博士論文

論文題目　Mean curvature flow with driving force
（駆動力付きの平均曲率流方程式）

氏名　張龍傑

Mean curvature flow with driving force

Longjie ZHANG

July, 2018

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Chapter 1

Introduction

1.1 Background

This thesis is intended to study the mean curvature flow with driving force. Precisely, we consider a family $\{\Gamma(t)\}_{t\geq 0}$ of hypersurfaces satisfying:

$$V = -\kappa + A \text{ on } \Gamma(t) \subset \mathbb{R}^{n+1}, \tag{1.1.1}$$

$$\Gamma(0) = \Gamma_0. \tag{1.1.2}$$

Here V is the outer normal velocity, κ is the mean curvature, A is a positive constant. We aim to consider the initial hypersurface Γ_0 has singularity.

This research is motivated by [14], the mean curvature flow with driving force under the Neumann boundary condition in a two-dimensional cylinder with periodically undulating boundary. In [14], they only consider the condition that for initial curve $\Gamma_0 = \{(x, y) \in \mathbb{R}^2 \mid y = u_0(x)\}$ with $|u_0'(x)| < M$ for some M. They show that the interior point of $\Gamma(t) = \{(x, y) \in \mathbb{R}^2 \mid y = u(x, t)\}$ never touches the boundary and $\Gamma(t)$ remains graph. Therefore, the problem can be studied by the classical quasilinear parabolic theory. If removing the assumption $|u_0'(x)| < M$, when $u(x, t)$

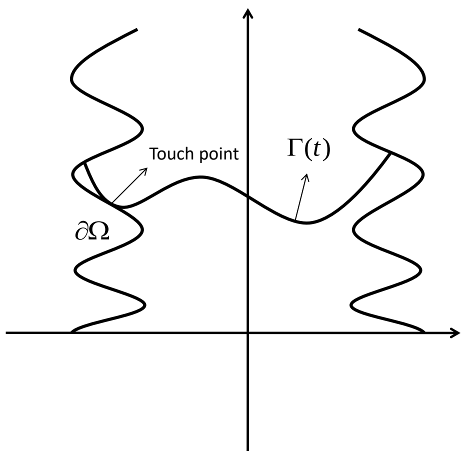

Figure 1.1: Curve touching

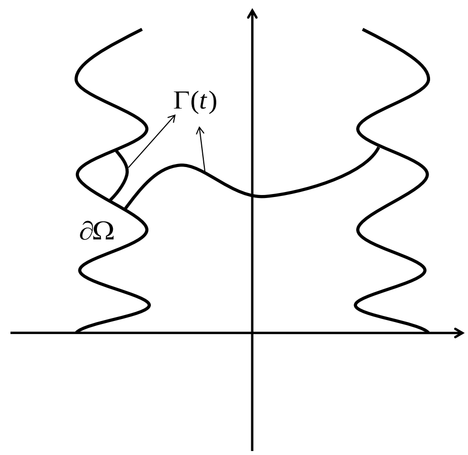

Figure 1.2: After touching

touches the boundary, the singularity will develop (Figure 1.1). Noting Figure 1.2, after touching, $\Gamma(t)$ possibly separates into two parts and become non-graph ($\Gamma(t)$ can not be represented by $y = u(x,t)$). This makes us analyze what will happen after touching boundary. Noting that $\Gamma(t)$ may become non-graph, we tend to use the level set method established by [5]. In Chapter 3 and Chapter 5, we consider our problem by level set method and identify whether the interface evolution is fattening or not.

For mean curvature flow with driving force, recently, there are some researches. In [13], Z. Liu shows that any solution starting as a convex, smooth, compact, embedded hypersurface remains so like the result of G. Huisken. Moreover, the solutions can be classified into three cases by its behaviors. In 2016, [10] consider the following free boundary problem called (Q)

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2},\ x \in (a(t), b(t)),\ 0 < t < T, \\ u(a(t), t) = 0,\ u(b(t), t) = 0,\ 0 \le t < T, \\ u_x(a(t), t) = \tan\theta_-(t),\ u_x(b(t), t) = -\tan\theta_+(t),\ 0 \le t < T, \\ u(x, 0) = u_0(x),\ a(0) \le x \le b(0), \end{cases} \tag{Q}$$

where $0 < \theta_\pm < \pi/2$. The family $\Gamma(t) = \{(x, y) \mid y = u(x, t),\ a(t) \le x \le b(t)\}$ moves by (1.1.1) in the plane and keeps the endpoints on the x-axis with the same fixed contact angles. They also classify the solutions into three cases and give the asymptotic behavior in each case.

In Chapter 4, we also give the results of the classification of the solutions in the plane, however, without assuming the convexity of the initial curve.

1.2 Main results

We consider the initial hypersurface Γ_0 has singularity at the origin and is symmetric to x, y-axis. Γ_0 is given as in Figure 1.3. (The precise setting is given in Chapter 3 and 5.) We want to identify the fattening phenomenon related to the singular angle γ. Moreover in Chapter 4, we classify the curvature flow with driving force in the plane into three cases and give the asymptotic behavior in each case.

In the case singular angle $\gamma = \pi/2$, we introduce our main assumptions $(A+)$, $(A-)$. Consider

$$V = -\kappa + A \ \text{ on } \Lambda^+(t) \subset \mathbb{R}^{n+1}, \tag{1.1.1*}$$

$$\Lambda^+(0) = \Lambda_0, \tag{1.1.2*}$$

where $\Lambda_0 = \Gamma_0 \cap \{x \ge 0\}$. (Figure 1.4). By level set method, as introduced in

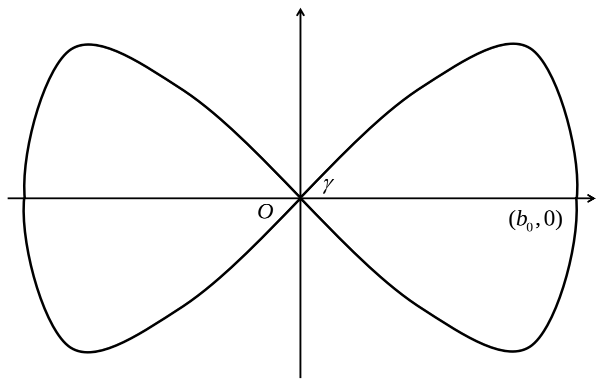

Figure 1.3: Initial hypersurface Γ_0

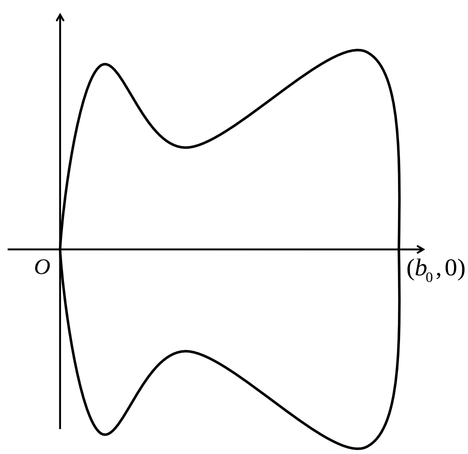

Figure 1.4: Initial curve Λ_0

Chapter 2, there exists unique viscosity solution ϕ of the following level set equation

$$\begin{cases} \phi_t = |\nabla\phi|\mathrm{div}(\frac{\nabla\phi}{|\nabla\phi|}) + A|\nabla\phi| \text{ in } \mathbb{R}^{n+1}\times(0,T), \\ \phi(x,y,0) = a_1(x,y), \end{cases}$$

where $a_1(x,y)$ is chosen such that $\Lambda_0 = \{(x,y) \mid a_1(x,y) = 0\}$ and $\{(x,y) \mid a_1(x,y) > 0\}$ is bounded. The results in appendix show that the zero set of ϕ is not fattening in a short time. Indeed, thanks to Theorem 2.3.7, the zero set of ϕ can be written into

$$\begin{aligned} \Lambda^+(t) &= \{(x,y)\in\mathbb{R}\times\mathbb{R}^n \mid \phi(x,y,t)=0\} \\ &= \{(x,y)\in\mathbb{R}\times\mathbb{R}^n \mid |y| = v(x,t), a_*(t)\le x\le b_*(t)\}, \end{aligned}$$

for $0 < t < T_*$. Moreover, (v, a_*, b_*) is the solution of the following free boundary problem

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} - \dfrac{n-1}{u} + A\sqrt{1+u_x^2},\ x\in(a_*(t),b_*(t)),\ 0<t<T_*, \\ u(a_*(t),t)=0,\ u(b_*(t),t)=0,\ 0\le t<T_*, \\ u_x(a_*(t),t)=\infty,\ u_x(b_*(t),t)=-\infty,\ 0\le t<T_*, \\ u(x,t)>0,\ x\in(a_*(t),b_*(t)),\ 0<t<T_*, \\ u(x,0)=u_0(x),\ 0\le x\le b_0. \end{cases} \tag{*}$$

Here a_* and b_* are called the end points of $\Lambda^+(t)$.

Assumption $(A+)$: There exists $\delta > 0$ such that $a_*(t) \ge 0$ for $0 \le t < \delta$.

Assumption $(A-)$: There exists $\delta > 0$ such that $a_*(t) < 0$ for $0 < t < \delta$.

In Chapter 3, we consider the flow in the plane with singular angle $\gamma = \pi/2$.

Theorem 3.1.1 shows that under the assumption $(A-)$, the outer evolution and inner evolution are away from origin. By uniqueness results we can prove they are

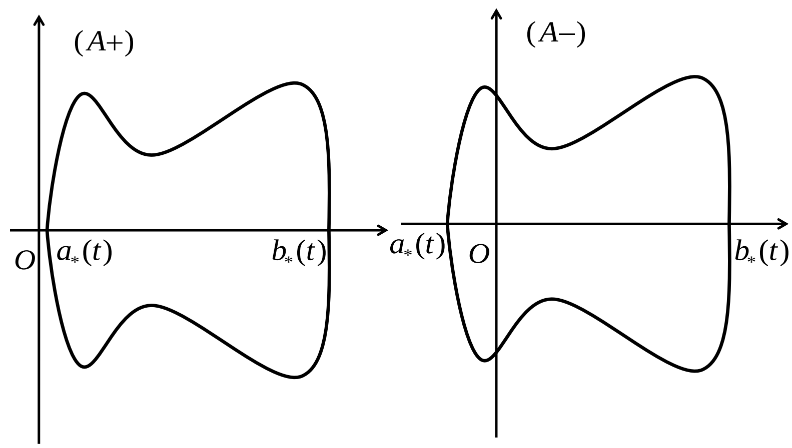

Figure 1.5: Main assumptions

coincide. Therefore, the interface evolution $\Gamma(t)$ of (1.1.1) with initial curve Γ_0 is not fattening.

In Theorem 3.1.2, assuming $(A+)$ holds, the inner evolution is separated and the outer evolution is connected. This means the interface evolution $\Gamma(t)$ of (1.1.1) with initial curve Γ_0 is fattening.

In Chapter 4, we continue to consider the curvature flow with driving force in the plane. Assume the solution $\Gamma(t)$ is given in Theorem 3.1.1. Theorem 4.1.1 and Theorem 4.1.2 classifies $\Gamma(t)$ into three cases: Expanding, Bounded and Shrinking.

Expanding. $\Gamma(t)$ remains embedded for all $t > 0$ and expands to infinity, as $t \to \infty$.

Bounded. $\Gamma(t)$ remains embedded and bounded for all $t > 0$. Moreover, $\Gamma(t)$ converges to a sphere with radius $1/A$, as $t \to \infty$.

Shrinking. $\Gamma(t)$ remains embedded until it contracts to a point at some finite time T. Moreover, in Theorem 4.1.3, we can prove the asymptotic behavior for $\Gamma(t)$ near

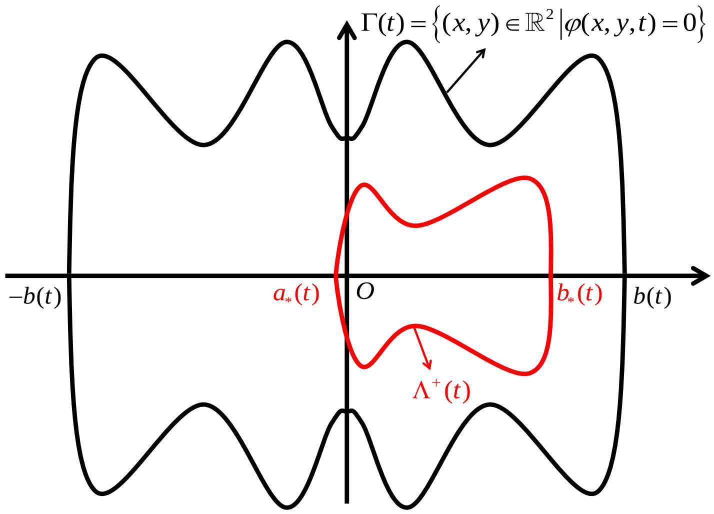

Figure 1.6: $a_*(t) < 0$ in Theorem 3.1.1

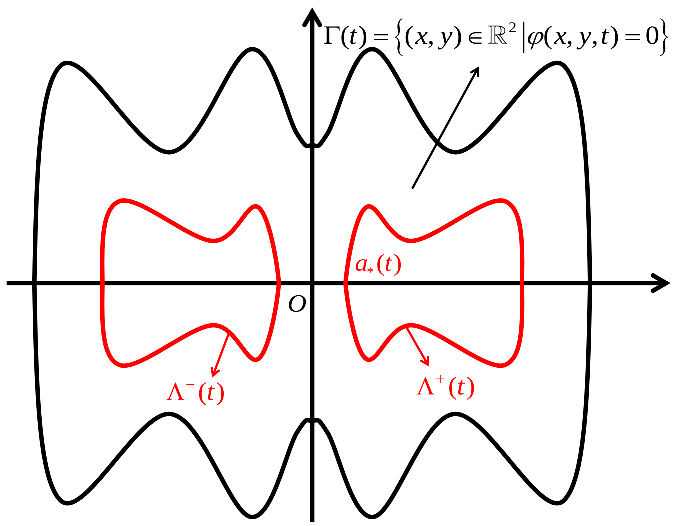

Figure 1.7: $a_*(t) \geq 0$ in Theorem 3.1.2

singular time T is a sphere. As a corollary, $\Gamma(t)$ becomes convex eventually.

To help readers understand the three cases, we give a simple example. Consider a family of circles $\partial B_{R(t)}$ (here we omit the center) evolving by $V = -\kappa + A$. Thus, $R(t)$ satisfies

$$\begin{cases} R'(t) = A - \frac{1}{R(t)},\ t > 0, \\ R(0) = R_0. \end{cases}$$

We can easily get that

Expanding. When $R_0 > 1/A$, $R(t) \uparrow \infty$, as $t \to \infty$.

Bounded. When $R_0 = 1/A$, $R(t) = 1/A$, for $0 \leq t < \infty$;

Shrinking. When $R_0 < 1/A$, there exists $T_{R_0} < \infty$ such that $R(t) \downarrow 0$, as $t \to T_{R_0}$;

In Chapter 5, we consider the mean curvature flow in higher dimensions and give the criteria in judging whether the interface evolution fattening or not.

For the singular angle $\gamma = \pi/2$, in Theorem 5.1.1, we prove the same results of Theorem 3.1.1 and Theorem 3.1.2 in higher dimensions.

In Theorem 5.1.2, for $n \geq 2$, we can find an angle $\alpha_n \in (0, \pi/2)$, as long as $0 \leq \gamma < \alpha_n$, the interface evolution $\Gamma(t)$ is not fattening. Moreover, $\Gamma(t)$ will separate into two disjoint components.

In Theorem 5.1.3, for $n = 1$, as long as $0 \leq \gamma < \pi/2$, the inner evolution always separates into two disjoint components. However, the outer evolution is connected. This means the interface evolution $\Gamma(t)$ is fattening.

At last, we can conclude the results of Chapter 3 and Chapter 5 into following tables:

Table 1.1: Singular angle $\gamma = \pi/2$

Assumption $(A+)$	$n = 1$	$n \geq 2$
Outer evolution	Connected	Connected
Inner evolution	Separated	Separated
Result	Fattening	Fattening

Table 1.2: Singular angle $\gamma = \pi/2$

Assumption $(A-)$	$n = 1$	$n \geq 2$
Outer evolution	Connected	Connected
Inner evolution	Connected	Connected
Result	Non-fattening	Non-fattening

Table 1.3: Singular angle $\gamma < \pi/2$

	$n = 1$, $0 \leq \gamma < \pi/2$	$n \geq 2$, $0 \leq \gamma < \alpha_n$
Outer evolution	Connected	Separated
Inner evolution	Separated	Separated
Result	Fattening	Non-fattening

Compare these results with the results in [2]. S. Angenent, T. Ilmanen and D.L. Chopp shows that for mean curvature flow $V = -\kappa$ in $\mathbb{R}^3$, there exists $\alpha \in (0, \pi/2)$ such that

(1). when singular angle $\gamma \in [\alpha, \pi/2]$, the interface evolution is fattening.

(2). when singular angle $\gamma \in [0, \alpha)$, the interface evolution is not fattening.

We note that when the singular angle $\gamma = \pi/2$, the interface evolution always becomes fattening. It is the most different from the case with driving force.

1.3 A short review of mean curvature flow

For the classical mean curvature flow:

$$V = -\kappa,$$

there are many results. Concerning this problem, G. Huisken [11] shows that any solution that starts out as a convex, smooth, compact, embedded surface remains so until it shrinks to a "round point" and its asymptotic shape is a sphere just before it disappears. He proves this result for hypersurfaces of $\mathbb{R}^{n+1}$ with $n \geq 2$, but M. Gage and R. Hamilton [7] show that it still holds when $n = 1$, the curves in the plane. M. Gage and R. Hamilton also show that embedded curve remains embedded, i.e. the curve will not intersect itself. M. Grayson [8] proves the remarkable fact that such family must become convex eventually. Thus, any embedded curve in the plane shrinks to "round point" under curve shortening flow. But in higher dimensions it is not true. M. Grayson [9] also shows that there exists a smooth flow that becomes singular before shrinking to a point. His example consisted of a barbell: two spherical surfaces connected by a sufficiently thin "neck". In this example, the inward curvature of the neck is so large that it will force the neck to pinch before shrinking. This result can be also proved by Angenent's doughnuts (seeing [3]). Moreover, in [1], A. Altschuler, S. B. Angenent and Y. Giga study the flow whose initial hypersurface is a compact, rotationally symmetric hypersurface but pinching on x-axis by level set method. They prove the hypersurface will separate into two smooth hypersurfaces after pinching.

1.4 Key methods

In this research, one of the most important tools is the level set method. Analytic foundation of the level set method is first established by L. C. Evans and J. Spruck [6] and independently by Y. G. Chen, Y. Giga and S. Goto [5] in 1991; see also [4]. Adjusting the theory of viscosity solutions in [6] the mean curvature flow equation is studied in detail while in [5], more general geometric evolution equations including the mean curvature flow equation with a driving force term are studied. They prove the existence and uniqueness of the interface evolution. However, the interface evolution possibly become fattening, first observed by [6]. Seeing our initial hypersurface has singularity at origin, we tend to use the level set method and identify the interface evolution is fattening or not.

Another important tool is the intersection number principle. But for the problem with driving force, the intersection number may increase. In [10], they give the extended intersection number principle to conquer this difficulty. Precisely, assume (u_1, a_1, b_1), (u_2, a_2, b_2) are the solutions of (Q) with contact angles $\theta^1_\pm$, $\theta^2_\pm$ and initial functions u^1_0, u^2_0. Let u^*_1, u^*_2 be the straight line extensions of u_1 and u_2 such that u^*_1, u^*_2 in $C^1(\mathbb{R} \times [0, T))$. They prove the intersection number between u^*_1 and u^*_2 is non-increasing provided that $\theta^1_\pm \neq \theta^2_\pm$. If $\theta^1_+ = \theta^2_+$, the intersection number will not increase provided that $b_1(t) \neq b_2(t)$ and decrease at t_0, where t_0 satisfies $b_1(t_0) = b_2(t_0)$. Similarly for $a(t)$. These results are called "extended intersection number principle". However, we cannot use the result directly. We will study the intersection number and give their applications in Chapter 2.

1.5 Organization of this thesis

The rest of this paper is organized as follows. In Chapter 2, we provide some preliminaries, which include level set method established by [5], a priori gradient estimates and the intersection number principle. In Chapter 3, we introduce the results in [17]. We study the curvature flow with driving force by the level set method and give the sufficient conditions for fattening and non-fattening. In Chapter 4, we continue to study the curvature flow with driving force in the plane. The solutions given in Chapter 3 can be classified into three cases and the asymptotic behaviors in each case are identified. To get the asymptotic behavior in Shrinking case, we need a distance comparison principle for curvature flow with driving force. G. Huisken first proves the principle for classical curve shortening flow in [12]. Seeing future, the distance comparison principle for curvature flow with driving force only holds under some special conditions. These results are included in [17] and [16]. In Chapter 5, we consider our problem in higher dimensions included in [15]. The criteria for fattening and non-fattening are given. In Chapter 6, we give the nonfattening results for α-domain.

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Chapter 2

Preliminaries

In this chapter, we give some useful preliminaries. In Section 2.1, we introduce the level set method and give the notion of the viscosity solution of level set equation. In Section 2.2, we give the interior estimates for the graph equation. In Section 2.3, we introduce the intersection number principle and give the application.

2.1 Level set method

First, we recall one of the main methods—level set method in this thesis. The level set method is first introduced by [5] and [7], [8] independently.

Let $\Gamma(t)$ be a smooth family of smooth, closed, compact, embedded hypersurfaces in $\mathbb{R}^N$ given by $\Gamma(t) = \{x|\psi(x,t) = 0, x \in \mathbb{R}^N\}$ for some ψ and $\{x \mid \psi(x,t) > 0\}$ is bounded. If $\Gamma(t)$ evolves by (1.1.1), we can see that $\psi(x,t)$ satisfies

$$\psi_t = |\nabla\psi|\mathrm{div}(\frac{\nabla\psi}{|\nabla\psi|}) + A|\nabla\psi| \text{ on } \{(x,t) \mid \psi(x,t) = 0\}.$$

Next we consider the equation in whole space

$$\psi_t = |\nabla\psi|\mathrm{div}(\frac{\nabla\psi}{|\nabla\psi|}) + A|\nabla\psi| \text{ in } \mathbb{R}^N \times (0,T). \tag{2.1.1}$$

Equation (2.1.1) is called the level set equation of (1.1.1). Theorem 4.3.1 in [9] proves the existence and uniqueness of the viscosity solution for (2.1.1) with $\psi(x,0) = \psi_0(x)$. Here $\psi_0(x)$ is a bounded and uniform continuous function.

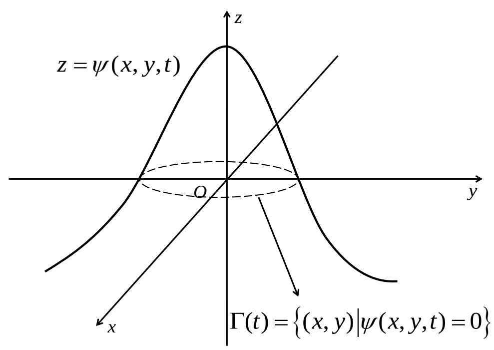

Figure 2.1: Level set method in $\mathbb{R}^2$

Level set method Using the solution of level set equation, we introduce the level set method.

Definition 2.1.1. (1) Let D_0 be a bounded open set in $\mathbb{R}^N$. A family of open sets $\{D(t) \mid D(t) \subset \mathbb{R}^N\}_{0<t<T}$ is called an (*generalized*) *open* (or *inner*) *evolution* of (1.1.1) with initial data D_0 if there exists a viscosity solution ψ of (2.1.1) that satisfies

$$D(t) = \{x \in \mathbb{R}^N \mid \psi(x,t) > 0\},\ D_0 = \{x \in \mathbb{R}^N \mid \psi(x,0) > 0\}.$$

(2) Let E_0 be a bounded closed set in $\mathbb{R}^N$. A family of closed sets $\{E(t) \mid E(t) \subset \mathbb{R}^N\}_{0<t<T}$ is called a (*generalized*) *closed* (or *outer*) *evolution* of (1.1.1) with initial

data E_0 if there exists a viscosity solution ψ of (2.1.1) that satisfies

$$E(t) = \{x \in \mathbb{R}^N \mid \psi(x,t) \geq 0\},\ E_0 = \{x \in \mathbb{R}^N \mid \psi(x,0) \geq 0\}.$$

The set $\Gamma(t) = E(t) \setminus D(t)$ is called an *(generalized) interface evolution* of (1.1.1) with initial data $\Gamma_0 = E_0 \setminus D_0$.

Remark 2.1.2. (1) For open set D_0 and $E_0 = \overline{D_0}$, we often choose

$$\psi(x,0) = \max\{\mathrm{sd}(x, \partial D_0), -1\}$$

where

$$\mathrm{sd}(x,\partial D_0) = \begin{cases} \mathrm{dist}(x,\partial D_0),\ x \in D_0, \\ -\mathrm{dist}(x,\partial D_0),\ x \notin D_0. \end{cases}$$

(2) Seeing that the choice of $\psi(x,0)$ is not unique, Theorem 4.2.8 in [9] implies that the open evolution $D(t)$ and closed evolution $E(t)$ are both independent of the choice of $\psi(x,0)$.

(3) Generally, even if $E_0 = \overline{D_0}$, we can not guarantee $E(t) = \overline{D(t)}$. If $E(t) \setminus D(t)$ has interior points for some t, we call the interface evolution is fattening. Respectively, if $E(t) = \overline{D(t)}$, for all $0 < t < T$, we say the interface evolution is not fattening.

(4) If D_0 and $\overline{D_0}$ are symmetric to x_i-axis, then it is also true for $D(t)$ and $E(t)$. Since level set equation (2.1.1) is invariance under orthogonal transformation.

We now list some fundamental properties of open evolution and closed evolution of (1.1.1). (All the results listed below can be found in Chapter 4 of [9])

Theorem 2.1.3. *(Semigroups) [9]. Denote $N(t)$ and $M(t)$ being the operators such that $N(t)D_0 = D(t)$ and $M(t)E_0 = E(t)$, for $t > 0$. Then we have $N(t)D(s) = D(t+s)$ and $M(t)E(s) = E(t+s)$, for any $t > 0$, $s > 0$.*

Theorem 2.1.4. *(Order preserving property or comparison principle) [9]. Let* D_0, D_0' *be two open sets in* $\mathbb{R}^N$ *and let* E_0, E_0' *be two closed sets in* $\mathbb{R}^N$. *Then*

(1) $N(t)D_0 \subset U(t)D_0'$, *if* $D_0 \subset D_0'$;

(2) $M(t)E_0 \subset M(t)E_0'$, *if* $E_0 \subset E_0'$;

(3) $N(t)D_0 \subset M(t)E_0'$, *if* $D_0 \subset E_0'$;

(4) $E_0 \subset D_0$ *and* $dist(E_0, \partial D_0) > 0$, *then* $M(t)E_0 \subset N(t)D_0$.

Theorem 2.1.5. *(Monotone convergence) [9].*

(1) Let $D(t)$ *and* $\{D_j(t)\}$ *be open evolutions with initial data* D_0 *and* D_{j0} *respectively. If* $D_{j0} \uparrow D_0$, *then* $D_j(t) \uparrow D(t)$, $t > 0$, *i.e.,* $\bigcup_{j \geq 1} D_j(t) = D(t)$;

(2) Let $E(t)$ *and* $\{E_j(t)\}$ *be closed evolutions with initial data* E_0 *and* E_{j0} *respectively. If* $E_{j0} \downarrow E_0$, *then* $E_j(t) \downarrow E(t)$, $t > 0$, *i.e.,* $\bigcap_{j \geq 1} E_j(t) = E(t)$.

Theorem 2.1.6. *(Continuity in time) [9]. Let* $D(t)$ *and* $E(t)$ *be open and closed evolutions, respectively.*

(1a) $D(t)$ *is a lower semicontinuous function of* $t \in [0, T)$, *in the sense that for any* $t_0 \geq 0$, *and sequence* $x_n \in (D(t_n))^c$ *with* $x_n \to x_0$, $t_n \to t_0$, *the limit* $x_0 \in (D(t_0))^c$. *If* $D(0)$ *is bounded so that* $\mathcal{C}_\epsilon(D(t_0))$ *is compact, this implies that for any* $t_0 \geq 0$, $\epsilon > 0$ *there is a* $\delta > 0$ *such that* $|t - t_0| < \delta$ *implies* $D(t) \supset \mathcal{C}_\epsilon(D(t_0))$.

(1b) $E(t)$ *is an upper semicontinuous function of* $t \in [0, T)$, *in the sense that for any* $t_0 \geq 0$, *and sequence* $x_n \in E(t_n)$ *with* $x_n \to x_0$, $t_n \to t_0$, *the limit* $x_0 \in E(t_0)$. *If* $E(0)$ *is bounded so that* $\mathcal{N}_\epsilon(E(t_0))$ *is compact, this implies that for any* $t_0 \geq 0$, $\epsilon > 0$ *there is a* $\delta > 0$ *such that* $|t - t_0| < \delta$ *implies* $E(t) \subset \mathcal{N}_\epsilon(E(t_0))$.

(2a) $D(t)$ *is a left upper semicontinuous in* t *in the sense that for any* $t_0 \in (0, T)$, $x_0 \in (D(t_0))^c$ *there is a sequence* $x_n \to x_0$ *and* $t_n \uparrow t_0$ *with* $x_n \in (D(t_n))^c$. *Moreover, for any* $t_0 \in (0, T)$, $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $t_0 - \delta < t < t_0$ *implies* $\mathcal{C}_\epsilon(D(t)) \subset D(t_0)$.

(2b) $E(t)$ *is a left lower semicontinuous in* t *in the sense that for any* $t_0 \in (0, T)$, $x_0 \in E(t_0)$ *there is a sequence* $x_n \to x_0$ *and* $t_n \uparrow t_0$ *with* $x_n \in E(t_0)$. *Moreover,*

for any $t_0 \in (0,T)$, $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $t_0 - \delta < t < t_0$ *implies* $\mathcal{N}_\epsilon(E(t)) \supset E(t_0)$.

Here $N_\epsilon(A) = \{x \in \mathbb{R}^N \mid d(x,A) < \epsilon\}$, for A is a closed subset in $\mathbb{R}^N$ and $C_\epsilon(A) = N_\epsilon(A^c)^c$, for A is an open subset in $\mathbb{R}^N$.

Theorem 2.1.7. *(Separate) Let* $\{D_1(t)\}_{0\le t<T}$ *be the open evolution of* $V = -\kappa + A$ *and* $\{D_2(t)\}_{0\le t<T}$ *be the open evolution of* $V = -\kappa - A$. *If* $D_1(0) \cap D_2(0) = \emptyset$, *then* $D_1(t) \cap D_2(t) = \emptyset$ *for* $0 \le t < T$.

The proof of this theorem is similar to Theorem 3.5 in [1]. We omit it.

Remark 2.1.8. For $A > 0$, even if $D_1(0)$ and $D_2(0)$ are disjoint, $D_1(t)$ and $D_2(t)$ may intersect. The basic reason is that the level set equation (2.1.1) is not orientation free(If u is a solution, there does not hold that $-u$ is also a solution for (2.1.1)).

In Chapter 3 and Chapter 5, to prove the fattening results, we need the following lemma. This lemma gives the construction of an open evolution containing two disjoint components.

Lemma 2.1.9. *Assume* $D_1(t)$ *and* $D_2(t)$ *being the open evolution of (2.1.1) with* $D_1(0) = U_1$ *and* $D_2(0) = U_2$. *And* $D(t)$ *is denoted as the open evolution of (2.1.1) with* $D(0) = U_1 \cup U_2$. *If* $D_1(t) \cap D_2(t) = \emptyset$ *for* $0 \le t \le T$, *then* $D(t) = D_1(t) \cup D_2(t)$, $0 \le t \le T$.

Under the condition $A = 0$, $D_1(t) \cap D_2(t) = \emptyset$ holds automatically provided that $D_1(0) \cap D_2(0) = \emptyset$. But for $A > 0$, it is not true. Therefore, we give the assumption $D_1(t) \cap D_2(t) = \emptyset$ for $0 \le t \le T$.

Proof. First, we assume $\delta =: \min\limits_{0\le t\le T} \operatorname{dist}(D_1(t), D_2(t)) > 0$. We define

$$a_i(x) = \max\{\operatorname{sd}(x, \partial D_i(0)), 0\},\ x \in \mathbb{R}^N,\ i = 1, 2.$$

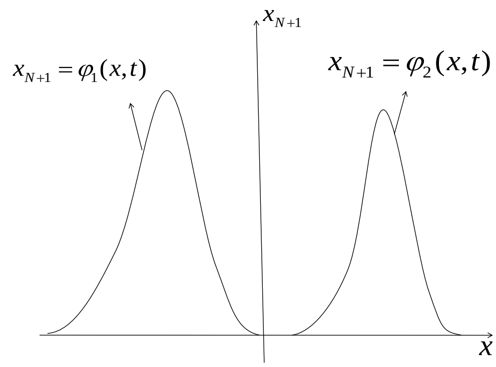

Figure 2.2: Proof of Lemma 2.1.9

Let φ_i be the viscosity solution of level set equation (2.1.1) with $\varphi(x,0) = a_i(x)$, $i = 1,2$. By the theory in [9], $\varphi_i(x,t) \in C_c(\mathbb{R}^N \times [0,T])$ and $D_i(t) = \{x \in \mathbb{R}^N \mid \varphi_i(x,t) > 0\}$. Moreover, $\varphi_i = 0$ in $\mathbb{R}^N \setminus D_i(t)$, $i = 1,2$.

Our assumption implies that suppφ_1 and suppφ_2 are seperated by δ. Denote $\varphi =: \max\{\varphi_1, \varphi_2\}$. For any open set B and $diam(B) < \delta$,

$$\varphi(x) = \varphi_1(x) \text{ or } \varphi_2(x),\ x \in B,$$

where $diam(B) = \sup\{|x-y| \mid x,y \in B\}$. Seeing the definition of viscosity solution, φ is also a viscosity solution of (2.1.1). Then $D(t) = \{x \mid \varphi(x,t) > 0\} = D_1(t) \cup D_2(t)$, for $0 \le t \le T$.

Next we prove the result only under the assumption $D_1(t) \cap D_2(t) = \emptyset$, $0 \le t \le T$. Consider $D_i^j(t) = \{x \mid \varphi_i(x,t) > \dfrac{1}{j}\}$.

We claim that $\min\limits_{0 \le t \le T} \text{dist}(D_1^j(t), D_2^j(t)) > 0$, for all j. If $\min\limits_{0 \le t \le T} \text{dist}(D_1^j(t), D_2^j(t)) = 0$, for some j, then there exist $t_0 \in [0,T]$ and sequences $\{x_m\} \subset D_1^j(t_0)$, $\{y_m\} \subset D_2^j(t_0)$ such that

$$|x_m - y_m| \to 0, \quad \varphi_1(x_m, t_0) > \frac{1}{j},\ \varphi_2(y_m, t_0) > \frac{1}{j}.$$

Then there exists x, such that $\lim_{m\to\infty} x_m = \lim_{m\to\infty} y_m = x$. Then

$$\varphi_1(x,t_0) \geq \frac{1}{j} > 0,\ \ \varphi_2(x,t_0) \geq \frac{1}{j} > 0.$$

Consequently, $x \in D_1(t_0) \cap D_2(t_0) \neq \emptyset$, contradiction. Then we have $\min_{0\leq t\leq T} \operatorname{dist}(D_1^j(t), D_2^j(t)) > 0$, for all j. By the argument in the first step, there holds $D^j(t) = D_1^j(t) \cup D_2^j(t)$ is the open evolution with initial openset $\{x \mid \varphi_1(x,0) > \frac{1}{j}\} \cup \{x \mid \varphi_2(x,0) > \frac{1}{j}\}$, for $0 \leq t \leq T$.

Noting $\bigcup_{j=1}^{\infty} D_1^j(0) \cup D_2^j(0) = U_1 \cup U_2$ and using Theorem 2.1.5, $D(t) = \bigcup_{j=1}^{\infty} D^j(t) = \bigcup_{j=1}^{\infty} D_1^j(t) \cup D_2^j(t) = D_1(t) \cup D_2(t)$, for $0 \leq t \leq T$. □

Theorem 2.1.10. *(Local smoothness for graphs) Suppose that ψ is a viscosity solution of (2.1.1). Assume in an open region $U \times (t_1, t_2)$,*

$$\{(x,t) \mid \psi = 0\} \cap U = \{(x,t) \mid x_N = g(x',t), x' \in U'\}$$

where $x' = (x_1, \cdots, x_{N-1})$, $U' = U \cap \{x_N = 0\}$ and g is continuous in $U' \times (t_1, t_2)$. Then the function g is a viscosity solution of

$$g_t = \left(\delta_{ij} - \frac{g_{x_i} g_{x_j}}{1 + |\nabla g|^2}\right) g_{x_i x_j} + A\sqrt{1 + |\nabla g|^2}$$

or

$$g_t = \left(\delta_{ij} - \frac{g_{x_i} g_{x_j}}{1 + |\nabla g|^2}\right) g_{x_i x_j} - A\sqrt{1 + |\nabla g|^2}.$$

If the normal velocity of $\{(x,t) \mid \psi = 0\} \cap U$ is upward (downward), we choose "+" ("−") in above graph equation.

Moreover, g is C^∞ in the region $U' \times (t_1, t_2)$.

The proof can be seen similarly in [8] (Theorem 5.1 and Theorem 5.4). Here we omit it.

2.2 A Priori estimates

In this section, we give the interior gradient estimate.

Graph equation Let $u(x,t)$ be some function on an open subset of $\mathbb{R}^n \times \mathbb{R}$, then the graph of $u(x,t)$ is a family of hypersurfaces in $\mathbb{R}^{n+1}$. The family of hypersurfaces moves by $V = -\kappa + A$ if and only if

$$u_t = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{1+|\nabla u|^2}\right) u_{x_i x_j} \pm A\sqrt{1+|\nabla u|^2},$$

where the signs of the last terms are determined by direction of the driving force.

Under the case $A = 0$,

$$u_t = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{1+|\nabla u|^2}\right) u_{x_i x_j},$$

The gradient estimate in entire space $\mathbb{R}^n$ is given by [6]. The local gradient estimate is also given by [8]. In this thesis, the local gradient estimate under the condition $A > 0$ is important. We prove it similarly as in [8].

Theorem 2.2.1. *For $u \in C^3(\Omega_T) \cap C^0(\overline{\Omega}_T)$, u satisfies*

$$u_t = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{1+|\nabla u|^2}\right) u_{x_i x_j} \pm A\sqrt{1+|\nabla u|^2}, \tag{2.2.1}$$

For the condition "+"("−"), we assume $u < 0$($u > 0$) in Ω_T, $u(0,T) = -v_0$($u(0,T) = v_0$). Then

$$|\nabla u(0,T)| \leq (3 + 16v_0)e^{2K},$$

where $K = 20v_0^2(4n + \frac{1}{T} + 4A + \frac{A}{2v_0}) + 2$, $\Omega_T = B_1(0) \times (0, 2T)$ *and*

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} .$$

Proof. We only prove the case "+". For the case "−" , we can consider "$-u$" to get the result. Denote $w = \sqrt{1 + |\nabla u|^2}$, $\nu^i = u_{x_i}/\sqrt{1 + |\nabla u|^2}$, $g^{ij} = \delta_{ij} - \nu^i \nu^j$. We define the operator L as

$$Lh = g^{ij} h_{x_i x_j} - h_t + A\nu^k h_{x_k}.$$

We let $h = \eta(x, t, u(x, t))w$, where η is a non-negative function and will be identified in future. By calculation,

$$\begin{aligned} Lh &= g^{ij}(w_{x_i x_j}\eta + w_{x_i}(\eta)_{x_j} + (\eta)_{x_i} w_{x_j} + w(\eta)_{x_i x_j}) \\ &- (\eta)_t w - \eta w_t + A\nu^k(w_{x_k}\eta + (\eta)_{x_k} w) \\ &= \eta Lw + wL\eta + 2g^{ij} w_{x_i}(\eta)_{x_j} \\ &= \eta Lw + wL\eta + 2g^{ij} w_{x_i}\left(\frac{h_{x_j} - w_{x_j}\eta}{w}\right). \end{aligned}$$

Then

$$Lh - 2g^{ij}\frac{w_{x_i}}{w} h_{x_j} = \eta\left(Lw - 2g^{ij}\frac{w_{x_i} w_{x_j}}{w}\right) + wL\eta.$$

We claim that

$$Lw - 2g^{ij}\frac{w_{x_i} w_{x_j}}{w} \geq 0.$$

Therefore, there holds

$$Lh - 2g^{ij}\frac{w_{x_i}}{w} h_{x_j} \geq wL\eta. \tag{2.2.2}$$

We begin to prove the claim. Seeing

$$w_{x_ix_j} = \nu^k u_{x_kx_ix_j} + \frac{1}{w}(u_{x_kx_i}u_{x_kx_j} - \nu^k\nu^l u_{x_kx_i}u_{x_lx_j}),$$

we have

$$\begin{aligned} g^{ij}w_{x_ix_j} \geq \nu^k g^{ij}u_{x_kx_ix_j} &= \nu^k((g^{ij}u_{x_ix_j})_{x_k} - g^{ij}_{x_k}u_{x_ix_j}) \\ &= \nu^k\left(u_{tx_k} - A\frac{u_{x_l}u_{x_lx_k}}{\sqrt{1+|\nabla u|^2}}\right) - \nu^k g^{ij}_{x_k}u_{x_ix_j}. \end{aligned}$$

Combining

$$g^{ij}_{x_k} = -\frac{1}{w}(\nu^j u_{x_ix_k} + \nu^i u_{x_jx_k}) + \frac{2u_{x_i}u_{x_j}}{w^3}w_{x_k},$$

$$\begin{aligned} \nu^k g^{ij}u_{x_kx_ix_j} &= w_t - A\nu^k w_{x_k} + \frac{\nu^k u_{x_ix_j}}{w}\left(\nu^j u_{x_ix_k} + \nu^i u_{x_jx_k} - \frac{2u_{x_i}u_{x_j}}{w^2}w_{x_k}\right) \\ &= w_t - A\nu^k w_{x_k} + \frac{2}{w}g^{ij}w_{x_i}w_{x_j}. \end{aligned}$$

Therefore

$$g^{ij}w_{x_ix_j} \geq w_t - A\nu^k w_{x_k} + \frac{2}{w}g^{ij}w_{x_i}w_{x_j}.$$

Then

$$Lw \geq \frac{2}{w}g^{ij}w_{x_i}w_{x_j}.$$

We complete the proof of the claim.

Next we choose $\eta = f \circ \phi(x, t, u(x,t))$,

$$\phi(x,t,z) = \left(\frac{z}{2v_0} + \frac{t}{T}(1-|x|^2)\right)^+$$

and

$$f(\phi) = e^{K\phi} - 1.$$

When $\phi > 0$, there holds

$$\phi_z = \frac{1}{2v_0},\ \phi_t = \frac{1-|x|^2}{T},\ \phi_{x_i} = -\frac{2t}{T}x_i,\ \phi_{x_ix_j} = -\frac{2t}{T}\delta_{ij}.$$

Consequently, when $\phi > 0$, $z < 0$, $0 < t < 2T$,

$$0 \le \phi \le 2,\ \sum \phi_{x_i}^2 \le \frac{4t^2}{T^2} \le 16.$$

By calculation,

$$\begin{aligned} L\eta &= g^{ij} f''(\phi_{x_i} + \phi_z u_{x_i})(\phi_{x_j} + \phi_z u_{x_j}) + g^{ij} f'(\phi_{x_ix_j} + \phi_z u_{x_ix_j}) \\ &- f'(\phi_t + \phi_z u_t) + A\nu^k f'(\phi_{x_k} + \phi_z u_{x_k}) \\ &\ge \frac{f''}{w^2}(\phi_{x_i} + \phi_z u_{x_i})^2 + f'(g^{ij}\phi_{x_ix_j} - \phi_t + A\nu^k \phi_{x_k}) + f'\phi_z Lu \end{aligned}$$

$$\begin{aligned} &= \frac{f''}{w^2}\left(-\frac{2t}{T}x_i + \frac{1}{2v_0}u_{x_i}\right)^2 + f'\left(-\frac{2t}{T}(n - \frac{|\nabla u|^2}{1+|\nabla u|^2}) - \frac{1-|x|^2}{T}\right. \\ &- \left. Ax_k \frac{2t}{T}\frac{u_{x_k}}{\sqrt{1+|\nabla u|^2}}\right) + f'\phi_z Lu. \end{aligned}$$

Combining

$$Lu = g^{ij}u_{x_ix_j} - u_t + A\nu^k u_{x_k} = -\frac{A}{\sqrt{1+|\nabla u|^2}},$$

there holds

$$\begin{aligned} L\eta &\ge \frac{f''}{w^2}\left(\frac{|\nabla u|^2}{8v_0^2} - 8\right) + f'\left(-4n - \frac{1}{T} - 4A\right) - f'\phi_z \frac{A}{\sqrt{1+|\nabla u|^2}} \\ &\ge \frac{f''}{w^2}\left(\frac{|\nabla u|^2}{8v_0^2} - 8\right) + f'\left(-4n - \frac{1}{T} - 4A - \frac{A}{2v_0}\right) \\ &= \frac{K^2 e^{K\phi}}{w^2}\left(\frac{|\nabla u|^2}{8v_0^2} - 8\right) + Ke^{K\phi}\left(-4n - \frac{1}{T} - 4A - \frac{A}{2v_0}\right). \end{aligned}$$

When $|\nabla u| \geq \max\{16v_0, 2\}$, we have

$$\frac{|\nabla u|^2}{16v_0^2} \geq 8, \ \frac{|\nabla u|^2}{16} \geq \frac{1+|\nabla u|^2}{20}.$$

Then

$$\begin{aligned} L\eta &\geq \frac{K^2 e^{K\phi}}{w^2}\frac{|\nabla u|^2}{16v_0^2} + Ke^{K\phi}\left(-4n - \frac{1}{T} - 4A - \frac{A}{2v_0}\right) \\ &\geq \frac{K^2 e^{K\phi}}{20v_0^2} + Ke^{K\phi}(-4n - \frac{1}{T} - 4A - \frac{A}{2v_0}) \\ &= Ke^{K\phi}\left(\frac{K}{20v_0^2} - 4n - \frac{1}{T} - 4A - \frac{A}{2v_0}\right) > 0, \end{aligned}$$

when we choose $K = 20v_0^2(4n + \frac{1}{T} + 4A + \frac{A}{2v_0}) + 2$, $\Omega_T = B_1(0) \times (0, 2T)$.

Therefore by (2.2.2), there holds

$$Lh - 2g^{ij}\frac{w_{x_i}}{w}h_{x_j} \geq 0 \text{ on } \{h > 0 \text{ or } |\nabla u| > \max\{16v_0, 2\}\}.$$

By maximum principle,

$$\begin{aligned} (e^{\frac{K}{2}} - 1)w(0,T) &= h(0,T) \leq \max_{h=0 \text{ and } |\nabla u| = \max\{16v_0, 2\}} h \\ &\leq (e^{2K} - 1)\max\{\sqrt{1 + (16v_0)^2}, \sqrt{5}\}. \end{aligned}$$

Consequently, $w(0,T) \leq e^{2K}(3 + 16v_0)$. □

Remark 2.2.2. (1) In Theorem 2.2.1, Ω_T can be replaced by $\Omega_T = B_R(x_0) \times (0, 2T)$ and $v_0 = u(x_0, T)$. Then the conclusion becomes

$$|\nabla u(x_0, T)| \leq e^{2K}(3 + 16\frac{v_0}{R}),$$

where $K = 20\frac{v_0^2}{R^2}\left(4n + \frac{R^2}{T} + \frac{4A}{R} + \frac{A}{2v_0}\right)+2$. We can set $v(x,t) = \frac{u(Rx + x_0, R^2 t)}{R}$, then we can use Theorem 2.2.1 for $v(x,t)$.

(2) When u is the solution of (2.2.1) for "+" without the assumption "$u < 0$", we can set

$$v = u - M - \epsilon$$

where $M = \sup_{\overline{\Omega}_T} |u|$ and $\epsilon > 0$. Using (1) in Remark 2.2.2 to v, we can deduce

$$|\nabla u(0,T)| \leq \left(3 + 16\frac{M - u(0,T) + \epsilon}{R}\right) e^{2\widetilde{K}_\epsilon},$$

where $\widetilde{K}_\epsilon = \frac{20(M - u(0,T) + \epsilon)^2}{R^2}\left(4n + \frac{R^2}{T} + \frac{4A}{R} + \frac{A}{2(M + \epsilon - u(0,T))}\right)+2$.

Tending $\epsilon \to 0$, we have

$$|\nabla u(0,T)| \leq \left(3 + 32\frac{M}{R}\right) e^{2\widetilde{K}},$$

where $\widetilde{K} = \frac{80M^2}{R^2}\left(4n + \frac{R^2}{T} + \frac{4A}{R}\right) + \frac{20AM}{R^2} + 2$.

Then we can get the next corollary by (2) in Remark 2.2.2 and the same method as in [1].

Corollary 2.2.3. *For $s_1 < s_2$, $\rho > 0$ and $x_0 \in \mathbb{R}^n$ we set*

$$\Omega = B_\rho(x_0) \times (s_1, s_2).$$

Suppose that $u \in C^3(\Omega)$ solves the equation (2.2.1) in Ω with $M = \sup_{\overline{\Omega}} |u| < \infty$. For any $\epsilon > 0$ there is a constant $C = C(M, \epsilon, n)$ such that

$$|\nabla u| \leq C \text{ on } \Omega_\epsilon = B_{\rho-\epsilon}(x_0) \times (s_1 + \epsilon^2, s_2).$$

Remark 2.2.4. (1) From Corollary 2.2.3 and [11], there exist $C_k(M, \epsilon, n)$ such that

$$|\nabla^k u| \leq C_k, \ (x,t) \in B_{\rho-2\epsilon}(x_0) \times (s_1 + 2\epsilon^2, s_2).$$

(2) Noting C and C_k are all independent of s_2, if the solution u exists for all $t > s_1$, s_2 can be chosen as ∞.

2.3 Intersection number principle

In this section, we introduce another important method—-intersection number principle and give their applications.

A short review of the research in the intersection number The Sturmian theorem states that the number of zeros(counted with multiplicity) of a solution of linear parabolic equation of the type

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u$$

doesn't increase with time, provided that u is defined on a rectangle $x_0 \leq x \leq x_1$, $0 < t < T$ and $u(x_j, t) \neq 0$ for $j = 0, 1$, for all $t \in (0, T)$. This result also holds for the number of sign changing rather than the number of zeros of $u(\cdot, t)$.

It is well known that the intersection number between two families of rotationally symmetric hypersurfaces $\Gamma_1(t)$ and $\Gamma_2(t)$ evolving by $V = -\kappa$ is non-increasing([3]). However, this result is not true in the case $A > 0$. Indeed seeing future, the intersection number between two families of rotationally symmetric hypersurfaces evolving by $V = -\kappa + A$ may increase. The intersection number between two families of rotationally symmetric hypersurfaces is defined as follows:

Intersection number for rotationally symmetric hypersurfaces For two rotationally symmetric hypersurfaces $\Gamma_1(t)$ and $\Gamma_2(t)$ are given by $\Gamma_1(t) = \{(x, y) \in$

$\mathbb{R} \times \mathbb{R}^n \mid r = u_1(x,t)\}$ and $\Gamma_2(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}^n \mid r = u_2(x,t)\}$. The intersection number between $\Gamma_1(t)$ and $\Gamma_2(t)$ denoted by $\mathcal{Z}[\Gamma_1(t), \Gamma_2(t)]$ is defined by the number of intersections between $u_1(\cdot,t)$ and $u_2(\cdot,t)$.

Extended intersection number principle First we introduce a more general result about the intersection number in the plane. Consider the following problem which we call (Q):

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \ x \in (a(t), b(t)), \ 0 < t < T, \\ u(a(t),t) = 0, \ u(b(t),t) = 0, \ 0 \le t < T, \\ u_x(a(t),t) = \tan\theta_-(t), \ u_x(b(t),t) = -\tan\theta_+(t), \ 0 \le t < T, \\ u(x,0) = u_0(x), \ a(0) \le x \le b(0), \end{cases} \tag{Q}$$

where $u_0 \in C[a(0), b(0)] \cap C^1(a(0), b(0))$ and $\theta_\pm(t)$ are smooth functions with values in $[0, \pi/2]$. Let

$$\gamma_1(t) := \begin{cases} \{(x,y) \mid y = \tan\theta_-(t)(x - a(t)), y < 0\}, \ \theta_-(t) < \pi/2 \\ \{(x,y) \mid x = a(t), y < 0\}, \ \theta_-(t) = \pi/2, \end{cases}$$

$$\gamma_2(t) := \begin{cases} \{(x,y) \mid y = -\tan\theta_+(t)(x - b(t)), y < 0\}, \ \theta_+(t) < \pi/2 \\ \{(x,y) \mid x = b(t), y < 0\}, \ \theta_+(t) = \pi/2, \end{cases}$$

and

$$\gamma_3(t) := \{(x,y) \mid y = u(x,t), a(t) \le x \le b(t)\}.$$

The extension curve of $u(\cdot,t)$ is given by

$$\gamma(t) := \gamma_1(t) \cup \gamma_2(t) \cup \gamma_3(t).$$

Proposition 2.3.1. *Let $u^1(x,t)$, $a^1(t) < x < b^1(t)$ be solution of (Q) for $\theta_\pm^1(t) \in [0, \pi/2)$, and $u^2(x,t)$, $a^2(t) < x < b^2(t)$ be solution of (Q) for $\theta_\pm^2(t) = \pi/2$, for $0 \le$*

$t < T$. Let $\gamma^i(t)$ be the extension curve of $u^i(x,t)$, respectively. Then $\mathcal{Z}[\gamma^1(t), \gamma^2(t)]$ is non-increasing in $t \in [0,T)$ and is finite for each $t \in [0,T)$. Moreover, $\mathcal{Z}[\gamma^1(t), \gamma^2(t)]$ will drop when $\gamma^1(t)$ intersects $\gamma^2(t)$ tangentially.

For the proof of this proposition, it is similar as the proof of the Proposition 2.4 in [10]. Here we omit it.

Remark 2.3.2. (1). Proposition 2.4 in [10] only give the results under $\theta^i_\pm \in (0, \pi/2)$, $i = 1, 2$.

(2). For $\theta^i_\pm = \pi/2$, $i = 1, 2$, the results in Proposition 2.3.1 are not true. We conclude the results in Remark 2.3.6.

We consider higher dimensional condition.

Horizontal and vertical graph equations If $\Gamma(t)$ is a family of rotationally symmetric hypersurfaces in $\mathbb{R}^{n+1}$, then parts of $\Gamma(t)$ may be represented either as horizontal graph, $r = u(x,t)$, or vertical graph, $x = v(r,t)$, where $(x, y_1, \cdots, y_n) \in \mathbb{R}^{n+1}$ and $r = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$.

If $\Gamma(t)$ is given as a horizontal graph, then $\Gamma(t)$ evolves by $V = -\kappa + A$ in $\mathbb{R}^{n+1}$ and the direction of the driving force points to the positive direction of $r = |y|$ axis if and only if u satisfies the horizontal graph equation

$$\frac{\partial u}{\partial t} = \frac{u_{xx}}{1+u_x^2} - \frac{n-1}{u} + A\sqrt{1+u_x^2}. \tag{2.3.1}$$

If $\Gamma(t)$ is given as a vertical graph, then $\Gamma(t)$ evolves by $V = -\kappa + A$ in $\mathbb{R}^{n+1}$ if and only if v satisfies the vertical graph equation

$$\frac{\partial v}{\partial t} = \frac{v_{rr}}{1+v_r^2} + \frac{n-1}{r} v_r + A\sqrt{1+v_r^2}, \tag{2.3.2}$$

or

$$\frac{\partial v}{\partial t} = \frac{v_{rr}}{1+v_r^2} + \frac{n-1}{r} v_r - A\sqrt{1+v_r^2}, \tag{2.3.3}$$

where the signs in the last terms are determined by the direction of the driving force (we choose "$+(-)$" when the direction of the driving force is rightward(leftward)).

Moreover, a smooth family of smooth, closed, embedded hypersurfaces given by $\Gamma(t) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid r = u(x,t), a(t) \leq x \leq b(t)\}$ evolves by $V = -\kappa + A$ if and only if u satisfies horizontal graph equation (2.3.1) and following:

$$u(b(t),t) = u(a(t),t) = 0,\ u_x(a(t),t) = -u_x(b(t),t) = \infty,\ 0 < t < T_U, \tag{2.3.4}$$

$$u(x,t) > 0,\ a(t) < x < b(t),\ 0 < t < T_U, \tag{2.3.5}$$

$$u(x,0) = u_0,\ a(0) \leq x \leq b(0). \tag{2.3.6}$$

Theorem 2.3.3. *Two smooth families of smooth, closed, embedded hypersurfaces given by $\Gamma_1(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}^n \mid r = u_1(x,t), a_1(t) \leq x \leq b_1(t)\}$, $\Gamma_2(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}^n \mid r = u_2(x,t), a_2(t) \leq x \leq b_2(t)\}$ evolve by $V = -\kappa + A$ in $\mathbb{R}^{n+1}$, $0 < t < T$. Then either $\Gamma_1 \equiv \Gamma_2$ for all $t \in (0,T)$, or the number of intersections of $\Gamma_1(t)$ and $\Gamma_2(t)$ is finite for all $t \in (0,T)$. In the second case, if $a_1(t)$, $b_1(t)$, $a_2(t)$ and $b_2(t)$ are all different and their order remains unchanged for all $t \in (0,T)$, this number is nonincreasing in time, and decreases whenever $\Gamma_1(t)$ and $\Gamma_2(t)$ have a tangential intersection.*

We only give the sketch of the proof. For example, if the order of a_1, b_1, a_2, b_2 is given by $a_1(t) < a_2(t) < b_1(t) < b_2(t)$, $0 < t < T$, the intersections are only in the interval $[a_2(t), b_1(t)]$. Since $u_1(a_2(t),t) - u_2(a_2(t),t) \neq 0$ and $u_1(b_1(t),t) - u_2(b_1(t),t) \neq 0$, $0 < t < T$, using Theorem D in [2], the intersection number between u_1 and u_2 is not increasing and decreases when tangentially intersecting in $[a_2(t), b_1(t)]$. Consequently, the intersection number between $\Gamma_1(t)$ and $\Gamma_2(t)$ is not increasing. We can prove the other conditions with the same method.

Using the intersection number principle, we can prove following gradient estimate. We postpone the intersection number arguments in Lemma 2.3.5.

Theorem 2.3.4. $\Gamma(t) = \{(x,y) \in \mathbb{R}^{n+1} \mid r = u(x,t), a_2(t) \le x \le b_2(t)\}$ *is a smooth family of closed, smooth, embedded hypersurfaces in* $\mathbb{R}^{n+1}$, $0 < t < T$. *If* $\Gamma(t)$ *evolves by* $V = -\kappa + A$ *in* $\mathbb{R}^{n+1}$, *there is a function* $\sigma\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$|u_x(x,t)| \le \sigma(t, u(x,t))$$

holds for $0 < t < T$, $a_2(t) < x < b_2(t)$. *The function* σ *only depends on* $M = \max\limits_{a_2(0)<x<b_2(0)} u(x,0)$ *and* T.

Proof. Let $w_0(r) \in C^\infty((0,+\infty))$, $w_0'(r) \ge 0$ and

$$x = w_0(r) = \begin{cases} 0, & 0 \le r < M+1 \\ 1, & r > M+2 \end{cases}.$$

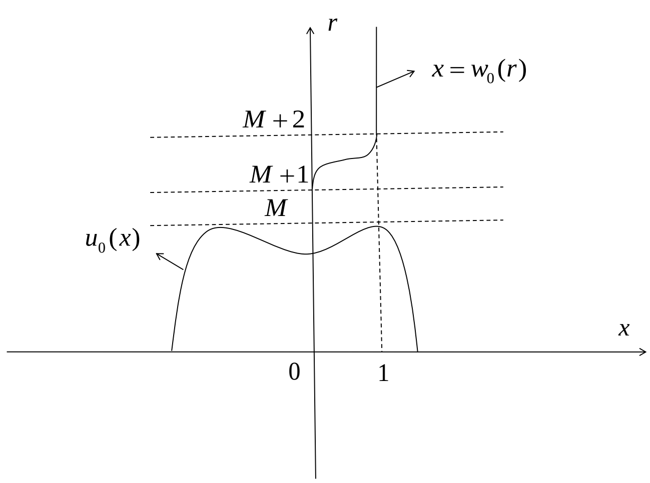

Figure 2.3: Proof of Theorem 2.3.4

We let w be the unique solution of the vertical equation (2.3.3) with the boundary condition

$$w_r(0,t) = 0,\ t \geq 0$$

and initial condition

$$w(r,0) = w_0(r),\ r \geq 0.$$

Differentiating (2.3.3) in r,

$$p_t = a^*(r,t)p_{rr} + b^*(r,t)p_r + c^*(r,t)p, \tag{2.3.7}$$

where $p = w_r,\ a^*(r,t) = 1/(1+w_r^2),\ b^*(r,t) = -2w_r w_{rr}/(1+w_r^2)^2 + (n-1)/r - Aw_r/\sqrt{1+w_r^2},\ c^*(r,t) = -(n-1)/r^2$.

By the maximum principle, we have for all $r, t > 0,\ w_r > 0$ and $\sup\limits_{r\geq 0} w_r(r,t)$ is nonincreasing in time. It follows from classical estimate for parabolic equation that all derivative of w are uniformly bounded for $r, t \geq 0$.

We claim that for any δ satisfying $0 < \delta < M + AT$, there exists $A_{\delta,T} > 0$ such that $A_{\delta,T}$ decreases with respect to δ and

$$p(r,t) \geq e^{-\frac{A_{\delta,T}}{t}} \tag{2.3.8}$$

for $\delta \leq r \leq M + AT$.

We only prove the claim for δ small by constructing a subsolution. Let $\underline{p}$ be the solution of

$$\underline{p}_t = a^*(r,t)\underline{p}_{rr} + b^*(r,t)\underline{p}_r + c^*(r,t)\underline{p},\ r > \delta/2,\ t > 0,$$

with boundary condition $\underline{p}(\delta/2,t) = 0$, $t \geq 0$ and initial data $\underline{p}(r,0) = w_0(r)$, $r \geq \delta/2$.

Since $|w_r| \leq \sup\limits_{r\geq 0} |w_0'(r)|$, $\sup\limits_{r\geq\delta/2,\ 0<t<T} |w_{rr}|$ depends only on δ and T. Seeing coefficients a^*, b^*, c^* depending on w_r, w_{rr} and r, a^*, b^*, c^* are all smooth and bounded for some constant depending on δ and T, for $r \geq \delta/2$, $0 < t < T$. Using the property of Green's function in half space, we can get the estimate (2.3.8) for $\underline{p}$ (seeing [4]).

Noting $p(r,0) = \underline{p}(r,0)$, $r \geq \delta/2$ and $p(\delta/2,t) > 0 = \underline{p}(\delta/2,t)$, then there holds that $p \geq \underline{p}$, for $r \geq \delta/2$, $t > 0$. Here we complete the proof of the claim.

Since $p(r,t) > 0$ for $r > 0$, the inverse of $x = w(r,t)$ exists, denoted by $r = v(x,t)$. Seeing the normal velocity of $x = w(r,t)$ is leftward, the normal velocity of $r = v(x,t)$ is upward. Therefore, $v(x,t)$ satisfies the horizontal graph equation (2.3.1) with the free boundary condition

$$v(a(t),t) = 0,\ v_x(a(t),t) = \infty,\ \lim_{x\to b(t)} v(x,t) = \infty,\ \lim_{x\to b(t)} v_x(x,t) = \infty,\ t > 0.$$

Let $\Sigma(t) = \{(x,y) \in \mathbb{R}^{n+1} \mid r = v(x,t), a(t) \leq x < b(t)\}$ and $\Sigma_\xi(t)$ denote the translation of $\Sigma(t)$ given by

$$x = w(r,t) + \xi.$$

$\Sigma_\xi(t)$ can be also represented by $r = v(x-\xi,t)$, $a(t)+\xi \leq x < b(t)+\xi$. Let $a_1(t)$ and $b_1(t)$ be the end point of $\Sigma_\xi(t)$, then $a_1(t) = \xi + a(t)$, $b_1(t) = \xi + b(t)$. Obviously, for $(x_0,t_0) \in (a_2(t_0), b_2(t_0)) \times (0,T)$, there exists $\xi \in \mathbb{R}$ such that

$$v(x_0-\xi,t_0) = u(x_0,t_0).$$

By the following Lemma 2.3.5, we can deduce that the graph of $u(x,t_0)$ intersects $v(x-\xi,t_0)$ only once.

Next we claim

$$v_x(x_0-\xi,t_0) \geq u_x(x_0,t_0).$$

If not, $v_x(x_0-\xi,t_0) < u_x(x_0,t_0)$, then there exists $\delta > 0$, such that

$$u(x,t_0) > v(x-\xi,t_0),$$

for all $x \in (x_0, x_0 + \delta)$. Since $\lim\limits_{x \to b_1(t)} v(x, t_0) = +\infty$ and $\max\limits_{a_2(t_0) \le x \le b_2(t_0)} u(x, t_0) < \infty$, $\Sigma_\xi(t_0)$ intersects $\Gamma(t_0)$ at least twice. This yields a contradiction.

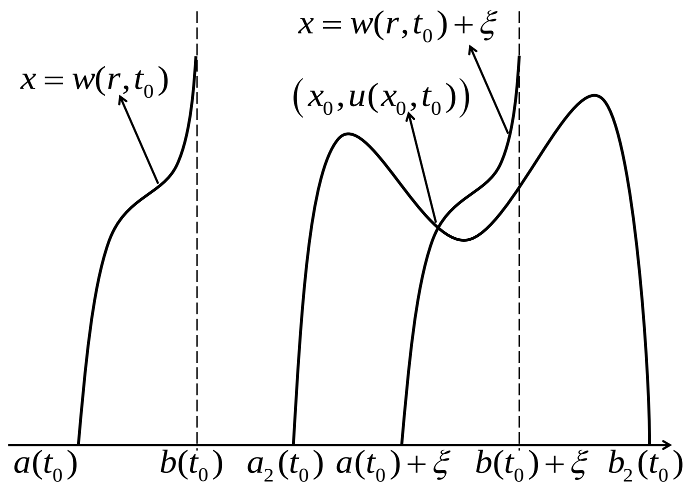

Figure 2.4: Proof of Theorem 2.3.4

By maximum principle, it is easy to see $r = u(x, t) < M + At < M + AT$, $a_2(t) \le x \le b_2(t)$, $0 < t < T$. Combining (2.3.8), there holds

$$u_x(x_0, t_0) \le \frac{1}{w_r(v(x_0 - \xi, t_0), t_0)} \le e^{\frac{A_{v(x_0 - \xi, t_0), T}}{t_0}} = e^{\frac{A_{u(x_0, t_0), T}}{t_0}} := \sigma(t_0, u(x_0, t_0)).$$

By considering the reflection $\widetilde{\Sigma}(0) = \{(x, y) \mid x = -w_0(r)\}$ and the equation (2.3.2) with $w_r(0, t) = 0,\ t \ge 0$ and $w(r, 0) = w_0(r),\ r \ge 0$, the bound for $-u_x(x_0, t_0)$ can be got similarly. $\square$

Lemma 2.3.5. *$\Sigma_\xi(t)$ and $\Gamma(t)$ is given in the proof of Theorem 2.3.4, then $\Sigma_\xi(t)$ intersects $\Gamma(t)$ at most once.*

Proof. By the same argument as Theorem 2.3.3, the intersection number between $\Sigma_\xi(t)$ and $\Gamma(t)$ is not increasing provided that $a_1(t)$, $b_1(t)$, $a_2(t)$ and $b_2(t)$ are all different and that their order remains unchanged. So we only prove this result when the order of $a_1(t)$, $b_1(t)$, $a_2(t)$ and $b_2(t)$ changes.

Case 1. Assume $a_1(t) < a_2(t) < b_1(t) < b_2(t)$, $t < t_2$ and $a_1(t) < a_2(t) < b_2(t) < b_1(t)$, $t > t_2$. And for $t < t_2$, $\Sigma_\xi(t)$ does not intersect $\Gamma(t)$. Then $\Sigma_\xi(t)$ does not intersect $\Gamma(t)$, for $t > t_2$.

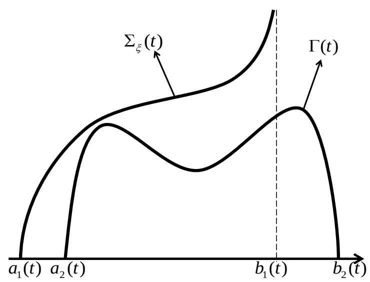

Figure 2.5: Case 1

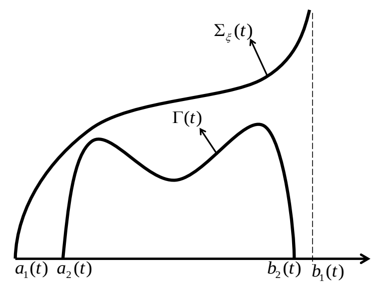

Figure 2.6: Case 1

Since $\lim_{x\to b_1(t_2)} v(x-\xi, t_2) = +\infty$ and $u(b_2(t_2), t_2) = 0$, there exists a positive δ independent of t, such that $v(x-\xi, t_2) > u(x, t_2)$, $b_1(t_2) - \delta < x < b_1(t_2)$. By continuity, there exists ϵ such that

$$v(b_1(t_2) - \delta - \xi, t) > u(b_1(t_2) - \delta, t),\ t_2 - \epsilon \le t < t_2 + \epsilon \tag{2.3.9}$$

and

$$v(x-\xi, t) > u(x, t),\ b_1(t_2) - \delta < x \le b_2(t),\ t_2 \le t < t_2 + \epsilon. \tag{2.3.10}$$

The assumptions in this case imply boundary condition

$$u(a_2(t), t) - v(a_2(t) - \xi, t) < 0,\ t_2 - \epsilon \le t < t_2 + \epsilon$$

and initial condition

$$u(x, t_2 - \epsilon) < v(x - \xi, t_2 - \epsilon),\ a_2(t_2 - \epsilon) \le x \le b_1(t_2) - \delta.$$

Combining the other boundary condition (2.3.9), using maximum principle in domain

$$\cup_{t_2-\epsilon\le t<t_2+\epsilon}\left([a_2(t), b_1(t_2)-\delta]\times\{t\}\right),$$

there holds

$$u(x,t) < v(x-\xi,t),\ a_2(t)\le x\le b_1(t_2)-\delta,\ t_2-\epsilon\le t<t_2+\epsilon.$$

Seeing (2.3.10), $u(x,t) < v(x-\xi,t)$, $a_2(t)\le x\le b_2(t)$, $t_2\le t<t_2+\epsilon$. It means that $\Sigma_\xi(t)$ does not intersect $\Gamma(t)$, for $t_2\le t<t_2+\epsilon$. So by Theorem 2.3.3, $\Sigma_\xi(t)$ does not intersect $\Gamma(t)$, for $t>t_2$.

Case 2. Assume $a_1(t)<a_2(t)$, $\Sigma_\xi(t)$ does not intersect $\Gamma(t)$, $t<t_3$ and $a_1(t_3)=a_2(t_3)$.

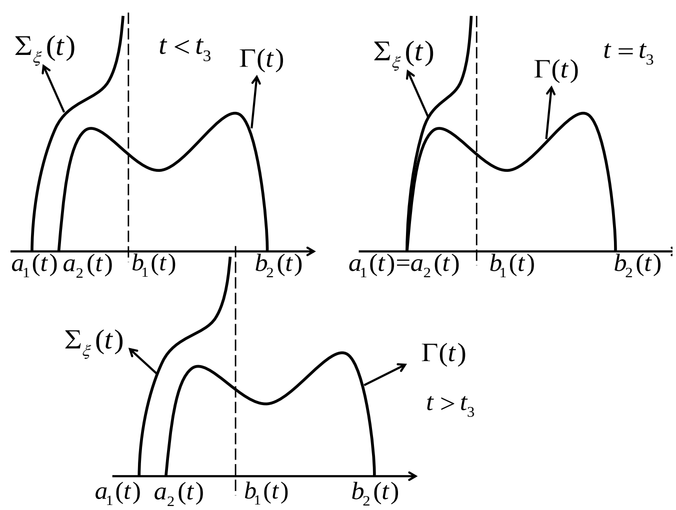

Figure 2.7: Case 2

Since $\lim\limits_{x\to a_2(t)} u_x(x,t)=\infty$, there exist δ_1 and ϵ such that $r=u(x,t)$ can be expressed as $x=h(r,t)$, $0\le r\le\delta_1$, $t_3-\epsilon<t<t_3+\epsilon$. The assumptions in this

case imply that

$$w(\delta_1, t) + \xi < h(\delta_1, t),\ t_3 - \epsilon < t < t_3 + \epsilon.$$

It is easy to see $w(r, t) + \xi$ and $h(r, t)$ satisfy the vertical graph equation

$$\begin{cases} w_t = \dfrac{w_{rr}}{1 + w_r^2} + \dfrac{n-1}{r} w_r - A\sqrt{1 + w_r^2},\ 0 \le r \le \delta_1,\ t_3 - \epsilon < t < t_3 + \epsilon, \\ w_r(0, t) = 0,\ t \ge 0, \end{cases}$$

and $w(r, t_3 - \epsilon) + \xi < h(r, t_3 - \epsilon)$. By strong maximum principle, $w(r, t) + \xi < h(r, t)$, for $0 \le r < \delta_1,\ t_3 - \epsilon < t < t_3 + \epsilon$. Contradiction to $a_1(t_3) = a_2(t_3)$. It means that this case does not happen.

Case 3. Assume $a_2(t) < b_2(t) < a_1(t) < b_1(t)$, $t < t_6$ and $a_2(t) < a_1(t) < b_2(t) < b_1(t)$, $t > t_6$.

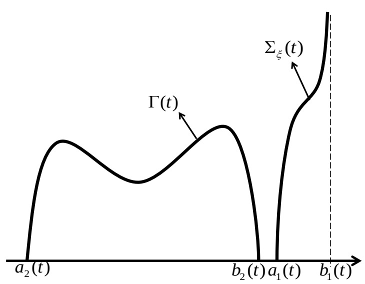

Figure 2.8: Case 3

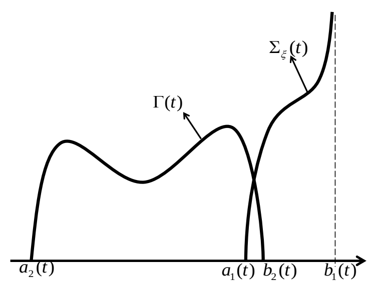

Figure 2.9: Case 3

Obviously, $\Sigma_\xi(t)$ dosen't intersect $\Gamma(t)$, $t < t_6$. Noting $\lim\limits_{x \to b_2(t)} u_x(x, t) = -\infty$ and $\lim\limits_{x \to a_1(t)} v_x(x - \xi, t) = \infty$, there exists ϵ such that

$$u_x(x, t) - v_x(x - \xi, t) < 0,\ a_1(t) \le x \le b_2(t),\ t_6 < t < t_6 + \epsilon.$$

Seeing $u(a_1(t), t) - v(a_1(t) - \xi, t) > 0$ and $u(b_2(t), t) - v(b_2(t) - \xi, t) < 0$, $u(x, t)$ intersects $v(x - \xi, t)$ only once in $[a_1(t), b_2(t)]$, $t_6 < t < t_6 + \epsilon$. Consequently, $\Sigma_\xi(t)$ intersects $\Gamma(t)$ only once, $t_6 < t < t_6 + \epsilon$. So by Theorem 2.3.3 we have $\Sigma_\xi(t)$

intersects $\Gamma(t)$ only once, $t > t_6$.

The other conditions can be investigated similarly as the three cases above. We note that the intersection number increases only in Case 3.

Then we can conclude that

1. if $a_1(0) < a_2(0)$, $\Sigma_\xi(t)$ does not intersect $\Gamma(t)$.
2. if $a_2(0) < a_1(0) < b_2(0)$, $\Sigma_\xi(t)$ intersects $\Gamma(t)$ at most once.
3. if $b_2(0) < a_1(0)$, $\Sigma_\xi(t)$ intersects $\Gamma(t)$ at most once.(Only in this case, the intersection number may increase)

We complete the proof. □

Remark 2.3.6. The intersection number between two closed, compact, embedded, rotationally symmetric hypersurfaces $\Gamma_1(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}^n \mid r = u_1(x,t), a_1(t) \leq x \leq b_1(t)\}$, $\Gamma_2(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}^n \mid r = u_2(x,t), a_2(t) \leq x \leq b_2(t)\}$ is denoted by $\mathcal{Z}(t) := \mathcal{Z}[\Gamma_1(t), \Gamma_2(t)]$. If $\Gamma_i(t)$ evolve by $V = -\kappa + A$ in $\mathbb{R}^{n+1}$, using Theorem 2.3.3 and the same methods in Lemma 2.3.5, we can similarly prove

(a). If $\mathcal{Z}(t) > 0$, $0 \leq t < t_0$, then $\mathcal{Z}(t)$ does not increase for $0 \leq t < t_0$.

(b). If $\mathcal{Z}(t_0) = 0$, then $\mathcal{Z}(t) \leq 1$, $t_0 < t < T$.

In this remark, observing the proof of Case 3 in Lemma 2.3.5, it also holds that the intersection number possibly increases once in the cases $a_1(0) > b_2(0)$ or $a_2(0) > b_1(0)$. The results in this remark can be proved similarly as Lemma 2.3.5.

Using the opinion in this remark similarly, since $\mathcal{Z}(0) \leq 1$ in Lemma 2.3.5, there holds $\mathcal{Z}(t) \leq 1$ for $0 < t < T$.

By the arguments of intersections, we can prove the following theorem.

Theorem 2.3.7. *Let $\Gamma(t)$, $t \in [0,T)$, be a family of smooth hypersurfaces evolving by $V = -\kappa + A$ in $\mathbb{R}^{n+1}$. If $\Gamma(0)$ is obtained by rotating the graph of a function around the x-axis, then so are the $\Gamma(t)$ for $t \in [0,T)$.*

For the proof of Theorem 2.3.7, we see that $\Gamma(t)$ is also rotationally symmetric because the equation is rotationally invariance. Since $\Gamma(0)$ is obtained by rotating the graph of a function around the x-axis, $\Gamma(0)$ can be written into $\Gamma(0) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid r = v_0(x)\}$ for some function $v_0(x)$. It means that all straight vertical plane $x = c$ intersects $\Gamma(0)$ at most once. Using the same argument in Lemma 2.3.5, we can prove that every vertical plane $x = c$ intersects $\Gamma(t)$ at most once. Then $\Gamma(t)$ can be also written into $\{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid r = u(x, t)\}$. We omit the details.

In our problem, seeing future, the hypersurface evolving by $V = -\kappa + A$ maybe intersect itself at x-axis. To conquer this difficulty, we refer to the definition of α-domain in [1].

Definition 2.3.8. We say a domain U is an α-domain if

(1). $U \subset \mathbb{R}^{n+1}$ is an open set of the form

$$U = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid r < u(x)\}.$$

(2). $I = \{x \in \mathbb{R} \mid u(x) > 0\}$ is a bounded, connected interval. Let the endpoints of I be $a_1 < a_2$.

(3). u is smooth on I;

(4). ∂U intersects each cylinder ∂C_ρ with $0 < \rho \leq \alpha$ twice and these intersections are transverse, where $C_\rho = \{(x, y) \in \mathbb{R}^{n+1} \mid r < \rho\}$.

We observe that the boundary ∂U of an α-domain U does not intersect itself at $y = 0$. The condition (3) implies ∂U is a smooth curve, except possibly at its endpoints $(a_1, \mathbf{0}), (a_2, \mathbf{0})$. The condition (4) implies that there exist $\delta_1, \delta_2 > 0$ such that

$$u(a_1 + \delta_1) = u(a_2 - \delta_2) = \alpha,$$

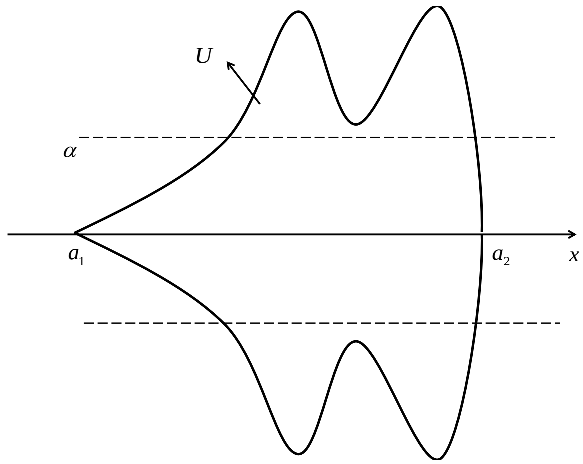

Figure 2.10: α-domain

and

$$u'(x) = \begin{cases} > 0, \ x \in (a_1, a_1 + \delta_1], \\ < 0, \ x \in [a_2 - \delta_2, a_2). \end{cases}$$

Therefore, the inverse of $u|_{[a_1,a_1+\delta_1]}$ and $u|_{[a_2-\delta_2,a_2]}$ exist, denoted by v_1, $v_2 : [0, \alpha] \to \mathbb{R}$. By the implicit function theorem, they are smooth in $(0, \alpha]$. Moreover, $v_1'(r) > 0$, $v_2'(r) < 0$, $(0 < r \leq \alpha)$ and

$$\partial U \cap C_\alpha = \{(x, y) \in \mathbb{R}^{n+1} \mid 0 \leq r \leq \alpha, \ x = v_i(r), \ i = 1, 2\}.$$

The two components of $\partial U \cap C_\alpha$ are called the left and right caps of ∂U.

If U is an α-domain, by Theorem 6.0.3, there exists T_U and $D(t)$ be the open evolution with $D(0) = U$ such that $\partial D(t)$ is smooth and $D(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| < u(x, t), a(t) < x < b(t)\}$ for $0 < t < T_U$. Moreover, (u, a, b) satisfies (2.3.1), (2.3.4), (2.3.5).

Next we prove Lemma 2.3.9 and 2.3.10 by the arguments of intersection number. Following Lemma 2.3.9 and 2.3.10 show the open evolution starting as α-domain does not intersect itself at x-axis in short time.

Lemma 2.3.9. *For $n = 1$, let U be an α-domain. Then there exists $t_U^\alpha > 0$ such that $D(t)$ denoted the open evolution with $D(0) = U$ is an $(\alpha + At)$-domain, $0 < t < \min\{t_U^\alpha, T_U\}$.*

Lemma 2.3.10. *For $n \geq 2$, there exists $t_U^\alpha > 0$ such that $D(t)$ is an $\alpha(t)$-domain for all $0 < t < \min\{t_U^\alpha, T_U\}$, where $\alpha(t)$ is the solution of the following equation*

$$\alpha'(t) = A - \frac{n-1}{\alpha(t)} \tag{2.3.11}$$

with initial data $\alpha(0) = \alpha$.

Using Proposition 2.3.1, we prove Lemma 2.3.9.

Proof of Lemma 2.3.9. Since U is an α-domain, using Theorem 6.0.3, there exists T_U such that $\partial D(t)$ is smooth and $\partial D(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| = u(x, t), a(t) \leq x \leq b(t)\}$ for $0 < t < T_U$. Here (u, a, b) satisfies

$$u_t = \frac{u_{xx}}{1 + u_x^2} + A\sqrt{1 + u_x^2}, \tag{(2.3.1*)}$$

with (2.3.4), (2.3.6).

Since U is not contained in the cylinder $\overline{C_\alpha}$, there exists a small ball $B_\epsilon(P) \subset U \setminus \overline{C_\alpha}$. By (1) in Theorem 2.1.4, there holds $B_{\epsilon(t)}(P) \subset D(t)$ for $0 < t < \delta_1$. Here $\epsilon(t)$ satisfies

$$\epsilon'(t) = A - \frac{1}{\epsilon(t)},\ 0 < t < \delta_1, \tag{2.3.12}$$

with $\epsilon(0) = \epsilon$. Since $B_\epsilon(P) \cap \overline{C_\alpha} = \emptyset$, by (1b) in Theorem 2.1.6, there exists $t_1 > 0$, such that $B_{\epsilon(t)}(P) \cap \overline{C_{\alpha+At}} = \emptyset$, $0 < t < t_1$. Then

$$B_{\epsilon(t)}(P) \subset D(t) \setminus \overline{C_{\alpha+At}},\ 0 < t < t_1.$$

This implies for all $0 < t_0 < t_U^\alpha$, $\rho < \alpha + At_0$, $y = \rho$ intersects $y = u(x, t_0)$ at least twice. Here $t_U^\alpha = \min\{\delta_1, t_1\}$.

On the other hand, for $0 < \rho < \alpha + At_0$, $0 < t_0 < T_U$, $y = \rho - At_0$ intersects $\gamma^1(0)$ exactly twice ($\gamma^1(t)$ is constructed in Proposition 2.3.1). Proposition 2.3.1 shows that $y = \rho$ intersects $\gamma^1(t_0)$ at most twice.

Therefore ∂C_ρ intersects $\partial D(t_0)$ exactly twice, $0 < t_0 < \min\{t_U^\alpha, T_U\}$.

Choosing $t_U = \min\{t_U^\alpha, T_U\}$, $D(t)$ is an $(\alpha + At)$-domain, $0 < t < t_U$. The proof is completed. □

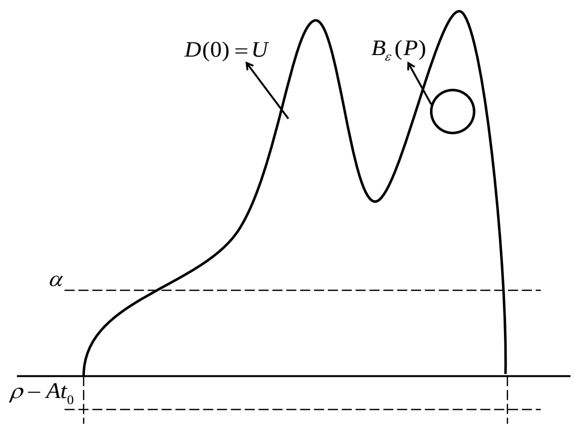

Figure 2.11: Proof of Lemma 2.3.9

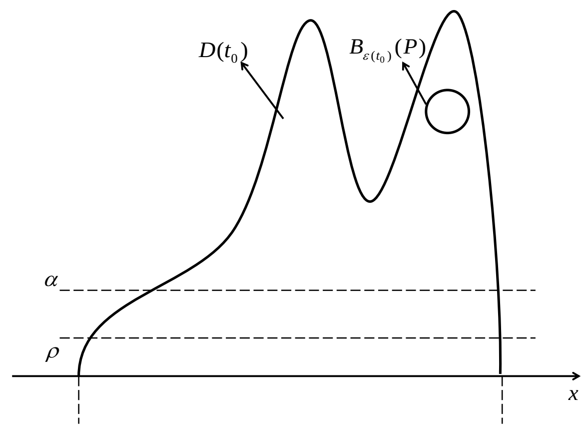

Figure 2.12: Proof of Lemma 2.3.9

We continue to prove Lemma 2.3.10. We note for the problem,

$$\begin{cases} \alpha'(t) = A - \dfrac{n-1}{\alpha(t)}, \ t > 0, \\ \alpha(0) = \alpha, \end{cases}$$

the following results hold.

(1) when $\alpha < (n-1)/A$, there exists $T_\alpha < \infty$ such that $\alpha(t) \downarrow 0$ as $t \to T_\alpha$ and $\lim_{t \to -\infty} \alpha(t) = (n-1)/A$;

(2) when $\alpha = (n-1)/A$, $\alpha(t) = (n-1)/A$ for $0 \le t < \infty$;

(3) when $\alpha > (n-1)/A$, $\alpha(t) \uparrow \infty$ as $t \to \infty$ and $\lim_{t \to -\infty} \alpha(t) = (n-1)/A$.

Proof of Lemma 2.3.10. Since U is not contained in the cylinder $\overline{C}_\alpha$, there is a small ball $B_\epsilon(P) \subset U \setminus \overline{C}_\alpha$. By (1) in Theorem 2.1.4, $B_{\epsilon(t)}(P) \subset D(t)$ for $0 < t < \delta_1$. Here $\epsilon(t)$ is the solution of the following equation

$$\epsilon'(t) = -A - \frac{n}{\epsilon(t)},\ 0 < t < \delta_1 \tag{2.3.13}$$

with initial data $\epsilon(0) = \epsilon$ and δ_1 is the maximal time for the solution existing.

Let δ_2 be the maximal existence time of the solution for equation (2.3.11) with initial data $\alpha(0) = \alpha$.

Theorem 2.1.7 implies $B_{\epsilon(t)}(P) \cap \overline{C}_{\alpha(t)} = \emptyset$ for $0 < t < t_U^\alpha$. Here

$$t_U^\alpha = \min\{\delta_1, \delta_2\}.$$

Therefore,

$$B_{\epsilon(t)}(P) \subset D(t) \setminus \overline{C_{\alpha(t)}},\ 0 < t < t_U^\alpha.$$

Then for all $0 < \rho < \alpha(t)$, ∂C_ρ must intersect $\partial D(t)$ at least twice for $0 < t < t_U^\alpha$.

On the other hand, Fix $0 < \rho < \alpha(t_0)$ and $0 < t_0 < \min\{t_U^\alpha, T_U\}$, and let $\rho(t)$ be the solution of the equation (2.3.11) with initial data $\rho(0) = \rho$. Next we prove that ∂C_ρ intersects $\partial D(t_0)$ at most twice.

By comparison principle for ordinary differential equation, we have $\rho(-t_0) < \alpha$. This means $y = \rho(-t_0)$ intersects $y = u(x, 0)$ only twice. Noting $\rho(t - t_0) > 0$ for $t \ge 0$,

$$y = \rho(t - t_0) > u(a(t), t) = 0,\ y = \rho(t - t_0) > u(b(t), t) = 0 \text{ for } t > 0.$$

By Theorem D in [2], $\mathcal{Z}_{[a(t),b(t)]}(\rho(t-t_0), u(\cdot,t))$ is not increasing for $t > 0$. Then

$$\mathcal{Z}_{[a(t),b(t)]}(\rho(t-t_0), u(\cdot,t)) \leq \mathcal{Z}_{[a(0),b(0)]}(\rho(-t_0), u(\cdot,0)) = 2.$$

Therefore ∂C_ρ intersects $\partial D(t_0)$ exactly twice, for $0 < t < \min\{t_U^\alpha, T_U\}$. Consequently, $D(t)$ is an $\alpha(t)$-domain for $0 < t < \min\{t_U^\alpha, T_U\}$. □

Proposition 2.3.11. *For t_U^α and T_U given in Lemma 2.3.9 and 2.3.10, there holds $t_U^\alpha \leq T_U$.*

To prove this proposition, we need the following lemma.

Lemma 2.3.12. *Assume that $D(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| < u(x,t), a(t) \leq x \leq b(t)\}$ is a ρ-domain for $0 < t < T$. Let $w_1 < w_2$ such that*

$$C_\rho \cap \partial D(t) = \{(x,y) \mid x = w_1(y,t) \text{ or } x = w_2(y,t)\}.$$

Then

$$\lim_{t\to T} w_1(y,t) = w_1(y,T) \qquad \text{and} \qquad \lim_{t\to T} w_2(y,t) = w_2(y,T)$$

exist and these convergences are uniformly convergent for $|y| \leq \frac{\rho}{2}$. Moreover, $v_1(r_1,T) < v_1(r_2,T)$ and $v_2(r_1,T) > v_2(r_2,T)$ for $0 < r_1 < r_2 < \frac{\rho}{2}$, where $v_1(r,t) = w_1(y,t)$ and $v_2(r,t) = w_2(y,t)$.

Proof. $w_1(y,t)$ and $w_2(y,t)$ satisfy the equation (2.2.1), respectively for "$\mp$". We only prove the result for $w_1(y,t)$. Since w_1 is uniformly bounded, Corollary 2.2.3 and Remark 2.2.4 imply that derivatives $\nabla_y^j w_1$, $j = 1, 2$, are uniformly bounded for $0 \leq |y| \leq \frac{\rho}{2}$, $\frac{T}{2} \leq t < T$. Consequently, $\frac{\partial w_1}{\partial t}$ is bounded for $0 \leq |y| \leq \frac{\rho}{2}$, $\frac{T}{2} \leq t < T$. So there exists $w_1(y,T)$ such that $w_1(y,t)$ converges to $w_1(y,T)$ uniformly for $0 \leq |y| \leq \frac{\rho}{2}$, as $t \to T$.

Note that the following hold

$$\frac{\partial v_1}{\partial r}(\frac{\rho}{2},t)>0,\ \frac{\partial v_1}{\partial r}(0,t)=0 \text{ for } 0<t<T$$

and

$$\frac{\partial v_1}{\partial r}(r,0)>0 \text{ for } 0<r<\frac{\rho}{2}.$$

In fact, differentiating (2.3.3) in r,

$$p_t = a(r,t)p_{rr} + b(r,t)p_r + c(r,t)p, \tag{2.3.7}$$

where $p = w_r$, $a(r,t) = 1/(1+w_r^2)$, $b(r,t) = -2w_r w_{rr}/(1+w_r^2)^2 + (n-1)/r - Aw_r/\sqrt{1+w_r^2}$, $c(r,t) = -(n-1)/r^2$. Strong maximum principle implies

$$\frac{\partial v_1}{\partial r}>0 \text{ for } 0<r<\frac{\rho}{2},\ 0<t\le T.$$

Therefore $v_1(r_1,T) < v_1(r_2,T)$ for $0<r_1<r_2<\frac{\rho}{2}$. Similarly, we can prove the conclusion for v_2. □

Proof of Proposition 2.3.11. If $T_U < t_U^\alpha$. By Lemma 2.3.10, there exists $\rho>0$ such that $D(t)$ is a ρ-domain for $0<t<T_U$.

We divide $\partial D(t)$ into two parts: $\partial D(t) = (\partial D(t)\cap\{r<\rho/2\})\cup(\partial D(t)\cap\{r\ge\rho/2\})$.

Step 1. $\partial D(t)\cap\{r<\rho/2\}$

Since $\partial D(t)$ is a ρ-domain, there exist $w_1<w_2$ such that $\partial D(t)\cap\{r<\rho\} = \{(x,y)\mid x=w_1(y,t),|y|<\rho\}\cup\{(x,y)\mid x=w_2(y,t),|y|<\rho\}$. By the same argument as in Lemma 2.3.12, $\nabla_y^j w_i$, $j=1,2$, $i=1,2$, are uniformly bounded for $0\le|y|\le\frac{\rho}{2}$, $\frac{T_U}{2}\le t<T_U$. Therefore, the mean curvature of $\partial D(t)\cap\{r<\rho/2\}$ is uniformly bounded for $\frac{T_U}{2}\le t<T_U$.

Step 2. $\partial D(t)\cap\{r\ge\rho/2\}$

Recalling $\partial D(t) = \{(x, y) \mid |y| = u(x, t), a(t) \le x \le b(t)\}$, by Lemma 2.3.12, $v_1(\rho/4, T_U) < v_1(\rho/2, T_U)$ and $v_2(\rho/4, T_U) > v_2(\rho/2, T_U)$. Then for any sufficiently small $\epsilon > 0$ for all t close to T_U there holds

$$[v_1(\rho/2, t), v_2(\rho/2, t)] \subset (v_1(\rho/4, t) + \epsilon, v_2(\rho/4, t) - \epsilon). \tag{2.3.14}$$

Theorem 2.3.4 shows that u_x is bounded for $|y| \ge \rho/4$. i.e. u_x is bounded in $[v_1(\rho/4, t), v_2(\rho/4, t)]$, t close to T_U. Remark 2.2.4 implies that u_{xx} is uniformly bounded in $(v_1(\rho/4, t) + \epsilon, v_2(\rho/4, t) - \epsilon)$, t close to T_U.

Therefore, (2.3.14) shows that u_x and u_{xx} are uniformly bounded for $x \in [v_1(\rho/2, t), v_2(\rho/2, t)]$, t close to T_U. Consequently, the curvature of $\partial D(t) \cap \{r \ge \rho/2\}$ is bounded for t close to T_U. Here we show that the curvature of $\partial D(t)$ is uniformly bounded as $t \uparrow T_U$. It contradicts that $\partial D(t)$ become singular at T_U. □

Remark 2.3.13. (1). In Lemma 2.3.10, $0 < t < \min\{t_U^\alpha, T_U\}$ is equivalent to $0 < t < t_U^\alpha$.

(2). Seeing the choice of t_U^α, $t_U^\alpha \le t_W^\alpha$, if $U \subset W$.

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Chapter 3

The criteria for fattening and non-fattening phenomenon in the plane

In this chapter, we want to introduce the results in [8]. In [8], we consider a family of axisymmetric curves evolving by its curvature with driving force in the plane. However, the initial curve is oriented singularly at origin. We investigate this problem by level set method and give some criteria to judge whether the interface evolution is fattening or not.

3.1 Introduction

Curvature flow with driving force, precisely, is given by

$$V = -\kappa + A \ \text{ on } \Gamma(t) \subset \mathbb{R}^2, \tag{1.1.1}$$

$$\Gamma(0) = \Gamma_0, \tag{1.1.2}$$

where V is the outer normal velocity of $\Gamma(t)$, κ is the curvature of $\Gamma(t)$ and the sign is chosen such that the problem is parabolic. The constant A called driving force is positive.

In this research, the initial curve Γ_0 is symmetric to x and y-axis. Let $\Lambda_0 = \Gamma_0 \cap \{x \geq 0\}$ be closed, smooth, embedded given by

$$\Lambda_0 = \{(x, y) \in \mathbb{R}^2 \mid |y| = u_0(x), 0 \leq x \leq b_0\}.$$

Here u_0 is even and $u_0 \in C[-b_0, b_0] \cap C^\infty\big((-b_0, 0) \cup (0, b_0)\big)$ for $b_0 > 0$. Consequently,

$$\Gamma_0 = \{(x, y) \in \mathbb{R}^2 \mid |y| = u_0(x), -b_0 \leq x \leq b_0\}$$

and Γ_0 is singular at origin (Figure 3.1).

By the assumption of Λ_0, there hold

$$u_0(x) > 0, \; 0 < x < b_0$$

and

$$u_0(0) = u_0(b_0) = 0, \; \lim_{x \to 0^+} u_0'(x) = -\lim_{x \to b_0^-} u_0'(x) = \infty.$$

Main assumptions. Before giving our main results, we first consider another problem.

$$V = -\kappa + A \text{ on } \Lambda^+(t) \subset \mathbb{R}^2, \tag{1.1.1*}$$

$$\Lambda^+(0) = \Lambda_0, \tag{1.1.2*}$$

(Figure 3.2). We consider this problem by level set method. As introduced in Chapter

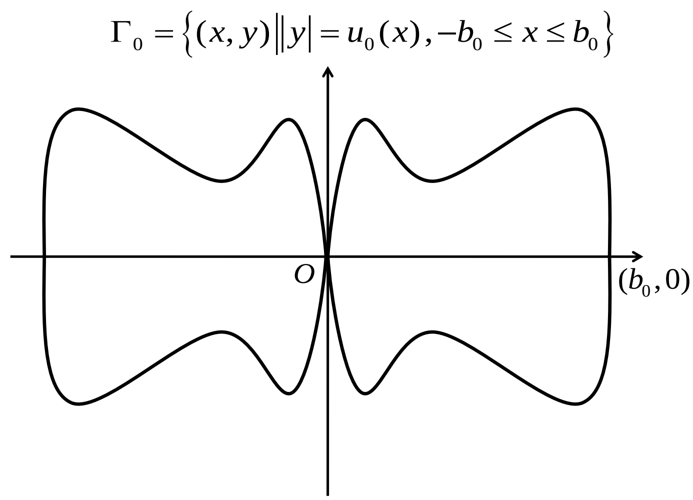

Figure 3.1: Initial curve Γ_0

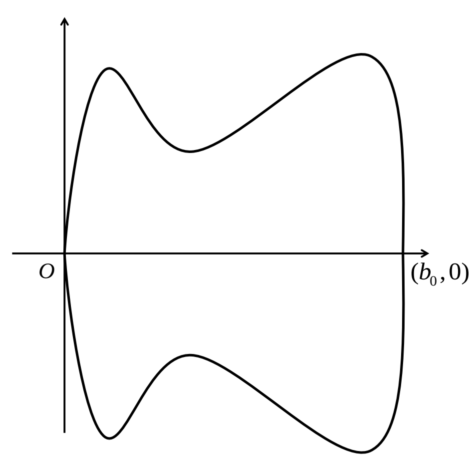

Figure 3.2: Initial curve Λ_0

2, there exists unique viscosity solution ϕ of the following level set equation

$$\begin{cases} \phi_t = |\nabla\phi|\mathrm{div}(\dfrac{\nabla\phi}{|\nabla\phi|}) + A|\nabla\phi| \text{ in } \mathbb{R}^2 \times (0,T), \\ \phi(x,y,0) = a_1(x,y), \end{cases}$$

where $a_1(x,y)$ is chosen such that $\Lambda_0 = \{(x,y) \mid a_1(x,y) = 0\}$ and $\{(x,y) \mid a_1(x,y) > 0\}$ is bounded. The results in appendix show that the zero set of ϕ is not fattening in a short time. Indeed, thanks to Theorem 2.3.7, the zero set of ϕ can be written into

$$\Lambda^+(t) = \{(x,y) \in \mathbb{R}^2 \mid \phi(x,y,t) = 0\} = \{(x,y) \in \mathbb{R}^2 \mid |y| = v(x,t), a_*(t) \le x \le b_*(t)\},$$

for $0 < t < T_*$. Moreover, (v, a_*, b_*) is the solution of the following free boundary problem

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2},\ x \in (a_*(t), b_*(t)),\ 0 < t < T_*, \\ u(a_*(t),t) = 0,\ u(b_*(t),t) = 0,\ 0 \le t < T_*, \\ u_x(a_*(t),t) = \infty,\ u_x(b_*(t),t) = -\infty,\ 0 \le t < T_*, \\ u(x,0) = u_0(x),\ 0 \le x \le b_0. \end{cases} \tag{*}$$

Here a_* and b_* are called the end points of $\Lambda^+(t)$.

Assumption $(A+)$: There exists $\delta > 0$ such that $a_*(t) \ge 0$ for $0 \le t < \delta$.

Assumption $(A-)$: There exists $\delta > 0$ such that $a_*(t) < 0$ for $0 < t < \delta$.

The assumptions $(A+)$ and $(A-)$ for $a_*(t)$ seem not to be understood easily. We explain the assumptions by giving some sufficient conditions.

The curvature of Λ_0 at origin is denoted by $\kappa(O)$. Noting that

$$\kappa(O) = -\lim_{x\to 0^+} u_0''/(1+(u_0')^2)^{3/2}$$

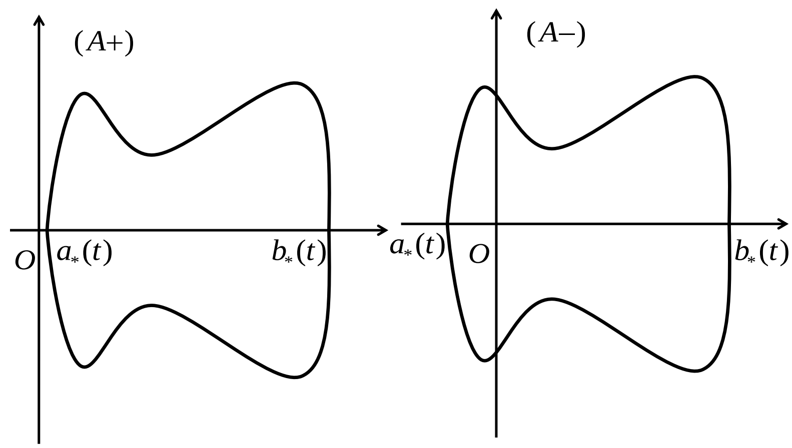

Figure 3.3: Main assumptions

and

$$a_*'(0) = \kappa(O) - A,$$

we can prove that

(a). if $\kappa(O) < A$, there holds $a_*(t) < 0$, for t small;

(b). if $\kappa(O) > A$, there holds $a_*(t) > 0$, for t small.

Here we give our main results.

Theorem 3.1.1. *Assuming the assumption* $(A-)$ *holds, then there exists* $T_1 > 0$ *such that the interface evolution* $\Gamma(t)$ *for (1.1.1) with initial curve* Γ_0 *is not fattening for* $0 \leq t < T_1$. *Moreover,* $\Gamma(t)$ *can be written into* $\Gamma(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| = u(x,t),\ -b(t) \leq x \leq b(t)\}$. *Here* (u, b) *is the unique solution of the following free* *boundary problem*

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2},\ -b(t) < x < b(t),\ 0 < t < T_1, \tag{3.1.1}$$

$$u(b(t),t) = u(-b(t),t) = 0,\ u_x(-b(t),t) = -u_x(b(t),t) = \infty,\ 0 < t < T_1, \tag{3.1.2}$$

$$u(x,0) = u_0(x),\ -b_0 \le x \le b_0. \tag{3.1.3}$$

Precisely, we say (u,b) is the solution of (3.1.1), (3.1.2) and (3.1.3), if

(1). $b(t)$ is a positive function and $b \in C([0,T_1)) \cap C^1((0,T_1))$.

(2). $u \in C(\overline{D}_{T_1}) \cap C^{2,1}(D_{T_1})$, where $\overline{D}_{T_1} = \cup_{0\le t<T_1}\big([-b(t),b(t)] \times \{t\}\big)$ and $D_{T_1} = \cup_{0<t<T_1}\big((-b(t),b(t)) \times \{t\}\big)$ (We must note that $\overline{D}_{T_1} \neq \overline{D_{T_1}}$).

(3). (u,b) satisfies (3.1.1), (3.1.2) and (3.1.3).

The definition of interface evolution and fattening are given in section 2.

Indeed, seeing future, the solution (u,b) given by Theorem 3.1.1 satisfies

$$u(x,t) > 0,\ -b(t) < x < b(t),\ 0 < t < T_1.$$

This impies that $\Gamma(t)$ is a family of embedded curves for $0 < t < T_1$. Let T be the maximal smooth time given by

$$T = \sup\{t \mid \Gamma(s) \text{ is smooth and embedding}, 0 < s < t\}.$$

We now give a sufficient result to have a fattening phenomenon.

Theorem 3.1.2. *(Fattening)*

Assuming the assumption $(A+)$ holds, the interface evolution $\Gamma(t)$ for (1.1.1) with initial data Γ_0 is fattening.

Theorem 3.1.1 and Theorem 3.1.2 can be explained by Figure 3.4 and 3.5. φ in

Figure 3.4 and 3.5 is the unique viscosity solution of

$$\begin{cases} \varphi_t = |\nabla\varphi|\mathrm{div}(\frac{\nabla\varphi}{|\nabla\varphi|}) + A|\nabla\varphi| \text{ in } \mathbb{R}^2 \times (0,T), \\ \varphi(x,y,0) = a_2(x,y), \end{cases}$$

where $a_2(x,y)$ satisfies $\Gamma_0 = \{(x,y) \mid a_2(x,y) = 0\}$ and $\Gamma_0 = \{(x,y) \mid a_2(x,y) > 0\}$ is bounded. Let $\Gamma(t) = \{(x,y) \mid \varphi(x,y,t) = 0\}$.

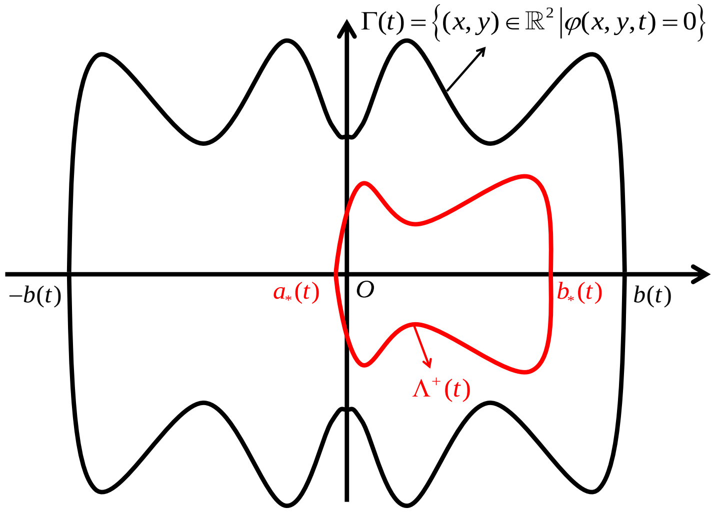

Figure 3.4: $a_*(t) < 0$ in Theorem 3.1.1

The role of $a_*(t)$ in main theorem. Let (u,b) be the unique solution of the free boundary problem

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \ -b(t) < x < b(t), \ 0 < t < T_1, \tag{3.1.1}$$

$$u(b(t),t) = u(-b(t),t) = 0, \ u_x(-b(t),t) = -u_x(b(t),t) = \infty, \ 0 < t < T_1, \tag{3.1.2}$$

$$u(x,0) = u_0(x), \ -b_0 \le x \le b_0. \tag{3.1.3}$$

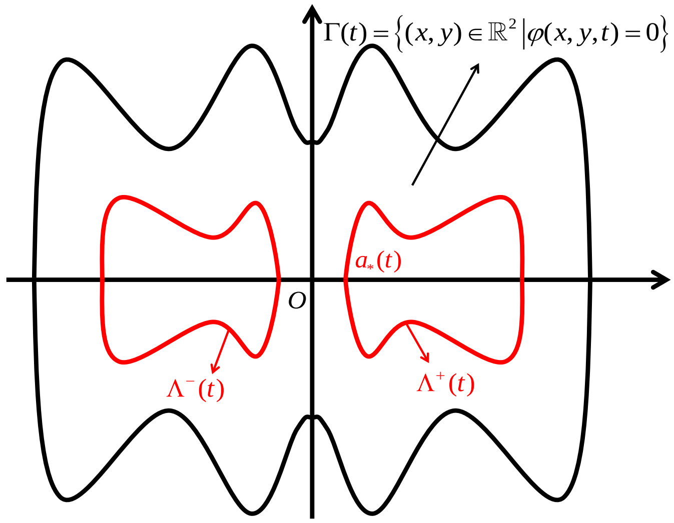

Figure 3.5: $a_*(t) \geq 0$ in Theorem 3.1.2

Obviously, the flow $\Gamma^*(t) = \{(x, y) \mid |y| = u(x, t),\ -b(t) \leq x \leq b(t)\}$ satisfies (1.1.1), (1.1.2) naturally.

On the other hand, let (v, a_*, b_*) be the solution of the problem (*). If $a_*(t) \geq 0$, $0 < t < \delta$, the flow

$$\Lambda^+(t) = \{(x, y) \mid |y| = v(x, t),\ a_*(t) \leq x \leq b_*(t)\}$$

does not intersect the flow

$$\Lambda^-(t) = \{(-x, y) \mid (x, y) \in \Lambda^+(t)\},$$

for $0 < t < \delta$. Denote $\Lambda(t) = \Lambda^+(t) \cup \Lambda^-(t)$. Obviously, $\Lambda(t)$ also satisfies (1.1.1). Seeing $\Gamma^*(0) = \Lambda(0) = \Gamma_0$, this means there exist two types of flows $\Gamma^*(t)$ and $\Lambda(t)$ evolving by $V = -\kappa + A$ with the same initial curve Γ_0. Therefore under this condition, the solution of the original problem (1.1.1), (1.1.2) is not unique. Indeed, seeing the proof of Theorem 3.1.2, the flow $\Gamma^*(t)$ is the boundary of closed evolution and the flow $\Lambda(t)$ is the boundary of open evolution.

If $a_*(t) < 0$, $0 < t < \delta$, $\Lambda^+(t) \cap \Lambda^-(t) \neq \emptyset$. Obviously, $\Lambda(t) = \Lambda^+(t) \cup \Lambda^-(t)$ does not satisfy (1.1.1). But $\Lambda^+(t)$ plays the role of a sub-solution (in the proof of Lemma 3.2.6). Using this sub-solution, the boundaries of the open evolution and closed evolution are away from the x-axis. Moreover, the derivatives and the second fundamental forms of them can be proved uniformly bounded. By the uniqueness result (Proposition 3.2.4), we can prove they are the same. Since the interface evolution $\Gamma(t)$ is imposed to be symmetric to the y-axis, $\Gamma(t)$ are perpendicular to the y-axis for $t > 0$, but this condition does not hold at time $t = 0$.

In our problem, if $A = 0$, since $a_*(t) \geq 0$ always holds, the interface evolution is fattening.

Motivation. This research is motivated by [7], the mean curvature flow with driving force under the Neumann boundary condition in a two-dimensional cylinder with periodically undulating boundary. In [7], they only consider the condition that for initial curve $\{(x, y) \in \mathbb{R}^2 \mid y = u_0(x)\}$ with $|u_0'(x)| < M$ for some M. They show that the interior point of $\Gamma(t) = \{(x, y) \in \mathbb{R}^2 \mid y = u(x, t)\}$ never touches the boundary and $\Gamma(t)$ remains graph. Therefore, the problem can be studied by the classical parabolic theory. If removing the assumption $|u_0'(x)| < M$, when $u(x, t)$ touches the boundary, the singularity will develop (Figure 3.6). Noting Figure 3.7, after touching, $\Gamma(t)$ possibly separates into two parts and become non-graph ($\Gamma(t)$ can't be represented by $y = u(x, t)$). This makes us analyze what will happen after touching boundary. Noting that $\Gamma(t)$ may become non-graph, we tend to use the level set method established by [2]; see also Evans and Spruck [3], [4], [5], for the mean curvature flow, where fattening phenomenon is first observed.

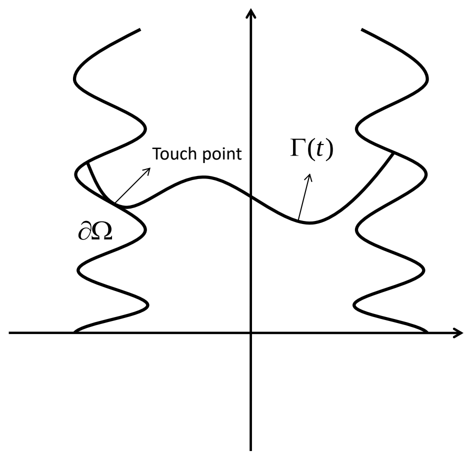

Figure 3.6: Curve touching

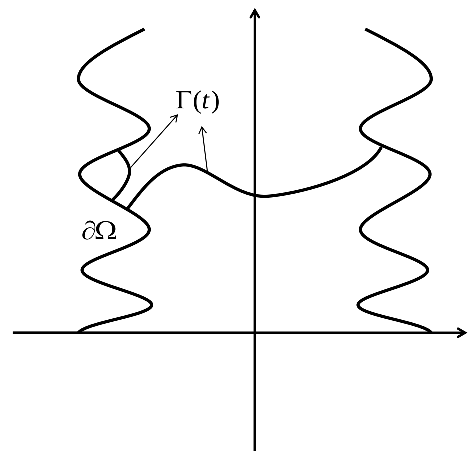

Figure 3.7: After touching

3.2 Identity for the outer evolution and inner evolution

In this section, we prove the fattening and non-fattening results. It is necessary to identify whether the outer evolution and inner evolution are identical.

Denote $U = \{(x, y) \in \mathbb{R}^2 \mid |y| < u_0(x), -b_0 \leq x \leq b_0\}$. By assumption of u_0 in Section 3.1, we know that $U \cap \{x \geq 0\}$ is an α-domain with smooth boundary, for some $\alpha > 0$.

We choose vector field $X \in C^1(\mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2)$ such that

(i) At any $P \in \partial U$ not on the x-axis has $\langle X, \mathbf{n}(P)\rangle > 0$, where $\mathbf{n}$ is inward unit normal vector at P.

(ii) We set $X((x, y)) = (0, -y/|y|)$, near $(0, 0)$ and set $X = (-1, 0)$ near $(b, 0)$, $X = (1, 0)$ near $(-b, 0)$.
We note that X has no definition at $(0, 0)$.

Since $X \neq 0$ on $\partial U \setminus \{(0,0)\}$, there exists a neighbourhood $V \supset \partial U$ such that $|X| \geq \delta > 0$ for some $\delta > 0$ in $V \setminus \{(0,0)\}$.

Proposition 3.2.1. *For ρ small enough, there exists a smooth curve $\Sigma \subset V \setminus \{(0,0)\}$ with*

(i) $X(P) \notin T_P\Sigma$ at all $P \in \Sigma$, i.e., Σ is transverse to the vector field X;

(ii) $\Sigma = \partial U$ in $\{(x,y) \mid |y| \geq 2\rho\}$;

(iii) $\Sigma \cap \{(x,y) \mid |y| \leq \rho\}$ consists of discs $\Delta_{\pm c} = \{(\pm c, y) \mid |y| \leq \rho\}$ and pipe $B_d = \{(x,y) \mid -d \leq x \leq d, |y| = \rho\}$.

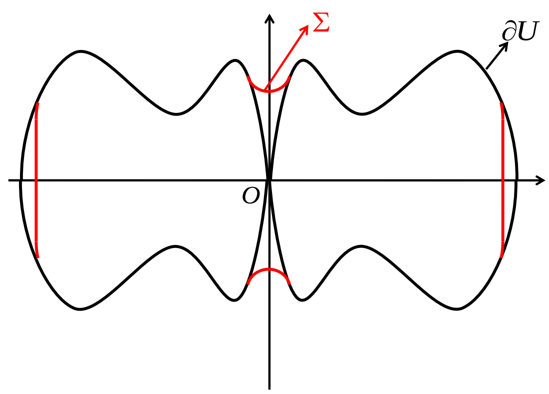

Figure 3.8: Proof of Proposition 3.2.1

Proof. Because $U \cap \{x \geq 0\}$ is an α-domain, there exist δ_j, γ_j and $0 < \delta_j < \gamma_j$ such that

$$u_0(\delta_j) = u_0(\gamma_j) = u_0(-\delta_j) = u_0(-\gamma_j) = \frac{\alpha}{2^j}$$

and

$$\partial U \cap C_\alpha = \{(x,y) \mid x = \pm v(y), |y| < \alpha\} \cup \{(x,y) \mid x = \pm w(y), |y| < \alpha\},$$

where $v, w \in C^\infty((-\alpha, \alpha))$ and $0 < v(y) < w(y)$ for $|y| < \alpha$. Here δ_j is decreasing and γ_j is increasing in j, respectively.

We let $w_j \in C^\infty((-\alpha/2^{j-1}, \alpha/2^{j-1}))$ be defined as following

$$w_j(y) = \begin{cases} \gamma_{j+2}, \ 0 \le |y| < \dfrac{\alpha}{2^{j+1}}, \\ w(y), \ \dfrac{\alpha}{2^j} < |y| < \dfrac{\alpha}{2^{j-1}}. \end{cases}$$

And $u_j \in C^\infty((-\delta_{j-1}, \delta_{j-1}))$ is defined as following

$$u_j(x) = \begin{cases} \dfrac{\alpha}{2^{j+1}}, \ x \in [0, \delta_{j+2}], \\ u_0(x), \ x \in [\delta_j, \delta_{j-1}). \end{cases}$$

Let Σ_j consist of three parts: $\{(x, y) \mid |y| = u_j(x), \ x \in (-\delta_j, \delta_j)\}$, $\{(x, y) \mid x = \pm w_j(y), |y| < \alpha/2^j\}$ and $\partial U \cap \{|y| \ge \alpha/2^j\}$. It is easy to see that for j sufficient large, $\Sigma_j \subset V \setminus \{(0, 0)\}$ satisfies (i), (ii), (iii) for $c = \gamma_{j+2}$, $\rho = \alpha/2^{j+1}$ and $d = \delta_{j+2}$. □

Denote $\sigma(P, t) : \Sigma \times (-\delta, \delta) \to V$ (V is given at the begining of this section and Σ is given by Proposition 3.2.1) the flow generated by vector field X in $\mathbb{R}^2$. Precisely, $\sigma(P, t)$ is defined as following:

$$\begin{cases} \dfrac{d\sigma(P, t)}{dt} = X(\sigma(P, t)), \ P \in \Sigma, \\ \sigma(P, 0) = P, \qquad P \in \Sigma. \end{cases}$$

Seeing (i) in Proposition 3.2.1, for any C^1 function $u : \Sigma \to \mathbb{R}$, "the image of u under σ"—$\{\sigma(P, u(P)) \mid P \in \Sigma\}$ is a C^1 curve. Conversely, for any curve $\Gamma \subset V$ being C^1 close to Σ, there exists a unique C^1 function $u : \Sigma \to \mathbb{R}$ such that $\Gamma = \{\sigma(P, u(P)) \mid P \in \Sigma\}$. In other words, the map $\sigma(\cdot, t)$ defines a new coordinate from Σ to V. Therefore, if $\Gamma(t) \subset V (0 < t < T)$ is a smooth family of smooth curves and C^1 close to Σ, there exists a unique function $u \in C^\infty(\Sigma \times (0, T))$ such that $\Gamma(t) = \{\sigma(P, u(P,t)) \mid P \in \Sigma\}$. Let z be the local coordinate on an open subset of Σ. If $\Gamma(t)$ evolves by $V = -\kappa + A$, under this coordinate u satisfies the following equation

$$\frac{\partial u}{\partial t} = a(z, u, u_z)\frac{\partial^2 u}{\partial z^2} + b(z, u, u_z). \tag{3.2.1}$$

Here a, b are smooth functions in their arguments Section 3 in [1]. And a is always positive so that (3.2.1) is a parabolic equation.

For example, $\sigma(\cdot, t)$ is the flow defined as above. We can easily deduce that

$$\sigma(P,t) = \begin{cases} (x, \rho - t),\ P \in B_d, \\ (-c + t, y),\ P \in \Delta_{-c}, \\ (c - t, y),\ P \in \Delta_c, \end{cases}$$

where we choose the local coordinates:

(1). $(x, \rho y)$ on B_d, for $|y| = 1$;

(2). $(\pm c, y)$ on $\Delta_{\pm c}$.

Assume $\Gamma(t)$ is symmetric to x-axis. Therefore, on B_d, u only depends on x, t and satisfies

$$u_t = a(x, u, u_x)u_{xx} + b(x, u, u_x), \tag{3.2.2}$$

where $a(x, u, u_x) = 1/(1 + u_x^2)$ and $b(x, u, u_x) = -A\sqrt{1 + u_x^2}$.

On $\Delta_{\pm c}$, u only depends on y, t. Then on $\Delta_{\pm c}$, u satisfies

$$u_t = a(y, u, u_y)u_{yy} + b(y, u, u_y), \tag{3.2.3}$$

where $a(y, u, u_y) = 1/(1 + u_y^2)$, $b(y, u, u_y) = -A\sqrt{1 + u_y^2}$.

Remark 3.2.2. In $\mathbb{R}^{n+1}$, b obtained above may not be smooth. For example,

$$u_t = \frac{u_{xx}}{1 + u_x^2} + \frac{n-1}{\rho - u} - A\sqrt{1 + u_x^2},$$

on $\{(x,y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| = \rho, -d < x < d\}$. In this case, $b = \frac{n-1}{\rho-u} - A\sqrt{1+u_x^2}$. It is easy to see b is not smooth at $u = \rho$. This is the most difficult point in higher dimension. We consider it in Chapter 5.

Lemma 3.2.3. *For smooth function $v(x,t)$ on $V \times (0,T)$, where V is a compact set, let $m(t)$ be*

$$m(t) = \max\{v(x,t) \mid x \in V\}.$$

Then there exists $P_t \in V$ such that $v(P_t,t) = m(t)$ and $m'(t) = v_t(P_t,t)$ for $t > 0$.

This is a well known result, for example, seeing [6].

Proposition 3.2.4. *Let $\Gamma_1(t)$, $\Gamma_2(t)$, $t \in [0,T]$ be two families of curves with $\sigma^{-1}(\Gamma_j(t))$ the graph of $u_j(\cdot,t)$ for certain $u_j \in C(\Sigma \times [0,T])$. Assume u_j are smooth on $\Sigma \times (0,T]$ and smooth on $\Sigma \setminus (\Delta_{\pm c} \cup B_d) \times [0,T]$. If $\Gamma_1(0) = \Gamma_2(0)$, then $\Gamma_1(t) = \Gamma_2(t)$, $0 \le t \le T$.*

The assumption "u_j are smooth on $\Sigma \times (0,T]$ and smooth on $\Sigma \setminus (\Delta_{\pm c} \cup B_d) \times [0,T]$" means that it is not necessary that the parts of $\Gamma_i(t)$ near origin and end points are smooth up to $t = 0$.

Proof. Consider $v(P,t) = u_1(P,t) - u_2(P,t)$. From our assumptions, we have $v \in C(\Sigma \times [0,T])$ and that v is smooth on $\Sigma \times (0,T]$ and $\Sigma \setminus (\Delta_{\pm c} \cup B_d) \times [0,T)$. Moreover, $v(P,0) \equiv 0$. Define $m(t) = \max\{v(P,t) \mid P \in \Sigma\}$. We want to show that $m'(t) \le Cm(t)$ for some constant C. Choose P_t as in Lemma 3.2.3 such that $m(t) = v(P_t,t)$ and $m'(t) = v_t(P_t,t)$.

Case 1. $P_t \in B_d$, since u_j satisfy the equation (3.2.2), v satisfies a parabolic equation

$$v_t = a_1(x,t)v_{xx} + b_1(x,t)v_x,$$

where $a_1(x,t)$ and $b_1(x,t)$ is smooth, and $a_1(x,t) > 0$. Since v attains its maximum at P_t, $v_x(P_t,t) = 0$ and $v_{xx}(P_t,t) \le 0$. Then $v_t(P_t,t) \le 0$. Considering Lemma 3.2.3, $m'(t) \le 0$.

Case 2. $P_t \in \Delta_{\pm c}$. We only consider $P_t \in \Delta_{-c}$. Then in the y-coordinates of Δ_{-c}, u_j satisfy (3.2.3). Therefore $v = u_1 - u_2$ satisfies a parabolic equation

$$v_t = a_2(y,t)v_{yy} + b_2(y,t)v_y.$$

Seeing $v_y(P_t,t) = 0$ and $v_{yy}(P_t,t) \leq 0$, $m'(t) \leq 0$.

Case 3. $P_t \in \Sigma \setminus (\Delta_{\pm c} \cup B_d)$. Then we can choose coordinate z on some neighbourhood of P_t on Σ and u_j satisfy (3.2.1). We may write this equation as $u_t = F(z,t,u,u_z,u_{zz})$. Then $v = u_1 - u_2$ satisfies

$$v_t = a_3(z,t)v_{zz} + b_3(z,t)v_z + c_3(z,t)v,$$

where

$$a_3 = a(z, u_1, (u_1)_z)$$

and

$$c_3(z,t) = \int_0^1 F_u(z,t,u^\theta,u_z^\theta,u_{zz}^\theta)d\theta,$$

where $u^\theta = (1-\theta)u_2 + \theta u_1$. As mentioned in equation (3.2.1), a_3 is positive.

By the assumption, outside of the disks $\Delta_{\pm c}$ and the pipe B_d, u_i are smooth up to $t = 0$, so the coefficient $c_3(z,t)$ is bounded, $0 \leq t \leq T$, saying by $|c_3(z,t)| \leq M < \infty$. The constant M may depend on the choice of local coordinate z. Noting Σ is compact, by easy covering argument, we can choose M independent of the choice of local coordinate. Since $v_z(P_t,t) = 0$, $v_{zz}(P_t,t) \leq 0$,

$$v_t(P_t,t) \leq c_3(P_t,t)v(P_t,t) \leq Mv(P_t,t).$$

Consequently, $m'(t) \leq Mm(t)$.

Combining the three cases, we have $m'(t) \leq Cm(t)$, for some constant $C > 0$.

Considering $m(0) = 0$, $m(t) \le 0$. Conversely, we can prove $M(t) = \min\{v(P,t) \mid P \in \Sigma\} \ge 0$. Therefore $u_1 \equiv u_2$. We complete the proof. □

Lemma 3.2.5. *There exists a sequence of closed sets E_j such that E_j° are $\alpha/2^j$-domains and $E_j \downarrow \overline{U}$. Where U is given at the beginning of the section and E° denotes the interior of the set E.*

Proof. We choose δ_j as in Proposition 3.2.1. We can construct $v_j \in C^\infty((-b_0, b_0))$ being even such that

$$v_j(x) = \begin{cases} \alpha/2^j, \ x \in (-\delta_j/2, \delta_j/2), \\ u_0(x), x \in [-b_0, -\delta_j] \cup [\delta_j, b_0], \end{cases}$$

$v_j(x) \ge u_0(x)$, $x \in [-b_0, b_0]$ and $v_j'(x) > 0$, $x \in (\delta_j/2, \delta_j)$. It is easy to see $v_j \downarrow u_0$ uniformly in $[-b_0, b_0]$.

Let $E_j = \{(x,y) \mid |y| \le v_j(x), \ -b_0 \le x \le b_0\}$. Since $v_j \downarrow u_0$ uniformly in $[-b_0, b_0]$, $E_j \downarrow \overline{U}$. It is easy to check E_j° are $\alpha/2^j$-domain. □

Lemma 3.2.6. *Let the same assumption in Theorem 3.1.1 be given. Then there exists $t_1 > 0$ such that, for all t_2 satisfying $0 < t_2 < t_1$, the second fundamental forms and derivatives of $\partial E_j(t)$ are uniformly bounded for $t_2 \le t \le t_1$, where $E_j(t)$ denote the closed evolution of $V = -\kappa + A$ with $E_j(0) = E_j$.*

Proof. Let $E_j(t) = \{(x,y) \mid |y| \le v_j(x,t)\}$.

Step 1. For all t_2 satisfying $0 < t_2 < \delta$ (δ given by Theorem 3.1.1), there exists a constant $c > 0$ such that

$$v_j(0,t) > c, \ t_2/2 < t < \delta.$$

Let $U^+(t)$ denote the bounded set with $\partial U^+(t) = \Lambda^+(t)$. Since $U^+(0) = U \cap \{x \ge 0\} \subset E_j = E_j(0)$, there holds $U^+(t) \subset E_j(t)$. By our assumption that $a_*(t) < 0$, for

$0 < t \leq \delta$, there holds $(0,0) \in U^+(t) \subset E_j(t)$, $0 < t < \delta$. For all $t_2 \in (0,\delta)$, there exists $c > 0$ such that $v_j(0,t) > c$, $t_2/2 \leq t \leq \delta$.

Step 2. Construction of auxiliary balls.

Since $U \cap \{x \geq 0\}$ is an α-domain, there exist $\beta_2 > \beta_1 > 0$ such that $u_0(\pm\beta_1) = u_0(\pm\beta_2) = \alpha$ and $u_0'(x) < 0$ for $x > \beta_2$, $u_0'(x) > 0$ for $0 < x < \beta_1$. There exist $p > \beta_1$ and $0 < q < \beta_2$ such that $u_0(\pm q) = u_0(\pm p) = \dfrac{\alpha}{2}$. We consider the points

$$Q = (-p, 0), \qquad P = (p, 0),$$

$$Q' = (-p, \alpha), \qquad P' = (p, \alpha).$$

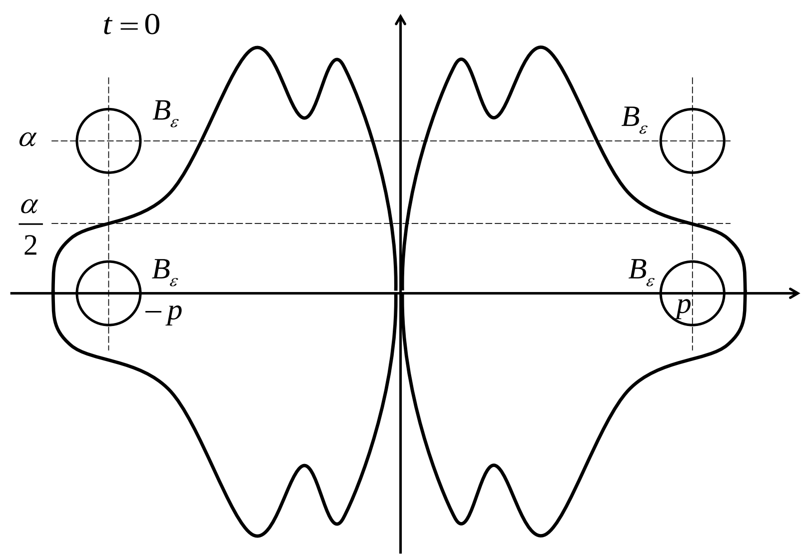

Figure 3.9: Proof of Lemma 3.2.6

Since $P \in U$ and $P' \in \overline{U}^c$, there exists ϵ such that $\overline{B_\epsilon(P)} \subset U$ and $\overline{B_\epsilon(P')} \subset \overline{U}^c$. Consequently, $\overline{B_\epsilon(P)} \cup \overline{B_\epsilon(Q)} \subset E^\circ$ and $\overline{B_\epsilon(P')} \cup \overline{B_\epsilon(Q')} \subset E^c$. Then for j large enough, $\overline{B_\epsilon(P)} \cup \overline{B_\epsilon(Q)} \subset E_j^\circ$ and $\overline{B_\epsilon(P')} \cup \overline{B_\epsilon(Q')} \subset E_j^c$. By (4) in Theorem 2.1.4,

$$\overline{B_{\epsilon(t)}(P)} \cup \overline{B_{\epsilon(t)}(Q)} \subset E_j(t)^\circ, \tag{3.2.4}$$

$0 < t < \delta_2$. By (1b) in Theorem 2.1.6, there exists $\delta_3 > 0$ such that

$$\overline{B_{\epsilon(t)}(P')} \cup \overline{B_{\epsilon(t)}(Q')} \subset E_j(t)^c, \tag{3.2.5}$$

for $0 < t < \delta_3$. Where $\epsilon(t)$ is the solution of (2.3.12) with $\epsilon(0) = \epsilon$, $0 < t < \delta_1$. Choose δ_2 independent of j such that $\epsilon(t) > \epsilon/2$, $0 < t < \delta_2$.

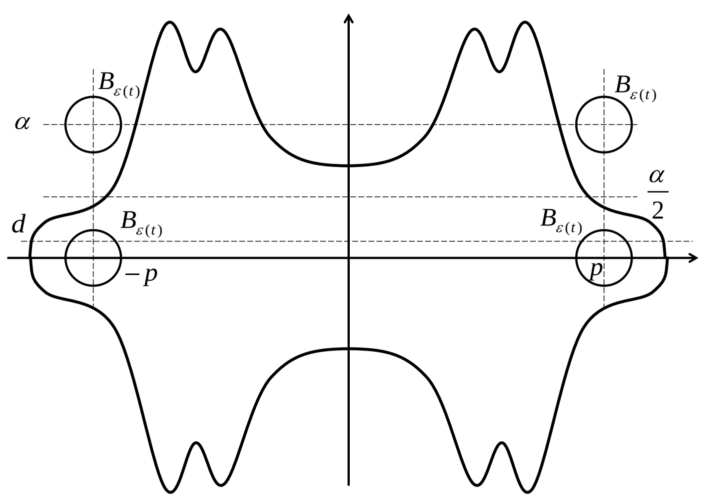

Figure 3.10: Proof of Lemma 3.2.6

Step 3. Divide $\partial E_j(t)$ into two parts by auxiliary balls.

Since for all $\rho < \alpha$, C_ρ intersects ∂E_j at most four times, by Proposition 2.3.1, there exists $t_0 > 0$ such that C_ρ intersects $\partial E_j(t)$ at most four times for $0 < t < t_0$. By continuity, we can deduce that there exists δ_4 such that for all $\rho < \alpha$, the equation $v_j(x,t) = \rho$ has just one root for $x > p$ for all $t < \delta_4$. By symmetry, it also holds for

$x < -p$.

Choosing $t_1 = \min\{t_0, \delta_2, \delta_3, \delta_4\}$, Step 1 and intersection number argument show that $E_j(t)^\circ$ are all c-domains, $t_2/2 < t < t_1$. Let $d < \min\{c, \epsilon/4\}$. By (3.2.4) in Step 2, we have $v_j(x,t) > d$, for $t_2/2 < t < t_1$, $|x-p| < \sqrt{\epsilon^2(t) - d^2}$ or $|x+p| < \sqrt{\epsilon^2(t) - d^2}$. Seeing $\epsilon(t) > \epsilon/2$, there holds

$$v_j(x,t) \geq d \text{ in } \Omega = (-p - \frac{\sqrt{3}}{4}\epsilon, p + \frac{\sqrt{3}}{4}\epsilon) \times (t_2/2, t_1).$$

For $x \leq -p$, by (3.2.5) in Step 2,

$$v_j(x,t) < \alpha/2 - \epsilon(t) < \alpha/2 - \epsilon/2,\ x \leq -p,\ 0 \leq t < t_1.$$

This is also true for $x \geq p$.

Step 4. The derivatives and second fundamental forms of $\partial E_j(t)$ are bounded in $\Omega' = [-p, p] \times (t_2, t_1)$.

Since $v_j(x,t) \geq d$ in $\Omega = (-p - \frac{\sqrt{3}}{4}\epsilon, p + \frac{\sqrt{3}}{4}\epsilon) \times (t_2/2, t_1)$, Theorem 2.3.4 implies that v_{jx} are uniformly bounded in Ω. By Remark 2.2.4, v_{jxx} are uniformly bounded in Ω'.

Step 5. The derivatives and second fundamental forms of $\partial E_j(t)$ are bounded for $x \leq -p$ and $x \geq p$, $t_2 < t < t_1$.

We only consider for $x \leq -p$. For $0 < t < t_1$, the part of $\partial E_j(t)$ on $x \leq -p$ can be represented by $x = w_j(y,t)$, for $|y| < \alpha/2$, $t \in (0, t_1)$. And w_j satisfy the equation (2.2.1) in the condition "$-$" and $n = 1$. Then Corollary 2.2.3 and Remark 2.2.4 imply that all $\frac{\partial^k}{\partial y^k} w_j(y,t)$, $k = 1, 2$, are uniformly bounded for $|y| \leq \alpha/2 - \epsilon/2$, $t_2 < t < t_1$ and for any $t_2 > 0$. Then the derivatives and second fundamental forms of $\partial E_j(t)$ are uniformly bounded for $x \leq -p$, $t_2 < t < t_1$.

The proof of this lemma is completed. □

Lemma 3.2.7. *There exist U_j being open and $U_j \cap \{x \geq 0\}$ being an α-domain such that $U_j \uparrow U$ as $j \to \infty$.*

Proof. Since $U \cap \{x > 0\}$ being α-domain, for $j \geq 1$, there exist δ_j satisfying $0 < \delta_j < \delta_0$ such that $u_0(\delta_j) = \alpha/2^j$, where δ_0 satisfies $u_0(\pm\delta_0) = \alpha$ and $u_0'(x) > 0$ for $0 < x < \delta_0$. We set $u_j \in C^\infty((-b_0, b_0))$ and even satisfying

$$u_j(x) = \begin{cases} 0, \ x = 0, \\ u_0(x), \ x \in [-b_0, -\delta_j] \cup [\delta_j, b_0], \end{cases}$$

and $u_j(x) \leq u_0$ for $x \in [-b_0, b_0]$, $u_j'(x) > 0$ for $x \in (0, \delta_j)$.

Let $U_j = \{(x, y) \mid |y| < u_j(x)\}$. Obviously $u_j \uparrow u_0$, then $U_j \uparrow U$. It is easy to check $U_j \cap \{x > 0\}$ are α-domain. □

Lemma 3.2.8. *Let the same assumption in Theorem 3.1.1 be given. Then there exists $t_1 > 0$ such that for all t_2 satisfying $0 < t_2 < t_1$, the second fundamental forms and derivatives of $\partial U_j(t)$ is uniformly bounded, $t_2 < t < t_1$, where $U_j(t)$ is the open evolution of $V = -\kappa + A$ with $U_j(0) = U_j$.*

This lemma can be proved similar as in Lemma 3.2.6.

Proof of Theorem 3.1.1. Seeing Lemma 3.2.6 and 3.2.8, $\partial U(t)$, $\partial E(t)$ are smooth curves and homeomorphic to the curve Σ given by Proposition 3.2.1. Consequently, $\partial U(t)$, $\partial E(t)$ satisfy the assumption of Proposition 3.2.4, $0 \leq t < T_1$, for some T_1 satisfying $0 < T_1 < t_1$. Here t_1 is given by Lemma 3.2.6 and 3.2.8. Then there holds $\partial U(t) = \partial E(t)$, $0 < t < T_1$. The proof of Theorem 3.1.1 is completed.

Moreover, by Theorem 2.1.10 and Theorem 2.3.7, $\Gamma(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| = u(x, t), -b(t) \leq x \leq b(t)\}$ and (u, b) is the solution of the following free boundary problem

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2},\ x \in (-b(t), b(t)),\ 0 < t < T, \\ u(-b(t), t) = 0,\ u(b(t), t) = 0,\ 0 \le t < T, \\ u_x(-b(t), t) = \infty,\ u_x(b(t), t) = -\infty,\ 0 \le t < T, \\ u(x, 0) = u_0(x),\ -b_0 \le x \le b_0. \end{cases}$$

□

Remark 3.2.9. Indeed, in Proposition 3.2.4, it is not necessary that the parts of $\Gamma_i(t)$ near end points are smooth up to $t = 0$. Therefore, we can remove the assumption that Γ_0 is smooth at end point $(-b_0, 0)$ and $(b_0, 0)$.

Proof of Theorem 3.1.2. It is sufficient to show that there is a ball B such that $B \subset E(t) \setminus U(t)$, for some t.

Closed evolution $E(t)$**.** Since E_j° (given by Lemma 3.2.5) are $\alpha/2^j$-domain with smooth boundary, by Lemma 2.3.9, there exists a positive time t_1, $t_1 < \delta$ (δ is given in Theorem 3.1.2) such that $E_j(t)^\circ$ are $(At + \alpha/2^j)$-domain for $0 < t < t_1$. Combining $E_j(t) \downarrow E(t)$, we have $E(t)^\circ$ is an At-domain, $0 < t < t_1$. Therefore $E(t)^\circ$ is an $At_1/2$-domain, $t_1/2 < t < t_1$.

Open evolution $U(t)$**.** Let $U^\pm(t)$ be the bounded open domain with $\partial U^\pm(t) = \Lambda^\pm(t)$. Thus the left end point of $U^+(t)$ and the right end point of $U^-(t)$ are $(a_*(t)$,0) and $(-a_*(t)$,0), respectively. By the assumption in this theorem $a_*(t) \ge 0$, $0 \le t < \delta$, it means that $-a_*(t) \le a_*(t)$, $0 \le t < \delta$. Therefore, $U^+(t) \cap U^-(t) = \emptyset$, $0 \le t < \delta$. From Lemma 2.1.9, the inner evolution $U(t)$ satisfies $U(t) = U^+(t) \cup U^-(t)$, for $0 \le t < \delta$.

By (2a) in Theorem 2.1.6 (the boundary of open evolution evolves continuously) and $a(t) \geq 0$, there exists $\delta_1 < \dfrac{At_1}{4}$ such that

$$B_{\delta_1}((0, \frac{At_1}{4})) \cap U(t) = \emptyset, \ \frac{t_1}{2} < t < t_1$$

and

$$B_{\delta_1}((0, \frac{At_1}{4})) \subset E(t), \ \frac{t_1}{2} < t < t_1.$$

Then $B_{\delta_1}((0, \dfrac{At_1}{4})) \subset \Gamma(t) = E(t) \setminus U(t)$, for $\dfrac{t_1}{2} < t < t_1$. □

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Chapter 4

Asymptotic behavior for curvature flow with driving force in the plane

In this chapter, we want to introduce the researches in [12] and [13]. We want to classify the solution given in Chapter 3 into three cases and consider the asymptotic behavior in each case.

4.1 Introduction

In [12] and [13], they classify the solution of the following problem and give their asymptotic behavior

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2},\ x \in (-b(t), b(t)),\ 0 < t < T, \tag{4.1.1}$$

$$u(-b(t), t) = 0,\ u(b(t), t) = 0,\ 0 \le t < T, \tag{4.1.2}$$

$$u_x(-b(t), t) = \infty,\ u_x(b(t), t) = -\infty,\ 0 \le t < T, \tag{4.1.3}$$

$$u(x, 0) = u_0(x),\ -b_0 \le x \le b_0, \tag{4.1.4}$$

where $u_0 \in C^\infty((-b_0, b_0)) \cap C([-b_0, b_0])$ is even and satisfies $u_0(x) > 0$, $-b_0 < x < b_0$. Moreover, we assume the curve $\Gamma_0 = \{(x, y) \mid |y| = u_0(x), -b_0 \le x \le b_0\}$ is smooth and embedded. The constant A called driving force is positive.

Recall constant T being the maximal time such that $\Gamma(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| = u(x, t), -b(t) \le x \le b(t)\}$ is smooth and embedded for $0 < t < T$.

Theorem 4.1.1. *(Classification) Denote*

$$h(t) = \max_{-b(t) \le x \le b(t)} u(x, t)$$

and

$$\Gamma(t) = \{(x, y) \mid |y| = u(x, t),\ -b(t) \le x \le b(t)\}.$$

Then $\Gamma(t)$ must fulfill one of the following situations.

(1). (Expanding) The maximal smooth time $T = \infty$ and both $h(t)$ and $b(t)$ tend to ∞, as $t \to \infty$.

(2). (Bounded) The maximal smooth time $T = \infty$ and both $h(t)$ and $b(t)$ are bounded from above and below by two positive constants, as $t \to \infty$.

(3). (Shrinking) The maximal smooth time $T < \infty$ and both $h(t)$ and $b(t)$ tend to 0, as $t \to T$.

Theorem 4.1.2. *(Asymptotic behavior)*
(1). (Expanding) The maximal smooth time $T = \infty$ and that both $h(t)$ and $b(t)$ tend to ∞ as $t \to \infty$. Then there exist $t_0 > 0$, $R_1(t)$, $R_2(t)$ such that

$$B_{R_1(t)}(O) \subset U(t) \subset B_{R_2(t)}(O),\ t > t_0,$$

where $U(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| < u(x, t),\ x > 0\}$. Moreover $\lim\limits_{t\to\infty} R_1(t)/t = \lim\limits_{t\to\infty} R_2(t)/t = A$.

(2). (Bounded) The maximal smooth time $T = \infty$ and that both $h(t)$ and

$b(t)$ are bounded from above and below by two positive constants for $t > 0$. Then $\lim_{t\to\infty} d_H(\Gamma(t), \partial B_{1/A}(O)) = 0$. Here d_H denotes the Hausdorff distance.

Next we give the asymptotic behavior for the Shrinking case.

If $T < \infty$, seeing Corollary 4.2.6, there exists t_0 such that $u(x,t)$ loses all its local minimum, $t_0 < t < T$ and $\Gamma(t)$ shrinks to the origin, as $t \to T$.

Noting that the initial function u_0 is even, $u(x,t)$ is also even. Therefore for every $t > t_0$, $u(x,t)$ is increasing for $x \in (-b(t), 0)$ and $u(x,t)$ is decreasing for $x \in (0, b(t))$. Moreover, $h(t) = u(0,t)$, $t > t_0$.

Results for Shrinking. Under the case $T < \infty$, we introduce the following similarity transformation (first used by [4]):

$$z = \frac{x}{\sqrt{2(T-t)}},\ \tau = -\frac{1}{2}\ln(T-t) \tag{4.1.5}$$

and

$$w(z,\tau) = \frac{1}{\sqrt{2}} e^{\tau} u(\sqrt{2}e^{-\tau} z, T - e^{-2\tau}). \tag{4.1.6}$$

We also define

$$r(\tau) = \frac{1}{\sqrt{2}} e^{\tau} h(T - e^{-2\tau}) \text{ and } q(\tau) = \frac{1}{\sqrt{2}} e^{\tau} b(T - e^{-2\tau}).$$

Obviously, $r(\tau) = w(0,\tau) = \max_{-q(\tau)\le z\le q(\tau)} w(z,\tau)$, $\tau > -\frac{1}{2}\ln(T - t_0)$. Then u satisfies (4.1.1), (4.1.2), (4.1.3), (4.1.4) if and only if w satisfies

$$w_\tau = \frac{w_{zz}}{1+w_z^2} - z w_z + w + \sqrt{2}Ae^{-\tau}\sqrt{1+w_z^2},\ z \in (-q(\tau), q(\tau)),\ \tau > \tau_0, \tag{4.1.7}$$

$$w(-q(\tau),\tau) = w(q(\tau),\tau) = 0,\ \tau > \tau_0, \tag{4.1.8}$$

$$w_z(-q(\tau),\tau) = \infty,\ w_z(q(\tau),\tau) = -\infty,\ \tau > \tau_0, \tag{4.1.9}$$

$$w_0(z) := w(z, \tau_0) = \frac{1}{\sqrt{2T}} u_0(\sqrt{2T}z), \ z \in [-b(0)/\sqrt{2T}, b(0)/\sqrt{2T}], \tag{4.1.10}$$

where $\tau_0 = -\frac{1}{2}\ln T$. The stationary problem for (4.1.7), (4.1.8), (4.1.9), (4.1.10) is given by

$$\frac{\varphi_{zz}}{1+\varphi_z^2} - z\varphi_z + \varphi = 0, \ z \in (-\overline{q}, \overline{q}), \tag{4.1.11}$$

$$\varphi(-\overline{q}) = \varphi(\overline{q}) = 0, \tag{4.1.12}$$

$$\varphi_z(-\overline{q}) = \infty, \ \varphi_z(\overline{q}) = -\infty, \tag{4.1.13}$$

for some $\overline{q}$. Obviously, $\varphi(z) = \sqrt{1-z^2}$ and $\overline{q} = 1$ are the unique solution of the above stationary problem (4.1.11)-(4.1.13).

Then we have the following theorem.

Theorem 4.1.3 (Asymptotic behavior)**.** *The solution $(w(z,\tau), q(\tau))$ of problem (4.1.7)-(4.1.10) converges to the unique solution $(\varphi(z), \overline{q})$ of (4.1.11)-(4.1.13) pointwise, as $\tau \to +\infty$, where w and φ are considered as 0 outside the interval.*

Furthermore, there exists t_1 such that $\Gamma(t)$ is strict convex for $t_1 < t < T$. Equivalently, $u_{xx}(x,t) < 0$, for $-b(t) < x < b(t)$, $t_1 < t < T$.

Remark 4.1.4. Indeed, we can prove the graph of $w(z,\tau)$ converges to the graph of $\varphi(z)$ under the Hausdorff distance.

Note that our result does not assume convexity for initial data as in [9], since we assume symmetry to the the initial curve.

The most important tool in proving the asymptotic behavior for Shrinking case is the comparison principle for extrinsic and intrinsic distances. Let the flow $\mathbf{G}$: $[0, L_*(t)] \times [0,T) \to \mathbb{R}^2$ be the smooth closed curves evolving by the classical curve shortening flow

$$\frac{\partial}{\partial t}\mathbf{G}(s,t) = \frac{\partial^2}{\partial s^2}\mathbf{G}(s,t),$$

where s denotes the arc length parameter, $L_*(t)$ denotes the perimeter of $G(\cdot,t)$. For any two points on mean curvature flow $\mathbf{G}(s,t)$, denoted by $\mathbf{G}(s_1,t)$, $\mathbf{G}(s_2,t)$. Denote $d = |\mathbf{G}(s_1,t) - \mathbf{G}(s_2,t)|$. $l = |s_2 - s_1|$ is the length of the curve between $\mathbf{G}(s_1,t)$, $\mathbf{G}(s_2,t)$. More precisely, l and d are called the intrinsic and extrinsic distances, respectively. The paper [7] shows that $m(t) = \min_{(s_1,s_2)\in[0,L_*(t)]\times[0,L_*(t)]}(d/l)(s_1,s_2,t)$ is non-decreasing in time. The ratio between extrinsic and intrinsic distance is also used by [10] and [11].

In our problem, if we let $\mathbf{F}$ satisfy

$$\Gamma(t) = \{(x,y)\in\mathbb{R}^2 \mid |y| = u(x,t), -b(t)\le x\le b(t)\} = \{\mathbf{F}(s,t)\in\mathbb{R}^2 \mid s\in[0,L(t)]\},$$

$L(t)$ denotes the perimeter of $\Gamma(t)$. Then $\mathbf{F}$ satisfies

$$\frac{\partial}{\partial t}\mathbf{F}(s,t) = \frac{\partial^2}{\partial s^2}\mathbf{F}(s,t) - A\mathbf{N},$$

where $\mathbf{N}$ denotes the unit inner normal vector. We will see the result in [7] does not hold in our problem. Seeing future, the curvature flow with driving force does not intersect itself interior, but could intersect itself exterior.

4.2 Formation of sigularity

In this section, we want to identify the singular formation of $\Gamma(t)$ at the singular time $T<\infty$. Recall

$$T = \sup\{t>0 \mid \Gamma(s) \text{ is smooth and embedding}, 0<s<t\}. \tag{4.2.1}$$

For curve shortening flow, Grayson shows that any curve shortening flow starting as a close smooth curve only shrinks to a point when becoming singular. But under the condition with driving force, there is no any result. I will show the formation of singularity under the asymmetric assumption.

Following Theorem 4.2.5 shows that axisymmetric flow will pinch at x-axis at singular time T. As a corollary, combining intersection number principle, we can show the flow given by Theorem 3.1.1 shrinks to origin at singular time T.

By Sturmian theorem, the numbers of local maxima and local minima are a finite nonincreasing function of time. It follows that, after a while, the numbers of local maxima and local minima are constants. After discarding an initial section of the solution, we may even assume that $x \mapsto u(x,t)$ has m local minima and $m+1$ local maxima. Let these minima and maxima be located at $\{\xi_j(t)\}_{1\le j\le m}$ and $\{\eta_j(t)\}_{0\le j\le m}$, respectively. And order the $\xi_j(t)$ and $\eta_j(t)$ so that

$$-b(t) < \eta_0(t) < \xi_1(t) < \eta_1(t) < \cdots < \xi_m(t) < \eta_m(t) < b(t). \tag{4.2.2}$$

Since the number of critical points of $u(\cdot,t)$ drops whenever $u(\cdot,t)$ has degenerate critical point, the minima and maxima of $u(\cdot,t)$ are all nondegenerate. By the implicit function theorem the $\xi_j(t)$ and $\eta_j(t)$ are therefore smooth functions of time.

Lemma 4.2.1. *The limits*

$$\lim_{t\to T} b(t) = b(T)$$

and

$$\lim_{t\to T} \xi_j(t) = \xi_j(T), \ \lim_{t\to T} \eta_j(t) = \eta_j(T)$$

exist.

Proof. We prove this lemma by the method from [1], first developed by [2]. However, in our proof, there is a little difference, since the intersection number between two flows evolving by $V = -\kappa + A$ may increase. The method in [1] should be modified.

First, we prove $\lim_{t\to T} b(t)$ exists. By the vertical equation

$$w_t = \frac{w_{rr}}{1+w_r^2} + A\sqrt{1+w_r^2},$$

we can derive $b'(t) = w_{rr}(0,t) + A \le A$ because of $w_{rr}(0,t) \le 0$. Then $b(t) - At$ is non-increasing. It is easy to see $b(t) - At$ is bounded for $t < T$. Therefore $\lim_{t\to T}(b(t) - At)$ exists. Consequently, $\lim_{t\to T} b(t)$ exists.

Next, we prove $\lim_{t\to T} \xi_j(t)$ exists. We assume

$$\limsup_{t\to T} \xi_j(t) > \liminf_{t\to T} \xi_j(t).$$

We can choose $x_0 \in (\liminf_{t\to T} \xi_j(t), \limsup_{t\to T} \xi_j(t))$ and $x_0 \neq 0$. Without loss of generality, we assume $-b(T) < x_0 < 0 < b(T)$. Since $\xi_j(t)$ is continuous in t, there exists a sequence $t_m \to T$ such that

$$\xi_j(t_m) = x_0 \text{ and } u_x(x_0, t_m) = 0.$$

We let $\widetilde{\Gamma}(t)$ be the reflection from $\Gamma(t)$ about $x = x_0$. Consequently, $\widetilde{a}(t) := 2x_0 - b(t)$ and $\widetilde{b}(t) := 2x_0 + b(t)$ are the end points of $\widetilde{\Gamma}(t)$. Obviously, $\widetilde{\Gamma}(t)$ also evolves by $V = -\kappa + A$ and $\widetilde{a}(T) < -b(T) < x_0 < \widetilde{b}(T) < b(T)$. For t being sufficiently close to T, $\widetilde{a}(t) < -b(t) < x_0 < \widetilde{b}(t) < b(t)$, i.e., the order of $\widetilde{a}(t)$, $\widetilde{b}(t)$, $-b(t)$, $b(t)$ does not change. Using Theorem 2.3.3, since $\widetilde{\Gamma}(t_m)$ intersects $\Gamma(t_m)$ at x_0 tangentially, the intersection number between $\widetilde{\Gamma}(t)$ and $\Gamma(t)$ will drop infinite times, for t close to T. But Theorem 2.3.3 shows that the intersection number between $\Gamma(t)$ and $\widetilde{\Gamma}(t)$ is finite (the choice of x_0 implies $\Gamma(t)$ is not identity to $\widetilde{\Gamma}(t)$). This yields a contradiction. □

Lemma 4.2.2. *If $\xi_j(T) < \eta_j(T)$, then for any compact interval $[c,d] \subset (\xi_j(T), \eta_j(T))$, there exists t_1 and $\delta > 0$ such that $u(x,t) \ge \delta$ for $x \in [c,d]$, $t \in [t_1, T)$. (Similarly for $\eta_{j-1}(T) < \xi_j(T)$, $-b(T) < \eta_0(T)$, $\eta_m(T) < b(T)$).*

Proof. Let $[a,b] \subset (\xi_j(T), \eta_j(T))$ be any compact interval, then there exists $t_1 < T$ such that $[a,b] \subset (\xi_j(t), \eta_j(t))$ and $u_x(x,t) > 0$, $x \in [a,b]$, $t \in (t_1, T)$. Denoting $\theta = \arctan u_x$, θ satisfies

$$\theta_t = \cos^2\theta\theta_{xx} + A\sin\theta\theta_x.$$

On the other hand, we let $\varphi(x,t) = \epsilon e^{-ct}\sin(\lambda(x-a))$, where $\lambda = \pi/(b-a)$, $c > A\lambda\pi + \lambda^2$, $0 < \epsilon < \pi$. Since $\varphi_{xx} \le 0$, $x \in [a,b]$ and seeing

$$\left| -A\lambda \frac{\sin(\epsilon e^{-ct}\sin(\lambda(x-a)))}{\sin(\lambda(x-a))}\cos(\lambda(x-a)) \right| \le A\lambda\pi,$$

there holds

$$\begin{aligned}
\varphi_t &- \cos^2\varphi\varphi_{xx} - A\sin\varphi\varphi_x \le \varphi_t - \varphi_{xx} - A\sin\varphi\varphi_x \\
&= \epsilon e^{-ct}\sin(\lambda(x-a))\left(-c + \lambda^2 - A\lambda\frac{\sin(\epsilon e^{-ct}\sin(\lambda(x-a)))}{\sin(\lambda(x-a))}\cos(\lambda(x-a))\right) \\
&\le \epsilon e^{-ct}\sin(\lambda(x-a))(-A\lambda\pi - \lambda^2 + \lambda^2 + A\lambda\pi) = 0,
\end{aligned}$$

for $x \in [a,b]$, $t \in (t_1, T)$. Since $u_x(x, t_1)$ is bounded from below for some positive constant in $[a,b]$, we can choose $\epsilon > 0$ small enough such that $\varphi(x,t_1) \le \theta(x,t_1)$. Seeing

$$\varphi(a,t) = 0 < \theta(a,t),\ \varphi(b,t) = 0 < \theta(b,t),\ t \in (t_1, T).$$

By maximum principle,

$$\theta(x,t) \ge \varphi(x,t),\ a < x < b,\ t_1 < t < T.$$

Consequently,

$$u_x \ge \arctan u_x = \theta \ge \epsilon e^{-ct}\sin(\lambda(x-a)),\ x \in [a,b],\ t \in (t_1, T).$$

$$u \geq \epsilon \frac{e^{-ct}}{\lambda}(1 - \cos(\lambda(x - a))),\ x \in [a, b],\ t \in (t_1, T).$$

Then for all $[c, d] \subset (a, b)$, u is uniformly bounded from below for $x \in [c, d],\ t \in [t_1, T)$. □

Lemma 4.2.3. *$\lim_{t \to T} u(x, t) = u(x, T)$ exists, and $u(x, t)$ converges uniformly to $u(x, T)$, for $x \in \mathbb{R}$, as $t \to T$. The function u is smooth at $(x, t) \in \mathbb{R} \times (0, T]$ provided that $u(x, t) > 0$. We interpret that $u(x, t) = 0$ outside $(a(t), b(t))$.*

Proof. By Lemma 4.2.2, for all $[c, d] \subset (\xi_{j-1}(T), \xi_j(T))$, $u(x, t) \geq \delta$, $x \in [c, d]$, $t \in [t_1, T)$. By Theorem 2.3.4, u_x is uniformly bounded on $[c, d] \times [t_1, T)$, which implies $\frac{\partial^i}{\partial x^i} u(x, t)$, $i = 1, 2$ are bounded on any compact subinterval of (c, d). On the other hand, from equation, $u_t(x, t)$ is uniformly bounded on such interval, so that $u(\cdot, t)$ converges uniformly on any such interval.

The same idea can be applied to the conditions in the intervals $(-b(T), \xi_1(T))$ and $(\xi_m(T), b(T))$. Since outside of $[-b(T), b(T)]$, $u(x, T)$ is considered to be 0, the result is true.

Except at $-b(T)$, $b(T)$ and $\xi_j(T)$s, $u(x, t)$ converges pointwise for every x not equaling $-b(T)$, $b(T)$, $\xi_j(T)$, as $t \to T$. The convergence is uniform on any interval that does not contain any of the points.

Next we want to prove the functions $u(\cdot, t)$ are equicontinuous for $T/2 < t < T$.

Assuming $x_1 < x_2$, if x_1, x_2 are both not in the interval $(-b(T), b(T))$, the conclusion is obvious. Assume $x_1 \in (-b(T), b(T))$.

Suppose that $|u(x_1, t) - u(x_2, t)| \geq \epsilon$. Then either $u(x_1, t) \geq \epsilon$ or $u(x_2, t) \geq \epsilon$ or both (since u is positive); we assume the first one. From Theorem 2.3.4, $|u_x| < \sigma(\epsilon/2, T/2)$ whenever $u(x, t) \geq \epsilon/2$, $T/2 < t < T$. Thus, if $u(x, t) \geq \epsilon/2$ on (x_1, x_2),

$$x_2 - x_1 \geq \frac{|u(x_1, t) - u(x_2, t)|}{\sigma(\epsilon/2, T/2)} \geq \frac{\epsilon}{\sigma(\epsilon/2, T/2)}.$$

If $u(x, t) < \epsilon/2$ some where in the interval (x_1, x_2), then there is a smallest x_3

satisfying $x_1 < x_3$ at which $u(x_3, t) = \epsilon/2$. On the interval (x_1, x_3), $u(x, t) \geq \epsilon/2$. Then

$$x_2 - x_1 \geq x_3 - x_1 \geq \frac{u(x_1, t) - u(x_3, t)}{\sigma(\epsilon/2, T/2)} \geq \frac{\epsilon}{2\sigma(\epsilon/2, T/2)}.$$

So for every $\epsilon > 0$, choose $\delta = \epsilon/(2\sigma(\epsilon/2, T/2))$ so that

$$|u(x_1, t) - u(x_2, t)| < \epsilon,$$

$|x_1 - x_2| < \delta$, for $T/2 < t < T$.

Thus $u(x, t)$ is equicontinuous. Noting that $u(x, t)$ converges to $u(x, T)$ in $\mathbb{R} \setminus \{\xi_i(T), -b(T), b(T)\}$ and $\mathbb{R} \setminus \{\xi_i(T), -b(T), b(T)\}$ is dense in $\mathbb{R}$, the proof is completed. □

Lemma 4.2.4. *If $u(\eta_0(T), T) > 0$ holds, then $-b(T) < \eta_0(T) < b(T)$.*

Proof. Since $u(\eta_0(T), T) > 0$, there exists $\delta > 0$ such that $\delta = \inf_{0 \leq t \leq T} u(\eta_0(t), t)$. We consider

$$x = v(|y|, t),$$

being the inverse function of $|y| = u(x, t)$ for $x \in (a(t), \eta_0(t))$ and let $w(y, t) = v(|y|, t)$. $w(y, t)$ satisfies the equation (2.2.1) for the condition "$-$" and "$n = 1$", $|y| < \delta$, $0 < t < T$. Clearly w is uniformly bounded, so Corollary 2.2.3 and Remark 2.2.4 imply that $\frac{\partial^k w}{\partial y^k}(y, t)$, $k = 1, 2$ are bounded for $|y| \leq \delta/2$, $T/2 \leq t < T$. So the limit function $w(y, T)$ obtained by Lemma 4.2.3 is smooth for $|y| \leq \delta/2$.

As the proof of Lemma 2.3.12, using maximum principle, $v_r(r, T) > 0$, $0 < t < \delta/2$. Consequently, $-b(T) = v(0, T) < v(\delta/2, T) < \eta_0(T)$. $\eta_0(T) < b(T)$ can be proved similarly. □

Theorem 4.2.5. *(Formation of singularities)*

1. If $m = 0$, $u(\eta_0(T), T) = 0$ and $b(T) = 0$. This implies that $\Gamma(t)$ shrinks to the origin O, as $t \to T$.

2. If $m \geq 1$, there is j such that $u(\xi_j(T), T) = 0$, $1 \leq j \leq m$.

Proof. 1. First, we prove for $m = 0$, i.e., $u(x, t)$ only has one maximum without local minimum. We prove this by contradiction.

Case 1. If $u(\eta_0(T), T) > 0$, from Lemma 4.2.4, $-b(T) < \eta_0(T) < b(T)$. $\Gamma(t)$ can be divided into three parts $\Delta_1(t)$, $\Delta_2(t)$ and $\Delta_3(t)$, for t being very close to T, where $\Delta_1(t)$ and $\Delta_2(t)$ are the left and right caps of $\Gamma(t)$, $\Delta_3(t)$ is the middle part of $\Gamma(t)$ away form x-axis. It is easy to show the derivatives and second fundamental forms of Δ_1, Δ_2 and Δ_3 are uniformly smooth for $t \to T$(We can similarly prove as in Lemma 2.3.12), which contradicts to $\Gamma(t)$ becoming singular at T.

Case 2. If $b(T) > 0$, there holds $-b(T) < \eta_0(T)$ or $\eta_0(T) < b(T)$, assuming $-b(T) < \eta_0(T)$. By Lemma 4.2.2, for every $[c, d] \subset (-b(T), \eta_0(T))$, $u(x, t) \geq \delta > 0$ in $[c, d] \times [t_1, T)$. Then $u(\eta_0(t), t) \geq \delta$, $t_1 \leq t < T$. Consequently, $u(\eta_0(T), T) \geq \delta$. By the same argument as in Case 1, we get a contradiction. Here we complete the proof under the condition $m = 0$.

2. For $m \geq 1$, if $u(\xi_j(T), T) > 0$, for any $1 \leq j \leq m$, we can divide $\Gamma(t)$ into three parts as above for t being close to T (seeing Figure 4.1). Then we can get contradiction similarly as in the case $m = 0$. So there is j such that $u(\xi_j(T), T) = 0$.

□

Corollary 4.2.6. *There is t_1 satisfying $0 < t_1 < T$ such that $u(x, t)$ loses all its local minima for $t \in [t_1, T)$. Moreover, $\Gamma(t)$ shrinks to a point, as $t \to T$.*

Proof. Denote $h(t) = \max_{-b(t)<x<b(t)} u(x, t)$. By Proposition 2.3.1, we can deduce that, for t satisfying $t_2 < t < T$ given, when $\rho < \min\{At_2, h(t)\}$, $y = \rho$ intersects $y = u(x, t)$ only twice.

We assume $u(x, t)$ does not lose its all local minima. From Theorem 4.2.5, there exists j, $1 \leq j \leq m$ such that $u(\xi_j(T), T) = 0$. So we can choose t_0 such that $t_2 < t_0 < T$ and $u(\xi_j(t_0), t_0) < At_2$. Obviously, $u(\xi_j(t_0), t_0) < h(t_0)$, then $u(\xi_j(t_0), t_0) <$

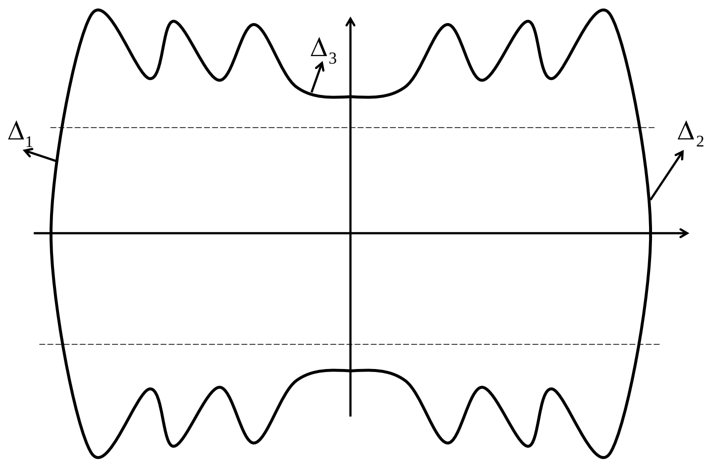

Figure 4.1: Case 2 in Theorem 4.2.5

$\min\{At_2, h(t_0)\}$. Consequently, $y = \rho = u(\xi_j(t_0), t_0)$ intersects $y = u(x, t_0)$ three times. Contradiction.

Therefore, there is t_1 such that $u(x,t)$ will lose its all local minima for $t \in [t_1, T)$. Seeing the proof in Theorem 4.2.5 for $m = 0$, $u(\eta_0(T), T) = 0$ and $b(T) = 0$. It means that $\Gamma(t)$ shrinks to a point, as $t \to T$. □

Remark 4.2.7. All the arguments in this section can be used to prove for any x-axisymmetric curve, since the fact that $u(\cdot, t)$ is even in $[-b(t), b(t)]$ is not used in the proofs.

4.3 Classification of the solutions

In this section, we will prove Theorem 4.1.1. Let

$$h(t) = \max_{-b(t) \le x \le b(t)} u(x,t), \quad l(t) = 2b(t).$$

Denote $U(t)$ being the open domain with $\partial U(t) = \Gamma(t)$.

Before proving the main results, we give a simple example for understanding the results. Consider a family of circles $\partial B_{R(t)}$ (here we omit the center) evolving by $V = -\kappa + A$. Therefore, $R(t)$ satisfies (2.3.12), precisely,

$$\begin{cases} R'(t) = A - \frac{1}{R(t)}, \; t > 0, \\ R(0) = R_0. \end{cases}$$

We can easily get that

(1). when $R_0 < 1/A$, there exists $T_{R_0} < \infty$ such that $R(t) \downarrow 0$, as $t \to T_{R_0}$;

(2). when $R_0 = 1/A$, $R(t) = 1/A$, for $0 \leq t < \infty$;

(3). when $R_0 > 1/A$, $R(t) \uparrow \infty$, as $t \to \infty$. Moreover, by L'Hospital rule,

$$\lim_{t\to\infty} R(t)/t = \lim_{t\to\infty} R'(t) = \lim_{t\to\infty} (A - 1/R(t)) = A.$$

This example shows a special case for our main results.

The following lemma says that $h(t_0)$ can become arbitrary large when $l(t_0)$ is large enough. We prove it by Proposition 2.3.1. Although the proof of Lemma 4.3.1 is similar as in [5], for the reader's convenience, we still give the proof for detail.

Lemma 4.3.1. *For any $\tau \in (0, T)$ and $M \in (0, A\tau/2)$, there exists $l_{M,\tau} > 0$ such that, when $l(t_0) > l_{M,\tau}$ for some $t_0 \in [\tau, T)$, it holds $h(t_0) > M$.*

Proof. For given $\tau \in (0, T)$ and $M \in (0, A\tau/2)$, we choose R_0 such that

$$R_0 \geq \frac{2}{A}.$$

Let $R(t)$ be the solution of (2.3.12) with $R(0) = R_0$. Since $R_0 > 1/A$, $R(t)$ increases in t. Therefore $R'(t) \geq A - 1/R_0 \geq A/2$. Integrating the inequality, there holds

$$R(\tau) \geq R_0 + A\tau/2 \geq R_0 + M.$$

So there exists $\tau_1 \in (0, \tau]$ such that

$$R(\tau_1) = R_0 + M.$$

Now we let

$$W(x,t) := \sqrt{R(t)^2 - x^2} - R_0, \quad x \in [\sigma_-(t), \sigma_+(t)], \ t \in (0, \tau_1],$$

where $\sigma_-(t) = -\sqrt{R(t)^2 - R_0^2}$ and $\sigma_+(t) = \sqrt{R(t)^2 - R_0^2}$. And we denote

$$\theta_\pm(t) = \arctan \frac{\sqrt{R(t)^2 - R_0^2}}{R_0}.$$

Obviously, $\pi/2 > \theta_\pm(t) > 0$. Therefore, $(W(x,t), \sigma_\pm(t))$ is the solution of (Q) with $\theta_\pm(t)$

We choose $l_{M,\tau} := \sigma_+(\tau_1) - \sigma_-(\tau_1) = 2\sqrt{R(\tau_1)^2 - R_0^2} = 2\sqrt{M^2 + 2R_0 M}$. Let $\gamma^1(t)$ and $\gamma^2(t)$ be the extension of $u(x,t)$ and $W(x,t)$ as in Proposition 2.3.1. So by Proposition 2.3.1, we can deduce

$$\mathcal{Z}(\gamma^1(t_0), \gamma^2(\tau_1)) \leq \mathcal{Z}(\gamma^1(t_0 - s), \gamma^2(\tau_1 - s)), \text{ for } s \in [0, \tau_1).$$

Since the extended curve $\gamma^2(\tau_1 - s)$ converges to the x-axis, as $s \to \tau_1$, the right-hand side of the above inequality equals 2 for s sufficiently close to τ_1. Consequently,

$$Z[\gamma^1(t_0), \gamma^2(\tau_1)] \leq 2.$$

Assuming $l(t_0) > l_{M,\tau}$, for some $t_0 \in [\tau, T)$, then $\sigma_\pm(\tau_1)$ satisfy

$$-b(t_0) < \sigma_1(\tau_-) < \sigma_+(\tau_1) < b(t_0).$$

Hence $\gamma^1(t_0)$ intersects $\gamma^2(\tau_1)$ twice below the x-axis. Therefore, $Z[\gamma^1(t_0), \gamma^2(\tau_1)] = 2$. This implies that $u(x, t_0) > W(x, \tau_1)$ on the interval $[\sigma_-(\tau_1), \sigma_+(\tau_1)]$. Consequently, $h(t_0) > M$. □

The following corollary shows that as long as $l(t)$ is unbounded, $\Gamma(t)$ will become "Expanding".

Corollary 4.3.2. *Assume $T = \infty$ and there exists a sequence $s_m \to \infty$ such that $l(s_m) \to \infty$, as $m \to \infty$. Then $l(t) \to \infty$ and $h(t) \to \infty$, as $t \to \infty$.*

Proof. We can use the same argument as in Lemma 4.3.1, there exist $C > 1/A$ and m_0 such that $u(x, s_{m_0}) > \sqrt{(C + R_0)^2 - x^2} - R_0$. Obviously, $\sqrt{(C + R_0)^2 - x^2} - R_0 > \sqrt{C^2 - x^2}$, $-C \le x \le C$. Therefore $u(x, s_{m_0}) \ge \sqrt{C^2 - x^2}$, $-C \le x \le C$. This implies that

$$B_C(O) \subset U(s_{m_0}),$$

recalling $\partial U(t) = \Gamma(t)$. Then by comparison principle, we get

$$B_{C(t)}(O) \subset U(t + s_{m_0}),\ t > 0.$$

Here $C(t)$ is the solution of (2.3.12) with $C(0) = C$. Seeing the choice of C and the example at the beginning of this section, we can deduce $C(t) \to \infty$, as $t \to \infty$. Therefore $h(t + s_{m_0}) > C(t) \to \infty$ and $l(t + s_{m_0}) > 2C(t) \to \infty$, as $t \to \infty$. □

Corollary 4.3.3. *If there exists a sequence $s_m \to \infty$, as $m \to \infty$ such that $h(s_m) \to 0$, as $m \to \infty$, then $l(s_m) \to 0$, as $m \to \infty$.*

Proof. Using Lemma 4.3.1 with $M = h(s_m)$,

$$l(s_m) \leq l_{M,\tau} = 2\sqrt{M^2 + 2R_0 M} = 2\sqrt{h(s_m)^2 + 2R_0 h(s_m)}$$

Then we have $l(s_m) \to 0$. □

Corollary 4.3.4. *If $T = \infty$, then $h(t)$ is bounded from below.*

Proof. If not, there exists another sequence $s_m \to \infty$ such that $h(s_m) \to 0$, as $s_m \to \infty$. By Corollary 4.3.3, $l(s_m) \to 0$, as $s_m \to \infty$. Then there exists s_{m_0} and $r < 1/A$ such that

$$U(s_{m_0}) \subset B_r((0,0)),$$

recalling $U(t)$ being the domain with $\partial U(t) = \Gamma(t)$. Then by Theorem 2.1.4, we have

$$U(t + s_{m_0}) \subset B_{r(t)}((0,0)),$$

where $r(t)$ is the solution of (2.3.12) with $r(0) = r$. Seeing the example at the beginning of this section, $B_{r(t)}((0,0))$ shrinks to origin in finite time. Then it is also for $U(t)$. This contradicts to $T = \infty$.

Hence $h(t)$ is bounded from below. □

Lemma 4.3.5. *Assume $T = \infty$ and there exists a sequence $s_m \to \infty$ such that $h(s_m) \to \infty$, as $m \to \infty$. Then $l(t) \to \infty$ and $h(t) \to \infty$, as $t \to \infty$.*

Proof. If $l(t)$ is unbounded, by Corollary 4.3.2, $h(t) \to \infty$ and $l(t) \to \infty$, $t \to \infty$. The result is true. Next we prove $l(t)$ is unbounded by contradiction. Assume $l(t)$ is bounded.

Step 1. We are going to prove that $\lim_{t\to\infty} b(t)$ exists. If $\liminf_{t\to\infty} b(t) < \limsup_{t\to\infty} b(t)$, we can choose x_0 such that

$$\liminf_{t\to\infty} b(t) < x_0 < \limsup_{t\to\infty} b(t).$$

We consider the function $u_1(x) = \sqrt{1/A^2 - (x + 1/A - x_0)^2}$. Obviously, $(u_1(x), x_0 - 2/A, x_0)$ is the solution of the problem (Q) with $\theta_\pm = \pi/2$.

Seeing $b(t) - x_0$ changes sign infinite many times as t varying over $[0, \infty)$, there exists a sequence $p_m \to \infty$ such that $u(x, p_m)$ intersects $u_1(x)$ tangentially at x_0.

Arguing as in Lemma 2.3.5, the intersection number between $u(x,t)$ and $u_1(x)$ drops at $x = b(p_m) = x_0$ and $t = p_m$. Therefore, the intersection number between $u(x,t)$ and $u_1(x)$ drops infinite many times. This yields a contradiction. Then we let $\nu := \lim_{t\to\infty} b(t)$.

Step 2. We deduce the contradiction.

Choose $t_1 = 4/A^2$. Since $h(s_m) \to \infty$, Lemma 2.3.9 implies for all $\rho < At_1$, $y = \rho$ intersects $y = u(x,t)$ only twice, $t > t_1$. Here we choose $\rho_0 = 2/A$. Then there exists $w(y,t) > 0$ such that

$$C_{\rho_0} \cap \Gamma(t) = \{(x,y) \mid x = w(y,t) \text{ or } x = -w(y,t)\},$$

recalling $C_\rho = \{(x,y) \in \mathbb{R}^2 \mid |y| < \rho\}$. Here $w(y,t)$ satisfies (2.2.1) under the condition "+" and $n = 1$, for $|y| < \rho_0$, $t > t_1$.

Since $w(0,t) = b(t)$ is bounded for $t > 0$, by Corollary 2.2.3 and Remark 2.2.4, $\frac{\partial^k w}{\partial y^k}(y,t)$, $k = 1,2,3$, are uniformly bounded for $|y| \le \rho_0/2$, $t > t_1 + \epsilon^2$. From equation, $\frac{\partial^k w}{\partial t^k} w(y,t)$, $k = 1,2$, are also bounded for $|y| < \rho_0/2$, $t > t_1 + \epsilon^2$. For any sequence $t_m \to \infty$ and any $[a,b] \subset [0,\infty)$, on a subsequence there exists $w_1(y,t)$ such that

$$w(\cdot, \cdot + t_{m_j}) \to w_1 \text{ in } C^{2,1}([-\rho_0/2, \rho_0/2] \times [a,b]),$$

as $j \to \infty$. The limit function $w_1(y,t)$ also satisfies (2.2.1) with the condition "+" and $n = 1$. Moreover, $w_1(0,t) = \lim_{j\to\infty} b(t + t_{m_j}) = \nu$ and $\frac{\partial}{\partial y} w_1(0,t) = 0$, $t \in [a,b]$.

Next, we consider the function $w_2(y) = \nu - 1/A + \sqrt{1/A^2 - y^2}$. $w_2(y)$ satisfies (2.2.1) with the condition "+" and $n = 1$. Moreover, $w_2(0) = \nu$ and $\frac{\partial}{\partial y} w_2(0) = 0$. So

$w_1(y,t)$ intersects $w_2(y)$ at $y=0$ tangentially for all $t \in [a,b]$. By intersection number principle, there holds $w_1(y,t) \equiv w_2(y)$, $|y| \le \rho_0/2$. Then we have $\frac{\partial w_1}{\partial y}(1/A,t) = \frac{\partial w_2}{\partial y}(1/A) = \infty$. However, from the definition of w_1, $\frac{\partial w_1}{\partial y}(y,t)$ is bounded for $|y| \le 1/A$ (w_1 is independent of t). This is a contradiction.

We complete the proof. □

Lemma 4.3.6. *Assume $T = \infty$. If $h(t)$ is bounded from above, then $h(t)$ and $l(t)$ are bounded from above and below.*

Proof. By Corollary 4.3.4 and 4.3.2, $h(t)$ is bounded from below and $l(t)$ is bounded from above.

Next, we prove $l(t)$ is bounded from below by contradiction. Assume there exists a sequence $s_m \to \infty$ such that $l(s_m) \to 0$.

Since $h(t)$ is bounded from below, by Lemma 2.3.9, there exist ρ_0 and t_1 such that for all $\rho < \rho_0$, $y = \rho$ intersects $y = u(x,t)$ only twice for $t > t_1$. Then we let $w(y,t) > 0$ such that

$$C_{\rho_0} \cap \Gamma(t) = \{(x,y) \mid x = w(y,t) \text{ or } x = -w(y,t)\}.$$

Arguing as Step 1 in Lemma 4.3.5, $\nu = \lim_{t\to\infty} b(t)$. By $l(s_m) \to 0$, we have $\nu = 0$. Arguing as Step 2 in Lemma 4.3.5,

$$w(\cdot,t) \to w_1 \text{ in } C^2([0,\rho_0/2]),$$

as $t \to \infty$. Here $w_1(y) = -1/A + \sqrt{1/A^2 - y^2} \le 0$. But seeing $w(y,t) > 0$, there holds $w_1(y) \ge 0$ for $|y| < \rho_0/2$. Contradiction.

Therefore $h(t)$ and $l(t)$ are bounded from below. □

Proof of Theorem 4.1.1. 1. Maximal smooth time $T < \infty$. As shown in Corollary 4.2.6, $l(t) \to 0$ and $h(t) \to 0$, as $t \to T$.

2. Maximal smooth time $T = \infty$.

2-1. $h(t)$ is unbounded, Lemma 4.3.5 shows that $l(t) \to \infty$, $h(t) \to \infty$, as $t \to \infty$. This yields the case "Expanding".

2-2. $h(t)$ is bounded, Lemma 4.3.6 shows that $l(t)$ and $h(t)$ are bounded from above and below. This yields the case "Bounded". □

4.4 Asymptotic behavior

In this section, we prove Theorem 4.1.2.

Proof of the Expanding case in Theorem 4.1.2. In this case, since $h(t)$ and $l(t)$ tend to infinity, using the same argument as in the proof of Corollary 4.3.2, there exist t_0 and $C > 1/A$ such that

$$B_C((0,0)) \subset U(t_0).$$

Therefore, $B_{C(t)}((0,0)) \subset U(t_0 + t)$, where $C(t)$ satisfies (2.3.12) with $C(0) = C$. Consequently, $B_{C(t-t_0)}((0,0)) \subset U(t)$, $t \geq t_0$,

On the other hand, seeing $U(0)$ being bounded, there exists $R > 1/A$ such that $U(0) \subset B_R((0,0))$. Then $U(t) \subset B_{R(t)}((0,0))$, where $R(t)$ also satisfies (2.3.12) with $R(0) = R$.

Denoting $R_1(t) = C(t-t_0)$ and $R_2(t) = R(t)$, $B_{R_1(t)}((0,0)) \subset U(t) \subset B_{R_2(t)}((0,0))$, $t > t_0$. By the example at the beginning of this section, we have $\lim_{t\to\infty} R_1(t)/t = \lim_{t\to\infty} R_2(t)/t = A$. We complete the proof. □

Before proving the asymptotic behavior of the condition "Bounded", we give the following lemma.

Lemma 4.4.1. *Under the condition "Bounded", there exists t_* such that $u_{xx}(x,t) <$*

0 for $x \in (-b(t), b(t))$, $t > t_*$. Recalling

$$\Gamma(t) = \{(x, y) \mid |y| = u(x, t),\ -b(t) \le x \le b(t)\}.$$

By Proposition 2.3.1, we can easily prove that there exists t_* such that for all straight line l, l intersects $\Gamma \cap \{y \ge 0\}$ at most twice. This proof is similar as Theorem 1.3 in [5]. Here we omit it.

Lemma 4.4.1 implies that $\Gamma(t)$ is convex for $t > t_*$. Noting $\Gamma(t)$ is symmetric to x,y-axis, $\Gamma(t)$ can be represented by polar coordinates. Let

$$\Gamma(t) = \{(x, y) \mid x = \rho(\theta, t)\cos\theta,\ y = \rho(\theta, t)\sin\theta,\ 0 \le \theta \le 2\pi\},\ t > t_*.$$

By calculation, it is easy to check ρ satisfies the following equation

$$\begin{cases} \rho_t = \dfrac{\rho_{\theta\theta}}{\rho^2 + \rho_\theta^2} - \dfrac{2\rho_\theta^2 + \rho^2}{\rho(\rho_\theta^2 + \rho^2)} + \dfrac{1}{\rho} A\sqrt{\rho_\theta^2 + \rho^2},\ 0 < \theta < 2\pi,\ t > t_*, \\ \rho(0, t) = \rho(2\pi, t),\ t > t_*, \\ \rho_\theta(0, t) = \rho_\theta(2\pi, t),\ t > t_*. \end{cases} \tag{4.4.1}$$

Lemma 4.4.2. *Under the condition "Bounded", there exist two positive constants ρ_1 and ρ_2 such that*

$$\rho_1 < \rho(\theta, t) < \rho_2,\ 0 \le \theta \le 2\pi,\ t > t^*.$$

Moreover, $\rho_\theta(\theta, t)$, $\rho_{\theta\theta}(\theta, t)$ and $\rho_{\theta\theta\theta}(\theta, t)$ are bounded for $0 \le \theta \le 2\pi,\ t > t^*$.

Proof. Combining $\Gamma(t)$ is convex and $b(t)$, $l(t)$ are bounded from above and below, obviously, ρ is bounded from above and below.

Then by quasilinear parabolic theory in [8], $\rho_\theta(\theta, t)$, $\rho_{\theta\theta}(\theta, t)$ and $\rho_{\theta\theta\theta}(\theta, t)$ are bounded for $0 \le \theta \le 2\pi,\ t > t^*$. □

Proof of the Bounded case in Theorem 4.1.2. For any sequence $s_m \to \infty$, by Lemma 4.4.2 and interval $[a,b] \subset [0,\infty)$, there exists $\rho^* \in C^{2,1}([0,2\pi]\times[a,b])$ such that on a subsequence

$$\rho(\cdot, \cdot + s_{m_j}) \to \rho^* \text{ in } C^{2,1}([0,2\pi]\times[a,b]).$$

Let $x = \rho^*(\theta,t)\cos\theta$ and $u^*(x,t) = \rho^*(\theta,t)\sin\theta$. And let $b^*(t) = \rho^*(0,t)$. Therefore, $(u^*, -b^*, b^*)$ is a solution of problem (Q) with $\theta_\pm(t) = \pi/2$. By the same method as Step 1 in Lemma 4.3.5, for some positive constant ν, the limit function $b^*(t) \equiv \nu$, $t \in [a,b]$.

Consider the function $u_* = \sqrt{1/A^2 - (x + 1/A - \nu)^2}$. Obviously, $(u_*, \nu - 2/A, \nu)$ is the solution of problem (Q) with $\theta_\pm(x) = \pi/2$. Obviously, $u^*(x,t)$ intersects $u_*(x)$ at $(\nu, 0)$ tangentially, for all $t \in [a,b]$. Then $u^*(x,t) \equiv u_*(x)$, $t \in [a,b]$. Since u^* is symmetric to y-axis, $\nu = 1/A$. Consequently, $\rho^* = 1/A$. Seeing the limit function ρ^* is independent of the choice of the subsequence s_{m_j}, we have

$$\rho(\cdot, t) \to \rho^* \text{ in } C^2([0,2\pi]).$$

Here we complete the proof $\square$

4.5 Comparison principle between extrinsic and intrinsic distances

To get the asymptotic behavior for Shrinking, in this section, we give a comparison principle between extrinsic and intrinsic distances.

First, we give some basic results for general mean curvature flow with driving force. For $A = 0$, the results are proved by Gage and Hamilton in [3]. Let M be an one-dimension Riemannian manifold and $\mathbf{F} : M \times [0,T) \to \mathbb{R}^2$ be a smooth map. $\mathbf{F}$

satisfies

$$\frac{\partial}{\partial t}\mathbf{F}(p,t) = \kappa\mathbf{N} - A\mathbf{N}, \tag{4.5.1}$$

where the sign of κ is determined by

$$\frac{\partial^2}{\partial s^2}\mathbf{F}(s,t) = \kappa\mathbf{N},$$

where we recall $\mathbf{N}$ is the unit inner normal velocity, s is the arc length parameter.

In this section, for convenience, we take $M = \mathbb{S}^1$ with parameter p. Let $\mathbf{F} : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a closed embedded curve moving by (4.5.1).

Using the arclength parameter s,

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial p},$$

where $v = |\partial\mathbf{F}/\partial p|$. The sign of κ will be determined by

$$\frac{\partial^2\mathbf{F}}{\partial s^2} = \kappa\mathbf{N}.$$

Let $\mathbf{T}$ be the unit tangent vector given by

$$\mathbf{T} = \frac{\partial\mathbf{F}/\partial p}{|\partial\mathbf{F}/\partial p|}.$$

The Frenet equations show that

$$\frac{1}{v}\frac{\partial}{\partial p}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\end{pmatrix} = \begin{pmatrix}0 & \kappa\\ -\kappa & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\end{pmatrix}.$$

Define θ by $\mathbf{T} = (\cos\theta, \sin\theta)$. We can deduce that

$$\frac{\partial s}{\partial \theta} = \frac{1}{\kappa}.$$

Lemma 4.5.1.

$$\frac{\partial v}{\partial t} = -\kappa^2 v + A\kappa v.$$

Proof. By (4.5.1) and the Frenet equations,

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial|\partial \mathbf{F}/\partial p|}{\partial t} = \langle \mathbf{T}, \mathbf{F}_{pt}\rangle = \langle \mathbf{T}, \mathbf{F}_{tp}\rangle = \langle \mathbf{T}, (\kappa\mathbf{N} - A\mathbf{N})_p\rangle \\ &= \langle \mathbf{T}, (\kappa - A)\mathbf{N}_p\rangle = \langle \mathbf{T}, -v\kappa(\kappa - A)\mathbf{T}\rangle = -\kappa^2 v + A\kappa v.\end{aligned}$$

□

Lemma 4.5.2. *Denote* $l = \int_{p_1}^{p_2} v dp = s(p_2) - s(p_1)$, $p_1, p_2 \in \mathbb{S}^1$, *then*

$$\frac{\partial l}{\partial t} = A\int_{s(p_1)}^{s(p_2)} \kappa ds - \int_{s(p_1)}^{s(p_2)} \kappa^2 ds.$$

In particular, $dL(t)/dt = 2\pi A - \int_0^{L(t)} \kappa^2 ds$, *where we recall* $L(t)$ *is the perimeter of the curve.*

Proof. Using $\partial v/\partial t = -\kappa^2 v + A\kappa v$ and $\partial\theta/\partial s = \kappa$, this lemma can be proved at once. □

We note that the arc length parameter s depends on t, then $\partial/\partial t$ does not commute with $\partial/\partial s$. The following lemma gives the relation between them.

Lemma 4.5.3.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + (\kappa^2 - A\kappa)\frac{\partial}{\partial s}.$$

Proof. Apply Lemma 4.5.1, we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s} &= \frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial}{\partial p}\right) = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{v}\right)\frac{\partial}{\partial p} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} - \frac{\partial v/\partial t}{v^2}\frac{\partial}{\partial p} \\ &= \frac{\partial}{\partial s}\frac{\partial}{\partial t} + (\kappa^2 - A\kappa)\frac{\partial}{\partial s}.\end{aligned}$$

□

The derivatives of **T** and **N** are related as follows:

Lemma 4.5.4.

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \kappa}{\partial s}\mathbf{N} \text{ and } \frac{\partial \mathbf{N}}{\partial t} = -\frac{\partial \kappa}{\partial s}\mathbf{T}.$$

Proof. By Lemma 4.5.3, (4.5.1) and Frenet equations,

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial t} &= \frac{\partial^2 \mathbf{F}}{\partial t \partial s} = \frac{\partial^2 \mathbf{F}}{\partial s \partial t} + (\kappa^2 - A\kappa)\frac{\partial \mathbf{F}}{\partial s} = \frac{\partial}{\partial s}(\kappa \mathbf{N} - A\mathbf{N}) + (\kappa^2 - A\kappa)\mathbf{T} \\ &= \frac{\partial \kappa}{\partial s}\mathbf{N} - (\kappa^2 - A\kappa)\mathbf{T} + (\kappa^2 - A\kappa)\mathbf{T} = \frac{\partial \kappa}{\partial s}\mathbf{N}.\end{aligned}$$

On the other hand,

$$0 = \frac{\partial}{\partial t}\langle \mathbf{T}, \mathbf{N}\rangle = \langle \frac{\partial \kappa}{\partial s}\mathbf{N}, \mathbf{N}\rangle + \langle \mathbf{T}, \frac{\partial \mathbf{N}}{\partial t}\rangle.$$

Note that $\partial \mathbf{N}/\partial t$ must be perpendicular to **N**. We complete the proof. □

Lemma 4.5.5.

$$\frac{\partial \theta}{\partial t} = \frac{\partial \kappa}{\partial s}$$

Proof. Since $\mathbf{T} = (\cos\theta, \sin\theta)$

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \theta}{\partial t}(-\sin\theta, \cos\theta).$$

On the other hand, we use the formula in Lemma 4.5.4 to calculate

$$\frac{\partial \mathbf{T}}{\partial t} = -\frac{\partial \kappa}{\partial s}\mathbf{N} = \frac{\partial \kappa}{\partial s}(-\sin\theta, \cos\theta).$$

Comparing components the proof is completed. □

Lemma 4.5.6. *Let $S(t)$ be the area enclosed by the curve $\mathbf{F}(\cdot, t)$. Then*

$$\frac{d}{dt}S(t) = -2\pi + AL(t).$$

Proof. By Gauss-Green's Theorem,

$$S(t) = -\frac{1}{2}\int_0^{L(t)} \langle \mathbf{F}, \mathbf{N} \rangle ds.$$

Using above lemmas, we get

$$\begin{aligned}
\frac{d}{dt}S(t) &= -\frac{1}{2}\int_0^{2\pi} \langle \frac{\partial \mathbf{F}}{\partial t}, v\mathbf{N} \rangle dp - \frac{1}{2}\int_0^{2\pi} \langle \mathbf{F}, \frac{\partial v}{\partial t}\mathbf{N} \rangle dp - \frac{1}{2}\int_0^{2\pi} \langle \mathbf{F}, v\frac{\partial \mathbf{N}}{\partial t} \rangle dp \\
&= -\frac{1}{2}\int_0^{2\pi} \langle \kappa\mathbf{N} - A\mathbf{N}, v\mathbf{N} \rangle dp - \frac{1}{2}\int_0^{2\pi} \langle \mathbf{F}, (-\kappa^2 v + A\kappa v)\mathbf{N} \rangle dp \\
&+ \frac{1}{2}\int_0^{2\pi} \langle \mathbf{F}, v\frac{\partial \kappa}{\partial s}\mathbf{T} \rangle dp = -\pi + \frac{1}{2}AL(t) - \frac{1}{2}\int_0^{L(t)} \langle \mathbf{F}, A\kappa\mathbf{N} \rangle ds \\
&+ \frac{1}{2}\int_0^{L(t)} \langle \mathbf{F}, \kappa^2\mathbf{N} \rangle ds - \frac{1}{2}\int_0^{L(t)} \kappa ds - \frac{1}{2}\int_0^{L(t)} \kappa^2 \langle \mathbf{F}, \mathbf{N} \rangle ds \\
&= -2\pi + \frac{1}{2}AL(t) - \frac{1}{2}\int_0^{L(t)} \langle \mathbf{F}, A\frac{\partial \mathbf{T}}{\partial s} \rangle ds = -2\pi + \frac{1}{2}AL(t) + \frac{A}{2}\int_0^{L(t)} ds \\
&= -2\pi + AL(t).
\end{aligned}$$

In the third and fifth equalities, we use the integral by parts. □

Next we are going to prove the comparison principle for extrinsic and intrinsic distances under mean curvature flow with driving force in a special case.

Theorem 4.5.7. *For our flow*

$$\Gamma(t) = \{(x, y) \mid |y| = u(x, t), -b(t) \leq x \leq b(t)\}, t_0 < t < T,$$

let $d = 2x$ *and* $l(x,t) = \int_{-x}^{x} \sqrt{1+u_x^2}dx$, $0 \le x \le b(t)$. *Then*

$$m(t) = \min_{0\le x\le b(t)} d/\psi$$

is strictly increasing provided that $m(t) < 1$, *for* $t_0 < t < T$, *where*

$$\psi = \frac{L}{\pi}\sin\frac{l\pi}{L},$$

where we recall t_0 *is defined in Section 4.1 such that* $u(x,t)$ *loses all its local minimum,* $t_0 < t < T$.

Remark 4.5.8. (1) The quantities d and l are the extrinsic and intrinsic distances between $(-x, u(x,t))$ and $(x, u(x,t))$ and $l \le L(t)/2$. Hence $d = 2x$ and $l = 2\int_0^x \sqrt{1+u_x^2}dx$.

(2) Noting that $\lim_{x\to 0^+} d/\psi = 1$, d/ψ can not attain its minimum which is less than 1 at $x = 0$.

Proof. Case 1: Let $0 < x_0 < b(t)$ be a minimum point of d/ψ defined through the relation

$$m(t) = (d/\psi)(x_0, t).$$

Then

$$\frac{\partial^2}{\partial x^2}\frac{d}{\psi}(x_0,t) \ge 0$$

and

$$0 = \frac{\partial}{\partial x}\frac{d}{\psi}(x_0,t) = \frac{2}{\psi} - \frac{2d\cos\alpha}{\psi^2}\sqrt{1+u_x^2},$$

where $\alpha = l(x_0,t)\pi/L$. Consequently,

$$\frac{1}{\sqrt{1+u_x^2}} = \frac{d}{\psi}\cos\alpha,$$

at $x = x_0$. Let $0 < \beta < \pi/2$ satisfy $\tan\beta = -u_x(x_0,t)$(recall $u_x(x_0,t) < 0$), then

$$\cos\beta = \frac{1}{\sqrt{1+u_x^2}}(x_0,t) = \left(\frac{d}{\psi}\cos\alpha\right)(x_0,t). \tag{4.5.2}$$

Since $d/\psi(x_0,t) < 1$, we observe that $0 < \alpha < \beta < \pi/2$. Moreover,

$$\begin{aligned} 0 &\leq \frac{\partial^2}{\partial x^2}\frac{d}{\psi}(x_0,t) = -\frac{4\cos\alpha}{\psi^2}\sqrt{1+u_x^2} - \frac{4\cos\alpha}{\psi^2}\sqrt{1+u_x^2} + \frac{8d}{\psi^3}\cos^2\alpha(1+u_x^2) \\ &+ \frac{4\pi d\sin\alpha}{L\psi^2}(1+u_x^2) - \frac{2d\cos\alpha}{\psi^2}\frac{u_x u_{xx}}{\sqrt{1+u_x^2}} = \frac{4\pi d\sin\alpha}{L\psi^2}(1+u_x^2) - \frac{2d\cos\alpha}{\psi^2}\frac{u_x u_{xx}}{\sqrt{1+u_x^2}} \\ &= \frac{4\pi^2 d}{L^2\psi}(1+u_x^2) - \frac{2d\cos\alpha}{\psi^2}\frac{u_x u_{xx}}{\sqrt{1+u_x^2}}, \end{aligned}$$

where we invoke (4.5.2) and $\psi = L/\pi\sin(l\pi/L)$. Consequently,

$$-\frac{2d\cos\alpha}{\psi^2}\frac{u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0,t) \geq -\frac{4\pi^2 d}{L^2\psi}(x_0,t). \tag{4.5.3}$$

$$\frac{\partial l}{\partial t}(x_0,t) = \frac{\partial}{\partial t}\left(\int_{-x}^{x}\sqrt{1+u_x^2}dx\right)(x_0,t) = \int_{-x_0}^{x_0}\frac{u_x}{\sqrt{1+u_x^2}}du_t = \frac{2u_x u_t}{\sqrt{1+u_x^2}}(x_0,t)$$

$$\begin{aligned} &- \int_{-x_0}^{x_0}\frac{u_t u_{xx}}{(1+u_x^2)^{3/2}}dx = \frac{2u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0,t) + 2Au_x(x_0,t) - \int_0^l \kappa^2 ds \\ &- 2A\arctan u_x(x_0,t) = \frac{2u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0,t) - 2A\tan\beta - \int_0^l \kappa^2 ds + 2A\beta, \end{aligned}$$

where we again invoke (4.5.2) and $\tan\beta = -u_x(x_0,t)$. Using the Hölder inequality, we have

$$l\int_0^l \kappa^2 ds \geq \left(\int_0^l \kappa ds\right)^2 = 4\beta^2 \tag{4.5.4}$$

and

$$L\int_0^L \kappa^2 ds \geq \left(\int_0^L \kappa ds\right)^2 = 4\pi^2. \tag{4.5.5}$$

$$\begin{aligned}
m'(t) &= \frac{d}{dt}\left(\frac{d}{\psi}\right)(x_0,t) = -\frac{d\sin\alpha}{\psi^2\pi}\left(2\pi A - \int_0^L \kappa^2 ds\right) + \frac{dl\cos\alpha}{\psi^2 L}\left(2\pi A - \int_0^L \kappa^2 ds\right) \\
&- \frac{d\cos\alpha}{\psi^2}\left(\frac{2u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0,t) - 2A\tan\beta - \int_0^l \kappa^2 ds + 2A\beta\right) \\
&= \frac{2Ad\cos\alpha}{\psi^2}\big((\tan\beta-\beta)-(\tan\alpha-\alpha)\big) - \frac{d\cos\alpha}{\psi^2}\frac{2u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0,t) + \frac{d\sin\alpha}{\psi^2\pi}\int_0^L \kappa^2 ds \\
&- \frac{dl\cos\alpha}{\psi^2 L}\int_0^L \kappa^2 ds + \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds > -\frac{4d\pi^2}{\psi L^2} + \frac{d\cos\alpha}{\psi^2\pi}(\tan\alpha-\alpha)\int_0^L \kappa^2 ds \\
&+ \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds \geq -\frac{4d\pi^2}{\psi L^2} + \frac{4\pi d\cos\alpha}{\psi^2 L}(\tan\alpha-\alpha) + \frac{4\beta^2 d\cos\alpha}{l\psi^2} = -\frac{4\pi d\alpha\cos\alpha}{\psi^2 L} \\
&+ \frac{4\beta^2 d\cos\alpha}{l\psi^2} = \frac{4\beta^2 d\cos\alpha}{l\psi^2} - \frac{4\alpha^2 d\cos\alpha}{l\psi^2} > 0,
\end{aligned}$$

where we use (4.5.2), (4.5.3), (4.5.4), (4.5.5) and $\tan\alpha-\alpha$ is increasing, $0<\alpha<\pi/2$.

Case 2: For $x_0 = b(t)$ such that

$$m(t) = (d/\psi)(x_0,t).$$

Since $u(x,t)$ is increasing for $-b(t) < x < 0$ and decreasing for $0 < x < b(t)$, $t_0 < t < T$, we let $x = v(y,t)$ be the inverse of $y = u(x,t)$ in the first quadrant. Consider

$$\mathcal{L}(y,t) = \begin{cases} 2\int_y^{h(t)} \sqrt{1+v_y^2(y,t)}dy, \ y>0, \\ L(t) - 2\int_y^{h(t)} \sqrt{1+v_y^2(y,t)}dy, \ y \leq 0, \end{cases}$$

recalling $h(t) = u(0,t) = \max\limits_{-b(t)<x<b(t)} u(x,t)$.

It is easy to see $l(x,t) = \mathcal{L}(u(x,t),t)$, for $0 \leq x \leq b(t)$, specially, $l(b(t),t) = \mathcal{L}(0,t)$. Since $y=0$ is an interior point and $\psi = \frac{L}{\pi}\sin\frac{\mathcal{L}\pi}{L}$ is smooth, we can prove this case similarly as in case 1. The proof is now complete. $\square$

Similarly, we can obtain

Theorem 4.5.9. *For our flow*

$$\Gamma(t) = \{(x, y) \mid |y| = u(x, t), -b(t) \le x \le b(t)\},\ t_0 < t < T,$$

where t_0 is the same as in Theorem 4.5.7. Let

$$d = 2y, \text{ and } l = 2\int_0^y \sqrt{1 + v_y^2(y, t)}dy,\ 0 \le y \le h(t),$$

where $v(y, t)$ is the inverse of $u(x, t)$ in the first quadrant as in the proof of Theorem 4.5.7. Then

$$m(t) = \min_{0 \le y \le h(t)} d/\psi$$

is strictly increasing provided that $m(t) < 1$, $t_0 < t < T$.

Using Theorems 4.5.7 and 4.5.9, we obtain

Corollary 4.5.10. *There exists a constant $C > 0$ such that*

$$d \ge Cl,\ t_0 < t < T,$$

where d and l are the extrinsic and intrinsic distances in Theorem 4.5.7 or 4.5.9. In particular,

$$h(t) \ge CL(t) \text{ and } b(t) \ge CL(t), t_0 < t < T.$$

Remark 4.5.11. To explain the geometric meaning in the proof of Theorem 4.5.7, we will give the calculation in geometric method for closed curve moving by (4.5.1).

Let $\mathbf{F} : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a family of closed embedded curves moving by (4.5.1). In this remark, we let

$$d(p_1, p_2, t) = |\mathbf{F}(p_1, t) - \mathbf{F}(p_2, t)|,\ l(p_1, p_2, t) = |s(p_1) - s(p_2)|,$$

where s denotes the arc length parameter at time t. ψ is also defined as in Theorem

4.5.7 by
$$\psi = \frac{L}{\pi} \sin \frac{l\pi}{L}.$$

We define
$$m(t) = \min_{(p_1,p_2)\in\mathbb{S}^1\times\mathbb{S}^1} d/\psi(p_1,p_2,t).$$

Assume that d/ψ attains its minimum at $(p_1,p_2) \in \mathbb{S}^1 \times \mathbb{S}^1$, i.e.,

$$m(t) = (d/\psi)(p_1,p_2,t) < 1.$$

Here we abuse the notation (p_1,p_2) to shorten the notations in the following argument.

Let s be the arc length parameter at time t and without loss of generality $0 \le s(p_1) < s(p_2) < L/2$ such that $l(p_1,p_2,t) = s(p_2) - s(p_1)$. Next we represent l, d by arclength parameter

$$l = s_2 - s_1 \text{ and } d = |\mathbf{F}(s_1,t) - \mathbf{F}(s_2,t)|.$$

Then

$$\frac{\partial}{\partial s_i}(d/\psi)(p_1,p_2,t) = 0,\ i = 1,2 \text{ and } \Big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\Big)_{2\times 2}(p_1,p_2,t) \ge 0.$$

Let
$$e_i := \frac{\partial \mathbf{F}}{\partial s_i}(p_1,p_2,t) \text{ and } \omega := \frac{\mathbf{F}(p_2,t) - \mathbf{F}(p_1,t)}{d(p_1,p_2,t)}.$$

Then there holds

$$0 = \frac{\partial}{\partial s_1}(d/\psi)(p_1,p_2,t) = -\frac{\langle \omega, e_1 \rangle}{\psi} + \frac{d}{\psi^2}\cos\alpha,$$

where $\alpha = l(p_1, p_2, t)\pi/L = (s(p_2) - s(p_1))\pi/L \in (0, \pi/2)$. Consequently,

$$\langle \omega, e_i \rangle = \frac{d}{\psi} \cos\alpha, \;\; i = 1, 2 \tag{4.5.6}$$

at (p_1, p_2, t). We can choose $0 < \beta < \pi/2$ such that

$$\cos\beta = \langle \omega, e_i \rangle = d/\psi \cos\alpha < \cos\alpha. \tag{4.5.7}$$

Then $\beta > \alpha$.

Since matrix $\big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\big)_{2\times 2}(p_1, p_2, t)$ is non-negative, then for every vector $\xi \in \mathbb{R}^2$ there holds

$$\xi\Big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\Big)_{2\times 2}(p_1, p_2, t)\xi^t \geq 0, \tag{4.5.8}$$

where ξ^t denotes the transposition of ξ.

In view of relations of (4.5.6), there are two possible cases:

Case 1: $e_1 = e_2$. We choose $\xi = (1, 1)$ in (4.5.8).

$$0 \leq (1,1)\Big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\Big)_{2\times 2}(p_1, p_2, t)(1,1)^t = \frac{1}{\psi}\langle \omega, (\kappa\mathbf{N})(p_2, t) - (\kappa\mathbf{N})(p_1, t)\rangle. \tag{4.5.9}$$

Case 2: $e_1 \neq e_2$. We choose $\xi = (1, -1)$ in (4.5.8).

$$\begin{aligned} 0 &\leq (1,-1)\Big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\Big)_{2\times 2}(p_1, p_2, t)(1,-1)^t \\ &= \frac{1}{\psi}\langle \omega, (\kappa\mathbf{N})(p_2, t) - (\kappa\mathbf{N})(p_1, t)\rangle + \frac{4\pi^2 d}{L^2\psi}. \end{aligned}$$

Then

$$-\frac{4\pi^2 d}{L^2\psi} \leq \frac{1}{\psi}\langle \omega, (\kappa\mathbf{N})(p_2, t) - (\kappa\mathbf{N})(p_1, t)\rangle. \tag{4.5.10}$$

Since there is no t derivative in above calculation, more precise calculation is necessary which is found in [7], Theorem 2.3. Here we safely omit it.

Therefore, by (4.5.1) and Lemma 4.5.2

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &= -\frac{d}{\psi^2}\frac{\partial\psi}{\partial t}+\frac{1}{\psi}\frac{\partial d}{\partial t} = -\frac{d}{\psi^2}\left(\frac{1}{\pi}\frac{dL}{dt}\sin\alpha+\frac{\partial l}{\partial t}\cos\alpha-\frac{l}{L}\frac{dL}{dt}\cos\alpha\right) \\
&+ \frac{1}{d\psi}\langle\omega, \frac{\partial}{\partial t}\mathbf{F}(p_2,t)-\frac{\partial}{\partial t}\mathbf{F}(p_1,t)\rangle = -\frac{d}{\psi^2}\left(\frac{1}{\pi}(2\pi A-\int_0^L\kappa^2 ds)\sin\alpha\right. \\
&+ \left.(A\int_0^l\kappa ds-\int_0^l\kappa^2 ds)\cos\alpha-\frac{l}{L}(2\pi A-\int_0^L\kappa^2 ds)\cos\alpha\right) \\
&+ \frac{1}{d\psi}\langle\omega,(\kappa-A)\mathbf{N}(p_2,t)-(\kappa-A)\mathbf{N}(p_1,t)\rangle = -\frac{2Ad}{\psi^2}\sin\alpha \\
&- \frac{dA}{\psi^2}\cos\alpha\int_0^l\kappa ds+\frac{2\pi dlA}{\psi^2 L}\cos\alpha-\frac{A}{\psi}\langle\omega,\mathbf{N}(p_2,t)-\mathbf{N}(p_1,t)\rangle \\
&+ \frac{1}{\psi}\langle\omega,(\kappa\mathbf{N})(p_2,t)-(\kappa\mathbf{N})(p_1,t)\rangle+\frac{d\sin\alpha}{\pi\psi^2}\int_0^L\kappa^2 ds+\frac{d\cos\alpha}{\psi^2}\int_0^l\kappa^2 ds \\
&- \frac{dl}{\psi^2 L}\cos\alpha\int_0^L\kappa^2 ds.
\end{aligned}$$

In the following step, we assume that

$$-\frac{A}{\psi}\langle\omega,\mathbf{N}(p_2,t)-\mathbf{N}(p_1,t)\rangle > 0. \tag{4.5.11}$$

Seeing Figure 4.2, there holds

$$-\frac{A}{\psi}\langle\omega,\mathbf{N}(p_2,t)-\mathbf{N}(p_1,t)\rangle = \frac{2A}{\psi}\sin\beta. \tag{4.5.12}$$

Case 1: $e_1 = e_2$. By calculation,

$$\frac{dA}{\psi^2}\cos\alpha\int_0^l\kappa ds = 0.$$

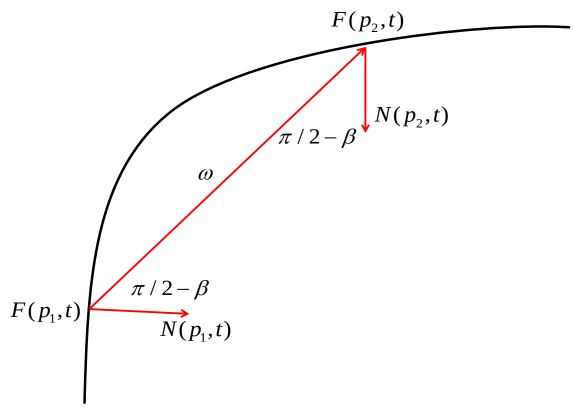

Figure 4.2: Assumption (4.5.11)

Then

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &\geq -\frac{2Ad}{\psi^2}\sin\alpha+\frac{2\pi dlA}{\psi^2 L}\cos\alpha+\frac{2A}{\psi}\sin\beta+\frac{d\sin\alpha}{\pi\psi^2}\int_0^L\kappa^2 ds+\frac{d\cos\alpha}{\psi^2}\int_0^l\kappa^2 ds\\ &- \frac{dl}{\psi^2 L}\cos\alpha\int_0^L\kappa^2 ds\geq\frac{2A}{\psi}\left(\sin\beta-\frac{d}{\psi}\sin\alpha\right)+\frac{d}{\pi\psi^2}(\sin\alpha-\alpha\cos\alpha)\int_0^L\kappa^2 ds\\ &> 0,\end{aligned}$$

where we use (4.5.7), (4.5.9), $d/\psi < 1$ and $\sin\alpha-\alpha\cos\alpha>0$, for $0<\alpha<\pi/2$.

Case 2: $e_1 \neq e_2$.

Using Hölder inequality,

$$l\int_0^l \kappa^2 ds \geq \left(\int_0^l \kappa ds\right)^2 = 4\beta^2$$

and

$$L\int_0^L \kappa^2 ds \geq \left(\int_0^L \kappa ds\right)^2 = 4\pi^2.$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &\geq -\frac{2Ad}{\psi^2}\sin\alpha - \frac{2\beta dA}{\psi^2}\cos\alpha + \frac{2\pi dlA}{\psi^2 L}\cos\alpha + \frac{2A}{\psi}\sin\beta + \frac{d\sin\alpha}{\pi\psi^2}\int_0^L \kappa^2 ds \\
&+ \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds - \frac{dl}{\psi^2 L}\cos\alpha\int_0^L \kappa^2 ds - \frac{4\pi^2 d}{L^2\psi} \\
&\geq \frac{2A}{\psi}\left(\sin\beta - \beta\cos\beta - (\frac{d}{\psi})(\sin\alpha - \alpha\cos\alpha)\right) + \frac{d}{\pi\psi^2}(\sin\alpha - \alpha\cos\alpha)\int_0^L \kappa^2 ds \\
&+ \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds - \frac{4\pi^2 d}{L^2\psi} \geq \frac{4\pi^2 d}{\pi L\psi^2}(\sin\alpha - \alpha\cos\alpha) + \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds - \frac{4\pi^2 d}{L^2\psi} \\
&= -\frac{4d\alpha^2\cos\alpha}{l\psi^2} + \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds \geq -\frac{4d\alpha^2\cos\alpha}{l\psi^2} + \frac{4d\beta^2\cos\alpha}{l\psi^2} > 0,
\end{aligned}
$$

where we use (4.5.7), (4.5.10), (4.5.12), $d/\psi < 1$, $\beta > \alpha$ and $\sin\alpha - \alpha\cos\alpha$ is increasing for $0 < \alpha < \pi/2$.

A sufficient condition for the assumption (4.5.11) is that the line connecting $\mathbf{F}(p_2, t)$ and $\mathbf{F}(p_1, t)$ lies in the domain surrounded by the curve. In Theorem 4.5.7, the conclusion that d/ψ is increasing provided that $d/\psi < 1$ is true in the direction $(2x_0, 0)$ instead of all directions, since the line connecting $(-x_0, u(x_0, t))$ and $(x_0, u(x_0, t))$ just enough lies in the domain surrounded by the curve $\Gamma(t)$. This is the key point under the condition $A > 0$. We cannot guarantee that d/ψ is non-decreasing in every direction even if d/ψ is very small. We construct such an example in Section 4.7.

4.6 Asymptotic behavior for Shrinking

Lemma 4.6.1. *For the shrinking case in Theorem C, there exist $C_1, C_2 > 0$ such that*

$$C_1 \leq \frac{b(t)}{\sqrt{T-t}} \leq C_2 \text{ and } C_1 \leq \frac{h(t)}{\sqrt{T-t}} \leq C_2, t_0 < t < T.$$

Proof. Since $u(x,t)$ has only one maximum at $x = 0$, it is easy to see that $0 \leq L(t) \leq 4h(t) + 4b(t) \to 0$, $0 \leq S(t) \leq 4b(t)h(t) \to 0$, $t \to T$. Using Lemma 4.5.6 and

$S(t) \to 0$, $L(t) \to 0$ as $t \to T$, there holds

$$S(t) = 2\pi(T-t) - A\int_t^T L(s)ds = 2\pi(T-t) + o(T-t).$$

By isoperimeter inequality $L(t)^2 \geq 4\pi S(t)$,

$$\liminf_{t\to T} \frac{L(t)^2}{T-t} \geq \lim_{t\to T} \frac{4\pi S(t)}{T-t} = 8\pi^2.$$

Using Corollary 4.5.10, there exists $C > 0$ such that

$$h(t) \geq CL(t) \text{ and } b(t) \geq CL(t).$$

Then there exists $C_1 > 0$ such that

$$\liminf_{t\to T} \frac{b(t)}{\sqrt{T-t}} \geq C_1 \text{ and } \liminf_{t\to T} \frac{h(t)}{\sqrt{T-t}} \geq C_1.$$

Using similarity transformation (4.1.5) and (4.1.6), there exists $\widetilde{C_1} > 0$ such that

$$r(\tau) \geq \widetilde{C_1} \text{ and } q(\tau) \geq \widetilde{C_1}.$$

We next prove upper bounds for $r(\tau)$, $q(\tau)$ by contradiction argument. Assume that if there exists a sequence $\tau_k \to \infty$ such that $r(\tau_k) \to \infty$. $\widetilde{S}(\tau)$ denotes the area enclosed by $w(z,\tau)$ and axis z. By calculation,

$$\widetilde{S}(\tau) = 2\int_0^{q(\tau)} w(z,\tau)dz = \frac{\int_0^{b(t)} u(x,t)dx}{T-t} = \frac{S(t)}{4(T-t)} \leq C,$$

for some C. Since $w(z,\tau_k)$ is even in z and $w(z,\tau_k)$ is monotone decreasing for $z>0$,

$$\widetilde{C_1} w(-\frac{\widetilde{C_1}}{2}, \tau_k) \leq \widetilde{S}(\tau_k) \leq C, \ \forall k.$$

Consequently, $w(-\widetilde{C_1}/2, \tau_k)$ is bounded for all k. Consider the extrinsic and intrinsic distances between $(-\widetilde{C_1}/2, w(-\widetilde{C_1}/2, \tau_k))$ and $(\widetilde{C_1}/2, w(\widetilde{C_1}/2, \tau_k))$ after transformation, denoted by $\widetilde{d}(\tau_k)$ and $\widetilde{l}(\tau_k)$, respectively. Then there hold $\widetilde{d}(\tau_k) = \widetilde{C_1}$ and $r(\tau_k) - w(-\widetilde{C_1}/2, \tau_k) < \widetilde{l}(\tau_k)$. By the argument above, since $w(-\widetilde{C_1}/2, \tau_k)$ is bounded, $\widetilde{l}(\tau_k) \to \infty$, as $k \to \infty$. Then $\widetilde{d}(\tau_k)/\widetilde{l}(\tau_k) \to 0$, as $k \to \infty$.

Consider the extrinsic and intrinsic distance between

$$(-\sqrt{2(T-t_k)}\widetilde{C}_1/2, u(-\sqrt{2(T-t_k)}\widetilde{C}_1/2, t_k)) \text{ and } (\sqrt{2(T-t_k)}\widetilde{C}_1/2, u(\sqrt{2(T-t_k)}\widetilde{C}_1/2, t_k)),$$

denoted by $d(t_k)$ and $l(t_k) < L(t_k)/2$, respectively. By calculation,

$$d(t_k) = \sqrt{2(T-t_k)}\widetilde{d}(\tau_k) \text{ and } l(t_k) = \sqrt{2(T-t_k)}\widetilde{l}(\tau_k).$$

Then $d(t_k)/l(t_k) = \widetilde{d}(\tau_k)/\widetilde{l}(\tau_k) \to 0$, as $k \to \infty$, which contradicts to Corollary 4.5.10. Therefore, $r(\tau)$ is bounded. Similarly it also holds for $q(\tau)$. Consequently,

$$C_1 \le \frac{b(t)}{\sqrt{T-t}} \le C_2 \text{ and } C_1 \le \frac{h(t)}{\sqrt{T-t}} \le C_2.$$

□

For the lemma above, it is obvious that there exist $D_1, D_2 > 0$ such that $D_1 < r(\tau) < D_2$ and $D_1 < q(\tau) < D_2$.

Since $w(z,\tau)$ is increasing for $-q(\tau) < z < 0$ and decreasing for $0 < z < q(\tau)$, $\tau > -\frac{1}{2}\ln(T-t_0)$, we can represent $w = w(z,\tau)$ under polar coordinate,

$$\begin{cases} z = \rho(\theta,\tau)\cos\theta, \\ w(z,\tau) = \rho(\theta,\tau)\sin\theta, \end{cases}$$

$0 \le \theta \le \pi$, $\tau > -\frac{1}{2}\ln(T-t_0)$. Consequently, $\rho(\theta,\tau)$ satisfies

$$\rho_\tau = \frac{\rho_{\theta\theta}}{\rho^2+\rho_\theta^2} - \frac{2\rho_\theta^2+\rho^2}{\rho(\rho_\theta^2+\rho^2)} + \rho + \frac{\sqrt{2}}{\rho}Ae^{-\tau}\sqrt{\rho_\theta^2+\rho^2},\ 0<\theta<\pi,\ \tau > -\frac{1}{2}\ln(T-t_0), \tag{4.6.1}$$

$$\rho_\theta(0,\tau) = \rho_\theta(\pi,\tau) = 0,\ \tau > -\frac{1}{2}\ln(T-t_0). \tag{4.6.2}$$

Lemma 4.6.2. *For any given $\epsilon > 0$, there exist positive constant C_k and B_k such that*

$$|\frac{\partial^k}{\partial\theta^k}\rho(\theta,\tau)| < C_k,\ |\frac{\partial^k}{\partial\tau^k}\rho(\theta,\tau)| < B_k,\ k=1,2,\cdots,\ 0\le\theta\le\pi,\ \tau \ge -\frac{1}{2}\ln(T-t_0)+\epsilon.$$

Proof. Firstly, we prove that there exist constants ρ_1, $\rho_2 > 0$ such that $\rho_1 \le \rho \le \rho_2$.

Since $r(\tau) < D_2$, $q(\tau) < D_2$ and $w(z,\tau)$ has only one maximum point at $x=0$, it is easy to get $\rho < \sqrt{2}D_2 := \rho_2$.

Consider the intrinsic and extrinsic distances, $\widetilde{l}(\tau)$ and $\widetilde{d}(\tau)$, respectively, between $(W(D_1/2,\tau), D_1/2)$ and $(-W(D_1/2,\tau), D_1/2)$, where $z = W(r,\tau)$ is the inverse of $r = w(z,\tau)$, for $z \ge 0$. By Corollary 4.5.10, $\widetilde{d}(\tau) \ge C\widetilde{l}(\tau)$. Note that $\widetilde{d}(\tau) = 2W(D_1/2,\tau)$ and $\widetilde{l}(\tau) \ge r(\tau) - D_1/2 \ge D_1/2$. Then $W(D_1/2,\tau) \ge CD_1/4$. Since $z = W(r,\tau)$ is decreasing with respective to r, $W(r,\tau) \ge W(D_1/2,\tau) \ge CD_1/4$, $0 \le r \le D_1/2$. It is easy to see $\rho > \min\{D_1/2, CD_1/4\} := \rho_1$.

Next, we are going to prove our main result. We extend ρ by even and periodic in θ. Using the interior estimates in [8], we can get

$$|\frac{\partial^k}{\partial\theta^k}\rho(\theta,\tau)| < C_k,\ |\frac{\partial^k}{\partial\tau^k}\rho(\theta,\tau)| < B_k,\ 0\le\theta\le\pi,\ \tau \ge -\frac{1}{2}\ln(T-t_0)+\epsilon.$$

□

Proof of Theorem 4.1.3. Firstly, We introduce the following Lyapunov functional borrowed from [6] (The Lyapunov functional also is used by [5]):

$$E[w(\cdot,\tau)] = \int_{-q(\tau)}^{q(\tau)} \exp\left\{-\frac{z^2+w^2(z,\tau)}{2}\right\}\sqrt{1+w_z^2(z,\tau)}dz.$$

We can compute that

$$\frac{d}{d\tau}E[w(\cdot,\tau)] = -\int_{-q(\tau)}^{q(\tau)} w_\tau^2(z,\tau)\exp\left\{-\frac{z^2+w^2(z,\tau)}{2}\right\}(1+w_z^2(z,\tau))^{-1/2}dz + J,$$

where

$$J = \sqrt{2}Ae^{-\tau}\int_{-q(\tau)}^{q(\tau)} \exp\left\{-\frac{z^2+w^2(z,\tau)}{2}\right\} w_\tau(z,\tau)dz.$$

We consider the following integral

$$\left|\int_{-q(\tau)}^{q(\tau)} \exp\left\{-\frac{z^2+w^2(z,\tau)}{2}\right\} w_\tau(z,\tau)dz\right| \le \int_{-q(\tau)}^{q(\tau)} \left|\frac{w_{zz}}{1+w_z^2} - zw_z + w + \sqrt{2}Ae^{-\tau}\sqrt{1+w_z^2}\right| dz$$

$$\le \left\{\int_{-q(\tau)}^{q(\tau)} \left|\frac{w_{zz}}{(1+w_z^2)^{3/2}}\right| + |z|\frac{|w_z|}{\sqrt{1+w_z^2}} + \frac{w}{\sqrt{1+w_z^2}} + \sqrt{2}A\right\}\sqrt{1+w_z^2}dz.$$

We note that $|q(\tau)|$, $|w(z,\tau)|$ are bounded. By Lemma 4.6.2, the curvature $|w_{zz}/(1+w_z^2)^{3/2}| = |(-\rho\rho_{\theta\theta}+2\rho_\theta^2+\rho^2)/(\rho_\theta^2+\rho^2)^{3/2}|$ is bounded, $0\le\theta\le\pi$, $\tau > -\frac{1}{2}\ln(T-t_0)+\epsilon$.

Then

$$|J| \le C_1\sqrt{2}Ae^{-\tau}\int_{-q(\tau)}^{q(\tau)}\sqrt{1+w_z^2}dz \le C_1\sqrt{2}Ae^{-\tau}(2r(\tau)+2q(\tau)) \le Ce^{-\tau},$$

for $\tau > -\frac{1}{2}\ln(T-t_0)+\epsilon$. Consequently,

$$\int_{-\frac{1}{2}\ln(T-t_0)+\epsilon}^{\infty} |J|d\tau < \infty.$$

We note that

$$E(w(\cdot,\tau)) \le 2r(\tau) + 2q(\tau) \le C,\ \tau > -\frac{1}{2}\ln(T-t_0)+\epsilon.$$

Therefore

$$\int_{-\frac{1}{2}\ln(T-t_0)+\epsilon}^{\infty}\int_{-q(\tau)}^{q(\tau)} w_\tau^2(z,\tau)\exp\left\{-\frac{z^2+w^2(z,\tau)}{2}\right\}(1+w_z(z,\tau))^{-1/2}dzd\tau < \infty.$$

Finally, it suffices to show that, for any sequence $\tau_n \to +\infty$, the sequence $(w(z,\tau_n), q(\tau_n))$ has a subsequence that converges to $(\varphi, \overline{q})$, as $n \to \infty$, where $(\varphi, \overline{q})$ is the solution of (4.1.11)-(4.1.13)(more precisely, the graph of $r = w(z,\tau_n)$ converges to the graph of $r = \varphi(z)$ under the Hausdorff distance).

We set

$$w_n(z,\tau) = w(z,\tau+\tau_n),\ \ q_n(\tau) = q(\tau+\tau_n),\ \rho_n(\theta,\tau) = \rho(\theta,\tau+\tau_n),\ \tau \in [a,a+1],$$

where $a > -\frac{1}{2}\ln(T-t_0)+\epsilon$. By Lemma 4.6.2, $\frac{\partial^k}{\partial\theta^k}\rho_n(\theta,\tau)$ and $\frac{\partial^j}{\partial\tau^j}\rho_n(\theta,\tau)$ are uniformly bounded for n, $\theta \in [0,\pi]$, $\tau \in [a,a+1]$, $k = 1,2,3$, $j = 1,2$. Then there exists $\rho^*(\theta,\tau)$ such that ρ_n converges to ρ^* in $C^{2,1}([0,\pi]\times[a,a+1])$ as $n \to \infty$. Consequently, $w_n(z,\tau)$ converges to $w^*(z,\tau)$ as $n \to \infty$, where $w^*(z,\tau) = \rho^*(\theta,\tau)\sin\theta$. Obviously, $w^*(z,\tau)$ satisfies

$$w_\tau = \frac{w_{zz}}{1+w_z^2} - zw_z + w,\ z \in (-q^*(\tau), q^*(\tau)),\ \tau \in [a,a+1], \tag{4.6.3}$$

$$w(-q^*(\tau),\tau) = w(q^*(\tau),\tau) = 0,\ \tau \in [a,a+1], \tag{4.6.4}$$

$$w_z(-q^*(\tau),\tau) = \infty,\ w_z(q^*(\tau),\tau) = -\infty,\ \tau \in [a,a+1], \tag{4.6.5}$$

where $q^*(\tau)$ denotes the limit of $q_n(\tau)$ defined as above.

We next prove $w^*_\tau(z,\tau)=0$. By the argument of Lyapunov function above,

$$\int_a^{a+1}\int_{-q(\tau+\tau_n)}^{q(\tau+\tau_n)} w_\tau^2(z,\tau+\tau_n)\exp\left\{-\frac{z^2+w^2(z,\tau+\tau_n)}{2}\right\}(1+w_z^2(z,\tau+\tau_n))^{-1/2}dzd\tau$$

$$\le \int_{\tau_n+a}^{\infty}\int_{-q(\tau)}^{q(\tau)} w_\tau^2(z,\tau)\exp\left\{-\frac{z^2+w^2(z,\tau)}{2}\right\}(1+w_z^2(z,\tau))^{-1/2}dzd\tau.$$

Using ρ_n converges to ρ^* in $C^{2,1}([0.\pi]\times[a,a+1])$ and letting $n\to\infty$,

$$\int_a^{a+1}\int_{-q^*(\tau)}^{q^*(\tau)} (w^*_\tau)^2(z,\tau)\exp\left\{-\frac{z^2+(w^*)^2(z,\tau)}{2}\right\}(1+(w^*_z)^2(z,\tau))^{-1/2}dzd\tau=0,$$

which implies $w^*_\tau\equiv 0$ for $-q^*(\tau)<z<q^*(\tau)$. So $(w^*,q(\tau))$ is a stationary solution of (4.6.3)-(4.6.5). Since the problem (4.1.11)-(4.1.13) is unique, $q^*(\tau)=\overline{q}$, where $\overline{q}$ is a constant. Therefore, we prove that $(w(z,\tau_n),q(\tau_n))$ converges to $(\varphi,\overline{q})$ up to a sequence. Therefore, we have $(w(z,\tau),q(\tau))\to(\varphi,\overline{q})$, as $\tau\to\infty$. Indeed, $(\varphi,\overline{q})=(\sqrt{1-z^2},1)$. The proof of Theorem 4.1.3 is complete.

Since $\Gamma(t)$ can be represented by $\mathbf{F}(p,t):\mathbb{S}^1\times[0,T)$. Seeing the proof of Theorem 4.1.3,

$$\kappa(p,\tau)=\frac{-w_{zz}}{(1+w_z^2)^{3/2}}\to 1,\ \text{uniformly on } \mathbb{S}^1\cap\{y\ge 0\},$$

as $\tau\to\infty$. Then for τ large enough $w_{zz}<0$ for $-q(\tau)<z<q(\tau)$. Consequently, seeing the relation between w and u, there exists t_1 such that $u_{xx}<0$, for $-b(t)<x<b(t)$, $t_1<t<T$. □

4.7 An example for $\min d/\psi = 0$

In this section we give an example that the comparison principle for extrinsic and intrinsic distances does not hold for $A > 0$. First, we give some curves.

$$\gamma_1 = \{(x,y) \mid (x - \frac{2}{A})^2 + y^2 = R^2, -L \leq y \leq R\}.$$

where $L > 1/A$ and $L < R < 2/A$.

$$\gamma_2 = \{(x,y) \mid |x - \frac{2}{A}| = \frac{1}{2}\sqrt{R^2 - L^2}, -2L - \delta < y < -L - \delta\},$$

where $0 < \delta < \min\{L/4, 2/A - \frac{1}{2}\sqrt{(2/A)^2 - L^2}\}$.

$$\gamma_3 = \{(x,y) \mid |y + 2L + 3\delta| = \delta, 0 \leq x < \frac{2}{A} - \frac{1}{2}\sqrt{R^2 - L^2} - \delta\}.$$

We connect γ_1, γ_2, γ_3 smoothly by short curves, called Γ_1. Extend Γ_1 by even, denoted by Γ_0. Let $\Gamma(t)$ be the maximal smooth solution of $V = -\kappa + A$ with initial curve Γ_0 and we show that the curve $\Gamma(t)$ will intersect itself in a finite time. By the construction of Γ_0, there exist $1/A < R_1 < R$ such that

$$B_{R_1}(2/A, 0) \subset U,\ B_{R_1}(-2/A, 0) \subset U,$$

where U is the domain surrounded by Γ_0. Let $R_1(t)$ be the solution of

$$R_1'(t) = A - \frac{1}{R_1(t)},$$

with $R(0) = R_1$. Then $\partial B_{R_1(t)}$ evolves by $V = -\kappa + A$ with ∂B_{R_1}. By comparison principle,

$$B_{R_1(t)}(2/A, 0) \subset U(t),\ B_{R_1(t)}(-2/A, 0) \subset U(t),$$

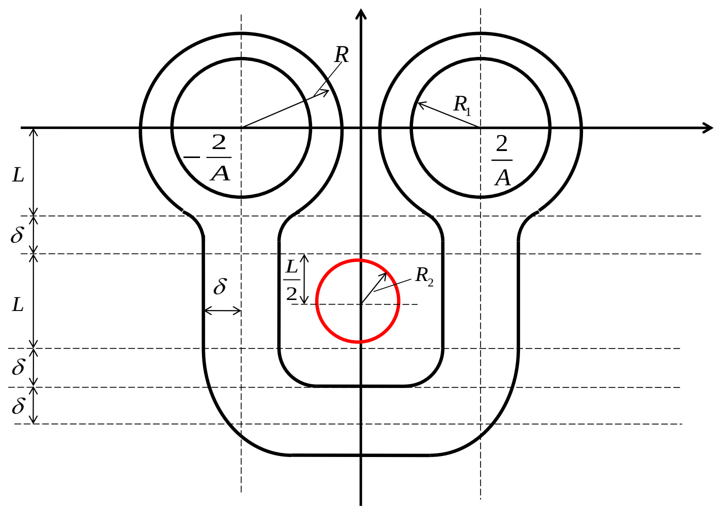

Figure 4.3: Initial curve Γ_0

where $U(t)$ is the domain surrounded by $\Gamma(t)$. Let $R_2(t)$ be the solution of

$$R_2'(t) = -A - \frac{1}{R_2(t)},$$

with $R_2(0) = R_2 := \min\{2/A - \sqrt{(2/A)^2 - L^2} - \delta, L/2\}$. Then $\partial B_{R_2(t)}$ evolves by $V = -\kappa - A$ with ∂B_{R_2}. Here we note the direction of the driving force must be reversed. Since $U \subset \mathbb{R}^2 \setminus B_{R_2}(0, -3L/2 - \delta)$, by comparison principle, $U(t) \subset \mathbb{R}^2 \setminus B_{R_2(t)}(0, -3L/2 - \delta)$, $0 \le t < t_2$, where t_2 is the maximal existence time of $R_2(t)$. Note that t_2 is independent on R and R_1. We can choose R and R_1 very closed to $2/A$ and seeing $R_1(t) \to \infty$ as $t \to \infty$, then there exists t_0, $t_0 < t_2$ such that

$$B_{R_1(t_0)}(2/A, 0) \cap B_{R_1(t_0)}(-2/A, 0) \neq \emptyset.$$

Combining $U(t) \subset \mathbb{R}^2 \setminus B_{R_2(t)}(0, -3L/2 - \delta)$, $0 \leq t < t_2$, this implies there exists t_1, $t_1 < t_0 < t_2$ such that $\Gamma(t_1)$ intersects itself at origin. It means that $m(t_1) = \min d/\psi = 0$.

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Chapter 5

Mean curvature flow with driving force in higher dimension

In this Chapter, we introduce the results in [4], mean curvature flow with drving force in higher dimensions. In [4], they give the criteria for fattening and non-fattening.

5.1 Introduction

We consider

$$V = -\kappa + A \ \text{ on } \Gamma(t) \subset \mathbb{R}^{n+1}, \tag{1.1.1}$$

$$\Gamma(0) = \Gamma_0, \tag{1.1.2}$$

where $\Gamma(t)$ is a smooth family of hypersurfaces embedded in $\mathbb{R}^{n+1}$, V is the outer normal velocity of $\Gamma(t)$, κ is the mean curvature of $\Gamma(t)$ and $A > 0$, called driving force, is a constant. Here the sign of κ is taken so that the problem is parabolic. For example, under the definition, the mean curvature of unit sphere in $\mathbb{R}^{n+1}$ is n.

In this chapter, we consider the initial data Γ_0 is smooth except for origin and

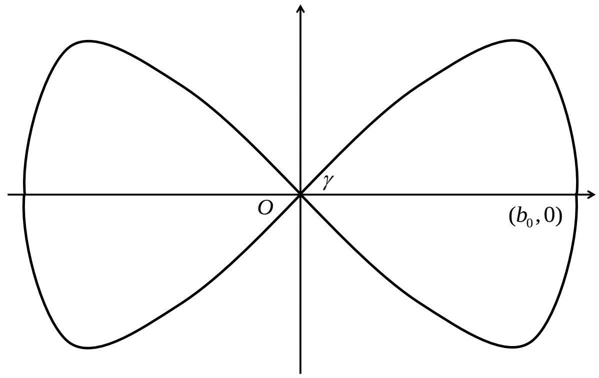

Figure 5.1: Initial hypersurface Γ_0

can be wrriten into

$$\Gamma_0 = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| = u_0(x), -b_0 \leq x \leq b_0\}, \tag{5.1.1}$$

where $u_0(x)$ is even and satisfies $u_0(x) > 0$, for $x \in (-b_0, 0) \cup (0, b_0)$, $u_0(0) = u_0(-b_0) = u_0(b_0) = 0$. Since Γ_0 is smooth at $(-b, 0, \cdots, 0)$ and $(b, 0, \cdots, 0)$, it is easy to see that $u_0'(-b_0) = -u_0'(b_0) = +\infty$.

Assume

$$\gamma := \lim_{x \to 0^+} \arctan u_0'(x) \in [0, \pi/2], \tag{5.1.2}$$

seeing Figure 5.1.

Main assumptions for $\gamma = \pi/2$. Under the condition $\gamma = \pi/2$, let $\Lambda_0 = \Gamma_0 \cap \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid x \geq 0\}$.

We consider another problem.

$$V = -\kappa + A \text{ on } \Lambda^+(t) \subset \mathbb{R}^{n+1}, \tag{1.1.1*}$$

$$\Lambda^+(0) = \Lambda_0 \tag{1.1.2*}$$

(Figure 3.2).

As in Chapter 3, let ϕ be the unique solution of

$$\begin{cases} \phi_t = |\nabla\phi|\text{div}(\frac{\nabla\phi}{|\nabla\phi|}) + A|\nabla\phi| \text{ in } \mathbb{R}^{n+1}\times(0,T), \\ \phi(x,y,0) = a_1(x,y), \end{cases}$$

where $a_1(x,y)$ satisfies $\Lambda_0 = \{(x,y) \mid a_1(x,y) = 0\}$ and $\{(x,y) \mid a_1(x,y) > 0\}$ is bounded. The results in appendix show that the zero set of ϕ is not fattening in a short time and the zero set of ϕ can be written into

$$\Lambda^+(t) = \{(x,y)\in\mathbb{R}^{n+1} \mid \phi(x,y,t) = 0\} = \{(x,y)\in\mathbb{R}^{n+1} \mid |y| = v(x,t), a_*(t) \le x \le b_*(t)\}$$

for $0 < t < T_*$. Moreover, (v, a_*, b_*) is the solution of the following free boundary problem

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} - \dfrac{n-1}{u} + A\sqrt{1+u_x^2},\ x\in(a_*(t),b_*(t)),\ 0<t<T_*, \\ u(a_*(t),t) = 0,\ u(b_*(t),t) = 0,\ 0\le t<T_*, \\ u_x(a_*(t),t) = \infty,\ u_x(b_*(t),t) = -\infty,\ 0\le t<T_*, \\ u(x,0) = u_0(x),\ 0\le x\le b_0, \\ u(x,t) > 0,\ x\in(a_*(t),b_*(t)),\ 0<t<T_*. \end{cases} \tag{**}$$

We call a_* and b_* the end points of $\Lambda^+(t)$.

Assumption (A+): There exists $\delta > 0$ such that $a_*(t) \ge 0$ for $0 \le t < \delta$.

Assumption (A−): There exists $\delta > 0$ such that $a_*(t) < 0$ for $0 < t < \delta$.

For example, if $\kappa(O) < A$, then, since

$$a_*'(0) = \kappa(O) - A < 0,$$

$a_*(t) < 0$ for any small $t > 0$. Similarly, if $\kappa(O) > A$, $a(t) > 0$ for any small $t > 0$. Here

$$\kappa(O) = \lim_{x\to 0^+}\left(-\frac{u_{xx}}{(1+u_x^2)^{3/2}} + \frac{n-1}{u\sqrt{1+u_x^2}}\right)$$

denotes the mean curvature of Λ_0 at the origin O.

Here we present our main results.

Theorem 5.1.1. *Let Γ_0 and γ be defined by (5.1.1) and (5.1.2).*

Assume $\gamma = \pi/2$, $n \geq 2$.

(1) If the assumption (A−) holds, then there exists $T > 0$ such that the interface evolution $\Gamma(t)$ for (1.1.1) with initial hypersurface Γ_0 is not fattening for $0 \leq t < T$.

(2) If the assumption (A+) holds, then the interface evolution $\Gamma(t)$ for (1.1.1) with initial hypersurface Γ_0 is fattening.

Theorem 5.1.2. *Let Γ_0 and γ be defined by (5.1.1) and (5.1.2).*

Then there exist $\alpha_n \in (0, \pi/2)$ ($n \geq 2$) such that $\alpha_n \to \pi/2$, as $n \to \infty$ and if $0 \leq \gamma < \alpha_n$, then there exists T_γ such that the interface evolution $\Gamma(t)$ for (1.1.1) with initial hypersurface Γ_0 is not fattening for $0 \leq t < T_\gamma$.

Theorem 5.1.3. *Let Γ_0 and γ be defined by (5.1.1) and (5.1.2).*

Assume $0 \leq \gamma < \pi/2$, for $n = 1$.

The interface evolution $\Gamma(t)$ for (1.1.1) in $\mathbb{R}^2$ with initial curve Γ_0 is fattening.

The definitions of fattening, non-fattening, outer-evolution, inner-evolution and interface evolution are given in section 2.

Theorem 5.1.1 can be explained by Figure 3.4 and Figure 3.5. φ in Figure 3.4 and Figure 3.5 is the unique viscosity solution of

$$\begin{cases} \varphi_t = |\nabla\varphi|\mathrm{div}(\dfrac{\nabla\varphi}{|\nabla\varphi|}) + A|\nabla\varphi| \text{ in } \mathbb{R}^{n+1}\times(0,T), \\ \varphi(x,y,0) = a_2(x,y) \text{ in } \mathbb{R}^{n+1}\times(0,T), \end{cases}$$

where $a_2(x, y)$ satisfies $\Gamma_0 = \{(x, y) \in \mathbb{R}^{n+1} \mid a_2(x, y) = 0\}$ and $\{(x, y) \in \mathbb{R}^{n+1} \mid a_2(x, y) > 0\}$ is bounded. Let $\Gamma(t) = \{(x, y) \in \mathbb{R}^{n+1} \mid \varphi(x, y, t) = 0\}$.

Motivation. In Chapter 3, we consider the mean curvature flow with driving force starting as singular initial curve in the plane and get the same results as in Theorem 5.1.1 under the condition $n = 1$. In this chapter, we give some criteria to judge whether the interface evolution starting as singular hypersurface is fattening or non-fattening in higher dimension. Combining the results in [5], we can conclude the results as the following tables. In the following tables, "Connected" means that the evolution is a connected set and "Separated" means that the evolution consists of two disjoint components.

Table 5.1: Singular angle $\gamma = \pi/2$

Assumption $(A+)$	$n = 1$	$n \geq 2$
Outer evolution	Connected	Connected
Inner evolution	Separated	Separated
Result	Fattening	Fattening

Table 5.2: Singular angle $\gamma = \pi/2$

Assumption $(A-)$	$n = 1$	$n \geq 2$
Outer evolution	Connected	Connected
Inner evolution	Connected	Connected
Result	Non-fattening	Non-fattening

Table 5.3: Singular angle $\gamma < \pi/2$

	$n = 1$, $0 \leq \gamma < \pi/2$	$n \geq 2$, $0 \leq \gamma < \alpha_n$
Outer evolution	Connected	Separated
Inner evolution	Separated	Separated
Result	Fattening	Non-fattening

The role of $a_*(t)$ in Theorem 5.1.1. Seeing Proposition 5.4.1, we can prove there exists a unique solution (u,b) of the following free boundary problem

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{n-1}{u} + A\sqrt{1+u_x^2},\ -b(t) < x < b(t),\ 0 < t < T_1, \tag{5.1.3}$$

$$u(b(t),t) = u(-b(t),t) = 0,\ u_x(-b(t),t) = -u_x(b(t),t) = \infty,\ 0 < t < T_1, \tag{5.1.4}$$

$$u(x,0) = u_0(x),\ -b_0 \le x \le b_0, \tag{5.1.5}$$

$$u(x,t) > 0,\ -b(t) < x < b(t),\ 0 < t < T_1. \tag{5.1.6}$$

Precisely, we say (u,b) is the solution of (5.1.3), (5.1.4), (5.1.5) and (5.1.6), if

(1) $b(t)$ is a positive function and $b \in C([0,T_1)) \cap C^1((0,T_1))$.

(2) $u \in C(\overline{D}_{T_1}) \cap C^{2,1}(D_{T_1})$, where $\overline{D}_{T_1} = \cup_{0\le t<T_1}\big([-b(t),b(t)] \times \{t\}\big)$ and $D_{T_1} = \cup_{0<t<T_1}\big((-b(t),b(t)) \times \{t\}\big)$ (We must note that $\overline{D}_{T_1} \ne \overline{D_{T_1}}$).

(3) (u,b) satisfies (5.1.3), (5.1.4), (5.1.5) and (5.1.6).

Obviously, the flow $\Gamma^*(t) = \{(x,y) \mid |y| = u(x,t),\ -b(t) \le x \le b(t)\}$ satisfies (1.1.1), (1.1.2) naturally.

Let (v, a_*, b_*) be the solution of the problem (*). If the assumption $(A+)$ holds, the flow

$$\Lambda^+(t) = \{(x,y) \mid |y| = v(x,t),\ a_*(t) \le x \le b_*(t)\}$$

does not intersect the flow

$$\Lambda^-(t) = \{(-x,y) \mid (x,y) \in \Lambda^+(t)\},$$

for $0 < t < \delta$. Denote $\Lambda(t) = \Lambda^+(t) \cup \Lambda^-(t)$. Obviously, $\Lambda(t)$ also satisfies (1.1.1). Seeing $\Gamma^*(0) = \Lambda(0) = \Gamma_0$, this means that there exist two types of flows $\Gamma^*(t)$ and $\Lambda(t)$ evolving by $V = -\kappa + A$ with the same initial curve Γ_0. Therefore under this condition, the solution of the original problem (1.1.1), (1.1.2) is not unique. Indeed, from the proof of Theorem 5.1.1, we see that the flow $\Gamma^*(t)$ is the boundary of the closed evolution and the flow $\Lambda(t)$ is the boundary of open evolution.

If $a_*(t) < 0$ for $0 < t < \delta$, $\Lambda^+(t) \cap \Lambda^-(t) \neq \emptyset$. Obviously, $\Lambda(t) = \Lambda^+(t) \cup \Lambda^-(t)$ does not satisfy (1.1.1). But $\Lambda^+(t)$ plays the role of a sub-solution (in the proof of Lemma 5.2.4). Using this sub-solution, the boundaries of the open evolution and closed evolution are away from the x-axis. Moreover, it can be proved that the derivatives and the second fundamental forms of them are uniformly bounded. By the uniqueness result (Proposition 5.2.2), we can prove they are coincide.

For classical mean curvature flow i.e. $A = 0$ and under the condition $\gamma = \pi/2$, since $a_*(t) \geq 0$ always holds, the interface evolution is fattening.

Background. In 1995, [1] considered the classical mean curvature flow in dimension n, $n \geq 2$. They proved that the singular formations for axisymmetric flow can only be shrinking or pinching. Moreover, they used level set method to show that after pinching, the interface evolution is non-fattening and separated into some disjoint connected components. Indeed, this result can be seen as a special condition $A = 0$ and $\gamma = 0$ in this paper.

Mean curvature with driving force under the condition $\gamma = \pi/2$ and $n = 1$, the curve in plane, has been considered in [5] recently. The same results as in Theorem 5.1.1 are given in [5]. In this paper, we give more general criteria to judge whether the interface evolution starting as singular initial hypersurface is fattening or non-fattening.

5.2 Singular angle $\gamma = \pi/2$

In this section, we consider the case $\gamma = \pi/2$ and prove Theorem 5.1.1.

Denote $U = \{(x, y) \in \mathbb{R}^{n+1} \mid |y| < u_0(x), -b_0 \leq x \leq b_0\}$. By assumption of u_0 in Section 5.1, we know that $U \cap \{x \geq 0\}$ is an α-domain with smooth boundary, for some $\alpha > 0$.

We choose vector field $X \in C^1(\mathbb{R}^{n+1} \setminus \{O\} \to \mathbb{R}^{n+1})$ such that

(i) At any $P \in \partial U$ not on the x-axis has $\langle X, \mathbf{n}(P)\rangle > 0$, $\mathbf{n}(P)$ is the inward unit normal vector to ∂U at P.

(ii) We set $X((x,y)) = (0, -y/|y|)$, near O and set $X = (-1, 0, \cdots, 0)$ near $(b, 0, \cdots, 0)$, $X = (1, 0, \cdots, 0)$ near $(-b, 0, \cdots, 0)$.
We note that X has no definition at O.

Since $X \neq 0$ on $\partial U \setminus \{O\}$ and $|X| = 1$ near O, by continuity, there exists a neighbourhood $V \supset \partial U$ such that $|X| \geq \delta > 0$ for some $\delta > 0$ in $V \setminus \{(0,0)\}$.

Proposition 5.2.1. *For ρ small enough, there exists a smooth hypersurface $\Sigma \subset V \setminus \{O\}$ with*

(i) $X(P) \notin T_P\Sigma$ at all $P \in \Sigma$, i.e., Σ is transverse to the vector field X;

(ii) $\Sigma = \partial U$ in $\{(x,y) \mid |y| \geq 2\rho\}$;

(iii) $\Sigma \cap \{(x,y) \mid |y| \leq \rho\}$ consists of discs $\Delta_{\pm c} = \{(\pm c, y) \mid |y| \leq \rho\}$ and pipe $B_d = \{(x,y) \mid -d \leq x \leq d, |y| = \rho\}$.

The proof of this proposition is similar as Proposition 3.2.1. We omit it.

Denote $\sigma(P,\alpha) : \Sigma \times (-\delta, \delta) \to V$ (V is given at the begining of this section and Σ is given by Proposition 5.2.1) the flow generated by vector field X in $\mathbb{R}^{n+1}$. Precisely, $\sigma(P,\alpha)$ is defined as following:

$$\begin{cases} \dfrac{d\sigma(P,\alpha)}{d\alpha} = X(\sigma(P,\alpha)), \ P \in \Sigma, \\ \sigma(P,0) = P, \qquad P \in \Sigma. \end{cases}$$

By (i) in Proposition 5.2.1, for any C^1 function $u : \Sigma \to \mathbb{R}$, "the image of u under σ":= $\{\sigma(P, u(P)) \mid P \in \Sigma\}$ is a C^1 hypersurface. Conversely, for any curve $\Gamma \subset V$ which is C^1 close to Σ, there exists a unique C^1 function $u : \Sigma \to \mathbb{R}$ such that $\Gamma = \{\sigma(P, u(P)) \mid P \in \Sigma\}$. In other words, the map $\sigma(\cdot, t)$ defines new coordinates from Σ to V. Therefore, if $\Gamma(t) \subset V (0 < t < T)$ is a smooth family of smooth hypersurfaces and C^1 close to Σ, there exists a unique function $u \in C^\infty(\Sigma \times (0,T))$ such that $\Gamma(t) = \{\sigma(P, u(P,t)) \mid P \in \Sigma\}$. Let $z = (z_1, \cdots, z_n)$ be the local coordinates on an open subset of Σ. If $\Gamma(t)$ evolves by $V = -\kappa + A$, under these coordinates u satisfies the following equation

$$\frac{\partial u}{\partial t} = a_{ij}(z, u, \nabla_z u)\nabla^2_{z_i z_j} u + b(z, u, \nabla_z u). \tag{5.2.1}$$

Here $\{a_{ij}\}$ is a smooth, positive matrix. Precisely, we can see Section 3 in [2]. Consequently, (5.2.1) is a parabolic equation.

For example, $\sigma(\cdot, \alpha)$ is the flow defined as above. We can easily deduce that

$$\sigma(P,\alpha) = \begin{cases} (x, \rho - \alpha),\ P \in B_d, \\ (-c + \alpha, y),\ P \in \Delta_{-c}, \\ (c - \alpha, y),\ P \in \Delta_c, \end{cases}$$

where we choose the local coordinates:

(1). $(x, \rho y)$ on B_d;

(2). $(\pm c, y)$ on $\Delta_{\pm c}$.

Suppose that $\Gamma(t)$ is symmetric to x-axis. Then, on B_d, u depends only on x, t and satisfies

$$u_t = \frac{u_{xx}}{1 + u_x^2} + \frac{n-1}{\rho - u} - A\sqrt{1 + u_x^2}. \tag{5.2.2}$$

In this case, $b = \frac{n-1}{\rho - u} - A\sqrt{1 + u_x^2}$. For $n \geq 2$, it is easy to see b is not smooth at $u = \rho$. This is the most significant difference between the condition $n = 1$ and condition $n \geq 2$.

On $\Delta_{\pm c}$, since u depends only on $y = (y_1, \cdots, y_n)$, u satisfies

$$u_t = \left(\delta_{ij} - \frac{u_{y_i} u_{y_j}}{1 + |\nabla u|^2}\right) u_{y_i y_j} - A\sqrt{1 + |\nabla u|^2}. \tag{5.2.3}$$

Proposition 5.2.2. *For $n \geq 2$, let $\Gamma_j(t)$, $t \in [0,T]$ be two families of hypersurfaces with $\sigma^{-1}(\Gamma_j(t))$ the graph of $u_j(\cdot,t)$ for certain $u_j \in C(\Sigma \times [0,T])$, $j = 1,2$. Let $D_j(t)$ be bounded open domain with $\partial D_j(t) = \Gamma_j(t)$ and assume that $D_j(t)$ are $\alpha(t)$-domain, $j = 1,2$. Moreover, assume that u_j are smooth on $\Sigma \times (0,T]$ and smooth on $\Sigma \setminus (\Delta_{\pm c} \cup B_d) \times [0,T]$. And suppose that $\rho - u_j$ are bounded from below on $\Sigma \setminus (\Delta_{\pm c} \cup B_d) \times [0,T]$. If*

(1). $\Gamma_j(t)$ evolves by (1.1.1);

(2). $\Gamma_1(0) = \Gamma_2(0)$;

(3). $\int_0^T \frac{1}{\alpha^2(t)} dt < \infty$,

then $\Gamma_1(t) = \Gamma_2(t)$ for $0 < t \leq T$.

Proof. Consider function $v(P,t) = u_1(P,t) - u_2(P,t)$. From our assumptions, there holds $v \in C(\Sigma \times [0,T])$ and v is smooth on $(\Sigma \setminus \Delta_{\pm c} \cup B_d) \times [0,T]$ and smooth on $\Sigma \times (0,T]$. Moreover $v(P,0) \equiv 0$. We define $M(t) = \max\{v(P,t) \mid P \in \Sigma\}$. Choose P_t as in Lemma 3.2.3 such that $M(t) = v(P_t,t)$ and $M'(t) = v_t(P_t,t)$.

Case 1. $P_t \in B_d$, u_j satisfy

$$u_t = \frac{u_{xx}}{1+u_x^2} + \frac{n-1}{\rho - u} - A\sqrt{1+u_x^2}. \tag{5.2.2}$$

Obviously, v satisfies the following equation

$$v_t = a^1(x,t)v_{xx} + b^1(x,t)v_x + c^1(x,t)v,$$

where $a^1(x,t) > 0$ and $c^1(x,t) = \frac{n-1}{(\rho-u_1)(\rho-u_2)}$. Since v attains its maximum at P_t, then $v_x(P_t,t) = 0$ and $v_{xx}(P_t,t) \leq 0$. So we have $v_t(P_t,t) \leq c^1(x,t)v$. By assumption that D_j are $\alpha(t)$-domain, then $\rho - u_j > \alpha(t)$, $j = 1,2$. Therefore,

$$v_t(P_t,t) \leq \frac{n-1}{\alpha^2(t)} v(P_t,t).$$

Consequently, $M'(t) \leq \frac{n-1}{\alpha^2(t)} M(t)$.

Case 2. $P_t \in \Sigma \setminus (\Delta_{\pm c} \cup B_d)$. Then we can choose coordinates z on some neighbourhood of P_t on Σ and u_j satisfy (5.2.1). We may write this equation as $u_t = F(z, t, u, \nabla u, \nabla^2 u)$. Then v satisfies

$$v_t = a^2_{ij}(z,t) v_{z_i z_j} + b^2_i(z,t) v_{z_i} + c^2(z,t) v$$

where

$$c^2(z,t) = \int_0^1 F_u(z, t, u^\theta, \nabla u^\theta, \nabla^2 u^\theta) d\theta,$$

where $u^\theta = (1-\theta)u_0 + \theta u_1$ and $\{a^2_{ij}\}$ is a positive definite.

Since v is smooth on $\Sigma \setminus (\Delta_{\pm c} \cup B_d) \times [0, T]$ and $\rho - u_j$ are bounded from below on $\Sigma \setminus (\Delta_{\pm c} \cup B_d)$, $0 < t < T$, then there exists a positive constant C such that $|c^2(z,t)| \leq C$. The constant C may depend on the choice of local coordinates z. By compactness of Σ, Σ has a finite covering consisting of neighborhoods of local coordinates, and we can choose C independent of the choice of local coordinates. Since $\nabla v(P_t, t) = 0$, $\{v_{z_i z_j}(P_t, t)\}$ is negative semi-definite,

$$v_t(P_t, t) \leq c(P_t, t) v(P_t, t) \leq C v(P_t, t).$$

Consequently, $M'(t) \leq C M(t)$.

Case 3. $P_t \in \Delta_{\pm c}$. We only consider $P_t \in \Delta_{-c}$. Then in the z-coordinates of Δ_{-c}, u_j satisfy the full graph equation

$$u_t = \left(\delta_{ij} - \frac{u_{y_i} u_{y_j}}{1 + |\nabla u|^2} \right) u_{y_i y_j} - A\sqrt{1 + |\nabla u|^2}. \tag{5.2.3}$$

Hence $v = u_1 - u_2$ satisfies a linear parabolic equation

$$v_t = a^3_{ij}(z,t) v_{z_i z_j} + b^3_i(z,t) v_{z_i},$$

where $\{a_{ij}^3\}$ is positive definite. Obviously, $\nabla v(P_t, t) = 0$ and $\{v_{z_i z_j}(P_t, t)\}$ is negative semi-definite. It follows that $M'(t) \leq 0$.

From the three cases above, if we put

$$r(t) = \frac{n-1}{\alpha^2(t)} + C,$$

then there holds $M'(t) \leq r(t)M(t)$. Consequently, by the assumption of $\alpha(t)$,

$$M(t) \leq M(0)e^{\int_0^t r(t)ds} = 0.$$

By considering $m(t) = \min\{v(P,t) \mid P \in \Sigma\}$, we can similarly prove $m(t) \geq 0$. Therefore $\Gamma_1(t) = \Gamma_2(t)$ for $0 \leq t \leq T$. □

Note that the intial curve in our problem is singular at x-axis. The assumption that "$D_j(t)$ are $\alpha(t)$ domain" in Proposition 5.2.2 means that $\Gamma_j(t)$ "escape" from origin with speed $\alpha(t)$. If the "escape speed" satisfies

$$\int_0^T \frac{1}{\alpha^2(t)} dt < \infty,$$

we can get the uniqueness.

Following Lemmas 5.2.3 and 5.2.4 can be proved by similar argument to the argument in $\mathbb{R}^2$ (Chapter 3). For reader's convenience, we give the proof of Lemma 5.2.4.

Lemma 5.2.3. *There exists a sequence of closed sets E_j such that E_j° are $\alpha/2^j$-domains and $E_j \downarrow \overline{U}$. Here U is given at the beginning of the section and E° denotes the interior of the set E.*

Lemma 5.2.4. *Let the same assumption of (1) in Theorem 5.1.1 be given. Then there exists $t_1 > 0$ such that, for all t_2 satisfying $0 < t_2 < t_1$, the second fundamental* *forms and derivatives of* $\partial E_j(t)$ *are uniformly bounded for* $t_2 \le t \le t_1$*, where* $E_j(t)$ *denote the closed evolution of* $V = -\kappa + A$ *with* $E_j(0) = E_j$ *and* E_j *are given in Lemma 5.2.3.*

Proof. Let $E_j(t) = \{(x,y) \mid |y| \le v_j(x,t),\ -c_j(t) \le x \le c_j(t)\}$.

Step 1. For all t_2 satisfying $0 < t_2 < \delta$ (δ given by assumption $(A-)$), there exists a constant $c > 0$ such that

$$v_j(0,t) > c,\ \ t_2/2 < t < \delta.$$

Let $U^+(t)$ denote the bounded set with $\partial U^+(t) = \Lambda^+(t)$. Since $U^+(0) = U \cap \{x \ge 0\} \subset E_j = E_j(0)$, $U^+(t) \subset E_j(t)$. By our assumption that $a_*(t) < 0$ for $0 < t \le \delta$, $O \in U^+(t) \subset E_j(t)$ for $0 < t < \delta$. For all $t_2 \in (0,\delta)$, there exists $c > 0$ such that $v_j(0,t) > c$ for $t_2/2 \le t \le \delta$.

Step 2. Construction of four auxiliary balls.

Since $U \cap \{x \ge 0\}$ is an α-domain, there exist $\beta_2 > \beta_1 > 0$ such that $u_0(\pm\beta_1) = u_0(\pm\beta_2) = \alpha$ and $u_0'(x) < 0$ for $x > \beta_2$, $u_0'(x) > 0$ for $0 < x < \beta_1$. There exist $p > \beta_1$ and $0 < q < \beta_2$ such that $u_0(\pm q) = u_0(\pm p) = \dfrac{\alpha}{2}$. We consider the points

$$Q = (-p,0), \qquad P = (p,0),$$

$$Q' = (-p,\alpha), \qquad P' = (p,\alpha).$$

Since $P \in U$ and $P' \in \overline{U}^c$, there exists ϵ such that $\overline{B_\epsilon(P)} \subset U$ and $\overline{B_\epsilon(P')} \subset \overline{U}^c$. Consequently, $\overline{B_\epsilon(P)} \cup \overline{B_\epsilon(Q)} \subset E^\circ$ and $\overline{B_\epsilon(P')} \cup \overline{B_\epsilon(Q')} \subset E^c$. Then for j large enough, $\overline{B_\epsilon(P)} \cup \overline{B_\epsilon(Q)} \subset E_j^\circ$ and $\overline{B_\epsilon(P')} \cup \overline{B_\epsilon(Q')} \subset E_j^c$. Comparison principle shows that

$$\overline{B_{\epsilon(t)}(P)} \cup \overline{B_{\epsilon(t)}(Q)} \subset E_j(t)^\circ \tag{5.2.4}$$

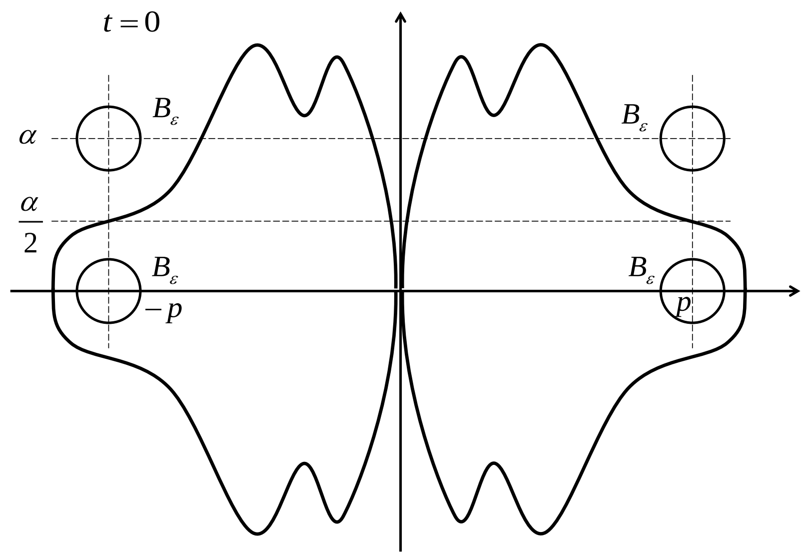

Figure 5.2: Proof of Lemma 5.2.4

for $0 < t < \delta_2$. By Theorem 2.1.7,

$$\overline{B_{\epsilon(t)}(P')} \cup \overline{B_{\epsilon(t)}(Q')} \subset E_j(t)^c \tag{5.2.5}$$

for $0 < t < \delta_2$. Here $\epsilon(t)$ is the solution of (2.3.13) with $\epsilon(0) = \epsilon$ on the interval $[0, \delta_1)$. Take δ_2 independent of j such that $\epsilon(t) > \epsilon/2$ for $0 < t < \delta_2$.

Step 3. Divide $\partial E_j(t)$ into two parts by auxiliary balls.

Since for all $\rho < \alpha/2$, C_ρ intersects ∂E_j at most four times, by the intersection number argument as in the proof of Lemma 2.3.10, there exists $t_0 > 0$ such that C_ρ intersects $\partial E_j(t)$ at most four times for $0 < t < t_0$. By continuity, we can deduce that there exists δ_4 such that for all $\rho < \alpha$, the equation $v_j(x,t) = \rho$ has just one root for $x > p$ for all $t < \delta_4$. By symmetry, it also holds for $x < -p$.

Put $t_1 = \min\{t_0, \delta_2, \delta_3, \delta_4\}$. Then Step 1 and intersection argument show that $E_j(t)^\circ$ are all c-domains for $t_2/2 < t < t_1$. Let $d < \min\{c, \epsilon/4\}$. By (5.2.4) in

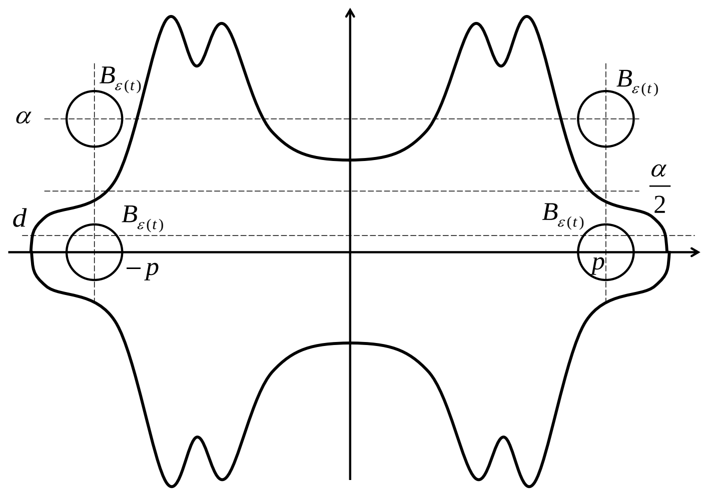

Figure 5.3: Proof of Lemma 5.2.4

Step 2, we have $v_j(x,t) > d$ for $t_2/2 < t < t_1$ and x with $|x-p| < \sqrt{\epsilon^2(t) - d^2}$ or $|x+p| < \sqrt{\epsilon^2(t) - d^2}$. By $\epsilon(t) > \epsilon/2$,

$$v_j(x,t) \geq d \text{ in } \Omega = (-p - \frac{\sqrt{3}}{4}\epsilon, p + \frac{\sqrt{3}}{4}\epsilon) \times (t_2/2, t_1).$$

For $x \leq -p$, by (5.2.5) in Step 2,

$$v_j(x,t) < \alpha/2 - \epsilon(t) < \alpha/2 - \epsilon/2 \text{ for } x \leq -p,\ 0 \leq t < t_1.$$

This is also true for $x \geq p$.

Step 4. The derivatives and second fundamental forms of $\partial E_j(t)$ are bounded in $\Omega' = [-p,p] \times (t_2, t_1)$.

Since $v_j(x,t) \geq d$ in $\Omega = (-p - \frac{\sqrt{3}}{4}\epsilon, p + \frac{\sqrt{3}}{4}\epsilon) \times (t_2/2, t_1)$, Theorem 2.3.4 implies that v_{jx} are uniformly bounded in Ω. By Remark 2.2.4, v_{jxx} are uniformly bounded in Ω'.

Step 5. The derivatives and second fundamental forms of $\partial E_j(t)$ are bounded for $x \le -p$ and $x \ge p$, $t_2 < t < t_1$.

We only consider for $x \le -p$. For $0 < t < t_1$, the part of $\partial E_j(t)$ on $x \le -p$ can be represented by $x = w_j(y,t)$ for $|y| < \alpha/2$, $t \in (0, t_1)$, and w_j satisfy the equation (2.2.1) in the condition "−" and $n = 1$. Then Corollary 2.2.3 and Remark 2.2.4 imply that all $\frac{\partial^k}{\partial y^k} w_j(y,t)$, $k = 1, 2$, are uniformly bounded for $|y| \le \alpha/2 - \epsilon/2$, $t_2 < t < t_1$ and for any $t_2 > 0$. Then the derivatives and second fundamental forms of $\partial E_j(t)$ are uniformly bounded for $x \le -p$, $t_2 < t < t_1$.

The proof of this lemma is completed. □

Lemma 5.2.5. *Let the same assumption in Theorem 5.1.1 hold. Then there exists $t_1 > 0$ such that for all $t_2 \in (0, t_1)$, the second fundamental forms and derivatives of $\partial U(t)$ is uniformly bounded for $t_2 < t < t_1$, where $U(t)$ is the open evolution of $V = -\kappa + A$ with $U(0) = U$.*

Lemma 5.2.5 is able to be proved as Lemma 5.2.4.

As mentioned in Proposition 5.2.2, in order to get the uniqueness, we must give the estimate of "escape speed". Let R_0 be taken small enough such that $B_{R_0}((R_0, 0, \cdots, 0)) \cup B_{R_0}((-R_0, 0, \cdots, 0)) \subset U$. In next lemma, we construct a sub-solution.

Lemma 5.2.6. *(Sub-solution) Take R_0 as above. Function $\underline{u}$ is even and defined by*

$$\underline{u}(x,t) = \begin{cases} \sqrt{(r(t)+R(t))^2 - R_0^2} - \sqrt{r^2(t) - x^2}, & 0 \le x < \frac{R_0 r(t)}{R(t)+r(t)}, \\ \sqrt{R^2(t) - (x - R_0)^2}, & \frac{R_0 r(t)}{R(t)+r(t)} \le x \le R_0 + R(t). \end{cases}$$

Here $r(t) = t^{3/4}$ and $R(t)$ satisfies $R' = A - n/R$, $R(0) = R_0$. Then there exists $t_* > 0$ such that $(\underline{u}, R_0 + R(t))$ is a sub-solution of (5.1.3), (5.1.4), (5.1.5), (5.1.6) for $0 < t < t_*$.

Proof. We can easily deduce that $|R(t) - R_0| = O(t)$, as $t \to 0$. Since $r(t) = t^{3/4} > R(t)$, for sufficient small t, $\underline{u}$ is well-defined for small t.

1. Positive: Obviously, $u_0(x,t) > 0$ for $-R_0 - R(t) < x < R_0 + R(t)$.

2. Initial condition: By the choice of R_0, $\underline{u}(x,0) = \sqrt{R_0 - (x+R_0)^2} \le u_0(x)$ for $0 \le x \le R_0$ and $R_0 \le b_0$.

3. Boundary condition: Obviously, at boundary,

$$\underline{u}(-R_0 - R(t), t) = \underline{u}(R_0 + R(t), t) = 0$$

and

$$\underline{u}_x(-R_0 - R(t), t) = -\underline{u}_x(R_0 + R(t), t) = \infty.$$

4. Interior: For $\frac{R_0 r(t)}{R(t)+r(t)} < x < R_0 + R(t)$ or $-R_0 - R(t) < x < -\frac{R_0 r(t)}{R(t)+r(t)}$, $\underline{u}(x,t) = \sqrt{R^2(t) - (x-R_0)^2}$ satisfies (5.1.3). Next we only need prove $\underline{u}$ is a sub-solution of (5.1.3) for $-\frac{R_0 r(t)}{R(t)+r(t)} < x < \frac{R_0 r(t)}{R(t)+r(t)}$ and t small. By calculation,

$$1 + \underline{u}_x^2 = \frac{r^2}{r^2 - x^2};$$

$$\underline{u}_{xx} = \frac{r^2}{(r^2 - x^2)^{3/2}};$$

$$\underline{u}_t = \frac{(r+R)(r'+R')}{\sqrt{(r+R)^2 - R_0^2}} - \frac{rr'}{\sqrt{r^2 - x^2}}.$$

Therefore,

$$\begin{aligned}
\underline{u}_t \quad &- \quad \frac{\underline{u}_{xx}}{1+\underline{u}_x^2} + \frac{n-1}{\underline{u}} - A\sqrt{1+\underline{u}_x^2} = \frac{(r+R)(r'+R')}{\sqrt{(r+R)^2 - R_0^2}} - \frac{rr'}{\sqrt{r^2-x^2}} - \frac{1}{\sqrt{r^2-x^2}} \\
&+ \quad \frac{n-1}{\sqrt{(r+R)^2 - R_0^2} - \sqrt{r^2-x^2}} - \frac{Ar}{\sqrt{r^2-x^2}} \le \frac{(r+R)(r'+R')}{\sqrt{(r+R)^2 - R_0^2}} - r' - \frac{1}{r} \\
&+ \quad \frac{n-1}{\sqrt{(r+R)^2 - R_0^2} - r}.
\end{aligned}$$

Since $|R'(t)|$ is bounded, $R(t)$ is bounded from above and below for small t, and $r(t) = t^{3/4}$, we can deduce that

$$\frac{1}{\sqrt{(r+R)^2 - R_0^2}} = \frac{1}{\sqrt{(r+R-R_0)(r+R+R_0)}} = O(t^{-3/8});$$

$$(r+R)(r'+R') = O(t^{-1/4})$$

as $t \to 0$. Consequently,

$$\begin{aligned} \underline{u}_t &- \frac{\underline{u}_{xx}}{1+\underline{u}_x^2} + \frac{n-1}{\underline{u}} - A\sqrt{1+\underline{u}_x^2} \leq C_1 t^{-5/8} - C_2 t^{-1/4} - C_3 t^{-3/4} + C_4 t^{-3/8} \\ &= t^{-3/4}(C_1 t^{1/8} - C_2 t^{1/2} - C_3 + C_4 t^{3/8}) < 0 \end{aligned}$$

for any sufficient small $t > 0$. It is easy to check that $\underline{u}$ is C^1 at $x = \frac{R_0 r(t)}{R(t)+r(t)}$. Then there exists $t_* > 0$ such that $\underline{u}$ is a sub-solution of (5.1.3) in viscosity sense for $0 < t < t_*$.

We complete the proof. □

Corollary 5.2.7. *Recall* $U = \{(x,y) \in \mathbb{R}^{n+1} \mid |y| < u_0(x),\ b_0 < x < b_0\}$ *and* $E(t)$ *be the outer evolution of* $\overline{U}$*. Then there exists* $t^* > 0$ *such that, for each* $t \in [0, t^*)$*,* $E(t)$ *can be described as follows,*

$$E(t) = \{(x,y) \in \mathbb{R}^{n+1} \mid |y| \leq v(x,t),\ -b_1(t) \leq x \leq b_1(t)\}.$$

Here (v, b_1) *is the uniqueness solution of (5.1.3), (5.1.4), (5.1.5), (5.1.6) on the interval* $[0, t^*)$*. Moreover, there exists* $\alpha(t)$ *with* $\int_0^{t^*} \frac{1}{\alpha^2(t)} dt < \infty$ *such that* $E(t)^\circ$ *is* $\alpha(t)$*-domain for* $t < t^*$*.*

Proof. Let $E_j(t)$ be given by Lemma 5.2.4. Since $\{(x,y) \in \mathbb{R}^{n+1} \mid |y| \leq \underline{u}(x,0)\} \subset E_j(0) = \{(x,y) \in \mathbb{R}^{n+1} \mid |y| \leq v_j(x,0),\ -c_j(0) \leq x \leq c_j(0)\})$, $\{(x,y) \in \mathbb{R} \times \mathbb{R}^N \mid |y| \leq \underline{u}(x,t)\} \subset E_j(t) = \{(x,y) \in \mathbb{R}^{n+1} \mid |y| \leq v_j(x,t),\ -c_j(t) \leq x \leq c_j(t)\}$ for

$0 < t < t_*$ and for all j.

By $r(t) = t^{3/4}$, $|R(t) - R_0| = O(t)$ and boundedness of $R(t)$ from below as $t \to 0$, there exists $t^* > 0$ such that

$$\begin{aligned} \underline{u}(0,t) &= \sqrt{(r(t)+R(t))^2 - R_0^2} - \sqrt{r^2(t)} = \sqrt{r(t)}\left(\sqrt{r(t) + 2R(t) + \frac{R(t)-R_0}{r(t)}} - \sqrt{r(t)}\right) \\ &\geq C' t^{3/8} - t^{3/4} \geq C t^{3/8}, \end{aligned}$$

for $t < t^*$. If we taken

$$\alpha(t) = Ct^{3/8} \text{ for } 0 \leq t < t^*,$$

then $\int_0^{t^*} \frac{1}{\alpha^2(t)} dt < \infty$. Therefore, $E_j(t)^\circ$ are all $\alpha(t)$-domains. Moreover, (v_j, c_j) satisfy (5.1.3), (5.1.4), (5.1.6). By $E_j(0) \downarrow E(0)$, Theorem 2.1.5 and the same method as in Lemma 5.2.4, we can show that $E_j(t) \downarrow E(t)$ and derivatives and second fundamental forms of $E_j(t)$ are uniformly bounded. $(v, b_1) = \lim_{j\to\infty} (v_j, c_j)$ is the solution of (5.1.3), (5.1.4), (5.1.5), (5.1.6). Moreover, $\{(x,y) \in \mathbb{R}^{n+1} \mid |y| \leq \underline{u}(x,t)\} \subset E(t)$ for $0 < t < t^*$. $E(t)^\circ$ is also an $\alpha(t)$-domain $0 < t < t^*$. The uniqueness of the solution follows from Proposition 5.2.2. □

Proof of (1) in Theorem 5.1.1. Let $E(t)$ and $U(t)$ be the closed and open evolution of (1.1.1) with $E(0) = \overline{U}$ and $U(0) = U$, respectively.

Let $U^+(t)$ denote the bounded open set with $\partial U^+(t) = \Lambda^+(t)$. Since $U^+(0) = U \cap \{x \geq 0\} \subset U = U(0)$, there holds $U^+(t) \subset U(t)$. By our assumption that $a_*(t) < 0$ for $0 < t \leq \delta$, $O \in U^+(t) \subset U(t)$ for $0 < t < \delta$, where δ is given in assumption $(A-)$. This means that $\partial U(t)$ escapes away from origin. Therefore,

$$\{(x,y) \in \mathbb{R}^{n+1} \mid |y| \leq \underline{u}(x,t)\} \subset U(t), \; 0 < t < T,$$

where $T = \min\{t^*, \delta, t_1\}$. (Recall t_1 is given by Lemma 5.2.5) Consequently, $U(t)$ is also an $\alpha(t)$-domain. For small t, we can easily check that $\partial E(t)$ and $\partial U(t)$ satisfy the assumption in Proposition 5.2.2. We get $\partial E(t) = \partial U(t)$ for $0 \le t < T$.

We complete the proof. □

Proof of (2) in Theorem 5.1.1. Choose $T_1 < \min\{t^*, \delta\}$. Here δ is given in assumption $(A+)$, and t^* is given in Corollary 5.2.7.

Let $U^{\pm}(t)$ be the bounded open domain with $\partial U^{\pm}(t) = \Lambda^{\pm}(t)$. Thus the left end point of $U^+(t)$ and the right end point of $U^-(t)$ are $(a_*(t), 0, \cdots, 0)$ and $(-a_*(t), 0, \cdot, 0)$, respectively. By assumption $(A+)$, $-a_*(t) \le a_*(t)$, $0 \le t < T_1$. Therefore, $U^+(t) \cap U^-(t) = \emptyset$, $0 \le t < \delta$. From Lemma 2.1.9, the inner evolution $U(t)$ satisfies $U(t) = U^+(t) \cup U^-(t)$, for $0 \le t < \delta$.

Corollary 5.2.7 shows that $E(t)^\circ$ is an $\alpha(t)$-domain for $0 < t < T_1$. By $\alpha(t) = Ct^{3/8}$, there exists small enough ρ such that the ball

$$B_\rho((0, C(\frac{T_1}{4})^{3/8}, 0, \cdots, 0)) \subset E(t), \frac{T_1}{2} < t < T_1$$

and

$$B_\rho((0, C(\frac{T_1}{4})^{3/8}, 0, \cdots, 0)) \cap U(t) = \emptyset, \ \frac{T_1}{2} < t < T_1.$$

This means that the interface evolution $\Gamma(t) = E(t) \setminus U(t)$ has interior. □

Remark 5.2.8. (1) In the proof of (1) in Theorem 5.1.1, we get the closed evolution and the open evolution are all connected sets. Therefore, they are homeomorphic. Moreover, using the unique result, we can prove that they are coincide.

(2) In the proof of (2) in Theorem 5.1.1, we get the closed evolution is connected and the open evolution is separated. Therefore, they are not homeomorphic.

5.3 Singular angle $\gamma < \pi/2$ with $n \geq 2$

In this section, we give the proof of Theorem 5.1.2. First, we introduce the following similarity transformation: for $T > 0$,

$$z = \frac{x}{\sqrt{2(T-t)}}, \ \tau = -\frac{1}{2}\ln(T-t) \tag{5.3.1}$$

and

$$w(z,\tau) = \frac{1}{\sqrt{2}}e^{\tau}u(\sqrt{2}e^{-\tau}z, T-e^{-2\tau}). \tag{5.3.2}$$

Then u satisfies

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{n-1}{u} + A\sqrt{1+u_x^2}, \tag{5.1.3}$$

if and only if w satisfies

$$w_\tau = \frac{w_{zz}}{1+w_z^2} - zw_z + w - \frac{n-1}{w} + \sqrt{2}Ae^{-\tau}\sqrt{1+w_z^2}. \tag{5.3.3}$$

In 1992, [3] shows that there is a torus shape self-similar solution of (1.1.1) for $A = 0$ called "Angenent shrinking doughnut". The self-similar solution remaining the shape of doughnut shrinks to a point. Moreover, in [1], using this self-similar solution, they prove that after a rotational hypersurface pinches, the hypersurface will be separated into two disjoint components. We also expect to prove Theorem 1.2 by using some self-similar solution of (1.1.1), however, it is difficult to find such solution. Therefore, we construct a compact self-similar super-solution of (5.1.3).

Proposition 5.3.1. *(Super-solution of equation (5.3.3)) Denote* $\overline{w} := C - \sqrt{\rho^2 - z^2}$, $-\rho \leq z \leq \rho$.

For $n > 2$*, for every* C*,* ρ *with* $C^2 + \rho^2 < n$ *and* $C > \rho > 1$*, there exists* $\tau_0(C,\rho)$ *such that* $\overline{w}$ *is a super-solution of (5.3.3) for* $-\rho < z < \rho$*,* $\tau > \tau_0$*.*

For $n = 2$*, Fix* $\theta \in (0,1)$*,* $\epsilon_0 \in (0, \frac{2}{9}\theta)$ *arbitrary. Then, for each* $1+\theta\epsilon_0 < \rho < C <$

$1+\epsilon_0$, *there exists* $\tau_0(C,\rho)$ *such that* $\overline{w}$ *is a super-solution of (5.3.3) for* $-\rho < z < \rho$, $\tau > \tau_0$.

Proof. By calculation,

$$\overline{w}_\tau = 0,$$

$$\overline{w}_z = \frac{z}{\sqrt{\rho^2 - z^2}}$$

and

$$\overline{w}_{zz} = \frac{\rho^2}{(\rho^2 - z^2)^{3/2}}.$$

For convenience, we put $q = \sqrt{\rho^2 - z^2}$ for $-\rho \le z \le \rho$.

$$\begin{aligned}
\overline{w}_\tau \quad &- \quad \frac{\overline{w}_{zz}}{1+\overline{w}_z^2} + z\overline{w}_z - \overline{w} + \frac{n-1}{\overline{w}} - \sqrt{2}Ae^{-\tau}\sqrt{1+\overline{w}_z^2} \\
&= \quad -\frac{1}{q} + \frac{n-1}{C-q} - C + q + \frac{\rho^2 - q^2}{q} - \sqrt{2}Ae^{-\tau}\frac{\rho}{q} \\
&= \quad -\frac{1}{q} + \frac{n-1}{C-q} - C + \frac{\rho^2}{q} - \sqrt{2}Ae^{-\tau}\frac{\rho}{q} \\
&= \quad \frac{1}{q(C-q)}\Big(Cq^2 - (\rho^2 + C^2 - n)q - C + \rho^2 C - \sqrt{2}Ae^{-\tau}\rho(C-q)\Big).
\end{aligned}$$

For $n > 2$, if $\rho^2 + C^2 < n$ and $\rho < C$, we can deduce $Cq^2 - (\rho^2 + C^2 - n)q - C + \rho^2 C$ of the right hand side of the formula above attains its minimum at $q = 0$. Consequently,

$$\overline{w}_\tau - \frac{\overline{w}_{zz}}{1+\overline{w}_z^2} + z\overline{w}_z - \overline{w} + \frac{n-1}{\overline{w}} - \sqrt{2}Ae^{-\tau}\sqrt{1+\overline{w}_z^2} \ge \frac{1}{q(C-q)}(-C + \rho^2 C - \sqrt{2}Ae^{-\tau}\rho C).$$

Therefore, if $\rho > 1$, then there exits $\tau_0(C,\rho)$ such that

$$\overline{w}_\tau - \frac{\overline{w}_{zz}}{1+\overline{w}_z^2} + z\overline{w}_z - \overline{w} + \frac{n-1}{\overline{w}} - \sqrt{2}Ae^{-\tau}\sqrt{1+\overline{w}_z^2} > 0$$

for $\tau > \tau_0$.

For $n = 2$, $Cq^2 - (\rho^2 + C^2 - 2)q - C + \rho^2 C$ attains its minimum at $\frac{\rho^2 + C^2 - 2}{2C}$.

Consequently,

$$\begin{aligned}\overline{w}_\tau &- \frac{\overline{w}_{zz}}{1+\overline{w}_z^2} + z\overline{w}_z - \overline{w} + \frac{1}{\overline{w}} - \sqrt{2}Ae^{-\tau}\sqrt{1+\overline{w}_z^2} \\ &\geq \frac{1}{q(C-q)}\left(-\frac{(C^2+\rho^2-2)^2}{4C} + (\rho^2-1)C - \sqrt{2}Ae^{-\tau}\rho C\right).\end{aligned}$$

Then fixed $\theta \in (0,1)$, for any $\epsilon_0 \in (0, \frac{2}{9}\theta)$ and $1+\theta\epsilon_0 < \rho < C < 1+\epsilon_0$,

$$-\frac{(C^2+\rho^2-2)^2}{4C} + (\rho^2-1)C \geq \frac{8\theta\epsilon_0 - 36\epsilon_0^2}{4(1+\theta\epsilon_0)} = \frac{2\theta\epsilon_0 - 9\epsilon_0^2}{(1+\theta\epsilon_0)} > 0.$$

Therefore, there exists $\tau_0(C,\rho) > 0$ such that

$$\overline{w}_\tau - \frac{\overline{w}_{zz}}{1+\overline{w}_z^2} + z\overline{w}_z - \overline{w} + \frac{1}{\overline{w}} - \sqrt{2}Ae^{-\tau}\sqrt{1+\overline{w}_z^2} > 0$$

for $\tau > \tau_0$. □

Remark 5.3.2. Under the condition $n > 2$, in the proof of Proposition 5.3.1, for convenience, we assume

$$\rho^2 + C^2 < n.$$

Indeed, it is not necessary. In the proof, we can use that $Cq^2 - (\rho^2+C^2-n)q - C + \rho^2 C$ attains its minimum at $q = \frac{\rho^2+C^2-n}{2C}$.

Corollary 5.3.3. *Let $\overline{w}$ and τ_0 be given by Proposition 5.3.1. Then for $T < e^{-\tau_0}$,*

$$\overline{u}(x,t;T) = \sqrt{2(T-t)}\,\overline{w}\left(\frac{x}{\sqrt{2(T-t)}}\right)$$

is a super-solution of equation (5.1.3) for $-\rho\sqrt{2(T-t)} < x < \rho\sqrt{2(T-t)}$, $0 < t < T$.

This result is obvious by Proposition 5.3.1. Here we omit the proof.

Remark 5.3.4. For $n = 2$, recall γ given in Section 1 and ϵ_0 given by Proposition 5.3.1. Then for all $0 \leq \gamma < \arctan\sqrt{(1+\epsilon_0)^2 - 1}$, we can choose ρ and C satisfying $1 < \rho < C < 1 + \epsilon_0$ such that

$$\gamma < \frac{\sqrt{C^2 - \rho^2}}{\rho}.$$

For $n > 2$, if $0 \leq \gamma < \arctan\sqrt{n-2}$, we can choose ρ, C satisfying $1 < \rho < C$ and $\rho^2 + C^2 < n$ such that

$$\gamma < \frac{\sqrt{C^2 - \rho^2}}{\rho}.$$

It is obvious that the cone

$$|y| = \frac{\sqrt{C^2 - \rho^2}}{\rho}|x|$$

is the envelop of the family of hypersurfaces $\{|y| = \lambda\overline{w}(x/\lambda)\}_{\lambda>0}$.

By the property of u_0, we can get

$$y = \frac{\sqrt{C^2 - \rho^2}}{\rho}|x| > u_0(x),$$

for small $|x|$.

Therefore, there exists $T(\rho, C)$ such that for all $0 \leq T < T(\rho, C)$,

$$\overline{u}(x, 0; T) > u_0(x),$$

$-\rho\sqrt{2T} < x < \rho\sqrt{2T}$.

Proof of Theorem 5.1.2. As mentioned in Remark 5.3.4, we choose

$$\alpha_n = \begin{cases} \arctan\sqrt{(1+\epsilon_0)^2 - 1}, & n = 2, \\ \arctan\sqrt{n-2}, & n > 2. \end{cases}$$

Obviously, $\alpha_n \to \pi/2$, as $n \to \infty$.

For every $\gamma < \alpha_n$, we choose C and ρ as in Remark 5.3.4. Let $T_\gamma < \min\{T(\rho, C), e^{-\tau_0(\rho,C)}\}$, where $\tau_0(\rho, C)$ is given by Proposition 5.3.1 and $T(\rho, C)$ is given by Remark 5.3.4. Next we show that for every $0 < t < T_\gamma$, the origin $O \in E(t)^c$. Let flow

$$\overline{\Gamma}(t;T) = \{(x, y) \mid |y| = \overline{u}(x, t, T), -\rho\sqrt{2(T-t)} \leq x \leq \rho\sqrt{2(T-t)}\},$$

$0 \leq t < T$. Remark 5.3.4 shows that for every $0 < t < T_\gamma$,

$$\overline{\Gamma}(0;t) \cap E(0) = \emptyset.$$

By comparison principle, we can easily show that

$$\overline{\Gamma}(s;t) \cap E(s) = \emptyset,\ 0 \leq s \leq t.$$

(Noting that $\overline{\Gamma}(s, t)$ is a hypersurface with boundary, we must consider the comparison principle in interior and boundary separately. By comparison principle, $\partial E(s)$ can not touch $\overline{\Gamma}(s, t)$ interior. At the boundary, gradient of $\overline{u}(x, s, t)$ is infinity. Theorem 2.3.4 and 2.3.7 implies that $\partial E(s)$ can not touch $\overline{\Gamma}(s, t)$ at the boundary. The details are left to the reader) Especially,

$$\{O\} \cap E(t) = \overline{\Gamma}(t;t) \cap E(t) = \emptyset.$$

Therefore for every $0 < t < T_\gamma$, there holds $O \in E(t)^c$.

Here we show that $E(t)$ is separated into two connected components, for $0 < t < T_\gamma$. Let $E^+(t)$ ($E^-(t)$) and $U^+(t)$ ($U^-(t)$) be the outer evolution and inner evolution of $V = -\kappa + A$ with $E^+(0) = \overline{U} \cap \{x \geq 0\}$ ($E^-(0) = \overline{U} \cap \{x \leq 0\}$) and $U^+(0) = U \cap \{x \geq 0\}$ ($U^-(0) = U \cap \{x \leq 0\}$), respectively.

Since $U \cap \{x \geq 0\}$ and $U \cap \{x \leq 0\}$ are α-domains and $E^+(t) \cap E^-(t) = \emptyset$, using

Theorem 6.0.3, we obtain

$$\partial E^+(t) = \partial U^+(t),\ \partial E^-(t) = \partial U^-(t),\ 0 < t < T_\gamma.$$

Therefore

$$\partial E(t) = \partial E^+(t) \cup \partial E^-(t) = \partial U^+(t) \cup \partial U^-(t) = \partial U(t),\ 0 < t < T_\gamma.$$

Here we complete the proof. □

Corollary 5.3.5. *Let u_0 be a function as in Section 5.1 and let γ be the constant in (5.1.2). For $n \geq 2$ and $0 \leq \gamma < \alpha_n$, there is no solution of the following free boundary problem,*

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{n-1}{u} + A\sqrt{1+u_x^2},\ -b(t) < x < b(t),\ 0 < t < T_1, \tag{5.1.3}$$

$$u(b(t),t) = u(-b(t),t) = 0,\ u_x(-b(t),t) = -u_x(b(t),t) = \infty,\ 0 < t < T_1, \tag{5.1.4}$$

$$u(x,0) = u_0(x),\ -b_0 \leq x \leq b_0, \tag{5.1.5}$$

$$u(x,t) > 0,\ -b(t) < x < b(t),\ 0 < t < T_1. \tag{5.1.6}$$

Proof. Assume that there exists the solution (u,b) of the free boundary problem. We can use the approximate argument similarly as in Lemma 5.2.3 to prove that the outer evolution $E(t)$ is written as follows

$$E(t) = \{(x,y) \mid \mathbb{R}^{n+1} \mid |y| \leq u(x,t), -b(t) \leq x \leq b(t)\}.$$

This contradicts that $E(t)$ is separated into two connected components. □

5.4 Singular angle $\gamma < \pi/2$ with $n = 1$

In this section, we give the proof of Theorem 5.1.3.

Proposition 5.4.1. *(Connected Outer evolution) Let u_0 be a function as in Section 1. In the plane, there is $T_1 > 0$ such that (u, b) is a unique solution of the following free boundary problem,*

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \ -b(t) < x < b(t), \ 0 < t < T_1, \tag{5.1.3*}$$

$$u(b(t), t) = u(-b(t), t) = 0, \ u_x(-b(t), t) = -u_x(b(t), t) = \infty, \ 0 < t < T_1, \tag{5.1.4}$$

$$u(x, 0) = u_0(x), \ -b_0 \le x \le b_0, \tag{5.1.5}$$

$$u(x, t) > 0, \ -b(t) < x < b(t), \ 0 < t < T_1. \tag{5.1.6}$$

Moreover, the outer evolution

$$E(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| \le u(x, t), -b(t) \le x \le b(t)\}, \ 0 < t < T_1.$$

Proof. Using the approximate argument as in Lemma 3.2.5, we can prove that there exists $T_1 > 0$ such that $E(t)^\circ$ is At-domain, $0 < t < T_1$. Moreover,

$$E(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| \le u(x, t), -b(t) \le x \le b(t)\}, \ 0 \le t < T_1.$$

Here (u, b) is the unique solution of (5.1.3*), (5.1.4), (5.1.5), (5.1.6). For the precise proof, we can see [5] similarly. Here we omit the details. □

Remark 5.4.2. Indeed, it is determined by the existence of the solution (u, b) of (5.1.3*), (5.1.4), (5.1.5), (5.1.6) whether the outer evolution is connected or separated.

Under the condition $n = 1$, saying roughly, if u satisfies

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \tag{5.1.3*}$$

and $u(x,0) = u_0(x) \geq 0$, by comparison principle, $u(x,t) > 0$, $t > 0$. This means that the problem always has a "positive" solution in the plane. This can be explained precisely by Lemma 2.3.9

However, under the conditon $n \geq 2$, the equation

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{n-1}{u} + A\sqrt{1+u_x^2} \tag{5.1.3}$$

has the "contraction power $-\frac{n-1}{u}$". We can not ensure that the problem has a "positive" solution with $u(x,0) = u_0(x) \geq 0$. Lemma 5.2.6 shows that if $\gamma = \pi/2$, this problem has a unique "positive" solution.

Proposition 5.4.3. *(Separated inner evolution) Suppose* $0 \leq \gamma < \pi/2$. *Let* (u, a, b) *be the solution of the following problem*

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2},\ a(t) < x < b(t),\ 0 < t < T_1, \tag{5.1.3**}$$

$$u(b(t),t) = u(a(t),t) = 0,\ u_x(a(t),t) = -u_x(b(t),t) = \infty,\ 0 < t < T_1, \tag{5.1.4*}$$

$$u(x,0) = u_0(x),\ 0 \leq x \leq b_0, \tag{5.1.5*}$$

$$u(x,t) > 0,\ a(t) < x < b(t),\ 0 < t < T_1. \tag{5.1.6*}$$

Then there exists $\delta > 0$ *such that* $a(t) > 0$, $0 < t < \delta$. *Moreover, the inner evolution* $U(t)$ *can be written as follows,*

$$U(t) = U^+(t) \cup U^-(t),\ 0 < t < \delta,$$

where $U^+(t) = \{(x,y) \in \mathbb{R}^2 \mid |y| < u(x,t),\ a(t) \le x \le b(t)\}$ *and* $U^-(t) = \{(-x,y) \mid (x,y) \in U^+(t)\}$.

Proof. We claim that there exists $\delta > 0$ such that $a(t) > 0$ for $0 < t < \delta$. If the claim holds, $U^+(t) \cap U^-(t) = \emptyset$, $0 < t < \delta$. Using Lemma 2.1.9, $U(t) = U^+(t) \cap U^-(t)$, $0 < t < \delta$.

We give the sketch of the proof of the claim.

Let $\gamma < \gamma_1 < \pi/2$. Define a family of circles

$$v_\lambda(y) = \lambda C - \sqrt{(\lambda C \cos(\pi/2 - \gamma_1))^2 - y^2}.$$

It is easy to find that the envelop of $\{v_\lambda\}_{\lambda>0}$ is $|y| = \tan\gamma_1 x$.

Let $\{(x,y) \mid x = v_0(y), -\delta_0 < y < \delta_0\}$ be the left cap of $\partial U \cap \{x \ge 0\}$.

By the choice of γ_1, if necessary, choose δ_0 smaller such that

$$v_0(y) \le \tan(\pi/2 - \gamma_1)|y|,\ -\delta_0 < y < \delta_0.$$

Consider the following inverse equation

$$v_t = \frac{v_{yy}}{1+v_y^2} - A\sqrt{1+v_y^2},\ -\delta_0 < y < \delta_0,$$

with $v(y,0) = \tan(\pi/2-\gamma_1)|y|$. Since the initial function is not smooth, we modify it by the family $\{v_\lambda\}_{\lambda>0}$ near the origin. Let $v_\lambda(y,t)$ be the solution with the modified initial data. We calculate

$$\frac{\partial}{\partial t} v_\lambda(0,0) = \frac{1}{\lambda C \cos(\pi/2 - \gamma_1)} - A \to \infty,$$

as $\lambda \to 0$. Therefore, there exists constant $C > 0$ such that $v_\lambda(0,t) > Ct$, for t small.

It is easy to see $v_\lambda(0,t) \to a(t)$, for every t small. Then

$$a(t) > Ct,\ 0 < t < \delta,$$

for some δ. Here we complete the proof.

□

Proof of Theorem 5.1.3. This result is an easy consequence of Proposition 5.4.1 and Proposition 5.4.3. □

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Chapter 6

Appendix

In this section, we want to prove that there exists unique smooth family of smooth hypersurfaces $\Gamma(t)$ satisfying

$$V = -\kappa + A, \text{ on } \Gamma(t) \subset \mathbb{R}^{n+1}, \tag{1.1.1}$$

where $\Gamma(0) = \partial U$ and U is an α-domain.

Seeing ∂U is not necessary smooth, we also use the level set method and prove the interface evolution is not fattening.

We choose smooth vector field $X : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that

(i) At any point $P \in \partial U$ not on the x-axis has $\langle X(P), \mathbf{n}(P)\rangle > 0$, $\mathbf{n}$ is inward unit normal vector at P.

(ii) Near the two end points of ∂U, X is constant vector with $X \equiv \pm e_0 = (\pm 1, 0, \cdots, 0)$.

Since $X \neq 0$ on the compact ∂U, there is an open neighbourhood $V \supset \partial U$ on which $|X| \geq \delta > 0$ for some $\delta > 0$.

Proposition 6.0.1. *For small enough $\rho > 0$ there exists a smooth hypersurface $\Sigma \subset V$ with*

(i) $X(P) \notin T_P\Sigma$ at all $P \in \Sigma$, i.e., Σ is transverse to the vector field X.

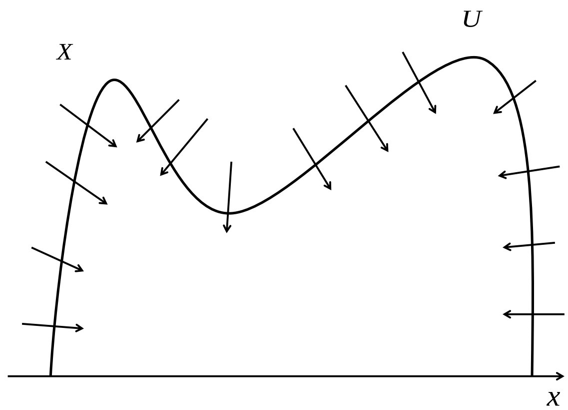

Figure 6.1: Vector field X

(ii) $\Sigma = \partial U$ *in* $\{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| \geq 2\rho\}$.

(iii) $\Sigma \cap \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| \leq \rho\}$ *consists of two flat disks* $\Delta_a = \{(a, y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| \leq \rho\}$ *and* $\Delta_b = \{(b, y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| \leq \rho\}$ *for some* $a < b$.

Seeing Figure 6.2, this proposition can be proved as in Proposition 3.2.1.

Let $\phi^t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}(t \in \mathbb{R})$, $t \in (-\delta, \delta)$ be the flow generated by vector field X on $\mathbb{R}^{n+1}$ determined by

$$\begin{cases} \dfrac{d\phi^t(P)}{dt} = X(\phi^t), \ \ P \in \Sigma, \\ \phi^0(P) = P, \qquad P \in \Sigma. \end{cases}$$

We denote $\sigma(P, s) := \phi^s(P)$. Suppose $\Gamma(t) \subset V(0 < t < T)$ are smooth hypersurfaces with $\sigma^{-1}(\Gamma(t))$ being the graph $u(\cdot, t)$ for $u : \Sigma \times [0, T) \to \mathbb{R}$. Let $z_1, z_2, \cdots, z_n$ be local coordinates on an open subset of Σ. If $\Gamma(t)$ evolving by $V = -\kappa + A$, then in these coordinates u satisfies the following parabolic equation

$$\frac{\partial u}{\partial t} = a_{ij}(z, u, \nabla_z u)\nabla^2_{z_i z_j} u + b(z, u, \nabla_z u). \tag{3.2.1}$$

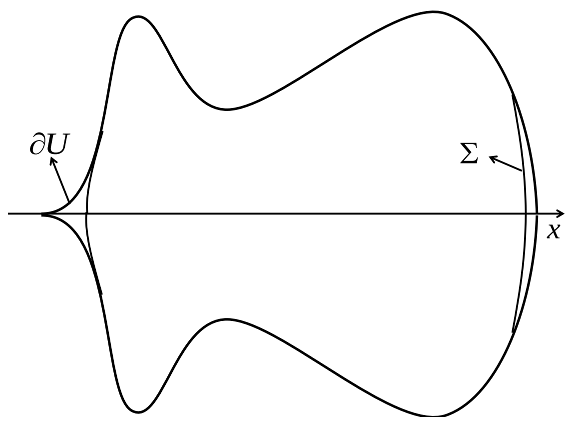

Figure 6.2: Proof of Proposition 6.0.1

For example, on Δ_a, by calculation, $\sigma(y_1,, y_2, \cdots, y_n, s) = (a - s, y_1, y_2, \cdots, y_n)$. Then u satisfies the "$-$" condition of (2.2.1).

Proposition 6.0.2. *For $n \geq 1$, let $\Gamma_1(t)$, $\Gamma_2(t)(0 \leq t < T)$ be two families of hypersurfaces and $\sigma^{-1}(\Gamma_j(t))$ be the graph of $u_j(\cdot, t)$ for certain $u_j \in C(\Sigma \times [0, T))$. Assume that the u_j are smooth on $\Sigma \times (0, T)$ as well as on $\Sigma \setminus (\Delta_a \cup \Delta_b) \times [0, T)$. Then if the $\Gamma_j(t)$ evolve by $V = -\kappa + A$ and if $\Gamma_1(0) = \Gamma_2(0)$, then there holds $\Gamma_1(t) = \Gamma_2(t)$ for $0 < t < T$.*

The proof is similar as in Proposition 3.2.4. Here we omit it.

Theorem 6.0.3. *If U is an α-domain with smooth boundary, let $D(t)$ and $E(t)$ be the open and closed evolutions of $V = -\kappa + A$ with $D(0) = U$ and $E(0) = \overline{U}$. Then there exists $T > 0$ such that $\partial D(t)$ and $\partial E(t)$ are smooth hypersurfaces for $0 < t \leq T$ and $\partial D(t) = \partial E(t)$. Moreover, denoting $\Sigma(t) = \partial D(t) = \partial E(t)$, $\Sigma(t)$ can be written into $\Sigma(t) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| = u(x, t), a(t) \leq x \leq b(t)\}$ and (u, a, b) is the*

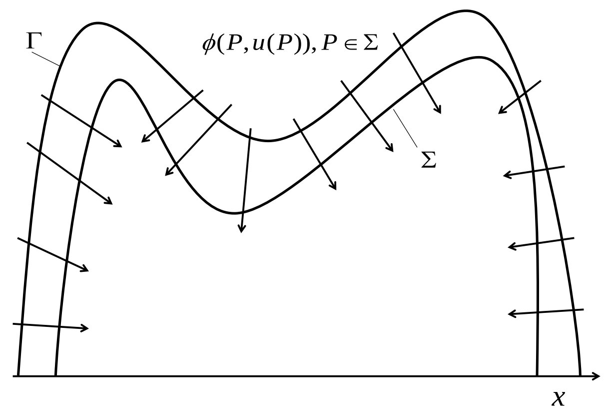

Figure 6.3: The transportation from Σ to Γ

solution of the following problem

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} - \dfrac{n-1}{u} + A\sqrt{1+u_x^2},\ x \in (a(t), b(t)),\ 0 < t < T, \\ u(a(t),t) = 0,\ u(b(t),t) = 0,\ 0 \le t < T, \\ u_x(a(t),t) = \infty,\ u_x(b(t),t) = -\infty,\ 0 \le t < T, \\ u(x,t) > 0,\ x \in (a(t), b(t)),\ 0 < t < T. \end{cases} \tag{**}$$

Proof. We only give the sketch of the proof. By approximate argument, $\partial D(t)$ and $\partial E(t)$ are smooth hypersurfaces and can be represented by $\sigma(P, u_j(P))$, for some u_j, $j = 1, 2$. Then we can use Proposition 6.0.2 to prove $\partial D(t) = \partial E(t)$. Therefore $\Gamma(t) = \partial E(t)$ can be represented by $\Gamma(t) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid |y| = u(x,t), a(t) \le x \le b(t)\}$. Using Theorem 2.1.10, (u, a, b) is the solution of (**). $\square$

Acknowledgments My deepest gratitude goes first and foremost to Professor Giga Yoshikazu, my supervisor, for his constant encouragement and guidance. And I express my hearty thanks to Professor Giga Yoshikazu, for his instructive advice and useful suggestions on this thesis and am grateful for his help in the completion of this thesis for letting me know several related useful papers.

I am also grateful to Professor Hiroshi Matano for his stimulating suggestions on the intersection number principle.