博士論文

論文題目　　Periods of tropical K3 hypersurfaces

（トロピカル K3 超曲面の周期）

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Periods of tropical K3 hypersurfaces

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Abstract

Let Δ be a smooth reflexive polytope in dimension 3 and f be a tropical polynomial whose Newton polytope is the polar dual of Δ. One can construct a 2-sphere B equipped with an integral affine structure with singularities by contracting the tropical K3 hypersurface defined by f. We write the complement of the singularity as $\iota \colon B_0 \hookrightarrow B$, and the local system of integral tangent vectors on B_0 as $\mathcal{T}_{\mathbb{Z}}$. Let further Y be an anti-canonical hypersurface of the toric variety associated with the normal fan of Δ, and $\mathrm{Pic}(Y)_{\mathrm{amb}}$ be the sublattice of $\mathrm{Pic}(Y)$ coming from the ambient space. We give a primitive lattice embedding $\mathrm{Pic}(Y)_{\mathrm{amb}} \hookrightarrow H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$, and compute the radiance obstruction of B, which sits in the subspace generated by the image of $\mathrm{Pic}(Y)_{\mathrm{amb}}$.

1 Introduction

Let M be a free $\mathbb{Z}$-module of rank 3 and $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. We set $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}(M, \mathbb{R})$. Let $\Delta \subset M_{\mathbb{R}}$ be a smooth reflexive polytope of dimension 3, and $\check{\Delta} \subset N_{\mathbb{R}}$ be the polar polytope of Δ. Let further Σ and $\check{\Sigma}$ be the normal fans to Δ and $\check{\Delta}$ respectively. We choose a refinement $\check{\Sigma}' \subset M_{\mathbb{R}}$ of $\check{\Sigma}$ such that the primitive generator of any 1-dimensional cone in $\check{\Sigma}'$ is contained in $\Delta \cap M$.

Let $A \subset N$ denote the subset consisting of all vertices of $\check{\Delta}$ and $0 \in N$. We consider a tropical Laurent polynomial

$$f(x) = \max_{n \in A} \{a(n) + n_1 x_1 + n_2 x_2 + n_3 x_3\}, \tag{1.1}$$

such that the function

$$A \to \mathbb{R}, \quad n \mapsto a(n) \tag{1.2}$$

induces a central subdivision of $\check{\Delta}$, i.e., every maximal dimensional simplex of the subdivision has the origin $0 \in N$ as its vertex. Let $V(f)$ be the tropical hypersurface defined by f in the tropical toric variety associated with $\check{\Sigma}'$. See Section 3.1 for the definition of tropical toric varieties.

We can construct a 2-sphere B equipped with an integral affine structure with singularities by contracting $V(f)$. See Section 4 for details about the construction. The same construction has already been performed in Gross–Siebert program [Gro05], [GS06]. There is also another construction by Haase and Zharkov, which was discovered independently [HZ02]. It is also known that maximally degenerating families of complex K3 surfaces with Ricci-flat Kähler metrics converge to 2-spheres with integral affine structures with singularities in the Gromov–Hausdorff limit [GW00].

In this paper, we compute the radiance obstruction of B. Radiance obstructions are invariants of integral affine manifolds, which were introduced in [GH84]. See Section 2 for its definition. Let $\iota\colon B_0 \hookrightarrow B$ denote the complement of singularities of B. Let further $\mathcal{T}_{\mathbb{Z}}$ be the local system on B_0 of integral tangent vectors. The cohomology group $H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$ has the cup product

$$\cup\colon H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}) \otimes H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}) \to H^2(B, \iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}}) \cong \mathbb{Z} \tag{1.3}$$

induced by the wedge product. Let Y be an anti-canonical hypersurface of the complex toric variety X_Σ associated with Σ, and

$$\mathrm{Pic}(Y)_{\mathrm{amb}} := \mathrm{Im}\left(\mathrm{Pic}(X_\Sigma) \hookrightarrow \mathrm{Pic}(Y)\right) \tag{1.4}$$

be the sublattice of $\mathrm{Pic}(Y)$ coming from the Picard group of the ambient space. We show the followings in this paper:

Theorem 1.1. *There is a primitive embedding*

$$\psi\colon\ \mathrm{Pic}(Y)_{\mathrm{amb}} \hookrightarrow H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}), \tag{1.5}$$

that preserves the pairing.

Each element n of $A \setminus \{0\}$ is the primitive generator of the 1-dimensional cone ρ_n of Σ. We write the restriction to Y of the toric divisor on X_Σ corresponding to ρ_n as D_n.

Theorem 1.2. *The radiance obstruction c_B of B is given by*

$$c_B = \sum_{n \in A \setminus \{0\}} \{a(n) - a(0)\}\, \psi(D_n). \tag{1.6}$$

It is known that the valuation of the j-invariant of an elliptic curve over a non-archimedean valuation field coincides with the cycle length of the tropical elliptic curve obtained by tropicalization [KMM08], [KMM09]. Theorem 1.1 and Theorem 1.2 are a generalization of this to the case of K3 hypersurfaces.

The definition of periods for general tropical curves was given in [MZ08]. It was also shown in [Iwa10] that the leading term of the period map of a degenerating family of Riemann surfaces is given by the period of the tropical curve obtained by tropicalization.

The period map is approximated by Schmid's nilpotent orbit [Sch73] in the limit to the degeneration point. The leading term of the nilpotent orbit is determined by the monodromy around the degeneration point. It was also shown in [GS10] that the wedge product of the radiance obstruction corresponds to the monodromy operator around the degeneration point in the case of Calabi–Yau varieties. Hence, we can see that the radiance obstruction gives the leading term of the period map of the corresponding family of Calabi–Yau varieties.

The organization of this paper is as follows: In Section 2, we recall the definitions of integral affine manifolds and their radiance obstructions. We also recall the definition of integral affine manifolds with singularities and we define their radiance obstructions. In Section 3, we recall some notions of tropical geometry, such as tropical toric varieties and tropical modifications. In Section 4, we explain the details about how to construct integral affine spheres with singularities from tropical K3 hypersurfaces. In Section 5, we discuss how dispersing singular points piling at one point affects the cohomology group $H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$ and the radiance obstruction c_B. The results in Section 5 will be used for proofs of the main theorems. In Section 6, we give proofs of Theorem 1.1 and Theorem 1.2. In Section 7, we discuss the relation with asymptotic behaviors of period maps of complex K3 hypersurfaces.

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2 Integral affine structures with singularities

Let M be a free $\mathbb{Z}$-module of rank n and $N := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be the dual lattice of M. We set $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{R})$, and $\operatorname{Aff}(M_{\mathbb{R}}) := M_{\mathbb{R}} \rtimes \operatorname{GL}(M)$.

Definition 2.1. An *integral affine manifold* is a real topological manifold B with an atlas of coordinate charts $\psi_i \colon U_i \to M_{\mathbb{R}}$ such that all transition functions $\psi_i \circ \psi_j^{-1}$ are contained in $\operatorname{Aff}(M_{\mathbb{R}})$.

Let B be an integral affine manifold. We give an affine bundle structure to the tangent bundle TB of B as follows: For each U_i and $x \in U_i$, we set an affine isomorphism

$$\theta_{i,x} \colon T_x B \to M_{\mathbb{R}}, \quad v \mapsto \psi_i(x) + d\psi_i(x) v, \tag{2.1}$$

and define an affine trivializations by

$$\theta_i \colon TU_i \to U_i \times M_{\mathbb{R}}, \quad (x, v) \mapsto (x, \theta_{i,x}(v)), \tag{2.2}$$

where $v \in T_x B$. This gives an affine bundle structure to TB. We write TB with this affine bundle structure as $T^{\mathrm{aff}} B$.

Let $\mathcal{T}_{\mathbb{Z}}$ be the local system on B of integral tangent vectors. We set $\mathcal{T} := \mathcal{T}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.2. We choose a sufficiently fine open covering $\mathcal{U} := \{U_i\}_i$ of B so that there is a flat section $s_i \in \Gamma(U_i, T^{\mathrm{aff}} B)$ for each U_i. When we set $c_B((U_i, U_j)) := s_j - s_i$ for each 1-simplex (U_i, U_j) of $\mathcal{U}$, the element c_B becomes a Čech 1-cocycle for $\mathcal{T}$. We call $c_B \in H^1(B, \mathcal{T})$ the *radiance obstruction* of B.

Definition 2.3. An *integral affine manifold with singularities* is a topological manifold B with an integral affine structure on $B_0 := B \setminus \Gamma$, where $\Gamma \subset B$ is a locally finite union of locally closed submanifolds of codimension greater than 2. We call Γ the singular locus of B.

We assume that integral affine manifolds with singularities satisfy the following condition. This was mentioned in [KS06, Section 3.1] as the fixed point property.

Condition 2.4. For any $x \in \Gamma$, there is a small neighborhood U such that the monodromy representation $\pi_1(U \setminus \Gamma) \to \mathrm{Aff}(M_{\mathbb{R}})$ has a fixed vector.

Let B be an integral affine manifold with singularities satisfying the above condition. We write the complement of the singular locus as $\iota \colon B_0 \hookrightarrow B$. Let further $\mathcal{T}_{\mathbb{Z}}$ be the local system on B_0 of integral tangent vectors. We set $\mathcal{T} := \mathcal{T}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ again.

Definition 2.5. We choose a sufficiently fine covering $\{U_i\}_i$ of B so that there is a flat section $s_i \in \Gamma(U_i \cap B_0, T^{\mathrm{aff}} B_0)$ for each U_i. This is possible as long as we assume Condition 2.4. When we set $c_B((U_i, U_j)) := s_j - s_i$, the element c_B becomes a Čech 1-cocycle for $\iota_* \mathcal{T}$. We call $c_B \in H^1(B, \iota_* \mathcal{T})$ the *radiance obstruction* of B.

Remark 2.6. The inclusion $\iota \colon B_0 \hookrightarrow B$ induces a map $\iota^* \colon H^1(B, \iota_* \mathcal{T}) \hookrightarrow H^1(B_0, \mathcal{T})$. Then we can see $\iota^*(c_B) = c_{B_0}$ from the definitions.

3 Tropical geometry

3.1 Tropical toric varieties and hypersurfaces

Let $(\mathbb{T}, +, \cdot)$ be the tropical semifield, where $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ and

$$a + b := \max\{a, b\}, \tag{3.1}$$
$$a \cdot b := a + b. \tag{3.2}$$

Here the addition $+$ in the right hand side of (3.2) means the usual addition. In the following of this section, all additions $+$ and multiplications $\cdot$ mean max and $+$ respectively unless otherwise mentioned.

Let M be a free $\mathbb{Z}$-module of rank n and $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be the dual lattice of M. We set $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R})$. We have a canonical $\mathbb{R}$-bilinear pairing

$$\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}. \tag{3.3}$$

For each cone $\sigma \in \mathcal{F}$, we set

$$\sigma^{\vee} := \{m \in M_{\mathbb{R}} \mid \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma\}, \tag{3.4}$$
$$\sigma^{\perp} := \{m \in M_{\mathbb{R}} \mid \langle m, n \rangle = 0 \text{ for all } n \in \sigma\}. \tag{3.5}$$

Let Σ be a fan in $N_{\mathbb{R}}$. We define the toric variety $X_{\Sigma}(\mathbb{T})$ associated with Σ over $\mathbb{T}$ as follows: For each cone $\sigma \in \Sigma$, we define X_{σ} as the set of monoid homomorphisms $\sigma^{\vee} \cap M \to (\mathbb{T}, \cdot)$

$$X_{\sigma}(\mathbb{T}) := \mathrm{Hom}(\sigma^{\vee} \cap M, \mathbb{T}) \tag{3.6}$$

with the compact open topology. For cones $\sigma, \tau \in \Sigma$ such that $\sigma \prec \tau$, we have a natural immersion,

$$X_{\sigma}(\mathbb{T}) \to X_{\tau}(\mathbb{T}), \quad (v \colon \sigma^{\vee} \cap M \to \mathbb{T}) \mapsto (\tau^{\vee} \cap M \subset \sigma^{\vee} \cap M \xrightarrow{v} \mathbb{T}), \tag{3.7}$$

where $\sigma \prec \tau$ means that σ is a face of τ. By gluing $\{X_\sigma(\mathbb{T})\}_{\sigma\in\Sigma}$ together, we obtain the tropical toric variety $X_\Sigma(\mathbb{T})$ associated with Σ,

$$X_\Sigma(\mathbb{T}) := \left(\coprod_{\sigma\in\Sigma} X_\sigma(\mathbb{T}) \right) \Big/ \sim . \tag{3.8}$$

We also define the torus orbit O_σ corresponding to σ by

$$O_\sigma(\mathbb{T}) := \operatorname{Hom}(\sigma^\perp \cap M, \mathbb{R}). \tag{3.9}$$

There is a projection map to the torus orbit

$$p_\sigma \colon X_\sigma(\mathbb{T}) \to O_\sigma(\mathbb{T}), \quad (w \colon \sigma^\vee \cap M \to \mathbb{T}) \mapsto (\sigma^\perp \cap M \subset \sigma^\vee \cap M \xrightarrow{w} \mathbb{T}). \tag{3.10}$$

See [Pay09] or [Kaj08] for more details about tropical toric varieties.

Consider a tropical Laurent polynomial

$$f = \sum_{m\in A} a_m x^m, \tag{3.11}$$

where $A \subset M$ be a finite subset and $a_m \in \mathbb{T}$. It gives rise to a piecewise linear function $f \colon N_\mathbb{R} \to \mathbb{T}$. The bend locus $V(f) \subset N_\mathbb{R}$ of f is called the tropical hypersurface defined by f. The tropical hypersurface defined by f in the tropical toric variety $X_\Sigma(\mathbb{T})$ is the closure of $V(f)$ in $X_\Sigma(\mathbb{T})$. Note that the tropical toric variety $X_\Sigma(\mathbb{T})$ contains the maximal dimensional torus orbit $O_{\{0\}}(\mathbb{T}) \cong N_\mathbb{R}$ as its open dense subset.

3.2 Tropical modifications

Tropical modifications are first introduced in [Mik06]. We briefly recall the idea of it. Let $(\mathbb{T}, \oplus, \odot)$ be the tropical hyperfield, where $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ and

$$a \oplus b := \begin{cases} \max\{a, b\}, & a \neq b, \\ \{t \in \mathbb{T} \mid t \leq a\}, & a = b, \end{cases} \tag{3.12}$$

$$a \odot b := a + b. \tag{3.13}$$

The addition $+$ in the right hand side of (3.13) means the usual addition.

Let $A \subset \mathbb{Z}^n$ be a finite subset. We consider a tropical polynomial

$$f(x) = \bigoplus_{n\in A} a_n \odot x^n, \tag{3.14}$$

which is a multi-valued function on $\mathbb{T}^n$ defined by

$$f(x) := \begin{cases} \sum_{n\in A} a_n x^n = a_{n_0} x^{n_0}, & \text{when } \exists n_0 \in A \text{ s.t. } a_{n_0} x^{n_0} > a_n x^n \ (\forall n \neq n_0), \\ \left\{ t \in \mathbb{T} \;\middle|\; t \leq \sum_{n\in A} a_n x^n \right\}, & \text{otherwise.} \end{cases} \tag{3.15}$$

We consider the graph $\Gamma_f \subset \mathbb{T}^{n+1}$ of the function f

$$\Gamma_f := \left\{ (x, y) \in \mathbb{T}^{n+1} \;\middle|\; y \in f(x) \right\}. \tag{3.16}$$

This coincides with the bend locus of

$$f'(x, y) := y + \sum_{n\in A} a_n x^n \tag{3.17}$$

in $\mathbb{T}^{n+1}$ and has a natural balanced polyhedral structure. Let $\delta_f \colon \Gamma_f \to \mathbb{T}^n$ be the projection forgetting the last component.

Definition 3.1. We call the balanced polyhedral complex Γ_f the *tropical modification* of $\mathbb{T}^n$ with respect to f. We also call the map $\delta_f \colon \Gamma_f \to \mathbb{T}^n$ the *contraction* with respect to f.

The graph of $\sum_{n \in A} a_n x^n$ is isomorphic to $\mathbb{T}^n$ as sets. Hence, we can think that associating Γ_f with $\mathbb{T}^n$ corresponds to replacing $+$ and $\cdot$ of $\sum_{n \in A} a_n x^n$ with $\oplus$ and $\odot$ respectively.

We can also define tropical modifications of general tropical varieties in affine spaces. We define tropical modifications of tropical manifolds that are not necessarily embedded in ambient spaces as maps between tropical manifolds which locally coincide with a tropical modification of an affine tropical variety. For a tropical manifold X, we regard a tropical manifold X' which relates to X by tropical modifications as a tropical manifold that is equivalent to X. We refer the reader to [Sha15] or [Kal15] for details.

4 Contractions of tropical hypersurfaces

In this section, all additions $+$ and multiplications $\cdot$ mean max and $+$ respectively unless otherwise mentioned. We also let M, N denote a free $\mathbb{Z}$-module of rank 3 and its dual lattice respectively.

4.1 A local model of contractions

We fix basis vectors e_1, e_2, e_3 of N. Consider the cone σ_k generated by $ke_1 + e_2, e_2 \in N$ in $N_{\mathbb{R}}$, where k is some positive integer. Then the dual cone $\sigma_k^\vee$ is generated by $e_1^*, -e_1^* + ke_2^*, \pm e_3^*$ and we have

$$
\begin{aligned}
X_{\sigma_k}(\mathbb{T}) &:= \operatorname{Hom}(\sigma_k^\vee \cap M, \mathbb{T}) && (4.1)\\
&= \left\{ (x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \;\middle|\; xy = z^k \right\}. && (4.2)
\end{aligned}
$$

We define the space $X_{k,l}$ by

$$
X_{k,l} := \left\{ (x, y, z, w) \in X_{\sigma_k}(\mathbb{T}) \;\middle|\; z = 0 + w^l \right\}, \tag{4.3}
$$

where l is also some positive integer. This space consists of two 2-dimensional faces

$$
\begin{aligned}
F_+ &:= \left\{ (x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \;\middle|\; xy = w^{kl}, z = w^l, w > 0 \right\}, && (4.4)\\
F_- &:= \left\{ (x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \;\middle|\; xy = z = 0 > w \right\}, && (4.5)
\end{aligned}
$$

and a 1-dimensional face

$$
L := \left\{ (x, y, z, w) \in \mathbb{T}^3 \times \mathbb{R} \;\middle|\; xy = z = w = 0 \right\}. \tag{4.6}
$$

Each of these faces has an integral affine structure induced from the ambient space $\mathbb{T}^3 \times \mathbb{R}$. We extend them and construct an integral affine structure with a singular point on $X_{k,l}$ as follows:

First, we choose a point $p = (x_0, y_0, 0, 0) \in L$. We set

$$
\begin{aligned}
U_x &:= X_{k,l} \setminus \{ (x, y, z, w) \in L \mid x \geq x_0 \}, && (4.7)\\
U_y &:= X_{k,l} \setminus \{ (x, y, z, w) \in L \mid x \leq x_0 \}. && (4.8)
\end{aligned}
$$

These give a covering of $X_{k,l} \setminus \{p\}$. Consider projections

$$p_x \colon U_x \to \mathbb{R}^2, \quad (x,y,z,w) \mapsto (x,w), \tag{4.9}$$
$$p_y \colon U_y \to \mathbb{R}^2, \quad (x,y,z,w) \mapsto (y,w). \tag{4.10}$$

The restrictions of p_x and p_y to $F_\pm$ are integral affine isomorphisms onto their images. Hence, we can extend the integral affine structures on $F_\pm$ to U_x and U_y so that projections p_x and p_y are integral affine isomorphisms onto their images. Here we have $U_x \cap U_y = F_+ \cup F_-$ and the integral affine structures on U_x and U_y coincide on F_+ and F_- with each other. Hence, we can extend the integral affine structures on U_x and U_y to an integral affine structure on $X_{k,l} \setminus \{p\}$. We can easily calculate the monodromy of the integral affine structure around p.

Lemma 4.1. *Consider a loop around the point p, which starts from a point in U_x, passes through F_-, U_y, and F_+ in this order, and comes back to the original point. The monodromy along this loop is given by the matrix*

$$\begin{pmatrix} 1 & kl \\ 0 & 1 \end{pmatrix}, \tag{4.11}$$

under the basis e_x, e_w corresponding to the coordinate (x,w) of U_x.

Proof. A point $(x,w) = (x_0, w_0)$ of U_x is shown as $(x,y,z,w) = (x_0, x_0^{-1}, 0, w_0)$ in F_-. If we see this in U_y, we have $(y,w) = (x_0^{-1}, w_0)$. This is shown as $(x,y,z,w) = (x_0 w^{kl}, x_0^{-1}, w_0^l, w_0)$ in F_+. If we see this in U_x again, we get $(x,w) = (x_0 w_0^{kl}, w_0)$. Hence, the monodromy transformation is given by $e_x \mapsto e_x, e_w \mapsto (kl)e_x + e_w$. □

Remark 4.2. In the above construction of $X_{k,l}$, there is an ambiguity in the choice of the position of $p \in L$.

Remark 4.3. When $k = l = 1$, the point p becomes a so-called focus-focus singularity. When k, l are not necessarily 1, the point p can be regarded as a concentration of kl focus-focus singularities. The invariant subspace with respect to the monodromy around p is generated by e_x. It coincides with the tangent space of L.

Remark 4.4. When $k = l = 1$, we have

$$X_{1,1} \cong \left\{ (x,y,w) \in \mathbb{T}^2 \times \mathbb{R} \;\middle|\; xy = 0 + w \right\}, \quad (x,y,z,w) \mapsto (x,y,w). \tag{4.12}$$

In [KS06, Section 8], a non-archimedean torus fibration corresponding to a surface containing a focus-focus singularity is constructed by using the algebraic surface defined by $(\alpha\beta - 1)\gamma = 1$. Here, the subtraction and multiplication mean the usual ones. When we set $\alpha = x, \beta = y, \gamma = w^{-1}$, the tropicalizaion of it coincides with $xy = 0 + w$, the equation defining $X_{1,1}$.

Consider replacing the right hand side $0 + w^l$ of the equation of (4.3) with $0 \oplus w^l$. Then the solution of the equation $z = 0 \oplus w^l$ in $X_{\sigma_k}(\mathbb{T})$ is the union of $X_{k,l}$ and the additional face

$$F_0 := \left\{ (x,y,z,w) \in \mathbb{T}^3 \times \mathbb{R} \;\middle|\; xy = z^k, z < 0, w = 0 \right\}. \tag{4.13}$$

This coincides with the tropical hypersurface $V(f)$ defined by $f = 0 + z + w^l$ in $X_{\sigma_k}(\mathbb{T})$. We can think that the surface $X_{k,l}$ is obtained by contracting the tropical hypersurfaces $V(f)$ to

x-direction and y-direction at the same time. We choose a point $p = (x_0, y_0, 0, 0) \in L$ and define a contraction map $\delta_{f,p} \colon V(f) \to X_{k,l}$ by

$$(x, y, z, w) \mapsto \begin{cases} (x, y, z, w) & (x, y, z, w) \in X_{k,l} \\ (x, x^{-1}, 0, 0) & x \geq x_0 \\ (y^{-1}, y, 0, 0) & y \geq y_0 \\ p = (x_0, y_0, 0, 0) & \text{otherwise.} \end{cases} \tag{4.14}$$

The face F_0 is contracted to the line L by this map. The tropical hypersurface $V(f)$ and the contraction $\delta_{f,p}$ are shown in Figure 4.1.

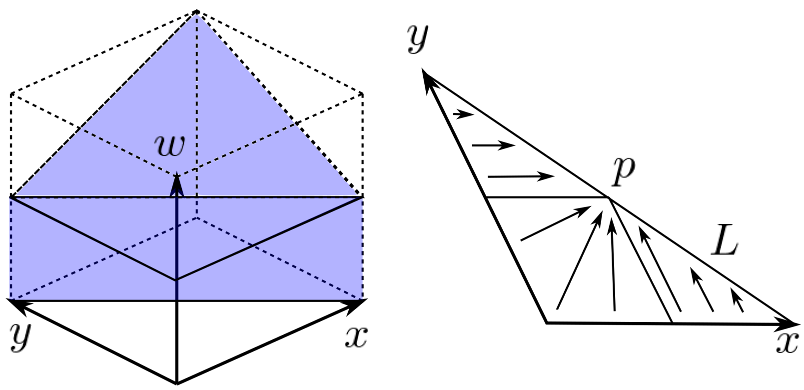

Figure 4.1: The tropical hypersurface $V(f)$ and the contraction of the face F_0

Associating $V(f)$ with $X_{k,l}$ is similar to tropical modifications which we recalled in Section 3.2 in the sense that we replace operations $+$ contained in a function with hyperoperations $\oplus$. In this article, we call associating the tropical hypersurface $V(f)$ with $X_{k,l}$ a tropical modification with respect to $0 + w^l$, and the map $\delta_{f,p} \colon V(f) \to X_{k,l}$ the contraction with respect to $0 + w^l$.

4.2 Contractions of tropical toric K3 hypersurfaces

Let $\Delta \subset M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ be a reflexive polytope of dimension 3, which is not necessarily smooth. We write the polar polytope of Δ as $\check{\Delta} \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, and the normal fans of $\Delta, \check{\Delta}$ as $\Sigma \subset N_{\mathbb{R}}, \check{\Sigma} \subset M_{\mathbb{R}}$ respectively. We choose a refinement $\check{\Sigma}' \subset M_{\mathbb{R}}$ of $\check{\Sigma}$ such that the primitive generator of any 1-dimensional cone in $\check{\Sigma}'$ is contained in $\Delta \cap M$. This gives rise to a crepant resolution of the toric variety associated with $\check{\Sigma}$.

Let $A \subset N$ denote the subset consisting of all vertices of $\check{\Delta}$ and $0 \in N$. We consider a tropical Laurent polynomial

$$f(x) = \sum_{n \in A} a(n) x^n, \tag{4.15}$$

such that the function

$$A \to \mathbb{R}, \quad n \mapsto a(n) \tag{4.16}$$

induces a central subdivision of $\check{\Delta}$. We consider the tropical hypersurface $V(f)$ defined by f in the tropical toric variety $X_{\check{\Sigma}'}(\mathbb{T})$.

The tropical hypersurface $V(f)$ intersects with the toric boundary of $X_{\check{\Sigma}'}(\mathbb{T})$ as follows: Let $\rho \in \check{\Sigma}'$ be a 1-dimensional cone, and F_ρ be the face of Δ which contains the primitive generator of ρ in its interior. Recall that there is a one-to-one correspondence between k-dimensional faces of Δ and $(2-k)$-dimensional faces of $\check{\Delta}$, given by

$$F \leftrightarrow F^* := \left\{ n \in \check{\Delta} \;\middle|\; \langle m, n \rangle = -1, \forall m \in F \right\}. \tag{4.17}$$

On the torus orbit $O_\rho(\mathbb{T}) \subset X_{\check{\Sigma}'}(\mathbb{T})$, the tropical hypersurface $V(f)$ is defined by

$$\sum_{n \in A \cap F_\rho^*} a(n) x^n. \tag{4.18}$$

The tropical hypersurface $V(f)$ intersects with the torus orbit $O_\rho(\mathbb{T})$ if and only if the number of elements of $A \cap F_\rho^*$ is greater than or equal to 2. This happens exactly when F_ρ is a vertex or an edge. Let $\sigma \in \check{\Sigma}'$ be a cone of dimension greater than 1, and $\{\rho_i\}_{i=0}^l \subset \check{\Sigma}'$ be the set of 1-dimensional faces of σ. On the torus orbit $O_\sigma(\mathbb{T}) \subset X_{\check{\Sigma}'}(\mathbb{T})$, the tropical hypersurface $V(f)$ is defined by

$$\sum_{n \in \bigcap_{i=0}^l A \cap F_{\rho_i}^*} a(n) x^n. \tag{4.19}$$

The tropical hypersurface $V(f)$ intersects with the torus orbit $O_\sigma(\mathbb{T})$ if and only if the number of elements of $\bigcap_{i=0}^l A \cap F_{\rho_i}^*$ is greater than or equal to 2. This happens when the dimension of σ is 2 and the primitive generators of its 1-dimensional faces ρ_1, ρ_2 are contained in a common edge of Δ.

We write the union of cells of $V(f)$ that do not intersect with the toric boundary as B. This is topologically a 2-sphere. In the following, we contract the tropical hypersurface $V(f)$ to the 2-sphere B, and equip B with an integral affine structure with singularities.

First, we choose positions of singular points. Let $\mathcal{P}$ be the natural polyhedral structure of B. For each cell $\tau \in \mathcal{P}$, we set

$$A_\tau := \left\{ n \in A \setminus \{0\} \mid f(x) = a(0) = a(n) x^n, \forall x \in \tau \right\}. \tag{4.20}$$

There is a one-to-one correspondence between $\mathcal{P}$ and proper faces of Δ given by $\tau \leftrightarrow F(\tau)$, where

$$F(\tau) := \left\{ m \in \Delta \mid \langle m, n \rangle = -1, \forall n \in A_\tau \right\}. \tag{4.21}$$

Let $\tau \in \mathcal{P}$ be a 1-dimensional cell and $v_0, v_1 \in \mathcal{P}$ be its endpoints. Let further $\{\rho_i\}_{i=0}^l$ be the set of 1-dimensional cones in $\check{\Sigma}'$ whose primitive generators are contained in $F(\tau)$. We write the primitive generator of ρ_i as $m_i \in M$. We renumber ρ_i $(0 \leq i \leq l)$ so that $m_0 = F(v_0)$ and $m_l = F(v_1)$, and m_i is nearer to m_0 than m_{i+1} for any $1 \leq i \leq l-1$. We choose l distinct points $p(\tau)_i$ $(1 \leq i \leq l)$ on the interior of τ so that the point $p(\tau)_i$ is nearer to the vertex v_0 than $p(\tau)_{i+1}$ for any $1 \leq i \leq l-1$. These points will be singular points of the integral affine structure of B. For each 1-dimensional cell $\tau \in \mathcal{P}$, we choose points $p(\tau)_i$ in this way and fix them.

For each point $p(\tau)_i$, we take an open neighborhood $U_{p(\tau)_i}$ of it. We also take an open neighborhood U_v for each vertex v of B. Here, we take these open sets so that they do not contain any other singular points or any other vertices of B, and all of these open sets $U_{p(\tau)_i}, U_v$ and interiors of all facets of B form a covering of B. We contract the tropical hypersurface $V(f)$ to B as follows:

- Around U_v

  Let $\rho \in \check{\Sigma}'$ be the cone whose primitive generator is $F(v)$, and $X_\rho(\mathbb{T}) \subset X_{\check{\Sigma}'}(\mathbb{T})$ be the tropical affine toric variety corresponding to ρ. Let further $V(f)_v$ be the star of v in $V(f)$. We consider the projection

$$p_\rho \colon X_\rho(\mathbb{T}) \to O_\rho(\mathbb{T}), \quad (w \colon \rho^\vee \cap M \to \mathbb{T}) \mapsto (\rho^\perp \cap M \subset \rho^\vee \cap M \xrightarrow{w} \mathbb{T}). \tag{4.22}$$

  We set $\tilde{U}_v := p_\rho^{-1}(p_\rho(U_v)) \cap V(f)_v$ and defined the map $\delta_v \colon \tilde{U}_v \xrightarrow{p_\rho} U_v$ as

$$\delta_v \colon \tilde{U}_v \xrightarrow{p_\rho} p_\rho(U_v) \cong U_v, \tag{4.23}$$

  where $p_\rho(U_v) \cong U_v$ is the inverse map of the bijection $p_\rho \colon U_v \to p_\rho(U_v)$. We equip U_v with the integral affine structure induced by the integral affine structure of $p_\rho(U_v) \subset O_\rho(\mathbb{T}) \cong \mathbb{R}^2$. The dominant part of f at v is given by

$$a(0) + \sum_{n \in A \cap F_\rho^*} a(n) x^n. \tag{4.24}$$

  By taking an appropriate coordinate, the function (4.24) can be rewritten as a function of the form $y + f_v$, where f_v is a function on $O_\rho(\mathbb{T})$. The map δ_v coincides with a restriction of the contraction of the hypersurface defined by $y + f_v$ with respect to the function f_v, which we considered in Section 3.2.

- Around $U_{p(\tau)_i}$

  We write the 2-dimensional cone whose 1-dimensional faces are ρ_{i-1} and ρ_i as $\sigma_i \in \check{\Sigma}'$ $(1 \leq i \leq l)$. Let $V(f)_\tau$ be the intersection of the star of τ in $V(f)$ and the subvariety $X_{\sigma_i}(\mathbb{T}) \subset X_{\check{\Sigma}'}(\mathbb{T})$. The tropical toric variety $X_{\sigma_i}(\mathbb{T})$ coincides with $X_{\sigma_{k_i}}(\mathbb{T})$ of Section 4.1, where k_i is the integral distance between the primitive generators m_{i-1}, m_i of ρ_{i-1}, ρ_i. On the other hand, the dominant part of f at τ is given by

$$a(0) + \sum_{n \in A_\tau} a(n) x^n. \tag{4.25}$$

  The set A_τ coincides with $F(\tau)^* \cap N$, and consists of two vertices of $\check{\Delta}$. By taking an appropriate coordinate, the function (4.25) can be rewritten as a function f_τ of the form $0 + z + w^l$, where l is the integral distance between the elements of A_τ. Let $V(f_\tau)$ be the tropical hypersurface defined by f_τ in $X_{\sigma_{k_i}}(\mathbb{T})$. We can embed $V(f)_\tau$ into $V(f_\tau)$. The open set $U_{p(\tau)_i} \subset V(f)_\tau$ is embedded into $X_{k_i,l}$ by the embedding. We define $\tilde{U}_{p(\tau)_i} \subset V(f)_\tau$ as the inverse image of $U_{p(\tau)_i}$ by the map

$$V(f)_\tau \subset V(f_\tau) \xrightarrow{\delta_{f_\tau, p(\tau)_i}} X_{k_i,l} \supset U_{p(\tau)_i}, \tag{4.26}$$

  where $\delta_{f_\tau, p(\tau)_i}$ is the contraction map of (4.14). We also define $\delta_{p(\tau)_i} \colon \tilde{U}_{p(\tau)_i} \to U_{p(\tau)_i}$ as the restriction of (4.26) to $\tilde{U}_{p(\tau)_i}$. We equip $U_{p(\tau)_i}$ with the integral affine structure induced from the integral affine structure of $X_{k_i,l} \supset U_{p(\tau)_i}$.

- Around facets

  We consider the identity map from the interior of each facet of B to itself. The interior of each facet has an integral affine structure induced from the ambient space $O_{\{0\}}(\mathbb{T}) \cong \mathbb{R}^3$.

The open sets $\tilde{U}_v, \tilde{U}_{p(\tau)_i}$ and interiors of all facets of B form a covering of $V(f)$. The above maps $\delta_v, \delta_{p(\tau)_i}$ and identity maps of interiors of facets coincides with each other on overlaps, since $\delta_v, \delta_{p(\tau)_i}$ are constructed by using the projections to the same direction on the overlap, and these maps is identical on facets of B. By gluing these maps together, we obtain a contraction map $\delta\colon V(f) \to B$ and an integral affine surface B with singular points on each edge. We regard B as a tropical K3 surface equivalent to the tropical K3 hypersurface $V(f)$.

Remark 4.5. There is an ambiguity in the choice of the position of each singular point $p(\tau)_i$. However, neither the cohomology group $H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$ nor the radiance obstruction c_B of B does not depend on this choice. We will reconsider this point in Remark 7.3.

Remark 4.6. Kontsevich and Soibelman constructed a 2-sphere with an integral affine structure with singularities by contracting a Clemens polytope of a degenerating family of K3 surfaces [KS06, Section 4.2.5]. Their contraction is quite similar to the above contraction of tropical hypersurfaces. Compare the local contraction given in [KS06, Section 4.2.5] to the contraction given in (4.14) of this article.

Example 4.7. Consider the polynomial

$$f(x,y,z) = 1 + x^3y^{-1}z^{-1} + x^{-1}y^3z^{-1} + x^{-1}y^{-1}z^3 + x^{-1}y^{-1}z^{-1}. \tag{4.27}$$

The Newton polytope $\check{\Delta} \subset N_{\mathbb{R}}$ of f is the simplex whose one side is 4. In this case, there are no further crepant refinements of $\check{\Sigma}$. We choose a point $p(\tau)$ on the interior of each edge τ of B, which will be a singular point. Let ρ_1 and ρ_2 be the 1-dimensional cones in the normal fan $\check{\Sigma}$ of $\check{\Delta}$ generated by $(1,0,0)$ and $(0,1,0)$ respectively. Let further v_1 and v_2 be the vertices of B such that $F(v_1) = (1,0,0), F(v_2) = (0,1,0)$, and τ be the edge of B connecting v_1 and v_2.

Around v_1, the tropical hypersurface $V(f)$ is locally defined by

$$1 + x^{-1}y^3z^{-1} + x^{-1}y^{-1}z^3 + x^{-1}y^{-1}z^{-1}, \tag{4.28}$$

and the contraction δ_{v_1} coincides with a restriction of the contraction with respect to the function f_{v_1} on $O_{\rho_1}(\mathbb{T})$ defined by

$$f_{v_1}(y,z) := y^3z^{-1} + y^{-1}z^3 + y^{-1}z^{-1}. \tag{4.29}$$

Around v_2, the tropical hypersurface $V(f)$ is locally defined by

$$1 + x^3y^{-1}z^{-1} + x^{-1}y^{-1}z^3 + x^{-1}y^{-1}z^{-1}, \tag{4.30}$$

and the contraction δ_{v_2} coincides with a restriction of the contraction with respect to the function f_{v_2} on $O_{\rho_2}(\mathbb{T})$ defined by

$$f_{v_2}(x,z) := x^3z^{-1} + x^{-1}z^3 + x^{-1}z^{-1}. \tag{4.31}$$

Around τ, the tropical hypersurface $V(f)$ is locally defined by

$$1 + x^{-1}y^{-1}z^3 + x^{-1}y^{-1}z^{-1}. \tag{4.32}$$

When we set $x' := 1x, y' := yz, z' := 1xyz, w' := z$, it is locally defined by

$$f_\tau(z', w') := 0 + z' + w'^4. \tag{4.33}$$

The contraction $\delta_{p(\tau)}$ coincides with a restriction of the contraction $\delta_{f_\tau, p(\tau)}$ of (4.14) ($k = 1, l = 4$). The open set U_τ is equipped with the integral affine structure of $X_{1,4}$.

These contractions are shown in Figure 4.2. Black points are chosen points as singular points. The red region shows the contraction δ_{v_1} and the blue region shows the contraction δ_{v_2}. The green region shows the contraction $\delta_{p(\tau)}$.

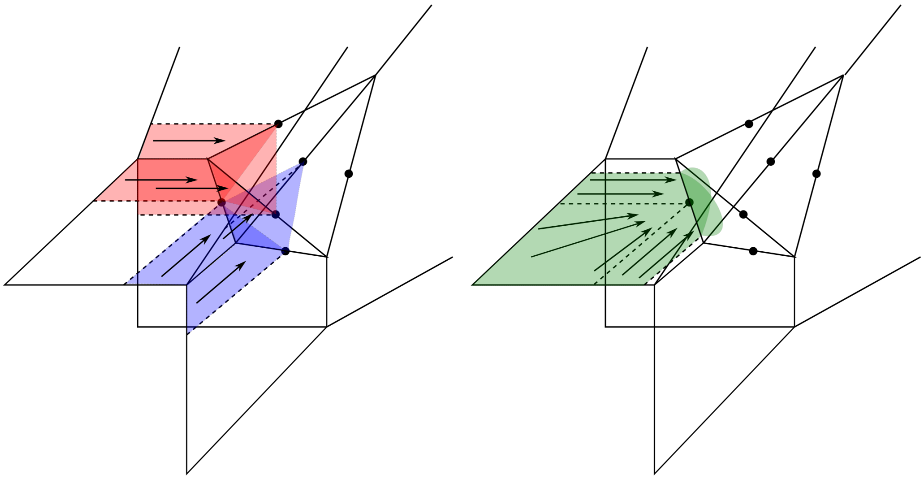

Figure 4.2: A contraction of the tropical hypersurface $V(f)$

5 Dispersions of focus-focus singularities

Let S be an integral affine surface with some singular points. We suppose that the monodromy around one of the singular points p of S is given by the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \tag{5.1}$$

under a local coordinate system (x, y) near p, where k is a non-zero integer. Here, the tangent vector e_x corresponding to the coordinate x is monodromy invariant, and the coordinate y is globally well-defined on a sufficiently small open neighborhood U of p. We write the line defined by $y = 0$ on U as L. We can construct another integral affine structure with singularities on U, which has just two singular points p_1 and p_2 on L whose monodromies are given by

$$\begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix}, \tag{5.2}$$

under the same coordinate system (x, y) respectively, where k_1, k_2 are non-zero integers such that $k_1 + k_2 = k$. By replacing the original integral affine structure with singularities on U with this new one, we can obtain another integral affine surface S' with singularities, since monodromies of both integral affine structures with respect to the loop along the boundary of U are the same.

Assume that the determinant of the monodromy matrix around any singular point of S is 1. Then we have $\iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}} \cong \mathbb{Z}$ for both S and S', where ι is the inclusion of the complement of singularities. The cohomology groups $H^1(S, \iota_* \mathcal{T}_{\mathbb{Z}})$ and $H^1(S', \iota_* \mathcal{T}_{\mathbb{Z}})$ have the cup product (1.3) induced by the wedge product. We also write the radiance obstructions of S and S' as c_S and $c_{S'}$ respectively.

Let $\mathcal{U}' = \{U_j\}_{j \in J'}$ be a sufficiently fine acyclic covering of S' for $\iota_* \mathcal{T}_{\mathbb{Z}}$ such that each open set have one singular point at most and each singular point is contained by only one open set. Let $U_{j_\alpha}, U_{j_\beta} \in \mathcal{U}'$ be the open sets containing p_1 and p_2 respectively. We set $U_{j_\gamma} := U_{j_\alpha} \cup U_{j_\beta}$, $J^\circ := J' \setminus \{j_\alpha, j_\beta\}$, and $J := J^\circ \cup \{j_\gamma\}$. We replace $\mathcal{U}'$ if necessary so that $U_{j_1} \cap U_{j_2}$ does not intersect with $U_{j_\alpha} \cap U_{j_\beta}$ for any $j_1, j_2 \in J^\circ$. The set of open sets $\mathcal{U} := \{U_j\}_{j \in J}$ is an acyclic covering of S for $\iota_* \mathcal{T}_{\mathbb{Z}}$.

We define a map $f\colon H^1(S,\iota_*\mathcal{T}_\mathbb{Z}) \to H^1(S',\iota_*\mathcal{T}_\mathbb{Z})$ by setting

$$f(\phi)\left((U_{j_1},U_{j_2})\right) := \phi\left(\left(U'_{j_1},U'_{j_2}\right)\right)\big|_{U_{j_1}\cap U_{j_2}} \tag{5.3}$$

for each $\phi \in Z^1(\mathcal{U},\iota_*\mathcal{T}_\mathbb{Z})$ and $j_1, j_2 \in J'$, where

$$U'_j := \begin{cases} U_j, & j \in J^\circ, \\ U_{j_\gamma}, & j \in \{j_\alpha, j_\beta\}. \end{cases} \tag{5.4}$$

Lemma 5.1. *The map $f\colon H^1(S,\iota_*\mathcal{T}_\mathbb{Z}) \to H^1(S',\iota_*\mathcal{T}_\mathbb{Z})$ is well-defined.*

Proof. Since we have

$$\begin{aligned} \delta(f(\phi))\left(((U_{j_1},U_{j_2},U_{j_3}))\right) &= f(\phi)\left((U_{j_2},U_{j_3})\right) - f(\phi)\left((U_{j_1},U_{j_3})\right) + f(\phi)\left((U_{j_1},U_{j_2})\right) && (5.5)\\ &= \phi\left(\left(U'_{j_2},U'_{j_3}\right)\right) - \phi\left(\left(U'_{j_1},U'_{j_3}\right)\right) + \phi\left(\left(U'_{j_1},U'_{j_2}\right)\right) && (5.6)\\ &= (\delta\phi)\left(\left(U'_{j_1},U'_{j_2},U'_{j_3}\right)\right) = 0, && (5.7) \end{aligned}$$

$f(\phi)$ is a cocyle. For any element $\theta \in C^0(\mathcal{U},\iota_*\mathcal{T}_\mathbb{Z})$, we take the element $\theta' \in C^0(\mathcal{U}',\iota_*\mathcal{T}_\mathbb{Z})$ defined by $\theta'(U_j) := \theta(U'_j)\big|_{U_j}$. Then we have

$$f(\delta\theta)\left((U_{j_1},U_{j_2})\right) = \delta\theta\left(\left(U'_{j_1},U'_{j_2}\right)\right)\big|_{U_{j_1}\cap U_{j_2}} = \theta(U'_{j_2})\big|_{U_{j_1}\cap U_{j_2}} - \theta(U'_{j_1})\big|_{U_{j_1}\cap U_{j_2}}, \tag{5.8}$$

$$\delta\theta'\left((U_{j_1},U_{j_2})\right) = \theta'(U_{j_2})|_{U_{j_1}\cap U_{j_2}} - \theta'(U_{j_1})|_{U_{j_1}\cap U_{j_2}} = \theta(U'_{j_2})\big|_{U_{j_1}\cap U_{j_2}} - \theta(U'_{j_1})\big|_{U_{j_1}\cap U_{j_2}}. \tag{5.9}$$

Hence, we obtain $f(\delta\theta) = \delta\theta'$. □

Proposition 5.2. *The map $f\colon H^1(S,\iota_*\mathcal{T}_\mathbb{Z}) \to H^1(S',\iota_*\mathcal{T}_\mathbb{Z})$ is a primitive embedding that preserves the pairing.*

Proof. First, we check that the map f is injective. Suppose there exists $\theta' \in C^0(\mathcal{U}',\iota_*\mathcal{T}_\mathbb{Z})$ such that $\delta(\theta') = f(\phi)$. We will construct an element $\theta \in C^0(\mathcal{U},\iota_*\mathcal{T}_\mathbb{Z})$ such that $\delta(\theta) = \phi$. Here since we have

$$\theta'(U_{j_\beta}) - \theta'(U_{j_\alpha}) = (\delta\theta')\left(\left(U_{j_\alpha},U_{j_\beta}\right)\right) = f(\phi)\left(\left(U_{j_\alpha},U_{j_\beta}\right)\right) = \phi\left(\left(U_{j_\gamma},U_{j_\gamma}\right)\right) = 0, \tag{5.10}$$

there is a section $s \in \Gamma(U_{j_\gamma},\iota_*\mathcal{T}_\mathbb{Z})$ such that $s|_{U_{j_\alpha}} = \theta'(U_{j_\alpha})$ and $s|_{U_{j_\beta}} = \theta'(U_{j_\beta})$. We define $\theta \in C^0(\mathcal{U},\iota_*\mathcal{T}_\mathbb{Z})$ by setting

$$\theta((U_j)) := \begin{cases} \theta'((U_j)), & j \in J^\circ, \\ s, & j = j_\gamma. \end{cases} \tag{5.11}$$

Then when $j_1, j_2 \in J^\circ$, we have

$$\delta\theta\left((U_{j_1},U_{j_2})\right) = \theta'(U_{j_2}) - \theta'(U_{j_1}) = \delta\theta'\left((U_{j_1},U_{j_2})\right) = f(\phi)\left((U_{j_1},U_{j_2})\right) = \phi\left((U_{j_1},U_{j_2})\right). \tag{5.12}$$

When $j_1 = j_\gamma, j_2 \in J^\circ$, we have

$$\begin{aligned} \delta\theta\left(\left(U_{j_\gamma},U_{j_2}\right)\right)\big|_{U_{j_\alpha}\cap U_{j_2}} &= \theta'(U_{j_2})|_{U_{j_\alpha}\cap U_{j_2}} - s\,|_{U_{j_\alpha}\cap U_{j_2}} && (5.13)\\ &= \theta'(U_{j_2})|_{U_{j_\alpha}\cap U_{j_2}} - \theta'(U_{j_\alpha})|_{U_{j_\alpha}\cap U_{j_2}} && (5.14)\\ &= \delta\theta'\left((U_{j_\alpha},U_{j_2})\right) = f(\phi)\left((U_{j_\alpha},U_{j_2})\right) = \phi\left(\left(U_{j_\gamma},U_{j_2}\right)\right)\big|_{U_{j_\alpha}\cap U_{j_2}}. && (5.15) \end{aligned}$$

Since we can also get $\delta\theta\left((U_{j_\gamma}, U_{j_2})\right)\big|_{U_{j_\beta}\cap U_{j_2}} = \phi\left((U_{j_\gamma}, U_{j_2})\right)\big|_{U_{j_\beta}\cap U_{j_2}}$ in the same way, we obtain

$$\delta\theta\left((U_{j_\gamma}, U_{j_2})\right) = \phi\left((U_{j_\gamma}, U_{j_2})\right). \tag{5.16}$$

Therefore, we obtain $\delta(\theta) = \phi$.

Next, we check that the map f preserves the pairing. We take a total orders of J°. By adding j_γ to J° as the minimum element, we obtain a total order of J. We also consider the total order of J' obtained by adding j_α, j_β to J° as the minimum and the second minimum elements respectively. For any $\phi_1, \phi_2 \in \check{H}^1(\mathcal{U}, \iota_*\mathcal{T}_\mathbb{Z})$, we can calculate as follows:

$$\begin{aligned}
\phi_1 \cup \phi_2 - f(\phi_1) \cup f(\phi_2) = & \sum_{\substack{j_1<j_2 \\ U_{j_\gamma}\cap U_{j_1}\cap U_{j_2}\neq\emptyset}} \phi_1((U_{j_\gamma}, U_{j_1})) \wedge \phi_2((U_{j_1}, U_{j_2})) && (5.17)\\
& - \sum_{\substack{j_1<j_2 \\ U_{j_\alpha}\cap U_{j_1}\cap U_{j_2}\neq\emptyset}} f(\phi_1)((U_{j_\alpha}, U_{j_1})) \wedge f(\phi_2)((U_{j_1}, U_{j_2})) && (5.18)\\
& - \sum_{\substack{j_1<j_2 \\ U_{j_\beta}\cap U_{j_1}\cap U_{j_2}\neq\emptyset}} f(\phi_1)((U_{j_\beta}, U_{j_1})) \wedge f(\phi_2)((U_{j_1}, U_{j_2})) && (5.19)\\
& - \sum_{\substack{j\in J^\circ \\ U_{j_\alpha}\cap U_{j_\beta}\cap U_{j}\neq\emptyset}} f(\phi_1)((U_{j_\alpha}, U_{j_\beta})) \wedge f(\phi_2)((U_{j_\beta}, U_{j})) && (5.20)\\
= & 0, && (5.21)
\end{aligned}$$

where $j_1, j_2 \in J^\circ$.

Lastly, we show that the map f is primitive. Consider the map

$$f \otimes \mathrm{id}_\mathbb{R} \colon H^1(S, \iota_*\mathcal{T}_\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{R} \to H^1(S', \iota_*\mathcal{T}_\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{R}, \quad \phi \otimes t \mapsto f(\phi) \otimes t, \tag{5.22}$$

which we will also write as f. Assume that there exists an element $\theta' \in C^0(\mathcal{U}', \iota_*\mathcal{T})$ such that $\delta(\theta') + f(\phi \otimes t) \in C^1(\mathcal{U}', \iota_*\mathcal{T}_\mathbb{Z})$. We will construct an element $\theta \in C^0(\mathcal{U}, \iota_*\mathcal{T})$ such that $\delta(\theta) + \phi \otimes t \in C^1(\mathcal{U}, \iota_*\mathcal{T}_\mathbb{Z})$. Since the monodromy invariant directions of p_1 and p_2 are the same, we can extend sections $\theta'(U_{j_\alpha}), \theta'(U_{j_\beta})$ to U_{j_γ}. Let $s_\alpha, s_\beta \in \Gamma(U_{j_\gamma}, \iota_*\mathcal{T}_\mathbb{Z})$ denote the extensions of $\theta'(U_{j_\alpha}), \theta'(U_{j_\beta})$ respectively. Since we have

$$\left(\delta(\theta') + f(\phi \otimes t)\right)\left((U_{j_\alpha}, U_{j_\beta})\right) = \theta'(U_{j_\beta})\big|_{U_{j_\alpha}\cap U_{j_\beta}} - \theta'(U_{j_\alpha})\big|_{U_{j_\alpha}\cap U_{j_\beta}} \in \iota_*\mathcal{T}_\mathbb{Z}(U_{j_\alpha} \cap U_{j_\beta}), \tag{5.23}$$

we can see $s_\alpha - s_\beta \in \iota_*\mathcal{T}_\mathbb{Z}(U_{j_\gamma})$. We define $\theta \in C^0(\mathcal{U}, \iota_*\mathcal{T}_\mathbb{Z})$ by setting

$$\theta((U_j)) := \begin{cases} \theta'((U_j)), & j \in J^\circ, \\ s_\alpha, & j = j_\gamma. \end{cases} \tag{5.24}$$

Then, in the case where $j_1, j_2 \in J^\circ$, we have

$$\begin{aligned}
\left(\delta\theta + \phi \otimes t\right)\left((U_{j_1}, U_{j_2})\right) &= \theta'(U_{j_2}) - \theta'(U_{j_1}) + (\phi \otimes t)\left((U_{j_1}, U_{j_2})\right) && (5.25)\\
&= \delta\theta'\left((U_{j_1}, U_{j_2})\right) + f(\phi \otimes t)\left((U_{j_1}, U_{j_2})\right) && (5.26)\\
&= \left(\delta\theta' + f(\phi \otimes t)\right)\left((U_{j_1}, U_{j_2})\right) \in \iota_*\mathcal{T}_\mathbb{Z}(U_{j_1} \cap U_{j_2}). && (5.27)
\end{aligned}$$

In the case where $j_1 = j_\gamma, j_2 \in J^\circ$, $U_{j_\alpha} \cap U_{j_2} \neq \emptyset$ or $U_{j_\beta} \cap U_{j_2} \neq \emptyset$. When $U_{j_\alpha} \cap U_{j_2} \neq \emptyset$, we have

$$(\delta\theta + \phi \otimes t) \left((U_{j_\gamma}, U_{j_2})\right)\big|_{U_{j_\alpha} \cap U_{j_2}} = \theta'(U_{j_2})|_{U_{j_\alpha} \cap U_{j_2}} - s_\alpha \,|_{U_{j_\alpha} \cap U_{j_2}} + (\phi \otimes t) \left((U_{j_\gamma}, U_{j_2})\right)\big|_{U_{j_\alpha} \cap U_{j_2}} \tag{5.28}$$

$$= \theta'(U_{j_2})|_{U_{j_\alpha} \cap U_{j_2}} - \theta'(U_{j_\alpha})|_{U_{j_\alpha} \cap U_{j_2}} + f(\phi \otimes t) \left((U_{j_\alpha}, U_{j_2})\right) \tag{5.29}$$

$$= (\delta\theta' + f(\phi \otimes t)) \left((U_{j_\alpha}, U_{j_2})\right) \in \iota_* \mathcal{T}_{\mathbb{Z}}(U_{j_\alpha} \cap U_{j_2}). \tag{5.30}$$

Hence, we obtain $(\delta\theta + \phi \otimes t) \left((U_{j_\gamma}, U_{j_2})\right) \in \iota_* \mathcal{T}_{\mathbb{Z}}(U_{j_\gamma} \cap U_{j_2})$. When $U_{j_\beta} \cap U_{j_2} \neq \emptyset$, we have

$$(\delta\theta + \phi \otimes t) \left((U_{j_\gamma}, U_{j_2})\right)\big|_{U_{j_\beta} \cap U_{j_2}} = \theta'(U_{j_2})|_{U_{j_\beta} \cap U_{j_2}} - s_\alpha \,|_{U_{j_\beta} \cap U_{j_2}} + (\phi \otimes t) \left((U_{j_\gamma}, U_{j_2})\right)\big|_{U_{j_\beta} \cap U_{j_2}} \tag{5.31}$$

$$\begin{aligned} &= \theta'(U_{j_2})|_{U_{j_\beta} \cap U_{j_2}} - s_\beta \,|_{U_{j_\beta} \cap U_{j_2}} \\ &\quad - (s_\alpha - s_\beta) \,|_{U_{j_\beta} \cap U_{j_2}} + f(\phi \otimes t) \left((U_{j_\beta}, U_{j_2})\right) \end{aligned} \tag{5.32}$$

$$\begin{aligned} &= \theta'(U_{j_2})|_{U_{j_\beta} \cap U_{j_2}} - \theta'(U_{j_\beta})\big|_{U_{j_\beta} \cap U_{j_2}} \\ &\quad - (s_\alpha - s_\beta) \,|_{U_{j_\beta} \cap U_{j_2}} + f(\phi \otimes t) \left((U_{j_\beta}, U_{j_2})\right) \end{aligned} \tag{5.33}$$

$$= (\delta\theta' + f(\phi \otimes t)) \left((U_{j_\beta}, U_{j_2})\right) - (s_\alpha - s_\beta) \,|_{U_{j_\beta} \cap U_{j_2}} \tag{5.34}$$

$$\in \iota_* \mathcal{T}_{\mathbb{Z}}(U_{j_\beta} \cap U_{j_2}). \tag{5.35}$$

Hence, we obtain $(\delta\theta + \phi \otimes t) \left((U_{j_\gamma}, U_{j_2})\right) \in \iota_* \mathcal{T}_{\mathbb{Z}}(U_{j_\gamma} \cap U_{j_2})$. Therefore, we have $\delta(\theta) + \phi \otimes t \in C^1(\mathcal{U}, \iota_* \mathcal{T}_{\mathbb{Z}})$. □

Proposition 5.3. *One has $f(c_S) = c_{S'}$.*

Proof. When we take a set of sections $\{s_j \in \Gamma(U_j, \iota_* \mathcal{T})\}_{j \in J}$, the radiance obstruction c_S of S is given by

$$c_S((U_{j_1}, U_{j_2})) = s_{j_2}|_{U_{j_1} \cap U_{j_2}} - s_{j_1}|_{U_{j_1} \cap U_{j_2}} \in \Gamma(U_{j_1} \cap U_{j_2}, \iota_* \mathcal{T}) \tag{5.36}$$

for any $j_1, j_2 \in J$. We set $s_{j_\alpha} := s_{j_\gamma} \,|_{U_{j_\alpha}}$ and $s_{j_\beta} := s_{j_\gamma} \,|_{U_{j_\beta}}$. We have

$$f(c_S)((U_{j_1}, U_{j_2})) = c_S((U'_{j_1}, U'_{j_2}))\big|_{U_{j_1} \cap U_{j_2}} = s_{j_2}|_{U_{j_1} \cap U_{j_2}} - s_{j_1}|_{U_{j_1} \cap U_{j_2}} \in \Gamma(U_{j_1} \cap U_{j_2}, \iota_* \mathcal{T}) \tag{5.37}$$

for any $j_1, j_2 \in J'$. This is just the radiance obstruction $c_{S'}$ of S' constructed from the set of sections $\{s_j \in \Gamma(U_j, \iota_* \mathcal{T})\}_{j \in J'}$. □

6 Proofs of Theorem 1.1 and Theorem 1.2

Let M be a free $\mathbb{Z}$-module of rank 3 and $N := \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice. We set $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \operatorname{Hom}(M, \mathbb{R})$. Let $\Delta \subset M_{\mathbb{R}}$ be a smooth reflexive polytope of dimension 3, and $\check{\Delta} \subset N_{\mathbb{R}}$ be the polar polytope of Δ. Let further Σ and $\check{\Sigma}$ be the normal fans to Δ and $\check{\Delta}$ respectively.

Let $A \subset N$ denote the subset consisting of all vertices of $\check{\Delta}$ and $0 \in N$. We consider a tropical Laurent polynomial

$$f(x) = \max_{n \in A} \{a(n) + n_1 x_1 + n_2 x_2 + n_3 x_3\}, \tag{6.1}$$

such that the function

$$A \to \mathbb{R}, \quad n \mapsto a(n) \tag{6.2}$$

induces a central subdivision of $\check{\Delta}$, i.e., every maximal dimensional simplex of the subdivision has the origin $0 \in N$ as its vertex. Let $V(f)$ be the tropical hypersurface defined by f in the tropical toric variety $X_{\check{\Sigma}}(\mathbb{T})$ associated with $\check{\Sigma}$. For the time being, we do not take a refinement of $\check{\Sigma}$.

Let further B be the 2-sphere with an integral affine structure with singularities obtained by contracting $V(f)$ in the way of Section 4.2, and $\mathcal{P}$ be its natural polyhedral structure. For each 1-dimensional cell of B, we choose its barycenter as a position of the singular point that should be on it. We write the complement of singularities of B as $\iota \colon B_0 \hookrightarrow B$. Let $\mathcal{T}_{\mathbb{Z}}$ be the local system on B_0 of integral tangent vectors. We set $\mathcal{T} := \mathcal{T}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

In the subsequent subsections, we give proofs of Theorem 1.1 and Theorem 1.2 in this setting. Note that the statements of Theorem 1.1 and Theorem 1.2 do not depend on the the choices of positions of singular points as mentioned in Remark 4.5. Furthermore, it is obvious from the way of construction of B that taking a refinement $\check{\Sigma}' \subset M_{\mathbb{R}}$ of $\check{\Sigma}$ such that the primitive generator of any 1-dimensional cone in $\check{\Sigma}'$ is contained in $\Delta \cap M$ can only disperse concentrations of focus-focus singularities on B, and do not change anything else. Therefore, if we prove the theorems in the above setting, Proposition 5.2 and Proposition 5.3 ensure that Theorem 1.1 and Theorem 1.2 hold also when we replace $\check{\Sigma}$ with $\check{\Sigma}'$.

6.1 Proof of Theorem 1.1

We consider the complex toric variety X_Σ associated with Σ. We write the group of toric divisors on X_Σ as

$$\mathrm{Div}_T(X_\Sigma) := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho, \tag{6.3}$$

where $\Sigma(1)$ is the set of 1-dimensional cones in Σ, and D_ρ is the toric divisor corresponding to $\rho \in \Sigma(1)$. We take the barycentric subdivision of $\mathcal{P}$ and let $\mathcal{U} := \{U_\tau\}_{\tau \in \mathcal{P}}$ be the covering of B, where U_τ is the open star of the barycenter of $\tau \in \mathcal{P}$. The covering $\mathcal{U}$ is acyclic for $\iota_* \mathcal{T}_{\mathbb{Z}}$ and $\iota_* \mathcal{T}$, and

$$\check{H}^1(\mathcal{U}, \iota_* \mathcal{T}_{\mathbb{Z}}) = H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}), \quad \check{H}^1(\mathcal{U}, \iota_* \mathcal{T}) = H^1(B, \iota_* \mathcal{T}). \tag{6.4}$$

There is a one-to-one correspondence between $\Sigma(1), A \setminus \{0\}$ and the set of facets of B given by

$$\rho \leftrightarrow n_\rho \leftrightarrow \sigma(\rho), \tag{6.5}$$

where $n_\rho \in N$ is the primitive generator of $\rho \in \Sigma(1)$ and $\sigma(\rho) \in \mathcal{P}$ is the maximal dimensional cell of B where $a(0)$ and $a(n_\rho) x^{n_\rho}$ attain the maximum of f.

Let $v \in \mathcal{P}$ be a vertex of B, and $\{\sigma(\rho_i)\}_{i=1}^3$ be the set of facets containing v. For a divisor $D = \sum_{\rho \in \Sigma(1)} k_\rho D_\rho \in \mathrm{Div}_T(X_\Sigma)$ and the vertex $v \in \mathcal{P}$, let $m(v, D)$ denote the element of M defined by

$$\langle m(v, D), n_{\rho_i} \rangle = -k_{\rho_i}, \quad 1 \leq i \leq 3. \tag{6.6}$$

Since the fan Σ is smooth, such an element $m(v, D)$ always uniquely exists.

Let $\mathcal{P}(0)$ denote the set of vertices in $\mathcal{P}$. Take an arbitrary map

$$\xi \colon \mathcal{P} \to \mathcal{P}(0), \tag{6.7}$$

such that $\xi(\tau)$ is a vertex of τ. We define a map

$$\psi_\xi \colon \mathrm{Div}_T(X_\Sigma) \to \check{H}^1(\mathcal{U}, \iota_* \mathcal{T}_\mathbb{Z}), \quad D \mapsto \psi_\xi(D), \tag{6.8}$$

by setting

$$\psi_\xi(D)\left((U_{\tau_0}, U_{\tau_1})\right) := m(\xi(\tau_1), D) - m(\xi(\tau_0), D) \tag{6.9}$$

for each 1-simplex (U_{τ_0}, U_{τ_1}) of $\mathcal{U}$. We will check that this map ψ_ξ gives the map of Theorem 1.1 in the following lemmas, from Lemma 6.1 to Lemma 6.5.

Lemma 6.1. *The map ψ_ξ is a well-defined group homomorphism.*

Proof. First, we check that $\psi_\xi(D)\left((U_{\tau_0}, U_{\tau_1})\right)$ is certainly a section of $\iota_* \mathcal{T}_\mathbb{Z}$ over $U_{\tau_0} \cap U_{\tau_1}$. Open sets U_{τ_0}, U_{τ_1} intersect if and only if $\tau_0 \prec \tau_1$ or $\tau_1 \prec \tau_0$. Assume $\tau_0 \prec \tau_1$.

Consider the case where τ_1 is a facet. Let $\rho \in \Sigma(1)$ and $n_\rho \in N$ be the elements corresponding to τ_1 under (6.5). Since points $m(\xi(\tau_0), D)$ and $m(\xi(\tau_1), D)$ are contained in the plane

$$\left\{ m \in M_\mathbb{R} \mid \langle m, n_\rho \rangle = -k_\rho \right\}, \tag{6.10}$$

the vector $m(\xi(\tau_1), D) - m(\xi(\tau_0), D)$ is contained in the plane

$$\left\{ m \in M_\mathbb{R} \mid \langle m, n_\rho \rangle = 0 \right\}. \tag{6.11}$$

On the other hand, the section $\iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_0} \cap U_{\tau_1})$ coincides with this subspace. Hence, we have $\psi_\xi(D)\left((U_{\tau_0}, U_{\tau_1})\right) \in \iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_0} \cap U_{\tau_1})$.

In the case where τ_1 is not a facet, τ_0 is a vertex and τ_1 is an edge. Let $\sigma(\rho_1)$ and $\sigma(\rho_2)$ be the facets of B containing τ_1 as their face. Since the points $m(\xi(\tau_0), D)$ and $m(\xi(\tau_1), D)$ are contained in the 1-dimensional space

$$\left\{ m \in M_\mathbb{R} \mid \langle m, n_{\rho_i} \rangle = -k_{\rho_i}, i = 1, 2 \right\}, \tag{6.12}$$

the vector $m(\xi(\tau_1), D) - m(\xi(\tau_0), D)$ is contained in the 1-dimensional subspace defined by

$$\left\{ m \in M_\mathbb{R} \mid \langle m, n_{\rho_i} \rangle = 0, i = 1, 2 \right\}. \tag{6.13}$$

On the other hand, the section $\iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_0} \cap U_{\tau_1})$ contains this subspace. Hence, we have $\psi_\xi(D)\left((U_{\tau_0}, U_{\tau_1})\right) \in \iota_* \mathcal{T}_\mathbb{Z}(U_{\tau_0} \cap U_{\tau_1})$.

Next, we show that $\psi_\xi(D)$ is a cocycle. For any 2-simplex $(U_{\tau_0}, U_{\tau_1}, U_{\tau_2})$ of $\mathcal{U}$, we have

$$\begin{aligned}\delta\left(\psi_\xi(D)\right)((U_{\tau_0}, U_{\tau_1}, U_{\tau_2})) &= \{m(\xi(\tau_2), D) - m(\xi(\tau_1), D)\} \\ &\quad - \{m(\xi(\tau_2), D) - m(\xi(\tau_0), D)\} \\ &\quad + \{m(\xi(\tau_1), D) - m(\xi(\tau_0), D)\} \\ &= 0. \end{aligned} \tag{6.14}$$

Lastly, we show that the map ψ_ξ is a group homomorphism. We will show $\psi_\xi(D + D') = \psi_\xi(D) + \psi_\xi(D')$ for any $D = \sum_\rho k_\rho D_\rho, D' = \sum_\rho k'_\rho D_\rho \in \mathrm{Div}_T(X_\Sigma)$. Let $v \in \mathcal{P}$ be a vertex, and $\{\sigma(\rho_i)\}_{i=1}^3$ be the set of facets containing v. Then the point $m(v, D + D')$ is defined by

$$\langle m(v, D + D'), n_{\rho_i}\rangle = -k_{\rho_i} - k'_{\rho_i}, \quad 1 \le i \le 3. \tag{6.15}$$

On the other hands, the points $m(v, D), m(v, D')$ are defined by

$$\langle m(v, D), n_{\rho_i}\rangle = -k_{\rho_i}, \quad \langle m(v, D'), n_{\rho_i}\rangle = -k'_{\rho_i}, \quad 1 \le i \le 3, \tag{6.16}$$

respectively. Hence, we have $m(v, D + D') = m(v, D) + m(v, D')$, and

$$\begin{aligned}\psi_\xi(D + D')((U_{\tau_0}, U_{\tau_1})) &= m(\xi(\tau_1), D + D') - m(\xi(\tau_0), D + D') \\ &= \{m(\xi(\tau_1), D) + m(\xi(\tau_1), D')\} - \{m(\xi(\tau_0), D) + m(\xi(\tau_0), D')\} \\ &= \{m(\xi(\tau_1), D) - m(\xi(\tau_0), D)\} + \{m(\xi(\tau_1), D') - m(\xi(\tau_0), D')\} \\ &= \psi_\xi(D)((U_{\tau_0}, U_{\tau_1})) + \psi_\xi(D')((U_{\tau_0}, U_{\tau_1})) \end{aligned} \tag{6.17}$$

for any 1-simplex (U_{τ_0}, U_{τ_1}) of $\mathcal{U}$. $\square$

Lemma 6.2. *The map ψ_ξ is independent of the choice of the map $\xi\colon \mathcal{P} \to \mathcal{P}(0)$.*

Proof. Let $\xi'\colon \mathcal{P} \to \mathcal{P}(0)$ be another map such that $\xi'(\tau) \prec \tau$ for any $\tau \in \mathcal{P}$. We show that $\psi_\xi(D) = \psi_{\xi'}(D)$ for any $D \in \mathrm{Div}_T(X_\Sigma)$. For each $D = \sum_\rho k_\rho D_\rho \in \mathrm{Div}_T(X_\Sigma)$, we define $\phi(D) \in C^0(\mathcal{U}, \iota_*\mathcal{T}_\mathbb{Z})$ by setting

$$\phi(D)((U_\tau)) := m(\xi'(\tau), D) - m(\xi(\tau), D) \tag{6.18}$$

for each 0-simplex (U_τ) of $\mathcal{U}$. We will show that the coboundary of $\phi(D)$ coincides with $\psi_{\xi'}(D) - \psi_\xi(D)$. First, we check that $\phi(D)$ is certainly an element of $C^0(\mathcal{U}, \iota_*\mathcal{T}_\mathbb{Z})$.

When τ is a vertex, $\xi(\tau) = \xi'(\tau) = \tau$, and we have $\phi(D)(U_\tau) = 0 \in \iota_*\mathcal{T}_\mathbb{Z}(U_{\tau_i})$.

When τ is an edge and is contained in facets $\sigma(\rho_1)$ and $\sigma(\rho_2)$, points $m(\xi(\tau), D), m(\xi'(\tau), D)$ are contained in the 1-dimensional space of (6.12). Hence, the vector $m(\xi'(\tau), D) - m(\xi(\tau), D)$ is contained in the 1-dimensional subspace of (6.13). On the other hand, the section $\iota_*\mathcal{T}_\mathbb{Z}(U_\tau)$ is the lattice of integral tangent vectors that are invariant under the monodromy transformation around the singular point on τ. That is the lattice contained in the subspace defined by (6.13). Hence, we have $\phi(D)\left((U_\tau)\right) \in \iota_*\mathcal{T}_\mathbb{Z}(U_\tau)$.

When τ is a facet, points $m(\xi(\tau), D), m(\xi'(\tau), D)$ are contained in the plane of (6.10), where ρ is the 1-dimensional cone corresponding to τ. Hence, the vector $m(\xi'(\tau), D) - m(\xi(\tau), D)$ is contained in the plane of (6.11). On the other hand, the section $\iota_*\mathcal{T}_\mathbb{Z}(U_\tau)$ is the lattice of integral tangent vectors on U_τ. That is the lattice contained in the subspace defined by (6.11). Hence, we have $\phi(D)\left((U_\tau)\right) \in \iota_*\mathcal{T}_\mathbb{Z}(U_\tau)$. Therefore, we have $\phi(D) \in C^0(\mathcal{U}, \iota_*\mathcal{T}_\mathbb{Z})$.

For any 1-simplex (U_{τ_0}, U_{τ_1}) of $\mathcal{U}$, one can get

$$\begin{aligned}(\psi_{\xi'}(D) - \psi_\xi(D))\left((U_{\tau_0}, U_{\tau_1})\right) &= \{m(\xi'(\tau_1), D) - m(\xi'(\tau_0), D)\} - \{m(\xi(\tau_1), D) - m(\xi(\tau_0), D)\} \\ &= \{m(\xi'(\tau_1), D) - m(\xi(\tau_1), D)\} - \{m(\xi'(\tau_0), D) - m(\xi(\tau_0), D)\} \\ &= (\delta\phi(D))((U_{\tau_0}, U_{\tau_1})). \qquad (6.19)\end{aligned}$$

Hence, we have $\psi_\xi(D) = \psi_{\xi'}(D)$. □

Recall that we have the exact sequence

$$M \to \mathrm{Div}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0, \qquad (6.20)$$

where the map $M \to \mathrm{Div}_T(X_\Sigma)$ is given by

$$m \mapsto D(m) := \sum_{\rho \in \Sigma(1)} \langle m, n_\rho \rangle \, D_\rho. \qquad (6.21)$$

Lemma 6.3. *The map ψ_ξ induces an injection*

$$\mathrm{Pic}(X_\Sigma) \hookrightarrow \check{H}^1(\mathcal{U}, \iota_* \mathcal{T}_\mathbb{Z}). \qquad (6.22)$$

Proof. First, we check that $\psi_\xi(D(m_0)) = 0$ for any $m_0 \in M$. Let $v \in \mathcal{P}$ be a vertex of B, and $\{\sigma(\rho_i)\}_{i=1}^3$ be the set of facets containing v. The element $m(v, D(m_0))$ satisfies

$$\langle m(v, D(m_0)), n_{\rho_i} \rangle = - \langle m_0, n_{\rho_i} \rangle \,, \quad 1 \le i \le 3. \qquad (6.23)$$

Therefore, we have $m(v, D(m_0)) = -m_0 \in M$ for any $v \in \mathcal{P}(0)$. From the definition of ψ_ξ, we can see that $\psi_\xi(D(m_0)) = 0$.

Next, we show that the induced map $\mathrm{Pic}(X_\Sigma) \to \check{H}^1(\mathcal{U}, \iota_* \mathcal{T}_\mathbb{Z})$ is injective. Assume that $\psi_\xi(D_0) = \delta(\phi)$ for some $D_0 \in \mathrm{Div}_T(X_\Sigma)$ and $\phi \in C^0(\mathcal{U}, \iota_* \mathcal{T}_\mathbb{Z})$. We show that there is some $m \in M$ such that $D_0 = D(m)$. Let $\tau \in \mathcal{P}$ be a 1-dimensional cell, and v_0, v_1 be its endpoints. Suppose $\xi(\tau) = v_1$. Then we have

$$\psi_\xi(D_0)((U_{v_0}, U_\tau)) = m(v_1, D_0)) - m(v_0, D_0) = \phi((U_\tau)) - \phi((U_{v_0})), \qquad (6.24)$$

$$\psi_\xi(D_0)((U_{v_1}, U_\tau)) = 0 = \phi((U_\tau)) - \phi((U_{v_1})). \qquad (6.25)$$

Here, $m(v_1, D_0)) - m(v_0, D_0)$ and $\phi((U_\tau))$ are parallel to the direction which is invariant under the monodromy around the singular point on τ. Hence, from (6.24), (6.25), it turns out that $\phi((U_{v_0}))$ and $\phi((U_{v_1}))$ also have to be parallel to this direction.

Let $\tau' \in \mathcal{P}$ be another 1-dimensional cell that has v_0 as its vertex. By the same argument, we can see that $\phi((U_{v_0}))$ has to be parallel also to the direction which is invariant under the monodromy around the singular point on τ'. Since these two monodromy invariant directions are linearly independent, $\phi((U_{v_0}))$ has to be zero. Similarly, we get $\phi((U_{v_1})) = 0$. Hence, by (6.24), (6.25), we obtain

$$\phi((U_\tau)) = 0, \quad m(v_1, D_0) = m(v_0, D). \qquad (6.26)$$

Since there is a sequence of edges of B connecting arbitrary two vertices of B, we can conclude that the element $m(v, D_0) \in M$ is the same for any $v \in \mathcal{P}(0)$. We write it as $m(D_0) \in M$.

We set $D_0 =: \sum_{\rho\in\Sigma(1)} k_\rho D_\rho$ and $D(-m(D_0)) =: \sum_{\rho\in\Sigma(1)} k'_\rho D_\rho$. For any $\rho \in \Sigma(1)$, we take a vertex $v \in \mathcal{P}(0)$ contained in the facet $\sigma(\rho) \in \mathcal{P}$. Then we have

$$\langle m(v, D_0), n_\rho\rangle = -k_\rho, \quad k'_\rho = \langle -m(D_0), n_\rho\rangle . \tag{6.27}$$

Since $m(v, D_0) = m(D_0)$, we obtain $k_\rho = k'_\rho$. Hence, we have $D_0 = D(-m(D_0))$. Therefore, the induced map $\mathrm{Pic}(X_\Sigma) \to \check{H}^1(\mathcal{U}, \iota_*\mathcal{T}_\mathbb{Z})$ is injective. □

This map $\mathrm{Pic}(X_\Sigma) \to H^1(B, \iota_*\mathcal{T}_\mathbb{Z})$ will be denoted by ψ.

Lemma 6.4. *The embedding $\psi\colon \mathrm{Pic}(X_\Sigma) \to H^1(B, \iota_*\mathcal{T}_\mathbb{Z})$ is primitive, i.e., the image of the map*

$$\psi\colon \mathrm{Pic}(X_\Sigma) \to H^1(B, \iota_*\mathcal{T}_\mathbb{Z}) \to H^1(B, \iota_*\mathcal{T}) = H^1(B, \iota_*\mathcal{T}_\mathbb{Z}) \otimes \mathbb{R} \tag{6.28}$$

coincides with $\mathrm{Im}(\psi) \otimes_\mathbb{Z} \mathbb{R} \cap H^1(B, \iota_*\mathcal{T}_\mathbb{Z})$.

Proof. We show that there exists a toric divisor $D \in \mathrm{Div}_T(X_\Sigma)$ such that $\psi_\xi(D) = \lambda$ for any $\lambda \in \mathrm{Im}(\psi) \otimes_\mathbb{Z} \mathbb{R} \cap H^1(B, \iota_*\mathcal{T}_\mathbb{Z})$. Choose an arbitrary vertex $v_0 \in \mathcal{P}(0)$. Let $\{\sigma(\rho_i)\}_{i=1}^3$ be the facets containing v_0, and $\{m_i\}_{i=1}^3$ be the basis of M such that $\langle m_i, n_{\rho_j}\rangle = \delta_{i,j}$ for any $1 \le j \le 3$. There exist $D' \in \mathrm{Div}_T(X_\Sigma)$ and $t \in \mathbb{R}$ such that $\psi_\xi(D') \otimes t = \lambda$. For $D' \otimes t =: \sum_\rho k_\rho D_\rho$ ($k_\rho \in \mathbb{R}$), we set

$$D := D' \otimes t - \sum_{i=1}^{3} k_{\rho_i} D(m_i) \in \mathrm{Div}_T(X_\Sigma) \otimes_\mathbb{Z} \mathbb{R}. \tag{6.29}$$

Then the coefficient of D_{ρ_i} in D is zero for $1 \le i \le 3$. Therefore, we have

$$\langle m(v_0, D), n_{\rho_i}\rangle = 0, \quad 1 \le i \le 3. \tag{6.30}$$

Hence, we have $m(v_0, D) = 0$. Moreover, since $D = D' \otimes t$ as elements of $\mathrm{Pic}(X_\Sigma) \otimes \mathbb{R}$, we have $\psi_\xi(D) = \psi_\xi(D') \otimes t = \lambda$. We will show $D \in \mathrm{Div}_T(X_\Sigma)$.

Since $\psi_\xi(D) = \lambda \in H^1(B, \iota_*\mathcal{T}_\mathbb{Z})$, there exist an element $\phi \in C^0(\mathcal{U}, \iota_*\mathcal{T})$ such that

$$\delta(\phi) + \psi_\xi(D) \in C^1(\mathcal{U}, \iota_*\mathcal{T}_\mathbb{Z}). \tag{6.31}$$

Let $v \in \mathcal{P}(0)$ be an arbitrary vertex, and $\tau_1, \tau_2 \in \mathcal{P}$ be two distinct edges containing v. Then we have

$$\begin{aligned}\{\delta(\phi) + \psi_\xi(D)\}\,((U_v, U_{\tau_i})) &= \phi(U_{\tau_i}) - \phi(U_v) + m(\xi(\tau_i), D) - m(v, D)\\ &\in \iota_*\mathcal{T}_\mathbb{Z}(U_v \cap U_{\tau_i})\end{aligned} \tag{6.32}$$

for $i = 1, 2$. Let $e_1, e_2 \in \iota_*\mathcal{T}_\mathbb{Z}(U_v)$ be sections that are parallel to the tangent direction of τ_1, τ_2 respectively and form a basis of $\iota_*\mathcal{T}_\mathbb{Z}(U_v) \cong \mathbb{Z}^2$. We set $\phi(U_v) =: a_1e_1 + a_2e_2$ ($a_i \in \mathbb{R}$). Since $\phi(U_{\tau_1})$ and the vector $m(\xi(\tau_1), D) - m(v, D)$ are parallel to e_1, it turns out from (6.32) that a_2 has to be an integer. Similarly we can get $a_1 \in \mathbb{Z}$. Hence, we have $\phi(U_v) \in \iota_*\mathcal{T}_\mathbb{Z}(U_v)$ for any $v \in \mathcal{P}(0)$.

Let $v_1, v_2 \in \mathcal{P}(0)$ be two arbitrary distinct vertices connected by an edge $\tau \in \mathcal{P}$. We have

$$\begin{aligned}\{\delta(\phi) + \psi_\xi(D)\}\,((U_{v_i}, U_\tau)) &= \phi(U_\tau) - \phi(U_{v_i}) + m(\xi(\tau), D) - m(v_i, D)\\ &\in \iota_*\mathcal{T}_\mathbb{Z}(U_{v_i} \cap U_\tau)\end{aligned} \tag{6.33}$$

for $i = 1, 2$. When $\xi(\tau) = v_1$, we have $\phi(U_\tau) \in \iota_* \mathcal{T}_{\mathbb{Z}}(U_\tau)$ from (6.33) of $i = 1$ and $\phi(U_{v_i}) \in \iota_* \mathcal{T}_{\mathbb{Z}}(U_{v_i})$. We also get $m(v_1, D) - m(v_2, D) \in M$ from $\phi(U_\tau) \in \iota_* \mathcal{T}_{\mathbb{Z}}(U_\tau)$ and (6.33) of $i = 2$. Similarly we get this also when $\xi(\tau) = v_2$. Hence, $m(v_1, D) - m(v_2, D) \in M$ for any vertices $v_1, v_2 \in \mathcal{P}(0)$.

Since there is a sequence of edges of B connecting any vertex of B and v_0 that we took in the beginning, and we have $m(v_0, D) = 0 \in M$, we can get $m(v, D) \in M$ for any $v \in \mathcal{P}(0)$. From the definition of $m(v, D)$, we can have $D \in \mathrm{Div}_T(X_\Sigma)$. □

Recall that the cohomology group $H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$ has the cup product induced by the wedge product

$$\cup \colon H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}) \otimes H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}) \to H^2(B, \iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}}). \tag{6.34}$$

Since any singular point of B has (4.11) as its monodromy matrix, we can see $\iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}} \cong \mathbb{Z}$, and hence

$$H^2(B, \iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}}) \cong H^2(B, \mathbb{Z}) \cong \mathbb{Z}. \tag{6.35}$$

Choosing $\iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}} \cong \mathbb{Z}$ amounts to choosing an orientation of B. Furthermore, we need to choose an orientation of B again in order to determine $H^2(B, \mathbb{Z}) \cong \mathbb{Z}$. Here, we choose the same orientation as we did for $\iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}} \cong \mathbb{Z}$. Then we obtain the pairing (1.3) of $H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$.

Let Y be an anti-canonical hypersurface of the complex toric variety X_Σ associated with Σ, and

$$\mathrm{Pic}(Y)_{\mathrm{amb}} := \mathrm{Im}\left(\mathrm{Pic}(X_\Sigma) \hookrightarrow \mathrm{Pic}(Y)\right) \tag{6.36}$$

be the sublattice of $\mathrm{Pic}(Y)$ coming from the Picard group of the ambient space.

Lemma 6.5. *The embedding $\psi \colon \mathrm{Pic}(Y)_{\mathrm{amb}} \hookrightarrow H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$ preserves the pairing.*

Proof. We show that $D_{\rho_1} \cdot D_{\rho_2} = \psi(D_{\rho_1}) \cup \psi(D_{\rho_2})$ for any $\rho_1, \rho_2 \in \Sigma(1)$. First, we show this in the case where $\rho_1 \neq \rho_2$. Recall that there is a one-to-one correspondence between $\Sigma(1)$ and facets of Δ given by $\rho \leftrightarrow F(\rho)$, where

$$F(\rho) := \{ m \in \Delta \mid \langle m, n_\rho \rangle = -1 \}. \tag{6.37}$$

A hypersurface in the toric variety X_Σ defined by a polynomial whose Newton polytope is Δ is an anti-canonical hypersuface. When $F(\rho_1) \cap F(\rho_2)$ is empty, $D_{\rho_1} \cdot D_{\rho_2} = 0$. When $F(\rho_1) \cap F(\rho_2)$ is not empty, $D_{\rho_1} \cdot D_{\rho_2}$ is equal to the integral length of the edge $F(\rho_1) \cap F(\rho_2)$. Let $l \in \mathbb{Z}_{\geq 0}$ denote the integral length of $F(\rho_1) \cap F(\rho_2)$. We choose two maps $\xi_i \colon \mathcal{P} \to \mathcal{P}(0)$ $(i = 1, 2)$ satisfying

- the condition of the map (6.7) : $\xi_i(\tau)$ is a vertex of τ for any $\tau \in \mathcal{P}$,
- $\xi_i(\tau) \notin \sigma(\rho_i)$ for any $\tau \not\subset \sigma(\rho_i)$.

We will show that $\psi_{\xi_1}(D_{\rho_1}) \cup \psi_{\xi_2}(D_{\rho_2}) = 0$ when $F(\rho_1) \cap F(\rho_2)$ is empty, and $\psi_{\xi_1}(D_{\rho_1}) \cup \psi_{\xi_2}(D_{\rho_2}) = l$ when $F(\rho_1) \cap F(\rho_2)$ is not empty.

When $\tau, \tau' \in \mathcal{P}$ are not in $\sigma(\rho_i)$, we have $\xi_i(\tau), \xi_i(\tau') \notin \sigma(\rho_i)$. From the definition of $m(v, D)$ (6.6), we can see $m(\xi_i(\tau), D_{\rho_i}) = m(\xi_i(\tau'), D_{\rho_i}) = 0$, and therefore

$$\psi_{\xi_i}(D_{\rho_i})((U_\tau, U_{\tau'})) = m(\xi_i(\tau'), D_{\rho_i}) - m(\xi_i(\tau), D_{\rho_i}) = 0. \tag{6.38}$$

Hence, for a 2-simplex $(U_{\tau_0}, U_{\tau_1}, U_{\tau_2})$ of $\mathcal{U}$,

$$\begin{aligned}\psi_{\xi_1}(D_{\rho_1}) \cup \psi_{\xi_2}(D_{\rho_2})((U_{\tau_0}, U_{\tau_1}, U_{\tau_2})) = &\{m(\xi_1(\tau_1), D_{\rho_1}) - m(\xi_1(\tau_0), D_{\rho_1})\} \\ &\wedge \{m(\xi_2(\tau_2), D_{\rho_2}) - m(\xi_2(\tau_1), D_{\rho_2})\}\end{aligned} \tag{6.39}$$

can be non-zero only when either τ_0 or τ_1 is contained in $\sigma(\rho_1)$ and either τ_1 or τ_2 is contained in $\sigma(\rho_2)$. One of τ_0, τ_1, τ_2 is a vertex of B. Assume τ_k is that vertex. Since $\tau_k \prec \tau_0, \tau_1, \tau_2$, we have $\tau_k \in \sigma(\rho_1) \cap \sigma(\rho_2)$ when (6.39) is not zero. When $\sigma(\rho_1) \cap \sigma(\rho_2)$ is empty, this never happens. Hence, we get $\psi_{\xi_1}(D_{\rho_1}) \cup \psi_{\xi_2}(D_{\rho_2}) = 0$ when $\sigma(\rho_1) \cap \sigma(\rho_2) = \emptyset$. In the following, we assume $\sigma(\rho_1) \cap \sigma(\rho_2) \neq \emptyset$, and directly compute the sum of (6.39).

All 2-simplices such that $\tau_k \in \sigma(\rho_1) \cap \sigma(\rho_2)$ are shown in Figure 6.1. (6.39) can be non-zero on these simplices. Let $v_0, \cdots, v_{10}$ denote the vertices of these 2-simplexes as shown in

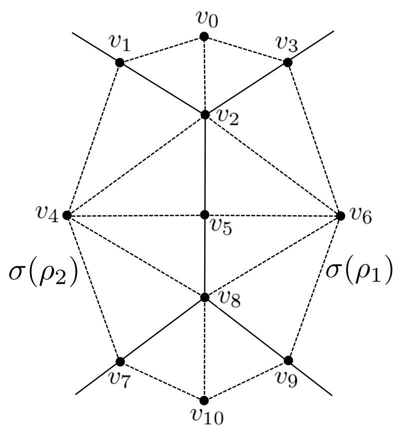

Figure 6.1: 2-simplexes where $\psi_{\xi_1}(D_{\rho_1}) \cup \psi_{\xi_2}(D_{\rho_2})$ can be non-zero

Figure 6.1. v_2, v_8 are vertices of B, and v_1, v_3, v_5, v_7, v_8 are barycenters of edges of B. v_4, v_6 denote the barycenters of $\sigma(\rho_2)$ and $\sigma(\rho_1)$ respectively. Let further τ_i $(0 \leq i \leq 10)$ denote the cell of B whose barycenter is v_i. For instance, $\tau_6 = \sigma(\rho_1)$, $\tau_4 = \sigma(\rho_2)$, and $\tau_5 = \sigma(\rho_1) \cap \sigma(\rho_2)$. We use the ascending order of index numbers i $(0 \leq i \leq 10)$ as a total order of $\{U_{\tau_i}\}_i$. We compute the sum of (6.39) by using this total order.

Let e_i $(1 \leq i \leq 3)$ be the integral tangent vectors at v_8 contained in $\mathbb{R}_{>0} \cdot \overrightarrow{v_8 v_9}$, $\mathbb{R}_{>0} \cdot \overrightarrow{v_8 v_7}$, $\mathbb{R}_{>0} \cdot \overrightarrow{v_8 v_5}$ respectively. Let further e_i $(i = 4, 5)$ be the integral tangent vectors at v_2 contained in $\mathbb{R}_{>0} \cdot \overrightarrow{v_2 v_3}$, $\mathbb{R}_{>0} \cdot \overrightarrow{v_2 v_1}$ respectively. By transporting e_1 and e_3 to v_2 along paths contained in $\sigma(\rho_1)$ and $\sigma(\rho_2)$ respectively, we set

$$e_4 = e_1 + s e_3, \quad e_5 = e_2 + t e_3, \tag{6.40}$$

where $s, t \in \mathbb{Z}$.

Consider the monodromy with respect to a loop that starts at v_8, passes through v_4, v_2, v_6 in this order, and comes back to v_8. From Lemma 4.1, we can see that it is given by

$$\begin{pmatrix} 1 & -l \\ 0 & 1 \end{pmatrix}, \tag{6.41}$$

under the basis (e_3, e_1), where l is the length of $\sigma(\rho_1) \cap \sigma(\rho_2)$. On the other hand, we can also calculate the monodromy of e_1 as follows: We have $e_1 = -e_2 - e_3$ in U_{v_8}. Since we have

$e_2 = e_5 - te_3$, e_1 becomes $(t-1)e_3 - e_5$ when it arrives at v_2. We also have $e_5 = e_3 - e_4$ in U_{v_2}, and $e_4 = e_1 + se_3$. Therefore, e_1 becomes $(s+t-2)e_3 + e_1$ when it is back to v_2. Hence we obtain

$$s + t - 2 = -l. \tag{6.42}$$

We choose maps $\xi_1, \xi_2 \colon \mathcal{P} \to \mathcal{P}(0)$ so that

$$\xi_1(\tau_i) = \begin{cases} \text{the endpoint of } \tau_1 \text{ that is not } v_2 & i = 0, 1, 4, \\ v_2 & i = 2, 3, 5, 6, \\ \text{the endpoint of } \tau_7 \text{ that is not } v_8 & i = 7, 10, \\ v_8 & i = 8, 9, \end{cases} \tag{6.43}$$

$$\xi_2(\tau_i) = \begin{cases} \text{the endpoint of } \tau_3 \text{ that is not } v_2 & i = 0, 3, 6, \\ v_2 & i = 1, 2, 4, 5, \\ v_8 & i = 7, 8, \\ \text{the endpoint of } \tau_9 \text{ that is not } v_8 & i = 9, 10. \end{cases} \tag{6.44}$$

Then we have

$$\psi_{\xi_1}(D_{\rho_1})((U_{\tau_{i_1}}, U_{\tau_{i_2}})) = \begin{cases} -e_5 & (i_1, i_2) = (0,2), (0,3), (1,2), (4,5), \\ e_5 & (i_1, i_2) = (2,4), \\ -e_2 & (i_1, i_2) = (4,8), (7,8), \\ e_5 - e_2 & (i_1, i_2) = (5,8), (6,8), (6,9), \\ e_2 & (i_1, i_2) = (8,10), (9,10), \\ 0 & \text{otherwise}, \end{cases} \tag{6.45}$$

$$\psi_{\xi_2}(D_{\rho_2})((U_{\tau_{i_1}}, U_{\tau_{i_2}})) = \begin{cases} -e_4 & (i_1, i_2) = (0,1), (0,2), \\ e_4 & (i_1, i_2) = (2,3), (2,6), (5,6), \\ e_4 - e_1 & (i_1, i_2) = (4,7), (4,8), (5,8), \\ -e_1 & (i_1, i_2) = (6,8), \\ e_1 & (i_1, i_2) = (7,10), (8,9), (8,10), \\ 0 & \text{otherwise}, \end{cases} \tag{6.46}$$

for $0 \le i_1 < i_2 \le 10$. We obtain

$$\psi_{\xi_1}(D_{\rho_1}) \cup \psi_{\xi_2}(D_{\rho_2})((U_{\tau_{i_1}}, U_{\tau_{i_2}}, U_{\tau_{i_3}})) = \begin{cases} -e_5 \wedge e_4 & (i_1, i_2, i_3) = (0,2,3), \\ -e_5 \wedge (e_4 - e_1) & (i_1, i_2, i_3) = (4,5,8), \\ (e_5 - e_2) \wedge e_1 & (i_1, i_2, i_3) = (6,8,9), \\ -e_2 \wedge e_1 & (i_1, i_2, i_3) = (7,8,10), \\ 0 & \text{otherwise}, \end{cases} \tag{6.47}$$

$$\sim \begin{cases} (2 - s - t)e_1 \wedge e_2 & (i_1, i_2, i_3) = (7,8,10), \\ 0 & \text{otherwise}, \end{cases} \tag{6.48}$$

for $0 \le i_1 < i_2 < i_3 \le 10$. This is equal to $2 - s - t$ in $H^2(B, \iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}}) \cong \mathbb{Z}$. Hence, from (6.42), we get $\psi(D_{\rho_1}) \cup \psi(D_{\rho_2}) = l$.

Lastly, we show that $D_{\rho_0} \cdot D_{\rho_0} = \psi(D_{\rho_0}) \cup \psi(D_{\rho_0})$ for any $\rho_0 \in \Sigma(0)$. Since there exists a primitive element $m \in M$ such that $D(m) = D_{\rho_0} + \sum_{\rho \neq \rho_0} a_\rho D_\rho$ ($a_\rho \in \mathbb{Z}$), we have

$$D_{\rho_0} \sim - \sum_{\rho \neq \rho_0} a_\rho D_\rho. \tag{6.49}$$

Hence, we obtain

$$D_{\rho_0} \cdot D_{\rho_0} = D_{\rho_0} \cdot \left(- \sum_{\rho \neq \rho_0} a_\rho D_\rho \right) = - \sum_{\rho \neq \rho_0} a_\rho D_{\rho_0} \cdot D_\rho \tag{6.50}$$

$$= - \sum_{\rho \neq \rho_0} a_\rho \psi(D_{\rho_0}) \cup \psi(D_\rho) = \psi(D_{\rho_0}) \cup \psi \left(- \sum_{\rho \neq \rho_0} a_\rho D_\rho \right) \tag{6.51}$$

$$= \psi(D_{\rho_0}) \cup \psi(D_{\rho_0}). \tag{6.52}$$

□

6.2 Proof of Theorem 1.2

We take a map of (6.7) $\xi \colon \mathcal{P} \to \mathcal{P}(0)$ such that $\xi(\tau) \prec \tau$. By enlarging each open set U_τ slightly, we assume that U_τ contains $\xi(\tau)$. We take each chart $\psi_\tau \colon U_\tau \to M_\mathbb{R}$ so that $\psi_\tau(\xi(\tau)) = 0 \in M_\mathbb{R}$. In order to specify the radiance obstruction of B, we choose the zero section $0 \in \Gamma(U_\tau \cap B_0, T^{\mathrm{aff}} B_0)$ for each U_τ. Then the radiance obstruction c_B is represented by the element of $C^1(\mathcal{U}, \iota_* \mathcal{T})$ given by

$$c_B \left((U_{\tau_0}, U_{\tau_1}) \right) := \xi(\tau_1) - \xi(\tau_0) \tag{6.53}$$

for each 1-simplex (U_{τ_0}, U_{τ_1}) of $\mathcal{U}$.

Let $v \in \mathcal{P}$ be a vertex, and $\{\sigma(\rho_i)\}_{i=1}^3$ be the set of facets of B containing v. Then the vertex $v \in M_\mathbb{R}$ is determined by

$$\langle m, n_{\rho_i} \rangle = -a(n_{\rho_i}) + a(0), \quad 1 \leq i \leq 3. \tag{6.54}$$

From the definition of the map ψ_ξ, we can see that

$$c_B = \psi_\xi \left(\sum_{\rho \in \Sigma(1)} \{a(n_\rho) - a(0)\} D_\rho \right) \tag{6.55}$$

$$= \sum_{\rho \in \Sigma(1)} \{a(n_\rho) - a(0)\} \psi(D_\rho). \tag{6.56}$$

7 Asymptotic behaviors of period maps

Let $K := \overline{\mathbb{C}\{t\}}$ be the convergent Puiseux series field, equipped with the standard non-archimedean valuation

$$\mathrm{val} \colon K \longrightarrow \mathbb{Q} \cup \{-\infty\}, \quad k = \sum_{j \in \mathbb{Q}} c_j t^j \mapsto - \min \{ j \in \mathbb{Q} \mid c_j \neq 0 \}. \tag{7.1}$$

Let M be a free $\mathbb{Z}$-module of rank 3 and $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. We set $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R} = \mathrm{Hom}(M, \mathbb{R})$. Let $\Delta \subset M_\mathbb{R}$ be a smooth reflexive polytope of dimension 3, and $\check{\Delta} \subset N_\mathbb{R}$ be the polar polytope of Δ. Let further $\Sigma \subset N_\mathbb{R}, \check{\Sigma} \subset M_\mathbb{R}$ be the normal fans of $\Delta, \check{\Delta}$ respectively. We choose a refinement $\check{\Sigma}' \subset M_\mathbb{R}$ of $\check{\Sigma}$ which gives rise to a projective crepant resolution $X_{\check{\Sigma}'} \to X_{\check{\Sigma}}$ of a toric variety associated with $\check{\Sigma}$.

We consider a Laurent polynomial $F = \sum_{n \in A} k_n x^n \in K[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}]$ over K whose Newton polytope is $\check{\Delta}$, where $A \subset N$ denotes the subset consisting of all vertices of $\check{\Delta}$ and $0 \in N$. We assume that the function

$$A \to \mathbb{R}, \quad n \mapsto \mathrm{val}(k_n) \tag{7.2}$$

induces a central subdivision of $\check{\Delta}$. Let $\mathrm{trop}(F)$ be the tropicalization of F, which is defined as the tropical polynomial

$$\mathrm{trop}(F)(x) := \max_{n \in A} \left\{ \mathrm{val}(k_n) + n_1 x_1 + n_2 x_2 + n_3 x_3 \right\}. \tag{7.3}$$

Let further B be the 2-sphere with an integral affine structure with singularities obtained by contracting the tropical hypersurface defined by $\mathrm{trop}(F)$ in the tropical toric variety $X_{\check{\Sigma}'}(\mathbb{T})$ associated with $\check{\Sigma}'$ in the way of Section 4.2. We write the radiance obstruction of B as $c_B \in H^1(B, \iota_* \mathcal{T})$.

Let $D \subset \mathbb{C}$ be the open unit disk. We consider the universal covering of $D \setminus \{0\}$

$$e \colon \mathbb{H} \to D \setminus \{0\}, \quad z \mapsto \exp(2\pi\sqrt{-1}z), \tag{7.4}$$

where $\mathbb{H}$ is the upper half plane. We set

$$H_R := \{ z \in \mathbb{H} \mid \mathfrak{Im}\, z > R \}, \tag{7.5}$$

where R is a positive real number such that $e(\sqrt{1}R)$ is smaller than the radius of convergence of k_n for any $n \in A$. For each element $z \in H_R$, we consider the polynomial $f_z \in \mathbb{C}[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}]$ obtained by substituting $e(z)$ to t in F. Let V_z be the complex hypersurface defined by f_z in the complex toric variety $X_{\check{\Sigma}'}$. This is a quasi-smooth K3 hypersurface. We describe the asymptotic behavior of the period of V_z in the limit $R \to \infty$ by using the radiance obstruction c_B. In the following, we assume $k_0 = 1$ by multiplying an element of K to F. Some parts of the following are borrowed from [Ued14, Section 7].

For a given element $\alpha = (a_n)_{n \in A \setminus \{0\}} \in (\mathbb{C}^\times)^{A \setminus \{0\}}$, we associate the polynomial

$$W_\alpha(x) = 1 + \sum_{n \in A \setminus \{0\}} a_n x^n. \tag{7.6}$$

We write the toric hypersurface defined by W_α in the complex toric variety $X_{\check{\Sigma}'}$ as Y_α. Let $(\mathbb{C}^\times)^{A \setminus \{0\}}_{\mathrm{reg}}$ be the set of $\alpha \in (\mathbb{C}^\times)^{A \setminus \{0\}}$ such that Y_α is $\check{\Sigma}'$-regular, i.e., the intersection of Y_α with any torus orbit of $X_{\check{\Sigma}'}$ is a smooth subvariety of codimension one. We consider the family of $\check{\Sigma}'$-regular hypersurfaces given by the second projection

$$\varphi \colon \mathfrak{Y} := \left\{ (x, \alpha) \in X_{\check{\Sigma}'} \times (\mathbb{C}^\times)^{A \setminus \{0\}}_{\mathrm{reg}} \;\middle|\; W_\alpha(x) = 0 \right\} \to (\mathbb{C}^\times)^{A \setminus \{0\}}_{\mathrm{reg}}, \tag{7.7}$$

and the action of $M \otimes_{\mathbb{Z}} \mathbb{C}^\times$ to this family given by

$$t \cdot (x, \alpha) := \left(t^{-1} x, (t^n a_n)_{n \in A \setminus \{0\}} \right), \tag{7.8}$$

where $t \in M \otimes_{\mathbb{Z}} \mathbb{C}^\times$. We write the quotient by this action as $\tilde{\varphi} \colon \tilde{\mathfrak{Y}} \to \mathcal{M}_{\mathrm{reg}}$, where $\mathcal{M}_{\mathrm{reg}} := (\mathbb{C}^\times)^{A \setminus \{0\}}_{\mathrm{reg}} / (M \otimes_{\mathbb{Z}} \mathbb{C}^\times)$. The space $\mathcal{M}_{\mathrm{reg}}$ can be regarded as a parameter space of $\check{\Sigma}'$-regular hypersurfaces whose Newton polytopes are $\check{\Delta}$. Let $(\mathscr{H}_B, \nabla^B, H_{B,\mathbb{Q}}, \mathscr{F}_B^\bullet, Q_B)$ be the residual B-model VHS of the family $\tilde{\varphi}\colon \tilde{\mathfrak{Y}} \to \mathcal{M}_{\mathrm{reg}}$ [Iri11, Definition 6.5].

When R is sufficiently large, the hypersurface V_z is $\check{\Sigma}'$-regular for any $z \in H_R$. We have a map l given by

$$l\colon H_R \to \mathcal{M}_{\mathrm{reg}}, \quad z \mapsto [(k_n(e(z)))_{n \in A \setminus \{0\}}], \tag{7.9}$$

where $k_n(e(z))$ is the complex number obtained by substituting $e(z)$ to t in k_n. We define a holomorphic form on V_z by

$$\Omega_z := \frac{\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}}{df_z}. \tag{7.10}$$

This defines a section of $l^* \mathscr{H}_B$ over H_R.

Let Y be an anti-canonical hypersurface of the complex toric variety X_Σ associated with Σ, and $\iota\colon Y \hookrightarrow X_\Sigma$ be the inclusion. Choose an integral basis $\{p_i\}_{i=1}^r$ of $\operatorname{Pic} X_\Sigma$ such that each p_i is nef. This determines a coordinate $q = (q_1, \cdots, q_r)$ on $\mathcal{M} := \operatorname{Pic} X_\Sigma \otimes_{\mathbb{Z}} \mathbb{C}^\times = \left(\mathbb{Z}^{A \setminus \{0\}}/M\right) \otimes_{\mathbb{Z}} \mathbb{C}^\times \supset \mathcal{M}_{\mathrm{reg}}$ and a coordinate $\tau = (\tau_1, \cdots, \tau_r)$ on $H^2_{\mathrm{amb}}(Y, \mathbb{C})$. Let $u_i \in H^2(X_\Sigma, \mathbb{Z})$ be the Poincaré dual of the toric divisor D_{ρ_i} corresponding to the one-dimensional cone $\rho_i \in \Sigma$, and $v = u_1 + \cdots + u_m$ be the anticanonical class. Givental's I-function is defined as the series

$$I_{X_\Sigma, Y}(q, z) = e^{p \log q / z} \sum_{d \in \mathrm{Eff}(X_\Sigma)} q^d \frac{\prod_{k=-\infty}^{\langle d, v\rangle} (v + kz) \prod_{j=1}^m \prod_{k=-\infty}^0 (u_j + kz)}{\prod_{k=-\infty}^0 (v + kz) \prod_{j=1}^m \prod_{k=-\infty}^{\langle d, u_j \rangle} (u_j + kz)},$$

which gives a multi-valued map from an open subset of $\mathcal{M} \times \mathbb{C}^\times$ to the cohomology ring $H^\bullet(X_\Sigma, \mathbb{C})$. Here, $\mathrm{Eff}(X_\Sigma)$ denotes the set of effective toric divisors on X_Σ. We write

$$I_{X_\Sigma, Y}(q, z) = F(q) + \frac{G(q)}{z} + \frac{H(q)}{z^2} + O(z^{-3}). \tag{7.11}$$

The mirror map $\varsigma\colon \mathcal{M} \to H^2_{\mathrm{amb}}(Y, \mathbb{C})$ is a multi-valued map defined by

$$\iota^* \left(\frac{G(q)}{F(q)} \right). \tag{7.12}$$

The residual B-model VHS is isomorphic to the ambient A-model VHS $(\mathscr{H}_A, \nabla^A, \mathscr{F}_A^\bullet, Q_A)$ [Iri11, Definition 6.2] via the mirror map ς [Iri11, Theorem 6.9]. Hence, the section of $l^* \mathscr{H}_B$ over H_R defined by Ω_z can also be regarded as a section of $(\varsigma \circ l)^* \mathscr{H}_A$ via the mirror isomorphism. Here we replace the real number R with a larger one if necessary.

We also choose elements $p_0 \in H^0_{\mathrm{amb}}(Y; \mathbb{Q})$ and $p_{r+1} \in H^4_{\mathrm{amb}}(Y; \mathbb{Q})$ so that we have $p_0 \cup p_{r+1} = 1$, and define sections

$$\tilde{p}_i := \exp(-\tau) \cup p_i, \quad 0 \leq i \leq r + 1, \tag{7.13}$$

of $\mathscr{H}_A$. These are flat sections with respect to the Dubrovin connection ∇^A. Note that the quantum cup product coincides with the ordinary cup product, since Y is a K3 surface. The sections $\{\tilde{p}_i\}_{i=0}^{r+1}$ form an integral structure of $H_{A,\mathbb{C}} := \operatorname{Ker} \nabla^A$. This is related to the integral structure $H^{\mathrm{amb}}_{A,\mathbb{Z}}$ defined in [Iri11, Definition 6.3] by a linear transformation.

We consider a map $\phi\colon l^* H_{A,\mathbb{C}}(H_R) \to (U \oplus \mathrm{Pic}(Y)_{\mathrm{amb}}) \otimes_{\mathbb{Z}} \mathbb{C}$ defined by

$$\tilde{p}_0 \mapsto 2\pi\sqrt{-1}e, \quad \tilde{p}_{r+1} \mapsto 2\pi\sqrt{-1}f, \quad \tilde{p}_i \mapsto -2\pi p_i \ (1 \le i \le r), \tag{7.14}$$

where U denotes the hyperbolic plane and (e, f) is its standard basis. This is an isomorphism preserving the pairing. We obtain the period map

$$H_R \to \mathbb{P}((U \oplus \mathrm{Pic}(Y)_{\mathrm{amb}}) \otimes_{\mathbb{Z}} \mathbb{C}) \tag{7.15}$$

determined by Ω_z via the mirror isomorphism and the map ϕ. The image of this map is contained in

$$\mathcal{D}' := \{[\sigma] \in \mathbb{P}((U \oplus \mathrm{Pic}(Y)_{\mathrm{amb}}) \otimes_{\mathbb{Z}} \mathbb{C}) \mid (\sigma, \sigma) = 0, (\sigma, \overline{\sigma}) > 0\}\,. \tag{7.16}$$

We also set

$$\mathcal{D} := \{\sigma \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes_{\mathbb{Z}} \mathbb{C} \mid (\mathfrak{Re}\,\tau, \mathfrak{Re}\,\tau) > 0\}\,. \tag{7.17}$$

Here we have an isomorphism $\mathcal{D}' \cong \mathcal{D}$ of complex manifolds given by

$$k_1 e + k_2 f + \sigma \mapsto \frac{-\sqrt{-1}}{k_1}\sigma, \tag{7.18}$$

where $k_1, k_2 \in \mathbb{C}$ and $\sigma \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes_{\mathbb{Z}} \mathbb{C}$. By this isomorphism, we obtain the period map $\mathscr{P}\colon H_R \to \mathcal{D}$.

Corollary 7.1. *The leading term of the period map $\mathscr{P}$ in the limit $R \to \infty$ is given by*

$$-2\pi\sqrt{-1}z \cdot \psi^{-1}(c_B), \tag{7.19}$$

where ψ denotes the map $\psi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}}\colon \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes_{\mathbb{Z}} \mathbb{R} \hookrightarrow H^1(B, \iota_* \mathcal{T})$.

Proof. From Theorem 1.2, we can see that the radiance obstruction c_B of B is given by

$$c_B = \sum_{n \in A \setminus \{0\}} \mathrm{val}(k_n)\psi(D_n), \tag{7.20}$$

where D_n is the restriction to Y of the toric divisor on X_Σ that corresponds to the 1-dimensional cone whose primitive generator is n.

The holomorphic form Ω_z corresponds to $F(q) \cdot 1 \in H^0_{\mathrm{amb}}(Y, \mathbb{C})$ under the mirror isomorphism [Iri11, Theorem 6.9], where $F(q)$ is the first term of the Givental's I-function (7.11). It turns out that the period map $\mathscr{P}\colon H_R \to \mathcal{D}$ is given by $z \mapsto \varsigma(l(z))$. The leading term of this is given by

$$\sum_{i=1}^{r} p_i \log q_i\left(l(z)\right). \tag{7.21}$$

Suppose $D_n = \sum_{i=1}^{r} b_{n,i} p_i$ in $\mathrm{Pic}(X_\Sigma)$, where $b_{n,i} \in \mathbb{Z}$. Then we have

$$q_i(l(z)) = \prod_{n \in A \setminus \{0\}} k_n(e(z))^{b_{n,i}}. \tag{7.22}$$

Hence, we obtain

$$\sum_{i=1}^{r} p_i \log q_i\left(l(z)\right) = \sum_{i=1}^{r} p_i \left(\sum_{n \in A \setminus \{0\}} b_{n,i} \log\left(k_n(e(z))\right) \right) \tag{7.23}$$

$$\sim - \sum_{i=1}^{r} p_i \left(\sum_{n \in A \setminus \{0\}} b_{n,i} \log e(z) \cdot \mathrm{val}(k_n) \right) \tag{7.24}$$

$$= - \log e(z) \sum_{n \in A \setminus \{0\}} \mathrm{val}(k_n) \sum_{i=1}^{r} b_{n,i} p_i \tag{7.25}$$

$$= -2\pi\sqrt{-1}z \sum_{n \in A \setminus \{0\}} \mathrm{val}(k_n) D_n \tag{7.26}$$

$$= -2\pi\sqrt{-1}z \cdot \psi^{-1}(c_B). \tag{7.27}$$

□

Corollary 7.1 implies that the radiance obstruction $\psi^{-1}(c_B) \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes_{\mathbb{Z}} \mathbb{R}$ can be regarded as the period of the tropical K3 hypersurface defined by $\mathrm{trop}(F)$. We can also obtain

$$\left(\psi^{-1}(c_B), \psi^{-1}(c_B)\right) \geq 0 \tag{7.28}$$

from Corollary 7.1 and the inequality $(\mathfrak{Re}\,\tau, \mathfrak{Re}\,\tau) > 0$ of (7.17). The following inequality (7.29) can be regarded as a tropical version of the Hodge–Riemann bilinear relation for K3 surfaces appearing in (7.16).

Corollary 7.2. *One has*

$$\left(\psi^{-1}(c_B), \psi^{-1}(c_B)\right) > 0. \tag{7.29}$$

Proof. From the assumption that the function $A \to \mathbb{R},\ n \mapsto \mathrm{val}(k_n)$ induces a central subdivision of $\check{\Delta}$, we can see that for any $n_0 \in A \setminus \{0\}$, there exists $m_0 \in M_{\mathbb{R}}$ such that

$$\langle m_0, n_0 \rangle = \mathrm{val}(k_{n_0}), \quad \langle m_0, n \rangle > \mathrm{val}(k_n) \quad \left(\forall n \in A \setminus \{0, n_0\}\right). \tag{7.30}$$

By subtracting the divisor $D(m_0)$ of (6.21), we get

$$\psi^{-1}(c_B) = \sum_{n \in A \setminus \{0\}} \mathrm{val}(k_n) D_n \sim \sum_{n \in A \setminus \{0, n_0\}} d_{n_0,n} D_n, \tag{7.31}$$

where $d_{n_0,n}$ is some negative real number. Since $D_{n_0} \cdot D_n \geq 0$ for any $n \in A \setminus \{0, n_0\}$ and there exists $n \in A \setminus \{0, n_0\}$ such that $D_{n_0} \cdot D_n > 0$, we can get

$$\left(D_{n_0}, \psi^{-1}(c_B)\right) = \left(D_{n_0}, \sum_{n \in A \setminus \{0, n_0\}} d_{n_0,n} D_n \right) < 0 \tag{7.32}$$

for any $n_0 \in A \setminus \{0\}$. Hence, we obtain

$$\left(\psi^{-1}(c_B), \psi^{-1}(c_B)\right) = \left(\sum_{n \in A \setminus \{0, n_0\}} d_{n_0, n} D_n, \psi^{-1}(c_B)\right) \tag{7.33}$$

$$= \sum_{n \in A \setminus \{0, n_0\}} d_{n_0, n} \left(D_n, \psi^{-1}(c_B)\right) \tag{7.34}$$

$$> 0. \tag{7.35}$$

□

Remark 7.3. As we saw in Remark 4.2 and Remark 4.5, there are ambiguities in the choices of positions of singular points when we contract tropical toric hypersurfaces, and the radiance obstruction does not depend on these choices. This means that moving singular points to monodromy invariant directions does not change the period of the tropical K3 surface B. We can infer that we should think that a tropical K3 surface which is obtained by moving singular points to monodromy invariant directions is "equivalent" to the original one.

Remark 7.4. The space

$$\{\sigma \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes_{\mathbb{Z}} \mathbb{R} \mid (\sigma, \sigma) > 0\} \tag{7.36}$$

is the period domain of tropical K3 hypersurfaces. This is the numerator of the moduli space of lattice polarized tropical K3 surfaces [HU18, Section 5]. In [OO18a], [OO18b], they construct Gromov–Hausdorff compactifications of polarized complex K3 surfaces by adding moduli spaces of lattice polarized tropical K3 surfaces to their boundaries.

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