dots,k_0$ are local parametrizations of Γ_h such that $\{\mu_h^k(U_k)\}_{k=1}^{k_0}$ is an open covering of Γ_h, and that $\{\eta_h^k\}_{k=1}^{k_0}$ is a partition of unity of Γ_h subordinate to the covering $\{\mu_h^k(U_k)\}_{k=1}^{k_0}$. Using these local parametrizations and partitions of unity we can localize integrals over Γ and Γ_h and express them as integrals over the same domains U_k, $k = 1,\dots,k_0$.

Proof. Let U be an open subset of $\mathbb{R}^2$ and $\mu\colon U \to \Gamma$ be a local parametrization of Γ. The Riemannian metric of Γ is locally given by $\theta = \nabla_s\mu(\nabla_s\mu)^T$ on U, where

$$\nabla_s\mu := \begin{pmatrix} \partial_{s_1}\mu_1 & \partial_{s_1}\mu_2 & \partial_{s_1}\mu_3 \\ \partial_{s_2}\mu_1 & \partial_{s_2}\mu_2 & \partial_{s_2}\mu_3 \end{pmatrix}.$$

Note that $\nabla_s\mu(s)n(\mu(s)) = 0$ for $s \in U$ since $\partial_{s_1}\mu(s)$ and $\partial_{s_2}\mu(s)$ are tangent to Γ at $\mu(s)$. Using μ we give a local parametrization $\mu_h\colon U \to \Gamma_h$ and the Riemannian metric θ_h of Γ_h by

$$\mu_h(s) := \mu(s) + h(\mu(s))n(\mu(s)), \quad \theta_h(s) := \nabla_s\mu_h(s)\{\nabla_s\mu_h(s)\}^T, \quad s \in U.$$

Hereafter we use the notation $\eta^\sharp(s) := \eta(\mu(s))$, $s \in U$ for a function η on Γ and suppress the argument $s \in U$. Then by a localization argument with partitions of unity of Γ and Γ_h mentioned above, the proof of (5.B.19) reduces to show

$$\sqrt{\det\theta_h} = J(\mu,h^\sharp)\sqrt{(1+|\tau_h^\sharp|^2)\det\theta} \quad \text{on} \quad U. \tag{5.B.20}$$

We prove (5.B.20) in two steps. First we show the equality

$$(1-|(\nabla_{\Gamma_h}\bar{h})\circ\mu_h|^2)\det\theta_h = J(\mu,h^\sharp)^2\det\theta \quad \text{on} \quad U, \tag{5.B.21}$$

where $\bar{h} := h\circ\pi$ is the constant extension of h in the normal direction of Γ. We differentiate both sides of $\mu_h = \mu + h^\sharp n^\sharp$ in s and use

$$\nabla_s n^\sharp(s) = \nabla_s\bigl(\bar{n}(\mu(s))\bigr) = \nabla_s\mu(s)\nabla\bar{n}(\mu(s)) = -\nabla_s\mu(s)W(\mu(s))$$

by (5.2.4) (note that $\mu(s) \in \Gamma$). Then

$$\nabla_s\mu_h = \nabla_s\mu(I_3 - h^\sharp W^\sharp) + \nabla_s h^\sharp \otimes n^\sharp \quad \text{on} \quad U.$$

Also, noting that $W^\sharp$ is a symmetric matrix we get

$$(\nabla_s\mu_h)^T = (I_3 - h^\sharp W^\sharp)(\nabla_s\mu) + n^\sharp \otimes \nabla_s h^\sharp \quad \text{on} \quad U.$$

From these equalities, $(\nabla_s\mu)n^\sharp = 0$, $W^\sharp n^\sharp = 0$, and

$$(\nabla_s h^\sharp \otimes n^\sharp)(n^\sharp \otimes \nabla_s h^\sharp) = |n^\sharp|^2 \nabla_s h^\sharp \otimes \nabla_s h^\sharp = \nabla_s h^\sharp \otimes \nabla_s h^\sharp$$

we deduce that

$$\theta_h = \nabla_s\mu_h(\nabla_s\mu_h)^T = \nabla_s\mu(I_3 - h^\sharp W^\sharp)^2(\nabla_s\mu)^T + \nabla_s h^\sharp \otimes \nabla_s h^\sharp$$

and thus

$$\det(\theta_h - \nabla_s h^\sharp \otimes \nabla_s h^\sharp) = \det[\nabla_s\mu(I_3 - h^\sharp W^\sharp)^2(\nabla_s\mu)^T] \quad \text{on} \quad U. \tag{5.B.22}$$

The formula $\det(I_2 + a \otimes b) = 1 + a \cdot b$ for $a, b \in \mathbb{R}^2$ yields

$$\begin{aligned} \det(\theta_h - \nabla_s h^\sharp \otimes \nabla_s h^\sharp) &= \det[I_3 - (\theta_h^{-1}\nabla_s h^\sharp) \otimes \nabla_s h^\sharp] \det\theta_h \\ &= \{1 - (\theta_h^{-1}\nabla_s h^\sharp) \cdot \nabla_s h^\sharp\} \det\theta_h. \end{aligned}$$

To calculate the right-hand side of (5.B.22) we set

$$A := \begin{pmatrix} \nabla_s\mu \\ (n^\sharp)^T \end{pmatrix}, \quad A_h := \begin{pmatrix} \nabla_s\mu(I_3 - h^\sharp W^\sharp) \\ (n^\sharp)^T \end{pmatrix}.$$

Here we see $n^\sharp \in \mathbb{R}^3$ as a column vector. Then by $(\nabla_s\mu)n^\sharp = 0$, $W^\sharp n^\sharp = 0$, and the symmetry of the matrix $W^\sharp$ we have $A_h = A(I_3 - h^\sharp W^\sharp)$ and

$$AA^T = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}, \quad A_h A_h^T = \begin{pmatrix} \nabla_s\mu(I_3 - h^\sharp W^\sharp)(\nabla_s\mu)^T & 0 \\ 0 & 1 \end{pmatrix}.$$

From these equalities and $\det(I_3 - h^\sharp W^\sharp) = J(\mu, h^\sharp)$ it follows that

$$\begin{aligned} \det[\nabla_s\mu(I_3 - h^\sharp W^\sharp)^2(\nabla_s\mu)^T] &= \det[A_h A_h^T] = \det[A(I_3 - h^\sharp W^\sharp)^2 A^T] \\ &= \det[(I_3 - h^\sharp W^\sharp)^2] \det[AA^T] \\ &= J(\mu, h^\sharp)^2 \det\theta. \end{aligned} \tag{5.B.23}$$

(Note that A and $I_3 - h^\sharp W^\sharp$ are 3×3 matrices.) Hence (5.B.22) reads

$$\{1 - (\theta_h^{-1}\nabla_s h^\sharp) \cdot \nabla_s h^\sharp\} \det\theta_h = J(\mu, h^\sharp)^2 \det\theta \quad \text{on} \quad U. \tag{5.B.24}$$

Now we recall that the tangential gradient of $\bar{h}$ on Γ_h is locally expressed as

$$\nabla_{\Gamma_h}\bar{h}(\mu_h(s)) = \sum_{i,j=1}^{2} \theta_h^{ij}(s) \frac{\partial(\bar{h} \circ \mu_h)}{\partial s_i}(s) \partial_{s_j}\mu_h(s), \quad s \in U.$$

Here θ_h^{ij} is the (i,j)-entry of θ_h^{-1}. Since $\bar{h} = h \circ \pi$ is the constant extension of h and $\pi(\mu_h(s)) = \mu(s)$, we have $\bar{h}(\mu_h(s)) = h(\mu(s)) = h^\sharp(s)$ and thus

$$(\nabla_{\Gamma_h}\bar{h}) \circ \mu_h = \sum_{i,j=1}^{2} \theta_h^{ij}(\partial_{s_i} h^\sharp)\partial_{s_j}\mu,$$

$$\left|(\nabla_{\Gamma_h}\bar{h}) \circ \mu_h\right|^2 = \sum_{i,j=1}^{2} \theta_h^{ij}(\partial_{s_i} h^\sharp)(\partial_{s_j} h^\sharp) = (\theta_h^{-1}\nabla_s h^\sharp) \cdot \nabla_s h^\sharp.$$

Applying this equality to the left-hand side of (5.B.24) we obtain (5.B.21).

Next we show that

$$1-\left|\nabla_{\Gamma_h}\bar{h}(y+h(y)n(y))\right|^2=\frac{1}{1+|\tau_h(y)|^2},\quad y\in\Gamma, \tag{5.B.25}$$

where τ_h is the tangential vector field on Γ given by (5.B.18). For the sake of simplicity, we set $\alpha:=\sqrt{1+|\tau_h|^2}$ and sometimes suppress the argument $y\in\Gamma$ of functions on Γ. By (5.2.12) and $d(y+h(y)n(y))=h(y)$ we have

$$\nabla\bar{h}(x)|_{x=y+h(y)n(y)}=\{I_3-h(y)W(y)\}^{-1}\nabla_\Gamma h(y)=\tau_h(y).$$

Since $\nabla_{\Gamma_h}\bar{h}=(I_3-\bar{n}_h\otimes\bar{n}_h)\nabla\bar{h}$ on Γ_h and $n_h=\alpha^{-1}(n-\tau_h)$ on Γ, we see by the above equality that

$$\nabla_{\Gamma_h}\bar{h}(y+h(y)n(y))=\{I_3-\alpha^{-2}(n-\tau_h)\otimes(n-\tau_h)\}\tau_h=\alpha^{-2}\{|\tau_h|^2n+(\alpha^2-|\tau_h|^2)\tau_h\}.$$

From this equality, $n\cdot\tau_h=0$, and $\alpha^2=1+|\tau_h|^2$ we deduce that

$$1-\left|\nabla_{\Gamma_h}\bar{h}(y+h(y)n(y))\right|^2=1-\alpha^{-4}(|\tau_h|^4+|\tau_h|^2)=1-\alpha^{-2}|\tau_h|^2.$$

Since $\alpha^{-2}=(1+|\tau_h|^2)^{-1}$, the above equality implies (5.B.25). Finally, we conclude by (5.B.21) and (5.B.25) that (5.B.20) is valid. $\square$

Now we assume that Γ is of class C^5 and prove the formulas and inequalities in Section 5.2.2 for the surface quantities on the boundary Γ_ε of the curved thin domain.

Proof of Lemma 5.2.9. First note that, since Γ is of class C^5, the Weingarten map $W\in C^3(\Gamma)^{3\times3}$ and the functions $g_0,g_1\in C^4(\Gamma)$ are bounded on Γ along with their first and second order tangential derivatives.

Let τ_ε^i and n_ε^i, $i=0,1$ be the vector fields on Γ given by (5.2.32) and (5.2.33). Then the first inequalities of (5.2.34) and (5.2.35) immediately follow from (5.2.9) and (5.2.10). To show the second inequalities of (5.2.34) and (5.2.35) we set

$$R_\varepsilon^i(y):=\{I_3-\varepsilon g_i(y)W(y)\}^{-1},\quad y\in\Gamma$$

and apply $\underline{D}_k$, $k=1,2,3$ to both sides of $R_\varepsilon^i(I_3-\varepsilon g_iW)=I_3$ on Γ to get

$$\underline{D}_kR_\varepsilon^i=\varepsilon R_\varepsilon^i\{(\underline{D}_kg_i)W+g_i\underline{D}_kW\}R_\varepsilon^i\quad\text{on}\quad\Gamma. \tag{5.B.26}$$

Then by (5.2.9) we see that there exists a constant $c>0$ independent of ε such that

$$|\underline{D}_kR_\varepsilon^i|\le c\varepsilon\quad\text{on}\quad\Gamma. \tag{5.B.27}$$

Applying (5.2.9), (5.2.10), and (5.B.27) to $\underline{D}_k\tau_\varepsilon^i=(\underline{D}_kR_\varepsilon^i)\nabla_\Gamma g_i+R_\varepsilon^i(\underline{D}_k\nabla_\Gamma g)$ we obtain

$$|\underline{D}_k\tau_\varepsilon^i|\le c,\quad|\underline{D}_k\tau_\varepsilon^i-\underline{D}_k\nabla_\Gamma g|\le|(\underline{D}_kR_\varepsilon^i)\nabla_\Gamma g_i|+|(R_\varepsilon^i-I_3)(\underline{D}_k\nabla_\Gamma g)|\le c\varepsilon$$

on Γ for all $k=1,2,3$. Hence the second inequalities of (5.2.34) and (5.2.35) follows. We further apply $\underline{D}_l$, $l=1,2,3$ to both sides of (5.B.26) and use (5.2.9) and (5.B.27) to get $|\underline{D}_l\underline{D}_kR_\varepsilon^i|\le c\varepsilon$ on Γ. From this inequality, (5.2.9), (5.B.27) we deduce that

$$\begin{aligned}|\underline{D}_l\underline{D}_k\tau_\varepsilon^i|\le|(\underline{D}_l\underline{D}_kR_\varepsilon^i)\nabla_\Gamma g_i|+|(\underline{D}_kR_\varepsilon^i)(\underline{D}_l\nabla_\Gamma g_i)|\\+|(\underline{D}_lR_\varepsilon^i)(\underline{D}_k\nabla_\Gamma g_i)|+|R_\varepsilon^i(\underline{D}_l\underline{D}_k\nabla_\Gamma g_i)|\le c.\end{aligned}$$

Hence the third inequality of (5.2.34) is valid.

Now let us prove the inequalities for n_ε^i. The first equality of (5.2.36) is a direct consequence of (5.2.33). Also, the other inequalities of (5.2.36) follow from (5.2.34). To prove (5.2.37) we see that $n_\varepsilon^0 + n_\varepsilon^1 = \varphi_\varepsilon n + \varepsilon \tau_\varepsilon$, where

$$\varphi_\varepsilon := \frac{1}{\sqrt{1+\varepsilon^2|\tau_\varepsilon^1|^2}} - \frac{1}{\sqrt{1+\varepsilon^2|\tau_\varepsilon^0|^2}}, \quad \tau_\varepsilon := -\frac{\tau_\varepsilon^1}{\sqrt{1+\varepsilon^2|\tau_\varepsilon^1|^2}} + \frac{\tau_\varepsilon^0}{\sqrt{1+\varepsilon^2|\tau_\varepsilon^0|^2}}.$$

From (5.2.34) it follows that

$$|\tau_\varepsilon| \le c, \quad |\nabla_\Gamma \tau_\varepsilon| \le c \quad \text{on} \quad \Gamma \tag{5.B.28}$$

with a positive constant c independent of ε. Let us estimate φ_ε and its tangential gradient. By the mean value theorem for the function $(1+s)^{-1/2}$, $s \ge 0$ and (5.2.34).

$$|\varphi_\varepsilon| \le \frac{\varepsilon^2}{2}\left||\tau_\varepsilon^1|^2 - |\tau_\varepsilon^0|^2\right| \le c\varepsilon^2 \quad \text{on} \quad \Gamma. \tag{5.B.29}$$

Also, since

$$\nabla_\Gamma\left(\frac{1}{\sqrt{1+\varepsilon^2|\tau_\varepsilon^i|^2}}\right) = -\frac{\varepsilon^2(\nabla_\Gamma\tau_\varepsilon^i)\tau_\varepsilon^i}{(1+\varepsilon^2|\tau_\varepsilon^i|^2)^{3/2}}, \quad i = 0, 1,$$

we use (5.2.34) to get

$$|\nabla_\Gamma \varphi_\varepsilon| \le c\varepsilon^2 \quad \text{on} \quad \Gamma. \tag{5.B.30}$$

Applying (5.B.28), (5.B.29), and (5.B.30) to $n_\varepsilon^0 + n_\varepsilon^1 = \varphi_\varepsilon n + \varepsilon\tau_\varepsilon$ and its tangential gradient matrix we obtain (5.2.37). $\square$

Proof of Lemma 5.2.10. We take an open subset U of $\mathbb{R}^2$ and a local parametrization $\mu\colon U \to \Gamma$. Then for $i = 0, 1$ the mapping

$$\mu_\varepsilon^i(s) := \mu(s) + \varepsilon g_i(\mu(s))n(\mu(s)), \quad s \in U$$

defines a local parametrization $\mu_\varepsilon^i\colon U \to \Gamma_\varepsilon^i$ and the pair $\{\partial_{s_1}\mu_\varepsilon^i(s), \partial_{s_2}\mu_\varepsilon^i(s)\}$ is a basis of the tangent plane of Γ_ε^i at $\mu_\varepsilon^i(s)$. Hence if we show

$$\bar{n}_\varepsilon^i(\mu_\varepsilon^i(s)) \cdot \partial_{s_k}\mu_\varepsilon^i(s) = 0, \quad s \in U,\ k = 1, 2,$$

then $\bar{n}_\varepsilon^i$ is normal to Γ_ε^i and thus $n_\varepsilon = \bar{n}_\varepsilon^i$ on Γ_ε^i, since $|n_\varepsilon^i| = 1$ and both n_ε and $\bar{n}_\varepsilon^i$ have the direction of $(-1)^{i+1}\bar{n}$. Moreover, since $\bar{n}_\varepsilon^i(\mu_\varepsilon^i(s)) = n_\varepsilon^i(\mu(s))$ is given by (5.2.33) (note that $\pi(\mu_\varepsilon^i(s)) = \mu(s)$), the above equality reduces to

$$n(\mu(s)) \cdot \partial_{s_k}\mu_\varepsilon^i(s) = \varepsilon\tau_\varepsilon^i(\mu(s)) \cdot \partial_{s_k}\mu_\varepsilon^i(s), \quad s \in U,\ k = 1, 2. \tag{5.B.31}$$

To prove (5.B.31) we differentiate both sides of

$$\mu_\varepsilon^i(s) = \mu(s) + \varepsilon\bar{g}_i(\mu(s))\bar{n}(\mu(s)), \quad s \in U$$

with respect to s_k, $k = 1, 2$ and use (5.2.4) (note that $\mu(s) \in \Gamma$) to get

$$\partial_{s_k}\mu_\varepsilon^i(s) = \{I_3 - \varepsilon g_i(\mu(s))W(\mu(s))\}\partial_{s_k}\mu(s) + \varepsilon\partial_{s_k}\mu(s) \cdot \nabla_\Gamma g_i(\mu(s))n(\mu(s)).$$

Here the first term on the right-hand side is tangent to Γ at $\mu(s)$ since μ is a local parametrization of Γ and $W = PW$ on Γ. Hence

$$n(\mu(s)) \cdot \partial_{s_k}\mu_\varepsilon^i(s) = \varepsilon\partial_{s_k}\mu(s) \cdot \nabla_\Gamma g_i(\mu(s)).$$

Also, since $\tau_\varepsilon^i = (I_3 - \varepsilon g_i W)^{-1}\nabla_\Gamma g_i$ is tangential on Γ and W is symmetric,

$$\begin{aligned}\varepsilon\tau_\varepsilon^i(\mu(s)) \cdot \partial_{s_k}\mu_\varepsilon^i(s) &= \varepsilon(I_3 - \varepsilon g_i W)^{-1}\nabla_\Gamma g_i(\mu(s)) \cdot (I_3 - \varepsilon g_i W)\partial_{s_k}\mu(s) \\ &= \varepsilon\partial_{s_k}\mu(s) \cdot \nabla_\Gamma g_i(\mu(s)).\end{aligned}$$

Here we suppressed the argument $\mu(s)$ of the function $I_3 - \varepsilon g_i W$. Therefore, we obtain (5.B.31) and conclude that the assertion $n_\varepsilon = \bar{n}_\varepsilon^i$ on Γ_ε^i, $i = 0, 1$ is valid. □

Proof of Lemma 5.2.11. For $i = 0, 1$ let τ_ε^i and n_ε^i be given by (5.2.32) and (5.2.33) and

$$\varphi_\varepsilon^i(x) := \frac{1}{\sqrt{1 + \varepsilon^2|\bar{\tau}_\varepsilon^i(x)|^2}} - 1, \quad x \in N.$$

By direct calculations and the inequalities (5.2.13), (5.2.15), and (5.2.34) we see that

$$|\partial_x^\alpha \varphi_\varepsilon^i(x)| \leq c\varepsilon^2 \quad \text{for all} \quad x \in N,\ |\alpha| = 0, 1, 2, \tag{5.B.32}$$

where $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$ with $\alpha_j \geq 0$, $j = 1, 2, 3$ and $c > 0$ is a constant independent of ε. Since $n_\varepsilon = \bar{n}_\varepsilon^i$ on Γ_ε^i by Lemma 5.2.10,

$$n_\varepsilon - (-1)^{i+1}\left(\bar{n} - \overline{\nabla_\Gamma g_i}\right) = (-1)^{i+1}\varphi_\varepsilon^i(\bar{n} - \varepsilon\bar{\tau}_\varepsilon^i) - (-1)^{i+1}\varepsilon\left(\bar{\tau}_\varepsilon - \overline{\nabla_\Gamma g_i}\right)$$

on Γ_ε^i. Applying (5.2.34), (5.2.35), and (5.B.32) to the right-hand side we obtain (5.2.40). Also, the inequalities (5.2.41) immediately follow from (5.2.40).

Let us prove (5.2.42). For $x \in N$ we set

$$\Phi_\varepsilon^i(x) := (-1)^{i+1}\left\{\varphi_\varepsilon^i(x)\bar{n}(x) - \frac{\varepsilon\bar{\tau}_\varepsilon^i(x)}{\sqrt{1 + \varepsilon^2|\bar{\tau}_\varepsilon^i(x)|^2}}\right\}.$$

Then we observe by direct calculations, (5.2.13), (5.2.15), (5.2.34), and (5.B.32) that

$$|\partial_x^\alpha \Phi_\varepsilon^i(x)| \leq c\varepsilon \quad \text{for all} \quad x \in N,\ |\alpha| = 0, 1, 2. \tag{5.B.33}$$

We differentiate $\bar{n}_\varepsilon^i(x) = (-1)^{i+1}\bar{n}(x) + \Phi_\varepsilon^i(x)$, $x \in N$ and apply (5.2.16) to get

$$\nabla\bar{n}_\varepsilon^i(x) = (-1)^i\{I_3 - d(x)\overline{W}(x)\}^{-1}\overline{W}(x) + \nabla\Phi_\varepsilon^i(x), \quad x \in N.$$

Since $\bar{n}_\varepsilon^i$ is an extension of n_ε (on Γ_ε^i) to N, the Weingarten map of Γ_ε^i is given by $W_\varepsilon = -P_\varepsilon\nabla\bar{n}_\varepsilon^i$ on Γ_ε. Thus, the above equality yields that

$$W_\varepsilon(x) = P_\varepsilon(x)\left\{(-1)^{i+1}\overline{R}_\varepsilon^i(x)\overline{W}(x) - \nabla\Phi_\varepsilon^i(x)\right\}, \quad x \in \Gamma_\varepsilon^i, \tag{5.B.34}$$

where $R_\varepsilon^i := (I_3 - \varepsilon g_i W)^{-1}$ on Γ. By this equality and (5.2.8) we observe that

$$\left|W_\varepsilon - (-1)^{i+1}\overline{W}\right| \leq \left|\left(P_\varepsilon - \overline{P}\right)\overline{R}_\varepsilon^i\overline{W}\right| + \left|\left(\overline{R}_\varepsilon^i - I_3\right)\overline{W}\right| + |P_\varepsilon\nabla\Phi_\varepsilon^i| \quad \text{on} \quad \Gamma_\varepsilon^i$$

and thus we obtain the first inequality of (5.2.42) by applying (5.2.9), (5.2.10), (5.2.41), and (5.B.33) to the right-hand side of the above inequality. The second inequality of (5.2.42) follows from the first one since $H = \mathrm{tr}[W]$ and $H_\varepsilon = \mathrm{tr}[W_\varepsilon]$.

Finally, let us show the inequality (5.2.43). Based on (5.B.34) we define an extension of the Weingarten map W_ε (on Γ_ε^i) to N by

$$\widetilde{W_\varepsilon^i}(x) := \overline{P_\varepsilon^i}(x)\left\{(-1)^{i+1}\overline{R_\varepsilon^i}(x)\overline{W}(x) - \nabla\Phi_\varepsilon^i(x)\right\}, \quad x \in N,$$

where $P_\varepsilon^i := I_3 - n_\varepsilon^i \otimes n_\varepsilon^i$ on Γ. For $x \in N$ let

$$E_\varepsilon^i(x) := (-1)^{i+1}\left\{\overline{P_\varepsilon^i}(x) - \overline{P}(x)\right\}\overline{R_\varepsilon^i}(x)\overline{W}(x),$$
$$F_\varepsilon^i(x) := (-1)^{i+1}\overline{P}(x)\left\{\overline{R_\varepsilon^i}(x) - I_3\right\}\overline{W}(x), \quad G_\varepsilon^i(x) := -\overline{P_\varepsilon^i}(x)\nabla\Phi_\varepsilon^i(x)$$

so that $\widetilde{W_\varepsilon^i} = (-1)^{i+1}\overline{W} + E_\varepsilon^i + F_\varepsilon^i + G_\varepsilon^i$ in N by (5.2.6). Hence by (5.2.12) we get

$$\partial_j\widetilde{W_\varepsilon^i} = \sum_{k=1}^3 (-1)^{i+1}\left[\left(I_3 - d\overline{W}\right)^{-1}\right]_{jk}\overline{\underline{D}_k W} + \partial_j E_\varepsilon^i + \partial_j F_\varepsilon^i + \partial_j G_\varepsilon^i \tag{5.B.35}$$

in N for $j = 1, 2, 3$. To estimate the derivatives of E_ε^i, F_ε^i, and G_ε^i, we observe by the equality $\bar{n}_\varepsilon^i = (-1)^{i+1}\bar{n} + \Phi_\varepsilon^i$ on N and the estimate (5.B.33) that

$$\left|\overline{P_\varepsilon^i} - \overline{P}\right| \le c\varepsilon, \quad \left|\partial_j\overline{P_\varepsilon^i} - \partial_j\overline{P}\right| \le c\varepsilon \quad \text{in} \quad N.$$

From these inequalities, (5.2.10), (5.2.13), (5.B.27), and (5.B.33) we deduce that

$$|\partial_j E_\varepsilon^i| \le c\varepsilon, \quad |\partial_j F_\varepsilon^i| \le c\varepsilon, \quad |\partial_j G_\varepsilon^i| \le c\varepsilon \quad \text{in} \quad N.$$

Applying (5.2.10) and the above inequalities to (5.B.35) we get

$$\left|\partial_j\widetilde{W_\varepsilon^i}(x) - (-1)^{i+1}\overline{\underline{D}_j W}(x)\right| \le c(|d(x)| + \varepsilon), \quad x \in N. \tag{5.B.36}$$

Now we recall that $\widetilde{W_\varepsilon^i}$ is an extension of W_ε (on Γ_ε^i) to N. Hence

$$\underline{D}_j^\varepsilon W_\varepsilon(x) = \sum_{k=1}^3 [P_\varepsilon(x)]_{jk}\partial_k\widetilde{W_\varepsilon^i}(x), \quad x \in \Gamma_\varepsilon^i$$

and from $P\nabla_\Gamma = \nabla_\Gamma$ it follows that

$$\left|\underline{D}_j^\varepsilon W_\varepsilon - \overline{\underline{D}_j W}\right| \le \sum_{k=1}^3\left(\left|\left[P_\varepsilon - \overline{P}\right]_{jk}\partial_k\widetilde{W_\varepsilon^i}\right| + \left|\overline{P}_{jk}\left\{\partial_k\widetilde{W_\varepsilon^i} - (-1)^{i+1}\overline{\underline{D}_k W}\right\}\right|\right)$$

on Γ_ε. Applying (5.2.41) and (5.B.36) with $|d| = \varepsilon|\bar{g}_i| \le c\varepsilon$ on Γ_ε^i to the right-hand side we conclude that (5.2.43) is valid. □

Proof of Lemma 5.2.12. Since

$$\overline{P}(y + \varepsilon g_0(y)n(y)) = \overline{P}(y + \varepsilon g_1(y)n(y)) = P(y), \quad y \in \Gamma,$$

we observe that

$$|P_\varepsilon(y + \varepsilon g_1(y)n(y)) - P_\varepsilon(y + \varepsilon g_0(y)n(y))| \le \sum_{i=0,1}\left|\left[P_\varepsilon - \overline{P}\right](y + \varepsilon g_i(y)n(y))\right|.$$

Applying (5.2.41) to the right-hand side we obtain the inequality (5.2.44) for $F_\varepsilon = P_\varepsilon$. The other inequalities are proved by (5.2.41)–(5.2.43) in the same way. □

Finally, let us prove the formula (5.3.15) in the proof of Lemma 5.3.3.

Proof of (5.3.15). In the proof of Lemma 5.3.3 we consider the mapping

$$\zeta(s) = \mu(s') + \varepsilon\{(1-s_3)g_0(\mu(s')) + s_3 g_1(\mu(s'))\}n(\mu(s))$$

for $s=(s',s_3)\in(0,1)$, $s'\in(0,1)^2$, where $\mu\colon(0,1)^2\to\Gamma$ is a local parametrization of Γ. Differentiating ζ in s we get (5.3.14), i.e.

$$\begin{aligned} &\partial_{s_i}\zeta(s) = \{I_3 - h_\varepsilon(s)W(\mu(s'))\}\partial_{s_i}\mu(s') + \eta_\varepsilon^i(s)n(\mu(s')), \quad i=1,2,\\ &\partial_{s_3}\zeta(s) = \varepsilon g(\mu(s'))n(\mu(s')) \end{aligned} \tag{5.B.37}$$

for $s\in(0,1)^3$, where

$$\begin{aligned} h_\varepsilon(s) &:= \varepsilon\{(1-s_3)g_0(\mu(s')) + s_3 g_1(\mu(s'))\},\\ \eta_\varepsilon^i(s) &:= \varepsilon\partial_{s_i}\mu(s')\cdot\{(1-s_3)\nabla_\Gamma g_0(\mu(s')) + s_3\nabla_\Gamma g_1(\mu(s'))\}, \quad i=1,2. \end{aligned}$$

For the sake of simplicity, we use the notation $\xi^\sharp(s') := \xi(\mu(s'))$, $s'\in(0,1)^2$ for a function ξ on Γ and suppress the arguments s and s'. Hence we refer to (5.B.37) as

$$\partial_{s_i}\zeta = (I_3 - h_\varepsilon W^\sharp)\partial_{s_i}\mu + \eta_\varepsilon^i n^\sharp, \quad \partial_{s_3}\zeta = \varepsilon g^\sharp n^\sharp \quad \text{on} \quad (0,1)^3,\ i=1,2. \tag{5.B.38}$$

By (5.B.38) we observe that the gradient matrix of ζ is of the form

$$\nabla_s\zeta = \begin{pmatrix} \partial_{s_1}\zeta_1 & \partial_{s_1}\zeta_2 & \partial_{s_1}\zeta_3\\ \partial_{s_2}\zeta_1 & \partial_{s_2}\zeta_2 & \partial_{s_2}\zeta_3\\ \partial_{s_3}\zeta_1 & \partial_{s_3}\zeta_2 & \partial_{s_3}\zeta_3 \end{pmatrix} = \begin{pmatrix} \nabla_s\mu(I_3 - h_\varepsilon W^\sharp) + \eta_\varepsilon\otimes n^\sharp\\ \varepsilon g^\sharp(n^\sharp)^T \end{pmatrix}.$$

Here we consider $n^\sharp\in\mathbb{R}^3$ as a column vector and define

$$\nabla_s\mu := \begin{pmatrix} \partial_{s_1}\mu_1 & \partial_{s_1}\mu_2 & \partial_{s_1}\mu_3\\ \partial_{s_2}\mu_1 & \partial_{s_2}\mu_2 & \partial_{s_2}\mu_3 \end{pmatrix}, \quad \eta_\varepsilon := \begin{pmatrix} \eta_\varepsilon^1\\ \eta_\varepsilon^2 \end{pmatrix}.$$

Since μ is a local parametrization of Γ, the vectors $\partial_{s_1}\mu$ and $\partial_{s_2}\mu$ are tangent to Γ and thus $(\nabla_s\mu)n^\sharp = 0$. Also, we have $W^\sharp n^\sharp = 0$, $(\eta_\varepsilon\otimes n^\sharp)n^\sharp = |n^\sharp|^2\eta_\varepsilon = \eta_\varepsilon$, and

$$(\eta_\varepsilon\otimes n^\sharp)(n^\sharp\otimes\eta_\varepsilon) = |n^\sharp|^2\eta_\varepsilon\otimes\eta_\varepsilon = \eta_\varepsilon\otimes\eta_\varepsilon.$$

From these equalities and the symmetry of the matrix $W^\sharp$ it follows that

$$\nabla_s\zeta(\nabla_s\zeta)^T = \begin{pmatrix} \nabla_s\mu(I_3 - h_\varepsilon W^\sharp)^2(\nabla_s\mu)^T + \eta_\varepsilon\otimes\eta_\varepsilon & \varepsilon g^\sharp\eta_\varepsilon\\ \varepsilon g^\sharp\eta_\varepsilon^T & \varepsilon^2(g^\sharp)^2 \end{pmatrix}.$$

Therefore, by elementary row operations we have

$$\begin{aligned} \det[\nabla_s\zeta(\nabla_s\zeta)^T] &= \det\begin{pmatrix} \nabla_s\mu(I_3 - h_\varepsilon W^\sharp)^2(\nabla_s\mu)^T + \eta_\varepsilon\otimes\eta_\varepsilon & \varepsilon g^\sharp\eta_\varepsilon\\ \varepsilon g^\sharp\eta_\varepsilon^T & \varepsilon^2(g^\sharp)^2 \end{pmatrix}\\ &= \det\begin{pmatrix} \nabla_s\mu(I_3 - h_\varepsilon W^\sharp)^2(\nabla_s\mu)^T & 0\\ \varepsilon g^\sharp\eta_\varepsilon^T & \varepsilon^2(g^\sharp)^2 \end{pmatrix}\\ &= \varepsilon^2(g^\sharp)^2\det[\nabla_s\mu(I_3 - h_\varepsilon W^\sharp)^2(\nabla_s\mu)^T]\\ &= \varepsilon^2(g^\sharp)^2 J(\mu,h_\varepsilon)^2\det\theta, \end{aligned}$$

where $\theta := \nabla_s\mu(\nabla_s\mu)^T$ and the last equality follows from the same calculations as in (5.B.23). Since $\det\nabla_s\zeta = \sqrt{\det[\nabla_s\zeta(\nabla_s\zeta)^T]}$, we conclude by the above equality that the formula (5.3.15) is valid. □

5.C Riemannian connection on a surface

In this appendix we introduce the notion of the Riemannian (or Levi-Civita) connection on a surface in $\mathbb{R}^3$ and give several formulas for the covariant derivatives.

Let Γ be a two-dimensional closed, connected, oriented, and sufficiently smooth (at least of class C^3) surface in $\mathbb{R}^3$. We use the same notations on the differential operators and the surface quantities on Γ as in Section 5.2.1. For $X \in C^1(\Gamma, T\Gamma)$ and $Y \in C(\Gamma, T\Gamma)$ we define the covariant derivative of X along Y as

$$\overline{\nabla}_Y X := P\left\{(Y \cdot \nabla)\widetilde{X}\right\} \quad \text{on} \quad \Gamma. \tag{5.C.1}$$

Here $\widetilde{X}$ is a C^1-extension of X to an open neighborhood of Γ with $\widetilde{X}|_\Gamma = X$. Note that, since Y is tangential on Γ, we have $(Y \cdot \nabla)\widetilde{X} = (Y \cdot \nabla_\Gamma)X$ and thus the value of $\overline{\nabla}_Y X$ does not depend on the choice of an extension of X. The directional derivative $(Y \cdot \nabla_\Gamma)X$ is expressed in terms of the covariant derivative and the Weingarten map.

Lemma 5.C.1. *For $X \in C^1(\Gamma, T\Gamma)$ and $Y \in C(\Gamma, T\Gamma)$ we have*

$$(Y \cdot \nabla)\widetilde{X} = (Y \cdot \nabla_\Gamma)X = \overline{\nabla}_Y X + (WX \cdot Y)n \quad \text{on} \quad \Gamma, \tag{5.C.2}$$

where $\widetilde{X}$ is any C^1-extension of X to an open neighborhood of Γ satisfying $\widetilde{X}|_\Gamma = X$.

Proof. Since $X \cdot n = 0$ and $-\nabla_\Gamma n = W$ on Γ,

$$(Y \cdot \nabla_\Gamma)X \cdot n = Y \cdot \nabla_\Gamma(X \cdot n) - X \cdot (Y \cdot \nabla_\Gamma)n = X \cdot (-\nabla_\Gamma n)^T Y = X \cdot W^T Y = WX \cdot Y$$

on Γ. Combining this with (5.C.1) we obtain (5.C.2). □

The formula (5.C.2) is called the Gauss formula (see e.g. [9, Section 4.2] and [33, Section VII.3]). Let us prove fundamental properties of $\overline{\nabla}$.

Lemma 5.C.2. *The following equalities hold:*

- *For $X \in C^1(\Gamma, T\Gamma)$, $Y, Z \in C(\Gamma, T\Gamma)$, and $\eta, \xi \in C(\Gamma)$,*

$$\overline{\nabla}_{\eta Y + \xi Z} X = \eta \overline{\nabla}_Y X + \xi \overline{\nabla}_Z X \quad \text{on} \quad \Gamma. \tag{5.C.3}$$

- *For $X \in C^1(\Gamma, T\Gamma)$, $Y \in C(\Gamma, T\Gamma)$, and $\eta \in C^1(\Gamma)$,*

$$\overline{\nabla}_Y(\eta X) = (Y \cdot \nabla_\Gamma \eta)X + \eta \overline{\nabla}_Y X \quad \text{on} \quad \Gamma. \tag{5.C.4}$$

- *For $X, Y \in C^1(\Gamma, T\Gamma)$ and $Z \in C(\Gamma, T\Gamma)$,*

$$Z \cdot \nabla_\Gamma(X \cdot Y) = \overline{\nabla}_Z X \cdot Y + X \cdot \overline{\nabla}_Z Y \quad \text{on} \quad \Gamma. \tag{5.C.5}$$

- *For $X, Y \in C^1(\Gamma, T\Gamma)$ and $\eta \in C^2(\Gamma)$,*

$$X \cdot \nabla_\Gamma(Y \cdot \nabla_\Gamma \eta) - Y \cdot \nabla_\Gamma(X \cdot \nabla_\Gamma \eta) = \left(\overline{\nabla}_X Y - \overline{\nabla}_Y X\right) \cdot \nabla_\Gamma \eta. \tag{5.C.6}$$

Proof. The identities (5.C.3) and (5.C.4) immediately follow from the definition (5.C.1) of the covariant derivative. Also, applying (5.C.2) and $X \cdot n = Y \cdot n = 0$ on Γ to the right-hand side of

$$Z \cdot \nabla_\Gamma(X \cdot Y) = (Z \cdot \nabla_\Gamma)X \cdot Y + X \cdot (Z \cdot \nabla_\Gamma)Y$$

we get (5.C.5). To prove (5.C.6) we use the formula (see e.g. [43, Lemma 2.2])

$$\underline{D}_i\underline{D}_j\eta - \underline{D}_j\underline{D}_i\eta = [W\nabla_\Gamma\eta]_i n_j - [W\nabla_\Gamma\eta]_j n_i \quad \text{on} \quad \Gamma,\ i,j = 1,2,3. \tag{5.C.7}$$

The left-hand side of (5.C.6) is of the form

$$\begin{aligned}\sum_{i,j=1}^{3} &\{X_i\underline{D}_i(Y_j\underline{D}_j\eta) - Y_i\underline{D}_i(X_j\underline{D}_j\eta)\} \\ &= \sum_{i,j=1}^{3}(X_i\underline{D}_iY_j - Y_i\underline{D}_iX_j)\underline{D}_j\eta + \sum_{i,j=1}^{3}(X_iY_j - X_jY_i)\underline{D}_i\underline{D}_j\eta.\end{aligned}$$

By (5.C.2) and $\nabla_\Gamma\eta \cdot n = 0$ on Γ, we have

$$\begin{aligned}\sum_{i,j=1}^{3}(X_i\underline{D}_iY_j - Y_i\underline{D}_iX_j)\underline{D}_j\eta &= \{(X \cdot \nabla_\Gamma)Y - (X \cdot \nabla_\Gamma)X\} \cdot \nabla_\Gamma\eta \\ &= \left(\overline{\nabla}_XY - \overline{\nabla}_YX\right) \cdot \nabla_\Gamma\eta.\end{aligned}$$

Also, using (5.C.7) and noting that $X \cdot n = Y \cdot n = 0$ on Γ we observe that

$$\begin{aligned}\sum_{i,j=1}^{3}(X_iY_j - X_jY_i)\underline{D}_i\underline{D}_j\eta &= \sum_{i,j=1}^{3}X_iY_j(\underline{D}_i\underline{D}_j\eta - \underline{D}_j\underline{D}_i\eta) \\ &= \sum_{i,j=1}^{3}X_iY_j([W\nabla\eta]_i n_j - [W\nabla_\Gamma\eta]_j n_i) \\ &= (X \cdot W\nabla_\Gamma\eta)(Y \cdot n) - (X \cdot n)(Y \cdot W\nabla_\Gamma\eta) = 0.\end{aligned}$$

Combining the above three equalities we obtain (5.C.6). □

By Lemma 5.C.2 we observe that the assignment $\overline{\nabla}\colon (X,Y) \mapsto \overline{\nabla}_YX$ defines the Riemannian (or Levi-Civita) connection on Γ (see e.g. [9, 30, 50] for the definition of the Riemannian connection). Note that the formula (5.C.6) represents the torsion-free condition $[X,Y] = \overline{\nabla}_XY - \overline{\nabla}_YX$ for $X, Y \in C^1(\Gamma, T\Gamma)$, where $[X,Y] = XY - YX$ is the Lie bracket of X and Y (see e.g. [63]).

Let U be a relatively open subset of Γ and τ_1 and τ_2 be C^1 tangential vector fields defined on U such that the pair $\{\tau_1(y), \tau_2(y)\}$ is an orthonormal basis of the tangent plane of Γ at each $y \in U$. (Since Γ is at least of class C^3, we can take such vector fields for sufficiently small U.) We call the pair $\{\tau_1, \tau_2\}$ a local orthonormal frame of the tangent plane of Γ on U. Note that

$$H = \mathrm{tr}[W] = \sum_{i=1,2} W\tau_i \cdot \tau_i \quad \text{on} \quad U \tag{5.C.8}$$

since $\{\tau_1, \tau_2, n\}$ is an orthonormal basis of $\mathbb{R}^3$ and $Wn = 0$. Let us express several quantities related to the tangential gradient matrix of tangential vector fields on Γ in terms of the covariant derivatives and the local orthonormal frame.

Lemma 5.C.3. *Let $\{\tau_1, \tau_2\}$ be a local orthonormal frame of the tangent plane of Γ on a relatively open subset U of Γ. For $X, Y \in C^1(\Gamma, T\Gamma)$ we have*

$$\mathrm{div}_\Gamma X = \sum_{i=1,2} \overline{\nabla}_i X \cdot \tau_i, \tag{5.C.9}$$

$$\nabla_\Gamma X : W = \sum_{i=1,2} \overline{\nabla}_i X \cdot W\tau_i = \sum_{i=1,2} W\overline{\nabla}_i X \cdot \tau_i, \tag{5.C.10}$$

$$\nabla_\Gamma X : \nabla_\Gamma Y = \sum_{i=1,2} \overline{\nabla}_i X \cdot \overline{\nabla}_i Y + WX \cdot WY, \tag{5.C.11}$$

$$\nabla_\Gamma X : (\nabla_\Gamma Y)P = \sum_{i=1,2} \overline{\nabla}_i X \cdot \overline{\nabla}_i Y, \tag{5.C.12}$$

$$W\nabla_\Gamma X : (\nabla_\Gamma Y)P = \sum_{i=1,2} \overline{\nabla}_{W\tau_i} X \cdot \overline{\nabla}_i Y \tag{5.C.13}$$

on U, where $\overline{\nabla}_i := \overline{\nabla}_{\tau_i}$ for $i = 1, 2$.

Proof. Throughout the proof we carry out calculations on U. First note that

$$\begin{aligned} (\nabla_\Gamma X)^T \tau_i &= (\tau_i \cdot \nabla_\Gamma) X = \overline{\nabla}_i X + (WX \cdot \tau_i) n, \quad i = 1, 2, \\ (\nabla_\Gamma X)^T n &= (n \cdot \nabla_\Gamma) X = 0 \end{aligned} \tag{5.C.14}$$

by (5.C.2) and $n \cdot \nabla_\Gamma X_j = 0$ for $j = 1, 2, 3$. Since $\{\tau_1, \tau_2, n\}$ forms an orthonormal basis of $\mathbb{R}^3$, we have

$$\mathrm{div}_\Gamma X = \mathrm{tr}[\nabla_\Gamma X] = \sum_{i=1,2} (\nabla_\Gamma X)^T \tau_i \cdot \tau_i + (\nabla_\Gamma X)^T n \cdot n.$$

The equality (5.C.9) follows from the above equality and (5.C.14). We also get (5.C.10) by using (5.C.14), $W^T = W$, and $Wn = 0$ to the right-hand side of

$$\nabla_\Gamma X : W = (\nabla_\Gamma X)^T : W^T = \sum_{i=1,2} (\nabla_\Gamma X)^T \tau_i \cdot W^T \tau_i + (\nabla_\Gamma X)^T n \cdot W^T n.$$

Let us prove (5.C.11). By (5.C.2) and $\overline{\nabla}_i X \cdot n = \overline{\nabla}_i Y \cdot n = 0$,

$$\begin{aligned} \nabla_\Gamma X : \nabla_\Gamma Y &= (\nabla_\Gamma X)^T : (\nabla_\Gamma Y)^T = \sum_{i=1,2} (\nabla_\Gamma X)^T \tau_i \cdot (\nabla_\Gamma Y)^T \tau_i + (\nabla_\Gamma X)^T n \cdot (\nabla_\Gamma Y)^T n \\ &= \sum_{i=1,2} \Big\{\overline{\nabla}_i X + (WX \cdot \tau_i) n\Big\} \cdot \Big\{\overline{\nabla}_i Y + (WY \cdot \tau_i) n\Big\} \\ &= \sum_{i=1,2} \overline{\nabla}_i X \cdot \overline{\nabla}_i Y + \sum_{i=1,2} (WX \cdot \tau_i)(WY \cdot \tau_i). \end{aligned}$$

Here the second term on the last line is equal to $WX \cdot WY$, since WX and WY are tangential vector fields on Γ and $\{\tau_1, \tau_2\}$ is an orthonormal basis of the tangent plane of Γ. Hence (5.C.11) is valid. Also, noting that $P^T = P$, $W^T = W$, and

$$\begin{aligned} P(\nabla_\Gamma Y)^T \tau_i &= P\Big\{\overline{\nabla}_i Y + (WY \cdot \tau_i) n\Big\} = \overline{\nabla}_i Y, \\ (\nabla_\Gamma X)^T W\tau_i &= (W\tau_i \cdot \nabla_\Gamma) X = \overline{\nabla}_{W\tau_i} X + (WX \cdot W\tau_i) n \end{aligned}$$

by (5.C.2), $Pn = 0$, and the fact that $\overline{\nabla}_i Y$ is tangential on Γ, we can prove (5.C.12) and (5.C.13) by the same calculations as above. $\square$

Next we give an integration by parts formula for integrals over Γ of the covariant derivatives along vector fields of a local orthonormal frame.

Lemma 5.C.4. *Let $\{\tau_1,\tau_2\}$ be a local orthonormal frame of the tangent plane of Γ on a relatively open subset U of Γ and $\overline{\nabla}_i := \overline{\nabla}_{\tau_i}$ for $i=1,2$. Suppose that $X\in C^2(\Gamma,T\Gamma)$ and $Y\in C^1(\Gamma,T\Gamma)$ are compactly supported in U. Then we have*

$$\sum_{i=1,2}\int_\Gamma\Bigl(\overline{\nabla}_i\overline{\nabla}_iX-\overline{\nabla}_{\overline{\nabla}_i\tau_i}X\Bigr)\cdot Y\,d\mathcal{H}^2=-\sum_{i=1,2}\int_\Gamma\overline{\nabla}_iX\cdot\overline{\nabla}_iY\,d\mathcal{H}^2. \tag{5.C.15}$$

Proof. The proof is basically the same as that of [50, Proposition 34]. We define a tangential vector field Z on Γ by

$$Z:=\begin{cases}\displaystyle\sum_{i=1,2}\Bigl(\overline{\nabla}_iX\cdot Y\Bigr)\tau_i & \text{on} \quad U,\\ 0 & \text{on} \quad \Gamma\setminus U.\end{cases}$$

Since $X\in C^2(\Gamma,T\Gamma)$ and $Y\in C^1(\Gamma,T\Gamma)$ are compactly supported in U, we have $Z\in C^1(\Gamma,T\Gamma)$. Moreover, since $\{\tau_1,\tau_2\}$ forms an orthonormal basis of the tangent plane of Γ, we see by (5.C.3) that $Z\cdot V=\overline{\nabla}_VX\cdot Y$ on Γ for all $V\in C(\Gamma,T\Gamma)$. From this fact and (5.C.5) we deduce that

$$\overline{\nabla}_iZ\cdot\tau_i=\tau_i\cdot\nabla_\Gamma(Z\cdot\tau_i)-Z\cdot\overline{\nabla}_i\tau_i=\tau_i\cdot\nabla_\Gamma\Bigl(\overline{\nabla}_iX\cdot Y\Bigr)-\overline{\nabla}_{\overline{\nabla}_i\tau_i}X\cdot Y$$

on U for $i=1,2$. Applying (5.C.5) again to the first term on the right-hand side we get

$$\overline{\nabla}_iZ\cdot\tau_i=\Bigl(\overline{\nabla}_i\overline{\nabla}_iX-\overline{\nabla}_{\overline{\nabla}_i\tau_i}X\Bigr)\cdot Y+\overline{\nabla}_iX\cdot\overline{\nabla}_iY \quad \text{on} \quad U.$$

By this equality and (5.C.9) we see that

$$\mathrm{div}_\Gamma Z=\sum_{i=1,2}\Bigl\{\Bigl(\overline{\nabla}_i\overline{\nabla}_iX-\overline{\nabla}_{\overline{\nabla}_i\tau_i}X\Bigr)\cdot Y+\overline{\nabla}_iX\cdot\overline{\nabla}_iY\Bigr\} \quad \text{on} \quad U. \tag{5.C.16}$$

Since X, Y, and Z are supported in U, we may assume that (5.C.16) holds on the whole surface Γ. Thus, integrating both sides of (5.C.16) over Γ and noting that the integral of the surface divergence of Z over Γ vanishes by the Stokes theorem (note that Z is tangential and Γ has no boundary), we conclude that the formula (5.C.15) is valid. $\square$

The formula (5.C.15) gives a relation between two Laplacians acting on tangential vector fields on Γ. For $X\in C^2(\Gamma,T\Gamma)$, $Y\in C^1(\Gamma,T\Gamma)$, and $Z\in C(\Gamma,T\Gamma)$, the second covariant derivative is defined as

$$\overline{\nabla}^2_{Z,Y}X:=\overline{\nabla}_Z\overline{\nabla}_YX-\overline{\nabla}_{\overline{\nabla}_ZY}X.$$

The trace of the second covariant derivative is called the connection Laplacian, i.e.

$$\mathrm{tr}\overline{\nabla}^2X:=\sum_{i=1,2}\overline{\nabla}^2_{\tau_i,\tau_i}X=\sum_{i=1,2}\Bigl(\overline{\nabla}_i\overline{\nabla}_iX-\overline{\nabla}_{\overline{\nabla}_i\tau_i}X\Bigr), \quad X\in C^2(\Gamma,T\Gamma).$$

Here $\{\tau_1, \tau_2\}$ is a local orthonormal frame and $\overline{\nabla}_i = \overline{\nabla}_{\tau_i}$ for $i = 1, 2$. On the other hand, the Bochner Laplacian $\Delta_B X$ of $X \in C^2(\Gamma, T\Gamma)$ is defined as a tangential vector field on Γ satisfying

$$\int_\Gamma \Delta_B X \cdot Y \, d\mathcal{H}^2 := -\sum_{i=1,2} \int_\Gamma \overline{\nabla}_i X \cdot \overline{\nabla}_i Y \, d\mathcal{H}^2 \tag{5.C.17}$$

for all $Y \in C^1(\Gamma, T\Gamma)$, where we assume that X and Y are appropriately localized by a partition of unity of Γ. (Note that there are other definitions of the connection and Bochner Laplacians in which one takes the opposite sign.) Then by (5.C.15) we observe that these two Laplacians agree, i.e.

$$\mathrm{tr}\overline{\nabla}^2 X = \Delta_B X, \quad X \in C^2(\Gamma, T\Gamma).$$

When the surface Γ is embedded in $\mathbb{R}^3$, we can also consider the Laplace–Beltrami operator acting on each component of a vector field $X = (X_1, X_2, X_3)$ on Γ:

$$\Delta_\Gamma X = (\Delta_\Gamma X_1, \Delta_\Gamma X_2, \Delta_\Gamma X_3), \quad \Delta_\Gamma = \sum_{i=1}^3 \underline{D}_i^2.$$

Let us give a relation between the two Laplacians Δ_B and Δ_Γ.

Lemma 5.C.5. *For $X \in C^2(\Gamma, T\Gamma)$ we have*

$$\Delta_B X = P\Delta_\Gamma X + W^2 X \quad \text{on} \quad \Gamma. \tag{5.C.18}$$

Proof. The relation (5.C.18) is proved in [43, Appendix B]. Here we give another proof of it. By a localization argument with a partition of unity of Γ, it is sufficient to show (5.C.18) on a relatively open subset U of Γ on which we can take a local orthonormal frame $\{\tau_1, \tau_2\}$. Let $Y \in C^1(\Gamma, T\Gamma)$ be an arbitrary vector field compactly supported in U. By (5.C.11) and (5.C.17) we have

$$\begin{aligned}\int_\Gamma \Delta_B X \cdot Y \, d\mathcal{H}^2 &= -\sum_{i=1,2} \int_\Gamma \overline{\nabla}_i X \cdot \overline{\nabla}_i Y \, d\mathcal{H}^2 \\ &= -\int_\Gamma \nabla_\Gamma X : \nabla_\Gamma Y \, d\mathcal{H}^2 + \int_\Gamma WX \cdot WY \, d\mathcal{H}^2.\end{aligned} \tag{5.C.19}$$

To the first integral on the last line we use (5.2.18) to get

$$\begin{aligned}\int_\Gamma \nabla_\Gamma X : \nabla_\Gamma Y \, d\mathcal{H}^2 &= \sum_{i,j=1}^3 \int_\Gamma (\underline{D}_i X_j)(\underline{D}_i Y_j) \, d\mathcal{H}^2 \\ &= -\sum_{i,j=1}^3 \int_\Gamma \{(\underline{D}_i^2 X_j) Y_j + (\underline{D}_i X_j) Y_j H n_i\} \, d\mathcal{H}^2 \\ &= -\int_\Gamma \Delta_\Gamma X \cdot Y \, d\mathcal{H}^2 - \int_\Gamma \{(n \cdot \nabla_\Gamma) X \cdot Y\} H \, d\mathcal{H}^2.\end{aligned}$$

Noting that the second integral on the last line vanishes by $(n \cdot \nabla_\Gamma)X = 0$, we apply this equality and $W^T = W$ to (5.C.19) to obtain

$$\int_\Gamma \Delta_B X \cdot Y \, d\mathcal{H}^2 = \int_\Gamma (\Delta_\Gamma X + W^2 X) \cdot Y \, d\mathcal{H}^2.$$

Now we set $Y := Pv$ in this equality with $v \in C^1(\Gamma)^3$ compactly supported in U. Then since $\Delta_B X$ and $W^2 X$ are tangential on Γ (note that $PW = W$), we have

$$\int_\Gamma \Delta_B X \cdot v \, d\mathcal{H}^2 = \int_\Gamma (P\Delta_\Gamma X + W^2 X) \cdot v \, d\mathcal{H}^2$$

for all $v \in C^1(\Gamma)^3$ compactly supported in U. Hence by the fundamental lemma of calculus of variations we conclude that (5.C.18) holds on U. $\square$

Note that the normal component of $\Delta_\Gamma X$ does not vanish in general even if X is tangential on Γ. Indeed, using $X \cdot n = 0$ and $-\nabla_\Gamma n = W$ we observe by direct calculations that

$$\Delta_\Gamma X \cdot n = \mathrm{div}_\Gamma(WX) + W : \nabla_\Gamma X = \mathrm{div}_\Gamma W \cdot X + 2W : \nabla_\Gamma X \quad \text{on} \quad \Gamma.$$

When Γ is a flat domain in $\mathbb{R}^2$, we have $W = -\nabla_\Gamma n = 0$ by $n = (0,0,1)$. Hence the normal component of $\Delta_\Gamma X$ vanishes and $\Delta_B X = \Delta_\Gamma X$ reduces to the usual Laplacian on $\mathbb{R}^2$ acting on each component of $X = (X_1, X_2)$.

Remark 5.C.6. By Lemma 5.2.6 the function space $C^2(\Gamma, T\Gamma)$ is dense in $H^m(\Gamma, T\Gamma)$, $m = 0, 1, 2$. Hence the formulas given in this appendix are also valid (a.e. on Γ) if we replace $C^m(\Gamma, T\Gamma)$, $m = 0, 1, 2$ with $H^m(\Gamma, T\Gamma)$.

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Chapter 6

Hamilton–Jacobi equations on an evolving surface

6.1 Introduction

In this chapter we are concerned with the existence, uniqueness and numerical approximation of Hamilton–Jacobi equations on moving hypersurfaces. Let $\Gamma(t)$, $t \in [0,T]$ be a family of smooth, closed, connected and oriented hypersurfaces in $\mathbb{R}^3$ and $S_T := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. We consider the following Hamilton–Jacobi equation on the evolving surfaces $\Gamma(t)$

$$\partial^\bullet u + H(x,t,\nabla_\Gamma u) = 0 \quad \text{on} \quad S_T. \tag{6.1.1}$$

In the above, $\partial^\bullet u = u_t + v_\Gamma \cdot \nabla u$ denotes the material derivative, v_Γ denotes the velocity of a parametrisation of $\Gamma(t)$, and $\nabla_\Gamma u = (I_3 - \nu \otimes \nu)\nabla u$ the tangential gradient of u, where ν is a unit normal field of $\Gamma(t)$ respectively. The precise definitions and assumptions on $H\colon S_T \times \mathbb{R}^3 \to \mathbb{R}$ will be given in Sections 2 and 3. The well-posedness theory is developed using the concept of viscosity solutions and existence is achieved through finite volume discretisations on evolving triangulations.

6.1.1 A motivating example

It is the purpose of this chapter to study the natural development of a theory of viscosity solutions to first order equations on evolving surfaces. One motivation for considering Hamilton–Jacobi equations of the form (6.1.1) is the following application. Consider the motion of a closed curve $\gamma(t) \subset \Gamma(t)$ according to the evolution law

$$V_\mu(x,t) = F(x,t) + \beta(x,t) \cdot \mu(x,t), \qquad x \in \gamma(t), \tag{6.1.2}$$

where V_μ denotes the velocity of $\gamma(t)$ in the direction of the conormal μ and $F\colon S_T \to \mathbb{R}$, and $\beta\colon S_T \to \mathbb{R}^3$ are a given function and vector field. Let us assume that

$$\gamma(t) = \{(x,t) \in S_T \mid u(x,t) = r\}$$

for some $r \in \mathbb{R}$ with a function $u\colon N_T \to \mathbb{R}$ satisfying $\nabla_\Gamma u(\cdot,t) \neq 0$ on $\gamma(t)$, where N_T is an open neighbourhood of S_T. Choosing parametrizations $\varphi(\cdot,t)\colon S^1 \to \mathbb{R}^3$ of $\gamma(t)$ we have that $u(\varphi(s,t),t) = r$ for $s \in S^1, t \in (0,T)$. If we differentiate both sides with respect to t we obtain

$$u_t(\varphi(s,t),t) + \varphi_t(s,t) \cdot \nabla u(\varphi(s,t),t) = 0,$$

or equivalently

$$\begin{aligned} 0 &= \partial^\bullet u(\varphi(s,t),t) + \big(\varphi_t(s,t) - v_\Gamma(\varphi_t(s,t),t)\big) \cdot \nabla u(\varphi(s,t),t) \\ &= \partial^\bullet u(\varphi(s,t),t) + \big(\varphi_t(s,t) - v_\Gamma(\varphi_t(s,t),t)\big) \cdot \nabla_\Gamma u(\varphi(s,t),t), \end{aligned}$$

since $\varphi(s,t) \in \Gamma(t)$ implies that $(\varphi_t(s,t) - v_\Gamma(\varphi(s,t),t)) \cdot \nu(\varphi(s,t),t) = 0$. Using that $\mu = \nabla_\Gamma u / |\nabla_\Gamma u|$ we obtain from (6.1.2) that at $x = \varphi(s,t)$

$$\begin{aligned} F(x,t) + \beta(x,t) \cdot \mu(x,t) = V_\mu(x,t) &= \varphi_t(s,t) \cdot \frac{\nabla_\Gamma u(x,t)}{|\nabla_\Gamma u(x,t)|} \\ &= -\frac{\partial^\bullet u(x,t)}{|\nabla_\Gamma u(x,t)|} + v_\Gamma(x,t) \cdot \frac{\nabla_\Gamma u(x,t)}{|\nabla_\Gamma u(x,t)|}. \end{aligned}$$

Formally the above calculations then show that the level sets of a solution u of (6.1.1) with

$$H(x,t,p) = F(x,t)\,|p| + \beta(x,t) \cdot p - v_\Gamma(x,t) \cdot p \tag{6.1.3}$$

evolve according to the evolution law (6.1.2).

6.1.2 Background

Partial differential equations on time evolving hypersurfaces arise in many applications in biology, fluids and materials science, for example see [6, 12, 13, 16] and the references cited therein. The theory of parabolic equations has been considered in [2, 3, 9, 26]. Existence and uniqueness of first order scalar conservation laws on moving hypersurfaces and Riemannian manifolds has been proved in [11, 20]. Viscosity solutions of Hamilton–Jacobi equations on Riemannian manifolds are considered in [24]. See [7] and [23] for level set approaches to the motion of curves on a stationary surface. Numerical transport on evolving surfaces by level set methods was considered in [1, 27]. The numerical analysis of advection diffusion equations on evolving surfaces via the evolving surface finite element method began in [9], see also [10, 19]. Finite volume schemes for diffusion and conservation laws on moving surfaces have been considered, respectively, in [21] and [14]. Another approach is to use diffuse interfaces, see [25].

6.1.3 Outline

The paper is organized as follows. We begin in Section 2 by establishing some notation and concepts relating to moving surfaces. In Section 3 we generalise the classical definition of viscosity solution (see e.g. [4, 5, 15]) to moving curved domains using surface derivative operators instead of the usual derivatives. In this setting we show that a comparison principle holds which yields uniqueness of a viscosity solution. As in the seminal work [8] we approach existence via a discretisation in space and time. To do so, we approximate the moving surfaces by triangulated surfaces so that we need to formulate our numerical scheme an unstructured meshes. Numerical schemes for Hamilton–Jacobi equations on unstructured meshes on flat domains have been proposed in [18] and [22]. In order to guarantee monotonicity of their schemes the authors in [18], [22] have to assume that the underlying triangulation is acute, which is a rather strong requirement and difficult to realise in the case of moving surfaces where the triangulation will vary from time step to time step. In order to address this issue we construct in Section 4 a finite volume scheme by adapting an idea introduced by Kim and Li in [17] to the case of evolving hypersurfaces. With this construction we are able to prove monotonicity and consistency assuming only regularity of the triangulation. In Section 5 we prove that the sequence of discrete solutions obtained via our scheme converges to a viscosity solution if the discretization parameters tend to zero. At the same time this gives an existence result for the Hamilton–Jacobi equation. Finally, we prove in Section 6 an $O(\sqrt{h})$ error bound between the viscosity solution and the numerical solution extending well–known error estimates for the flat case to the case of moving hypersurfaces.

6.2 Preliminaries

6.2.1 Tangential derivatives of functions on fixed surfaces

Let Γ be a smooth, closed (i.e. compact without boundary) and orientable hypersurface in $\mathbb{R}^3$ with outward unit normal field ν. For a differentiable function f on Γ we define the tangential gradient by

$$\nabla_\Gamma f(x) := P_\Gamma(x)\nabla \tilde{f}(x), \quad x \in \Gamma, \tag{6.2.1}$$

where $\tilde{f}$ is a smooth extension of f to an open neighbourhood N of Γ satisfying $\tilde{f} = f$ on $N \cap \Gamma$ and $P_\Gamma(x) := I_3 - \nu(x) \otimes \nu(x)$ is the orthogonal projection onto the tangent plane of Γ at x. Here I_3 is the (3×3) identity matrix and $\nu \otimes \nu = (\nu_i \nu_j)_{i,j}$ is the tensor product of ν. It is well–known that $\nabla_\Gamma f(x)$ is independent of the particular extension $\tilde{f}$. Furthermore, we define by $\Delta_\Gamma f := \nabla_\Gamma \cdot \nabla_\Gamma f$ the Laplace–Beltrami operator of f. We denote by d the signed distance function to Γ oriented in such a way that it increases in the direction of ν. There exists an open neighbourhood U of Γ such that d is smooth in U and such that for every $x \in U$ there exists a unique $\pi(x) \in \Gamma$ with

$$x = \pi(x) + d(x)\nu(\pi(x)) \quad \text{ and } \quad \nabla d(x) = \nu(\pi(x)). \tag{6.2.2}$$

For a given function $f : \Gamma \to \mathbb{R}$ we can define $f_c : U \to \mathbb{R}$ via $f_c(x) := f(\pi(x))$, which extends f constantly in the normal direction to Γ. It is not difficult to verify that

$$\nabla f_c(x) = \nabla_\Gamma f(x), \quad x \in \Gamma, \tag{6.2.3}$$

$$\|\nabla f_c\|_{B(U)} \le c\|\nabla_\Gamma f\|_{B(\Gamma)}, \tag{6.2.4}$$

$$\|\nabla^2 f_c\|_{B(U)} \le c\left(\|\nabla_\Gamma f\|_{B(\Gamma)} + \|\nabla^2_\Gamma f\|_{B(\Gamma)}\right), \tag{6.2.5}$$

provided that the derivatives of f exist. Here, $\|f\|_{B(D)} := \sup_{x\in D} |f(x)|$.

6.2.2 Time dependent surfaces

Let us next turn to the case of time dependent surfaces and assume that Γ_0 is a closed, connected, oriented and smooth hypersurface in $\mathbb{R}^3$. We consider a family $\{\Gamma(t)\}_{t\in[0,T]}$, $T > 0$ of evolving hypersurfaces given via a smooth flow map $\Phi\colon \Gamma_0 \times [0, T] \to \mathbb{R}^3$ such that $\Phi(\cdot, t)$ is a diffeomorphism of Γ_0 onto $\Gamma(t)$ satisfying

$$\frac{\partial \Phi}{\partial t}(X, t) = v_\Gamma(\Phi(X, t), t), \quad \Phi(X, 0) = X, \tag{6.2.6}$$

for all $X \in \Gamma_0, t \in (0, T)$. Here we say that v_Γ is the velocity field of $\Gamma(t)$. Let $d(\cdot, t)$ be the signed distance function to $\Gamma(t)$ increasing in the direction of $\nu(\cdot, t)$, where $\nu(\cdot, t)$ is the unit

outward normal of $\Gamma(t)$. For each $t \in [0,T]$ there exists a bounded open subset $N(t) \subset \mathbb{R}^3$ such that d is smooth in $N_T := \bigcup_{t\in(0,T)}(N(t) \times \{t\})$ and such that for every $x \in N(t)$ there exists a unique $\pi(x,t) \in \Gamma(t)$ satisfying (6.2.2).
Next, for a differentiable function f on S_T, the material derivative of f along the velocity v_Γ is defined as

$$\partial^\bullet f(\Phi(X,t),t) = \frac{d}{dt}\Big(f(\Phi(X,t),t)\Big), \quad (X,t) \in \Gamma_0 \times (0,T).$$

The material derivative is also expressed as

$$\partial^\bullet f(x,t) = \partial_t \tilde{f}(x,t) + v_\Gamma(x,t) \cdot \nabla \tilde{f}(x,t), \quad (x,t) \in S_T, \tag{6.2.7}$$

where $\tilde{f}$ is an arbitrary extension of f to N_T satisfying $\tilde{f}|_{S_T} = f$.

6.2.3 Triangulated surface

In order to approximate the evolving surfaces $\Gamma(t)$ we choose a family of triangulations $(\mathcal{T}_h(t))_{0<h<h_0}$ of $\Gamma(t)$ and set

$$\Gamma_h(t) := \bigcup_{K(t)\in\mathcal{T}_h(t)} K(t) \quad \text{and} \quad h := \max_{t\in[0,T]} \max_{K(t)\in\mathcal{T}_h(t)} h_{K(t)},$$

where $h_{K(t)} = \operatorname{diam} K(t)$ for each triangle $K(t)$. We assume that there exists a constant $\gamma > 0$ such that

$$\forall t \in [0,T]\ \forall K(t) \in \mathcal{T}_h(t) \qquad h_{K(t)} \leq \gamma \rho_{K(t)}, \tag{6.2.8}$$

where $\rho_{K(t)}$ is the radius of the largest circle contained in $K(t)$. We denote by $\nu_h(\cdot,t)$ the unit normal to $\Gamma_h(t)$ oriented in the direction in which the signed distance $d(\cdot,t)$ increases. It is well–known that for all $K(t) \subset \Gamma_h(t)$

$$\|d(\cdot,t)\|_{B(K(t))} \leq Ch^2_{K(t)}, \tag{6.2.9}$$

$$\|\nu_{h|K(t)} - \nu(\cdot,t)\|_{B(K(t))} \leq Ch_{K(t)}, \tag{6.2.10}$$

where we can think of $\nu(\cdot,t)$ as being extended to a neighbourhood of $\Gamma(t)$ via $\nu(x,t) = \nabla d(x,t)$ (cf. (6.2.2)).
We assume that the vertices of the triangulation are advected with the velocity v_Γ and thus the number of the vertices, which we refer to as $M \in \mathbb{N}$, is fixed in time. For $i = 1,\ldots,M$ we call the i-th vertex simply i and write $x_i^0 \in \Gamma(0)$ for its point at $t = 0$. By the assumption on the motion of the vertices, the position of i at time $t \in [0,T]$ is given by $x_i(t) = \Phi(x_i^0,t) \in \Gamma(t)$ so that the triangulated surfaces $\Gamma_h(t)$ are interpolations of $\Gamma(t)$. In particular, $\Gamma_h(t) \subset N(t)$ if h_0 is sufficiently small and we assume that $\pi_h(\cdot,t) := \pi(\cdot,t)|_{\Gamma_h(t)}$ is a homeomorphism of $\Gamma_h(t)$ onto $\Gamma(t)$ for each $t \in [0,T]$. Writing $\pi_h^{-1}(\cdot,t)$ for the inverse map, we define the lift of a function $\eta\colon \Gamma_h(t) \to \mathbb{R}$ onto $\Gamma(t)$ by

$$\eta^l(x) := \eta(\pi_h^{-1}(x,t)), \quad x \in \Gamma(t).$$

For each $t \in [0,T]$ we introduce the finite element space

$$V_h(t) = \{u_h \in C^0(\Gamma_h(t)) \mid u_h|_{K(t)} \text{ is linear affine for each } K(t) \in \mathcal{T}_h(t)\}$$

together with its standard nodal basis $\chi_1(\cdot,t),\dots,\chi_M(\cdot,t)$, where $\chi_i(\cdot,t)\in V_h(t)$ satisfies $\chi_i(x_j(t),t)=\delta_{ij}$.
For a function $\eta\in C^0(\Gamma(t))$ we define the linear interpolation $I_h^t\eta\in V_h(t)$ by

$$I_h^t\eta(x):=\sum_{i=1}^M \eta(x_i(t))\chi_i(x,t),\quad x\in\Gamma_h(t).$$

Lemma 6.2.1. *Suppose that $\eta\colon\Gamma(t)\to\mathbb{R}$, $t\in[0,T]$ is Lipschitz continuous, i.e. there exists a constant $L_U>0$ such that*

$$|\eta(x)-\eta(y)|\le L_U|x-y|,\quad x,y\in\Gamma(t). \tag{6.2.11}$$

Then we have

$$\|\eta-[I_h^t\eta]^l\|_{B(\Gamma(t))}\le Ch. \tag{6.2.12}$$

Proof. Fix $x\in\Gamma(t)$. Then there exists $\tilde{x}\in\Gamma_h(t)$ such that $x=\pi_h(\tilde{x},t)$, say $\tilde{x}\in K(t)$ for some $K(t)\in\mathcal{T}_h(t)$. Assuming for simplicity that the vertices of $K(t)$ are $x_1(t),x_2(t)$ and $x_3(t)$ we may write

$$\eta(x)-[I_h^t\eta]^l(x)=\eta(x)-\sum_{i=1}^3\eta(x_i(t))\chi_i(\tilde{x},t)=\sum_{i=1}^3\big(\eta(x)-\eta(x_i(t))\big)\chi_i(\tilde{x},t),$$

since $\sum_{i=1}^3\chi_i(\tilde{x},t)=1$. Combining this relation with the fact that $\chi_i(\tilde{x},t)\ge 0$, (6.2.11), (6.2.2) and (6.2.9) we deduce that

$$\begin{aligned}|\eta(x)-[I_h^t\eta]^l(x)| &\le \max_{i=1,2,3}|\eta(x)-\eta(x_i(t))|\\ &\le L_U\max_{i=1,2,3}|x-x_i(t)|=L_U\max_{i=1,2,3}|\pi(\tilde{x},t)-x_i(t)|\\ &\le L_U\max_{i=1,2,3}\big(|\tilde{x}-x_i(t)|+|d(\tilde{x},t)|\big)\\ &\le L_U\big(h_{K(t)}+Ch_{K(t)}^2\big)\le Ch_{K(t)}\le Ch.\end{aligned}$$

□

6.3 Viscosity solutions: Uniqueness

We consider the Hamilton–Jacobi equation

$$\begin{cases}\partial^\bullet u(x,t)+H(x,t,\nabla_\Gamma u(x,t))=0, & (x,t)\in S_T,\\ u(x,0)=u_0(x), & x\in\Gamma(0).\end{cases} \tag{6.3.1}$$

Here $H\colon\overline{S_T}\times\mathbb{R}^3\to\mathbb{R}$ is a Hamiltonian and $u_0\colon\Gamma(0)\to\mathbb{R}$ is an initial value. Throughout this chapter we suppose that there exist positive constants $L_{H,1}$ and $L_{H,2}$ such that

$$|H(x,t,p)-H(y,s,p)|\le L_{H,1}(|x-y|+|t-s|)(1+|p|), \tag{6.3.2}$$
$$|H(x,t,p)-H(x,t,q)|\le L_{H,2}|p-q| \tag{6.3.3}$$

for all $(x,t),(y,s)\in\overline{S_T}$ and $p,q\in\mathbb{R}^3$. Furthermore, we assume for the velocity field that $v_\Gamma\in C^1(\overline{S_T})$. Note that the Hamiltonian in (6.1.3) satisfies the above conditions provided that F and β are Lipschitz on $\overline{S_T}$.
For $\Gamma=\Gamma(t)$ with each fixed $t\in[0,T]$ or $\Gamma=\overline{S_T}$, we denote by $USC(\Gamma)$ (resp. $LSC(\Gamma)$) the set of all upper (resp. lower) semicontinuous functions on Γ. In what follows we shall work in the framework of discontinuous viscosity solutions.

Definition 6.3.1. Let u_0 be a function on $\Gamma(0)$. A locally bounded function $u \in USC(\overline{S_T})$ (resp. $u \in LSC(\overline{S_T})$) is called a viscosity subsolution (resp. supersolution) of (6.3.1) if $u(x,0) \leq u_0(x)$ (resp. $u(x,0) \geq u_0(x)$) for all $x \in \Gamma(0)$ and, for any $\varphi \in C^1(\overline{S_T})$, if $u - \varphi$ takes a local maximum (resp. minimum) at $(x_0, t_0) \in \overline{S_T}$ with $t_0 > 0$, then

$$\partial^\bullet \varphi(x_0,t_0) + H(x_0, t_0, \nabla_\Gamma \varphi(x_0,t_0)) \leq 0 \quad (\text{resp. } \geq 0). \tag{6.3.4}$$

If u is a sub- and supersolution, then we call u a viscosity solution to (6.3.1).

By the definition above, a viscosity solution is continuous and satisfies $u(x,0) = u_0(x)$, $x \in \Gamma(0)$. In Section 6.5 we prove that the upper and lower weak limits of a sequence of approximate solutions are a subsolution and supersolution, respectively, and then obtain a viscosity solution by showing that the upper weak limit agrees with the lower weak limit. For this argument and the uniqueness of a viscosity solution the following comparison principle is crucial.

Theorem 6.3.2. *Let u be a subsolution and v be a supersolution of* (6.3.1)*. Suppose that $u(\cdot,0) \leq v(\cdot,0)$ on $\Gamma(0)$. Then $u \leq v$ on $\overline{S_T}$.*

Proof. We essentially use a standard argument that is e.g. outlined in [5, Section 5]. Let us define for $\eta > 0$ the function $u_\eta(x,t) := u(x,t) - \eta t$. Clearly, $u_\eta \in USC(\overline{S_T})$ and $u_\eta(\cdot,0) \leq v(\cdot,0)$ on $\Gamma(0)$. Since $v \in LSC(\overline{S_T})$ we have $u_\eta - v \in USC(\overline{S_T})$ so that $\sigma_\eta := \max_{\overline{S_T}}(u_\eta - v)$ exists. Let us suppose that $\sigma_\eta > 0$. We use the doubling of variables technique and define for $0 < \alpha \ll 1$

$$\Psi_\alpha(x,t,y,s) := u_\eta(x,t) - v(y,s) - \frac{|x-y|^2 + |t-s|^2}{\alpha^2}, \quad (x,t,y,s) \in \overline{S_T} \times \overline{S_T}.$$

The function Ψ_α is upper semicontinuous on $\overline{S_T} \times \overline{S_T}$ and hence attains a maximum at some point $(\bar{x}, \bar{t}, \bar{y}, \bar{s}) \in \overline{S_T} \times \overline{S_T}$, where we suppress the dependence on α. It is shown in [5, Lemma 5.2] that

$$\frac{|\bar{x}-\bar{y}|^2}{\alpha^2}, \ \frac{|\bar{t}-\bar{s}|^2}{\alpha^2} \to 0, \qquad \text{as } \alpha \to 0, \tag{6.3.5}$$

$$\bar{t}, \bar{s} > 0, \qquad \text{for small } \alpha > 0. \tag{6.3.6}$$

We define for (x,t), $(y,s) \in \mathbb{R}^4$ the functions

$$\varphi^1(x,t) := v(\bar{y},\bar{s}) + \frac{|x-\bar{y}|^2 + |t-\bar{s}|^2}{\alpha^2}, \ \varphi^2(y,s) := u_\eta(\bar{x},\bar{t}) - \frac{|\bar{x}-y|^2 + |\bar{t}-s|^2}{\alpha^2}.$$

Clearly, the restriction of φ^i, $i = 1,2$ to $\overline{S_T}$ belongs to $C^1(\overline{S_T})$. Since u is a subsolution to (6.3.1) and $u(x,t) - (\varphi^1(x,t) + \eta t) = (u_\eta - \varphi^1)(x,t) = \Psi_\alpha(x,t,\bar{y},\bar{s})$ takes a maximum at $(x,t) = (\bar{x},\bar{t}) \in \overline{S_T}$ with $\bar{t} > 0$, we have

$$\partial^\bullet \varphi^1(\bar{x},\bar{t}) + H(\bar{x},\bar{t},\nabla_\Gamma \varphi^1(\bar{x},\bar{t})) \leq -\eta.$$

Observing that by (6.2.1) and (6.2.7)

$$\nabla_\Gamma \varphi^1(x,t) = \frac{2}{\alpha^2} P_\Gamma(x,t)(x-\bar{y}), \quad \partial^\bullet \varphi^1(x,t) = \frac{2}{\alpha^2}(t-\bar{s}) + \frac{2}{\alpha^2} v_\Gamma(x,t) \cdot (x-\bar{y})$$

we deduce

$$\frac{2(\bar{t}-\bar{s})}{\alpha^2} + \frac{2}{\alpha^2} v_\Gamma(\bar{x},\bar{t}) \cdot (\bar{x}-\bar{y}) + H\big(\bar{x},\bar{t},\frac{2}{\alpha^2} P_\Gamma(\bar{x},\bar{t})(\bar{x}-\bar{y})\big) \leq -\eta. \tag{6.3.7}$$

Since v is a supersolution and $(v-\varphi^2)(y,s) = -\Psi_\alpha(\bar{x},\bar{t},y,s)$ takes a minimum at $(y,s) = (\bar{y},\bar{s}) \in \overline{S_T}$ with $\bar{s} > 0$, it follows that

$$\partial^\bullet \varphi^2(\bar{y},\bar{s}) + H(\bar{y},\bar{s},\nabla_\Gamma \varphi^2(\bar{y},\bar{s})) \geq 0$$

and we obtain similarly as above

$$-\frac{2(\bar{t}-\bar{s})}{\alpha^2} - \frac{2}{\alpha^2} v_\Gamma(\bar{y},\bar{s}) \cdot (\bar{x}-\bar{y}) - H\big(\bar{y},\bar{s},\frac{2}{\alpha^2} P_\Gamma(\bar{y},\bar{s})(\bar{x}-\bar{y})\big) \leq 0. \tag{6.3.8}$$

We deduce from (6.3.7) and (6.3.8) that

$$\begin{aligned}\bar{A} := &\frac{2}{\alpha^2}\{v_\Gamma(\bar{x},\bar{t}) - v_\Gamma(\bar{y},\bar{s})\} \cdot (\bar{x}-\bar{y}) \\ &+ H\left(\bar{x},\bar{t},\frac{2}{\alpha^2} P_\Gamma(\bar{x},\bar{t})(\bar{x}-\bar{y})\right) - H\left(\bar{y},\bar{s},\frac{2}{\alpha^2} P_\Gamma(\bar{y},\bar{s})(\bar{x}-\bar{y})\right) \leq -\eta. \end{aligned} \tag{6.3.9}$$

Since v_Γ, P_Γ are smooth on $\overline{S_T}$ we obtain with the help of (6.3.2), (6.3.3) and (6.3.5)

$$\begin{aligned}|\bar{A}| &\leq \frac{2}{\alpha^2}\big(|v_\Gamma(\bar{x},\bar{t}) - v_\Gamma(\bar{y},\bar{s})| + L_{H,2}|P_\Gamma(\bar{x},\bar{t}) - P_\Gamma(\bar{y},\bar{s})|\big)|\bar{x}-\bar{y}| \\ &\quad + L_{H,1}(|\bar{x}-\bar{y}| + |\bar{t}-\bar{s}|)\left(1 + \frac{2}{\alpha^2}|P_\Gamma(\bar{x},\bar{t})(\bar{x}-\bar{y})|\right) \\ &\leq C\frac{|\bar{x}-\bar{y}|^2 + |\bar{t}-\bar{s}|^2}{\alpha^2} + \alpha^2 \to 0, \quad \alpha \to 0\end{aligned}$$

contradicting (6.3.9). Hence, $\sigma_\eta \leq 0$, so that $u_\eta \leq v$ on $\overline{S_T}$. The result now follows upon sending $\eta \to 0$. □

Corollary 6.3.3 (Uniqueness of a viscosity solution)**.** *For any initial value $u_0 \in C(\Gamma(0))$ there exists at most one viscosity solution to* (6.3.1).

6.4 Finite volume scheme

Let us next turn to the approximation of (6.3.1). As mentioned already in the introduction, our scheme is based on the finite volume scheme for Hamilton–Jacobi equations in a flat and stationary domain introduced by Kim and Li in [17].

Let N be a positive integer, $\tau := T/N$ a time step, $t^n = n\tau, n = 0,\dots,N$ and $x_i^n = x_i(t^n)$, $V_h^n = V_h(t^n)$. In order to derive our scheme we start from the following viscous approximation of (6.3.1)

$$\partial^\bullet u(x,t) + H(x,t,\nabla_\Gamma u(x,t)) = \varepsilon \Delta_\Gamma u(x,t), \quad (x,t) \in S_T, \tag{6.4.1}$$

where $0 < \epsilon \ll 1$. Let us fix $i \in \{1,\dots,M\}$ and consider a time-dependent set $V_i(t) \subset \Gamma(t)$ centered at $x_i(t)$. Integrating (6.4.1) for $t = t^n$ over $V_i(t^n)$ we find that

$$\int_{V_i(t^n)} \partial^\bullet u \, d\mathcal{H}^2 + \int_{V_i(t^n)} H(\cdot,t^n,\nabla_\Gamma u)\, d\mathcal{H}^2 = \varepsilon \int_{V_i(t^n)} \Delta_\Gamma u \, d\mathcal{H}^2. \tag{6.4.2}$$

Here, $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. Let us consider the first term on the left-hand side of (6.4.2). Using the transport theorem (see e.g. [10, Theorem 5.1]) and approximating $\int_{V_i(t)} u\, d\mathcal{H}^2$ by $u(x_i(t),t)|V_i(t)|$ ($|V_i(t)| = \mathcal{H}^2(V_i(t))$) we obtain

$$\begin{aligned}\int_{V_i(t^n)} \partial^\bullet u\, d\mathcal{H}^2 &= \frac{d}{dt}\int_{V_i(t)} u\, d\mathcal{H}^2_{|t=t^n} - \int_{V_i(t^n)} \nabla_\Gamma \cdot v_\Gamma\, u\, d\mathcal{H}^2 \\ &\approx \frac{u(x_i^{n+1},t^{n+1})|V_i(t^{n+1})| - u(x_i^n,t^n)|V_i(t^n)|}{\tau} - \int_{V_i(t^n)} \nabla_\Gamma \cdot v_\Gamma\, u\, d\mathcal{H}^2.\end{aligned}$$

Since $\frac{d}{dt}|V_i(t)| = \int_{V_i(t)} \nabla_\Gamma \cdot v_\Gamma\, d\mathcal{H}^2$ we may approximate

$$|V_i(t^{n+1})| \approx |V_i(t^n)| + \tau \int_{V_i(t^n)} \nabla_\Gamma \cdot v_\Gamma\, d\mathcal{H}^2$$

so that

$$\int_{V_i(t^n)} \partial^\bullet u\, d\mathcal{H}^2 \approx \frac{u(x_i^{n+1},t^{n+1}) - u(x_i^n,t^n)}{\tau}\,|V_i(t^n)|.$$

Finally, after applying the Gauss theorem for hypersurfaces to the integral on the right-hand side of (6.4.2) we obtain

$$\frac{u(x_i^{n+1},t^{n+1}) - u(x_i^n,t^n)}{\tau}\,|V_i(t^n)| + \int_{V_i(t^n)} H(\cdot,t^n,\nabla_\Gamma u)\, d\mathcal{H}^2 \approx \varepsilon \int_{\partial V_i(t^n)} \frac{\partial u}{\partial \mu}\, d\mathcal{H}^1, \quad (6.4.3)$$

where μ denotes the outer unit conormal to $\partial V_i(t^n)$. In order to turn (6.4.3) into a numerical scheme we construct a suitable discrete version $V^{n,i} \subset \Gamma_h(t^n)$ of $V_i(t^n)$. Let $\mu_i \in \mathbb{N}$ be the number of triangles that have the common vertex i, which is independent of n. The other vertices of the triangles with common vertex i are denoted by i_j, $j = 1,\ldots,\mu_i$, which we enumerate in clockwise direction. We write $T_j^{n,i} \in \mathcal{T}_h(t^n)$ for the triangle with vertices i, i_j, and i_{j+1} and $E_j^{n,i}$, $\bar{E}_j^{n,i}$ for the edges of $T_j^{n,i}$ connecting the vertices i and i_j and the vertices i_j and i_{j+1}, respectively (see Figure 6.1, left).

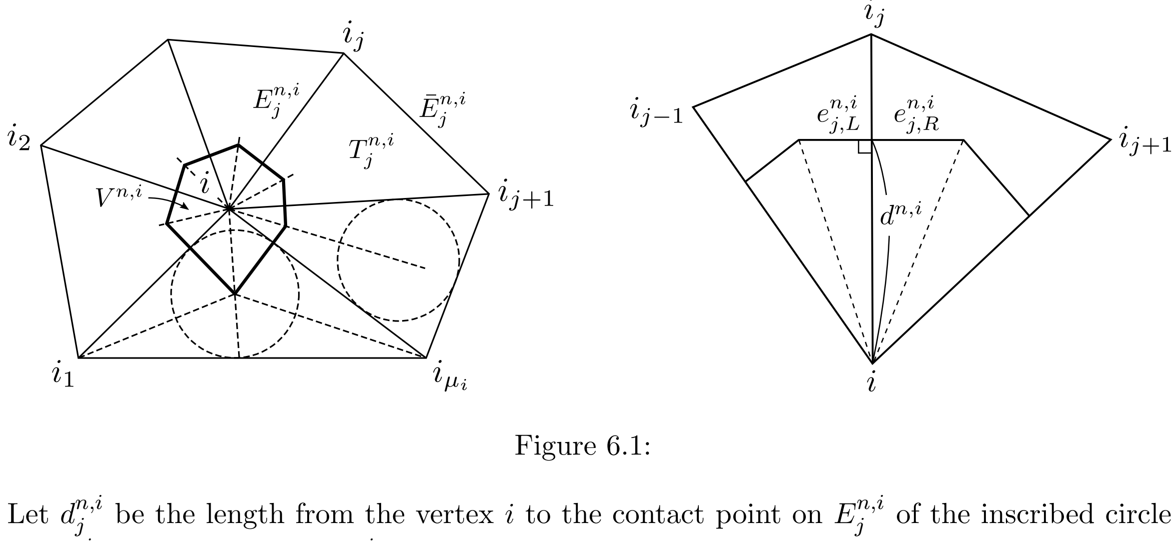

Figure 6.1:

Let $d_j^{n,i}$ be the length from the vertex i to the contact point on $E_j^{n,i}$ of the inscribed circle of $T_j^{n,i}$ and $d^{n,i} := \min\{d_j^{n,i} \mid j = 1,\ldots,\mu_i\}$. We define the volume $V^{n,i} \subset \Gamma_h(t^n)$ as a polygonal region surrounded by line segments perpendicular to each edge $E_j^{n,i}$ and whose distances from the vertex i are all equal to $d^{n,i}$. The parts of the edge of $V^{n,i}$ perpendicular to $E_j^{n,i}$ and lying in $T_{j-1}^{n,i}$ and $T_j^{n,i}$ are denoted by $e_{j,L}^{n,i}$ and $e_{j,R}^{n,i}$ with their length $h_{j,L}^{n,i}$ and

$h^{n,i}_{j,R}$, respectively (see Figure 6.1, right). Note that in view of (6.2.8) there exist constants $0 < \alpha_1 < \alpha_2$ and $C > 0$ such that

$$\alpha_1 \leq \frac{h^{n,i}_{j,L} + h^{n,i}_{j,R}}{|E^{n,i}_j|} \leq \alpha_2, \quad h_{T^{n,i}_j} \leq C d^{n,i} \tag{6.4.4}$$

for all $n = 0, 1, \ldots, N$, $i = 1, \ldots, M$, and $j = 1, \ldots, \mu_i$.
If we look for a discrete solution of the form $u^n_h = \sum_{i=1}^M u^n_i \chi_i(\cdot, t^n) \in V^n_h$, then (6.4.3) motivates the following relation:

$$\frac{u^{n+1}_i - u^n_i}{\tau}|V^{n,i}| + \sum_{j=1}^{\mu_i} |V^{n,i} \cap T^{n,i}_j| H\left(x^n_i, t^n, \nabla_{\Gamma_h} u^n_h|_{T^{n,i}_j}\right) = \varepsilon^n_i \sum_{j=1}^{\mu_i} \frac{u^n_{i_j} - u^n_i}{|E^{n,i}_j|}(h^{n,i}_{j,L} + h^{n,i}_{j,R})$$

for suitably chosen $\varepsilon^n_i > 0$. Here, $\nabla_{\Gamma_h} u^n = (I_3 - \nu^{n,i}_j \otimes \nu^{n,i}_j)\nabla u^n_h$ with $\nu^{n,i}_j = \nu_{h|T^{n,i}_j}$. Note that $\nu^{n,i}_j$ and hence $\nabla_{\Gamma_h} u^n$ is constant on $T^{n,i}_j$. To summarize, our numerical scheme for the Hamilton–Jacobi equation (6.3.1) looks as follows. For a given $u_0 \colon \Gamma(0) \to \mathbb{R}$, set

$$u^0_h := I^0_h u_0 = \sum_{i=1}^M u^0_i \chi_i(\cdot, 0) \in V^0_h, \quad u^0_i := u_0(x^0_i). \tag{6.4.5}$$

For $n = 0, 1, \ldots, N-1$, if $u^n_h = \sum_{i=1}^M u^n_i \chi_i(\cdot, t^n) \in V^n_h$ is given, then we define

$$u^{n+1}_h = S^n_h(u^n_h) := \sum_{i=1}^M u^{n+1}_i \chi_i(\cdot, t^{n+1}) \in V^{n+1}_h, \tag{6.4.6}$$

where

$$u^{n+1}_i = [S^n_h(u^n_h)]_i := u^n_i - \tau H^n_i(u^n_i, u^n_{i_1}, \ldots, u^n_{i_{\mu_i}}), \quad i = 1, \ldots, M. \tag{6.4.7}$$

Here $H^n_i(u^n_i, u^n_{i_1}, \ldots, u^n_{i_{\mu_i}})$ is the numerical Hamiltonian given by

$$\begin{aligned} &H^n_i(u^n_i, u^n_{i_1}, \ldots, u^n_{i_{\mu_i}}) \\ &:= \sum_{j=1}^{\mu_i} \frac{|V^{n,i} \cap T^{n,i}_j|}{|V^{n,i}|} H\left(x^n_i, t^n, \nabla_{\Gamma_h} u^n_h|_{T^{n,i}_j}\right) - \frac{\varepsilon^n_i}{|V^{n,i}|} \sum_{j=1}^{\mu_i} \frac{u^n_{i_j} - u^n_i}{|E^{n,i}_j|}(h^{n,i}_{j,L} + h^{n,i}_{j,R}). \end{aligned} \tag{6.4.8}$$

Let us derive several properties of the finite volume scheme (6.4.5)–(6.4.8). It is easy to see that the scheme is invariant under translation with constants, i.e.

$$S^n_h(u^n_h + c) = S^n_h(u^n_h) + c \tag{6.4.9}$$

for any $u^n_h \in V^n_h$ and $c \in \mathbb{R}$. We proceed by proving that the scheme is montone.

Lemma 6.4.1 (Monotonicity)**.** *There exist positive constants C_1 and C_2 depending only on γ and $L_{H,2}$ such that, if*

$$\varepsilon^n_i = C_1 \max_j h_{T^{n,i}_j}, \quad \tau \leq C_2 \min_{i,j} |E^{n,i}_j| \tag{6.4.10}$$

and u^n_h, $v^n_h \in V^n_h$ satisfy $u^n_h \leq v^n_h$ on $\Gamma_h(t^n)$, then $S^n_h(u^n_h) \leq S^n_h(v^n_h)$ on $\Gamma_h(t^{n+1})$.

Proof. Let $u_h^n,\ v_h^n \in V_h^n$ be of the form

$$u_h^n = \sum_{i=1}^{M} u_i^n \chi_i(\cdot, t^n), \quad v_h^n = \sum_{i=1}^{M} v_i^n \chi_i(\cdot, t^n) \quad \text{on} \quad \Gamma_h(t^n).$$

Note that $u_h^n \leq v_h^n$ on $\Gamma_h(t^n)$ is equivalent to $u_i^n \leq v_i^n$ for all $i = 1, \dots, M$ since the nodal basis functions χ_i are piecewise linear affine and satisfy $\chi_i(x_j(t), t) = \delta_{ij}$. By the same reason it is sufficient to establish that

$$[S_h^n(u_h^n)]_i \leq [S_h^n(v_h^n)]_i \quad \text{for all} \quad i = 1, \dots, M \tag{6.4.11}$$

in order to prove our claim. For $i = 1, \dots, M$, by (6.4.7) and (6.4.8) we have

$$[S_h^n(v_h^n)]_i - [S_h^n(u_h^n)]_i = v_i^n - u_i^n + \tau(I_1 + I_2 + I_3), \tag{6.4.12}$$

where $I_1 + I_2 + I_3 = -H_i^n(v_i^n, v_{i_1}^n, \dots, v_{i_{\mu_i}}^n) + H_i^n(u_i^n, u_{i_1}^n, \dots, u_{i_{\mu_i}}^n)$ with

$$\begin{aligned}
I_1 &:= -\sum_{j=1}^{\mu_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} \left\{ H\left(\nabla_{\Gamma_h} v_h^n|_{T_j^{n,i}}\right) - H\left(\nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}\right) \right\}, \\
I_2 &:= \frac{\varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\mu_i} \frac{v_{i_j}^n - u_{i_j}^n}{|E_j^{n,i}|} (h_{j,L}^{n,i} + h_{j,R}^{n,i}), \\
I_3 &:= -\frac{\varepsilon_i^n (v_i^n - u_i^n)}{|V^{n,i}|} \sum_{j=1}^{\mu_i} \frac{(h_{j,L}^{n,i} + h_{j,R}^{n,i})}{|E_j^{n,i}|}.
\end{aligned}$$

In the definition of I_1 we suppressed x_i^n and t^n of H. Let us estimate I_1, I_2, and I_3. By (6.3.3) and an inverse inequality

$$\begin{aligned}
&\left| H\left(\nabla_{\Gamma_h} v_h^n|_{T_j^{n,i}}\right) - H\left(\nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}}\right) \right| \leq L_{H,2} \left| \nabla_{\Gamma_h} v_h^n|_{T_j^{n,i}} - \nabla_{\Gamma_h} u_h^n|_{T_j^{n,i}} \right| \\
&\quad \leq C |\nabla (v_h^n - u_h^n)_{|T_j^{n,i}}| \leq C |E_j^{n,i}|^{-1} \, \|v_h^n - u_h^n\|_{B(T_j^{n,i})} \\
&\quad \leq C |E_j^{n,i}|^{-1} \{(v_i^n - u_i^n) + (v_{i_j}^n - u_{i_j}^n) + (v_{i_{j+1}}^n - u_{j_{i+1}}^n)\},
\end{aligned}$$

since u_h^n, v_h^n are linear on $T_j^{n,i}$ and $v_h^n - u_h^n \geq 0$. Using that $\sum_{j=1}^{\mu_i} |V^{n,i} \cap T_j^{n,i}| / |V^{n,i}| = 1$ as well as

$$|V^{n,i} \cap T_j^{n,i}| = \frac{1}{2} d^{n,i} (h_{j,R}^{n,i} + h_{j+1,L}^{n,i}) \leq |E_j^{n,i}| \max_j h_{T_j^{n,i}}, \quad j = 1, \dots, \mu_i, \tag{6.4.13}$$

we get

$$|I_1| \leq \frac{C}{\min_j |E_j^{n,i}|} (v_i^n - u_i^n) + \frac{C}{|V^{n,i}|} \max_j h_{T_j^{n,i}} \sum_{j=1}^{\mu_i} (v_{i_j}^n - u_{i_j}^n). \tag{6.4.14}$$

Next, from (6.4.4) and the fact that $u_{i_j}^n \leq v_{i_j}^n$ for $j = 1, \dots, \mu_i$ we infer that

$$I_2 \geq \frac{\alpha_1 \varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\mu_i} (v_{i_j}^n - u_{i_j}^n). \tag{6.4.15}$$

In view of the relation $|V^{n,i}| = \sum_{j=1}^{\mu_i} \frac{1}{2} d^{n,i}(h_{j,L}^{n,i} + h_{j,R}^{n,i})$ and (6.2.8) we obtain

$$\begin{aligned}\frac{1}{|V^{n,i}|}\sum_{j=1}^{\mu_i}\frac{h_{j,L}^{n,i}+h_{j,R}^{n,i}}{|E_j^{n,i}|} &\le \frac{1}{|V^{n,i}|}\frac{1}{\min_j |E_j^{n,i}|}\sum_{j=1}^{\mu_i}(h_{j,L}^{n,i}+h_{j,R}^{n,i}) = \frac{2}{d^{n,i}}\frac{1}{\min_j |E_j^{n,i}|}\\ &\le \frac{C}{\max_j h_{T_j^{n,i}} \min_j |E_j^{n,i}|} \le \frac{CC_1}{\varepsilon_i^n}\frac{1}{\min_j |E_j^{n,i}|},\end{aligned}$$

where we used (6.4.10) in the last step. Hence

$$I_3 \ge -\frac{CC_1}{\min_j |E_j^{n,i}|}(v_i^n - u_i^n). \tag{6.4.16}$$

From (6.4.12), (6.4.14), (6.4.15), and (6.4.16) it follows that

$$\begin{aligned}&[S_h^n(v_h^n)]_i - [S_h^n(u_h^n)]_i\\ &\qquad \ge \big(1 - \frac{\tau C(1+C_1)}{\min_j |E_j^{n,i}|}\big)(v_i^n - u_i^n) + \frac{1}{|V^{n,i}|}(\alpha_1\varepsilon_i^n - C\max_j h_{T_j^{n,i}})\sum_{j=1}^{\mu_i}(v_{i_j}^n - u_{i_j}^n)\end{aligned}$$

which yields (6.4.11) if we choose $C_1 = C/\alpha_1$ and $C_2 = 1/C(1+C_1)$ in (6.4.10). □

In what follows we write $I_h^n\varphi$ instead of $I_h^{t^n}\varphi$, i.e.

$$I_h^n\varphi = \sum_{i=1}^{M}\varphi_i^n\chi_i(\cdot,t^n) \in V_h^n, \quad \varphi_i^n = \varphi(x_i^n,t^n).$$

Lemma 6.4.2 (Consistency)**.** *Suppose that* (6.4.10) *is satisfied. Then there exists a constant $C_3 > 0$ depending only on γ, $L_{H,2}$ such that*

$$\begin{aligned}&\left|\frac{\varphi_i^{n+1} - [S_h^n(I_h^n\varphi)]_i}{\tau} - \{\partial^\bullet\varphi(x_i^n,t^n) + H(x_i^n,t^n,\nabla_\Gamma\varphi(x_i^n,t^n))\}\right|\\ &\qquad\qquad \le C_3 h\left(\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2\varphi\|_{B(\overline{S_T})} + \|(\partial^\bullet)^2\varphi\|_{B(\overline{S_T})}\right)\end{aligned} \tag{6.4.17}$$

for all $\varphi \in C^2(\overline{S_T})$, $n = 0,1,\dots,N-1$, and $i = 1,\dots,M$. Here, $(\partial^\bullet)^2\varphi$ is the second-order material derivative of φ.

Proof. Using (6.4.7) and (6.4.8) we have

$$\frac{\varphi_i^{n+1} - [S_h^n(I_h^n\varphi)]_i}{\tau} = \frac{\varphi_i^{n+1} - \varphi_i^n}{\tau} + H_i^n(\varphi_i^n, \varphi_{i_1}^n, \dots, \varphi_{i_{\mu_i}}^n).$$

Let us set

$$\begin{aligned}I_1 &:= \frac{\varphi_i^{n+1}-\varphi_i^n}{\tau} - \partial^\bullet\varphi(x_i^n,t^n),\\ I_2 &:= \sum_{j=1}^{\mu_i}\frac{|V^{n,i}\cap T_j^{n,i}|}{|V^{n,i}|}H(x_i^n,t^n,\nabla_{\Gamma_h}I_h^n\varphi|_{T_j^{n,i}}) - H(x_i^n,t^n,\nabla_\Gamma\varphi(x_i^n,t^n)),\\ I_3 &:= -\frac{\varepsilon_i^n}{|V^{n,i}|}\sum_{j=1}^{\mu_i}\frac{\varphi_{i_j}^n - \varphi_i^n}{|E_j^{n,i}|}(h_{j,L}^{n,i}+h_{j,R}^{n,i}),\end{aligned}$$

so that

$$\frac{\varphi_i^{n+1} - [S_h^n(I_h^n\varphi)]_i}{\tau} - \{\partial^\bullet\varphi(x_i^n, t^n) + H(x_i^n, t^n, \nabla_\Gamma\varphi(x_i^n, t^n))\} = I_1 + I_2 + I_3 \qquad (6.4.18)$$

and estimate I_1, I_2, and I_3 separately. From $\varphi_i^n = \varphi(x_i^n, t^n) = \varphi(\Phi(x_i^0, t^n), t^n)$ and the definition of the material derivative it follows that

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\tau} = \frac{1}{\tau}\int_{t^n}^{t^{n+1}} \frac{d}{dt}\big(\varphi(\Phi(x_i^0, s), s)\big)\, ds = \frac{1}{\tau}\int_{t^n}^{t^{n+1}} \partial^\bullet\varphi(\Phi(x_i^0, s), s)\, ds.$$

Applying the definition of the material derivative again we obtain

$$\begin{aligned} I_1 &= \frac{1}{\tau}\int_{t^n}^{t^{n+1}} \{\partial^\bullet\varphi(\Phi(x_i^0, s), s) - \partial^\bullet\varphi(\Phi(x_0^i, t^n), t^n)\} ds \\ &= \frac{1}{\tau}\int_{t^n}^{t^{n+1}} \int_{t^n}^{s} (\partial^\bullet)^2\varphi(\Phi(x_i^0, \tilde{s}), \tilde{s})\, d\tilde{s}\, ds. \end{aligned}$$

Since $\varphi \in C^2(\overline{S_T})$, the second-order material derivative $(\partial^\bullet)^2\varphi$ is bounded on $\overline{S_T}$. Hence by the above equality, $t^{n+1} - t^n = \tau$, and (6.4.10) we obtain

$$|I_1| \le \frac{(t^{n+1} - t^n)^2}{\tau}\|(\partial^\bullet)^2\varphi\|_{B(\overline{S_T})} = \tau\|(\partial^\bullet)^2\varphi\|_{B(\overline{S_T})} \le Ch\|(\partial^\bullet)^2\varphi\|_{B(\overline{S_T})}. \qquad (6.4.19)$$

Next we estimate I_2. From now on, we suppress t^n in all functions and x_i^n in the Hamiltonian. Clearly,

$$I_2 = \sum_{j=1}^{\mu_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|}\left\{H\left(\nabla_{\Gamma_h} I_h^n\varphi|_{T_j^{n,i}}\right) - H(\nabla_\Gamma\varphi(x_i^n))\right\}. \qquad (6.4.20)$$

For each $j = 1, \dots, \mu_i$, the inequality (6.3.3) yields

$$\left|H\left(\nabla_{\Gamma_h} I_h^n\varphi|_{T_j^{n,i}}\right) - H(\nabla_\Gamma\varphi(x_i^n))\right| \le L_{H,2}\left|\nabla_{\Gamma_h} I_h^n\varphi|_{T_j^{n,i}} - \nabla_\Gamma\varphi(x_i^n)\right|. \qquad (6.4.21)$$

Abbreviating $\varphi^{-l}(x) := \varphi(\pi_h(x)), x \in \Gamma_h$ we may write

$$\begin{aligned} &\nabla_{\Gamma_h} I_h^n\varphi|_{T_j^{n,i}} - \nabla_\Gamma\varphi(x_i^n) \\ &\qquad = \big(\nabla_{\Gamma_h} I_h^n\varphi|_{T_j^{n,i}} - \nabla_{\Gamma_h}\varphi^{-l}(x_i^n)\big) + \big(\nabla_{\Gamma_h}\varphi^{-l}(x_i^n) - \nabla_\Gamma\varphi(x_i^n)\big) \equiv A + B. \quad (6.4.22) \end{aligned}$$

Since $I_h^n\varphi_{|T_j^{n,i}}$ is the linear interpolation of $\varphi^{-l}_{|T_j^{n,i}}$ we obtain

$$|A| \le Ch\|\nabla^2_{\Gamma_h}\varphi^{-l}\|_{B(T_j^{n,i})} \le Ch\big(\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} + \|\nabla^2_\Gamma\varphi\|_{B(\overline{S_T})}\big). \qquad (6.4.23)$$

On the other hand, we infer from (4.18) in [10] and the relations $\pi_h(x_i^n) = x_i^n, d(x_i^n) = 0$, $\nu(x_i^n) \cdot \nabla_\Gamma\varphi(x_i^n) = 0$ that

$$B = (I_3 - \nu_j^{n,i} \otimes \nu_j^{n,i})\nabla_\Gamma\varphi(x_i^n) - \nabla_\Gamma\varphi(x_i^n) = \big(\nu(x_i^n) \otimes \nu(x_i^n) - \nu_j^{n,i} \otimes \nu_j^{n,i}\big)\nabla_\Gamma\varphi(x_i^n),$$

so that by (6.2.10)

$$|B| \le 2\|\nu - \nu_h\|_{B(T_j^{n,i})}\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} \le Ch\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})}. \qquad (6.4.24)$$

Combining (6.4.20)–(6.4.24) we obtain

$$|I_2| \le Ch\left(\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2\varphi\|_{B(\overline{S_T})}\right). \tag{6.4.25}$$

Finally, let us write

$$I_3 = \frac{\varepsilon_i^n}{|V^{n,i}|}(J_1 + J_2), \tag{6.4.26}$$

where

$$\begin{aligned} J_1 &:= -\sum_{j=1}^{\mu_i} \frac{h_{j,L}^{n,i} + h_{j,R}^{n,i}}{|E_j^{n,i}|}\{(\varphi_{i_j}^n - \varphi_i^n) - \nabla_\Gamma\varphi(x_i^n)\cdot(x_{i_j}^n - x_i^n)\}, \\ J_2 &:= -\sum_{j=1}^{\mu_i}\left(\nabla_\Gamma\varphi(x_i^n)\cdot\frac{x_{i_j}^n - x_i^n}{|E_j^{n,i}|}\right)(h_{j,L}^{n,i} + h_{j,R}^{n,i}). \end{aligned} \tag{6.4.27}$$

Extending φ constantly in normal direction via φ_c and recalling (6.2.3) we have

$$\begin{aligned} \varphi_{i_j}^n - \varphi_i^n - \nabla_\Gamma\varphi(x_i^n)\cdot(x_{i_j}^n - x_i^n) &= \varphi_c(x_{i_j}^n) - \varphi_c(x_i^n) - \nabla\varphi_c(x_i^n)\cdot(x_{i_j}^n - x_i^n) \\ &= \int_0^1\{\nabla\varphi_c(x_i^n + s(x_{i_j}^n - x_i^n)) - \nabla\varphi_c(x_i^n)\}\,ds\cdot(x_{i_j}^n - x_i^n) \\ &= \int_0^1\left(\int_0^s \nabla^2\varphi_c(x_i^n + \tilde{s}(x_{i_j}^n - x_i^n))(x_{i_j}^n - x_i^n)\,d\tilde{s}\right)ds\cdot(x_{i_j}^n - x_i^n). \end{aligned}$$

Thus, we deduce from (6.2.5) and (6.4.4) that

$$|J_1| \le C\left(\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2\varphi\|_{B(\overline{S_T})}\right)\sum_{j=1}^{\mu_i}|E_j^{n,i}|^2. \tag{6.4.28}$$

To estimate J_2 we observe that

$$0 = \int_{V^{n,i}} \mathrm{div}_{\Gamma_h} p\,d\mathcal{H}^2 = \sum_{j=1}^{\mu_i}\int_{V^{n,i}\cap T_j^{n,i}} \mathrm{div}_{\Gamma_h} p\,d\mathcal{H}^2 \tag{6.4.29}$$

for the constant vector $p = \nabla_\Gamma\varphi(x_i^n) \in \mathbb{R}^3$. For each $j = 1, \ldots, \mu_i$, $V^{n,i}\cap T_j^{n,i}$ is a flat quadrilateral whose sides consist of the edges $e_{j,R}^{n,i}$, $e_{j+1,L}^{n,i}$, and

$$S_{j,L}^{n,i} := E_j^{n,i}\cap\partial(V^{n,i}\cap T_j^{n,i}), \quad S_{j,R}^{n,i} := E_{j+1}^{n,i}\cap\partial(V^{n,i}\cap T_j^{n,i}).$$

The unit outward co-normal $\mu_j^{n,i}$ to $\partial(V^{n,i}\cap T_j^{n,i})$ (i.e. the unit outward normal to $\partial(V^{n,i}\cap T_j^{n,i})$ that is tangent to $T_j^{n,i}$) is given by

$$\mu_j^{n,i} = \begin{cases} \mu_{j,E}^{n,i} & \text{on } e_{j,R}^{n,i}, \\ \mu_{j+1,E}^{n,i} & \text{on } e_{j+1,L}^{n,i}, \\ \mu_{j,L}^{n,i} & \text{on } S_{j,L}^{n,i}, \\ \mu_{j+1,R}^{n,i} & \text{on } S_{j,R}^{n,i}, \end{cases}$$

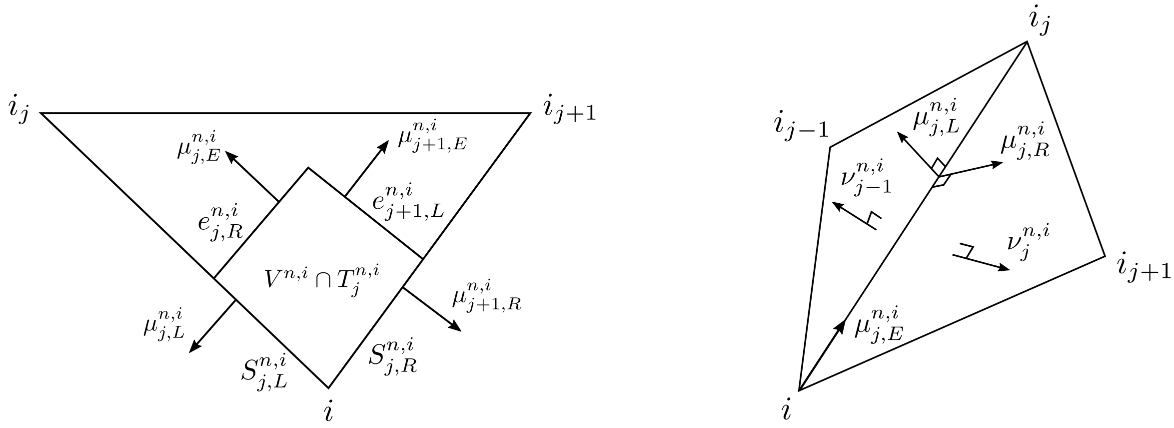

Figure 6.2:

where (see Figure 6.2)

$$\mu^{n,i}_{j,E} := \frac{x^n_{i_j} - x^n_i}{|x^n_{i_j} - x^n_i|} \quad \text{and} \quad \mu^{n,i}_{j,L} := \nu^{n,i}_j \times \mu^{n,i}_{j,E}, \quad \mu^{n,i}_{j,R} := -\nu^{n,i}_{j-1} \times \mu^{n,i}_{j,E}. \tag{6.4.30}$$

Here, $\times$ denotes the vector product in $\mathbb{R}^3$. Using the divergence theorem for integrals over a flat quadrilateral we have

$$\begin{aligned}
\int_{V^{n,i}\cap T^{n,i}_j} \mathrm{div}_{\Gamma_h} p \, d\mathcal{H}^2 &= \int_{e^{n,i}_{j,R}} p\cdot \mu^{n,i}_{j,E}\, d\mathcal{H}^1 + \int_{e^{n,i}_{j+1,L}} p\cdot \mu^{n,i}_{j+1,E}\, d\mathcal{H}^1 \\
&\quad + \int_{S^{n,i}_{j,L}} p\cdot \mu^{n,i}_{j,L}\, d\mathcal{H}^1 + \int_{S^{n,i}_{j,R}} p\cdot \mu^{n,i}_{j+1,R}\, d\mathcal{H}^1 \\
&= p\cdot \{h^{n,i}_{j,R}\,\mu^{n,i}_{j,E} + h^{n,i}_{j+1,L}\,\mu^{n,i}_{j+1,E} + d^{n,i}(\mu^{n,i}_{j,L} + \mu^{n,i}_{j+1,R})\},
\end{aligned}$$

since $|e^{n,i}_{j,R}| = h^{n,i}_{j,R}$, $|e^{n,i}_{j+1,L}| = h^{n,i}_{j+1,L}$ and $|S^{n,i}_{j,L}| = |S^{n,i}_{j,R}| = d^{n,i}$ by the definition of the volume $V^{n,i}$. Summing up both sides of the above equality over $j = 1,\dots,\mu_i$ we obtain from (6.4.29)

$$\begin{aligned}
0 &= \sum_{j=1}^{\mu_i} (p\cdot \mu^{n,i}_{j,E})(h^{n,i}_{j,L} + h^{n,i}_{j,R}) + d^{n,i}\sum_{j=1}^{\mu_i} p\cdot(\mu^{n,i}_{j,L} + \mu^{n,i}_{j,R}) \\
&= -J_2 + d^{n,i}\sum_{j=1}^{\mu_i} \nabla_\Gamma \varphi(x^n_i)\cdot(\mu^{n,i}_{j,L} + \mu^{n,i}_{j,R}).
\end{aligned}$$

Here the last line follows from $p = \nabla_\Gamma\varphi(x^n_i)$, (6.4.30), and (6.4.27). Hence

$$|J_2| = |d^{n,i}\sum_{j=1}^{\mu_i} \nabla_\Gamma\varphi(x^n_i)\cdot(\mu^{n,i}_{j,L} + \mu^{n,i}_{j,R})| \le C d^{n,i}\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} \max_j |\mu^{n,i}_{j,L} + \mu^{n,i}_{j,R}|. \tag{6.4.31}$$

Note that, contrary to the case of a flat stationary domain considered in [17], the equality $\mu^{n,i}_{j,L} = -\mu^{n,i}_{j,R}$ does not hold in general because the triangles $T^{n,i}_{j-1}$ and $T^{n,i}_j$ do not lie in the same plane. Instead we deduce from (6.4.30) and $|\mu^{n,i}_{j,E}| = 1$

$$\begin{aligned}
|\mu^{n,i}_{j,L} + \mu^{n,i}_{j,R}| &= |(\nu^{n,i}_j - \nu^{n,i}_{j-1}) \times \mu^{n,i}_{j,E}| \le |\nu^{n,i}_j - \nu^{n,i}_{j-1}| \\
&\le |\nu^{n,i}_j - \nu(x^n_i,t^n)| + |\nu(x^n_i,t^n) - \nu^{n,i}_{j-1}| \le Ch
\end{aligned} \tag{6.4.32}$$

by (6.2.10). Inserting (6.4.28), (6.4.31) with (6.4.32) into (6.4.26) and taking into account (6.4.4) as well as (6.4.10) we derive

$$\begin{aligned} |I_3| &\leq C\frac{\varepsilon_i^n}{|V^{n,i}|}\left(\sum_{j=1}^{\mu_i}|E_j^{n,i}|^2 + d^{n,i}h\right)\left(\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2\varphi\|_{B(\overline{S_T})}\right) \\ &\leq Ch\left(\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2\varphi\|_{B(\overline{S_T})}\right). \end{aligned} \tag{6.4.33}$$

The result now follows from (6.4.18) together with (6.4.19), (6.4.25) and (6.4.33). □

6.5 Convergence to viscosity solutions

The purpose of this section is to prove that the approximate solution generated by the scheme (6.4.5)–(6.4.8) converges to a viscosity solution of the Hamilton–Jacobi equation (6.3.1) providing at the same time an existence result for this problem. We start with a technical result that compares the nodal values of a solution of the scheme with those at the initial time, see Lemma 2.3 in [17] for a similar result in the flat case.

Lemma 6.5.1. *Suppose that $v_h^n = \sum_{i=1}^M v_i^n\chi(\cdot,t^n) \in V_h^n$ is a solution of $v_h^{n+1} = S_h^n(v_h^n), n = 0,\ldots,N-1$ with initial data $v_h^0(x_i^0) = v_0(x_i^0), i = 1,\ldots,M$, where $v_0 : \Gamma(0) \to \mathbb{R}$ is Lipschitz continuous with constant L_0. If (6.4.10) holds, then there exists a constant $C_4 > 0$ depending on γ, H and L_0 such that*

$$\max_{i=1,\ldots,M}|v_i^n - v_i^0| \leq C_4 t^n, \quad n = 0,1,\ldots,N. \tag{6.5.1}$$

Proof. Let us denote by $v_0^\sharp$ the push-forward of v_0 i.e. $v_0^\sharp(x,t) := v_0(\Phi^{-1}(x,t)), (x,t) \in \overline{S_T}$ and by $I_h^n v_0^\sharp \in V_h^n$ its interpolant. Since $x_i^n = \Phi(x_i^0,t^n)$ we have

$$[I_h^n v_0^\sharp]_i = I_h^n v_0^\sharp(x_i^n) = v_0^\sharp(x_i^n,t^n) = v_0(\Phi^{-1}(x_i^n,t^n)) = v_0(x_i^0),\ i = 1,\ldots,M. \tag{6.5.2}$$

Note that the right-hand side is independent of n. We claim that there exists a constant $R \geq 0$ such that

$$|\nabla I_h^n v_0^\sharp| \leq R \quad \text{on } \Gamma_h(t^n). \tag{6.5.3}$$

To see this, let us fix a triangle $K(t^n) \subset \Gamma_h(t^n)$ whose vertices are denoted for simplicity by x_1^n, x_2^n and x_3^n. By transforming onto the unit triangle, using (6.5.2), the Lipschitz continuity of v_0 and Φ^{-1} as well as (6.2.8) we obtain

$$\begin{aligned} |\nabla I_h^n v_{0|K(t^n)}^\sharp| &\leq \frac{C}{\rho_{K(t^n)}}\max_{i=2,3}|I_h^n v_0^\sharp(x_i^n) - I_h^n v_0^\sharp(x_1^n)| = \frac{C}{\rho_{K(t^n)}}\max_{i=2,3}|v_0(x_i^0) - v_0(x_1^0)| \\ &\leq \frac{CL_0}{\rho_{K(t^n)}}\max_{i=2,3}|x_i^0 - x_1^0| = \frac{CL_0}{\rho_{K(t^n)}}\max_{i=2,3}|\Phi^{-1}(x_i^n,t^n) - \Phi^{-1}(x_1^n,t^n)| \\ &\leq \frac{CL_0}{\rho_{K(t^n)}}\max_{i=2,3}|x_i^n - x_1^n| \leq CL_0\gamma =: R \end{aligned}$$

proving (6.5.3). Recalling the definition (6.4.8) of the numerical Hamiltonian we deduce with the help of (6.5.3) and (6.4.10) that

$$\begin{aligned}
&|H_i^n([I_h^n v_0^\sharp]_i, [I_h^n v_0^\sharp]_{i_1}, \dots, [I_h^n v_0^\sharp]_{i_{\mu_i}})| \\
&\le \sum_{j=1}^{\mu_i} \frac{|V^{n,i} \cap T_j^{n,i}|}{|V^{n,i}|} \big| H\big(x_i^n, t^n, \nabla_{\Gamma_h} I_h^n v_0^\sharp|_{T_j^{n,i}}\big)\big| + \frac{\varepsilon_i^n}{|V^{n,i}|} \sum_{j=1}^{\mu_i} \frac{|[I_h^n v_0^\sharp]_{i_j} - [I_h^n v_0^\sharp]_i|}{|E_j^{n,i}|} (h_{j,L}^{n,i} + h_{j,R}^{n,i}) \\
&\qquad \le \max_{(x,t)\in \overline{S_T}, |p|\le R} |H(x,t,p)| + C \frac{\max_j (h_j^{n,i})^2}{|V^{n,i}|} \le: C_4
\end{aligned} \tag{6.5.4}$$

where C_4 can be chosen independently of i and n.
Now let us show by induction with respect to $n = 0, 1, \dots, N$ that

$$v_i^n \le [I_h^n v_0^\sharp]_i + C_4 t^n \quad \text{for all} \quad i = 1, \dots, M. \tag{6.5.5}$$

Since $v_i^0 = v_0(x_i^0) = [I_h^0 v_0^\sharp]_i$ the inequality (6.5.5) holds for $n = 0$. Let us assume that (6.5.5) is true for some $n \in \{0, 1, \dots, N-1\}$ so that $v_h^n \le I_h^n v_0^\sharp + C_4 t^n$ on $\Gamma_h(t^n)$. Applying Lemma 6.4.1 together with (6.4.9) we infer that

$$v_h^{n+1} = S_h^n(v_h^n) \le S_h^n(I_h^n v_0^\sharp + C_4 t^n) = S_h^n(I_h^n v_0^\sharp) + C_4 t^n$$

on $\Gamma_h(t^{n+1})$, and hence by (6.4.7), (6.4.8), and (6.5.4)

$$\begin{aligned}
v_i^{n+1} &\le [S_h^n(I_h^n v_0^\sharp)]_i + C_4 t^n = [I_h^n v_0^\sharp]_i - \tau H_i^n([I_h^n v_0^\sharp]_i, [I_h^n v_0^\sharp]_{i_1}, \dots, [I_h^n v_0^\sharp]_{i_{\mu_i}}) + C_4 t^n \\
&\le [I_h^n v_0^\sharp]_i + C_4 \tau + C_4 t^n = [I_h^{n+1} v_0^\sharp]_i + C_4 t^{n+1}
\end{aligned}$$

for all $i = 1, \dots, M$, where we used (6.5.2) in the last step. Hence we see by induction that (6.5.5) holds for all $n = 0, 1, \dots, N$. By the same argument we can show that $[I_h^n v_0^\sharp]_i - C_4 t^n \le v_i^n$ for all $n = 0, 1, \dots, N$ and $i = 1, \dots, M$. Finally, (6.5.2), (6.5.5), and the above inequality yield (6.5.1). ☐

Let us denote by $u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t_n) \in V_h^n$, $n = 0, 1, \dots, N$ the finite element function on $\Gamma_h(t_n)$ given by the numerical scheme (6.4.5)–(6.4.8). Now we define an approximate solution $u_h^l : \overline{S_T} \to \mathbb{R}$ by

$$u_h^l(x,t) = \sum_{i=1}^M u_i^n \chi_i^l(x,t), \quad t \in [t^n, t^{n+1}), \, x \in \Gamma(t) \tag{6.5.6}$$

for $n = 0, 1, \dots, N-1$ (we include $t = t^N = T$ when $n = N-1$), where u_0 is a given function on $\Gamma(0)$. For $(x,t) \in \overline{S_T}$ set

$$\bar{u}(x,t) := \limsup_{\substack{h\to 0 \\ \overline{S_T} \ni (y,s) \to (x,t)}} u_h^l(y,s), \quad \underline{u}(x,t) := \liminf_{\substack{h\to 0 \\ \overline{S_T} \ni (y,s) \to (x,t)}} u_h^l(y,s). \tag{6.5.7}$$

It follows from [4, Section V.2.1, Proposition 2.1] that $\bar{u} \in USC(\overline{S_T})$ and $\underline{u} \in LSC(\overline{S_T})$. Our aim is to show that $\bar{u}$ (resp. $\underline{u}$) is a subsolution (resp. supersolution) to (6.3.1). As a first step we prove

Lemma 6.5.2. *Let $\bar{u}$ and $\underline{u}$ be given by* (6.5.6)–(6.5.7). *Assume that* (6.4.10) *is satisfied and that $u_0 \in C(\Gamma(0))$. Then $\bar{u}(\cdot, 0) = \underline{u}(\cdot, 0) = u_0$ on $\Gamma(0)$.*

Proof. Fix $x_0 \in \Gamma(0)$. By (6.5.7) it immediately follows that $\underline{u}(x_0, 0) \le \bar{u}(x_0, 0)$. Therefore, if the inequality

$$\bar{u}(x_0, 0) \le u_0(x_0) \le \underline{u}(x_0, 0) \tag{6.5.8}$$

holds, then we get $\bar{u}(x_0, 0) = \underline{u}(x_0, 0) = u_0(x_0)$. Let us prove (6.5.8). Since $\Gamma(0)$ is compact in $\mathbb{R}^3$, the function $u_0 \in C(\Gamma(0))$ is bounded and uniformly continuous on $\Gamma(0)$. Hence setting

$$\omega_0(r) := \sup\{|u_0(x) - u_0(x_0)| \mid x \in \Gamma(0),\ |x - x_0| \le r\}, \quad r \in [0, \infty),$$

we see that $\omega_0(0) = 0$ and ω_0 is bounded, nondecreasing, continuous at $r = 0$. From this fact and the proof of [15, Lemma 2.1.9 (i)] there exists a bounded, nondecreasing, and continuous function ω on $[0, \infty)$ satisfying $\omega(0) = 0$ and $\omega_0 \le \omega$ on $[0, \infty)$. Fix an arbitrary $\delta > 0$. By the above properties of ω we may take a constant $A_\delta > 0$ such that $\omega(r) \le \delta + A_\delta r^2$ for all $r \in [0, \infty)$. From this inequality and $|u_0(x) - u_0(x_0)| \le \omega_0(|x - x_0|) \le \omega(|x - x_0|)$ it follows that

$$u_0(x) \le u_0(x_0) + \delta + A_\delta |x - x_0|^2 \quad \text{for all } x \in \Gamma(0). \tag{6.5.9}$$

Now we construct $v_h^n = \sum_{i=1}^M v_i^n \chi_i(\cdot, t^n) \in V_h^n$, $n = 0, 1, \dots, N$ by (6.4.5)–(6.4.8) from the initial value $v_0(x) := A_\delta |x - x_0|^2$, $x \in \Gamma(0)$. Then by interpolating both sides of (6.5.9) on $\Gamma_h(0)$ and observing that $u_0(x_0) + \delta$ is constant we have

$$u_h^0 \le u_0(x_0) + \delta + v_h^0 \quad \text{on} \quad \Gamma_h(0).$$

Combining this inequality with Lemma 6.4.1 and (6.4.9) we obtain

$$u_h^1 = S_h^0(u_h^0) \le S_h^0(u_0(x_0) + \delta + v_h^0) = u_0(x_0) + \delta + S_h^0(v_h^0) = u_0(x_0) + \delta + v_h^1 \quad \text{on} \quad \Gamma_h(t^1)$$

and then inductively $u_h^n \le u_0(x_0) + \delta + v_h^n$ on $\Gamma_h(t^n)$ for $n = 0, 1, \dots, N$, or

$$u_i^n \le u_0(x_0) + \delta + v_i^n \le u_0(x_0) + \delta + v_i^0 + C_4 t^n \tag{6.5.10}$$

for $n = 0, 1, \dots, N$, $i = 1, \dots, M$, where we applied Lemma 6.5.1 to v_h^n. Multiplying both sides by $\chi_i^l(\cdot, t), t \in [t^n, t^{n+1})$ and summing them over $i = 1, \dots, M$ we infer with the help of (6.5.2) (with t instead of t^n)

$$u_h^l(x, t) \le u_0(x_0) + \delta + [I_h^t v_0^\sharp]^l(x) + C_4 t \quad \text{for all} \quad (x, t) \in \overline{S_T}. \tag{6.5.11}$$

Since $v_0^\sharp(x_0, 0) = v_0(x_0) = 0$ and $v_0^\sharp$ is Lipschitz continuous on $\overline{S_T}$ we may estimate

$$\begin{aligned} |[I_h^t v_0^\sharp]^l(x)| &\le |[I_h^t v_0^\sharp]^l(x) - v_0^\sharp(x, t)| + |v_0^\sharp(x, t) - v_0^\sharp(x_0, 0)| \\ &\le \|v_0^\sharp(\cdot, t) - [I_h^t v_0^\sharp]^l\|_{B(\Gamma(t))} + C(|x - x_0| + t) \le C(h + |x - x_0| + t), \end{aligned}$$

where we also used Lemma 6.2.1. Combining this estimate with (6.5.11) we infer

$$\bar{u}(x_0, 0) = \limsup_{\substack{h \to 0 \\ \overline{S_T} \ni (x,t) \to (x_0, 0)}} u_h^l(x, t) \le u_0(x_0) + \delta.$$

Since $\delta > 0$ is arbitrary, it follows that $\bar{u}(x_0, 0) \le u_0(x_0)$. By the same argument we can show $u_0(x_0) \le \underline{u}(x_0, 0)$. Hence (6.5.8) is valid and the lemma follows. □

Lemma 6.5.3. *Under the same assumptions as in Lemma 6.5.2, $\bar{u}$ (resp. $\underline{u}$) is a subsolution (resp. supersolution) to* (6.3.1)*.*

Proof. We know from Lemma 6.5.2 that $\bar{u}(x,0) = \underline{u}(x,0) = u_0(x)$, $x \in \Gamma(0)$ so that it remains to verify (6.3.4).
Let us suppose first that $\varphi \in C^2(\overline{S_T})$ and that $\bar{u} - \varphi$ takes a local maximum at $(x_0, t_0) \in \overline{S_T}$ with $t_0 > 0$. Since $\bar{u}$ is bounded on $\overline{S_T}$ we may assume by a standard argument that $\bar{u} - \varphi$ has a strict global maximum at (x_0, t_0). Let φ_h^l be given by

$$\varphi_h^l(x,t) := \sum_{i=1}^{M} \varphi_i^n \chi_i^l(x,t), \quad t \in [t^n, t^{n+1}),\, x \in \Gamma(t), \tag{6.5.12}$$

where $\varphi_i^n := \varphi(x_i^n, t^n)$, $i = 1, \ldots, M$ and we include $t = t^N = T$ if $n = N-1$. We claim that

$$(\bar{u} - \varphi)(x,t) = \limsup_{\substack{h \to 0 \\ \overline{S_T} \ni (y,s) \to (x_0,t_0)}} (u_h^l - \varphi_h^l)(y,s). \tag{6.5.13}$$

In order to see this, we note that in view of the Lipschitz continuity of φ on $\overline{S_T}$ it is sufficient to show that $\varphi_h^l \to \varphi$ uniformly on $\overline{S_T}$. But,

$$\begin{aligned}
\|\varphi_h^l - \varphi\|_{B(\overline{S_T})} \le{}& \sup_{t \in [0,T]} \|\varphi(\cdot,t) - [I_h^t \varphi]^l\|_{B(\Gamma(t))} \\
&+ \max_{n=0,\ldots,N-1} \sup_{x \in \Gamma(t), t^n \le t \le t^{n+1}} \Big| \sum_{i=1}^{M} (\varphi(x_i(t),t) - \varphi(x_i^n, t^n)) \chi_i^l(x,t) \Big| \\
\le{}& Ch + \max_{n=0,\ldots,N-1} \sup_{i=1,\ldots,M, t^n \le t \le t^{n+1}} |\varphi(x_i(t),t) - \varphi(x_i^n,t^n)| \le C(h+\tau)
\end{aligned}$$

by Lemma 6.2.1, the fact that $x_i(t) = \Phi(x_i^0, t)$ and the Lipschitz continuity of φ and Φ. Thus, (6.5.13) holds so that there exist $h_j > 0$ and $(y_j, s_j) \in \overline{S_T}$, $j \in \mathbb{N}$ with $h_j \to 0$, $(y_j, s_j) \to (x_0, t_0)$, and $(u_{h_j}^l - \varphi_{h_j}^l)(y_j, s_j) \to (\bar{u} - \varphi)(x_0, t_0)$ as $j \to \infty$. For each $j \in \mathbb{N}$, the function $u_{h_j}^l - \varphi_{h_j}^l$ is of the form

$$(u_{h_j}^l - \varphi_{h_j}^l)(x,t) = \sum_{i=1}^{M} (u_i^n - \varphi_i^n) \chi_i^l(x,t), \;\; x \in \Gamma(t), t \in [t^n, t^{n+1}), n = 0, \ldots, N-1.$$

Let us choose $n_j \in \{0, 1, \ldots, N\}$ and $i_j \in \{1, \ldots, M\}$ such that

$$u_{i_j}^{n_j} - \varphi_{i_j}^{n_j} = \max\{u_i^n - \varphi_i^n \mid n = 0, \ldots, N,\, i = 1, \ldots, M\}$$

and use $\chi_i(x,t) \ge 0$, $i = 1, \ldots, M$ and $\sum_{i=1}^{M} \chi_i^l(x,t) = 1$ to get

$$(u_{h_j}^l - \varphi_{h_j}^l)(x,t) \le (u_{i_j}^{n_j} - \varphi_{i_j}^{n_j}) \sum_{i=1}^{M} \chi_i^l(x,t) = (u_{h_j}^l - \varphi_{h_j}^l)(x_{i_j}^{n_j}, t^{n_j}) \tag{6.5.14}$$

for all $(x,t) \in \overline{S_T}$. In particular, for all $j \in \mathbb{N}$,

$$(u_{h_j}^l - \varphi_{h_j}^l)(y_j, s_j) \le (u_{h_j}^l - \varphi_{h_j}^l)(x_{i_j}^{n_j}, t^{n_j}).$$

Since $(x_{i_j}^{n_j}, t^{n_j})$ belongs to the compact set $\overline{S_T}$, we may assume (up to a subsequence) that there exists $(\bar{x}, \bar{t}) \in \overline{S_T}$ such that $(x_{i_j}^{n_j}, t^{n_j}) \to (\bar{x}, \bar{t})$ as $j \to \infty$. Then by the above inequality and (6.5.13) we have

$$(\bar{u} - \varphi)(x_0, t_0) = \lim_{j\to\infty} (u_{h_j}^l - \varphi_{h_j}^l)(y_j, s_j) \leq \limsup_{j\to\infty} (u_{h_j}^l - \varphi_{h_j}^l)(x_{i_j}^{n_j}, t^{n_j}) \leq (\bar{u} - \varphi)(\bar{x}, \bar{t})$$

where the last inequality follows from the fact that $\bar{u} - \varphi \in USC(\overline{S_T})$. Recalling that $\bar{u} - \varphi$ takes a strict global maximum at (x_0, t_0) we infer that $(\bar{x}, \bar{t}) = (x_0, t_0)$. In particular, since $\lim_{j\to\infty} t^{n_j} = \bar{t} = t_0 > 0$ we have for sufficiently large j that $t^{n_j} > 0$ i.e. $n_j \geq 1$. Thus we can set $(x, t) = (x_i^{n_j-1}, t^{n_j-1})$ in (6.5.14) to obtain

$$(u_{h_j}^l - \varphi_{h_j}^l)(x_i^{n_j-1}, t^{n_j-1}) \leq \delta_j := u_{i_j}^{n_j} - \varphi_{i_j}^{n_j},$$

or equivalently, $u_i^{n_j-1} \leq \varphi_i^{n_j-1} + \delta_j$ for $i = 1, \ldots, M$. From this we see that

$$u_{h_j}^{n_j-1} \leq I_{h_j}^{n_j-1}\varphi + \delta_j \quad \text{on} \quad \Gamma_{h_j}(t^{n_j-1}),$$

and then by Lemma 6.4.1 and (6.4.9)

$$u_{h_j}^{n_j} = S_{h_j}^{n_j-1}(u_{h_j}^{n_j-1}) \leq S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1}\varphi + \delta_j) = S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1}\varphi) + \delta_j \;\text{ on } \Gamma_{h_j}(t^{n_j}).$$

Inserting $x = x_{i_j}^{n_j} \in \Gamma_{h_j}(t^{n_j})$ into this inequality we get

$$u_{i_j}^{n_j} \leq [S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1}\varphi)]_{i_j} + \delta_j = [S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1}\varphi)]_{i_j} + u_{i_j}^{n_j} - \varphi_{i_j}^{n_j}$$

by the definition of δ_j and hence,

$$\varphi_{i_j}^{n_j} - [S_{h_j}^{n_j-1}(I_{h_j}^{n_j-1}\varphi)]_{i_j} \leq 0. \tag{6.5.15}$$

Since $\varphi \in C^2(\overline{S_T})$, we can combine (6.5.15) with Lemma 6.4.2 to derive

$$\partial^\bullet \varphi(x_{i_j}^{n_j-1}, t^{n_j-1}) + H(x_{i_j}^{n_j-1}, t^{n_j-1}, \nabla_\Gamma \varphi(x_{i_j}^{n_j-1}, t^{n_j-1})) \leq C_\varphi h_j \tag{6.5.16}$$

and observing that

$$|(x_{i_j}^{n_j}, t^{n_j}) - (x_{i_j}^{n_j-1}, t^{n_j-1})| \leq C\tau \leq Ch_j \to 0,\, j \to \infty$$

we obtain (6.3.4) by sending $j \to \infty$ in (6.5.16).
Finally, let $\varphi \in C^1(\overline{S_T})$ and suppose that $\bar{u} - \varphi$ takes a local maximum at $(x_0, t_0) \in \overline{S_T}$, $t_0 > 0$. As in the first part of the proof, we may assume that $\bar{u} - \varphi$ takes a strict global maximum at (x_0, t_0). Let us approximate φ by a sequence $(\varphi_\delta) \subset C^2(\overline{S_T})$ such that $\varphi_\delta \to \varphi$ in $C^1(\overline{S_T})$ as $\delta \to 0$. For a suitable subsequence there exist $(x_\delta, t_\delta) \in \overline{S_T}$ such that $(x_\delta, t_\delta) \to (x_0, t_0)$ and $\bar{u} - \varphi_\delta$ takes a global maximum at (x_δ, t_δ). In particular, $t_\delta > 0$ for sufficiently small $\delta > 0$. It follows from the first part of the proof that

$$\partial^\bullet \varphi_\delta(x_\delta, t_\delta) + H(x_\delta, t_\delta, \nabla_\Gamma \varphi_\delta(x_\delta, t_\delta)) \leq 0.$$

Letting $\delta \to 0$ in the above inequality we see that φ satisfies (6.3.4) at (x_0, t_0), so that $\bar{u}$ is a subsolution to (6.3.1). In the same way one shows that $\underline{u}$ is a supersolution. $\square$

Finally, let us prove the existence of a viscosity solution to (6.3.1).

Theorem 6.5.4. *Suppose that $u_0 \in C(\Gamma(0))$. Then there exists a unique viscosity solution to* (6.3.1).

Proof. The uniqueness of a viscosity solution was already shown in Corollary 6.3.3. Let us prove the existence. Since $u_0 \in C(\Gamma(0))$, Lemmas 6.5.2 and 6.5.3 imply that $\bar{u}$ and $\underline{u}$ constructed by (6.5.6)–(6.5.7) are a subsolution and supersolution to (6.3.1), respectively, and satisfy $\bar{u}(\cdot,0) = \underline{u}(\cdot,0) = u_0$ on $\Gamma(0)$. Hence we can apply the comparison principle (see Theorem 6.3.2) to the subsolution $\bar{u}$ and the supersolution $\underline{u}$ to get $\bar{u} \leq \underline{u}$ on $\overline{S_T}$. Moreover, by (6.5.7) we easily see that $\underline{u} \leq \bar{u}$ on $\overline{S_T}$. Therefore, $u := \bar{u} = \underline{u}$ is a viscosity solution to (6.3.1). □

6.6 Error bound

In this section we prove an error estimate between the viscosity solution to (6.3.1) and the numerical solution given by the scheme (6.4.5)–(6.4.8).

Theorem 6.6.1. *Let u be the viscosity solution to* (6.3.1) *with initial value u_0. For $h > 0$ and $n = 0, 1, \ldots, N$, let $u_h^n = \sum_{i=1}^M u_i^n \chi_i(\cdot, t^n) \in V_h^n$ be the finite element function constructed from u_0 by* (6.4.5)–(6.4.8). *Assume that* (6.4.10) *is satisfied and that u is Lipschitz continuous on $\overline{S_T}$ in the sense that*

$$|u(x,t) - u(y,s)| \leq L_U(|x-y| + |t-s|) \tag{6.6.1}$$

for all $(x,t), (y,s) \in \overline{S_T}$, where $L_U > 0$ is a constant independent of (x,t) and (y,s). Then there exist $h_0 > 0$ and a constant $C > 0$ independent of h such that

$$\max_{1\leq i\leq M,\, 0\leq n\leq N} |u(x_i^n, t^n) - u_i^n| \leq Ch^{1/2} \quad \text{for all} \quad h \in (0, h_0). \tag{6.6.2}$$

Proof. The argument is similar to that in the proof of the comparison principle (see Theorem 6.3.2). Let us define

$$\Psi(x,t,i,n) := u(x,t) - \rho\sqrt{h}\, t - u_i^n - \frac{|x - x_i^n|^2 + |t - t^n|^2}{\sqrt{h}} \tag{6.6.3}$$

for $(x,t) \in \overline{S_T}$, $i \in \{1, \ldots, M\}$ and $n \in \{0, 1, \ldots, N\}$. Here, the constant $\rho > 0$ is subject to $\rho\sqrt{h} \leq 1$ and will be chosen later. Clearly,

$$\begin{aligned} \max_{1\leq i\leq M,\, 0\leq n\leq N} (u(x_i^n, t^n) - u_i^n) &= \max_{1\leq i\leq M,\, 0\leq n\leq N} (\Psi(x_i^n, t^n, i, n) + \rho\sqrt{h}\, t^n) \\ &\leq \max_{(x,t)\in\overline{S_T},\, i=1,\ldots,M,\, n=0,\ldots,N} \Psi(x,t,i,n) + \rho\sqrt{h}\, T \\ &= \Psi(x_0, t_0, i_0, n_0) + \rho\sqrt{h}\, T \end{aligned} \tag{6.6.4}$$

for some $(x_0, t_0) \in \overline{S_T}$, $i_0 \in \{1, \ldots, M\}$ and $n_0 \in \{0, 1, \ldots, N\}$. In particular, we have $\Psi(x_{i_0}^{n_0}, t^{n_0}, i_0, n_0) \leq \Psi(x_0, t_0, i_0, n_0)$, i.e.

$$u(x_{i_0}^{n_0}, t^{n_0}) - \rho\sqrt{h}\, t^{n_0} - u_{i_0}^{n_0} \leq u(x_0, t_0) - \rho\sqrt{h}\, t_0 - u_{i_0}^{n_0} - \frac{|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2}{\sqrt{h}}.$$

From this, (6.6.1), and the fact that $\rho\sqrt{h} \le 1$ it follows that

$$\begin{aligned}\frac{|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2}{\sqrt{h}} &\le u(x_0, t_0) - u(x_{i_0}^{n_0}, t^{n_0}) + \rho\sqrt{h}(t^{n_0} - t_0) \\ &\le L_U(|x_0 - x_{i_0}^{n_0}| + |t_0 - t^{n_0}|) + |t_0 - t^{n_0}| \\ &\le C(|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2)^{1/2}\end{aligned}$$

and hence

$$\frac{(|x_0 - x_{i_0}^{n_0}|^2 + |t_0 - t^{n_0}|^2)^{1/2}}{\sqrt{h}} \le C. \tag{6.6.5}$$

Now let us consider several possible cases.
Case 1: $t_0 > 0$ and $n_0 \ge 1$. By exploiting the fact that u is a subsolution we obtain as in (6.3.7)

$$\frac{2}{\sqrt{h}}(t_0 - t^{n_0}) + \frac{2}{\sqrt{h}} v_\Gamma(x_0, t_0) \cdot (x_0 - x_{i_0}^{n_0}) + H\big(x_0, t_0, \frac{2}{\sqrt{h}} P_\Gamma(x_0, t_0)(x_0 - x_{i_0}^{n_0})\big) \le -\rho\sqrt{h}. \tag{6.6.6}$$

On the other hand, since $\Psi(x_0, t_0, i, n_0 - 1) \le \Psi(x_0, t_0, i_0, n_0), i = 1, \dots, M$ we infer

$$\varphi_i^{n_0-1} - u_i^{n_0-1} \le \varphi_{i_0}^{n_0} - u_{i_0}^{n_0}, \quad i = 1, \dots, M,$$

where

$$\varphi_i^n = \varphi(x_i^n, t^n) \quad \text{and} \quad \varphi(x, t) = -\frac{|x_0 - x|^2 + (t_0 - t)^2}{\sqrt{h}}.$$

Hence, $I_h^{n_0-1}\varphi \le u_h^{n_0-1} + \varphi_{i_0}^{n_0} - u_{i_0}^{n_0}$ on $\Gamma_h(t^{n_0-1})$ so that we deduce with the help of Lemma 6.4.1, (6.4.9) and the definition of the scheme

$$S_h^{n_0-1}(I_h^{n_0-1}\varphi) \le S_h^{n_0-1}(u_h^{n_0-1}) + \varphi_{i_0}^{n_0} - u_{i_0}^{n_0} = u_h^{n_0} + \varphi_{i_0}^{n_0} - u_{i_0}^{n_0}.$$

Evaluating the above inequality for $x = x_{i_0}^{n_0}$ we find that

$$[S_h^{n_0-1}(I_h^{n_0-1}\varphi)]_{i_0} \le \varphi_{i_0}^{n_0},$$

from which we infer that

$$-\partial^\bullet\varphi(x_{i_0}^{n_0}, t^{n_0}) - H\big(x_{i_0}^{n_0}, t^{n_0}, \nabla_\Gamma\varphi(x_{i_0}^{n_0}, t^{n_0})\big) \le A + B, \tag{6.6.7}$$

where

$$\begin{aligned}A &= -\partial^\bullet\varphi(x_{i_0}^{n_0-1}, t^{n_0-1}) - H\big(x_{i_0}^{n_0-1}, t^{n_0-1}, \nabla_\Gamma\varphi(x_{i_0}^{n_0-1}, t^{n_0-1})\big) + \frac{\varphi_{i_0}^{n_0} - [S_h^{n_0-1}(I_h^{n_0-1}\varphi)]_{i_0}}{\tau}, \\ B &= [\partial^\bullet\varphi(x_{i_0}^{n_0-1}, t^{n_0-1}) - \partial^\bullet\varphi(x_{i_0}^{n_0}, t^{n_0})] \\ &\quad + [H\big(x_{i_0}^{n_0-1}, t^{n_0-1}, \nabla_\Gamma\varphi(x_{i_0}^{n_0-1}, t^{n_0-1})\big) - H\big(x_{i_0}^{n_0}, t^{n_0}, \nabla_\Gamma\varphi(x_{i_0}^{n_0}, t^{n_0})\big)].\end{aligned}$$

We deduce from Lemma 6.4.2 that

$$|A| \le C_3 h \left(\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})} + \|\nabla_\Gamma^2\varphi\|_{B(\overline{S_T})} + \|(\partial^\bullet)^2\varphi\|_{B(\overline{S_T})}\right) \le C\sqrt{h} \tag{6.6.8}$$

since

$$\partial^{\bullet}\varphi(x,t) = -\frac{2}{\sqrt{h}}(t-t_0) - \frac{2}{\sqrt{h}} v_\Gamma(x,t)\cdot(x-x_0), \tag{6.6.9}$$

$$\nabla_\Gamma \varphi(x,t) = -\frac{2}{\sqrt{h}} P_\Gamma(x,t)(x-x_0). \tag{6.6.10}$$

Using (6.6.9), (6.6.10), (6.3.2), (6.3.3) and the Lipschitz continuity of v_Γ we further obtain

$$\begin{aligned} |B| &\le \Big(\frac{C}{\sqrt{h}} + L_{H,1}\big(1+\|\nabla_\Gamma\varphi\|_{B(\overline{S_T})}\big)\Big)\big(|x_{i_0}^{n_0}-x_{i_0}^{n_0-1}| + |t^{n_0}-t^{n_0-1}|\big) \\ &\quad + L_{H,2}|\nabla_\Gamma\varphi(x_{i_0}^{n_0},t^{n_0}) - \nabla_\Gamma\varphi(x_{i_0}^{n_0-1},t^{n_0-1})| \\ &\le \frac{C}{\sqrt{h}}\,\tau \le C\sqrt{h}, \end{aligned} \tag{6.6.11}$$

where we used (6.4.10) for the last inequality. If we insert (6.6.8) and (6.6.11) into (6.6.7) and use again (6.6.9), (6.6.10) we obtain

$$\begin{aligned} &-\frac{2}{\sqrt{h}}(t_0-t^{n_0}) - \frac{2}{\sqrt{h}} v_\Gamma(x_{i_0}^{n_0},t^{n_0})\cdot(x_0-x_{i_0}^{n_0}) \\ &\qquad - H\big(x_{i_0}^{n_0},t^{n_0},\frac{2}{\sqrt{h}}P_\Gamma(x_{i_0}^{n_0},t^{n_0})(x_0-x_{i_0}^{n_0})\big) \le C\sqrt{h}. \end{aligned} \tag{6.6.12}$$

We sum up both sides of (6.6.6) and (6.6.12) and employ the Lipschitz continuity of v_Γ as well as (6.3.2), (6.3.3) to get

$$\begin{aligned} \rho\sqrt{h} &\le C\sqrt{h} + \frac{2}{\sqrt{h}}\{v_\Gamma(x_{i_0}^{n_0},t^{n_0}) - v_\Gamma(x_0,t_0)\}\cdot(x_0-x_{i_0}^{n_0}) \\ &\quad + H\big(x_{i_0}^{n_0},t^{n_0},\frac{2}{\sqrt{h}}P_\Gamma(x_{i_0}^{n_0},t^{n_0})(x_0-x_{i_0}^{n_0})\big) - H\big(x_0,t_0,\frac{2}{\sqrt{h}}P_\Gamma(x_0,t_0)(x_0-x_{i_0}^{n_0})\big) \\ &\le C\sqrt{h} + \frac{C(|x_0-x_{i_0}^{n_0}| + |t_0-t^{n_0}|)|x_0-x_{i_0}^{n_0}|}{\sqrt{h}} \\ &\quad + L_{H,1}(|x_0-x_{i_0}^{n_0}| + |t_0-t^{n_0}|)\left(1+\frac{2}{\sqrt{h}}|P_\Gamma(x_0,t_0)(x_0-x_{i_0}^{n_0})|\right) \\ &\quad + \frac{2L_{H,2}}{\sqrt{h}}|P_\Gamma(x_0,t_0)-P_\Gamma(x_{i_0}^{n_0},t^{n_0})||x_0-x_{i_0}^{n_0}| \\ &\le C\sqrt{h} + C\frac{|x_0-x_{i_0}^{n_0}|^2 + |t_0-t^{n_0}|^2}{\sqrt{h}} + C\big(|x_0-x_{i_0}^{n_0}| + |t_0-t^{n_0}|\big) \\ &\le C\sqrt{h} \end{aligned}$$

in view of (6.6.5). Choosing $\rho > C$ we obtain a contradiction so that this case cannot occur.
Case 2: $t_0 = 0$ and $n_0 \ge 0$. Since $u(x_0,t_0) = u(x_0,0) = u_0(x_0)$ we obtain with the help of (6.6.1), Lemma 6.5.1 and (6.6.5)

$$\begin{aligned} \Psi(x_0,t_0,i_0,n_0) = \Psi(x_0,0,i_0,n_0) &\le u(x_0,0) - u_{i_0}^{n_0} = u_0(x_0) - u_0(x_{i_0}^0) + u_{i_0}^0 - u_{i_0}^{n_0} \\ &\le L_U|x_0-x_{i_0}^0| + C_4 t^{n_0} \le C\big(|x_0-x_{i_0}^{n_0}| + |x_{i_0}^{n_0}-x_{i_0}^0|\big) + C_4 t^{n_0} \\ &\le C\big(|x_0-x_{i_0}^{n_0}| + |t_0-t^{n_0}|\big) \le C\sqrt{h}. \end{aligned} \tag{6.6.13}$$

Case 3: $t_0 \geq 0$ and $n_0 = 0$. Using once more (6.6.1) and (6.6.5) we derive

$$\begin{aligned}\Psi(x_0, t_0, i_0, n_0) &= \Psi(x_0, t_0, i_0, 0) \\ &\leq u(x_0, t_0) - u_{i_0}^0 = u(x_0, t_0) - u(x_{i_0}^0, 0) \\ &\leq L_U\big(|x_0 - x_{i_0}^0| + t_0\big) = L_U\big(|x_0 - x_{i_0}^{n_0}| + |t_0 - t^{n_0}|\big) \\ &\leq C\sqrt{h}.\end{aligned} \tag{6.6.14}$$

In conclusion we infer that from (6.6.4), (6.6.13), (6.6.14) and the fact that Case 1 cannot occur that

$$\max_{1 \leq i \leq M,\, 0 \leq n \leq N} (u(x_i^n, t^n) - u_i^n) \leq C\sqrt{h}.$$

In an analogous way we bound $\max_{1 \leq i \leq M,\, 0 \leq n \leq N}(u_i^n - u(x_i^n, t^n))$ which completes the proof of the theorem. □

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