博士論文（要約）

論文題目　On depth in the local Langlands correspondence

(局所Langlands対応における深度の概念について)

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SUMMARY OF "ON DEPTH IN THE LOCAL LANGLANDS CORRESPONDENCE"

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ABSTRACT. In this paper, we investigate the local Langlands correspondence for classical groups over p-adic fields from the viewpoint of the depth, which is an invariant of representations. In the first and second parts of this paper, we establish a general result on a depth preserving property of the local Langlands correspondence for classical groups under the assumption that the residual characteristic p is large enough. In the third part, we study the structures of L-packets consisting of simple supercuspidal representations, which are the representations with the minimal positive depth, under the assumption that p is odd.

CONTENTS

1. BACKGROUND

Let $\mathbf{G}$ be a connected reductive group over a p-adic field F. One of the central themes in representation theory of $G = \mathbf{G}(F)$ is the conjectural *local Langlands correspondence* for $\mathbf{G}$. To be more precise, let $\Pi(\mathbf{G})$ denote the set of equivalence classes of irreducible smooth representations of G, and $\Phi(\mathbf{G})$ the set of conjugacy classes of L-parameters of $\mathbf{G}$, which are admissible homomorphisms from $W_F \times \mathrm{SL}_2(\mathbb{C})$ to ${}^L\mathbf{G}$. Here ${}^L\mathbf{G}$ is the L-group of $\mathbf{G}$, which is the semi-direct product $\widehat{\mathbf{G}} \rtimes W_F$ of the Langlands dual group $\widehat{\mathbf{G}}$ of $\mathbf{G}$ and the Weil group W_F of F. Then the local Langlands correspondence for $\mathbf{G}$ predicts that there exists a "natural" map from the set $\Pi(\mathbf{G})$ to the set $\Phi(\mathbf{G})$ with finite fibers (called L-packets). In other words, the local Langlands correspondence gives a natural partition of the set $\Pi(\mathbf{G})$ into finite sets parametrized by L-parameters:

$$\Pi(\mathbf{G}) = \bigsqcup_{\phi \in \Phi(\mathbf{G})} \Pi_{\phi}^{\mathbf{G}}.$$

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The local Langlands correspondence for fully general connected reductive groups has not been known at present. However, thanks to recent developments based on works of a lot of people, for several groups, the correspondence was established. In particular, for

- general linear groups ([HT01]),
- quasi-split symplectic or special orthogonal groups ([Art13]),
- quasi-split unitary groups ([Mok15]), and
- non-quasi-split unitary groups ([KMSW14]),

the local Langlands correspondence has been established.

The aim of this paper is to investigate the "naturality" of the local Langlands correspondence for these groups beyond their characterizations. For example, in the case of general linear groups, the correspondence is characterized by the theory of ε-factors and L-factors of representations. In the cases of other classical groups listed above, the correspondence is characterized by the theory of endoscopy. However it is known that the local Langlands correspondence for these groups satisfies a lot of properties other than such characterizations.

One example for such phenomena is a *depth preserving property* of the local Langlands correspondence for general linear groups. To be more precise, let us recall the notion of the *depth* of representations. When $\mathbf{G} = \mathrm{GL}_1$, the group $G = F^\times$ has a maximal open compact subgroup $\mathcal{O}_F^\times$ (the unit group) and its filtration $\{1 + \mathfrak{p}_F^n\}_{n \in \mathbb{Z}_{>0}}$ (the higher unit groups). As its generalization, for a tamely ramified connected reductive group $\mathbf{G}$ over F, we can define various open compact subgroups (called *parahoric subgroups*) of G and their filtrations (called the *Moy–Prasad filtrations*). By using these subgroups of G, for each irreducible representation π of G, we can define its depth "$\mathrm{depth}(\pi)$", which expresses how large subgroups having an invariant part in the representation are. On the other hand, for the inertia subgroup I_F of the Weil group W_F of F, we can define its ramification filtration $\{I_F^\bullet\}$. Then, by noting that an L-parameter of $\mathbf{G}$ is an admissible homomorphism from $W_F \times \mathrm{SL}_2(\mathbb{C})$ to the L-group of $\mathbf{G}$, we can define the depth "$\mathrm{depth}(\phi)$" of an L-parameter ϕ, which measures how deep the ramification of the L-parameter ϕ is. Then it is known that the local Langlands correspondence for GL_N preserves the depth. Namely, we have the following:

Theorem 1.1 ([Yu09, 2.3.6] and [ABPS16b, Proposition 4.5]). *Let π be an irreducible smooth representation of $\mathrm{GL}_N(F)$ and ϕ the L-parameter of GL_N corresponding to π under the local Langlands correspondence for GL_N. Then we have*

$$\mathrm{depth}(\pi) = \mathrm{depth}(\phi).$$

Note that, when $N = 1$, this is nothing but the well-known property of the local class field theory about the correspondence between the higher unit groups $\{1 + \mathfrak{p}_F^n\}_{n \in \mathbb{Z}_{>0}}$ and the ramification filtration $\{I_F^{\mathrm{ab},n}\}_{n \in \mathbb{Z}_{>0}}$.

Therefore it is a natural attempt to investigate the relation between the depth of representations and that of L-parameters under the local Langlands correspondence for other classical groups. At present, there is no complete description of the behavior of the depth under the local Langlands correspondence for general classical groups except for some small groups (see, for example, [ABPS16a] for the details). However, in a recent paper [GV17], Ganapathy and Varma gave the following partial answer to this problem:

Theorem 1.2 ([GV17, Corollary 10.6.4])**.** *Let* $\mathbf{H}$ *be a quasi-split symplectic or special orthogonal group over* F*. We assume that the residual characteristic is large enough. Let* ϕ *be a tempered* L*-parameter of* $\mathbf{H}$*, and* $\Pi_{\phi}^{\mathbf{H}}$ *the* L*-packet of* $\mathbf{H}$ *for* ϕ*. Then we have*

$$\max\bigl\{\mathrm{depth}(\pi) \bigm| \pi \in \Pi_{\phi}^{\mathbf{H}}\bigr\} \leq \mathrm{depth}(\phi).$$

In the first part of this paper (Part 1), we improve this inequality to the equality, under the assumption that the residual characteristic p is large enough compared to the size of the classical group. Furthermore, in the case of quasi-split unitary groups, we prove that the representations in each L-packet have the same depth. The key of our proof is an analysis of the *endoscopic character relation*, which is the characterization of the local Langlands correspondence for classical groups, via harmonic analysis on p-adic reductive groups.

In the second part (Part 2), we extend the above result for quasi-split unitary groups to non-quasi-split unitary groups. The key point in this generalization is to utilize the *local theta correspondence*, which can transfer representations of non-quasi-split unitary groups to those of quasi-split unitary groups. By combining several known results on the local theta correspondence, we can reduce the problem for the non-quasi-split case to that for the quasi-split case, which is already proved in Part 1.

In the final part (Part 3), we tackle the depth preserving problem of the local Langlands correspondence from a slightly different viewpoint, that is, an explicit description of the local Langlands correspondence. In Parts 1 and 2, in order to use general results of harmonic analysis on p-adic reductive groups, we have to assume that the residual characteristic p is large enough. However, by focusing on some special class of representations, there is a case that we can avoid such a restriction on the residual characteristic. More specifically, in Part 3, we investigate the local Langlands correspondence for *simple supercuspidal representations*, which are the representations with the minimal positive depth. In this part, by computing the endoscopic character relation precisely, we study the structures of L-packets consisting of simple supercuspidal representations without appealing to the results in Parts 1 and 2. By such a precise description of L-packets, in particular, we can conclude that the depth preserving property holds for simple supercuspidal representations only under the assumption that p is odd.

2. Outline of Part 1

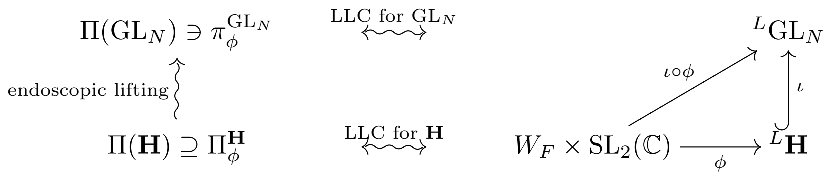
Our main theorem in Part 1 is the following, which is an improvement of the result of Ganapathy–Varma (Theorem 1.2):

Theorem 2.1. *Let* $\mathbf{H}$ *be a quasi-split classical* (*namely, symplectic, special orthogonal, or unitary*) *group over* F*. We assume that the residual characteristic is large enough. Let* ϕ *be an* L*-parameter of* $\mathbf{H}$*, and* $\Pi_{\phi}^{\mathbf{H}}$ *the* L*-packet of* $\mathbf{H}$ *for* ϕ*. Then we have*

$$\max\bigl\{\mathrm{depth}(\pi) \bigm| \pi \in \Pi_{\phi}^{\mathbf{H}}\bigr\} = \mathrm{depth}(\phi).$$

From now on, let $\mathbf{H}$ be a quasi-split classical group over F. Before we explain the sketch of our proof, we recall the *endoscopic character relation*, which is used to formulate the naturality of the local Langlands correspondence for $\mathbf{H}$. First, we can regard $\mathbf{H}$ as an *endoscopic group* of a (twisted) general linear group GL_N

over F (strictly speaking, when $\mathbf{H}$ is a unitary group, we have to consider the Weil restriction of GL_N with respect to a quadratic extension associated to $\mathbf{H}$). In particular, we have an embedding ι from the L-group of $\mathbf{H}$ to that of GL_N. Here the size N of the general linear group depends on each classical group. Now let us take an L-parameter ϕ of $\mathbf{H}$. By the theory of Langlands classification, we can extend the local Langlands correspondence for tempered representations to nontempered representations formally. Therefore, to consider the naturality of the local Langlands correspondence, we may assume that ϕ is tempered. Then, by noting that ϕ is a homomorphism from $W_F \times \mathrm{SL}_2(\mathbb{C})$ to ${}^L\mathbf{H}$, we obtain an L-parameter of GL_N by composing ϕ with the embedding ι. From these L-parameters, we get representations of two different groups. One is the representation $\pi_\phi^{\mathrm{GL}_N}$ of $\mathrm{GL}_N(F)$ corresponding to $\iota \circ \phi$ under the local Langlands correspondence for GL_N (note that, for GL_N, each L-packet is a singleton). The other is an L-packet $\Pi_\phi^{\mathbf{H}}$, which is a finite set of representations of H, corresponding to ϕ under the local Langlands correspondence for $\mathbf{H}$.



In this situation, we say that $\pi_\phi^{\mathrm{GL}_N}$ is the *endoscopic lift* of $\Pi_\phi^{\mathbf{H}}$ from $\mathbf{H}$ to GL_N. Then the endoscopic character relation is the following equality satisfied by the *twisted character* $\Theta_{\phi,\theta}^{\mathrm{GL}_N}$ of $\pi_\phi^{\mathrm{GL}_N}$ and the *characters* Θ_π of representations π belonging to $\Pi_\phi^{\mathbf{H}}$:

$$\Theta_{\phi,\theta}^{\mathrm{GL}_N}(f) = \sum_{\pi \in \Pi_\phi^{\mathbf{H}}} \Theta_\pi(f^H).$$

Here f is any test function of $\mathrm{GL}_N(F)$ and f^H is its Langlands–Shelstad–Kottwitz transfer to $H = \mathbf{H}(F)$. The important point here is that the composition with the L-embedding does not change the depth of L-parameters. Namely, by this formulation of the naturality of the local Langlands correspondence for $\mathbf{H}$ and the depth preserving property of the local Langlands correspondence for GL_N, the depth preserving problem of the local Langlands correspondence for $\mathbf{H}$ is equivalent to that of the endoscopic lifting from $\mathbf{H}$ to GL_N. We tackle the latter problem by investigating the endoscopic character relations via harmonic analysis on p-adic reductive groups.

To explain the strategy of our proof of Theorem 2.1, we recall Ganapathy–Varma's method used in the proof of Theorem 1.2 in [GV17]. The key tools in their proof are the following DeBacker's two results:

(1) Description of the radii of the character expansions of irreducible smooth representations ("homogeneity", established in [DeB02a]).
(2) Parametrization of nilpotent orbits via the Bruhat–Tits theory (established in [DeB02b]).

Let us recall them. First, for every irreducible smooth representation π of H, we have its *character* Θ_π, which is an invariant distribution on H. In general, it is very complicated and difficult to describe the behavior of the character Θ_π. However, in some "small neighborhood" of the origin, we can express the character Θ_π as a linear combination of the nilpotent orbital integrals of the Fourier transforms. More precisely, if we have an appropriate exponential map $\mathfrak{c}_{\mathbf{H}}$ from the Lie algebra $\mathfrak{h}$ to H, then, for every function f on the Lie algebra supported on a "small neighborhood" of the origin, we have

$$\Theta_\pi(f \circ \mathfrak{c}_{\mathbf{H}}^{-1}) = \sum_{\mathcal{O} \in \mathrm{Nil}(\mathfrak{h})} c_{\mathcal{O}} \cdot \widehat{\mu_{\mathcal{O}}}(f)$$

(this is called the *character expansion* of the character of a representation, and established by Harish-Chandra ([HC99])). Then the following question about this character expansion naturally arises: what is the optimal size of a "small neighborhood"? In [DeB02a], DeBacker gave an answer to this question by using the Bruhat–Tits theory. To be more precise, we put r to be the depth of an irreducible smooth representation π and H_{r+} to be the union of the $(r+)$-th Moy–Prasad filtrations of parahoric subgroups. Then DeBacker proved that the character expansion is valid on $\mathfrak{c}_{\mathbf{H}}^{-1}(H_{r+})$ under some assumptions on the residual characteristic p. On the other hand, in another paper [DeB02b], DeBacker established a parametrization of nilpotent orbits via the Bruhat–Tits theory under some assumptions on the residual characteristic. By using this parametrization, we can recover the depth of an irreducible smooth representation from the radius of its character expansion. Namely we can show that if Θ_π has a character expansion on $\mathfrak{c}_{\mathbf{H}}^{-1}(H_{s+})$ for some positive number $s \in \mathbb{R}$, then the depth of π is not greater than s. In other words, we can say that the depth of an irreducible smooth representation gives an optimal radius of the character expansion.

On the other hand, for twisted characters of irreducible smooth representations, the theory of the character expansion can be formulated as follows: for every function f on the Lie algebra of GL_N supported on a "small neighborhood" of the origin, we have

$$\Theta_{\phi,\theta}^{\mathrm{GL}_N}(f \circ \mathfrak{c}^{-1}) = \sum_{\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_\theta)} c_{\mathcal{O}} \cdot \widehat{\mu_{\mathcal{O}}}(f_\theta).$$

Here $\mathfrak{c}$ is a kind of exponential map, $\mathfrak{g}_\theta$ is the Lie algebra of the group $\mathbf{G}_\theta$ which is the identity component of the fixed part of an involution θ of $\mathbf{G}$, and f_θ is a function on $\mathfrak{g}_\theta$ which is a *semisimple descent* of f. For this expansion of twisted characters, in [AK07], Adler and Korman established a result which is analogous to that of DeBacker under some assumptions on the residual characteristic of the same type as DeBacker's one. Namely, they described the size of a "small neighborhood" where the character expansion is valid in terms of the depth of the representation.

Now we recall Ganapathy–Varma's method. Their idea is to compare the depth of a tempered L-packet $\Pi_\phi^{\mathbf{H}}$ and its endoscopic lift $\pi_\phi^{\mathrm{GL}_N}$ by comparing the radii of the character expansions for $\Pi_\phi^{\mathbf{H}}$ and $\pi_\phi^{\mathrm{GL}_N}$ via the endoscopic character relation. Under the assumption that the residual characteristic is large enough to satisfy the assumptions of DeBacker's results and Adler–Korman's result, Ganapathy and Varma proved Theorem 1.2 in the following way:

(1) The radius of the character expansion of the twisted character $\Theta_{\phi,\theta}^{\mathrm{GL}_N}$ of $\pi_\phi^{\mathrm{GL}_N}$ is given by $\mathrm{depth}(\pi_\phi^{\mathrm{GL}_N})+$ (Adler–Korman's result).
(2) By using the endoscopic character relation, we know that the maximum of the radii of the character expansions of the characters of representations belonging to $\Pi_\phi^{\mathbf{H}}$ is smaller than $\mathrm{depth}(\pi_\phi^{\mathrm{GL}_N})+$.

(3) By using DeBacker's parametrization of the nilpotent orbits, we can conclude that the maximum of the depth of representations belonging to $\Pi_\phi^{\mathbf{H}}$ is smaller than $\mathrm{depth}(\pi_\phi^{\mathrm{GL}_N})+$.

Then it is natural to consider the converse direction of this argument by swapping the roles of GL_N and $\mathbf{H}$, that is:

(1)′ The maximum of radii of the character expansions of the characters of representations π belonging to $\Pi_\phi^{\mathbf{H}}$ is given by the maximum of $\mathrm{depth}(\pi)+$ (DeBacker's result).

(2)′ By using the endoscopic character relation, we know that the radius of the character expansion of the twisted character $\Theta_{\phi,\theta}^{\mathrm{GL}_N}$ of $\pi_\phi^{\mathrm{GL}_N}$ is smaller than $\max\{\mathrm{depth}(\pi)+\}$.

(3)′ By using DeBacker's parametrization of the nilpotent orbits, we conclude that the depth of $\pi_\phi^{\mathrm{GL}_N}$ is smaller than $\max\{\mathrm{depth}(\pi)+\}$.

However, we cannot so immediately imitate Ganapathy–Varma's arguments. The problem is in the step (3)′. That is, the behavior of the characteristic functions of the Moy–Prasad filtrations of parahoric subgroups under the semisimple descent is not so clear.

In Part 1 of this paper, in order to carry out the step (3)′, we investigate the semisimple descents for the characteristic functions of the Moy–Prasad filtrations of parahoric subgroups of general linear groups by a group-theoretic computation. Then, as a consequence of such a computation, we can complete the above arguments of the converse direction and get the following converse inequality:

$$\max\bigl\{\mathrm{depth}(\pi)\bigm|\pi\in\Pi_\phi^{\mathbf{H}}\bigr\}\geq\mathrm{depth}(\phi).$$

In particular, by combining this with Theorem 1.2, we get the equality (Theorem 2.1).

When $\mathbf{H}$ is a unitary group $\mathrm{U}_{E/F}(N)$ associated to a quadratic extension E of F, the semisimple descent coincides with the Langlands–Shelstad–Kottwitz transfer. Thus, by the above computation of the semisimple descents for the characteristic functions of the Moy–Prasad filtrations, we get the following generalization of the *fundamental lemma* to a positive depth direction:

Theorem 2.2. *We assume that the residual characteristic p is not equal to 2. We take a point x of the Bruhat–Tits building of $\mathbf{H}$ and we identify it with a point of the Bruhat–Tits building of $\mathbf{G}=\mathrm{Res}_{E/F}\mathrm{GL}_N$ canonically. Let $r\in\mathbb{R}_{>0}$. Let $H_{x,r}$ and $G_{x,r}$ be the r-th Moy–Prasad filtrations with respect to the point x. Then $\mathrm{vol}(H_{x,r})^{-1}\mathbb{1}_{H_{x,r}}\in C_c^\infty(H)$ is a transfer of $\mathrm{vol}(G_{x,r})^{-1}\mathbb{1}_{G_{x,r}}\in C_c^\infty(G)$.*

We remark that a similar assertion for $r=0$ (namely, the fundamental lemma for parahoric subgroups) in the case where E is unramified over F is proved in [Kot86] (see also [Hai09]). This theorem is not only interesting itself, but also having an application to the depth preserving problem of the endoscopic lifting. We can immediately deduce the following theorem from Theorem 2.2 by using the endoscopic character relation:

Theorem 2.3. *We assume that the residual characteristic p is not equal to 2. Let ϕ be an L-parameter of $\mathbf{H}$, and $\Pi_\phi^{\mathbf{H}}$ the L-packet of $\mathbf{H}$ for ϕ. Then we have*

$$\min\bigl\{\mathrm{depth}(\pi)\bigm|\pi\in\Pi_\phi^{\mathbf{H}}\bigr\}\geq\mathrm{depth}(\phi).$$

In particular, by combining it with Theorem 1.2*, we have*

$$\mathrm{depth}(\pi) = \mathrm{depth}(\phi)$$

for every $\pi \in \Pi_{\phi}^{\mathbf{H}}$ *under the assumption that the residual characteristic* p *is greater than* $2N+1$.

We finally remark that we cannot expect that the inequality in Theorem 1.2 holds for a general connected reductive group. For example, in [RY14, Section 7.4] Reeder and Yu constructed a candidate of the L-parameters corresponding to "simple supercuspidal representations" of $\mathrm{SU}_p(\mathbb{Q}_p)$ for an odd prime p, by assuming Hiraga–Ichino–Ikeda's formal degree conjecture. In this example, the depth of simple supercuspidal representations and the depth of their L-parameters are given by $\frac{1}{2p}$ and $\frac{1}{2(p+1)}$, respectively. See also [ABPS16a, Section 3.3].

3. Outline of Part 2

Our aim in this part is to extend Theorem 2.3 to non-quasi-split unitary groups. Namely, our main result in Part 2 is the following:

Theorem 3.1. *Let* $\mathbf{G}$ *be a non-quasi-split unitary group in* N *variables over* F*. We assume that the residual characteristic* p *of* F *is greater than or equal to* $2N+3$*. Then, for every irreducible smooth representation* π *of* $\mathbf{G}(F)$ *and its corresponding* L*-parameter* ϕ *of* $\mathbf{G}$*, we have*

$$\mathrm{depth}(\pi) = \mathrm{depth}(\phi).$$

To show this, we utilize the *local theta correspondence*. Let V (resp. V') be an ϵ-hermitian (resp. ϵ'-hermitian) space over F such that $\epsilon\epsilon' = -1$, and $\mathrm{U}(V)$ (resp. $\mathrm{U}(V')$) the corresponding unitary group over F. Then the local theta correspondence gives a correspondence between representations of $\mathrm{U}(V)$ and those of $\mathrm{U}(V')$. For an irreducible smooth representation π of $\mathrm{U}(V)$, we denote the corresponding representation of $\mathrm{U}(V')$ by $\theta(\pi)$. Here note that $\theta(\pi)$ is possibly zero and that, if $\theta(\pi)$ is not zero, then it is irreducible and smooth. The key point in our proof is that the local theta correspondence has the following two properties:

Pan's depth preserving theorem: The local theta correspondence preserves the depth of representations ([Pan02]). Namely, if $\theta(\pi)$ is not zero, then we have

$$\mathrm{depth}\big(\theta(\pi)\big) = \mathrm{depth}(\pi).$$

Gan–Ichino's description of the L-parameter of the local theta lift: If $\theta(\pi)$ is not zero, then the L-parameter $\theta(\phi)$ of $\theta(\pi)$ can be described by using the L-parameter ϕ of π ([GI14, Appendix C]).

By combining these properties with the observation that *every unitary group in odd variables is quasi-split*, we can reduce the problem to the quasi-split case (Theorem 2.3). More precisely, for a given non-quasi-split unitary group $\mathbf{G}$, we assume that $\mathbf{G}$ is realized by an ϵ-hermitian space V over F (necessarily V has even dimension). We put $2n$ to be the dimension of this space V and let V' be a $(2n+1)$-dimensional $(-\epsilon)$-hermitian space over F. Then the corresponding unitary group $\mathrm{U}(V')$ is quasi-split. Thus, for an irreducible smooth representation π of $\mathbf{G} = \mathrm{U}(V)$, we can compare its depth with the depth of the L-parameter ϕ of π in the following way:

(1) By Pan's theorem, the depth of π is equal to the depth of its theta lift $\theta(\pi)$.

(2) Since $\mathrm{U}(V')$ is quasi-split, by Theorem 2.3, the depth of $\theta(\pi)$ is equal to the depth of its L-parameter $\theta(\phi)$.
(3) By Gan–Ichino's description, we can compare the depth of $\theta(\phi)$ with that of ϕ.

$$\begin{array}{ccc}
\pi\colon \text{representation of } \mathrm{U}(V) & \overset{\text{LLC for } \mathrm{U}(V)}{\leftrightsquigarrow} & \phi\colon L\text{-parameter of } \mathrm{U}(V) \\
\theta \wr\downarrow & & \theta \wr\downarrow \\
\theta(\pi)\colon \text{representation of } \mathrm{U}(V') & \overset{\text{LLC for } \mathrm{U}(V')}{\leftrightsquigarrow} & \theta(\phi)\colon L\text{-parameter of } \mathrm{U}(V')
\end{array}$$

However, in carrying out this strategy, we have to take care of the difference between the normalization of the local theta correspondence used in Pan's result and that in Gan–Ichino's result. Recall that the local theta correspondence is obtained by considering the Weil representation of the metaplectic group $\mathrm{Mp}(W)$ for the symplectic space $W := V \otimes V'$. More precisely, the metaplectic group $\mathrm{Mp}(W)$ is a covering group of the symplectic group $\mathrm{Sp}(W)$ and contains covering groups $\widetilde{\mathrm{U}(V)}$ of $\mathrm{U}(V)$ and $\widetilde{\mathrm{U}(V')}$ of $\mathrm{U}(V)$. Then, by restricting the Weil representation of $\mathrm{Mp}(W)$ to $\widetilde{\mathrm{U}(V)} \times \widetilde{\mathrm{U}(V')}$, we get a correspondence between representations of $\widetilde{\mathrm{U}(V)}$ and those of $\widetilde{\mathrm{U}(V')}$.

$$\begin{array}{ccc}
\widetilde{\mathrm{U}(V)} \times \widetilde{\mathrm{U}(V')} & \longrightarrow & \mathrm{Mp}(W) \\
\downarrow & & \downarrow \\
\mathrm{U}(V) \times \mathrm{U}(V') & \longrightarrow & \mathrm{Sp}(W)
\end{array}$$

To make this to be a correspondence between representations of $\mathrm{U}(V)$ and those of $\mathrm{U}(V')$, we have to choose splittings of the coverings $\widetilde{\mathrm{U}(V)} \to \mathrm{U}(V)$ and $\widetilde{\mathrm{U}(V')} \to \mathrm{U}(V')$. The important point here is that such splittings are *not* canonical. Namely, there are several ways to construct them. The depth preserving result of Pan is based on Pan's splittings constructed in [Pan01] by using the generalized lattice model of the Weil representation. On the other hand, the result of Gan–Ichino is based on Kudla's splitting constructed in [Kud94] by using the Schrödinger model of the Weil representation. Therefore, in order to combine these two results, we have to compute the difference between these two kinds of splittings.

4. Outline of Part 3

The aim of this part is to determine the structures of L-packets consisting of simple supercuspidal representations and their corresponding L-parameters for quasi-split classical groups over p-adic fields.

To be more precise, let $\mathbf{G}$ be a quasi-split classical group over F. Then, as explained in Section 2, we can regard $\mathbf{G}$ as an *endoscopic group* of a (twisted) general linear group GL_N over F. Here the size N of the general linear group depends on each classical group. For example, for

- the symplectic group Sp_{2n} of size $2n$,
- the quasi-split special orthogonal group SO_{2n}^{μ} which is of size $2n$ and corresponds to a ramified quadratic character μ of $F^{\times}$,
- the split special orthogonal group SO_{2n+2} of size $2n+2$, and

- the quasi-split special orthogonal group $\mathrm{SO}_{2n+2}^{\mathrm{ur}}$ which is of size $2n+2$ and corresponds to the unramified quadratic character μ_{ur} of $F^{\times}$

(these are groups which will be treated in this part), their dual groups and corresponding general linear groups are given by the following:

$\mathbf{G}$	Sp_{2n}	SO_{2n}^{μ}	$\mathrm{SO}_{2n+2}^{(\mathrm{ur})}$
$\widehat{\mathbf{G}}$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{SO}_{2n}(\mathbb{C})$	$\mathrm{SO}_{2n+2}(\mathbb{C})$
GL_N	GL_{2n+1}	GL_{2n}	GL_{2n+2}

Then, by the local Langlands correspondence, for each tempered L-parameter of $\mathbf{G}$, we have the L-packet $\Pi_{\phi}^{\mathbf{G}}$ and its endoscopic lift $\pi_{\phi}^{\mathrm{GL}_N}$ to GL_N satisfying the endoscopic character relation. Although we introduced the endoscopic character relation as an equality of distribution characters in Section 2, we can also write it in terms of functions as follows:

$$\Theta_{\phi,\theta}^{\mathrm{GL}_N}(g) = \sum_{h\leftrightarrow g/\sim} \frac{D_{\mathbf{G}}(h)^2}{D_{\mathrm{GL}_N,\theta}(g)^2} \Delta_{\mathbf{G},\mathrm{GL}_N}(h,g) \sum_{\pi\in\Pi_{\phi}^{\mathbf{G}}} \Theta_{\pi}(h).$$

Here we do not explain the precise meaning of each term in this equality. However, we emphasize that the relation between $\pi_{\phi}^{\mathrm{GL}_N}$ and $\Pi_{\phi}^{\mathbf{G}}$ is characterized by this identity since we have linear independence of the characters of representations.

Now we wish to describe the local Langlands correspondence for $\mathbf{G}$ explicitly. Then we can divide this problem of explicit description of the local Langlands correspondence for $\mathbf{G}$ into the following two problems:

(1) Describe the endoscopic lifting from $\mathbf{G}$ to GL_N explicitly by investigating the endoscopic character relation.
(2) Describe the local Langlands correspondence for GL_N explicitly.

In this part, we consider these problems for *simple supercuspidal representations*, which were introduced by Gross and Reeder in [GR10] (and also by Reeder and Yu in [RY14]), of quasi-split classical groups. From now on, we assume that the residual characteristic p is not equal to 2. Simple supercuspidal representations are supercuspidal representations obtained by the compact induction of *affine generic characters* of the pro-unipotent radical of the Iwahori subgroup, and characterized as the representations having the *minimal positive depth*. In general, the depth is a non-negative rational number, and the minimal positive depth for each group is given by the following:

$\mathbf{G}$	GL_N	Sp_{2n}	SO_{2n}^{μ}	$\mathrm{SO}_{2n+2}^{(\mathrm{ur})}$
minimal positive depth	$\frac{1}{N}$	$\frac{1}{2n}$	$\frac{1}{2n}$	$\frac{1}{2n}$

Since the construction of simple supercuspidal representations is very explicit, after making some choices (for example, fixing a uniformizer of F), we can easily parametrize the equivalence classes of them by a concrete set, which is denoted by $\mathrm{SSC}(\mathbf{G})$ in this part. Roughly speaking, an element of $\mathrm{SSC}(\mathbf{G})$ consists of data of

- a central character,
- an affine generic character on the pro-unipotent radical of the Iwahori subgroup of G, and
- a way to extend the affine generic character to its intertwining subgroup,

and described as follows:

group $\mathbf{G}$	parametrizing set $\mathrm{SSC}(\mathbf{G})$
GL_N	$(k^\times)^\vee \times k^\times \times \mathbb{C}^\times$
Sp_{2n}	$\mu_2 \times \{0,1\} \times k^\times$
SO^{μ}_{2n}	$\mu_2 \times k^\times$
$\mathrm{SO}^{(\mathrm{ur})}_{2n+2}$	$\mu_2 \times \{0,1\} \times k^\times \times \mu_2$

Here μ_2 is the set $\{\pm1\}$ of signs, and $(k^\times)^\vee$ is the set of characters of the multiplicative group $k^\times$ of the residue field k of F. For $X \in \mathrm{SSC}(\mathbf{G})$, we denote the corresponding simple supercuspidal representation of G by $\pi^{\mathbf{G}}_X$. We write $\omega^{\mathbf{G}}_X$ for the central character of $\pi^{\mathbf{G}}_X$.

Now we state our main theorem in Part 3.

Theorem 4.1 (Main theorem in Part 3)**.** *We assume that p is not equal to 2. Let $\pi^{\mathbf{G}}_X$ be a simple supercuspidal representation of G corresponding to an element X of $\mathrm{SSC}(\mathbf{G})$. Let $\phi \in \Phi(\mathbf{G})$ be the L-parameter of $\pi^{\mathbf{G}}_X$ (namely, the L-packet $\Pi^{\mathbf{G}}_\phi$ of ϕ contains $\pi^{\mathbf{G}}_X$).*

The case where $\mathbf{G} = \mathrm{Sp}_{2n}$: *The order of the L-packet $\Pi^{\mathbf{G}}_\phi$ is two and $\Pi^{\mathbf{G}}_\phi$ equals the orbit of $\pi^{\mathbf{G}}_X$ with respect to the action of the adjoint group of $\mathbf{G}$. Moreover the endoscopic lift $\pi^{\mathrm{GL}_{2n+1}}_\phi$ of $\Pi^{\mathbf{G}}_\phi$ to GL_{2n+1} is given by the parabolic induction of*

$$\pi^{\mathrm{GL}_{2n}}_Y \boxtimes \omega^{\mathrm{GL}_{2n}}_Y$$

for some $Y \in \mathrm{SSC}(\mathrm{GL}_{2n})$ which is explicitly described in terms of $X \in \mathrm{SSC}(\mathrm{Sp}_{2n})$.

The case where $\mathbf{G} = \mathrm{SO}^{\mu}_{2n}$: *The L-packet $\Pi^{\mathbf{G}}_\phi$ is a singleton. Moreover the endoscopic lift $\pi^{\mathrm{GL}_{2n}}_\phi$ of $\Pi^{\mathbf{G}}_\phi$ to GL_{2n} is a simple supercuspidal representation $\pi^{\mathrm{GL}_{2n}}_Y$ corresponding to an element $Y \in \mathrm{SSC}(\mathrm{GL}_{2n})$, which is explicitly described in terms of $X \in \mathrm{SSC}(\mathrm{SO}^{\mu}_{2n})$.*

The case where $\mathbf{G} = \mathrm{SO}^{(\mathrm{ur})}_{2n+2}$: *The order of the L-packet $\Pi^{\mathbf{G}}_\phi$ is two and $\Pi^{\mathbf{G}}_\phi$ equals the orbit of $\pi^{\mathbf{G}}_X$ with respect to the action of the adjoint group of $\mathbf{G}$. Moreover the endoscopic lift of $\Pi^{\mathbf{G}}_\phi$ to GL_{2n+2} is given by the parabolic induction of*

$$\begin{cases} \pi^{\mathrm{GL}_{2n}}_Y \boxtimes (\omega^{\mathrm{GL}_{2n}}_Y \otimes \mu_{\mathbf{G}}) \boxtimes \mathbb{1} & \text{if } \zeta = 1, \\ \pi^{\mathrm{GL}_{2n}}_Y \boxtimes (\omega^{\mathrm{GL}_{2n}}_Y \otimes \mu_{\mathrm{ur}} \otimes \mu_{\mathbf{G}}) \boxtimes \mu_{\mathrm{ur}} & \text{if } \zeta = -1. \end{cases}$$

Here $\mu_{\mathbf{G}}$ is the quadratic character of $F^\times$ corresponding to $\mathbf{G}$, ζ is the fourth parameter of the data $X \in \mathrm{SSC}(\mathrm{SO}^{(\mathrm{ur})}_{2n+2})$, and $Y \in \mathrm{SSC}(\mathrm{GL}_{2n})$ is an element described explicitly in terms of X.

We note that, in the case of even special orthogonal groups, the local Langlands correspondence has been established modulo the action of the outer automorphism. However, in this introduction, we ignore the difference between the set of irreducible smooth representations and the set of their orbits under the action of the outer automorphism for simplicity.

Theorem 4.1 gives a complete answer to the problem (1) for simple supercuspidal representations of $\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}^{\mu}_{2n}(F)$, and $\mathrm{SO}^{(\mathrm{ur})}_{2n+2}(F)$. Moreover, since the L-parameters of simple supercuspidal representations of general linear groups have been explicitly determined by the works of Bushnell–Henniart ([BH05]) and Imai–Tsushima ([IT15]), by combining Theorem 4.1 with them, we also get an answer to the problem (2) for the lifted representations.

Before we explain the outline of the proof of Theorem 4.1, we remark on several preceding works:

- In our previous works ([Oi16a, Oi16b]), we considered the same problem for split special orthogonal groups of odd degrees and unramified quasi-split unitary groups, and got results of the same type.
- In the case of split special orthogonal groups of odd degrees, the endoscopic lifts of simple supercuspidal representations had already been determined by Adrian ([Adr16]) before our work ([Oi16a]), under some assumption on the residual characteristic p. His method is based on a computation of the twisted local γ-factors of simple supercuspidal representations, and totally different from our one.
- For tamely ramified connected reductive groups, Kaletha constructed a candidate of L-packets consisting of *epipelagic* representations in [Kal15]. Since a simple supercuspidal representation is a special case of epipelagic representations, the L-packets constructed by him includes our ones. Moreover he showed various expected properties of the local Langlands correspondence for his L-packets. In particular, he proved the stability of L-packets under some assumption on the residual characteristic. Thus we can say that, in the cases of quasi-split classical groups, his L-packets coincide with those of Arthur and Mok. On the other hand, the endoscopic character relation for twisted endoscopy has not been checked for his L-packets yet. In other words, the endoscopic lifts of his L-packets to general linear groups have not been determined yet. Therefore Theorem 4.1 does not follow from his results. We also emphasize that, in our method, we only have to assume that the residual characteristic p is odd.

Now we first explain the rough idea of the proof of Theorem 4.1. The starting point is a computation of characters of simple supercuspidal representations. By using the character formula for supercuspidal representations, we can write the character of a simple supercuspidal representation as a group-theoretical sum of the values of an affine generic character. In particular, at "shallowest" elements of pro-p Iwahori subgroups (which we will call *affine generic* elements), we can write the characters of simple supercuspidal representations in terms of *Kloosterman sums.* Our basic strategy is to combine such a computation with the endoscopic character relation. That is, we get the values of the twisted character of $\pi_{\phi}^{\mathrm{GL}_N}$ by combining such a computation with the endoscopic character relation, and then recover $\pi_{\phi}^{\mathrm{GL}_N}$ from its twisted character. However, in carrying out such a procedure, there are several difficulties.

First, a priori, there is a possibility that $\Pi_{\phi}^{\mathbf{G}}$ contains a representation which is totally different from simple supercuspidal representations. Therefore we first have to determine the structure of $\Pi_{\phi}^{\mathbf{G}}$. To accomplish this, we utilize various properties of the local Langlands correspondence.

Second, we do not have a full character formula for the twisted characters of the representations which are obtained by the parabolic induction from "non-θ-stable" parabolic subgroups (here θ is an involution of $\mathbf{G}$ used to define the twisted character). In particular, we do not have a way to compute the twisted characters of the lifted representations of $\mathrm{GL}_{2n+1}(F)$ in the case where $\mathbf{G} = \mathrm{Sp}_{2n}$ in Theorem 4.1. To resolve this difficulty, we first study the standard endoscopy of Sp_{2n} and reduce the problem to the case where $\mathbf{G} = \mathrm{SO}_{2n}^{\mu}$.

Let us explain the more detailed outline of the proof of Theorem 4.1. From now on, we put $\mathbf{G} := \mathrm{Sp}_{2n}$. Let $\pi_X^{\mathbf{G}}$ and ϕ be as in Theorem 4.1. Then the proof of Theorem 4.1 can be divided into four parts as follows:

Step 1. Determine the structure of $\Pi_\phi^{\mathbf{G}}$: The first step is to determine the structure of $\Pi_\phi^{\mathbf{G}}$. To do this, we first note that, by the *stability* of L-packets, $\Pi_\phi^{\mathbf{G}}$ consists of orbits with respect to the action of the adjoint group $\mathbf{G}_{\mathrm{ad}}$ of $\mathbf{G}$. According to a result of Kaletha in [Kal13], every G_{ad}-orbit of simple supercuspidal representations consists of exactly two simple supercuspidal representations, only one of which is generic (with respect to a fixed Whittaker data). Thus $\Pi_\phi^{\mathbf{G}}$ contains at least two simple supercuspidal representations, one of which is generic. Moreover, by combining this observation with a result of Mœglin and Xu ([Mœg11, Xu17]), we know that every member of $\Pi_\phi^{\mathbf{G}}$ is supercuspidal.

On the other hand, by using a constancy and a non-vanishing property of the characters of simple supercuspidal representations at affine generic elements of the Iwahori subgroup, we can show that if the order of $\Pi_\phi^{\mathbf{G}}$ is greater than two, then $\Pi_\phi^{\mathbf{G}}$ contains either an irreducible depth-zero supercuspidal representation or another simple supercuspidal representation. However we can eliminate these possibilities as follows. First, by the above Kaletha's result and the uniqueness of a generic representation in an L-packet (uniqueness part of so-called *Shahidi's generic packet conjecture*, which is established by [Var17] and [Ato17]), $\Pi_\phi^{\mathbf{G}}$ has no more simple supercuspidal representation. Second, by the *constancy of the formal degree* of representations in each L-packet (which is proved in [Sha90]), in order to show that $\Pi_\phi^{\mathbf{G}}$ does not have an irreducible depth-zero supercuspidal representation, it suffices to show that the formal degree of a simple supercuspidal representation is not equal to those of irreducible depth-zero supercuspidal representations. However, since irreducible depth-zero supercuspidal representations are obtained by the compact induction of irreducible cuspidal representations of the reductions of maximal parahoric subgroups of G ([MP96]), we can check this easily by studying the dimensions of irreducible cuspidal representations of reductive groups over finite fields.

Step 2. Reduce the case of Sp_{2n} to the case of SO_{2n}^{μ}: The second step is to study the relationship between simple supercuspidal L-packets of Sp_{2n} and those of ramified even special orthogonal groups SO_{2n}^{μ}. Since the order of $\Pi_\phi^{\mathbf{G}}$ is two, we know that $\Pi_\phi^{\mathbf{G}}$ is the endoscopic lift of an L-packet $\Pi_\phi^{\mathbf{H}}$ of a proper endoscopic group $\mathbf{H}$ of $\mathbf{G}$ (i.e., $\mathbf{H} \neq \mathbf{G}$). In fact, by the argument in Step 1, we can show that this group $\mathbf{H}$ is the special orthogonal group SO_{2n}^{μ} of degree $2n$ corresponding to a ramified quadratic character μ of $F^\times$ which is determined by the data $X \in \mathrm{SSC}(\mathrm{Sp}_{2n})$.

On the other hand, we note that $(\mathbf{G}, \mathbf{H})$ is a *dual pair* (strictly speaking, we should consider the orthogonal group rather than the special orthogonal group, but we ignore this difference in this introduction). Then $\Pi_\phi^{\mathbf{G}}$ can also be regarded as (a character twist of) the *theta lift* of $\Pi_\phi^{\mathbf{H}}$, by the compatibility of the theta lifting and the endoscopic lifting, which is known by [GI14] (or a special case of [AG17]). As the theta correspondence preserves the depth of representations ([Pan02]), we can conclude that $\Pi_\phi^{\mathbf{H}}$ consists of only one representation, which is simple supercuspidal. Let us denote this representation by $\pi_{X'}^{\mathbf{H}}$, for an element X' of $\mathrm{SSC}(\mathbf{H})$.

The final task in this step is to determine the data X' by computing the endoscopic character relation for $\Pi_\phi^{\mathbf{G}}$ and $\Pi_\phi^{\mathbf{H}}$. Then we know that every simple supercuspidal representation of H can be obtained by such a "descent" of simple supercuspidal representations of G, and the case of Sp_{2n} of Theorem 4.1 is reduced to the case of SO_{2n}^μ.

Step 3. Determine the endoscopic lift of $\Pi_\phi^{\mathbf{H}}$ to GL_{2n}: The third step is to determine the endoscopic lift $\pi_\phi^{\mathrm{GL}_{2n}}$ of $\Pi_\phi^{\mathbf{H}}$ to GL_{2n} and complete the proof of the case of SO_{2n}^μ of Theorem 4.1.

We first show that the depth of $\pi_\phi^{\mathrm{GL}_{2n}}$ is not greater than $\frac{1}{2n}$ (in particular $\pi_\phi^{\mathrm{GL}_{2n}}$ is either simple supercuspidal or depth-zero supercuspidal). Since the representation $\pi_\phi^{\mathrm{GL}_{2n+1}}$ is obtained by the parabolic induction of the tensor product of $\pi_\phi^{\mathrm{GL}_{2n}}$ and its central character, the depth of $\pi_\phi^{\mathrm{GL}_{2n}}$ is equal to that of $\pi_\phi^{\mathrm{GL}_{2n+1}}$. Thus it suffices to show that the depth of $\pi_\phi^{\mathrm{GL}_{2n+1}}$ is not greater than $\frac{1}{2n}$. In order to evaluate the depth of $\pi_\phi^{\mathrm{GL}_{2n+1}}$, it is enough to show that its twisted character is constant on a sufficiently large open compact set. By using the endoscopic character relation for $(\Pi_\phi^{\mathbf{G}}, \pi_\phi^{\mathrm{GL}_{2n+1}})$, we can reduce it to constancy of the characters of representations belonging to $\Pi_\phi^{\mathbf{H}}$ on an open compact coset of the Iwahori subgroup of H, and we can check it easily. The key point of this argument is to consider the endoscopic character relation for $(\Pi_\phi^{\mathbf{G}}, \pi_\phi^{\mathrm{GL}_{2n+1}})$, not for $(\Pi_\phi^{\mathbf{H}}, \pi_\phi^{\mathrm{GL}_{2n}})$. The reason why we consider the pair $(\mathrm{GL}_{2n+1}, \mathbf{G})$ rather than $(\mathrm{GL}_{2n}, \mathbf{H})$ is that the Kottwitz–Shelstad transfer factor, which appears in the endoscopic character relation, for $(\mathrm{GL}_{2n+1}, \mathbf{G})$ is trivial while that for $(\mathrm{GL}_{2n}, \mathbf{H})$ is not trivial. Namely, for the pair $(\mathrm{GL}_{2n}, \mathbf{H})$, the above argument fails because the Kottwitz–Shelstad transfer factor may not be constant on a coset of an Iwahori subgroup.

After we evaluate the depth of $\pi_\phi^{\mathrm{GL}_{2n}}$, we eliminate the possibility that $\pi_\phi^{\mathrm{GL}_{2n}}$ is depth-zero and determine $\pi_\phi^{\mathrm{GL}_{2n}}$ by computing the endoscopic character relation for $(\mathrm{GL}_{2n}, \mathbf{H})$ at affine generic elements.

$$\mathrm{GL}_{2n+1} \xleftarrow[\text{endoscopy}]{\text{twisted}} \mathbf{G} \xleftarrow[\text{endoscopy}]{\text{standard}} \mathbf{H} \xrightarrow[\text{endoscopy}]{\text{twisted}} \mathrm{GL}_{2n}$$

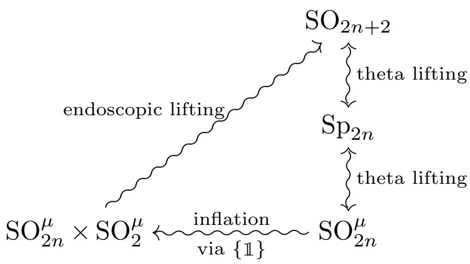
$$\pi_\phi^{\mathrm{GL}_{2n+1}} \xleftarrow[\text{lifting}]{\text{endoscopic}} \Pi_\phi^{\mathbf{G}} \xleftarrow[\text{lifting}]{\text{endoscopic}} \Pi_\phi^{\mathbf{H}} \xrightarrow[\text{lifting}]{\text{endoscopic}} \pi_\phi^{\mathrm{GL}_{2n}}$$

Step 4. Deduce the case of $\mathrm{SO}_{2n+2}^{(\mathrm{ur})}$ from the case of SO_{2n}^μ: The final step is to construct the simple supercuspidal L-packets of split or unramified quasi-split even special orthogonal groups, and determine their endoscopic lifts to general linear groups. For simplicity, here we consider only the split case.

In order to construct simple supercuspidal L-packets, we consider the theta lifting. More precisely, if we take a ramified quadratic character μ of $F^\times$, then $(\mathrm{SO}_{2n}^\mu, \mathrm{Sp}_{2n})$ and $(\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+2})$ are dual pairs, so we can construct representations of SO_{2n+2} from those of SO_{2n}^μ by considering the theta lifting twice. On the other hand, $\mathrm{SO}_{2n}^\mu \times \mathrm{SO}_2^\mu$ is an endoscopic group of SO_{2n+2}, and the theta lift of $\Pi_\phi^{\mathrm{SO}_{2n}^\mu}$ to SO_{2n+2} coincides with the endoscopic lift of the L-packet $\Pi_\phi^{\mathrm{SO}_{2n}^\mu} \times \{\mathbb{1}\}$

to SO_{2n+2} up to a character twist, by the compatibility of the theta lifting and the endoscopic lifting. Again by using the depth-preserving property of the theta lifting, we can show that the lifted L-packet consists of two simple supercuspidal representations. Finally, by computing the endoscopic character relation for $(\mathrm{SO}_{2n}^{\mu} \times \mathrm{SO}_{2}^{\mu}, \mathrm{SO}_{2n+2})$, we can determine the lifted L-packet.

In fact, this construction gives only the half of the simple supercuspidal representations of $\mathrm{SO}_{2n+2}(F)$. However, we can get the other half by twisting these simple supercuspidal L-packets by the *spinor norm* character of $\mathrm{SO}_{2n+2}(F)$. This completes the proof of Theorem 4.1.



Finally, we comment on applications of our results. We can use the results in this part as a touchstone in verifying a lot of conjectural properties of the local Langlands correspondence.

For example, we can check the *formal degree conjecture* for our L-packets. The formal degree conjecture was formulated by Hiraga–Ichino–Ikeda in [HII08] and asserts that there is an explicit relation between the special value of the adjoint γ-factor of a discrete L-parameter ϕ and the formal degree of the representations in the L-packet of ϕ. This conjecture is proved for several groups, for example, general linear groups ([HII08]), odd special orthogonal groups ([ILM17]), and some small classical groups such as the unitary group of degree 3 ([GI14]). However, for other groups such as symplectic groups, the formal degree conjecture has still been open. We consider this conjecture for simple supercuspidal representations. Since simple supercuspidal representations are obtained by the compact induction, we can compute their formal degree quite easily. On the other hand, as we mentioned before, we have an explicit description of the L-parameters of simple supercuspidal representations as a consequence of Theorem 4.1 and the works of Bushnell–Henniart and Imai–Tsushima. By using this description, a computation of the special value of the adjoint γ-factors of such L-parameters is reduced to a simple problem of representation theory of finite groups. In the end of this part, we carry out such a computation and confirm that the formal degree conjecture holds for simple supercuspidal L-packets of the quasi-split classical groups, under some assumption on p (including some cases of "bad" primes).

Another example is the *depth preserving property* of the local Langlands correspondence, which is the main theme of this paper. By Theorem 4.1, we can conclude that the depth preserving property holds for simple supercuspidal representations. The important point here is that, in Theorem 4.1, we only have to assume that the residual characteristic p is odd. Namely, simple supercuspidal representations give us an example of the depth preservation in a case which is not covered by the cases treated in a general result of Part 1.

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