博士論文

Constructions of various solutions for parabolic equations in mathematical biology and phase-field models
(数理生物学とフェイズフィールドモデルにおける放物型方程式に対する様々な解の構成)

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Abstract

In this dissertation we consider well-posedness and constructions of various solutions for the parabolic evolution equations by theories of analytic semigroups and maximal L^p regularity. The equations we consider are the bidomain equations which represent electrophysiological wave propagation in the heart, the phase-field Navier–Stokes equations which represent the deformation of the vesicle membrane in incompressible viscous fluids, the Cahn–Hilliard equation which represent the spinodal decomposition of binary mixture, and the abstract parabolic evolution equations. The bidomain equations are the model of biology and the phase-field Navier–Stokes equations and the Cahn–Hilliard equations are so-called phase-field models.

In Chapter 1 we consider the semigroup generated by the principal part of the bidomain equations. In general, solutions of the linear evolution equations can be analyzed by the semigroup which is the solution operator corresponding from the initial data to the solution at some time. It is important to characterize whether the semigroup is analytic or not. Analytic semigroups represent the smoothing effect in parabolic evolution equations. The bidomain equations is complicated since they have three unknown functions $u_{i,e}$ and u. Bourgault et al. introduced a bidomain operator and they regarded the equation into a reaction diffusion system ([Bourgault et al. 2009]). They proved that the bidomain operator is a self-adjoint operator and a non-negative operator in L^2 space. It means that the operator A generates an analytic semigroup e^{-tA}. In this chapter, we consider the bidomain operator in L^p spaces for $1 < p \leq \infty$. We prove an L^∞ resolvent estimate by a contradiction argument and a blow-up argument. From the inequality obtained by the negation of its conclusion, we show that one holds the inequality by compactness, but the other breaks down the inequality by uniqueness, which leads a contradiction. The estimates from L^2 and L^∞ imply an L^p resolvent estimate for $1 < p < \infty$ by the interpolation and the duality. We properly introduce the bidomain operator in L^p spaces and characterize the resolvent set of the operator. The main theorem is that the bidomain operator generates an analytic semigroup in L^p spaces. For the non-linear bidomain equations, we construct a local well-posedness theorem by a general theory of analytic semigroups. This chapter consists of a joint work with Professor Giga (1).

In Chapter 2 we consider the time periodic problem about the bidomain equations since the heart periodically beats in time. Under the assumption that the operator A generates an exponential stable analytic semigroup e^{-tA}, we construct DaPrato-Grisvard type maximal L^p-$D_A(\theta,p)$ regularity in a real interpolation spaces for an abstract linear parabolic equation $u' + Au = f$, where $D_A(\theta,p) := (X, D(A))_{\theta,p}$. This means that for any time periodic function $f \in L^p(\mathbb{T}; D_A(\theta,p))$, there exists a unique time periodic solution u which have the same regularity $u', Au \in L^p(\mathbb{T}; D_A(\theta,p))$, where $L^p(\mathbb{T}; D_A(\theta,p))$ is a class of X-valued $L^p(\mathbb{T})$ function with $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$ for a period $T > 0$. This is a linear theory about the time periodic problem. The proof is based

on the time periodic solution formula $u(t) = \int_{-\infty}^{t} e^{-(t-s)A} f(s)ds$. For the non-linear bidomain equations, we prove that the equations admit a unique periodic solution by Banach's fixed point theorem provided the source terms are small. This chapter consists of a joint work with Professor Hieber, Dr Tolksdorf and Mr. Kress (2).

In Chapter 3 we continue discussing the time periodic problem about the bidomain equations. In Chapter 2 we need the smallness conditions since the bidomain equations are considered as a perturbed equation of their linearized equations. In this chapter we restrict that the nonlinear term is FitzHugh–Nagumo type and the function space is L^2, but we prove that the equations admit a time periodic solution without assuming the smallness conditions on the source. When we use Galerkin method, we apply Brouwer's fixed point theorem for Poincaré map. This guarantees the existence of a weak time periodic solution. Moreover we regard the initial data as this periodic solution, then by global well-posedness result for initial value problem, the weak solutions agree with the strong solutions. This is a regularity theorem, which implies the existence of the strong time periodic solutions in maximal L^2-L^2 spaces. This chapter consists of a joint work with Professor Giga and Mr. Kress (3).

In Chapter 4 we consider the well-posedness for the phase-field Navier–Stokes equations. In previous works, it was proved the existence of the global weak solutions ([Du et al. 2007]) and the unique existence of the local strong solutions ([Takahashi et al. 2012]). However the former is not known its uniqueness and regularity and the latter is analyzed as a semi-linear evolution equation although the coupling part should be the principal part. Therefore its regularity class of the solutions is not suitable. We treat the equation as a quasi-linear evolution equation which means the coupling part is the principal part. We prove the linear operator has maximal L^p-L^q regularity property and prove the unique existence of the local strong solutions and the continuity on the initial data. Moreover we have that the solution is real analytic in time and space, thus this is a classical solution. At last it is shown that the variational strict stable solution is exponentially stable provided the product of the viscosity coefficient and the mobility constant is large. This chapter consists of a paper (4).

In Chapter 5 we consider the global existence and uniqueness of the solutions for the Cahn–Hilliard equation. Let the order parameter u and the chemical potential μ. In previous works, the boundary condition was Neumann condition for μ so that it derives the volume conservation law $\frac{d}{dt}\int_\Omega u\,dx = 0$. Since the equation is fourth order, we need two boundary conditions. The other boundary condition is Neumann condition for u so that it derives that the energy $E_\Omega(u)$ decrease, or a dynamic boundary condition for u so that it derives the energy $E_\Omega(u) + E_{\partial\Omega}(u)$ decrease. Here $E_{\partial\Omega}(u)$ is a energy from the boundary. However when the substances permeate the boundary, volume preservation is not necessarily achieved. Gal and Goldstein et al. introduced new boundary conditions which model the boundaries are permeable walls and non-permeable walls, respectively ([Gal 2006], [Goldstein et al. 2011]). The former derives $\frac{d}{dt}(\int_\Omega u\,dx + \int_{\partial\Omega} u\,dS) = -c\int_\Gamma \mu \frac{dS}{b}$ with the constants $b > 0, c \geq 0$ and the latter derives the total volume conservation law $\frac{d}{dt}(\int_\Omega u\,dx + \int_{\partial\Omega} u\,dS) = 0$. In permeable walls, the case $c = 0$ implies the total volume conservation law. We consider these two boundary conditions including the case $c = 0$. We apply the linear theory of maximal L^p regularity which corresponds to higher order equations and the dynamic boundary condition (cf. [Denk et al. 2008]). It characterizes the solvability and the classes of data by a necessary and a sufficient condition. For the non-linear Cahn–Hilliard equation, we use Banach's fixed point theorem, energy estimates and a-priori estimate. We are able to generalize the global solvability for $p \neq 2$ although above previous works are L^2 framework. Moreover since the spaces of initial functions are optimal, we are able to classify the necessary of the compatibility conditions by p although above previous works need the compatibility conditions. This chapter consists of a paper (5).

In Chapter 6 we consider the well-posedness for the abstract parabolic evolution equations by means of maximal L_p regularity with time weights. The (classical) maximal L_p regularity property is the solvability and the estimate for the abstract linear evolution equations $u' + Au = f$ in $L_p(0, T; X)$. The remarkable application of the maximal L_p regularity is the solvability of the quasi-linear parabolic evolution equations $u' + A(u)u = F(u)$. However when we use this theory, in general, we need to take the initial data in the real interpolation space $(X, D(A))_{1-1/p,p}$ which is the trace space of the solution space at $t = 0$. The theory of maximal L_p regularity with time weight is a generalization of this initial data while keeping the solution class excepting for the behavior near $t = 0$. In a series of papers by J. Prüss, they extended the initial data to $(X, D(A))_{\mu-1/p,p}$ for $\mu_c \leq \mu \leq 1$, where the critical weight μ_c is determined by the non-linearities F. Moreover they give a sufficient condition to be a global solution when the equation is a semi-linear and the non-linear term is the bi-linear. In this chapter we extend this general local well-posedness theory to the quasi-linear parabolic evolution equations $u' + A(t, u) = F(t, u)$ with the time-dependent operator and the non-linear term. Under the case $A(t, u) = A(t)$ and the assumptions used in the local well-posedness with some technical assumption, we give a sufficient condition in order that the local solution becomes a global solution. This is a generalization of above bilinear non-linearities. As a example of the local well-posedness theorem, we applied the theory to the quasi-linear heat equations $u' - a(t, u)\Delta u = F(t, u) + |\nabla u|^\kappa$ with $\kappa > 2$.

All chapters are based on the papers below, respectively. All sections, notations and theorems, etc are cited only in each chapter where they appear.

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Contents

Chapter 1

On a resolvent estimate for bidomain operators and its applications

We study bidomain equations that are commonly used as a model to represent the electrophysiological wave propagation in the heart. We prove existence, uniqueness and regularity of a strong solution in L^p spaces. For this purpose we derive an L^∞ resolvent estimate for the bidomain operator by using a contradiction argument based on a blow-up argument. Interpolating with the standard L^2-theory, we conclude that bidomain operators generate C_0-analytic semigroups in L^p spaces, which leads to construct a strong solution to a bidomain equation in L^p spaces.

Keywords: bidomain model; resolvent estimates; blow-up argument

1.1 Introduction

The bidomain model is a system related to intra- and extra-cellular electric potentials and some ionic variables. Mathematically, bidomain equations can be written as two partial differential equations coupled with a system of m ordinary differential equations:

$$\partial_t u + f(u,w) - \nabla \cdot (\sigma_i \nabla u_i) = s_i \quad \text{in } (0,\infty)\times\Omega, \tag{1.1.1}$$
$$\partial_t u + f(u,w) + \nabla \cdot (\sigma_e \nabla u_e) = -s_e \quad \text{in } (0,\infty)\times\Omega, \tag{1.1.2}$$
$$\partial_t w + g(u,w) = 0 \quad \text{in } (0,\infty)\times\Omega, \tag{1.1.3}$$
$$u = u_i - u_e \quad \text{in } (0,\infty)\times\Omega, \tag{1.1.4}$$
$$\sigma_i \nabla u_i \cdot n = 0,\ \sigma_e \nabla u_e \cdot n = 0 \quad \text{on } (0,\infty)\times\partial\Omega, \tag{1.1.5}$$
$$u(0) = u_0,\ w(0) = w_0 \quad \text{in } \Omega. \tag{1.1.6}$$

Here, functions u_i and u_e are intra- and extra-cellular electric potentials, u is the transmembrane potential (or the action potential) and $w = w(t,x) \in \mathbb{R}^m (m \in \mathbb{N})$ is some ionic variables (current, gating variables, concentrations, etc.). All these functions are unknown. On the other hand, the physical region occupied by the heart $\Omega \subset \mathbb{R}^d$, conductivity matrices $\sigma_{i,e} = \sigma_{i,e}(x)$, external applied current sources $s_{i,e} = s_{i,e}(t,x)$, total transmembrane ionic currents $f : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$ and initial data u_0 and w_0 are given. The symbol n denotes the unit outward normal vector to

$\partial\Omega$. The reader is referred to the books [12] and [22] about mathematical physiology including bidomain models.

There are some literature about well-posedness of bidomain equations. First pioneering work is due to P. Colli-Franzone and G. Savaré [13]. They introduced a variational formulation and derived existence, uniqueness and some regularity results in Hilbert spaces. Here, they assumed nonlinear terms f, g are forms of $f(u,w) = k(u) + \alpha w$, $g(u,w) = -\beta u + \gamma w$ $(\alpha, \beta, \gamma \geq 0)$ with a suitable growth condition on k. Examples include cubic-like FitzHugh-Nagumo model, which is the most fundamental electrophysiological model. However, other realistic models cannot be handled by their approach because nonlinear terms are limited. Later M. Veneroni [40] extended to their results by using fixed point argument and established well-posedness of more general and more realistic ionic models. These two papers discussed strong solutions by deriving further regularity of weak solutions. In 2009, Y. Bourgault, Y. Coudiére and C. Pierre [11] showed well-posedness of a strong solution in L^2 spaces. They transformed bidomain equations into an abstract evolution equation of the form

$$\begin{cases} \partial_t u + Au + f(u,w) = s, \\ \partial_t w + g(u,w) = 0 \end{cases}$$

by introducing the bidomain operator A in L^2 and modified source term s. Formally the bidomain operator is the harmonic mean of two elliptic operators, i.e. $(A_i^{-1} + A_e^{-1})^{-1}$ or $A_i(A_i + A_e)^{-1}A_e$, where $A_{i,e}\cdot$ is the elliptic operator $-\nabla \cdot (\sigma_{i,e}\nabla\ \cdot\ )$ with the homogeneous Neumann boundary condition. They proved that the bidomain operator is a non-negative self-adjoint operator by considering corresponding weak formulations. Since their framework is in L^2, well-posedness was only proved for $d \leq 3$ in L^2 spaces.

The main goal of this chapter is to establish L^p-theory $(1 < p < \infty)$ and L^∞-theory for the bidomain operator with applications to bidomain equations. More explicitly, we shall prove that the bidomain operator forms an analytic semigroup e^{-tA} both in L^p and L^∞. By this result we are able to construct a strong solution in L^p for any space dimension d (by taking p large if necessary). Our result allows any locally Lipschitz nonlinear terms.

To derive analyticity it is sufficient to derive resolvent estimates. For L^p resolvent estimates a standard way is to use the Agmon's method (e.g. [24], [37]). The main idea of the method is as follows. If we have a $W^{2,p}(\Omega \times \mathbb{R})$ a priori estimate for the operator $A - e^{i\theta}\partial_{tt}$, then A has an L^p resolvent estimate. Unfortunately, it seems difficult to derive such a $W^{2,p}$ a priori estimate because of nonlocal structure of the bidomain operator. Thus we argue in a different way.

We first establish an L^∞ resolvent estimate for the bidomain operator by a contradiction argument including a blow-up argument. We then derive an L^p resolvent estimate for $2 \leq p \leq \infty$ by interpolating L^2 and L^∞ results. The L^p-theory for $1 < p < 2$ is established by a duality argument. Note that a standard idea to derive an L^∞ resolvent estimate due to Masuda-Stewart (see the third next paragraph) does not apply because their method is based on an L^p resolvent estimate, which we would like to prove.

A blow-up argument was first introduced by E. De Giorgi [14] in order to study regularity of a minimal surface. It is also efficient to derive a priori estimates for solutions of a semilinear elliptic problem [17] and a semilinear parabolic problem [18],

[16]. Recently, K. Abe and the first author [1], [2] showed that the Stokes operator is a generator of an analytic semigroup on $C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$ (the space of smooth solenoidal vector fields with compact support in Ω) for some class of domains Ω including bounded and exterior domains by using a blow-up argument for a nonstationary problem. For a direct proof extending the Masuda-Stewart method for resolvent estimates, see [3]. Suzuki [35] showed analyticity of semigroups generated by higher order elliptic operators in L^∞ spaces by a blow-up method even if the domain has only uniformly C^1 regularity for resolvent equations. Our approach is similar to his approach, but boundary conditions are different and our equations are systems. For the Dirichlet boundary condition, we can easily take a cut-off function and a test function. However, for the Neumann boundary condition, we have to take a cut-off function and a test function carefully so that we does not violate boundary conditions.

Our method is based on a contradiction argument together with a blow-up argument. Let us explain a heuristic idea. Suppose that we would like to prove that

$$|\lambda|\|u\|_\infty \leq C\|s\|_\infty$$

with some $C > 0$ independent of sufficiently large λ, u and s which satisfy the resolvent equation $\lambda u + Au = s$ in Ω. Here, $\|\cdot\|_\infty$ denotes the $L^\infty(\Omega)$ norm. Suppose that the estimate were false. Then there would exists a sequence $\{\lambda_k\}_{k=1}^\infty$, $|\lambda_k| \to \infty$ and $\{u_k, s_k\}$ satisfy $\lambda_k u_k + Au_k = s_k$ in Ω such that $|\lambda_k|\|u_k\|_\infty > k\|s_k\|_\infty$. By normalizing u_k to introduce $v_k = u_k/\|u_k\|_\infty$, we observe that $\|v_k\|_\infty = 1$. We take $\{x_k\}_{k=1}^\infty \subset \Omega$ such that $|v_k(x_k)| > 1/2$. We rescale $w_k(x) = v_k(x_k + x/|\lambda_k|^{1/2})$. This function solves the equation $e^{i\theta_k}w_k + A_k w_k = t_k$ in Ω_k with $A_k \to A_0$ if A has a nice scaling property, where $e^{i\theta_k} = \lambda_k/|\lambda_k|$, $t_k(x) = s_k(x_k + x/|\lambda_k|^{1/2})/|\lambda_k|\|u_k\|_\infty$ and $\Omega_k := |\lambda_k|^{1/2}(\Omega - x_k)$. Here, A_0 is the bidomain operator with a constant coefficient. Since $|\lambda_k| \to \infty$, the rescaled domain Ω_k converges to either the whole space or the half space. If w_k converges to some w, then w solves the limit equation $e^{i\theta_\infty}w + A_0 w = 0$ since $\|t_k\|_\infty < 1/k$. If the convergence is strong enough, then the assumption $|w_k(0)| > 1/2$ implies $|w(0)| \geq 1/2$. However, if the solution of the limit equation $e^{i\theta_\infty}w + A_0 w = 0$ is unique, i.e. $w = 0$, then we get a contradiction. The key step is a local 'Compactness' of the blow-up sequence $\{w_k\}_{k=1}^\infty$ near zero to conclude $|w(0)| \geq 1/2$ and 'Uniqueness' of a blow-up limit.

Let us explain some literatures for L^∞-theory. For the Laplace operator or general elliptic operators it is well known that the corresponding semigroup is analytic in L^∞-type spaces. K. Yosida [42] considered the second order elliptic operator on $\mathbb{R}$. It was difficult to extend his method for multi-dimensional elliptic operators. K. Masuda [25], [26] (see also [27]) first proved the analyticity of the semigroup generated by a general elliptic operator (even for higher-order elliptic operators) in $C_0(\mathbb{R}^d)$, the space of continuous functions vanishing at the space-infinity. For general domains, H. B. Stewart treated Dirichlet conditions [33] and general boundary conditions [34]. Their methods are based on a localization with L^p results and interpolation inequalities. The reader may refer to the comprehensive book written by A. Lunardi [24, Chapter 3] for the Masuda-Stewart method which applies to many other cases. However, in our situations, we cannot apply these methods since we do not have L^p estimates.

Originally, bidomain equations were derived at a microscopic level. The cardiac cellular structure of the tissue can be viewed by disjoint unions of two regions separated by the interface, i.e. $\Omega = \Omega_i \cup \Omega_e \cup \Gamma$, where Ω_i and Ω_e are disjoint intra- and extra cellular domains and $\overline{\Gamma} = \partial\Omega_i \cap \partial\Omega_e$ is their interface called the active membrane. When we consider this model, the intra- and extra cellular potential $u_{i,e}$ are functions in $\overline{\Omega_{i,e}}$ respectively, and transmembrane potential $u = u_i - u_e$ is the function on $\overline{\Gamma}$. Bidomain equations are replaced to equations on Ω_i, Ω_e and Γ in this microscopic model. The dynamics inside the heart is much complicated. There are only a few papers (e.g. [13], [38]) because of standard techniques and results on reaction diffusion equation systems cannot be directly applied. H. Matano and Y. Mori [28] showed existence and uniqueness of a global classical solution for 3D cable model which is one of the microscopic cellular model by proving a uniform L^∞ bound of solutions.

Conversely at a macroscopic model, the cardiac tissue can be represented by a continuous model (called "bi"domain model), i.e. $\Omega = \Omega_i = \Omega_e = \Gamma$ though each point of the heart Ω is one of the interior part Ω_i or exterior part Ω_e or their boundary Γ. Formal derivation from microscopic model to macroscopic model was shown by a homogenization process when a periodic cardiac structure [22], [23]. The authors of [31] showed a rigorous mathematical derivation of the macroscopic model by using the tools of the Γ-convergence theory. The paper [7] studied the asymptotic behavior of the family of vectorial integral functionals, which is concerned with bidomain model, in the framework of Γ-convergence. The bidomain model is also used to analyze nonconvex mean curvature flow as a diffuse interface approximation [6], [10], [9]. Nonconvexity leads to the gradient flow of a nonconvex functional, which corresponds in general to an ill-posed parabolic problem. To study an ill-posed problem, it is often efficient to regularize it, for example by adding some higher order term, and then passing to the limit as the regularizing parameter goes to zero. However, papers [6], [10], [9] introduced completely different regularization, namely, to use bidomain equations, where hidden anisotrophy plays a key role. Recently in [29], interesting phenomena about stability of traveling wave solutions was found for bidomain Allen-Cahn equations, which is quite different from classical Allen-Cahn equations. This is also relevant to the hidden anisotropy of the bidomain model.

The outline of this chapter is as follows. In Section 1.2 after preparing a few notations, we state an L^∞ resolvent estimate for bidomain equations, which is a key estimate of analyticity in L^p and L^∞ spaces. In Section 1.3 we give our proof of an L^∞ resolvent estimate by using a blow-up argument. In Section 1.4 the system of bidomain equations is replaced by a single equation by using bidomain operators in L^p spaces. Then we show existence and uniqueness of the solution. The method is based on a continuity method [20]. We also establish L^p and L^∞ resolvent estimates for bidomain operators based in our analysis in Section 1.3. In Section 1.5 to solve original problem (1.1.1)-(1.1.6) in L^p we define bidomain operators and domains of their fractional powers in order to handle nonlinear terms f, g having only locally Lipschitz continuity. From an L^p resolvent estimate, we show bidomain operators are sectorial operators and then we derive existence, uniqueness and regularity of a strong solution to (1.1.1)-(1.1.6) in L^p spaces. In the appendix, we collect the L^1 boundedness of Fourier multiplier, which was left in Section 1.3.

1.2 Resolvent estimate for bidomain operators

1.2.1 Preliminaries, notations and definitions

In this subsection we give a rigorous setting in order to state an L^∞ resolvent estimate. We first recall the definition of uniformly C^k-domain for $k \geq 1$ and function spaces $W^{2,p}_{loc}(\overline{\Omega})$.

Let $B(x_0, r)$ be an open ball with center x_0, radius $r > 0$, i.e. $B(x_0, r) = \{x \in \mathbb{R}^d \mid |x - x_0| < r\}$.

Definition 1.1 (Uniformly C^k-domain)**.** Let $\Omega \subset \mathbb{R}^d$ be a domain with $d \geq 2$. We say that Ω is a uniformly C^k-domain ($k \geq 1$) if there exist $K > 0$ and $r > 0$ such that for each point $x_0 \in \partial\Omega$ there exists a C^k function γ of $d-1$ variables x' such that -upon relabeling, reorienting and rotation the coordinates axes if necessary- we have

$$\Omega \cap B(x_0, r) = \left\{x = (x', x_d) \in B(x_0, r) \mid x_d > \gamma(x')\right\},$$
$$\|\gamma\|_{C^k(\mathbb{R}^{d-1})} = \sup_{|\alpha| \leq k,\ x' \in \mathbb{R}^{d-1}} |\partial^\alpha_{x'} \gamma(x')| \leq K.$$

Definition 1.2. We say $u \in W^{2,p}_{loc}(\overline{\Omega})$ if there exists $v \in W^{2,p}_{loc}(\mathbb{R}^d)$ such that $u = v$ a.e. in $\overline{\Omega}$.

Here $W^{2,p}$ is L^p Sobolev space of order 2 and $W^{2,p}_{loc}$ is their localized version.

The conductivity matrices $\sigma_{i,e}$ are functions of the space variable $x \in \overline{\Omega}$ with coefficients $C^1(\overline{\Omega})$ and satisfy the uniform ellipticity condition. Namely, we assume that there exist constants $0 < \underline{\sigma} < \overline{\sigma}$ such that

$$\underline{\sigma}|\xi|^2 \leq \langle \sigma_{i,e}(x)\xi, \xi\rangle \leq \overline{\sigma}|\xi|^2 \tag{1.2.1}$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^d$. Let $a = a(x)$ denote unit tangent vector at the point $x \in \partial\Omega$. Set the longitudinal conductances $k^l_{i,e} : \partial\Omega \to \mathbb{R}$ and the transverse conductances $k^t_{i,e} : \partial\Omega \to \mathbb{R}$ along the fibers. Commonly used conductance tensors are of the form ([16])

$$\sigma_{i,e}(x) = k^t_{i,e}(x)I + (k^l_{i,e}(x) - k^t_{i,e}(x))a(x) \otimes a(x) \ \ (x \in \partial\Omega).$$

By this form we have the normal n is the eigenvector of $\sigma_{i,e}$ whose eigenvalue is $k^t_{i,e}(x)$:

$$\sigma_{i,e}(x)n(x) = k^t_{i,e}(x)n(x) \ (x \in \partial\Omega).$$

Under these assumptions of $\sigma_{i,e}$, we have the property of boundary conditions:

$$\sigma_{i,e}\nabla u \cdot n = 0 \Leftrightarrow \nabla u \cdot n = 0 \quad \text{on } \partial\Omega. \tag{1.2.2}$$

Source terms $s_{i,e}$ also have important property. In physiology no current flow outside through boundary $\partial\Omega$ and the intra- and extra-cellular media communicate electrically through the transmembrane. Hereafter we assume current conservation;

$$\int_\Omega (s_i(t) + s_e(t))dx = 0 \ (t \geq 0).$$

This is nothing but the compatibility condition for bidomain equations. This averaging zero condition is used when we transform the system of bidomain equations (1.1.1)-(1.1.6) into single equation (1.5.7)-(1.5.8).

1.2.2 Resolvent estimate

We consider the following resolvent equations

$$(*)\begin{cases} \lambda u - \nabla\cdot(\sigma_i \nabla u_i) = s & \text{in } \Omega,\\ \lambda u + \nabla\cdot(\sigma_e \nabla u_e) = s & \text{in } \Omega,\\ u = u_i - u_e & \text{in } \Omega,\\ \sigma_i\nabla u_i\cdot n = 0,\ \sigma_e\nabla u_e\cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

corresponding to (1.1.1)-(1.1.6). These equations come from the Laplace transformation of linear part of bidomain equations.

Let us state an L^∞ resolvent estimate. We set $\Sigma_{\theta,M} := \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\operatorname{Arg}\lambda| < \theta, M < |\lambda|\}$ and $N(u, u_i, u_e, \lambda)$ of the form

$$N(u,u_i,u_e,\lambda) := \sup_{x\in\Omega}\Big(|\lambda||u(x)| + |\lambda|^{1/2}\left(|\nabla u(x)| + |\nabla u_i(x)| + |\nabla u_e(x)|\right)\Big).$$

Theorem 1.3 (L^∞ resolvent estimate for bidomain equations). *Let $\Omega\subset\mathbb{R}^d$ be a uniformly C^2-domain and $\sigma_{i,e}\in C^1(\overline{\Omega},\mathbb{S}^d)$ satisfy* (1.2.1) *and* (1.2.2). *Then for each $\varepsilon\in(0,\pi/2)$ there exist $C>0$ and $M>0$ such that*

$$N(u,u_i,u_e,\lambda) \le C\|s\|_{L^\infty(\Omega)}$$

for all $\lambda\in\Sigma_{\pi-\varepsilon,M}$, $s\in L^\infty(\Omega)$ and strong solutions $u, u_{i,e}\in\bigcap_{n<p<\infty} W^{2,p}_{loc}(\overline{\Omega})\cap W^{1,\infty}(\Omega)$ of $(*)$.

Remark 1.4. (i) It is impossible to derive an estimate $|\lambda|\|u_{i,e}\|_\infty \le C\|s\|_{L^\infty(\Omega)}$ because if (u,u_i,u_e) is a triplet of strong solutions then so is (u, u_i+c, u_e+c) for all $c\in\mathbb{R}$.
(ii) By the Sobolev embedding theorem [5],

$$\bigcap_{n<p<\infty} W^{2,p}_{loc}(\overline{\Omega})\cap W^{1,\infty}(\Omega) \subset \bigcap_{0<\alpha<1} C^{1+\alpha}(\overline{\Omega}).$$

Hence (u,u_i,u_e) are C^1 functions and the left-hand side of the resolvent estimate makes sense.

1.3 Proof of an L^∞ resolvent estimate

Proof of Theorem 1.3. We divide the proof into five steps. The first two steps are reformulation of equations and estimates. The last three steps (compactness, characterization of the limit and uniqueness) are crucial.

Step 1 (Normalization)

We argue by contradiction. Suppose that the statement were false. Then there would exist $\varepsilon \in (0, \pi/2)$, for any $k \in \mathbb{N}$ there would exist $\lambda_k = |\lambda_k| e^{i\theta_k} \in \Sigma_{\pi-\varepsilon,k}$, $s_k \in L^\infty(\Omega)$ and $u_k, u_{ik}, u_{ek} \in \bigcap_{n<p<\infty} W^{2,p}_{loc}(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ which are strong solutions of resolvent equations

$$\begin{cases} \lambda_k u_k - \nabla \cdot (\sigma_i \nabla u_{ik}) = s_k & \text{in } \Omega, \\ \lambda_k u_k + \nabla \cdot (\sigma_e \nabla u_{ek}) = s_k & \text{in } \Omega, \\ u_k = u_{ik} - u_{ek} & \text{in } \Omega, \\ \sigma_i \nabla u_{ik} \cdot n = 0, \ \sigma_e \nabla u_{ek} \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

with an L^∞ estimate $N(u_k, u_{ik}, u_{ek}, \lambda_k) > k \|s_k\|_{L^\infty(\Omega)}$.

We set

$$\begin{pmatrix} v_k \\ v_{ik} \\ v_{ek} \\ \tilde{s}_k \end{pmatrix} := \frac{1}{N(u_k, u_{ik}, u_{ek}, \lambda_k)} \begin{pmatrix} |\lambda_k| u_k \\ |\lambda_k| u_{ik} \\ |\lambda_k| u_{ek} \\ s_k \end{pmatrix}.$$

Then we get normalized resolvent equations of the form

$$\begin{cases} e^{i\theta_k} v_k - \frac{1}{|\lambda_k|} \nabla \cdot (\sigma_i \nabla v_{ik}) = \tilde{s}_k & \text{in } \Omega, \\ e^{i\theta_k} v_k + \frac{1}{|\lambda_k|} \nabla \cdot (\sigma_e \nabla v_{ek}) = \tilde{s}_k & \text{in } \Omega, \\ v_k = v_{ik} - v_{ek} & \text{in } \Omega, \\ \sigma_i \nabla v_{ik} \cdot n = 0, \ \sigma_e \nabla v_{ek} \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

with estimates $\frac{1}{k} > \|\tilde{s}_k\|_{L^\infty(\Omega)}$ and

$$\begin{aligned} &N\left(\frac{v_k}{|\lambda_k|}, \frac{v_{ik}}{|\lambda_k|}, \frac{v_{ek}}{|\lambda_k|}, \lambda_k\right) \\ =&\sup_{x\in\Omega} \left(|v_k(x)| + |\lambda_k|^{-1/2} \left(|\nabla v_k(x)| + |\nabla v_{ik}(x)| + |\nabla v_{ek}(x)| \right) \right) \\ =&1. \end{aligned}$$

Step 2 (Rescaling)

Secondly, we rescale variables near maximum points of normalized N. By definition of supremum there exists $\{x_k\}_{k=1}^\infty \subset \Omega$ such that

$$|v_k(x_k)| + |\lambda_k|^{-1/2} \left(|\nabla v_k(x_k)| + |\nabla v_{ik}(x_k)| + |\nabla v_{ek}(x_k)| \right) > \frac{1}{2}$$

for all $k \in \mathbb{N}$. We rescale functions $\{(w_k, w_{ik}, w_{ek})\}_{k=1}^\infty$, $\{t_k\}_{k=1}^\infty$, matrices $\{(\sigma_{ik}, \sigma_{ek})\}_{k=1}^\infty$ and domain Ω_k with respect to x_k. Namely, we set

$$\begin{pmatrix} w_k \\ w_{ik} \\ w_{ek} \end{pmatrix} (x) := \begin{pmatrix} v_k \\ v_{ik} \\ v_{ek} \end{pmatrix} \left(x_k + \frac{x}{|\lambda_k|^{1/2}} \right),$$

$$
\begin{aligned}
t_k(x) :=& \tilde{s}_k \left(x_k + \frac{x}{|\lambda_k|^{1/2}} \right), \\
\sigma_{ik}(x) := \sigma_i \left(x_k + \frac{x}{|\lambda_k|^{1/2}} \right), &\ \ \sigma_{ek}(x) := \sigma_e \left(x_k + \frac{x}{|\lambda_k|^{1/2}} \right), \\
\Omega_k :=& |\lambda_k|^{1/2}(\Omega - x_k).
\end{aligned}
$$

By changing variables $\Omega \ni x \mapsto |\lambda_k|^{1/2}(x - x_k) \in \Omega_k$, we notice that our equations and our estimates can be rewritten of the form

$$
\begin{cases}
e^{i\theta_k} w_k - \nabla \cdot (\sigma_{ik} \nabla w_{ik}) = t_k & \text{in } \Omega_k, \\
e^{i\theta_k} w_k + \nabla \cdot (\sigma_{ek} \nabla w_{ek}) = t_k & \text{in } \Omega_k, \\
w_k = w_{ik} - w_{ek} & \text{in } \Omega_k, \\
\sigma_{ik} \nabla w_{ik} \cdot n_k = 0, \ \sigma_{ek} \nabla w_{ek} \cdot n_k = 0 & \text{on } \partial\Omega_k,
\end{cases}
$$

with estimates

$$
\begin{aligned}
&\frac{1}{k} > \|t_k\|_{L^\infty(\Omega_k)}, \\
&|w_k(0)| + |\nabla w_k(0)| + |\nabla w_{ik}(0)| + |\nabla w_{ek}(0)| > \frac{1}{2}, \\
&\sup_{x \in \Omega_k} \left(|w_k(x)| + |\nabla w_k(x)| + |\nabla w_{ik}(x)| + |\nabla w_{ek}(x)| \right) = 1,
\end{aligned}
$$

where n_k denotes the unit outer normal vector to Ω_k. Here, we remark that unknown functions w_{ik} and w_{ek} are defined up to an additive constant. So without loss of generality we may assume that $w_{ik}(0) := 0$.

Step 3 (Compactness)

In this step, we will show local uniform boundedness for $\{(w_k, w_{ik}, w_{ek})\}_{k=1}^\infty$. If these sequences are bounded, one can take subsequences $\{(w_{k_l}, w_{ik_l}, w_{ek_l})\}_{l=1}^\infty$ which uniformly convergences in the norm C^1 on each compact set. We need to divide two cases. One is the whole space case and the other is the half space case up to translation and rotation.
We set $d_k = \operatorname{dist}(0, \partial\Omega_k) = |\lambda_k|^{1/2} \operatorname{dist}(x_k, \partial\Omega)$ and $D := \liminf_{k \to \infty} d_k$.

Case(3-i) $D = \infty$

In this case the limit Ω_k is $\mathbb{R}^d$ in the sense that for any $R > 0$ there is k_0 such that $B(0, R) \subset \Omega_k$ for $k_0 \leq k$. For the convergence of the domain, see [1], [4], [35]. Let cut-off function $\rho \in C_0^\infty(\mathbb{R}^d)$ be such that $\rho(x) \equiv 1$ for $|x| \leq 1$ and $\rho(x) \equiv 0$ for $|x| \geq 3/2$. Here and hereafter by $C_0^k(E)$ we mean the space of all k-times continuously differentiable function with compact support in the set E. We localize functions w_k, w_{ik}, w_{ek} as follows

$$
\begin{pmatrix} w_k^\rho \\ w_{ik}^\rho \\ w_{ek}^\rho \end{pmatrix} := \rho \begin{pmatrix} w_k \\ w_{ik} \\ w_{ek} \end{pmatrix} \quad \text{in } \Omega_k.
$$

By multiplying rescaled resolvent equations by ρ, we consider the following localized equations

$$e^{i\theta_k} w_k^\rho - \nabla \cdot (\sigma_{ik} \nabla w_{ik}^\rho) = t_k \rho + I_{ik} \quad \text{in } \Omega_k, \tag{1.3.1}$$
$$e^{i\theta_k} w_k^\rho + \nabla \cdot (\sigma_{ek} \nabla w_{ek}^\rho) = t_k \rho + I_{ek} \quad \text{in } \Omega_k, \tag{1.3.2}$$
$$w_k^\rho = w_{ik}^\rho - w_{ek}^\rho \quad \text{in } \Omega_k, \tag{1.3.3}$$
$$\sigma_{ik} \nabla w_{ik}^\rho \cdot n_k = 0, \ \sigma_{ek} \nabla w_{ek}^\rho \cdot n_k = 0 \quad \text{on } \partial\Omega_k, \tag{1.3.4}$$

where

$$I_{ik} = - \sum_{1 \le m,n \le d} \Big\{ \big((\sigma_{ik})_{mn}\big)_{x_m} \rho_{x_n} w_{ik} + (\sigma_{ik})_{mn} \rho_{x_m x_n} w_{ik} + (\sigma_{ik})_{mn} \rho_{x_n} (w_{ik})_{x_m} + (\sigma_{ik})_{mn} \rho_{x_m} (w_{ik})_{x_n} \Big\},$$

$$I_{ek} = \sum_{1 \le m,n \le d} \Big\{ \big((\sigma_{ek})_{mn}\big)_{x_m} \rho_{x_n} w_{ek} + (\sigma_{ek})_{mn} \rho_{x_m x_n} w_{ek} + (\sigma_{ek})_{mn} \rho_{x_n} (w_{ek})_{x_m} + (\sigma_{ek})_{mn} \rho_{x_m} (w_{ek})_{x_n} \Big\}$$

are lower order terms of w_{ik} and w_{ek}. Here, we take sufficiently large k such that $B(0,2) \subset \Omega_k$.

Take some $p > n$ and apply $W^{2,p}(\Omega_k)$ a priori estimate for second order elliptic operators $-\nabla \cdot (\sigma_{ik} \nabla \cdot)$, which have the Neumann boundary (1.3.4). By (1.3.1) there exists $C > 0$ independent of $k \in \mathbb{N}$ such that

$$\begin{aligned} & \|w_{ik}^\rho\|_{W^{2,p}(\Omega_k)} \\ \le & C \left(\|w_{ik}^\rho\|_{L^p(\Omega_k)} + \|w_k^\rho\|_{L^p(\Omega_k)} + \|t_k \rho\|_{L^p(\Omega_k)} + \|I_{ik}\|_{L^p(\Omega_k)} \right) \\ \le & C |B(0,2)|^{1/p} \left(\|w_{ik}^\rho\|_{L^\infty(\Omega_k)} + \|w_k^\rho\|_{L^\infty(\Omega_k)} + \|t_k \rho\|_{L^\infty(\Omega_k)} + \|I_{ik}\|_{L^\infty(\Omega_k)} \right) \\ =: & C |B(0,2)|^{1/p} \left(I + II + III + IV \right), \end{aligned}$$

where we use Hölder inequality in the second inequality. The first term I is uniformly bounded in k since $w_{ik}(0) = 0$ and $\|\nabla w_{ik}\|_{L^\infty(\Omega_k)} \le 1$. The second term II and the third term III are also uniformly bounded in k since $\|w_k\|_{L^\infty(\Omega_k)} \le 1$, $\|\rho\|_{L^\infty(\Omega_k)} \le 1$ and $\|t_k\|_{L^\infty(\Omega_k)} < 1/k$. Finally the forth term IV is also uniformly bounded in k since

$$\begin{aligned} IV &\le C(d, \sup_k \|\sigma_{ik}\|_{W^{1,\infty}(\Omega_k)}) \|w_{ik}\|_{W^{1,\infty}(\Omega_k)} \\ &\le C. \end{aligned}$$

Here, the constant C may differ from line to line. Therefore the sequence $\{w_{ik}^\rho\}_{k=1}^\infty$ is uniformly bounded in $W^{2,p}(\Omega_k)$. Functions $\{w_{ek}^\rho\}_{k=1}^\infty$ and $\{w_k^\rho\}_{k=1}^\infty$ are also uniformly bounded in $W^{2,p}(\Omega_k)$ since the same calculation as above and (1.3.3). Here, Ω_k depends on $k \in \mathbb{N}$. By zero extension from Ω_k to $\mathbb{R}^d$, we have $\{(w_k^\rho, w_{ik}^\rho, w_{ek}^\rho)\}_{k=1}^\infty$ is uniform bounded in the norm $\left(W^{2,p}(\mathbb{R}^d)\right)^3$. Thus we are able to take subsequences

$\{(w^{\rho}_{k_l}, w^{\rho}_{ik_l}, w^{\rho}_{ek_l})\}_{l=1}^{\infty}$ and $w, w_i, w_e \in W^{2,p}(\mathbb{R}^d)$ such that

$$\begin{pmatrix} w^{\rho}_{k_l} \\ w^{\rho}_{ik_l} \\ w^{\rho}_{ek_l} \end{pmatrix} \to \begin{pmatrix} w \\ w_i \\ w_e \end{pmatrix} \text{ in the norm } C^1(\mathbb{R}^d) \text{ as } l \to \infty,$$

by Rellich's compactness theorem [5]. Since

$$|w_{k_l}(0)| + |\nabla w_{k_l}(0)| + |\nabla w_{ik_l}(0)| + |\nabla w_{ek_l}(0)| > \frac{1}{2},$$

we get

$$|w(0)| + |\nabla w(0)| + |\nabla w_i(0)| + |\nabla w_e(0)| \geq \frac{1}{2}.$$

Case(3-ii) $D < \infty$

By $D = \liminf_{k\to\infty} d_k < \infty$, there exists a subsequence $\{d_{k_l}\}$ such that $\lim_{l\to\infty} d_{k_l} = D$. We abbreviated d_{k_l} and write d_k. We may assume that Ω_k tends to $\mathbb{R}^d_{+,D} := \{(x', x_d) \in \mathbb{R}^d \mid x_d > -D\}$ as $k \to \infty$ in C^2-sense up to translation and rotation. Indeed, since Ω is uniformly C^2, there is a unique nearest point of $z_k \in \partial\Omega$ from x_k for sufficiently large k (cf. [20]). Moreover, by rotation with respect to x_k, the domain Ω is represented locally near z_k as the domain occupied above the graph of a C^2 function γ_k of $d-1$ variable x' such that x_d-direction corresponds to the direction from z_k to x_k. This implies that $\nabla_{x'}\gamma_k(z'_k) = 0$, where $z_k = (z'_k, z_{kd}), x_k = (x'_k, x_{kd})$; see Definition 1.1. By translation we may assume that $x_{kd} = 0$. By uniformity of C^2-regularity we may assume that the size of neighborhood where the representation is valid and C^2-bound for γ_k can be taken independent of k. Under this setting Ω_k is represented locally near the origin as the domain occupied above the graph of $\tilde{\gamma}_k(x') = |\lambda_k|^{1/2}\gamma_k(x'_k + x'/|\lambda_k|^{1/2})$ and the size of region where the representation is valid is of order $|\lambda_k|^{-1/2}$. By definition this $\tilde{\gamma}_k$ converges locally uniformly in $\mathbb{R}^{d-1}$ to a constant function $x_d = -D$ up to second order derivatives as $k \to \infty$ since $\nabla_{x'}\tilde{\gamma}_k(0) = 0$ and $\tilde{\gamma}_k(0) = -|\lambda_k|^{1/2}(x_{kd} - z_{kd}) = -d_k \to -D$ as $k \to \infty$.

We would like to take a cut-off function similar to the Case(3-i). However, we need to keep the Neumann boundary condition. We refer the papers [1], [4], [35] to take ρ_k satisfying $\partial\rho_k/\partial n_k = 0$ so that it converges to some function ρ. Assume that $\rho \in C^2_0(\overline{\mathbb{R}^d_{+,D}})$ with $\operatorname{supp}\rho \subset B(0,R)$ and $\partial\rho/\partial x_d = 0$ on $x_d = -D$. Take a sequence of functions $\rho_k \in C^2_0(\overline{\Omega_k})$ such that $\partial\rho_k/\partial n_k = 0$ on $\partial\Omega_k$, $\operatorname{supp}\rho_k \subset B(0, 4R/3)$ and that ρ_k converges to ρ uniformly in $\Omega_k \cap \mathbb{R}^d_{+,D}$ up to second derivatives. Fortunately, such ρ_k exists as constructed in [4, p.27, Appendix B].

We argue in the same way as Case(3-i), we localize (w_k, w_{ik}, w_{ek}) by multiplying ρ_k. Note that the Neumann boundary condition is fulfilled for $(w^{\rho_k}_{ik}, w^{\rho_k}_{ek})$ thanks to the condition $\partial\rho_k/\partial n_k = 0$ on $\partial\Omega_k$, which yields $W^{2,p}(\Omega_k)$ a priori estimate.

Then we can take subsequences $\{(w_{k_l}^{\rho_{k_l}}, w_{ik_l}^{\rho_{k_l}}, w_{ek_l}^{\rho_{k_l}})\}_{l=1}^{\infty}$ and $w, w_i, w_e \in W^{2,p}(\overline{\mathbb{R}^d_{+,D}})$ such that

$$\begin{pmatrix} w_{k_l}^{\rho_{k_l}} \\ w_{ik_l}^{\rho_{k_l}} \\ w_{ek_l}^{\rho_{k_l}} \end{pmatrix} \to \begin{pmatrix} w \\ w_i \\ w_e \end{pmatrix} \text{ in the norm } C^1(\overline{\mathbb{R}^d_{+,D}}) \text{ as } l \to \infty.$$

As in Case (3-i), we get the same inequality.

In this step, we are able to conclude that $w \not\equiv 0$ and $w_{i,e}$ are not constants on some neighborhood near the origin.

Step 4 (Characterization of the limit)

Let us explain resolvent equations of w_{k_l}, w_{ik_l}, w_{ek_l} tend to the limit equation

$$\begin{cases} e^{i\theta_\infty} w - \nabla \cdot (\sigma_{i\infty} \nabla w_i) = 0 & \text{in } \Omega_\infty, \\ e^{i\theta_\infty} w + \nabla \cdot (\sigma_{e\infty} \nabla w_e) = 0 & \text{in } \Omega_\infty, \\ w = w_i - w_e & \text{in } \Omega_\infty, \\ \sigma_{i\infty} \nabla w_i \cdot n_\infty = 0, \ \sigma_{e\infty} \nabla w_e \cdot n_\infty = 0 & \text{on } \partial\Omega_\infty, \end{cases} \tag{1.3.5}$$

in the weak sense, where $\theta_\infty = \lim_{k\to\infty} \theta_k$, $\sigma_{i\infty}, \sigma_{e\infty}$ are constant coefficients matrices defined as below which satisfy uniform ellipticity condition and n_∞ is unit outer normal vector $(0, \cdots, 0, -1)$ when $\Omega_\infty = \mathbb{R}^d_{+,D}$. If $\Omega_\infty = \mathbb{R}^d$, we do not need to consider boundary conditions.

We have $w, w_i, w_e \in \bigcap_{n<p<\infty} W^{2,p}_{loc}(\overline{\Omega_\infty}) \cap W^{1,\infty}(\Omega_\infty)$ and

$$\begin{pmatrix} w_{k_l} \\ \nabla w_{ik_l} \\ \nabla w_{ek_l} \end{pmatrix} \to \begin{pmatrix} w \\ \nabla w_i \\ \nabla w_e \end{pmatrix} \text{ weak} * \text{ in } L^\infty(\Omega_\infty) \text{ as } l \to \infty$$

since $\sup_{x\in\Omega_k} (|w_k(x)| + |\nabla w_k(x)| + |\nabla w_{ik}(x)| + |\nabla w_{ek}(x)|) = 1.$

Case(4-i) $\Omega_\infty = \mathbb{R}^d$

Proposition 1.5. *The limit $w, w_i, w_e \in \bigcap_{n<p<\infty} W^{2,p}_{loc}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ satisfy that for any $\phi_{i,e} \in C_0^\infty(\mathbb{R}^d)$*

$$\begin{cases} e^{i\theta\infty}(w, \phi_i)_{L^2(\mathbb{R}^d)} + (\sigma_{i\infty} \nabla w_i, \nabla \phi_i)_{L^2(\mathbb{R}^d)} = 0, \\ e^{i\theta\infty}(w, \phi_e)_{L^2(\mathbb{R}^d)} - (\sigma_{e\infty} \nabla w_e, \nabla \phi_e)_{L^2(\mathbb{R}^d)} = 0, \\ w = w_i - w_e, \end{cases}$$

where $\theta_\infty = \lim_{k\to\infty} \theta_k$ and $\sigma_{i\infty}, \sigma_{e\infty}$ are constant coefficients matrices which satisfy uniform ellipticity condition. Here, $(\cdot, \cdot)_{L^2(\mathbb{R}^d)}$ denotes L^2-inner product.

Proof of Proposition 1.5. For each function $\eta \in C_0^\infty(\mathbb{R}^d)$, there exists $k_\eta \in \mathbb{N}$ such that $\operatorname{supp}\eta \subset \Omega_k$ for $k_\eta \leq k$. Since $\operatorname{supp}\eta$ is compact, there exist w_{k_l}, w_{ik_l}, w_{ek_l} such that

$$\begin{pmatrix} w_{k_l} \\ w_{ik_l} \\ w_{ek_l} \end{pmatrix} \to \begin{pmatrix} w \\ w_i \\ w_e \end{pmatrix} \text{ weakly on } W^{2,p}(\operatorname{supp}\eta) \text{ as } l \to \infty$$

for all $n < p < \infty$. Now we have to determine $\sigma_{i\infty}$ and $\sigma_{e\infty}$. For a matrix $A = \{a_{mn}\}_{1\leq m,n\leq d}$, set $\|A\| := \max_{1\leq m,n\leq d}|a_{mn}|$. Since σ_i is uniformly continuous, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\left|x_k + \frac{x}{|\lambda_k|^{1/2}} - x_k\right| = \left|\frac{x}{|\lambda_k|^{1/2}}\right| < \delta$ then $\left\|\sigma_i\left(x_k + \frac{x}{|\lambda_k|^{1/2}}\right) - \sigma_i(x_k)\right\| = \|\sigma_{ik}(x) - \sigma_{ik}(0)\| < \varepsilon$. We can take $k_0 \in \mathbb{N}$ such that $\left|\frac{x}{|\lambda_k|^{1/2}}\right| < \delta$ for $k_0 \leq k$ since $x \in \operatorname{supp}\eta$ and $|\lambda_k| \to \infty$. Since $\|\sigma_i(x_k)\| \leq \sup_{x\in\Omega}\|\sigma_i(x)\|$, there exists a subsequence $\{\sigma_{ik_l}\}_{l=1}^\infty$ and a constant matrix $\sigma_{i\infty}$ such that $\sigma_{ik_l}(0) = \sigma_i(x_{k_l}) \to \sigma_{i\infty}$ $(l \to \infty)$. Then for $k_0 \leq k$

$$\begin{aligned} \|\sigma_{ik_l}(x) - \sigma_{i\infty}\| &\leq \|\sigma_{ik_l}(x) - \sigma_{ik_l}(0)\| + \|\sigma_{ik_l}(0) - \sigma_{i\infty}\| \\ &\leq \varepsilon + \|\sigma_{ik_l}(0) - \sigma_{i\infty}\| \\ &\to \varepsilon \ (l \to \infty). \end{aligned}$$

Since $\varepsilon > 0$ and $x \in \operatorname{supp}\eta$ are arbitrary, we get $\|\sigma_{ik_l} - \sigma_{i\infty}\| \to 0$ $(l \to \infty)$. The above calculation is also valid for σ_e. Naturally, $\sigma_{i\infty}$ and $\sigma_{e\infty}$ are positive definite constant matrices.

We consider the weak formulation of the resolvent equation under oblique boundary condition. For any test functions $\phi_{i,e} \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{cases} e^{i\theta_{k_l}}(w_{k_l}, \phi_i)_{L^2(\Omega_{k_l})} + (\sigma_{ik_l}\nabla w_{ik_l}, \nabla\phi_i)_{L^2(\Omega_{k_l})} = (t_{k_l}, \phi_i)_{L^2(\Omega_{k_l})}, \\ e^{i\theta_{k_l}}(w_{k_l}, \phi_e)_{L^2(\Omega_{k_l})} - (\sigma_{ek_l}\nabla w_{ek_l}, \nabla\phi_e)_{L^2(\Omega_{k_l})} = (t_{k_l}, \phi_e)_{L^2(\Omega_{k_l})}, \\ w_{k_l} = w_{k_l} - w_{k_l}. \end{cases}$$

As $l \to \infty$,

$$\begin{cases} e^{i\theta\infty}(w, \phi_i)_{L^2(\mathbb{R}^d)} + (\sigma_{i\infty}\nabla w_i, \nabla\phi_i)_{L^2(\mathbb{R}^d)} = 0, \\ e^{i\theta\infty}(w, \phi_e)_{L^2(\mathbb{R}^d)} - (\sigma_{e\infty}\nabla w_e, \nabla\phi_e)_{L^2(\mathbb{R}^d)} = 0, \\ w = w_i - w_e. \end{cases}$$

□

Case(4-ii) $\Omega_\infty = \mathbb{R}^d_{+,D}$

Proposition 1.6. *The limit $w, w_i, w_e \in \bigcap_{n<p<\infty} W^{2,p}_{loc}(\overline{\mathbb{R}^d_{+,D}}) \cap W^{1,\infty}(\mathbb{R}^d_{+,D})$ satisfy that for any $\phi_{i,e} \in C_0^\infty(\mathbb{R}^d)|_{\mathbb{R}^d_{+,D}}$*

$$\begin{cases} e^{i\theta\infty}(w, \phi_i)_{L^2(\mathbb{R}^d_{+,D})} + (\sigma_{i\infty}\nabla w_i, \nabla\phi_i)_{L^2(\mathbb{R}^d_{+,D})} = 0, \\ e^{i\theta\infty}(w, \phi_e)_{L^2(\mathbb{R}^d_{+,D})} - (\sigma_{e\infty}\nabla w_e, \nabla\phi_e)_{L^2(\mathbb{R}^d_{+,D})} = 0, \\ w = w_i - w_e, \end{cases}$$

where $\theta_\infty = \lim_{k\to\infty}\theta_k$ *and* $\sigma_{i\infty}, \sigma_{e\infty}$ *are constant coefficients matrices which satisfy* (1.2.1) *and* (1.2.2).

We can prove this proposition by similar calculation to Case (4-i).

Step 5 (Uniqueness)

In this last step we prove that limit functions are unique. The method is to reduce existence of solution to dual problems and use the fundamental lemma of calculus of variation. In order to solve the dual problem we use the Fourier transform. In the half space case we extend to the whole space. However, we have to pay attention to the boundary condition. We overcome the difficulty by using the condition (1.2.2).

Case(5-i) $\Omega_\infty = \mathbb{R}^d$

Lemma 1.7. *Let* $w, w_i, w_e \in \bigcap_{n<p<\infty} W^{2,p}_{loc}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ *satisfy*

$$\begin{cases} e^{i\theta_\infty}(w,\phi_i)_{L^2(\mathbb{R}^d)} + (\sigma_{i\infty}\nabla w_i, \nabla\phi_i)_{L^2(\mathbb{R}^d)} = 0, \\ e^{i\theta_\infty}(w,\phi_e)_{L^2(\mathbb{R}^d)} - (\sigma_{e\infty}\nabla w_e, \nabla\phi_e)_{L^2(\mathbb{R}^d)} = 0, \\ w = w_i - w_e, \end{cases} \tag{1.3.6}$$

for all $\phi_{i,e} \in C_0^\infty(\mathbb{R}^d)$*, then* $w = 0$ *and* $w_i = w_e =$*constant.*

Proof of Lemma 1.7. Equations (1.3.6) implies the following equations

$$\begin{cases} \big(w_i, e^{i\theta_\infty}\phi_i - \nabla\cdot(\sigma_{i\infty}\nabla\phi_i)\big)_{L^2(\mathbb{R}^d)} - (w_e, e^{i\theta_\infty}\phi_i)_{L^2(\mathbb{R}^d)} = 0, \\ (w_i, e^{i\theta_\infty}\phi_e)_{L^2(\mathbb{R}^d)} - \big(w_e, e^{i\theta_\infty}\phi_e - \nabla\cdot(\sigma_{e\infty}\nabla\phi_e)\big)_{L^2(\mathbb{R}^d)} = 0, \end{cases}$$

$$\begin{aligned} \big(w_i, e^{i\theta_\infty}(\phi_i - \phi_e) - &\nabla\cdot(\sigma_{i\infty}\nabla\phi_i)\big)_{L^2(\mathbb{R}^d)} \\ &- \big(w_e, e^{i\theta_\infty}(\phi_i - \phi_e) + \nabla\cdot(\sigma_{e\infty}\nabla\phi_e)\big)_{L^2(\mathbb{R}^d)} = 0. \end{aligned}$$

We set $\mathcal{T} := \{(\phi_i,\phi_e) \in \mathcal{S}'(\mathbb{R}^d)^2 \mid \phi_i - \phi_e, \nabla\cdot(\sigma_{i\infty}\nabla\phi_i), \nabla\cdot(\sigma_{e\infty}\nabla\phi_e) \in L^1(\mathbb{R}^d)\}$. Here $\mathcal{S}'(\mathbb{R})$ is the space of all tempered distributions in the sense of L. Schwartz. Define a smooth cut-off function χ_R such that $\chi_R \equiv 1$ on $B(0,R/2)$ and $\chi_R \equiv 0$ on $B(0,R)^c$, and a Friedrich's mollifier F_R such that $\operatorname{supp} F_R \subset B(0,1/R)$. Since for any $(\phi_i,\phi_e) \in \mathcal{T}$ the sequences $\{((\chi_R\phi_i) * F_R, (\chi_R\phi_e) * F_R)\}_{R>0} \subset (C_0^\infty(\mathbb{R}^d))^2$ converges (ϕ_i,ϕ_e) in $\mathcal{T}$, we are able to take (ϕ_i,ϕ_e) in $\mathcal{T}$ as test functions. We consider the dual problem of the limit equation. For all $\psi_{i,e} \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d}(\psi_i - \psi_e)dx = 0$, we would like to find solutions $(\phi_i,\phi_e) \in \mathcal{T}$ such that

$$\begin{aligned} e^{i\theta_\infty}(\phi_i - \phi_e) - \nabla\cdot(\sigma_{i\infty}\nabla\phi_i) &= \psi_i && \text{in } \mathbb{R}^d, \\ e^{i\theta_\infty}(\phi_i - \phi_e) + \nabla\cdot(\sigma_{e\infty}\nabla\phi_e) &= \psi_e && \text{in } \mathbb{R}^d. \end{aligned}$$

We will prove the existence of these solutions in the next lemma. A key issue is whether $\phi_{i,e}$ satisfies a necessary decay condition as included in $\mathcal{T}$. For the moment, assume that we are able to take the solutions $\phi_{i,e}$. Then, we have that for all $\psi_{i,e} \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d}(\psi_i - \psi_e)dx = 0$,

$$(w_i, \psi_i)_{L^2(\mathbb{R}^d)} - (w_e, \psi_e)_{L^2(\mathbb{R}^d)} = 0.$$

Let $\psi_i = \psi_e$ then $(w, \psi_i)_{L^2(\mathbb{R}^d)} = 0$ for all $\psi_i \in C_0^\infty(\mathbb{R}^d)$. By fundamental lemma of calculus of variations, we get $w \equiv 0$. Let $\psi_e \equiv 0$ then $(w_i, \psi_i)_{L^2(\mathbb{R}^d)} = 0$ for all $\psi_i \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \psi_i dx = 0$. This means $w_i \equiv \text{constant}$. Obviously $w_e = w_i$ since $w = w_i - w_e$. □

Lemma 1.8. *For all $\psi_{i,e} \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d}(\psi_i - \psi_e)dx = 0$, there exist the solutions $(\phi_i, \phi_e) \in \mathcal{T}$ such that*

$$\begin{aligned} e^{i\theta_\infty}(\phi_i - \phi_e) - \nabla \cdot (\sigma_{i\infty} \nabla \phi_i) &= \psi_i && \text{in } \mathbb{R}^d, \\ e^{i\theta_\infty}(\phi_i - \phi_e) + \nabla \cdot (\sigma_{e\infty} \nabla \phi_e) &= \psi_e && \text{in } \mathbb{R}^d. \end{aligned}$$

Proof. We are able to solve these equations by the Fourier transform. We solve the following equations

$$\begin{cases} (e^{i\theta_\infty} + \langle \sigma_{i\infty}\xi, \xi\rangle)\mathcal{F}\phi_i - e^{i\theta_\infty}\mathcal{F}\phi_e = \mathcal{F}\psi_i \\ e^{i\theta_\infty}\mathcal{F}\phi_i - (e^{i\theta_\infty} + \langle \sigma_{e\infty}\xi, \xi\rangle)\mathcal{F}\phi_e = \mathcal{F}\psi_e, \end{cases}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse.

Solutions are formally of the form

$$\begin{aligned} \phi_i &= \mathcal{F}^{-1}\left(\frac{1}{e^{i\theta_\infty} + p(\xi)}\left(q_e(\xi)\mathcal{F}\psi_i + \frac{e^{i\theta_\infty}}{\langle(\sigma_{i\infty} + \sigma_{e\infty})\xi, \xi\rangle}(\mathcal{F}\psi_i - \mathcal{F}\psi_e)\right)\right) (\in \mathcal{S}'(\mathbb{R}^d)) \\ \phi_e &= \mathcal{F}^{-1}\left(\frac{-1}{e^{i\theta_\infty} + p(\xi)}\left(q_i(\xi)\mathcal{F}\psi_e - \frac{e^{i\theta_\infty}}{\langle(\sigma_{i\infty} + \sigma_{e\infty})\xi, \xi\rangle}(\mathcal{F}\psi_i - \mathcal{F}\psi_e)\right)\right) (\in \mathcal{S}'(\mathbb{R}^d)) \end{aligned}$$

and

$$\begin{aligned} \phi_i - \phi_e &= \mathcal{F}^{-1}\left(\frac{1}{e^{i\theta_\infty} + p(\xi)}\left(q_e(\xi)\mathcal{F}\psi_i + q_i(\xi)\mathcal{F}\psi_e\right)\right), \\ \nabla \cdot (\sigma_{i\infty}\nabla\phi_i) &= \mathcal{F}^{-1}\left(\frac{-\langle\sigma_{i\infty}\xi, \xi\rangle}{e^{i\theta_\infty} + p(\xi)}q_e(\xi)\mathcal{F}\psi_i - \frac{e^{i\theta_\infty}}{e^{i\theta_\infty} + p(\xi)}q_i(\xi)(\mathcal{F}\psi_i - \mathcal{F}\psi_e)\right), \\ \nabla \cdot (\sigma_{e\infty}\nabla\phi_e) &= \mathcal{F}^{-1}\left(\frac{\langle\sigma_{e\infty}\xi, \xi\rangle}{e^{i\theta_\infty} + p(\xi)}q_i(\xi)\mathcal{F}\psi_e - \frac{e^{i\theta_\infty}}{e^{i\theta_\infty} + p(\xi)}q_e(\xi)(\mathcal{F}\psi_i - \mathcal{F}\psi_e)\right), \end{aligned}$$

with

$$p(\xi) := \frac{\langle\sigma_{i\infty}\xi, \xi\rangle\langle\sigma_{e\infty}\xi, \xi\rangle}{\langle(\sigma_{i\infty} + \sigma_{e\infty})\xi, \xi\rangle},\ q_i(\xi) := \frac{\langle\sigma_{i\infty}\xi, \xi\rangle}{\langle(\sigma_{i\infty} + \sigma_{e\infty})\xi, \xi\rangle},\ q_e(\xi) := \frac{\langle\sigma_{e\infty}\xi, \xi\rangle}{\langle(\sigma_{i\infty} + \sigma_{e\infty})\xi, \xi\rangle}.$$

Let $q^{(k)}$ be a positively homogeneous function of degree k and $q(\xi) = q_i(\xi)$ or $q_e(\xi)$. We consider $\mathcal{M}_1\Psi_1 := \mathcal{F}^{-1}\left(\frac{1}{e^{i\theta_\infty}+p(\xi)}q^{(2)}(\xi)\mathcal{F}\Psi_1\right)$ for $\Psi_1 \in C_0^\infty(\mathbb{R}^d)$ and $\mathcal{M}_2\Psi_2 := \mathcal{F}^{-1}\left(\frac{1}{e^{i\theta_\infty}+p(\xi)}q(\xi)\mathcal{F}\Psi_2\right)$ for $\Psi_2 \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d}\Psi_2 dx = 0$. We notice that the solutions $\nabla \cdot (\sigma_{i\infty}\nabla\phi_i)$ and $\nabla \cdot (\sigma_{e\infty}\nabla\phi_e)$ are the sum of this form. We shall prove that

$\mathcal{M}_1\Psi_1$ and $\mathcal{M}_2\Psi_2$ belong L^1 space. Then we conclude that $\phi_i - \phi_e \in L^1(\mathbb{R}^d)$ since $e^{i\theta_\infty}(\phi_i - \phi_e) = \nabla\cdot(\sigma_{i\infty}\nabla\phi_i) + \psi_i \in L^1(\mathbb{R}^d)$.

We notice that $\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1} \in L^1(\mathbb{R}^d)$ whose proof is postponed in the appendix.

First, we prove $\mathcal{M}_1\Psi_1 \in L^1(\mathbb{R}^d)$ for any $\Psi_1 \in C_0^\infty(\mathbb{R}^d)$. Let $R > 0$ be $\operatorname{supp}\Psi_1 \subset R$. Then we have

$$
\begin{aligned}
&\|\mathcal{M}_1\Psi_1\|_{L^1(\mathbb{R}^d)}\\
\le&\|\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1} * \mathcal{F}^{-1}(q^{(2)}(\xi)\mathcal{F}\Psi_1)\|_{L^1(\mathbb{R}^d)}\\
\le&\|\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1}\|_{L^1(\mathbb{R}^d)}\cdot\\
&\qquad\Big(\|\mathcal{F}^{-1}(q^{(2)}(\xi)\mathcal{F}\Psi_1)\|_{L^1(B(0,R))} + \|\mathcal{F}^{-1}(q^{(2)}(\xi)\mathcal{F}\Psi_1)\|_{L^1(B(0,R)^c)}\Big)\\
\le&C\|\mathcal{F}^{-1}(q^{(2)}(\xi)\Psi_1)\|_{L^\infty(\mathbb{R}^d)} + C\left\|\int_{\mathbb{R}^d}\tilde{q}^{(-2-d)}(x-y)\Psi_1(y)dy\right\|_{L^1(B(0,R)^c)}\\
\le&C\|q^{(2)}(\xi)\Psi_1\|_{L^1(\mathbb{R}^d)} + C\int_{\operatorname{supp}\Psi_1}\|\tilde{q}^{(-2-d)}(x-y)\|_{L^1_x(B(0,R)^c)}|\Psi_1(y)|dy\\
\le&C\|q^{(2)}(\xi)\Psi_1\|_{L^1(\mathbb{R}^d)} + C\|\Psi_1\|_{L^1(\mathbb{R}^d)}\\
<&+\infty
\end{aligned}
$$

by Young's inequality, Fubini's theorem and $\Psi_1 \in C_0^\infty(\mathbb{R}^d)$ since $\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1} \in L^1(\mathbb{R}^d)$. Here $\tilde{q}^{(-2-d)}$ is the inverse Fourier transform of $q^{(2)}$, which is a positively homogeneous function of degree $-2-d$ observed by the following calculation: for $r > 0$,

$$
\tilde{q}^{(-2-d)}(rx) = \int_{\mathbb{R}^d} e^{irx\cdot\xi}q^{(2)}(\xi)d\xi = \int_{\mathbb{R}^d} e^{ix\cdot\eta}q^{(2)}(r^{-1}\eta)\eta^{-d}d\eta = r^{-2-d}\mathcal{F}^{-1}q^{(2)}(x)
$$

by a changing variable $r\xi = \eta$.

Second, we prove $\mathcal{M}_2\Psi_2 \in L^1(\mathbb{R}^d)$ for any $\Psi_2 \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d}\Psi_2 dx = 0$. We notice that $\mathcal{F}^{-1}(q^{(0)}(\xi)\mathcal{F}\Psi_2) \in L^1(\mathbb{R}^d)$ in the appendix. We thus conclude that $\mathcal{M}_2\Psi_2 \in L^1(\mathbb{R}^d)$ by applying Young's inequality with $\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1} \in L^1(\mathbb{R}^d)$:

$$
\|\mathcal{M}_2\Psi_2\|_{L^1(\mathbb{R}^d)} \le \|\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1}\|_{L^1(\mathbb{R}^d)}\|\mathcal{F}^{-1}(q^{(0)}(\xi)\mathcal{F}\Psi_2)\|_{L^1(\mathbb{R}^d)} < +\infty.
$$

Therefore from the L^1-finiteness of $\mathcal{M}_1\Psi_1$ and $\mathcal{M}_2\Psi_2$, we see $\phi_i - \psi_e, \nabla\cdot(\sigma_{i\infty}\nabla\phi_i), \nabla\cdot(\sigma_{e\infty}\nabla\phi_e) \in L^1(\mathbb{R}^d)$ i.e. $(\phi_i, \phi_e) \in \mathcal{T}$. □

Case(5-ii) $\Omega_\infty = \mathbb{R}^d_{+,D}$

It is enough to show the case $\mathbb{R}^d_+$ by changing variables.

Lemma 1.9. *Let* $w, w_i, w_e \in \bigcap_{n<p<\infty} W^{2,p}_{loc}(\overline{\mathbb{R}^d_+}) \cap W^{1,\infty}(\mathbb{R}^d_+)$ *satisfy*

$$
\begin{cases}
e^{i\theta\infty}(w, \phi_i)_{L^2(\mathbb{R}^d_+)} + (\sigma_{i\infty}\nabla w_i, \nabla\phi_i)_{L^2(\mathbb{R}^d_+)} = 0,\\
e^{i\theta\infty}(w, \phi_e)_{L^2(\mathbb{R}^d)} - (\sigma_{e\infty}\nabla w_e, \nabla\phi_e)_{L^2(\mathbb{R}^d_+)} = 0,\\
w = w_i - w_e,
\end{cases}
\tag{1.3.7}
$$

for all $\phi_{i,e} \in C_0^\infty(\mathbb{R}^d)|_{\mathbb{R}^d_+}$, *then* $w = 0$ *and* $w_i = w_e$ *=constant.*

Proof of lemma 1.9. Equations (1.3.7) implies the following equations;

$$\begin{cases} \big(w_i, e^{i\theta_\infty}\phi_i - \nabla\cdot(\sigma_{i\infty}\nabla\phi_i)\big)_{L^2(\mathbb{R}^d_+)} - (w_e, e^{i\theta_\infty}\phi_i)_{L^2(\mathbb{R}^d_+)} = 0, \\ (w_i, e^{i\theta_\infty}\phi_e)_{L^2(\mathbb{R}^d_+)} - \big(w_e, e^{i\theta_\infty}\phi_e - \nabla\cdot(\sigma_{e\infty}\nabla\phi_e)\big)_{L^2(\mathbb{R}^d_+)} = 0, \end{cases}$$

for all $\phi_i \in C_0^\infty(\mathbb{R}^d)|_{\mathbb{R}^d_+}$ such that $\sigma_{i\infty}\nabla\phi_i\cdot n_\infty = 0$ and $\phi_e \in C_0^\infty(\mathbb{R}^d)|_{\mathbb{R}^d_+}$ such that $\sigma_{e\infty}\nabla\phi_e\cdot n_\infty = 0$, and

$$\begin{aligned} \big(w_i, e^{i\theta_\infty}(\phi_i - \phi_e) - &\nabla\cdot(\sigma_{i\infty}\nabla\phi_i)\big)_{L^2(\mathbb{R}^d_+)} \\ &- \big(w_e, e^{i\theta_\infty}(\phi_i - \phi_e) + \nabla\cdot(\sigma_{e\infty}\nabla\phi_e)\big)_{L^2(\mathbb{R}^d_+)} = 0. \end{aligned}$$

The problem can be reduced to the whole space. Let $Ew_{i,e}$ be an even extension to the whole space $\mathbb{R}^d$, i.e.

$$Ew_{i,e}(x) := \begin{cases} w_{i,e}(x', x_d) & (x_d \geq 0) \\ w_{i,e}(x', -x_d) & (x_d < 0). \end{cases}$$

Matrices $\sigma_{i\infty}$ and $\sigma_{e\infty}$ are constant so we extend these to whole space $\mathbb{R}^d$, which we simply write by $\sigma_{i\infty}$ and $\sigma_{e\infty}$. Since $\sigma_{i\infty}\nabla w_i\cdot n_\infty = \nabla w_i\cdot n_\infty = 0$, $\sigma_{e\infty}\nabla w_e\cdot n_\infty = \nabla w_e\cdot n_\infty = 0$ and $w_{i,e} \in \bigcap_{n<p<\infty} W^{2,p}_{loc}(\overline{\mathbb{R}^d_+}) \cap W^{1,\infty}(\mathbb{R}^d_+)$, we have $Ew_{i,e} \in \bigcap_{n<p<\infty} W^{2,p}_{loc}(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$. For arbitrary $\varphi_{i,e} \in C_0^\infty(\mathbb{R}^d)$, let $\varphi_{i,e}^{\text{even}}$ and $\varphi_{i,e}^{\text{odd}}$ be the even and odd parts of $\varphi_{i,e}$, i.e.

$$\begin{aligned} \varphi_{i,e}^{\text{even}}(x) &:= \frac{\varphi_{i,e}(x', x_d) + \varphi_{i,e}(x', -x_d)}{2}, \\ \varphi_{i,e}^{\text{odd}}(x) &:= \frac{\varphi_{i,e}(x', x_d) - \varphi_{i,e}(x', -x_d)}{2}. \end{aligned}$$

For simplicity, set a linear operator $L_i(\varphi_i, \varphi_e) := e^{i\theta_\infty}(\varphi_i - \varphi_e) - \nabla\cdot(\sigma_{i\infty}\nabla\varphi_i)$ and $L_e(\varphi_i, \varphi_e) := e^{i\theta_\infty}(\varphi_i - \varphi_e) + \nabla\cdot(\sigma_{e\infty}\nabla\varphi_e)$. From the assumption of σ_i, note that $\sigma_{i\infty}$ have the form of $\sigma_{i\infty} = \begin{pmatrix} \tilde{\sigma}_{i\infty} & 0 \\ 0 & \tau_i \end{pmatrix}$ for some constant $(d-1)\times(d-1)$ matrix $\tilde{\sigma}_{i\infty}$ and $\tau_i > 0$ because $(0, \cdots, 0, -1)$ is eigenvector of $\sigma_{i\infty}$. So we have that $L_i(\varphi_i^{\text{even}}, \varphi_e^{\text{even}})$ is even function and $L_i(\varphi_e^{\text{odd}}, \varphi_e^{\text{odd}})$ is odd function. Consider L_e same as L_i. Naturally, L_e also has the same property. Then we have

$$\begin{aligned} &(Ew_i, L_i(\varphi_i, \varphi_e))_{L^2(\mathbb{R}^d)} - (Ew_e, L_e(\varphi_i, \varphi_e))_{L^2(\mathbb{R}^d)} \\ =&(Ew_i, L_i(\varphi_i^{\text{even}}, \varphi_e^{\text{even}}))_{L^2(\mathbb{R}^d)} + (Ew_i, L_i(\varphi_i^{\text{odd}}, \varphi_e^{\text{odd}}))_{L^2(\mathbb{R}^d)} \\ &- (Ew_e, L_e(\varphi_i^{\text{even}}, \varphi_e^{\text{even}}))_{L^2(\mathbb{R}^d)} - (Ew_e, L_e(\varphi_i^{\text{odd}}, \varphi_e^{\text{odd}}))_{L^2(\mathbb{R}^d)} \\ =&(Ew_i, L_i(\varphi_i^{\text{even}}, \varphi_e^{\text{even}}))_{L^2(\mathbb{R}^d)} - (Ew_e, L_e(\varphi_i^{\text{even}}, \varphi_e^{\text{even}}))_{L^2(\mathbb{R}^d)} \\ =&2\left\{(w_i, L_i(\varphi_i^{\text{even}}, \varphi_e^{\text{even}}))_{L^2(\mathbb{R}^d_+)} - (w_e, L_e(\varphi_i^{\text{even}}, \varphi_e^{\text{even}}))_{L^2(\mathbb{R}^d_+)}\right\}. \end{aligned}$$

The function φ_i^{even} satisfies $\sigma_{i\infty}\nabla\varphi_i^{\text{even}} \cdot n_\infty = \nabla\varphi_i^{\text{even}} \cdot n_\infty = 0$. Function φ_e^{even} also satisfies same boundary condition. Since the last term of above calculation equals to zero, we conclude that for any $\varphi_{i,e} \in C_0^\infty(\mathbb{R}^d)$

$$(Ew_i, L_i(\varphi_i, \varphi_e))_{L^2(\mathbb{R}^d)} - (Ew_e, L_e(\varphi_i, \varphi_e))_{L^2(\mathbb{R}^d)} = 0.$$

This means $Ew_i = Ew_e = \text{constant}$ by the Case(4-i). Therefore we have $w = 0$ and $w_i = w_e = \text{constant}$. □

Results of Step 3 and Step 5 are contradictory, so the proof of Theorem 1.3 is now complete. □

1.4 Bidomain operators

1.4.1 Definition of bidomain operators in L^p spaces

In this subsection we define bidomain operators in L^p spaces for $1 < p < \infty$. To avoid technical difficulties we assume that Ω is a bounded C^2-domain. We reformulate resolvent equations corresponding to the parabolic and elliptic system as are derived in [11]. The new system contains only u and u_e as unknown functions. Since $u_i = u + u_e$ by (1.1.4), the new system is of the form:

(1.4.1) $$\lambda u - \nabla \cdot (\sigma_i \nabla u) - \nabla \cdot (\sigma_i \nabla u_e) = s \quad \text{in } \Omega,$$

(1.4.2) $$-\nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e)\nabla u_e) = 0 \quad \text{in } \Omega,$$

(1.4.3) $$\sigma_i \nabla u \cdot n + \sigma_i \nabla u_e \cdot n = 0 \quad \text{on } \partial\Omega,$$

(1.4.4) $$\sigma_i \nabla u \cdot n + (\sigma_i + \sigma_e)\nabla u_e \cdot n = 0 \quad \text{on } \partial\Omega.$$

Let $1 < p < \infty$ and Ω be a bounded C^2-domain. Set $L^p_{av}(\Omega) := \{u \in L^p(\Omega) \mid \int_\Omega u dx = 0\}$ and the operator P_{av} defined by $P_{av}u := u - \frac{1}{|\Omega|}\int_\Omega u dx$, which is the orthogonal projection. Evidently, $L^p_{av}(\Omega)$ is a closed subspace in $L^p(\Omega)$ and P_{av} is a bounded linear operator on $L^p(\Omega)$. We similarly define a function space $W^{2,p}_{av}(\Omega)$, i.e. $W^{2,p}_{av}(\Omega) = W^{2,p}(\Omega) \cap L^p_{av}(\Omega)$. We define an operator $A_{i,e}$ in $L^p_{av}(\Omega)$ with the domain $D(A_{i,e})$ corresponding to a uniformly elliptic operator $-\nabla \cdot (\sigma_{i,e}\nabla \cdot)$ with the oblique boundary condition. It is explicitly defined as

$$u \in D(A_{i,e}) := \left\{u \in W^{2,p}_{av}(\Omega) \mid \sigma_{i,e}\nabla u \cdot n = 0 \text{ a.e. in } \partial\Omega\right\} \subset L^p_{av}(\Omega),$$
$$A_{i,e}u := -\nabla \cdot (\sigma_{i,e}\nabla u).$$

Lemma 1.10 ([32]). *Let $1 < p < \infty$ and let Ω be a bounded C^2-domain. Assume that $\sigma_{i,e} \in C^1(\overline{\Omega})$ satisfies* (1.2.1). *Then the operator A_i is densely defined closed linear operator on $L^p_{av}(\Omega)$ and for any $f \in L^p_{av}(\Omega)$ there uniquely exists $u \in D(A_i)$ such that $A_i u = f$. The operator A_e also has the same property.*

If we assume that $\sigma_{i,e}\nabla u \cdot n = 0$ is equivalent to $\nabla u \cdot n = 0$, then $D(A_i) = \left\{u \in W^{2,p}_{av}(\Omega) \mid \nabla u \cdot n = 0 \text{ a.e. in } \partial\Omega\right\} = D(A_e)$. So we are able to define the operator $A_i + A_e$ with the domain $D(A_i)(= D(A_e))$ and we observe that inverse operator

$(A_i + A_e)^{-1}$ on L^p_{av} is a bounded linear operator. Under $\int_\Omega u_e dx = 0$, which is often used assumption to study bidomain equations, from (1.4.2),

$$\begin{aligned} &A_i P_{av} u + (A_i + A_e) u_e = 0 \\ \Leftrightarrow &(A_i + A_e) u_e = -A_i P_{av} u \quad (\in L^p_{av}(\Omega)) \\ \Leftrightarrow &u_e = -(A_i + A_e)^{-1} A_i P_{av} u \ (\in D(A_i)). \end{aligned}$$

We substitute this into (1.4.1) to set

$$\begin{aligned} &\lambda u + A_i P_{av} u - A_i (A_i + A_e)^{-1} A_i P_{av} u = s \\ \Leftrightarrow &\lambda u + A_i (A_i + A_e)^{-1} A_e P_{av} u = s. \end{aligned}$$

We are ready to define bidomain operators A.

Definition 1.11 ([11, Definition 12($p = 2$)])**.** For $1 < p < \infty$, we define the bidomain operator $A : D(A) := \{u \in W^{2,p}(\Omega) \mid \nabla u \cdot n = 0 \text{ a.e. in } \partial\Omega\} \subset L^p(\Omega) \to L^p(\Omega)$ by

$$A = A_i (A_i + A_e)^{-1} A_e P_{av}. \tag{1.4.5}$$

Under $\int_\Omega u_e dx = 0$, equations (1.4.1)-(1.4.4) for the function u can be written in a single resolvent equation of the form

$$(\lambda + A) u = s \quad \text{in } \Omega. \tag{1.4.6}$$

Once we solve this equation, we are able to derive $u_e = -(A_i + A_e)^{-1} A_i P_{av} u$.

1.4.2 Resolvent set of bidomain operators

We study existence and uniqueness of the solution for bidomain equations (1.4.6). We derive $W^{2,p}$ a priori estimate for fixed λ by $W^{2,p}$ a priori estimate for the usual elliptic operator A_e. To define the bidomain operator A, we now assume that Ω is a bounded C^2-domain and $\sigma_{i,e} \in C^1(\overline{\Omega})$ satisfy (1.2.1) and (1.2.2), which will be used throughout.

Theorem 1.12 (A priori estimate for bidomain operators)**.** *Let $1 < p < \infty$. For each $\lambda \in \Sigma_{\pi,0}$ there exists $C_\lambda > 0$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \le C_\lambda \left(\|(\lambda + A)u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right)$$

for all $u \in D(A)$.

Proof. We operate $(A_i + A_e) A_i^{-1} P_{av}$ to $(\lambda + A)u = s$ to get $(\lambda + A_e) P_{av} u = (A_i + A_e) A_i^{-1} P_{av} s - \lambda A_e A_i^{-1} P_{av} u$. Since A_e has a resolvent estimate [36], for each $\varepsilon \in (0, \pi/2)$ there exists $C > 0$ such that

$$\begin{aligned} &|\lambda| \|P_{av} u\|_{L^p(\Omega)} + |\lambda|^{1/2} \|\nabla P_{av} u\|_{L^p(\Omega)} + \|\nabla^2 P_{av} u\|_{L^p(\Omega)} \\ &\le C \|(A_i + A_e) A_i^{-1} P_{av} s - \lambda A_e A_i^{-1} P_{av} u\|_{L^p(\Omega)} \\ &\le C \|s\|_{L^p(\Omega)} + C |\lambda| \|u\|_{L^p(\Omega)} \cdots (**) \end{aligned}$$

for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$. Here, note that $(A_i + A_e)A_i^{-1}P_{av}$ and $A_e A_i^{-1}P_{av}$ are bounded operators in $L^p(\Omega)$. From above inequality we have for any $\lambda \in \Sigma_{\pi,0}$ there exists $C_\lambda > 0$ independent of u (may depend on λ) such that

$$\|u\|_{W^{2,p}(\Omega)} \leq C_\lambda \left(\|(\lambda + A)u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right).$$

□

By this theorem we observe that the bidomain operator A in L^p spaces is a densely defined closed linear operator.

Let A_p be the bidomain operator in L^p spaces. We characterize the resolvent set of bidomain operator A_p in L^p spaces from the previous result [11] that the bidomain operator A_2 is non-negative self-adjoint operator in L^2 spaces, i.e. $\Sigma_{\pi,0} \subset \rho(-A_2)$.

Lemma 1.13. *Let $1 < p < \infty$. Let $\lambda \in \Sigma_{\pi,0}$. Assume that $(\lambda + A_p)u = 0$ implies $u = 0$, then the inequality $\|u\|_{W^{2,p}(\Omega)} \leq C_\lambda \|(\lambda + A_p)u\|_{L^p(\Omega)}$ holds, where $C_\lambda > 0$ is the constant independent of $u \in D(A_p)$.*

Proof. We argue by contradiction. If the inequality were false, there would exist a sequence $\{u_k\}_{k=1}^\infty \subset D(A_p)$ satisfying

$$\|u_k\|_{W^{2,p}(\Omega)} = 1, \ \ \|(\lambda + A_p)u_k\|_{L^p(\Omega)} < 1/k.$$

By the compactness of the imbedding $W^{2,p}(\Omega) \to W^{1,p}(\Omega)$ (Rellich's compactness theorem), there exists a subsequence $\{u_{k_l}\}_{l=1}^\infty$ converging strongly in $W^{1,p}(\Omega)$ to a function $u \in D(A_p)$. Define $\tilde{u}_{k_l} = (A_i + A_e)^{-1}A_e P_{av} u_{k_l}$, $\tilde{u} = (A_i + A_e)^{-1}A_e P_{av} u$ and the conjugate exponent p' of p, $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$. We have $\{\tilde{u}_{k_l}\}_{l=1}^\infty$ are uniform bounded in $W^{2,p}(\Omega)$ converging to a function $\tilde{u} \in D(A_p)$. Since

$$\int_\Omega \lambda u_{k_l} v + \sigma_i \nabla \tilde{u}_{k_l} \cdot \nabla v \to \int_\Omega \lambda u v + \sigma_i \nabla \tilde{u} \cdot \nabla v$$

for all $v \in L^{p'}(\Omega)$, we must have $\int_\Omega \lambda u v + \sigma_i \nabla \tilde{u} \cdot \nabla v = 0$ for all $v \in L^{p'}(\Omega)$. Hence $(\lambda + A_p)u = 0$. The uniqueness implies $u = 0$. However, the estimate in Theorem 1.12 implies

$$1 = \|u_{k_l}\|_{W^{2,p}(\Omega)} \leq C \left(\|(\lambda + A_p)u_{k_l}\|_{L^p(\Omega)} + \|u_{k_l}\|_{L^p(\Omega)} \right).$$

Sending $l \to \infty$ implies $1 \leq C \liminf_{k \to \infty} \|u_{k_l}\|_{L^p(\Omega)}$. This would contradict that $u_{k_l} \to u = 0$ strongly in $W^{1,p}(\Omega)$. □

Theorem 1.14. *Let $2 \leq p < \infty$. Then for any $\lambda \in \Sigma_{\pi,0}$ and $s \in L^p(\Omega)$, there uniquely exists $u \in D(A_p)$ such that $(\lambda + A_p)u = s$.*

Proof. If $\lambda \in \Sigma_{\pi,0}$ and $u \in D(A_p)$ satisfy $(\lambda + A_p)u = 0$ then $u = 0$ since $u \in D(A_p) \subset D(A_2)$ and $\lambda \in \rho(-A_2)$. For existence of a solution to a bidomain equation we use the continuity method [20]. For each $t \in [0, 1]$ we set

$$L_t := \lambda + A_i(tA_i + A_e)^{-1}A_e P_{av} : D(A_p) \to L^p(\Omega).$$

By Lemma 1.13 we see there is a constant $C_\lambda > 0$ such that $\|u\|_{W^{2,p}(\Omega)} \leq C_\lambda \|L_t u\|_{L^p(\Omega)}$ for all $u \in D(A_p)$ and $t \in [0,1]$. Suppose that $L_{\tilde{t}} : D(A_p) \to L^p(\Omega)$ is onto for some $\tilde{t} \in [0,1]$, then $L_{\tilde{t}}$ is one-to-one. Hence there exists inverse mapping $L_{\tilde{t}}^{-1} : L^p(\Omega) \to D(A_p)$. For $t \in [0,1]$ and $s \in L^p(\Omega)$, the equation $L_t u = s$ is equivalent to the equation

$$\begin{aligned}
L_t u &= s \\
L_{\tilde{t}} u &= s + (L_{\tilde{t}} - L_t)u \\
&= s + (t-\tilde{t})A_i(\tilde{t}A_i + A_e)^{-1}A_i(tA_i + A_e)^{-1}A_e P_{av} u \\
u &= L_{\tilde{t}}^{-1}\{s + (t-\tilde{t})A_i(\tilde{t}A_i + A_e)^{-1}A_i(tA_i + A_e)^{-1}A_e P_{av} u\}.
\end{aligned}$$

Set the mapping $T : D(A_p) \to D(A_p)$ and $\delta > 0$ of the form

$$\begin{aligned}
Tu &= L_{\tilde{t}}^{-1}\{s + (t-\tilde{t})A_i(\tilde{t}A_i + A_e)^{-1}A_i(tA_i + A_e)^{-1}A_e P_{av} u\}, \\
\delta &= \left\{ \sup_{t,\tilde{t}\in[0,1]} \|L_{\tilde{t}}^{-1}\{A_i(\tilde{t}A_i + A_e)^{-1}A_i(tA_i + A_e)^{-1}A_e P_{av}\|_{\mathcal{L}(W^{2,p}(\Omega))} \right\}^{-1}.
\end{aligned}$$

The mapping T is a contraction mapping if $|t-\tilde{t}| < \delta$ and hence the mapping $L_t : D(A_p) \to L^p(\Omega)$ is onto for all $t \in [0,1]$ satisfying $|t-\tilde{t}| < \delta$ because of δ is independent of $t, \tilde{t}$. By dividing the interval $[0,1]$ into subintervals of length less than δ, we see that the mapping L_t is onto for all $t \in [0,1]$ because of $L_0 = \lambda + A_i P_{av} : D(A_p) \to L^p(\Omega)$ is onto when $\lambda \in \Sigma_{\pi,0}$. □

Lemma 1.15. *Let $1 < p < \infty$. The adjoint of the bidomain operator A_p is $A_{p'}$.*

Proof. Let $u \in D(A_p)$, $v \in D(A_{p'})$ and $2 \leq p < \infty$. For simplicity, we write $\langle \cdot,\cdot \rangle := \langle \cdot,\cdot \rangle_{L^p(\Omega)\times L^{p'}(\Omega)}$.

$$\begin{aligned}
&\langle A_p u, v\rangle \\
&= \langle A_i P_{av} u - A_i(A_i + A_e)^{-1}A_i P_{av} u, v\rangle \\
&= \langle A_i P_{av} u - A_i(A_i + A_e)^{-1}A_i P_{av} u, v\rangle \\
&\qquad - \langle A_i P_{av} u - (A_i + A_e)(A_i + A_e)^{-1}A_i P_{av} u, (A_i + A_e)^{-1}A_i P_{av} v\rangle \\
&= \langle A_i P_{av} u - A_i(A_i + A_e)^{-1}A_i P_{av} u, v - (A_i + A_e)^{-1}A_i P_{av} v\rangle \\
&\qquad + \langle A_e(A_i + A_e)^{-1}A_i P_{av} u, (A_i + A_e)^{-1}A_i P_{av} v\rangle \\
&= \langle u - (A_i + A_e)^{-1}A_i P_{av} u, A_i P_{av} v - A_i(A_i + A_e)A_i P_{av} v\rangle \\
&\qquad + \langle (A_i + A_e)^{-1}A_i P_{av} u, A_e(A_i + A_e)^{-1}A_i P_{av} v\rangle \\
&= \langle u, A_{p'} v\rangle \\
&\quad - \langle (A_i + A_e)^{-1}A_i P_{av} u, A_i P_{av} v - A_i(A_i + A_e)^{-1}A_i P_{av} v - A_e(A_i + A_e)^{-1}A_i P_{av} v\rangle \\
&= \langle u, A_{p'} v\rangle.
\end{aligned}$$

So we get $A_p \subset A_{p'}^*$. In order to show $D(A_p) \supset D(A_{p'}^*)$, we first show that $\lambda \in \rho(-A_p)$ implies $\lambda \in \rho(-A_{p'}^*)$. Remark that $D(A_2) \subset D(A_{p'})$ and $A_{p'} u = A_2 u$ $(u \in D(A_2))$. For $\lambda \in \rho(-A_p)$, $(\lambda + A_{p'})D(A_{p'}) \supset (\lambda + A_{p'})D(A_2) = (\lambda + A_2)D(A_2) = L^2(\Omega)$. So $R\left(\lambda + A_{p'}\right)$ is dense in $L^{p'}(\Omega)$. Therefore $\lambda + A_{p'}^*$ is one-to-one in $L^p(\Omega)$. Since $A_p \subset A_{p'}^*$ and $\lambda + A_p$ is surjection in $L^p(\Omega)$, we get $\lambda + A_{p'}^*$ is surjection. This means $\lambda \in \rho(-A_{p'}^*)$.

Take $u \in D(A^*_{p'})$ and for some $\lambda \in \rho(-A_p) \cap \rho(-A^*_{p'}) \neq \emptyset$, then

$$\begin{aligned} v &:= (\lambda + A_p)^{-1}(\lambda + A^*_{p'})u \in D(A_p) \\ (\lambda + A_p)v &= (\lambda + A^*_{p'})u \\ (\lambda + A^*_{p'})v &= (\lambda + A^*_{p'})u \\ v &= u. \end{aligned}$$

Therefore $D(A^*_{p'}) \subset D(A_p)$ and $A^*_{p'} = A_p$. Since $A_{p'}$ is a closed linear operator, we have $A_{p'} = A^{**}_{p'} = A^*_p$. This means for all $1 < p < \infty$ the adjoint of the bidomain operator A_p is $A_{p'}$. □

So we have for all $1 < p < \infty$, $\rho(-A_p) = \rho(-A^*_p) = \rho(-A_{p'}) = \Sigma_{\pi,0}$.

Our Theorem 1.14 implies existence and uniqueness of the resolvent bidomain equation since it is equivalent to the equation (1.4.6).

Theorem 1.16 (Existence and Uniqueness). *Let $1 < p < \infty$, Ω be a bounded C^2-domain and $\sigma_{i,e} \in C^1(\overline{\Omega}, \mathbb{S}^d)$ satisfy* (1.2.1) *and* (1.2.2). *Then for any $\lambda \in \Sigma_{\pi,0}$, $s \in L^p(\Omega)$, the resolvent problem*

$$\begin{cases} \lambda u - \nabla \cdot (\sigma_i \nabla u_i) = s & \text{in } \Omega, \\ \lambda u + \nabla \cdot (\sigma_e \nabla u_e) = s & \text{in } \Omega, \\ u = u_i - u_e & \text{in } \Omega, \\ \sigma_i \nabla u_i \cdot n = 0, \ \sigma_e \nabla u_e \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u, u_{i,e} \in W^{2,p}(\Omega)$ satisfying $\int_\Omega u_e dx = 0$.

1.4.3 Analyticity of semigroup generated by bidomain operators

We will study bidomain equations in the framework of an analytic semigroup, so let us recall the definition of a sectorial operator. Let X be a complex Banach space and $A : D(A) \subset X \to X$ be a linear operator, may not have a dense domain.

Definition 1.17. The operator A is said to be a sectorial operator with angle $\theta (\in [0, \pi/2)$ if for each $\varepsilon \in (0, \pi/2)$ there exist $C > 0$ and $M \geq 0$ such that

$$(1)\ \rho(-A) \supset \Sigma_{\pi-\theta,M}, \quad (2) \sup_{\lambda \in \Sigma_{\pi-\theta-\varepsilon,M}} |\lambda| \|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C.$$

We do not assume that the operator A has a dense domain. So it may happen that the analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ generated by the operator A may not be strongly continuous, that is for each $x \in X$ the function $t \mapsto e^{-tA}x$ is not necessarily continuous on $[0, \infty)$. We call $\{e^{-tA}\}_{t \geq 0}$ C_0-analytic semigroup if for each $x \in X$, $t \mapsto e^{-tA}x$ is continuous on $[0, \infty)$. We have that if the operator A is a sectorial operator with angle θ, then $t \mapsto e^{-tA}$ is analytic in $[0, \infty)$ and it can be extended holomorphically in a sector with opening angle $2(\pi/2 - \theta)$. For sectorial operators, it is known that

$$\{e^{-tA}\}_{t \geq 0} : \text{strongly continuous} \Leftrightarrow \forall x \in X, \lim_{t \to 0} e^{-tA}x = x \Leftrightarrow \overline{D(A)} = X.$$

Therefore, $\{e^{-tA}\}_{t\geq 0}$ is C_0-analytic semigroup if and only if the operator A is a sectorial operator with dense domain $D(A)$ in X (See [24]).

Let us go back to consider bidomain operators. Note that [11] showed the bidomain operator A is a non-negative self-adjoint operator in $L^2(\Omega)$ so that it is a sectorial operator. Namely, $\rho(-A_2) \supset \Sigma_{\pi,0}$ and for each $\varepsilon \in (0, \pi/2)$ there exists $C > 0$ such that

$$\sup_{\lambda\in\Sigma_{\pi-\varepsilon,0}} |\lambda|\|u\|_{L^2(\Omega)} \leq C\|s\|_{L^2(\Omega)}$$

for all $s \in L^2(\Omega)$. We derived an L^∞ resolvent estimate (Theorem 1.3); for each $\varepsilon \in (0, \pi/2)$ there exist $C > 0$ and $M \geq 0$ such that $\rho(-A) \supset \Sigma_{\pi,M}$

$$\sup_{\lambda\in\Sigma_{\pi-\varepsilon,M}} |\lambda|\|u\|_{L^\infty(\Omega)} \leq C\|s\|_{L^\infty(\Omega)}$$

and for all $s \in L^\infty(\Omega)$.

By using Riesz–Thorin interpolation theorem, we are able to derive an L^p resolvent estimate, i.e. for each $\varepsilon \in (0, \pi/2)$ and $2 \leq p \leq \infty$ there exist $C > 0$ and $M \geq 0$ such that $\rho(-A_p) \supset \Sigma_{\pi,M}$ and that

$$\sup_{\lambda\in\Sigma_{\pi-\varepsilon,M}} |\lambda|\|u\|_{L^p(\Omega)} \leq C\|s\|_{L^p(\Omega)}$$

and for all $s \in L^p(\Omega)$.

For $2 \leq p < \infty$ and its conjugate exponent $p'(\in (1, 2])$, we have

$$\|(\lambda + A_{p'})^{-1}\|_{\mathcal{L}(L^{p'}(\Omega))} = \|((\lambda + A_p)^{-1})^*\|_{\mathcal{L}(L^{p'}(\Omega))} = \|(\lambda + A_p)^{-1}\|_{\mathcal{L}(L^p(\Omega))} \leq \frac{C}{|\lambda|}.$$

We derived the resolvent estimate for bidomain operators $-A_p$ in L^p spaces for the sufficiently large λ. However, in the next theorem, we estimate the resolvent for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$ and higher order derivatives $\|\nabla u\|_{L^p(\Omega)}$ and $\|\nabla^2 u\|_{L^p(\Omega)}$, which is similar to an elliptic operator in L^p spaces.

Theorem 1.18 (L^p resolvent estimates for bidomain operators)**.** *Let $1 < p < \infty$. For each $\varepsilon \in (0, \pi/2)$ there exists $C > 0$ depending only on ε such that the unique solution $u \in D(A_p)$ of the resolvent equation $(\lambda + A_p)u = s$ satisfies*

$$|\lambda|\|u\|_{L^p(\Omega)} + |\lambda|^{1/2}\|\nabla u\|_{L^p(\Omega)} + \|\nabla^2 u\|_{L^p(\Omega)} \leq C\|s\|_{L^p(\Omega)}$$

for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$ and $s \in L^p(\Omega)$.

Proof. We divide the resolvent estimate $(\lambda + A_p)u = s$ into $(\lambda + A_p)u_1 = P_{av}s$ and $(\lambda + A_p)u_2 = s - P_{av}s$. Note that $u = u_1 + u_2$, $P_{av}s \in L^p_{av}(\Omega)$, $s - P_{av}s$ is a constant and the origin 0 belongs to $\rho(-A_p|_{L^p_{av}(\Omega)})$. For each $\varepsilon \in (0, \pi/2)$ we fix $M \geq 0$ which is the constant in the above explanation. Since $(\lambda + A_p)^{-1}P_{av}s = (\lambda + A_p|_{L^p_{av}(\Omega)})^{-1}P_{av}s$ and the resolvent operator $(\lambda + A_p|_{L^p_{av}(\Omega)})^{-1}$ is uniform bounded in a compact subset $\overline{\Sigma_{\pi-\varepsilon,0}} \cap \overline{B(0, 2M)}$, we have there exists $C > 0$ depending on ε such that

$$\begin{aligned}\|u_1\|_{L^p(\Omega)} &= \|(\lambda + A_p)^{-1}P_{av}s\|_{L^p(\Omega)}\\ &= \|(\lambda + A_p|_{L^p_{av}(\Omega)})^{-1}P_{av}s\|_{L^p(\Omega)}\end{aligned}$$

$$\leq \frac{C}{|\lambda|+1}\|P_{av}s\|_{L^p(\Omega)}$$

for all $\lambda \in \Sigma_{\pi-\varepsilon,0} \cap B(0,2M)$. On the other hand we have $u_2 = \frac{1}{\lambda}(s - P_{av}s)$, so there exists $C > 0$ such that

$$\begin{aligned}\|u_2\|_{L^p(\Omega)} &= \|\frac{1}{\lambda}(s - P_{av}s)\|_{L^p(\Omega)} \\ &\leq \frac{C}{|\lambda|}\|s - P_{av}s\|_{L^p(\Omega)}\end{aligned}$$

for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$. We use the operator P_{av} is a bounded linear operator and combine two estimates. We have that there exists $C > 0$ such that $\|u\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|}\|s\|_{L^p(\Omega)}$ for all $\lambda \in \Sigma_{\pi-\varepsilon,0} \cap B(0,2M)$. Since we have already proved the resolvent estimate for $|\lambda| > M$, the resolvent estimate holds for all $\lambda \in \Sigma_{\pi-\varepsilon,0}$. Estimates for higher order derivatives it follows from the key estimate $(**)$ of the proof of Theorem 1.12. □

We can also define the bidomain operator in $L^\infty(\Omega)$. When the domain Ω is bounded, $L^\infty(\Omega)$ is contained in $\bigcap_{n<p<\infty} L^p(\Omega)$. So for all $s \in L^\infty(\Omega)$ we can take a unique solution of (1.4.1)-(1.4.4) $u, u_{i,e} \in \bigcap_{n<p<\infty} W^{2,p}(\Omega)$ satisfying $\int_\Omega u_e dx = 0$. Here, note that we cannot expect a $W^{2,\infty}(\Omega)$ solution such as a usual elliptic problem.

For $\lambda \in \Sigma_{\pi-\varepsilon,M}$ let $R_\infty(\lambda)$ be the solution operator from $s \in L^\infty(\Omega)(\subset \bigcap_{n<p<\infty} L^p(\Omega))$ to $u \in \bigcap_{n<p<\infty} W^{2,p}(\Omega)(\subset L^\infty(\Omega))$ such that u is a solution of the resolvent bidomain equation (1.4.1)-(1.4.4). We warn that the abstract equation (1.4.6) is not available for L^∞ at this moment. The operator $R_\infty(\lambda)$ is a bounded operator whose operator norm is dominated by $C/|\lambda|$, i.e.,

$$\|R_\infty(\lambda)s\|_{L^\infty(\Omega)} \leq \frac{C}{|\lambda|}\|s\|_{L^\infty(\Omega)}.$$

The operator $R_\infty(\lambda)$ may be regarded as a bijection operator from $L^\infty(\Omega)$ to $R_\infty(\lambda)L^\infty(\Omega)$. The operator $R_\infty : \Sigma_{\pi-\varepsilon,M} \to \mathcal{L}(L^\infty(\Omega))$ satisfy the following resolvent equation;

$$R_\infty(\lambda) - R_\infty(\mu) = (\mu - \lambda)R_\infty(\lambda)R_\infty(\mu) \ \ (\lambda, \mu \in \Sigma_{\pi-\varepsilon,M}).$$

Namely the operator $R_\infty(\lambda)$ is a pseudo-resolvent. We use the following proposition.

Proposition 1.19 ([8, Proposition B.6.]). *Set a subset $U \subset \mathbb{C}$ and a Banach space X. Let a function $R : U \to \mathcal{L}(X)$ be a pseudo-resolvent. Then*
(a) $\mathrm{Ker}\, R(\lambda)$ *and* $\mathrm{Ran}\, R(\lambda)$ *are independent of* $\lambda \in U$.
(b) *There is an operator A on X such that $R(\lambda) = (\lambda + A)^{-1}$ for all $\lambda \in U$ if and only if* $\mathrm{Ker}\, R(\lambda) = \{0\}$.

By this proposition, there exists an operator A_∞ with the domain $D(A_\infty) = R_\infty(\lambda)L^\infty(\Omega)$ $(\subset \cap_{n<p<\infty} W^{2,p}(\Omega))$ such that $(\lambda + A_\infty)^{-1}s = u$, i.e. $(\lambda + A_\infty)u = s$. We call A_∞ the bidomain operator in $L^\infty(\Omega)$. We have the bidomain operator A_∞ in $L^\infty(\Omega)$ is a sectorial operator. However, it is easy to see that $D(A_\infty)$ is not dense. Indeed, $\bigcap_{n<p<\infty} W^{2,p}(\Omega) \subset C(\overline{\Omega})$ and hence $\overline{D(A_\infty)}^{L^\infty(\Omega)} \subset C(\overline{\Omega})$, where $\overline{D(A_\infty)}^{L^\infty(\Omega)}$ is the closure of $D(A_\infty)$ in the $L^\infty(\Omega)$ norm. Since $C(\overline{\Omega})$ is not dense in $L^\infty(\Omega)$, $D(A_\infty)$ is not dense in $L^\infty(\Omega)$. We restrict the dense domain $\overline{D(A_\infty)}^{L^\infty(\Omega)}$. We also have

$\overline{D(A_\infty)}^{L^\infty(\Omega)} = \{u \in \cap_{n<p<\infty} W^{2,p}(\Omega) \mid u, A_\infty u \in L^\infty(\Omega), \nabla u \cdot n = 0 \text{ a.e. on } \partial\Omega\}$ (see [24]). So we consider again such that

$$\begin{aligned} &D(\tilde{A}_\infty) := \{u \in D(A_\infty) \mid u, A_\infty u \in UC(\overline{\Omega})\}, \\ &\tilde{A}_\infty u := A_\infty u, \end{aligned}$$

where $UC(\overline{\Omega})$ denotes the space of all the uniformly continuous functions in $\overline{\Omega}$. Then the operator $\tilde{A}_\infty$ is a densely defined sectorial operator in $UC(\overline{\Omega})$. Our resolvent estimates (Theorem 1.18 for L^p, Theorem 1.3 for L^∞) yields the following theorem.

Theorem 1.20 (Analyticity of bidomain operators)**.** *For $1 < p < \infty$ bidomain operators A_p in $L^p(\Omega)$ generate bounded C_0-analytic semigroups with angle $\pi/2$. The operator A_∞ generates a non-C_0-analytic semigroup with angle $\pi/2$ in $L^\infty(\Omega)$, and the operator $\tilde{A}_\infty$ generates a C_0-analytic semigroup with angle $\pi/2$ in $UC(\overline{\Omega})$.*

1.5 Strong solutions in L^p spaces

By discussion in the previous section, we are able to study nonstationary state bidomain equations by using the bidomain operator A. Let us state the definition of a strong solution. Assume that Ω is a bounded C^2-domain, $1 < p < \infty$, $s_{i,e} \in C^\nu_{loc}([0,\infty); L^p(\Omega))$ (for some $0 < \nu < 1$) such that $s_i(t) + s_e(t) \in L^p_{av}(\Omega)$ $(\forall t \geq 0)$ and $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ are locally Lipschitz continuous functions. Before giving the definition of a strong solution, we recall parabolic-elliptic type bidomain equations.

(1.5.1)	$\partial_t u + f(u,w) - \nabla \cdot (\sigma_i \nabla u) - \nabla \cdot (\sigma_i \nabla u_e) = s_i$	in $(0,\infty) \times \Omega$,
(1.5.2)	$-\nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e) \nabla u_e) = s_i + s_e$	in $(0,\infty) \times \Omega$,
(1.5.3)	$\partial_t w + g(u,w) = 0$	in $(0,\infty) \times \Omega$,
(1.5.4)	$\sigma_i \nabla u \cdot n + \sigma_i \nabla u_e \cdot n = 0$	on $(0,\infty) \times \partial\Omega$,
(1.5.5)	$\sigma_i \nabla u \cdot n + (\sigma_i + \sigma_e) \nabla u_e \cdot n = 0$	on $(0,\infty) \times \partial\Omega$,
(1.5.6)	$u(0) = u_0, \ w(0) = w_0$	in Ω.

Definition 1.21 ([11, Definition 18] Strong solution)**.** For $\tau > 0$ consider the functions $z : t \in [0,\tau) \mapsto z(t) = (u(t), w(t)) \in Z := L^p(\Omega) \times B^m$ ($B = L^\infty(\Omega)$ or $C^\nu(\Omega)$) and $u_e : t \in [0,\tau) \mapsto u_e(t) \in L^p(\Omega)$. Given $z_0 = (u_0, w_0) \in Z$, we say that (u, u_e, w) is a strong solution to $(1.5.1)$ to $(1.5.6)$ if

(1) $z : [0,\tau) \to Z$ is continuous and $z(0) = (u_0, w_0)$ in Z,
(2) $z : (0,\tau) \to Z$ is Fréchet differentiable,
(3) $t \in [0,\tau) \mapsto \Big(f\big(u(t), w(t)\big), g\big(u(t), w(t)\big)\Big) \in Z$ is well-defined, locally ν-Hölder continuous on $(0,\tau)$ and is continuous at $t = 0$,
(4) for all $t \in (0,\tau)$, $u(t) \in W^{2,p}(\Omega)$, $u_e(t) \in W^{2,p}_{av}(\Omega)$,

and (u, u_e, w) verify $(1.5.1)$-$(5.6.1)$ for all $t \in (0,\tau)$ and for a.e. $x \in \Omega$, and the boundary conditions $(1.5.4)$ and $(1.5.5)$ for all $t \in (0,\tau)$ and for a.e. $x \in \partial\Omega$.

Let us consider bidomain equations as an abstract parabolic evolution equation on some Cartesian product spaces. We set

$$\begin{aligned}
&\mathcal{A}z := (Au, 0) \text{ for } z = (u, w) \in D(\mathcal{A}) := D(A) \times B^m,\\
&F : z \in Z \mapsto (f(z), g(z)) \in Z,\\
&S : t \in [0, \infty) \mapsto (s(t), 0) = \big(s_i(t) - A_i(A_i + A_e)^{-1}(s_i(t) + s_e(t)), 0\big) \in Z,\\
&\mathcal{F}(t, z) = S(t) - F(z).
\end{aligned}$$

If one collects all calculation, then bidomain equations is transformed into

$$\text{(1.5.7)} \qquad \frac{dz}{dt}(t) + \mathcal{A}z(t) = \mathcal{F}(t, z(t)) \qquad \text{in } Z,$$

$$\text{(1.5.8)} \qquad u_e(t) = (A_i + A_e)^{-1}\{(s_i(t) + s_e(t)) - A_i P_{av} u(t)\} \qquad \in D(A_e),$$

$$z(0) = z_0 \qquad \text{in } Z.$$

Lemma 1.22 ([11, Lemma 19]). *The function $z = (u, w)$ with u_e is a strong solution* (1.5.1)-(1.5.6) *if and only if conditions* (1)-(3) *of Definition* 1.21 *and condition* (4') *below is satisfied;*

(4') *for all $t \in (0, \tau)$, $u(t) \in D(A)$ satisfies* (1.5.7) *and* (1.5.8).

We will use the general theory in Henry's book [21]. We have to control the non-linear term f, g. The key idea is to use fractional powers $\mathcal{A}^\alpha$ and related space Z^α with $0 \le \alpha \le 1$.

Definition 1.23 ([21]). If $\mathcal{A}$ is a sectorial operator in a Banach space Z and if there is $a \ge 0$ such that $\operatorname{Re}\sigma(\mathcal{A} + a) > 0$, then for each $\alpha > 0$ we define the operator

$$(\mathcal{A} + a)^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(\mathcal{A}+a)t} dt.$$

For $\alpha > 0$, we see $(\mathcal{A} + a)^{-\alpha}$ is a bounded linear operator on Z which is one-to-one. By using this operator with fractional power, we define the domain Z^α of fractional power;

$$\begin{aligned}
&Z^\alpha := R((\mathcal{A} + a)^{-\alpha}) \qquad (\alpha > 0),\\
&\|x\|_{Z^\alpha} := \|((\mathcal{A} + a)^{-\alpha})^{-1} x\|_Z.
\end{aligned}$$

For $\alpha = 0$, we define $Z^0 := Z$, $\|x\|_{Z^0} := \|x\|_Z$.

Remark 1.24 ([21]).

- Different choices of a give equivalent norms on Z^α.
- $(Z^\alpha, \|\cdot\|_{Z^\alpha})$ is a Banach space, $Z^1 = D(\mathcal{A})$ and for $0 \le \beta \le \alpha \le 1$, Z^α is a dense subspace of Z^β with continuous inclusion.

Lemma 1.25 ([21, Theorem 1.6.1]). *If $B = L^\infty(\Omega)$ and f, g are locally Lipschitz continuous on $\mathbb{R} \times \mathbb{R}^m$, then*

$$Z^\alpha \subset L^\infty(\Omega) \times B^m \quad \text{if } \frac{d}{2p} < \alpha \le 1,$$

and in that case, $F : z \in Z^\alpha \mapsto F(z) \in Z$ is locally Lipschitz continuous.
If $B = C^\nu(\Omega)$ and f, g are C^2 functions on $\mathbb{R} \times \mathbb{R}^m$, then

$$Z^\alpha \subset C^\nu(\Omega) \times B^m \quad \text{if } \frac{1}{2}\Big(\nu + \frac{d}{p}\Big) < \alpha \leq 1,$$

and in that case, $F : z \in Z^\alpha \mapsto F(z) \in Z$ is locally Lipschitz continuous.

We are ready to state existence and uniqueness of the strong solution for bidomain equations. When $p = 2$, $d = 2, 3$, this was proved in [11, Theorem 20] so our result is regarded as an extension of their result.

Theorem 1.26 (Local existence and uniqueness)**.** *Let $0 \leq \alpha < 1$ and $1 < p < \infty$ satisfying the relation in Lemma* 1.25*. Then for any $z_0 = (u_0, w_0) \in Z^\alpha$, there exists $T > 0$ such that bidomain equations have a unique strong solution on $[0, T)$.*

Proof. It is enough to show

- $\mathcal{A}$ is a sectorial operator,
- $\mathcal{F} : [0, \infty) \times Z^\alpha \to Z$ is a locally Hölder continuous function in t and a locally Lipschitz continuous function in z,

because of existence and uniqueness theorem [21, Theorem 3.3.3]. First part is obvious since $\mathcal{A} = (A, 0)$, A is a sectorial operator and 0 is a bounded linear operator. Note that a bounded linear operator is a sectorial operator and direct sum of a sectorial operator is a sectorial operator [21]. Second part follows from the calculation as below. We need to show $s : [0, \infty) \to L^p(\Omega)$ is locally ν-Hölder continuous in time. For any compact set $M \subset [0, \infty)$ there exists $C > 0$ such that for all $t_1, t_2 \in M$, we have

$$\begin{aligned}
&\|s(t_1) - s(t_2)\|_{L^p(\Omega)} \\
=&\|s_i(t_1) - s_i(t_2) - A_i(A_i + A_e)^{-1}(s_i(t_1) - s_i(t_2) + s_e(t_1) - s_e(t_2))\|_{L^p(\Omega)} \\
\leq&\|s_i(t_1) - s_i(t_2)\|_{L^p(\Omega)} + C(\|s_i(t_1) - s_i(t_2)\|_{L^p(\Omega)} + \|s_e(t_1) - s_e(t_2)\|_{L^p(\Omega)}) \\
\leq&C|t_1 - t_2|^\nu.
\end{aligned}$$

Here, we invoked the fact that $A_i(A_i + A_e)^{-1}$ is a bounded linear operator and that $s_{i,e}$ are locally ν-Hölder continuous functions. □

We conclude this chapter by studying regularity of a strong solution. Let $0 < \nu < 1$, Ω be a bounded $C^{2+\nu}$-domain, f, g be C^2 regularity, and coefficient of $\sigma_{i,e}$ be $C^{1+\nu}(\overline{\Omega})$.

Theorem 1.27 (Regularity of a strong solution)**.** *Consider the case $B = C^\nu(\Omega)$ in Definition* 1.21 *and $0 \leq \alpha < 1$ defined by Lemma* 1.25*. Assume that $s_{i,e} \in C^\nu_{loc}([0, \infty); L^p(\Omega))$ such that $s_{i,e}(t) \in C^\nu(\Omega)$ and $\int_\Omega (s_i(t) + s_e(t))dx = 0 (\forall t \geq 0)$. For $z_0 = (u_0, w_0) \in Z^\alpha$ the unique strong solution z of bidomain equations defined on $[0, T)$ for some $T > 0$ satisfies furthermore:*

(1) *For any $x \in \overline{\Omega}$, $u(x, \cdot) \in C^1((0, T); \mathbb{R})$ and $w(x, \cdot) \in C^1((0, T); \mathbb{R}^m)$.*
(2) *For any $t \in (0, T)$, $u(\cdot, t), u_{i,e}(\cdot, t) \in C^2(\overline{\Omega})$.*

Proof. We see that $t \in (0,T) \mapsto z(t) \in C^\nu(\Omega) \times (C^\nu(\Omega))^m$ is continuous (Fréchet) differentiable. This actually implies that $(t,x) \in (0,T) \times \overline{\Omega} \mapsto z(x,t) = (u(x,t), w(t,x))$ is continuously differentiable in t. By [21, Theorem 3.5.2], we have $t \in (0,T) \mapsto z(t) \in Z^\nu$ is continuously (Fréchet) differentiable. This means $du/dt(t) \in C^\nu(\Omega)$. From (1.5.7),

$$P_{av}u(t) = A_e^{-1}(A_i + A_e)A_i^{-1}\left\{-\frac{du}{dt}(t) - f(u(t), w(t)) + s(t)\right\}.$$

By elliptic regularity theorem for Hölder spaces, $P_{av}u(\cdot,t)$ is $(2+\nu)$-Hölder continuous since $-du/dt(t) - f(u(t),w(t)) + s(t)$ is ν-Hölder continuous. Therefore $u(\cdot,t)$ is in $C^2(\overline{\Omega})$. The function u_e is also in $C^2(\overline{\Omega})$ by (1.5.8). □

1.6 Appendix

In this appendix we show $\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1}$ belongs $L^1(\mathbb{R}^d)$ and for any $\Psi_2 \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \Psi_2 dx = 0$, $\mathcal{F}^{-1}(q^{(0)}(\xi)\mathcal{F}\Psi_2)$ belongs $L^1(\mathbb{R}^d)$, where $p(\xi) = \frac{\langle \sigma_{i\infty}\xi,\xi\rangle\langle \sigma_{e\infty}\xi,\xi\rangle}{\langle(\sigma_{i\infty}+\sigma_{e\infty})\xi,\xi\rangle}$ and $q^{(0)}(\xi) = q_i(\xi) = \frac{\langle \sigma_{i\infty}\xi,\xi\rangle}{\langle(\sigma_{i\infty}+\sigma_{e\infty})\xi,\xi\rangle}$. For $q^{(0)}(\xi) = q_e(\xi) = \frac{\langle \sigma_{e\infty}\xi,\xi\rangle}{\langle(\sigma_{i\infty}+\sigma_{e\infty})\xi,\xi\rangle}$, the L^1 finiteness of $\mathcal{F}^{-1}(q^{(0)}(\xi)\mathcal{F}\Psi_2)$ is the same as $q^{(0)}(\xi) = q_i(\xi)$.

Proof that $\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1} \in L^1(\mathbb{R}^d)$. By the chain rule, we have that for any multi-index α with $|\alpha| \geq 1$,

$$\begin{aligned}
&\partial_\xi^\alpha (e^{i\theta_\infty} + p(\xi))^{-1} \\
&= \sum_{\ell=1}^{|\alpha|} (-1)^\ell \ell! (e^{i\theta_\infty} + p(\xi))^{-1-\ell} \left(\sum_{\substack{\alpha_1+\cdots+\alpha_\ell=\alpha \\ |\alpha_1|,\cdots,|\alpha_\ell|\geq 1}} C^\ell_{\alpha_1,\cdots,\alpha_\ell} \partial_\xi^{\alpha_1} p(\xi) \cdots \partial_\xi^{\alpha_\ell} p(\xi) \right) \\
&= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\alpha_1+\cdots+\alpha_\ell=\alpha \\ |\alpha_1|,\cdots,|\alpha_\ell|\geq 1}} C^\ell_{\alpha_1,\cdots,\alpha_\ell} (e^{i\theta_\infty} + p(\xi))^{-1-\ell} q^{(2-|\alpha_1|)}(\xi) \cdots q^{(2-|\alpha_\ell|)}(\xi) \\
&= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\alpha_1+\cdots+\alpha_\ell=\alpha \\ |\alpha_1|,\cdots,|\alpha_\ell|\geq 1}} C^\ell_{\alpha_1,\cdots,\alpha_\ell} (e^{i\theta_\infty} + p(\xi))^{-1-\ell} q^{(2\ell-|\alpha|)}(\xi)
\end{aligned}$$

holds, where constants change from line to line and $q^{(k)}(\xi)$ is a positively homogeneous function of degree k. We apply this formula to estimate L^∞ norm of $x^\alpha \mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1}$. We take α such that $|\alpha| = d+1$ to estimate behavior of $\mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1}$ near space infinity and that $|\alpha| = d-1$ to estimate behavior near the origin. We observe that

$$\begin{aligned}
&\left\| \sum_{|\alpha|=d\pm 1} x^\alpha \mathcal{F}^{-1}(e^{i\theta_\infty} + p(\xi))^{-1} \right\|_{L^\infty(\mathbb{R}^d)} \\
\leq &\left\| \sum_{|\alpha|=d\pm 1} \mathcal{F}^{-1}(i\partial_\xi)^\alpha (e^{i\theta_\infty} + p(\xi))^{-1} \right\|_{L^\infty(\mathbb{R}^d)}
\end{aligned}$$

$$\leq \sum_{|\alpha|=d\pm1}\left(\left\|\partial_\xi^\alpha(e^{i\theta_\infty}+p(\xi))^{-1}\right\|_{L^1(B(0,1))}+\left\|\partial_\xi^\alpha(e^{i\theta_\infty}+p(\xi))^{-1}\right\|_{L^1(B(0,1)^c)}\right)$$

$$\leq \sum_{|\alpha|=d\pm1}\sum_{\ell=1}^{|\alpha|} C_\alpha^\ell(\|(e^{i\theta_\infty}+p(\xi))^{-1-\ell}q^{(2\ell-|\alpha|)}(\xi)\|_{L^1(B(0,1))}$$

$$+\|(e^{i\theta_\infty}+p(\xi))^{-1-\ell}q^{(2\ell+2)}(\xi)q^{(-2-|\alpha|)}(\xi)\|_{L^1(B(0,1)^c)}).$$

Since for $\ell=1,2,\cdots,d\pm1$, $-d<2\ell-(d\pm1)$ and $(e^{i\theta_\infty}+p(\xi))^{-1}$ is bounded, the first term in the right-hand side is finite. Since for $\ell=1,2,\cdots,d\pm1$, $-2-(d\pm1)<-d$ and $(e^{i\theta_\infty}+p(\xi))^{-1-\ell}q^{(2\ell+2)}(\xi)$ is also bounded, the second term in the right-hand side is finite. We thus conclude that $\left\|\sum_{|\alpha|=d\pm1}x^\alpha\mathcal{F}^{-1}(e^{i\theta_\infty}+p(\xi))^{-1}\right\|_{L^\infty(\mathbb{R}^d)}<+\infty$. This implies that

$$|\mathcal{F}^{-1}(e^{i\theta_\infty}+p(\xi))^{-1}(x)|\leq\begin{cases}C|x|^{-d+1} & \text{for } |x|\leq1,\\ C|x|^{-d-1} & \text{for } 1\leq|x|.\end{cases}$$

We thus conclude that $\mathcal{F}^{-1}(e^{i\theta_\infty}+p(\xi))^{-1}$ is in $L^1(\mathbb{R}^d)$. □

The proof for $\mathcal{F}^{-1}q^{(0)}(\xi)\mathcal{F}\Psi_2\in L^1(\mathbb{R}^d)$ is essentially reduced to next proposition.

Proposition 1.28. *For any $f\in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d}f(x)dx=0$, the solution $u(\in\mathcal{S}'(\mathbb{R}^d))$ of $-\Delta u=f$ in $\mathbb{R}^d$ defined by $u(x)=\mathcal{F}^{-1}(|\xi|^{-2}\mathcal{F}f)(x)$ satisfies that for any $j,k=1,2,\cdots,d$ there exists $C>0$ such that for any $x\in\mathbb{R}^d$,*

$$|\partial_j\partial_k u(x)|\leq\frac{C}{(1+|x|)^{d+1}}.$$

In particular $\partial_j\partial_k u\in L^1(\mathbb{R}^d)$.

Proof. We have $\partial_j\partial_k u\in L^\infty(\mathbb{R}^d)$:

$$\|\partial_j\partial_k u\|_{L^\infty(\mathbb{R}^d)}=\left\|\mathcal{F}^{-1}\frac{i\xi_j i\xi_k}{|\xi|^2}\mathcal{F}f\right\|_{L^\infty(\mathbb{R}^d)}\leq\left\|\frac{i\xi_j i\xi_k}{|\xi|^2}\right\|_{L^\infty(\mathbb{R}^d)}\|\mathcal{F}f\|_{L^1(\mathbb{R}^d)}<+\infty.$$

Set $\tilde{q}^{(-d)}(x):=\mathcal{F}^{-1}\frac{i\xi_j i\xi_k}{|\xi|^2}$, then $\nabla\tilde{q}^{(-d)}(x)$ is a positively homogeneous function of degree $-d-1$ and $|\nabla\tilde{q}^{(-d)}(x)|\leq C|x|^{-(d+1)}$. Let $R>0$ be sufficiently large so that $\operatorname{supp} f\subset B(0,R/2)$. For any $x\in B(0,R)^c$,

$$\begin{aligned}|\partial_j\partial_k u(x)|&=\left|\int_{\mathbb{R}^d}\tilde{q}^{(-d)}(x-y)f(y)dy\right|\\&=\left|\int_{\mathbb{R}^d}(\tilde{q}^{(-d)}(x-y)-\tilde{q}^{(-d)}(x))f(y)dy\right|\\&=\left|\int_{\operatorname{supp} f}\int_0^1\langle\nabla\tilde{q}^{(-d)}(x-ty),y\rangle dtf(y)dy\right|\\&=\int_{\operatorname{supp} f}\int_0^1\frac{|y||f(y)|}{|x-ty|^{d+1}}dtdy\end{aligned}$$

In the above calculation we have used the fact $\int_{\mathbb{R}^d} \tilde{q}^{(-d)}(x)f(y)dy = 0$ since $\int_{\mathbb{R}^d} f(y)dy = 0$. By supp $f \subset B(0, R/2)$ and $x \in B(0,R)^c$, we see $|x| \le |x-ty|+t|y| \le |x-ty|+R/2 < |x-ty|+|x|/2$ so that $|x-ty|^{-(d+1)} \le C|x|^{-(d+1)}$. So we have

$$|\partial_j \partial_k u(x)| \le \frac{C}{|x|^{d+1}}$$

for all $x \in B(0,R)^c$. Since we know that $\partial_j\partial_k u \in L^\infty(\mathbb{R}^d)$, we now conclude

$$|\partial_j \partial_k u(x)| \le \frac{C}{(1+|x|)^{d+1}}.$$

□

We transform the symbol $q^{(0)}(\xi) = q_i(\xi)$ into $\frac{i\xi_j i\xi_k}{|\xi|^2}$ so that we get the L^1 boundedness in the next proposition.

Proposition 1.29. *For any $\Psi_2 \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \Psi_2(x)dx = 0$ there exists $C > 0$ such that for any $x \in \mathbb{R}^d$,*

$$|\mathcal{F}^{-1} q_i(\xi)\mathcal{F}\Psi_2(x)| \le \frac{C}{(1+|x|)^{d+1}}$$

In particular $\mathcal{F}^{-1} q_i(\xi)\mathcal{F}\Psi_2 \in L^1(\mathbb{R}^d)$.

Proof. We are able to take a diagonal matrix $D = \operatorname{diag}((\lambda_1)^{-1/2}, \cdots, (\lambda_d)^{-1/2})$ and a orthogonal matrix $P = (p_1, \cdots, p_d)$ such that $\eta = DP\xi$ and $\langle(\sigma_{i\infty}+\sigma_{e\infty})\xi, \xi\rangle = |\eta|^2$, where $\{(\lambda_\ell, p_\ell)\}_{\ell=1}^d$ are the pair of eigenvalues and eigenfunctions of $\sigma_{i\infty}+\sigma_{e\infty}$. Then we have that the following calculation:

$$\begin{aligned}
&\mathcal{F}^{-1} q_i(\xi)\mathcal{F}\Psi_2(x) \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} \frac{\langle \sigma_{i\infty}\xi, \xi\rangle}{\langle(\sigma_{i\infty}+\sigma_{e\infty})\xi, \xi\rangle} d\xi \right) \Psi_2(y)dy \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{i(x-y)\cdot(DP)^{-1}\eta} \frac{\langle \sigma_{i\infty}(DP)^{-1}\eta, (DP)^{-1}\eta\rangle}{|\eta|^2} \det D^{-1}\, d\eta \right) \Psi_2(y)dy \\
&= \sum_{j,k} C_{j,k}\partial_j\partial_k \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{iDP(x-y)\cdot\eta} \frac{1}{|\eta|^2} d\eta \right) \Psi_2(y)dy \\
&= \sum_{j,k} C_{j,k}\partial_j\partial_k \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{iDP(x-y)\cdot\eta} \frac{1}{|\eta|^2} d\eta \right) \widetilde{\Psi_2}(DPy)dy
\end{aligned}$$

for some $C_{j,k} \in \mathbb{R}$ and $\widetilde{\Psi_2} \in C_0^\infty(\mathbb{R}^d)$ such that $\Psi_2(y) = \widetilde{\Psi_2}(DPy)$ and $\int_{\mathbb{R}^d} \widetilde{\Psi_2}(x)dx = 0$. This implies that $\mathcal{F}^{-1}q_i(\xi)\mathcal{F}\Psi_2(x) = \sum_{j,k} C_{j,k}\partial_j\partial_k(-\Delta)^{-1}\widetilde{\Psi_2}(DPx)$ and

$$|\mathcal{F}^{-1} q_i(\xi)\mathcal{F}\Psi_2(x)| \le \frac{C}{(1+|DPx|)^{d+1}} \le \frac{C}{(1+|x|)^{d+1}}$$

and then $\mathcal{F}^{-1} q_i(\xi)\mathcal{F}\Psi_2 \in L^1(\mathbb{R}^d)$. □

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Chapter 2

Strong Time Periodic Solutions to the Bidomain Equations with FitzHugh–Nagumo Type Nonlinearities

Consider the bidomain equations subject to ionic transport described by the models of FitzHugh–Nagumo, Aliev–Panfilov, or Rogers–McCulloch. It is proved that this set of equations admits a unique, *strong* T-periodic solution provided it is innervated by T-periodic intra- and extracellular currents. The approach relies on a new periodic version of the classical Da Prato–Grisvard theorem on maximal L^p-regularity in real interpolation spaces.

Keywords: bidomain model; periodic solutions; maximal regularity in real interpolation spaces

2.1 Introduction

The bidomain system is a well established system of equations describing the electrical activities of the heart. For a detailed description of this model as well as its derivation from general principles, we refer, e.g., to [9, 18] and the monograph by Keener and Sneyd [20]. The system is given by

$$\text{(BDE)} \qquad \begin{cases} \partial_t u + F(u,w) - \nabla \cdot (\sigma_i \nabla u_i) = I_i & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + F(u,w) + \nabla \cdot (\sigma_e \nabla u_e) = -I_e & \text{in } (0,\infty)\times\Omega, \\ \partial_t w + G(u,w) = 0 & \text{in } (0,\infty)\times\Omega, \end{cases}$$

subject to the boundary conditions

$$\sigma_i \nabla u_i \cdot \nu = 0, \quad \sigma_e \nabla u_e \cdot \nu = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{2.1.1}$$

and the initial data

$$u(0) = u_0, \quad w(0) = w_0 \quad \text{in} \quad \Omega. \tag{2.1.2}$$

Here $\Omega \subset \mathbb{R}^n$ denotes a domain describing the myocardium, the functions u_i and u_e model the intra- and extracellular electric potentials, $u := u_i - u_e$ denotes the transmembrane potential, and ν denotes the outward unit normal vector to $\partial\Omega$. The anisotropic properties of the intra- and extracellular tissue parts will be described by the conductivity matrices $\sigma_i(x)$ and $\sigma_e(x)$. Furthermore, I_i and I_e stand for the intra- and extracellular stimulation current, respectively.

The variable w, the so-called gating variable, corresponds to the ionic transport through the cell membrane. On a microscopic level, the intra- and extracellular quantities are defined on disjoint domains Ω_i and Ω_e of Ω. After a homogenization procedure described rather rigorously, e.g., in [10, 11], one obtains the macroscopic model above, where the intra- and extracellular components are defined on all of Ω. The behavior of the ionic current through the cell membrane, described by the variable w, is coupled with the transmembrane voltage u by the equation in the third line of (BDE).

Mathematical models describing the propagation of impulses in electrophysiology have a long tradition starting with the classical model by Hodgkin and Huxley in the 1950s, see, e.g., the recent survey article of Stevens [34]. In this article, we consider various models for the ionic transport including the models by FitzHugh–Nagumo, Aliev–Panfilov, and Rogers–McCulloch. The *FitzHugh–Nagumo model* reads as

$$\begin{aligned} F(u,w) &= u(u-a)(u-1) + w = u^3 - (a+1)u^2 + au + w, \\ G(u,w) &= bw - cu, \end{aligned}$$

where $0 < a < 1$ and b, $c > 0$. In the *Aliev–Panfilov model* the functions F and G are given by

$$\begin{aligned} F(u,w) &= ku(u-a)(u-1) + uw = ku^3 - k(a+1)u^2 + kau + uw, \\ G(u,w) &= ku(u-1-a) + dw, \end{aligned}$$

whereas for the *Rogers–McCulloch model* we have

$$\begin{aligned} F(u,w) &= bu(u-a)(u-1) + uw = bu^3 - b(a+1)u^2 + bau + uw, \\ G(u,w) &= dw - cu. \end{aligned}$$

The coefficients in these models satisfy the conditions $0 < a < 1$ and b, c, d, $k > 0$.

Despite its importance in cardiac electrophysiology, not many analytical results on the bidomain equations are known until today. Note that the so-called bidomain operator is a very *non local* operator, which makes the analysis of this equation seriously more complicated compared, e.g., to the classical Allen–Cahn equation.

The rigorous mathematical analysis of this system started with the work of Colli-Franzone and Savaré [11], who introduced a variational formulation of the problem and showed the global existence and uniqueness of weak and strong solutions for FitzHugh-Nagumo model. Veneroni [37] extended the latter result to more general models for the ionic transport including the Luo and Rudy I model [26].

In 2009, a new approach to this system was presented by Bourgault, Cordière, and Pierre in [5]. They introduced for the first time the so-called bidomain operator within the L^2-setting and showed that it is a non-negative and self-adjoint operator. By making use of the theory of evolution equations they further showed the existence and uniqueness of a local strong solution and the existence of a global, weak solution to the system above for a large class of ionic models including the FitzHugh–Nagumo, Aliev–Panfilov, and Rogers–McCulloch models above. In [23], the uniqueness and regularity of the weak solution were proved.

For results concerning the optimal control problem subject to the monodomain approximation, in which the conductivity matrices satisfy $\sigma_i = \lambda\sigma_e$ for some $\lambda > 0$, we refer to a series of papers by Kunisch et al. [6,21,22,29], see also [35].

A new impetus to the field was recently given by Giga and Kajiwara [16], who investigated the bidomain equations within the L^p-setting for $1 < p \leq \infty$. They showed that the bidomain operator is the generator of an analytic semigroup on $\mathrm{L}^p(\Omega)$ for $p \in (1, \infty]$ and constructed a local, strong solution to the bidomain system within this setting.

All these results mainly concern the well-posedness of the bidomain equations and results on the dynamics of the solution are even more rare. We refer here to the very recent work of Mori and Matano [28], who studied for the first time the stability of front solutions of the bidomain equations.

In this context it is now a very natural question to ask, whether the bidomain equations admit time periodic solutions. Periodic solutions can be formulated in various regularity classes, ranging from weak over mild to strong solutions.

In this chapter, we consider the situation where the bidomain model, combined with one of the models for the ionic transport above, is innervated by *periodic intra- and extracellular currents* I_i and I_e. It is then our aim to show that in this case the innervated system admits a *strong* time periodic solution of period T provided the outer forces I_i and I_e are both time-periodic of period $T > 0$.

Let us emphasize, that we consider here the *full* bidomain model taking into account the anisotropic phenomena and not only the so-called monodomain approximation. A function space related to a fixed point argument for the Poincaré map in the strong sense is naturally linked to a space of maximal regularity. This leads us to the scale of real interpolation spaces and our approach is then based on a periodic version of the classical Da Prato–Grisvard theorem [12]. A different approach within the L^p-setting based on a semilinear version of a result by Arendt and Bu [4] on strong periodic solutions of linear equations would require additional properties of the bidomain operator, which, however, seem to be unknown.

Some more specific words about the strategy of our approach are in order. The bidomain system is first reformulated into a coupled system. In this coupled system a 2×2 operator matrix $\mathcal{A}$ involving the bidomain operator A in one of its components will represent the linear part of (BDE). Given a Banach space X and a T-periodic function $f : \mathbb{R} \to X$ whose restriction to $(0, T)$ belongs to $\mathrm{L}^p(0, T; X)$, we understand by a strong T-periodic solution to the bidomain equations with right-hand side $(f, 0)$ a T-periodic tupel $(u, w) \in \mathrm{L}^p(0, T; X)$ satisfying $(u', w') \in \mathrm{L}^p(0, T; X)$ and $\mathcal{A}(u, w) \in \mathrm{L}^p(0, T; X)$. This means in particular that (u, w) admit the property of maximal L^p-regularity. In order to obtain a T-periodic solution to (BDE) within this regularity class, we choose as underlying Banach space the real interpolation space $D_A(\theta, p)$ for $\theta \in (0, 1)$, $1 \leq p < \infty$, and A being again the bidomain operator. Our approach to T-periodic solutions for the linearized equation is then based on a periodic version of the classical Da Prato–Grisvard theorem, which we develop in Section 2.4. Having this at hand, we apply then the contraction mapping principle in the space of maximal regularity to find a strong

T-periodic solution of the nonlinear problem in a neighborhood of stable equilibrium points.

This chapter is organized as follows: While Section 3.2 is devoted to fix some notation and to collect some known results, our main results on strong T-periodic solutions to the bidomain equations subject to a large class of models for the ionic transport are presented in Section 2.3. The following Section 2.4 presents a periodic version of the Da Prato–Grisvard theorem, which will be extended in Section 2.5 to the semilinear setting. In Section 2.6 we apply our previous results to the bidomain equations subject to various models for the ionic transport including the models by FitzHugh–Nagumo, Aliev–Panfilov, and Rogers–McCulloch.

2.2 Preliminaries

In the whole article, let the space dimension $n \geq 2$ be fixed and let $\Omega \subset \mathbb{R}^n$ denote a bounded domain with boundary $\partial\Omega$ of class C^2. For the conductivity matrices σ_i and σ_e we make the following assumptions.

Assumption E. The conductivity matrices $\sigma_i, \sigma_e : \overline{\Omega} \to \mathbb{R}^{n\times n}$ are symmetric matrices and are functions of class $\mathrm{C}^1(\overline{\Omega})$. Ellipticity is imposed by means of the following condition: there exist constants $\underline{\sigma}$, $\overline{\sigma}$ with $0 < \underline{\sigma} < \overline{\sigma}$ such that

$$\underline{\sigma}|\xi|^2 \leq \langle \sigma_i(x)\xi, \xi\rangle \leq \overline{\sigma}|\xi|^2 \quad \text{and} \quad \underline{\sigma}|\xi|^2 \leq \langle \sigma_e(x)\xi, \xi\rangle \leq \overline{\sigma}|\xi|^2 \tag{2.2.1}$$

for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^n$. Moreover, it is assumed that

$$\begin{aligned} \sigma_i \nabla u_i \cdot \nu = 0 \qquad &\Leftrightarrow \qquad \nabla u_i \cdot \nu = 0 \qquad \text{on } \partial\Omega, \\ \sigma_e \nabla u_e \cdot \nu = 0 \qquad &\Leftrightarrow \qquad \nabla u_e \cdot \nu = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.2.2}$$

It is known due to [8] that (3.2.2) is a biological reasonable assumption.

Next, we define the bidomain operator in the L^q-setting for $1 < q < \infty$. To this end, let $\mathrm{L}^q_{av}(\Omega) := \{u \in \mathrm{L}^q(\Omega) : \int_\Omega u \,\mathrm{d}x = 0\}$ and let P_{av} be the orthogonal projection from $\mathrm{L}^q(\Omega)$ to $\mathrm{L}^q_{av}(\Omega)$, i.e., $P_{av}u := u - \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x$. We then introduce the operators A_i and A_e by

$$\begin{aligned} A_{i,e}u &:= -\nabla \cdot (\sigma_{i,e}\nabla u), \\ D(A_{i,e}) &:= \left\{u \in \mathrm{W}^{2,q}(\Omega) \cap \mathrm{L}^q_{av}(\Omega) : \sigma_{i,e}\nabla u \cdot \nu = 0 \text{ a.e. on } \partial\Omega\right\} \subset \mathrm{L}^q_{av}(\Omega), \end{aligned}$$

where $A_{i,e}$ and $\sigma_{i,e}$ indicates that either A_i and σ_i or A_e and σ_e are considered. Due to condition (3.2.2) we obtain $D(A_i) = D(A_e)$ and thus, it is possible to define the sum $A_i + A_e$ of A_i and A_e with the domain $D(A_i) = D(A_e)$. Note that the inverse operator $(A_i + A_e)^{-1}$ on $\mathrm{L}^q_{av}(\Omega)$ is a bounded linear operator.

Following [16] we define the bidomain operator as follows. Let σ_i and σ_e satisfy Assumption E. Then the bidomain operator A is defined as

$$A = A_i(A_i + A_e)^{-1}A_eP_{av} \tag{2.2.3}$$

with domain

$$D(A) := \{u \in \mathrm{W}^{2,q}(\Omega) : \nabla u \cdot \nu = 0 \text{ a.e. on } \partial\Omega\}.$$

The following resolvent estimates for A were proven by Giga and Kajiwara in [16]. Here, denote for $\theta \in (0, \pi]$ the sector $\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \theta\}$.

Proposition 2.1 ([16, Theorem 4.7, Theorem 4.9])**.** *Let $1 < q < \infty$, Ω be a bounded C^2-domain and let σ_i and σ_e satisfy Assumption E. Then, for $\lambda \in \Sigma_\pi$ and $f \in \mathrm{L}^q(\Omega)$, the resolvent problem*

$$(\lambda + A)u = f \quad \text{in } \Omega \tag{2.2.4}$$

has a unique solution $u \in D(A)$. Moreover, for each $\varepsilon \in (0, \pi/2)$ there exists a constant $C > 0$ such that for all $\lambda \in \Sigma_{\pi-\varepsilon}$ and all $f \in \mathrm{L}^q(\Omega)$ the unique solution $u \in D(A)$ satisfies

$$|\lambda|\|u\|_{\mathrm{L}^q(\Omega)} + |\lambda|^{1/2}\|\nabla u\|_{\mathrm{L}^q(\Omega)} + \|\nabla^2 u\|_{\mathrm{L}^q(\Omega)} \le C\|f\|_{\mathrm{L}^q(\Omega)}.$$

Observe that the proposition above implies in particular that $-A$ generates a bounded analytic semigroup e^{-tA} on $\mathrm{L}^q(\Omega)$.

Under the assumption of the conservation of currents, i.e.,

$$\int_\Omega (I_i(t) + I_e(t))\,\mathrm{d}x = 0, \quad t \ge 0 \tag{2.2.5}$$

and assuming moreover $\int_\Omega u_e\,\mathrm{d}x = 0$, the bidomain equations (BDE) may be equivalently rewritten as an evolution equation [5,16] of the form

$$\text{(ABDE)} \qquad \begin{cases} \partial_t u + Au + F(u,w) = I, & \text{in } (0,\infty), \\ \partial_t w + G(u,w) = 0, & \text{in } (0,\infty), \\ u(0) = u_0, & \\ w(0) = w_0, & \end{cases}$$

where

$$I := I_i - A_i(A_i + A_e)^{-1}(I_i + I_e) \tag{2.2.6}$$

is the modified source term. The functions u_e and u_i can be recovered from u by virtue of the following relations

$$\begin{aligned} u_e &= (A_i + A_e)^{-1}\{(I_i + I_e) - A_i P_{av} u\}, \\ u_i &= u + u_e. \end{aligned}$$

Our main results on the unique existence of strong T-periodic solutions to (PABDE) are formulated in the real interpolation space $D_A(\theta, p)$ between $D(A)$ and the underlying space $\mathrm{L}^q(\Omega)$. This choice of spaces is motivated by our aim to prove the existence and uniqueness of T-periodic solutions to the bidomain equations in the *strong*, and not only in the mild sense. The classical Da Prato–Grisvard theorem ensures the maximal

L^p-regularity for parabolic evolution equations in these spaces and our approach is based on a *periodic version of the Da Prato–Grisvard theorem*.

More specifically, let X be a Banach space and $-\mathcal{A}$ be the generator of a bounded analytic semigroup $\mathrm{e}^{-t\mathcal{A}}$ on X with domain $D(\mathcal{A})$. For $\theta \in (0,1)$ and $1 \leq p < \infty$, we denote by $D_{\mathcal{A}}(\theta,p)$ space defined as

$$D_{\mathcal{A}}(\theta,p) := \Big\{x \in X : [x]_{\theta,p} := \Big(\int_0^\infty \|t^{1-\theta}\mathcal{A}\mathrm{e}^{-t\mathcal{A}}x\|_X^p \,\frac{\mathrm{d}t}{t}\Big)^{1/p} < \infty\Big\}. \tag{2.2.7}$$

When equipped with the norm $\|x\|_{\theta,p} := \|x\| + [x]_{\theta,p}$, the space $D_{\mathcal{A}}(\theta,p)$ becomes a Banach space. For details and more on interpolation spaces we refer, e.g., to [24,25]. It is well-known that $D_{\mathcal{A}}(\theta,p)$ coincides with the real interpolation space $(X, D(\mathcal{A}))_{\theta,p}$ and that the respective norms are equivalent. If $0 \in \rho(\mathcal{A})$, then the real interpolation space norm is equivalent to the homogeneous norm $[\cdot]_{\theta,p}$, see [17, Corollary 6.5.5]. Consider in particular the bidomain operator A in $X = L^q(\Omega)$ for $1 < q < \infty$. Then, following Amann [2, Theorem 5.2], the space $(X, D(A))_{\theta,p}$ can be characterized as

$$(\mathrm{L}^q(\Omega), D(A))_{\theta,p} = \mathrm{B}^{2\theta}_{q,p}(\Omega), \quad 1 \leq p \leq \infty, \tag{2.2.8}$$

provided $2\theta \in (0, 1+1/q)$. Here $\mathrm{B}^s_{q,p}(\Omega)$ denotes, as usual, the Besov space of order $s \geq 0$.

For $0 < T < \infty$, we define the solution space $\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}$ as

$$\mathbb{E}^{\mathrm{per}}_{\mathcal{A}} := \{u \in \mathrm{W}^{1,p}(0,T; D_{\mathcal{A}}(\theta,p)) : \mathcal{A}u \in \mathrm{L}^p(0,T; D_{\mathcal{A}}(\theta,p)) \text{ and } u(0) = u(T)\}$$

with norm

$$\|u\|_{\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}} := \|u\|_{\mathrm{W}^{1,p}(0,T;D_{\mathcal{A}}(\theta,p))} + \|\mathcal{A}u\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))},$$

which corresponds to the data space

$$\mathbb{F}_{\mathcal{A}} := \mathrm{L}^p(0,T; D_{\mathcal{A}}(\theta,p)).$$

In our situation, where A denotes the bidomain operator, the solution space for the transmembrane potential u reads as

$$\mathbb{E}^{\mathrm{per}}_{A} = \{u \in \mathrm{W}^{1,p}(0,T; D_A(\theta,p)) : Au \in \mathrm{L}^p(0,T; D_A(\theta,p)) \text{ and } u(0) = u(T)\}.$$

The solution space for the gating variable w is defined as

$$\mathbb{E}^{\mathrm{per}}_{w} := \{w \in \mathrm{W}^{1,p}(0,T; D_A(\theta,p)) : w(0) = w(T)\}.$$

Then, the solution space for the periodic bidomain system is defined as the product space

$$\mathbb{E} := \mathbb{E}^{\mathrm{per}}_{A} \times \mathbb{E}^{\mathrm{per}}_{w}.$$

Finally, for a Banach space X we denote by $\mathbb{B}^X(u^*, R)$ the closed ball in X with center $u^* \in X$ and radius $R > 0$, i.e.,

$$\mathbb{B}^X(u^*, R) := \{u \in X : \|u - u^*\|_X \le R\}.$$

2.3 Main results for various models

In this section we state our main results concerning the existence and uniqueness of strong T-periodic solutions to the bidomain equations subject to various models of the ionic transport. Notice that the respective models treated here are slightly more general as described in the introduction, as an additional parameter $\varepsilon > 0$ is introduced, that incorporates the phenomenon of fast and slow diffusion.

Additionally to Assumption E on the conductivity matrices of the bidomain operator A, we require the following regularity and periodicity conditions on the forcing term I.

Assumption P. Let $1 \le p < \infty$ and $n < q < \infty$ satisfy $1/p + n/(2q) \le 3/4$. Assume $I : \mathbb{R} \to D_A(\theta, p)$ is a T-periodic function satisfying $I_{|(0,T)} \in \mathbb{F}_A$ for some $\theta \in (0, 1/2)$ and $T > 0$.

Remark 2.2. If Ω has a C^4-boundary and if the conductivity matrices σ_i and σ_e lie in $\mathrm{W}^{3,\infty}(\Omega; \mathbb{R}^{n\times n})$, then Assumption P is satisfied by virtue of (3.2.5) if $I_i, I_e : \mathbb{R} \to D_A(\theta, p)$ are T-periodic functions satisfying $I_{i|_{(0,T)}}$ and $I_{e|_{(0,T)}} \in \mathbb{F}_A$. Indeed, this follows by real interpolation since $A_i(A_i + A_e)^{-1}$ is bounded on $\mathrm{L}^q_{av}(\Omega)$ and from $D(A) \cap \mathrm{L}^q_{av}(\Omega)$ into $\mathrm{W}^{2,q}(\Omega) \cap \mathrm{L}^q_{av}(\Omega)$.

We start with the most classical model due to FitzHugh and Nagumo.

2.3.1 The periodic bidomain FitzHugh–Nagumo model

For $T > 0$, $0 < a < 1$, and $b, c, \varepsilon > 0$, the periodic bidomain FitzHugh–Nagumo equations are given by

$$\text{(2.3.1)} \qquad \begin{cases} \partial_t u + \varepsilon A u = I - \dfrac{1}{\varepsilon}[u^3 - (a+1)u^2 + au + w] & \text{in } \mathbb{R}\times\Omega, \\ \partial_t w = cu - bw & \text{in } \mathbb{R}\times\Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R}\times\Omega, \\ w(t) = w(t+T) & \text{in } \mathbb{R}\times\Omega. \end{cases}$$

This system has three equilibrium points, the trivial one $(u_1, w_1) = (0, 0)$ and two others given by (u_2, w_2) and (u_3, w_3), where

(2.3.2)
$$u_2 = \frac{1}{2}(a+1-d), \quad w_2 = \frac{c}{2b}(a+1-d), \quad u_3 = \frac{1}{2}(a+1+d), \quad w_3 = \frac{c}{2b}(a+1+d),$$

and $d = \sqrt{(a+1)^2 - 4(a+\frac{c}{b})}$. We assume that the following stability condition $(\mathrm{S_{FN}})$ on the coefficients is satisfied:

$$(\mathrm{S_{FN}}) \qquad c < b\left(\frac{(a-1)^2}{4} - a\right) \quad \text{and} \quad u_3 > \frac{1}{3}\left(a + 1 + \sqrt{(a+1)^2 - 3a}\right).$$

Our result on strong periodic solutions to the bidomain FitzHugh–Nagumo equations reads then as follows.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded C^2-domain and suppose that Assumptions E and P hold true.*

a) Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$, the equation (2.3.1) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \mathbb{B}^{\mathbb{E}}((0,0), R)$.*

b) If condition $(\mathrm{S_{FN}})$ *is satisfied, then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$, the equation* (2.3.1) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \mathbb{B}^{\mathbb{E}}((u_3, w_3), R)$.*

2.3.2 The periodic bidomain Aliev–Panfilov model

For $T > 0$, $0 < a < 1$, and $d, k, \varepsilon > 0$, the periodic bidomain Aliev–Panfilov equations are given by

$$(2.3.3) \qquad \begin{cases} \partial_t u + \varepsilon A u = I - \dfrac{1}{\varepsilon}[ku^3 - k(a+1)u^2 + kau + uw] & \text{in } \mathbb{R} \times \Omega, \\ \partial_t w = -(ku(u-1-a) + dw) & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega, \\ w(t) = w(t+T) & \text{in } \mathbb{R} \times \Omega. \end{cases}$$

This system has only one stable equilibrium point, namely the trivial solution $(u_1, w_1) = (0, 0)$. Our theorem on the existence and uniqueness of strong, periodic solutions to the periodic bidomain Aliev–Panfilov equations reads as follows.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded C^2-domain and suppose that Assumptions E and P hold true. Then, there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$, the equation* (2.3.3) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \mathbb{B}^{\mathbb{E}}((0,0), R)$.*

2.3.3 The periodic bidomain Rogers–McCulloch model

For $T > 0$, $0 < a < 1$, and $b, c, d, \varepsilon > 0$, the periodic bidomain Rogers–McCulloch equations are given by

$$
\begin{cases}
\partial_t u + \varepsilon A u = I - \frac{1}{\varepsilon}[bu^3 - b(a+1)u^2 + bau + uw] & \text{in } \mathbb{R} \times \Omega, \\
\partial_t w = cu - dw & \text{in } \mathbb{R} \times \Omega, \\
u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega, \\
w(t) = w(t+T) & \text{in } \mathbb{R} \times \Omega.
\end{cases}
\tag{2.3.4}
$$

This system has three equilibrium points, the trivial one $(u_1, w_1) = (0, 0)$ and two others given by (u_2, w_2) and (u_3, w_3), where

$$
u_2 = \frac{1}{2}(a + 1 - \frac{c}{bd} - e), \quad w_2 = \frac{c}{2d}(a + 1 - \frac{c}{bd} - e), \tag{2.3.5}
$$

$$
u_3 = \frac{1}{2}(a + 1 - \frac{c}{bd} + e), \quad w_3 = \frac{c}{2d}(a + 1 - \frac{c}{bd} + e), \tag{2.3.6}
$$

and $e = \sqrt{\big(a + 1 - \frac{c}{bd}\big)^2 - 4a}$. We assume that the following stability condition ($\mathrm{S_{RM}}$) on the coefficients is satisfied:

$$
\sqrt{\big(a + 1 - \frac{c}{bd}\big)^2 - 4a} - \frac{c}{bd} > 0. \tag{$\mathrm{S_{RM}}$}
$$

Our theorem on the existence and uniqueness of strong periodic solutions to the periodic bidomain Rogers–McCulloch equations reads as follows.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded C^2-domain and suppose that Assumptions E and P hold true.*

a) Then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$, the equation (2.3.4) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \mathbb{B}^{\mathbb{E}}((0, 0), R)$.*

b) If condition ($\mathrm{S_{RM}}$) *is satisfied, then there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$, the equation* (2.3.4) *admits a unique T-periodic strong solution (u, w) with $(u, w)_{|(0,T)} \in \mathbb{B}^{\mathbb{E}}((u_3, w_3), R)$.*

2.3.4 The periodic bidomain Allen–Cahn equation

For $T > 0$, the periodic bidomain Allen–Cahn equation is given by

$$
\begin{cases}
\partial_t u + Au = I + u - u^3 & \text{in } \mathbb{R} \times \Omega, \\
u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega.
\end{cases}
\tag{2.3.7}
$$

This system has three equilibrium points, $u_1 = -1$, $u_2 = 0$, and $u_3 = 1$ and our theorem on the existence and uniqueness of strong, periodic solutions to the periodic bidomain Allen–Cahn equation reads as follows.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded C^2-domain and suppose that Assumptions E and P hold true.*

a) Then, there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$ the equation (2.3.7) *admits a unique T-periodic strong solutions u with $u_{|(0,T)} \in \mathbb{B}^{\mathbb{E}_A^{\mathrm{per}}}(-1, R)$.*

b) Then, there exist constants $R > 0$ and $C(R) > 0$ such that if $\|I\|_{\mathbb{F}_A} < C(R)$ the equation (2.3.7) *admits a unique T-periodic strong solutions u with $u_{|(0,T)} \in \mathbb{B}^{\mathbb{E}_A^{\mathrm{per}}}(1, R)$.*

2.4 A periodic version of the Da Prato–Grisvard theorem

Let X be a Banach space and $-\mathcal{A}$ be the generator of a bounded analytic semigroup on X. Assume that $\theta \in (0,1)$, $1 \leq p < \infty$, and $0 < T < \infty$. Then, for $f \in \mathrm{L}^p(0,T; D_{\mathcal{A}}(\theta,p))$ we consider

(2.4.1)
$$u(t) := \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s, \qquad 0 < t < T.$$

Then, u is the unique mild solution to the abstract Cauchy problem

(ACP)
$$\begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & 0 < t < T \\ u(0) = 0 \end{cases}$$

and fulfills, thanks to the classical Da Prato and Grisvard theorem [12], the following maximal regularity estimate.

Proposition 2.7 ([12, Da Prato, Grisvard])**.** *Let $\theta \in (0,1)$, $1 \leq p < \infty$, and $0 < T < \infty$. Then there exists a constant $C > 0$ such that for all $f \in \mathrm{L}^p(0,T; D_{\mathcal{A}}(\theta,p))$, the function u given by* (2.4.1) *satisfies $u(t) \in D(\mathcal{A})$ for almost every $0 < t < T$ and*

$$\|\mathcal{A}u\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))} \leq C\|f\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

We remark at this point that the theorem above implies that the mild solution u to (ACP) is in fact a strong solution satisfying $u'(t) + \mathcal{A}u(t) = f(t)$ for almost every $0 < t < T$.

The proof of our main results are based on the following periodic version of the Da Prato–Grisvard theorem, which is also of independent interest. To this end, we define the periodicity of measurable functions as follows. For some $0 < T < \infty$, we say a measurable function $f : \mathbb{R} \to X$ is called *periodic of period T* if $f(t) = f(t+T)$ holds true for almost all $t \in (-\infty, \infty)$.

For $\theta \in (0,1)$, $1 \leq p < \infty$, and $0 < T < \infty$ assume that $f : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$ is periodic of period T. Then the periodic version of (ACP) reads as

(PACP)
$$\begin{cases} u'(t) + \mathcal{A}u(t) = f(t), & t \in \mathbb{R}, \\ u(t) = u(t+T), & t \in \mathbb{R}. \end{cases}$$

Formally, a candidate for a solution u of (PACP) is given by

(2.4.2)
$$u(t) := \int_{-\infty}^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s.$$

The following lemma shows that, under certain assumptions on $\mathcal{A}$ and f, u is indeed well-defined, continuous and periodic.

Lemma 2.8. *Let $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ be a T-periodic function satisfying $f_{|(0,T)} \in \mathrm{L}^p(0, T; D_{\mathcal{A}}(\theta, p))$ and assume that $0 \in \rho(\mathcal{A})$. Then, the function u defined by* (2.4.2) *is well-defined, satisfies $u \in \mathrm{C}(\mathbb{R}; D_{\mathcal{A}}(\theta, p))$, and is T-periodic.*

Proof. Let $k_0 \in \mathbb{Z}$ be such that $-k_0T < t \le -(k_0 - 1)T$. Using Hölder's inequality, the periodicity of f, and the exponential decay of $\mathrm{e}^{-t\mathcal{A}}$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{t} \|\mathrm{e}^{-(t-s)\mathcal{A}} f(s)\|_{D_{\mathcal{A}}(\theta,p)} \,\mathrm{d}s \\
&\quad = \int_{-k_0T}^{t} \|\mathrm{e}^{-(t-s)\mathcal{A}} f(s)\|_{D_{\mathcal{A}}(\theta,p)} \,\mathrm{d}s + \sum_{k=k_0}^{\infty} \int_{-(k+1)T}^{-kT} \|\mathrm{e}^{-(t-s)\mathcal{A}} f(s)\|_{D_{\mathcal{A}}(\theta,p)} \,\mathrm{d}s \\
&\quad \le C\Big(\int_{0}^{t+k_0T} \|f(s)\|^p_{D_{\mathcal{A}}(\theta,p)} \,\mathrm{d}s \Big)^{\frac{1}{p}} + C \sum_{k=k_0}^{\infty} \mathrm{e}^{-\omega kT} \int_0^T \|\mathrm{e}^{-(T-s)\mathcal{A}} f(s)\|_{D_{\mathcal{A}}(\theta,p)} \,\mathrm{d}s \\
&\quad \le C\Big(1 + \sum_{k=k_0}^{\infty} \mathrm{e}^{-\omega kT}\Big)\Big(\int_0^T \|f(s)\|^p_{D_{\mathcal{A}}(\theta,p)} \,\mathrm{d}s \Big)^{\frac{1}{p}}
\end{aligned}$$

for some $\omega > 0$. It follows that u is well-defined. For the continuity of u we write for $h > 0$

$$u(t+h) - u(t) = \int_t^{t+h} \mathrm{e}^{-(t+h-s)\mathcal{A}} f(s) \,\mathrm{d}s + \int_{-\infty}^{t} \mathrm{e}^{-(t-s)\mathcal{A}} [\mathrm{e}^{-h\mathcal{A}} - \mathrm{Id}] f(s) \,\mathrm{d}s.$$

By the boundedness of the semigroup it suffices to consider the second integral. This resembles the expression from the first part of the proof but with f being replaced by $[\mathrm{e}^{-h\mathcal{A}} - \mathrm{Id}]f$. Thus,

$$\Big\| \int_{-\infty}^{t} \mathrm{e}^{-(t-s)\mathcal{A}} [\mathrm{e}^{-h\mathcal{A}} - \mathrm{Id}] f(s) \,\mathrm{d}s \Big\|_{D_{\mathcal{A}}(\theta,p)} \le C\Big(\int_0^T \|[\mathrm{e}^{-h\mathcal{A}} - \mathrm{Id}] f(s)\|^p_{D_{\mathcal{A}}(\theta,p)} \,\mathrm{d}s \Big)^{\frac{1}{p}}$$

and the right-hand side tends to zero as $h \to 0$ by Lebesgue's theorem. The periodicity of u directly follows by using the transformation $s' = s + T$ and the periodicity of f. □

We now state the periodic version of the Da Prato–Grisvard theorem.

Theorem 2.9. *Let X be a Banach space and $-\mathcal{A}$ be the generator of a bounded analytic semigroup on X with $0 \in \rho(\mathcal{A})$. Let $\theta \in (0, 1)$, $1 \le p < \infty$, and $0 < T < \infty$.*

Then there exists a constant $C > 0$ such that for all periodic functions $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ with $f_{|(0,T)} \in \mathrm{L}^p(0, T; D_{\mathcal{A}}(\theta, p))$ the function u defined by (2.4.2) *lies in $\mathrm{C}(\mathbb{R}; D_{\mathcal{A}}(\theta, p))$, is periodic of period T, satisfies $u(t) \in D(\mathcal{A})$ for almost every $t \in \mathbb{R}$, and satisfies*

$$\|\mathcal{A}u\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))} \le C\|f\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

Proof. The continuity and periodicity of u are proven in Lemma 2.8. Let $t \in [0,T)$ and use the transformation $s' = s + (k+1)T$ for $k \in \mathbb{N}_0$ as well as that f is periodic to write

$$u(t) = \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s + \sum_{k=0}^{\infty} \mathrm{e}^{-(t+kT)\mathcal{A}} \int_0^T \mathrm{e}^{-(T-s)\mathcal{A}} f(s)\,\mathrm{d}s. \tag{2.4.3}$$

In the following, use the notation

$$\mathfrak{u} := \int_0^T \mathrm{e}^{-(T-s)\mathcal{A}} f(s)\,\mathrm{d}s.$$

Since Proposition 2.7 implies

$$\int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s \in D(\mathcal{A}) \qquad (\text{a.e. } t \in (0,T))$$

and

$$\Big\| t \mapsto \mathcal{A} \int_0^t \mathrm{e}^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}s \Big\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))} \leq C \|f\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))},$$

by the exponential decay of the semigroup, it suffices to prove the estimate

$$\|t \mapsto \mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))} \leq C \|f\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))}.$$

Step 1.

Let $\gamma_1, \gamma_2 \in (0,1)$ with $\gamma_1 + \gamma_2 = 1$ and $1/p' < \gamma_2 < 1 - \theta + 1/p'$, where p' denotes the Hölder conjugate exponent to p. Then, the boundedness and the analyticity of the semigroup, followed by a linear transformation and Hölder's inequality imply

$$\begin{aligned}
\|\mathcal{A}\mathrm{e}^{-\tau\mathcal{A}}\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X &\leq C \int_0^T \frac{1}{(T+\tau+t-s)^{\gamma_1}} \frac{1}{(T+\tau+t-s)^{\gamma_2}} \|\mathcal{A}\mathrm{e}^{-(T+\tau+t-s)/2\mathcal{A}} f(s)\|_X \,\mathrm{d}s \\
&= C \int_t^{T+t} \frac{1}{(\tau+s)^{\gamma_1}} \frac{1}{(\tau+s)^{\gamma_2}} \|\mathcal{A}\mathrm{e}^{-(\tau+s)/2\mathcal{A}} f(T+t-s)\|_X \,\mathrm{d}s \\
&\leq C(\tau+t)^{1/p'-\gamma_2} \bigg(\int_t^{T+t} \frac{1}{(\tau+s)^{\gamma_1 p}} \|\mathcal{A}\mathrm{e}^{-(\tau+s)/2\mathcal{A}} f(T+t-s)\|_X^p \,\mathrm{d}s \bigg)^{\frac{1}{p}}.
\end{aligned}$$

Notice that $1/p' < \gamma_2$ was eminent in the calculation above. Next, $t > 0$ implies

(2.4.4)

$$\|\mathcal{A}\mathrm{e}^{-\tau\mathcal{A}}\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X \leq C\tau^{1/p'-\gamma_2} \bigg(\int_t^{T+t} \frac{1}{(\tau+s)^{\gamma_1 p}} \|\mathcal{A}\mathrm{e}^{-(\tau+s)/2\mathcal{A}} f(T+t-s)\|_X^p \,\mathrm{d}s \bigg)^{\frac{1}{p}}.$$

Step 2.

An application of (2.4.4) and Fubini's theorem yields

$$\int_0^T \|\mathcal{A}\mathrm{e}^{-\tau\mathcal{A}}\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X^p \,\mathrm{d}t$$

$$\leq C\tau^{p(1/p'-\gamma_2)} \int_0^{2T} \int_{\max\{0,s-T\}}^{\min\{T,s\}} \frac{1}{(\tau+s)^{\gamma_1 p}} \|\mathcal{A}\mathrm{e}^{-(\tau+s)/2\mathcal{A}} f(T+t-s)\|_X^p \,\mathrm{d}t\,\mathrm{d}s.$$

Notice that the inner integral can be estimated by using $\min\{T,s\} \leq s$. The transformation $t' = T+t-s$ delivers then the estimate

(2.4.5)

$$\|t \mapsto \mathcal{A}\mathrm{e}^{-\tau\mathcal{A}}\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_{\mathrm{L}^p(0,T;X)}^p \leq C\tau^{p(1/p'-\gamma_2)} \int_0^{2T} \int_{\max\{0,T-s\}}^{T} \frac{1}{(\tau+s)^{\gamma_1 p}} \|\mathcal{A}\mathrm{e}^{-(\tau+s)/2\mathcal{A}} f(t)\|_X^p \,\mathrm{d}t\,\mathrm{d}s.$$

Step 3.

Use Fubini's theorem first and then (2.4.5) to estimate the full norm by

$$\int_0^T [\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}]_{\theta,p}^p \,\mathrm{d}t \leq C \int_0^\infty \tau^{\gamma-1} \int_0^{2T} \int_{\max\{0,T-s\}}^{T} \frac{1}{(\tau+s)^{\gamma_1 p}} \|\mathcal{A}\mathrm{e}^{-(\tau+s)/2\mathcal{A}} f(t)\|_X^p \,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\tau,$$

where $\gamma = p(1+1/p'-\theta-\gamma_2)$. Apply Fubini's theorem followed by the substitution $s' = \tau + s$ to get

$$\int_0^T [\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}]_{\theta,p}^p \,\mathrm{d}t \leq C \int_0^T \int_0^\infty \tau^{\gamma-1} \int_{T+\tau-t}^{2T+\tau} \frac{1}{s^{\gamma_1 p}} \|\mathcal{A}\mathrm{e}^{-s/2\mathcal{A}} f(t)\|_X^p \,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}t.$$

Finally, use Fubini's theorem in order to calculate the τ-integral (here $\gamma_2 < 1-\theta+1/p'$ is essential) and note that $t-T$ is negative and γ positive to get

$$\begin{aligned}\int_0^T [\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}]_{\theta,p}^p \,\mathrm{d}t &\leq \frac{C}{\gamma} \int_0^T \int_{T-t}^\infty \frac{1}{s^{\gamma_1 p}} \|\mathcal{A}\mathrm{e}^{-s/2\mathcal{A}} f(t)\|_X^p (s+t-T)^\gamma \,\mathrm{d}s\,\mathrm{d}t \\ &\leq \frac{C}{\gamma} \int_0^T \int_{T-t}^\infty s^{\gamma-\gamma_1 p} \|\mathcal{A}\mathrm{e}^{-s/2\mathcal{A}} f(t)\|_X^p \,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

The proof is concluded by definition γ and of the real interpolation space norm, since this gives

$$\int_0^T [\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}]_{\theta,p}^p \,\mathrm{d}t \leq \frac{2^{p(1-\theta)}C}{2\gamma} \|f\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))}^p.$$

Step 4.

In this step, we estimate $\int_0^T \|\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X \,\mathrm{d}t$. It is known, see [17, Corollary 6.6.3], that $D_{\mathcal{A}}(\vartheta,1) \hookrightarrow D(\mathcal{A}^\vartheta)$ and that $D_{\mathcal{A}}(\theta,p) \hookrightarrow D_{\mathcal{A}}(\vartheta,1)$ for every $0 < \vartheta < \theta$. Thus,

$$D_{\mathcal{A}}(\theta,p) \hookrightarrow D(\mathcal{A}^\vartheta).$$

Now, let $\vartheta_1, \vartheta_2, \vartheta_3 \in (0,1)$ with $\vartheta_1+\vartheta_2+\vartheta_3 = 1$, $\vartheta_1 < \theta$, $\vartheta_2 p' < 1$ and $\vartheta_3 p < 1$, where p' denotes the Hölder conjugate exponent to p. Then, the bounded analyticity of $e^{-t\mathcal{A}}$, Hölder's inequality and the above embedding imply

$$\|\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X = \|\mathcal{A}^{\vartheta_3}\mathrm{e}^{-t\mathcal{A}} \int_0^T \mathcal{A}^{\vartheta_2}\mathrm{e}^{-(T-s)\mathcal{A}}\mathcal{A}^{\vartheta_1} f(s)\,\mathrm{d}s\|_X$$

$$
\begin{aligned}
&\leq Ct^{-\vartheta_3} \int_0^T (T-s)^{-\vartheta_2} \|A^{\vartheta_1} f(s)\|_X \, \mathrm{d}s \\
&\leq Ct^{-\vartheta_3} \left(\int_0^T (T-s)^{-\vartheta_2 p'} \, \mathrm{d}s \right)^{\frac{1}{p'}} \left(\int_0^T \|A^{\vartheta_1} f(s)\|_X^p \, \mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq Ct^{-\vartheta_3} \|f\|_{\mathrm{L}^p(0,T:D_{\mathcal{A}}(\theta,p))}.
\end{aligned}
$$

Consequently,

$$
\int_0^T \|\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\mathfrak{u}\|_X \, \mathrm{d}t \leq c\|f\|_{\mathrm{L}^p(0,T:D_{\mathcal{A}}(\theta,p))}. \qquad \square
$$

We conclude this section by showing that, under the assumptions of Theorem 2.9, u defined by (2.4.2) indeed is the unique strong solution to (PACP).

Proposition 2.10. *Under the hypotheses of Theorem 2.9 the function u defined by* (2.4.2) *is the unique strong solution to* (PACP)*, i.e., u is the unique periodic function of period T in $\mathrm{C}(\mathbb{R}; X)$ that is for almost every $t \in \mathbb{R}$ differentiable in t, satisfies $u(t) \in D(\mathcal{A})$, and $\mathcal{A}u \in \mathrm{L}^p(0,T;X)$, and u solves*

$$
u'(t) + \mathcal{A}u(t) = f(t).
$$

Proof. First of all, u is periodic by Lemma 2.8 and since $D_{\mathcal{A}}(\theta,p)$ continuously embeds into X the very same lemma implies $u \in \mathrm{C}(\mathbb{R};X)$.

Assume first that $f_{|(0,T)} \in \mathrm{L}^p(0,T;D(\mathcal{A}))$. Then, by a direct calculation, u defined by (2.4.2) is differentiable, satisfies $u(t) \in D(\mathcal{A})$, and solves

$$
u'(t) + \mathcal{A}u(t) = f(t)
$$

for every $t \in \mathbb{R}$. The density of $\mathrm{L}^p(0,T;D(\mathcal{A}))$ in $\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))$ and the estimate proven in Theorem 2.9 imply that all these properties carry over to all right-hand sides in $\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))$ (but only for almost every $t \in \mathbb{R}$) by an approximation argument.

For the uniqueness, assume that $v \in \mathrm{C}(\mathbb{R};X)$ with $v', \mathcal{A}v \in \mathrm{L}^p(0,T;X)$ is another periodic function of period T which satisfies the equation for almost every $t \in \mathbb{R}$. Let $w := u - v$. Then w satisfies

$$
w'(t) = -\mathcal{A}w(t) \qquad (\text{a.e. } t \in \mathbb{R}).
$$

In this case, for $t > 0$, w can be written by means of the semigroup as $w(t) = \mathrm{e}^{-t\mathcal{A}}(u(0) - v(0))$. Now, the exponential decay of the semigroup and the periodicity of w imply that w must be zero for all $t \in \mathbb{R}$. $\square$

Remark 2.11. Combining Theorem 2.9 and Proposition 2.10 shows that for each periodic f with period T and $f_{|(0,T)} \in \mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))$ also $u'_{|(0,T)} \in \mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))$. The same is true for u since $0 \in \rho(\mathcal{A})$. Summarizing, there exists a constant $C > 0$ such that

$$
\|u\|_{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}} \leq C\|f\|_{\mathrm{L}^p(0,T;D_{\mathcal{A}}(\theta,p))}, \tag{2.4.6}
$$

where $\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}$ is defined as in the end of Section 3.2.

2.5 Time periodic solutions for semilinear equations

In this section, we use the periodic version of the Da Prato–Grisvard theorem to construct time periodic solutions to semilinear parabolic equations by employing Banach's fixed point theorem. The framework that is presented here includes all the models from Section 2.3.

2.5.1 An abstract existence theorem for general types of nonlinearities

Let $-\mathcal{A}$ be the generator of a bounded analytic semigroup $\mathrm{e}^{-t\mathcal{A}}$ on a Banach space X with the domain $D(\mathcal{A})$ and $0 \in \rho(\mathcal{A})$. For $T > 0$, $\theta \in (0,1)$, and $1 \le p < \infty$ let $f : \mathbb{R} \to D_{\mathcal{A}}(\theta, p)$ be periodic of period T with $f_{|(0,T)} \in \mathrm{L}^p(0,T; D_{\mathcal{A}}(\theta,p))$. We are aiming for the strong solvability of

$$
\text{(NACP)} \qquad \begin{cases} u'(t) + \mathcal{A}u(t) = F[u](t) + f(t) & (t \in \mathbb{R}) \\ u(t) = u(t+T) & (t \in \mathbb{R}) \end{cases}
$$

under some smallness assumptions on f. The solution u will be constructed in the space of maximal regularity $\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}$ defined in the end of Section 3.2. Recall the corresponding data space

$$
\mathbb{F}_{\mathcal{A}} = \mathrm{L}^p(0,T; D_{\mathcal{A}}(\theta,p))
$$

and let $\mathbb{B}_\rho := \mathbb{B}^{\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}}(0,\rho)$ for some $\rho > 0$. For the nonlinear term F, we make the following standard assumption.

Assumption N. There exists $R > 0$ such that the nonlinear term F is a mapping from $\mathbb{B}_R$ into $\mathbb{F}_{\mathcal{A}}$ and satisfies

$$
F \in \mathrm{C}^1(\mathbb{B}_R; \mathbb{F}_{\mathcal{A}}), \quad F(0) = 0, \quad \text{and} \quad DF(0) = 0,
$$

where $DF : \mathbb{B}_R \to \mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}, \mathbb{F}_{\mathcal{A}})$ denotes the Fréchet derivative.

The following theorem proves existence and uniqueness of solutions to (NACP) in the class $\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}$ for small forcings f.

Theorem 2.12. *Let $T > 0$, $0 < \theta < 1$, $1 \le p < \infty$, and F and $R > 0$ subject to Assumption N. Then there is a constant $r \le R$ and $c = c(T,\theta,p,r) > 0$ such that if $f : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$ is T-periodic with $\|f\|_{\mathbb{F}_{\mathcal{A}}} \le c$, then there exists a unique solution $u : \mathbb{R} \to D_{\mathcal{A}}(\theta,p)$ of* (NACP) *with the same period T and $u_{|(0,T)} \in \mathbb{B}_r$.*

Proof. Let $S : \mathbb{B}_R \to \mathbb{E}_{\mathcal{A}}^{\mathrm{per}}, v \mapsto u_v$ be the solution operator of the linear equation

$$
u_v'(t) + \mathcal{A}u_v(t) = F[v(t)] + f(t) \quad \text{in } (0,T)
$$

with $u_v(0) = u_v(T)$. This is well-defined since $F[v] \in \mathbb{F}_{\mathcal{A}}$ by Assumption N, so that, by Proposition 2.10 and Remark 2.11, u_v uniquely exists and lies in $\mathbb{E}_{\mathcal{A}}^{\mathrm{per}}$.

We prove that this solution operator is a contraction on $\mathbb{B}_r$ for some $r \le R$. Let $M > 0$ denote the infimum of all constants C satisfying (2.4.6). Choose $r > 0$ small enough such that

$$\sup_{w\in\mathbb{B}_r} \|DF[w]\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\text{per}},\mathbb{F}_{\mathcal{A}})} \leq \frac{1}{2M},$$

which is possible by Assumption N. By virtue of (2.4.6) as well as the mean value theorem, estimate for any $v \in \mathbb{B}_r$ and f satisfying $\|f\|_{\mathbb{F}_{\mathcal{A}}} \leq r/(2M) =: c$,

$$\|S(v)\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}} \leq M(\|F[v]\|_{\mathbb{F}_{\mathcal{A}}} + \|f\|_{\mathbb{F}_{\mathcal{A}}}) \leq M(\sup_{w\in\mathbb{B}_r} \|DF[w]\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\text{per}},\mathbb{F}_{\mathcal{A}})}\|v\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}} + \|f\|_{\mathbb{F}_{\mathcal{A}}}) \leq r.$$

So $S(\mathbb{B}_r) \subset \mathbb{B}_r$. Similarly, for any $v_1, v_2 \in \mathbb{B}_r$,

$$\|S(v_1) - S(v_2)\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}} \leq M \sup_{w\in\mathbb{B}_r} \|DF[w]\|_{\mathcal{L}(\mathbb{E}_{\mathcal{A}}^{\text{per}},\mathbb{F}_{\mathcal{A}})}\|v_1 - v_2\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}} \leq \frac{1}{2}\|v_1 - v_2\|_{\mathbb{E}_{\mathcal{A}}^{\text{per}}}.$$

Consequently, the solution operator S is a contraction on $\mathbb{B}_r$ and the contraction mapping theorem is applicable. The solution to (NACP) is defined as follows. Let u be the unique fixed point of S. Since $Su = u$, u satisfies $u(0) = u(T)$ and thus can be extended periodically to the whole real line. This function solves (NACP). ☐

2.5.2 Two special examples

A short glimpse towards the models presented in Subsections 2.3.1-2.3.4 reveals that one of the following situations occurs:

- The bidomain operator A appears only in the first but not in the second equation of the bidomain models and the nonlinearity depends linearly on the gating variable w. (Subsections 2.3.1-2.3.3)
- The ODE and the gating variable w are omitted. (Subsection 2.3.4)

As a consequence, in the first situation the operator associated with the linearization of the bidomain models can be written as an operator matrix whose first component of the domain embeds into a $\mathrm{W}^{2,q}$-space. Since the dynamics of the gating variable is described only by an ODE, there appears no smoothing in the spatial variables of w. However, as we aim to employ Theorem 2.12 and as the nonlinearity of the first equation depends linearly on w, at least in the models of Aliev–Panfilov and Rogers–McCulloch, w must be contained in $D_A(\theta, p)$. Otherwise one cannot view the nonlinearity as a suitable right-hand side as it is done in Subsection 2.5.1. Hence, we choose $D_A(\theta, p)$ as the ground space for the gating variable.

To describe this situation in our setup, assume in the following, that $-\mathcal{A}$ is the generator of a bounded analytic semigroup on a Banach space $X = X_1 \times X_2$, with domain $D(\mathcal{A}) = D(A_1) \times D(A_2)$, and $0 \in \rho(\mathcal{A})$. We further set for some $1 < q < \infty$, $1 \leq p < \infty$, and $\theta \in (0, 1)$

$$X_1 = \mathrm{L}^q(\Omega), \quad D(A_1) = D(A), \quad \text{and} \quad X_2 = D(A_2) = D_A(\theta, p).$$

Furthermore, define two types of nonlinearities as follows: For $a_1, a_2, a_3, a_4 \in \mathbb{R}$ let

$$F_1[u_1, u_2] := \begin{pmatrix} a_1 u_1^2 + a_2 u_1^3 + a_3 u_1 u_2 \\ a_4 u_1^2 \end{pmatrix}$$

and for b_1, $b_2 \in \mathbb{R}$ let

$$F_2[u_1] := b_1 u_1^2 + b_2 u_1^3.$$

Here, F_1 will be a prototype of the nonlinearities considered in Subsections 2.3.1-2.3.3 and F_2 for the one considered in Subsection 2.3.4. For the moment, the condition $0 \in \rho(\mathcal{A})$ seems inappropriate as $0 \notin \rho(A)$. However, we will linearize the bidomain equations around suitable stable stationary solutions and in this situation $0 \in \rho(\mathcal{A})$ will be achieved.

In the following, we concentrate only on F_1, since the results for F_2 may be proved in a similar way. To derive conditions on p, q, and θ ensuring that F_1 satisfies Assumption N, the following two lemmas are essential. The first one is a consequence of the mixed derivative theorem, see, e.g., [14] and reads as follows.

Lemma 2.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain, $T > 0$, $1 < p, q < \infty$, and $\sigma \in [0, 1]$. Then the following continuous embedding is valid*

$$\mathrm{W}^{1,p}(0, T; \mathrm{L}^q(\Omega)) \cap \mathrm{L}^p(0, T; \mathrm{W}^{2,q}(\Omega)) \subset \mathrm{W}^{\sigma,p}(0, T; \mathrm{W}^{2(1-\sigma),q}(\Omega)).$$

Lemma 2.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain, $1 \le p < \infty$, $1 < q < \infty$, $q \le r, s \le \infty$, $1/r + 1/s = 1/q$, and $\theta \in (0, 1/2)$. Then there exists a constant $C > 0$ such that for*

$$\|uv\|_{\mathrm{B}^{2\theta}_{q,p}(\Omega)} \le C \|u\|_{\mathrm{W}^{1,s}(\Omega)} \|v\|_{\mathrm{B}^{2\theta}_{r,p}(\Omega)} \qquad (u \in \mathrm{W}^{1,s}(\Omega), v \in \mathrm{B}^{2\theta}_{r,p}(\Omega)).$$

Proof. Assume first that $v \in \mathrm{W}^{1,r}(\Omega)$. By Hölder's inequality it follows that

$$\|uv\|_{\mathrm{L}^q(\Omega)} \le \|u\|_{\mathrm{L}^s(\Omega)} \|v\|_{\mathrm{L}^r(\Omega)} \quad \text{and} \quad \|uv\|_{\mathrm{W}^{1,q}(\Omega)} \le 2 \|u\|_{\mathrm{W}^{1,s}(\Omega)} \|v\|_{\mathrm{W}^{1,r}(\Omega)}.$$

Now, real interpolation delivers the desired inequality. □

In the following proposition we elaborate the conditions on p, q, and θ that ensure that F maps $\mathbb{E}^{\mathrm{per}}_{\mathcal{A}}$ into $\mathbb{F}_{\mathcal{A}}$.

Proposition 2.15. *Let $1 \le p < \infty$, $n < q < \infty$ satisfy $1/p + n/(2q) \le 3/4$ and $\theta \in (0, 1/2)$ there exists a constant $C > 0$ such that*

$$\|F_1(u_1, u_2)\|_{\mathbb{F}_{\mathcal{A}}} \le C \big(\|u_1\|^2_{\mathbb{E}^{\mathrm{per}}_{A_1}} + \|u_1\|^3_{\mathbb{E}^{\mathrm{per}}_{A_1}} + \|u_1\|_{\mathbb{E}^{\mathrm{per}}_{A_1}} \|u_2\|_{\mathbb{E}^{\mathrm{per}}_{A_2}} \big)$$

for all $u_1 \in \mathbb{E}^{\mathrm{per}}_{A_1}$ and $u_2 \in \mathbb{E}^{\mathrm{per}}_{A_2}$.

Proof. We start with the first component of F_1. By (2.2.8) we have $D_A(\theta, p) = \mathrm{B}^{2\theta}_{q,p}(\Omega)$ and Lemma 2.14 implies

$$\|u_1 u_2\|^p_{\mathrm{L}^p(0,T; D_A(\theta,p))} \le C \|u_1 u_2\|^p_{\mathrm{L}^p(0,T; \mathrm{B}^{2\theta}_{q,p}(\Omega))} \le C \int_0^T \|u_1\|^p_{\mathrm{W}^{1,\infty}(\Omega)} \|u_2\|^p_{\mathrm{B}^{2\theta}_{q,p}(\Omega)} \, \mathrm{d}t,$$

by choosing $r = q$, $s = \infty$ in Lemma 2.14. Using that $\mathrm{W}^{1,p}(0,T;\mathrm{B}^{2\theta}_{q,p}(\Omega)) \subset \mathrm{L}^{\infty}(0,T;\mathrm{B}^{2\theta}_{q,p}(\Omega))$ delivers

$$\|u_1u_2\|^p_{\mathrm{L}^p(0,T;D_A(\theta,p))} \leq C\|u_2\|^p_{\mathrm{W}^{1,p}(0,T;\mathrm{B}^{2\theta}_{q,p}(\Omega))}\|u_1\|^p_{\mathrm{L}^p(0,T;\mathrm{W}^{1,\infty}(\Omega))}.$$

Finally, note that $D(A_1) \subset \mathrm{W}^{2,q}(\Omega) \subset \mathrm{W}^{1,\infty}(\Omega)$ if $n < q$. Next, by the continuous embedding $\mathrm{W}^{1,q}(\Omega) \subset \mathrm{B}^{2\theta}_{q,p}(\Omega)$, Hölder's inequality and the mixed derivative theorem, we obtain for $\alpha \in \{2,3\}$

$$\|u_1^{\alpha}\|_{\mathrm{L}^p(0,T;D_A(\theta,p))} \leq C\|u_1\|^{\alpha}_{\mathrm{L}^{\alpha p}(0,T;\mathrm{W}^{1,\alpha q}(\Omega))} \leq C\|u_1\|^{\alpha}_{\mathrm{W}^{\sigma,p}(0,T;\mathrm{W}^{2(1-\sigma),q}(\Omega))}.$$

provided $\sigma \in [0,1]$ satisfies

$$\sigma - 1/p \geq -1/(\alpha p), \quad \text{and} \quad 2(1-\sigma) - n/q \geq 1 - n/(\alpha q).$$

The condition $1/p + n/(2q) \leq 3/4$ guarantees the existence of σ for $\alpha \in \{2,3\}$. The second component of F_1 was already estimated above. □

Finally, by definition of F_1 it is clear that $F_1(0,0) = 0$. Moreover, due to the polynomial structure of F_1 it is clear that F_1 is Fréchet differentiable with $DF_1(0,0) = 0$. Hence, we have the following proposition.

Proposition 2.16. *With the definitions of this subsection the nonlinearities F_1 and F_2 satisfy Assumption N.*

2.6 Proofs of the Main Theorems

Before treating the models described in Section 2.3, we remark that the linear part of the bidomain systems will be represented as an operator matrix and it will be eminent that the negative of this operator matrix generates a bounded analytic semigroup. This will be proven in the following lemma.

Lemma 2.17. *Let $-B$ be the generator of a bounded analytic semigroup on a Banach space X_1 with $0 \in \rho(B)$, $1 \leq p < \infty$, and $\theta \in (0,1)$. Let $X_2 = D_B(\theta,p)$ and define for $d > 0$ and $b, c \geq 0$ the operator $\mathcal{A} : X := X_1 \times X_2 \to X$ with domain $D(\mathcal{A}) := D(B) \times X_2$ by*

$$\mathcal{A} := \begin{pmatrix} B & b \\ -c & d \end{pmatrix}.$$

Then $-\mathcal{A}$ generates a bounded analytic semigroup on X with $0 \in \rho(\mathcal{A})$.

Proof. Let Σ_ω, $\omega \in (\pi/2, \pi]$, be a sector that satisfies $\rho(-B) \subset \Sigma_\omega$ with

$$\|\lambda(\lambda + B)^{-1}\|_{\mathcal{L}(X_1)} \leq C \qquad (\lambda \in \Sigma_\omega).$$

First note that $0 \in \rho(\mathcal{A})$; its inverse being

$$\mathcal{A}^{-1} = \begin{pmatrix} d & -b \\ c & B \end{pmatrix} (bc + dB)^{-1}.$$

Note that the choice $X_2 = D_B(\theta, p)$ is used here as $\mathcal{A}^{-1}$ is only an operator from $X_1 \times X_2$ onto $D(B) \times X_2$ if $D(B) \subset X_2 \subset X_1$ and if $B(bc + dB)^{-1}$ maps X_2 into X_2. By the definition of $D_B(\theta, p)$ in (2.2.7) this latter is satisfied.

For the resolvent problem let $\lambda \in \Sigma_\beta$, $\beta \in (\pi/2, \omega)$ to be chosen. Then,

$$(\lambda + \mathcal{A})^{-1} = (\lambda + d)^{-1} \begin{pmatrix} \lambda + d & -b \\ c & \lambda + B \end{pmatrix} \Big(\lambda + \frac{bc}{\lambda + d} + B\Big)^{-1}$$

whenever $\lambda + \frac{bc}{\lambda+d} \in \rho(-B)$. To determine the angle β for which $\lambda + \frac{bc}{\lambda+d} \in \rho(-B)$ distinguish between the cases $|\lambda| < M$ and $|\lambda| \geq M$ for some suitable constant $M > 0$. Notice that only the case $b, c > 0$ is of interest. Let $C_\omega > 0$ be a constant depending solely on ω such that $|\lambda + d| \geq C_\omega(|\lambda| + d)$. Choose M such that $|\lambda| \geq M$ if and only if

$$C_\omega \sin(\omega - \beta)[|\lambda|^2 + d|\lambda|] \geq 2bc. \tag{2.6.1}$$

This implies

$$\Big|\frac{bc}{\lambda + d}\Big| \leq \frac{bc}{C_\omega(|\lambda| + d)} \leq \frac{|\lambda| \sin(\omega - \beta)}{2}$$

and thus that $\lambda + \frac{bc}{\lambda+d} \in \Sigma_\omega$. Moreover,

$$\Big|\lambda + \frac{bc}{d + \lambda}\Big| \geq |\lambda| \Big(1 - \frac{\sin(\omega - \beta)}{2}\Big). \tag{2.6.2}$$

Next, choose β that close to $\pi/2$ such that

$$M \sin(\beta - \pi/2) \leq \frac{bcd}{bc + (d + M)^2}. \tag{2.6.3}$$

Notice that M itself depends on β, however, it depends only uniformly on its distance to ω by (2.6.1). In the case $|\lambda| < M$ the validity of (2.6.3) together with trigonometric considerations implies that $\operatorname{Re}\big(\lambda + \frac{bc}{d+\lambda}\big) \geq 0$ proving that under conditions (2.6.1) and (2.6.3) we have $\lambda + \frac{bc}{d+\lambda} \in \Sigma_\omega$ whenever $\lambda \in \Sigma_\beta$. We conclude that $\lambda \in \rho(-\mathcal{A})$. To obtain the resolvent estimate, we calculate

$$\begin{aligned}
&\|\lambda(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \\
\leq &\Big\|\lambda\Big(\lambda + \frac{bc}{\lambda + d} + B\Big)^{-1}\Big\|_{\mathcal{L}(X_1)} + \Big|\frac{\lambda b}{\lambda + d}\Big| \Big\|\Big(\lambda + \frac{bc}{\lambda + d} + B\Big)^{-1}\Big\|_{\mathcal{L}(X_2, X_1)} \\
&+ \Big|\frac{\lambda c}{\lambda + d}\Big| \Big\|\Big(\lambda + \frac{bc}{\lambda + d} + B\Big)^{-1}\Big\|_{\mathcal{L}(X_1, X_2)} + \Big|\frac{\lambda}{\lambda + d}\Big| \Big\|(\lambda + B)\Big(\lambda + \frac{bc}{\lambda + d} + B\Big)^{-1}\Big\|_{\mathcal{L}(X_2)}.
\end{aligned}$$

The first term on the right-hand side is directly handled by the resolvent estimate of B. The second is treated by this resolvent estimate as well and by noting that $X_2 \subset X_1$. The fourth term is estimated by using that the definition of X_2 in (2.2.7) implies resolvent estimates in X_2 (the resolvent commutes with the semigroup appearing in (2.2.7)). For the third term, the estimate follows from the invertibility of B and the interpolation inequality $\|x\|_{X_2} \le C\|x\|_{X_1}^{1-\theta}\|Bx\|_{X_1}^{\theta}$. Altogether, this yields

$$\begin{aligned}&\|\lambda(\lambda+\mathcal{A})^{-1}\|_{\mathcal{L}(X)}\\ \le& C\bigg(|\lambda| + \Big|\frac{\lambda b}{\lambda+d}\Big| + \Big|\frac{\lambda c}{\lambda+d}\Big|\Big|\lambda+\frac{bc}{\lambda+d}\Big|^{\theta} + \Big|\frac{\lambda^2}{\lambda+d}\Big|\bigg)\Big|\lambda+\frac{bc}{\lambda+d}\Big|^{-1} + C\Big|\frac{\lambda}{\lambda+d}\Big|.\end{aligned}$$

The resolvent estimate for $|\lambda| \ge M$ follows by means of the uniform boundedness of the term $|\lambda/(\lambda+d)|$ and by (2.6.2).

For $|\lambda| < M$ the function $\lambda \mapsto \lambda(\lambda+\mathcal{A})^{-1}$ is continuous on $\overline{\Sigma_\beta} \cap \overline{B(0,M)}$ since $0 \in \rho(\mathcal{A})$. This implies the resolvent estimate also for small λ. □

Now, we are ready to prove the main results presented in Section 2.3. To do so, the equilibrium points of the nonlinearities are calculated for the respective models. Afterwards, the solutions to the bidomain models are written as the sum of the equilibrium solution and a perturbation. This results in an equation for the perturbation which is shown via Theorem 2.12 to have strong periodic solutions for suitable equilibrium points.

2.6.1 The periodic bidomain FitzHugh–Nagumo equation

Recall the periodic bidomain FitzHugh–Nagumo equation

$$\begin{cases} \partial_t u + \varepsilon A u = I - \frac{1}{\varepsilon}[u^3 - (a+1)u^2 + au + w] & \text{in } \mathbb{R}\times\Omega, \\ \partial_t w = cu - bw & \text{in } \mathbb{R}\times\Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R}\times\Omega, \\ w(t) = w(t+T) & \text{in } \mathbb{R}\times\Omega. \end{cases} \tag{2.6.4}$$

In order to calculate the equilibrium points, we consider

$$u^3 - (a+1)u^2 + au + w = 0, \tag{2.6.5}$$

$$cu - bw = 0. \tag{2.6.6}$$

Then, the equilibrium points are $(u_1, w_1) = (0,0)$ and assuming $c < b\big(\frac{(a+1)^2}{4} - a\big)$, we obtain furthermore

$$(u_2, w_2) = \left(\frac{1}{2}(a+1-d), \frac{c}{2b}(a+1-d)\right), \tag{2.6.7}$$

$$(u_3, w_3) = \left(\frac{1}{2}(a+1+d), \frac{c}{2b}(a+1+d)\right), \tag{2.6.8}$$

with $d = \sqrt{(a+1)^2 - 4(a+\frac{c}{b})}$. In the following, we use the results from Sections 2.4 and 2.5 to obtain periodic solutions in a neighborhood of these equilibrium points. For this purpose, we use Taylor expansion at the equilibrium points and perform the following change of variables

$$\begin{pmatrix} v \\ z \end{pmatrix} := \begin{pmatrix} u - u_i \\ w - w_i \end{pmatrix}$$

for $i = 1, 2, 3$. Then, functions F and G describing the ionic transport defined as in the introduction read as follows

$$F(v, z) = \frac{1}{\varepsilon}[v^3 + (3u_i - a - 1)v^2 + (3u_i^2 - 2(a + 1)u_i + a)v + z],$$
$$G(v, z) = -cv + bz.$$

Plugging this into equation (2.6.4) and shifting the linear parts of F and G to the left-hand side yields

(2.6.9)
$$\begin{cases} \partial_t \begin{pmatrix} v \\ z \end{pmatrix} + \begin{pmatrix} \varepsilon A + \frac{1}{\varepsilon}[3u_i^2 - 2(a+1)u_i + a] & \frac{1}{\varepsilon} \\ -c & b \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} = \begin{pmatrix} I - \frac{1}{\varepsilon}[v^3 + (3u_i - a - 1)v^2] \\ 0 \end{pmatrix}, \\ v(t) = v(t+T), \\ z(t) = z(t+T). \end{cases}$$

First of all, notice that Proposition 2.16 implies that the nonlinearity in (2.6.9) satisfies Assumption N. Next, regarding the system with respect to the equilibrium point $(0, 0)$, then $-(\varepsilon A + \frac{a}{\varepsilon})$ generates a bounded analytic semigroup by Proposition 2.1 and since $0 \in \rho(\varepsilon A + \frac{a}{\varepsilon})$, we may apply Lemma 2.17 to conclude that the negative of the operator matrix in (2.6.9) has zero in its resolvent set and generates a bounded analytic semigroup. Consequently, Theorem 2.12 is applicable in the case of the equilibrium point $(0, 0)$ and delivers a unique strong periodic solution (v, z) to (2.6.9) in the desired function space for small periodic forcings I.

For the second equilibrium point we have $3u_2^2 - 2(a + 1)u_2 + a < 0$. Since $0 \in \sigma(A)$ the operator $-(\varepsilon A + \frac{1}{\varepsilon}[3u_2^2 - 2(a + 1)u_2 + a])$ does not generate a bounded analytic semigroup so that Lemma 2.17 is not applicable.

If

$$u_3 > \frac{a + 1 + \sqrt{(a + 1)^2 - 3a}}{3},$$

we obtain $3u_3^2 - 2(a + 1)u_3 + a > 0$. Thus, $-(\varepsilon A + \frac{1}{\varepsilon}[3u_3^2 - 2(a + 1)u_3 + a])$ generates a bounded analytic semigroup by Proposition 2.1 and $0 \in \rho(\varepsilon A + \frac{1}{\varepsilon}[3u_3^2 - 2(a+1)u_3 + a])$. Hence, we can apply Lemma 2.17 to conclude that the negative of the operator matrix in (2.6.9) has zero in its resolvent set and generates a bounded analytic semigroup. Consequently, Theorem 2.12 is applicable in this case of the equilibrium point (u_3, w_3) and delivers a unique strong periodic solution (v, z) to (2.6.9) in the desired function spaces for small periodic forcings I. This proves Theorem 2.3.

2.6.2 The periodic bidomain Aliev–Panfilov equation

Recall the periodic bidomain Aliev–Panfilov equation

$$\text{(2.6.10)} \qquad \begin{cases} \partial_t u + \varepsilon A u = I - \dfrac{1}{\varepsilon}[ku^3 - k(a+1)u^2 + kau + uw] & \text{in } \mathbb{R} \times \Omega, \\ \partial_t w = -(ku(u-1-a) + dw) & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega, \\ w(t) = w(t+T) & \text{in } \mathbb{R} \times \Omega. \end{cases}$$

In order to calculate the equilibrium points, we consider

$$\text{(2.6.11)} \qquad ku^3 - k(a+1)u^2 + kau + uw = 0,$$

$$\text{(2.6.12)} \qquad ku(u-1-a) + dw = 0.$$

Then, the equilibrium points are $(u_1, w_1) = (0, 0)$ and, if we assume $\frac{(a+1)^2}{4} + \frac{da}{1-d} > 0$, furthermore

$$\text{(2.6.13)} \qquad (u_2, w_2) = \left(\frac{a+1}{2} - e, -ku_2^2 + k(a+1)u_2 - ka \right),$$

$$\text{(2.6.14)} \qquad (u_3, w_3) = \left(\frac{a+1}{2} + e, -ku_3^2 + k(a+1)u_3 - ka \right).$$

with $e = \sqrt{\frac{(a+1)^2}{4} + \frac{da}{1-d}}$. In the following, we want to use the results from Sections 2.4 and 2.5 to obtain periodic solutions in a neighborhood of these equilibrium points. For this purpose, we use Taylor expansion at the equilibrium points and perform the following change of variables

$$\begin{pmatrix} v \\ z \end{pmatrix} := \begin{pmatrix} u - u_i \\ w - w_i \end{pmatrix}$$

for $i = 1, 2, 3$. Then, functions F and G describing the ionic transport defined as in the introduction read as follows

$$F(v, z) = \frac{1}{\varepsilon}[kv^3 + (3ku_i - k(a+1))v^2 + (3ku_i^2 - 2k(a+1)u_i + ka + w_i)v + u_i z + vz],$$
$$G(v, z) = (2ku_i - k(a+1))v + dz + kv^2.$$

Plugging this into equation (2.6.10) and shifting the linear parts of F and G to the left-hand side yields

$$\text{(2.6.15)} \qquad \begin{cases} \partial_t \begin{pmatrix} v \\ z \end{pmatrix} + \begin{pmatrix} \varepsilon A + \frac{1}{\varepsilon}[3ku_i^2 - 2k(a+1)u_i + ka + w_i] & \frac{u_i}{\varepsilon} \\ 2ku_i - k(a+1) & d \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} \\ \qquad = \begin{pmatrix} I - \frac{1}{\varepsilon}[kv^3 + (3ku_i - k(a+1))v^2 + vz] \\ -kv^2 \end{pmatrix}, \\ v(t) = v(t+T), \\ z(t) = z(t+T). \end{cases}$$

According to Proposition 2.16, the nonlinearity in (2.6.15) satisfies Assumption N. Moreover, considering the system for the equilibrium point $(0,0)$, then $-(\varepsilon A + \frac{ka}{\varepsilon})$ generates a bounded analytic semigroup by Proposition 2.1 and since $0 \in \rho(\varepsilon A + \frac{ka}{\varepsilon})$, we can apply Lemma 2.17 to conclude that the negative of the operator matrix in (2.6.15) has zero in its resolvent set and generates a bounded analytic semigroup. Consequently, Theorem 2.12 is applicable in the case of the equilibrium point $(0,0)$ and delivers a unique strong periodic solution (v,z) to (2.6.15) in the desired function space for small periodic forcings I.

For the second equilibrium point we see that $u_2 < 0$, so that the component in the upper right component of the operator matrix is negative. Therefore, we cannot apply Lemma 2.17 for (u_2, w_2).

Similarly, for (u_3, w_3) it is

$$2ku_3 - k(a+1) = 2ke > 0.$$

Hence, Lemma 2.17 is not applicable in this case. Altogether, Theorem 2.4 follows.

2.6.3 The periodic bidomain Rogers–McCulloch equation

Recall the periodic bidomain Rogers–McCulloch equation

$$\text{(2.6.16)} \qquad \begin{cases} \partial_t u + \varepsilon A u = I - \dfrac{1}{\varepsilon}[bu^3 - b(a+1)u^2 + bau + uw] & \text{in } \mathbb{R} \times \Omega, \\ \qquad \partial_t w = cu - dw & \text{in } \mathbb{R} \times \Omega, \\ \qquad u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega, \\ \qquad w(t) = w(t+T) & \text{in } \mathbb{R} \times \Omega. \end{cases}$$

In order to calculate the equilibrium points, we consider

$$\text{(2.6.17)} \qquad bu^3 - b(a+1)u^2 + bau + uw = 0,$$

$$\text{(2.6.18)} \qquad cu - dw = 0.$$

Then, the equilibrium points are $(u_1, w_1) = (0,0)$ and, if we assume $\big(a + 1 - \frac{c}{bd}\big)^2 - 4a > 0$, furthermore

$$\text{(2.6.19)} \qquad (u_2, w_2) = \left(\frac{1}{2}\Big(a + 1 - \frac{c}{bd} - e\Big), \frac{c}{2d} \cdot \Big(a + 1 - \frac{c}{bd} - e\Big)\right),$$

$$\text{(2.6.20)} \qquad (u_3, w_3) = \left(\frac{1}{2}\Big(a + 1 - \frac{c}{bd} + e\Big), \frac{c}{2d} \cdot \Big(a + 1 - \frac{c}{bd} + e\Big)\right).$$

with $e = \sqrt{\big(a + 1 - \frac{c}{bd}\big)^2 - 4a}$. In the following, we want to use the results from Sections 2.4 and 2.5 to obtain periodic solutions in a neighborhood of these equilibrium points. For this purpose, we use Taylor expansion at the equilibrium points and perform the following change of variables

$$\begin{pmatrix} v \\ z \end{pmatrix} := \begin{pmatrix} u - u_i \\ w - w_i \end{pmatrix}$$

for $i = 1, 2, 3$. Then, functions F and G describing the ionic transport defined as in Section 3.1 read as follows

$$
\begin{aligned}
F(v,z) &= \frac{1}{\varepsilon}[bv^3 + (3bu_i - b(a+1))v^2 + (3bu_i^2 - 2b(a+1)u_i + ba + w_i)v + u_i z + vz],\\
G(v,y) &= -cv + dz.
\end{aligned}
$$

Plugging this into equation (2.6.16) and shifting the linear parts of F and G to the left-hand side yields

$$
\text{(2.6.21)} \qquad
\begin{cases}
\partial_t \begin{pmatrix} v \\ z \end{pmatrix} + \begin{pmatrix} \varepsilon A + \frac{1}{\varepsilon}[3bu_i^2 - 2b(a+1)u_i + ba + w_i] & \frac{u_i}{\varepsilon} \\ -c & d \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} \\
\qquad = \begin{pmatrix} I - \frac{1}{\varepsilon}[bv^3 + (3bu_i - b(a+1))v^2 + vz] \\ 0 \end{pmatrix}, \\
v(t) = v(t+T), \\
z(t) = z(t+T).
\end{cases}
$$

According to Proposition 2.16, the nonlinearity in (2.6.21) satisfies Assumption N. Next, considering the equilibrium point $(0,0)$, the operator $-(\varepsilon A + \frac{ba}{\varepsilon})$ generates a bounded analytic semigroup by Proposition 2.1 and since $0 \in \rho(\varepsilon A + \frac{ba}{\varepsilon})$, we can apply Lemma 2.17 to conclude that the negative of the operator matrix in (2.6.21) has zero in its resolvent set and generates a bounded analytic semigroup. Consequently, Theorem 2.12 is applicable in the case of the equilibrium point $(0,0)$ and delivers a unique strong periodic solution (v,z) to (2.6.21) in the desired function space for small forcings I.

Next, equation (2.6.17) implies $w_i = -bu_i^2 + b(a+1)u_i - ba$ for $i = 2, 3$. Then

$$3bu_i^2 - 2b(a+1)u_i + ba + w_i = u_i(2bu_i - b(a+1)).$$

Hence, for the second equilibrium point we either have $3bu_2^2 - 2b(a+1)u_2 + ba + w_2 < 0$, then $-(\varepsilon A + \frac{1}{\varepsilon}[3bu_2^2 - 2b(a+1)u_2 + ba + w_2])$ does not generate a bounded analytic semigroup, or $u_2 < 0$. Therefore, we cannot apply Lemma 2.17 for (u_2, w_2).

If we assume

$$\sqrt{\left(a+1-\frac{c}{bd}\right)^2 - 4a} - \frac{c}{bd} > 0,$$

we obtain $3bu_3^2 - 2b(a+1)u_3 + ba + w_3 > 0$ and $u_3 > 0$. Thus, $-(\varepsilon A + \frac{1}{\varepsilon}[3bu_3^2 - 2b(a+1)u_3 + ba + w_3])$ generates a bounded analytic semigroup by Proposition 2.1 and $0 \in \rho(\varepsilon A + \frac{1}{\varepsilon}[3bu_3^2 - 2b(a+1)u_3 + ba + w_3])$. Hence, we can apply Lemma 2.17 to conclude that the negative of the operator matrix in (2.6.21) has zero in its resolvent and generates a bounded analytic semigroup. Thus, Theorem 2.12 is applicable in this case for (u_3, w_3) and delivers a unique strong periodic solution (v,z) in the desired function space for small forcings I. This delivers Theorem 2.5.

2.6.4 The periodic bidomain Allen–Cahn equation

Recall the periodic bidomain Allen–Cahn equation

$$\begin{cases} \partial_t u + Au = I + u - u^3 & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega. \end{cases} \tag{2.6.22}$$

The equilibrium points of this system are $u_1 = -1$, $u_2 = 0$, and $u_3 = 1$. In the following, we want to use the results from Sections 2.4 and 2.5 to obtain periodic solutions in a neighborhood of these equilibrium points. For this purpose, we use Taylor expansion at the equilibrium points and perform the change of variables $v = u - u_i$ for $i = 1, 2, 3$. Then, the function $F(u) = u^3 - u$ reads as follows

$$F(v) = v^3 + 3u_i v^2 - (1 - 3u_i^2)v, \qquad i = 1, 2, 3.$$

Plugging this into equation (2.6.22) and shifting the linear parts of F to the left-hand side yields

$$\begin{cases} \partial_t v + (A - 1 + 3u_i^2)v = I - v^3 - 3u_i v^2 & \text{in } \mathbb{R} \times \Omega, \\ u(t) = u(t+T) & \text{in } \mathbb{R} \times \Omega \end{cases} \tag{2.6.23}$$

for $i = 1, 2, 3$. According to Proposition 2.16, the nonlinearity in (2.6.23) satisfies Assumption N. Since $-(A+2)$ generates a bounded analytic semigroup by Proposition 2.1 and since $0 \in \rho(A+2)$, Theorem 2.12 is applicable in the case of the equilibrium points u_1 and u_3 and delivers a unique strong periodic solution v to (2.6.23) in the desired function space for small forcings I. Thus, we obtain Theorem 2.6.

2.7 Discussion: biological significance of our mathematical results

Anatomically, the human heart consists of four chambers. The two lower ones are called ventricles, the two upper ones are the atria. The ventricles possess thick walls and their contraction pushes blood through the human body. The atria, having thin walls, collect the blood and their contraction delivers it back to the ventricles. Excitation of the heart starts at the sinoatrial node located in the atrium. The cells in this node *periodically* initiate excitation waves. The wave then propagates through the atria causing atrial contraction. The excitation enters, after a certain delay, the ventricles and causes the contraction of the ventricles.

Cardiac arrhytmias is the result of abnormal excitation of the heart. Its mechanisms are strongly related to wave propagation phenomena. The waves in question possess the property of refractoriness, meaning that after excitation, the cardiac cell requires some time to recover its original properties. This time is called the refractory period. If the travel time of a wave propagating around an obstacle is longer than the refractory period a rotation will take place yielding in the production of vortices. Such a rotation implies periodic excitation of the heart with a frequency much faster than the one by the sinus mode and so-called tachycardia may occur. The above vortices are called *spiral waves*.

The cardiac modeling involves, of course, not only the understanding of a single cardiac cell but also the knowledge of how the changes in a single cardiac cell will manifest in the whole organ.

Modeling single cardiac cells has a long tradition in mathematical biology. Almost all cardiac cell models are based on the model of Hodgkin and Huxley in 1952. Their general aim was to understand the mechanism which governs movements of ions through the cell membrane during electrical activity. Like all biological cells, cardiac cells are surrounded by a membrane. This membrane acts as an interface separating intracellular and extracellular media and prevents free passage of the ion from the one side to the other one. Ion channels allow particular species to cross this membrane. This *gating* process is regulated by the electrical potential difference across the membrane. Biological cells maintain stable equilibria membrane potentials. If a current is applied, the media may then be excitable or not. The Hodgkin–Huxley model describes the electrical activities of the cell by a system of ordinary differential equation for four state variables, the membrane potential V and the three gating variables m, n, h controlling the permeability of the membrane for behavior for certain classes of ions. This system was investigated numerically by Hodgkin and Huxley. The results of an investigation of this ODE-system by means of methods from dynamical systems may be interpreted from a biological point of view as follows: If the change of the membrane potential is sufficiently small, the cell does not react at all. However, if the excitation is above a certain threshold value, the cell fires a so-called spike. Moreover, if the excitation values lie in a certain, well-defined interval, the system allows for *periodic solutions* meaning that the cell is periodically firing spikes.

Aiming for a complete picture of the dynamical behavior of the Hodgkin–Huxley system, it is desirable to have a system at hand, which is on the one hand easier to handle from a mathematical perspective but shows nevertheless all the dynamical features of the Hodgkin–Huxley system. The FitzHugh–Nagumo model, a simplification of the Hodgkin–Huxley model, shows the same dynamical behavior as the Hodgkin–Huxley model, however, it consists only of two state variables. This has the advantage that its dynamical behavior can be investigated in a phase plane and in particular, aiming for an explanation for the existence of periodic solutions, the Poincaré–Bendixson theorem is available in this two-dimensional setting. Note that the latter does not hold in a higher dimensional setting. The FitzHugh–Nagumo model takes into account the different time scaling of the variables m, n, and h, distinguishes between slow reactions for m and h and fast reactions for n and setting m and h to be constant within the fast scale, one arrives at the system described in the introduction.

The Rogers–McCulloch model, also discussed in the introduction, is a modification of the FitzHugh–Nagumo model taking into account the hyperpolarization of the refractory part of the action potential. The Aliev–Panfilov models another modification accounting for the restitution property. For a comprehensive list of various single cell models, see, e.g., [31].

The phase plane analysis of the FitzHugh–Nagumo model is well understood: every stationary point is asymptotically stable. The existence of periodic solutions to this system is shown by means of the Poincaré–Bendixson theorem. Indeed, one can show that the FitzHugh–Nagumo model possesses a periodic solution if and only if the values of the current I applied are located in a certain interval and the coefficients involved satisfy certain inequalities. It is interesting to note that periodic solutions to the Hodgkin–Huxley model were indeed measured experimentally by Hodgkin and Huxley.

Starting from this understanding of a single cardiac cell, it is now a major task to build a *macroscopic model* describing the propagation of a wave through cardiac *tissues*. In cardiac tissues, the excitable cells are related to each other via so-called gap junctions. A first model representing the tissue as a resistive network yielded in the *monodomain equations*, in which the diffusive current is modeled by the term $\mathrm{div}(D\nabla)$ for a constant diffusion tensor D based on the assumptions of equal anisotropy ratios $\sigma_e = \alpha\sigma_i$, where σ_i and σ_e denote the conductivity parameters of the intra- and extracellular domains. The latter is, however, an unrealistic assumption and was introduced only for reasons of mathematical simplicity.

The so-called *bidomain model*, the main object of our analysis, was introduced for the first time by L. Tung in his PhD thesis [36]. Its aim is to describe the spatial distribution of macroscopic potentials as they are measured on the surface of the heart. Intra- and extracellular quantities are considered in the bidomain model via an elaborate homogenization procedure in an averaged sense. For details see, e.g., [32]. As a result, the bidomain system considers the cardiac tissue as a two-phase medium, where the intra- and extracellular domains occupy the same macroscopic space and overlap at every point. Each domain is considered as a homogeneous structure. The coupling between the two domains is provided by the transmembrane current.

It is now a very interesting question to ask in mathematical biology, whether the dynamical behavior of the macroscopic bidomain model qualitatively is in accordance with the measurements and whether one is able to describe, understand, and predict the dynamical behavior of this system by mathematical methods including periodic solutions. Secondly, it would be very interesting to determine in which way the macroscopic bidomain model resembles the dynamical behavior of a single cell model. Results on the dynamics of the bidomain system are extremely rare.

A first step in this direction was done recently by Mori and Matano [28], who studied for the first time the stability of front solutions of the bidomain equations, however, under restrictive assumptions on the conductivity parameters of the intra- and extracellular domains. The main difficulty is that the bidomain equation is given by a coupled system of partial differential equations for the intra- and extracellular currents, u_i and u_e, and which is in addition coupled to an ordinary differential equation. Writing this system as an evolution equation for $u = u_i - u_e$, its dynamical behavior is determined by spectral properties of an operator matrix involving the bidomain operator. The latter is a *very* nonlocal operator.

From a biological perspective, it would be very interesting to know whether the bidomain model subject to various models of ionic transport allow - as in the situation of the Hodgkin–Huxley or FitzHugh–Nagumo model, respectively, for single cells - again for periodic solutions. Another interesting direction is the investigation of dynamical instabilities resulting in the generation of spiral waves, as discussed above. A very interesting result concerns an instability that occurs as a result of period-doubling bifurcation for a map describing the periodically forced cardiac cell, see [7]. There necessary conditions for onset of instability leading to spiral waves have been investigated. These conditions are related to measurable characteristics of cardiac tissue and to the period of the stimulation of cardiac cells. These conditions were also tested in experimental research, see [19].

Our investigations show that, similarly to the case of the FitzHugh–Nagumo model for a single cell, the macroscopic bidomain model allows - in accordance with measurements for cardiac tissue - also for periodic solutions, in our case, however, in a neighborhood of a stable equilibrium point. They are characterized by the values of the coefficients of the underlying models of ionic transport, see Section 2.3 for a precise formulation. The existence of periodic solutions to the bidomain equations in a neighborhood of an instable equilibrium point remains a very interesting topic of our further investigations, mathematically and in particular from a biological perspective, since spiral waves are intimately connected to arrhythmias.

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Chapter 3

Strong time-periodic solutions to the bidomain equations with arbitrary large forces

We prove the existence of strong time-periodic solutions to the bidomain equations with arbitrary large forces. We construct weak time-periodic solutions by a Galerkin method combined with Brouwer's fixed point theorem and a priori estimate independent of approximation. We then show their regularity so that our solution is a strong time-periodic solution in L^2 spaces. Our strategy is based on the weak-strong uniqueness method.

Keywords: bidomain model; periodic solutions; weak-strong uniqueness, large data

3.1 Introduction

In this chapter, we consider the bidomain system which is a well-established system describing the electrical wave propagation in the heart. The system is given by

$$\text{(BDE)} \qquad \begin{cases} \partial_t u - \operatorname{div}(\sigma_i \nabla u_i) + f(u,w) = s_i & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + \operatorname{div}(\sigma_e \nabla u_e) + f(u,w) = -s_e & \text{in } (0,\infty)\times\Omega, \\ \partial_t w + g(u,w) = 0 & \text{in } (0,\infty)\times\Omega, \\ u = u_i - u_e & \text{in } (0,\infty)\times\Omega, \\ \sigma_i \nabla u_i \cdot \nu = 0,\ \sigma_e \nabla u_e \cdot \nu = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0) = u_0,\ w(0) = w_0 & \text{in } \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^d$ denotes a domain describing the myocardium and the outward unit normal vector to $\partial\Omega$ is denoted by ν. The unknown functions u_i and u_e model the intra- and extracellular electric potentials, and u denotes the transmembrane potential. The variable w, the so-called gating variable, corresponds to the ionic transport through the cell membrane. The anisotropic properties of the intra- and extracellular tissue parts are described by the conductivity matrices $\sigma_i(x)$ and $\sigma_e(x)$, whereas $s_i(t,x)$ and $s_e(t,x)$ denote the intra- and extracellular stimulation current, respectively. The ionic transport is described by the nonlinearities f and g. In this article, we will consider a large class of ionic models including those by FitzHugh–Nagumo, Rogers–McCulloch, and

Aliev–Panfilov. Note, that we will look at the Aliev–Panfilov model in a slightly modified form as considered, e.g., in [13]. The *FitzHugh–Nagumo model* reads as

$$\begin{aligned} f(u,w) &= u(u-a)(u-1)+w = u^3-(a+1)u^2+au+w, \\ g(u,w) &= -\varepsilon(ku-w), \end{aligned}$$

with $0 < a < 1$ and $k, \varepsilon > 0$.
In the *Rogers–McCulloch model* the functions f and g are given by

$$\begin{aligned} f(u,w) &= bu(u-a)(u-1)+uw = bu^3-b(a+1)u^2+bau+uw, \\ g(u,w) &= -\varepsilon(ku-w), \end{aligned}$$

with $0 < a < 1$ and $b, k, \varepsilon > 0$.
For the modified *Aliev–Panfilov model* we have

$$\begin{aligned} f(u,w) &= bu(u-a)(u-1)+uw = bu^3-b(a+1)u^2+bau+uw, \\ g(u,w) &= \varepsilon(ku(u-1-d)+w) \end{aligned}$$

with $0 < a, d < 1$ and $b, k, \varepsilon > 0$. To get weak time-periodic solutions for the Aliev–Panfilov model we will need the further assumption $b > k$. For a detailed description of the bidomain model we refer to the monographs by Keener and Sneyd [20] and Colli Franzone, Pavarino, and Scacchi [9].

Since the bidomain model describes electrical activities in the heart, it is a natural question to ask whether it admits time-periodic solutions. Therefore, consider the situation where the bidomain model is innervated by periodic intra- and extracellular stimulation currents s_i and s_e. Recently, Hieber, Kajiwara, Kress, and Tolksdorf [9] proved the existence and uniqueness of a strong T-periodic solution to the innervated model in real interpolation spaces provided the external forces s_i and s_e are both time-periodic of period $T > 0$. In their approach, they furthermore assumed that the external forces satisfy a suitable *smallness condition*.

It is the goal of this chapter to prove the existence of time-periodic solutions without assuming any smallness condition on the external forces. We employ the method given by Galdi, Hieber, and Kashiwabara [12] for the case of the primitive equations. First, the existence of weak time-periodic solutions for the three nonlinear dynamic models mentioned above is shown by using a Galerkin approximation combined with Brouwer's fixed point theorem. Then, we use the global well-posedness result by Colli Franzone and Savaré [11] and consider the *weak* time-periodic solution as a *weak* solution to the initial value problem. Finally, we apply a weak-strong uniqueness argument to get a strong-time periodic solution without assuming any smallness condition for the external applied currents in case of the FitzHugh–Nagumo model.

The bidomain model was first introduced by Tung [36] in 1978. Despite its central importance in cardiac electrophysiology, the rigorous mathematical analysis started not until the work of Colli Franzone and Savaré [11] in 2002. They introduced a variational formulation of the bidomain problem and proved the existence and uniqueness of weak and strong solutions to the bidomain equations with FitzHugh–Nagumo type nonlinearities. A slightly more detailed review of their results is given in Section 3.4. Veneroni [37] extended their results to more general ionic models including the Luo and Rudy I model. In 2009, Bourgault, Cordière, and Pierre [5] presented a new approach to the bidomain system. They introduced the so-called bidomain operator within the L^2-setting and showed that it is a non-negative and self-adjoint operator. By using the bidomain operator, they transformed the bidomain system into an abstract evolution equation and showed the existence and uniqueness of a local strong solution and the existence of a global weak solution for a large class of ionic models including the three models introduced above. Later, Kunisch and Wagner [23] showed uniqueness and further regularity for these weak solutions. Giga and Kajiwara [16] investigated the bidomain system within the L^p-setting and showed that the bidomain operator is the generator of an analytic semigroup on $L^p(\Omega)$ for $p \in (1, \infty]$. Recently, Hieber and Prüss proved the maximal L_p-L_q regularity for the bidomain operator in [15] and proved the global well-posedness in [5]. They considered the case $s_{i,e} = 0$ with FitzHugh-Nagumo type non-linearities. More recently, the bidomain equations were treated as a kind of gradient system in [2]. They proved the global well-posedness results in L^2 spaces and energy spaces. Their paper also treated the case $s_{i,e} = 0$.

For results concerning the dynamics of the solution, we refer to the work of Mori and Matano [28]. They studied the stability of front solutions of the bidomain equations.

On a microscopic level, the cardiac cellular structure is described by two disjoint domains Ω_i and Ω_e, which denote the intra- and extracellular space, respectively, and which are separated by the the active membrane $\overline{\Gamma} = \partial\Omega_i \cap \partial\Omega_e$. The intra- and extracellular quantities are defined on the corresponding domains and the transmembrane potential u is a function on $\overline{\Gamma}$. After a homogenization procedure, see, e.g., [10,11], the macroscopic model of the bidomain equations is obtained. Here all membrane, intra-, and extracellular quantities are defined everywhere on Ω.

This chapter is organized as follows: We start in Section 3.2 with collecting known facts concerning the bidomain operator. In Section 3.3, we construct a weak time-periodic solution to the bidomain equations. When we construct the weak time-periodic solutions, we need some growth conditions on the nonlinear terms f, g. Fortunately, all three models mentioned above fulfill these conditions, which are confirmed in the appendix. A global well-posedness result is reviewed in Section 3.4 and invoked to obtain a strong time-periodic solution for the FitzHugh–Nagumo model. We do not treat the other two models introduced above since the global well-posedness for the *initial value problem* is not proved in a suitable L^2 setting.

3.2 Preliminaries

In this section, we fix some notation and formally introduce the bidomain operator in a weak as well as in a strong setting. In the whole article, let $\Omega \subset \mathbb{R}^d$ denote a bounded domain whose boundary $\partial\Omega$ is of class C^2. For convenience, we use the following notation for the function spaces which we will use throughout this article

$$V = H^1(\Omega),\ H = L^2(\Omega),\ V' = (H^1(\Omega))',$$

where the spaces are endowed with their usual norms and they are Hilbert spaces. Furthermore, we set $Q = (0,T) \times \Omega$. The canonical pair of V' and V is denoted by ${}_{V'}\langle \cdot, \cdot \rangle_V$.

We assume that the conductivity matrices σ_i and σ_e satisfy the following assumption.

Assumption C. The conductivity matrices $\sigma_i, \sigma_e : \overline{\Omega} \to \mathbb{R}^{d\times d}$ are symmetric matrices and are functions of class $C^1(\overline{\Omega})$. Ellipticity is imposed by means of the following condition: there exist constants $\underline{\sigma}$, $\overline{\sigma}$ with $0 < \underline{\sigma} < \overline{\sigma}$ such that

$$\underline{\sigma}|\xi|^2 \le {}^t\xi\sigma_i(x)\xi \le \overline{\sigma}|\xi|^2 \quad \text{and} \quad \underline{\sigma}|\xi|^2 \le {}^t\xi\sigma_e(x)\xi \le \overline{\sigma}|\xi|^2 \tag{3.2.1}$$

for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^d$. Moreover, it is assumed that

$$\begin{array}{llll} \sigma_i \nabla u_i \cdot \nu = 0 & \Leftrightarrow & \nabla u_i \cdot \nu = 0 & \text{on } \partial\Omega, \\ \sigma_e \nabla u_e \cdot \nu = 0 & \Leftrightarrow & \nabla u_e \cdot \nu = 0 & \text{on } \partial\Omega. \end{array} \tag{3.2.2}$$

It is known due to [8] that (3.2.2) is biological reasonable.

First, we want to introduce the bidomain operator in a weak setting as well as the corresponding bidomain bilinear form. Therefore, we define $V_{av}(\Omega) := \{u \in V : \int_\Omega u \, \mathrm{d}x = 0\}$. Following [5], we define the bilinear forms

$$a_i(u,v) := \int_\Omega \sigma_i \nabla u \cdot \nabla v \, \mathrm{d}x, \quad a_e(u,v) := \int_\Omega \sigma_e \nabla u \cdot \nabla v \, \mathrm{d}x$$

for all $(u,v) \in V_{av} \times V_{av}$. Due to (3.2.1) these bilinear forms are symmetric, continuous and uniformly elliptic on $V_{av} \times V_{av}$. Then, we define the weak operators A_i and A_e from V_{av} onto V'_{av} by

$$\langle A_i u, v \rangle := a_i(u,v), \quad \langle A_e u, v \rangle := a_e(u,v)$$

for all $(u,v) \in V_{av} \times V_{av}$. Let P_{av} be the orthogonal projection from V to $V_{av}(\Omega)$, i.e., $P_{av}u := u - \frac{1}{|\Omega|}\int_\Omega u \, \mathrm{d}x$ and denote its transpose by $P_{av}^T : V'_{av} \to V'$. Now we are able to define the weak bidomain operator and the corresponding bidomain bilinear form as

$$\begin{aligned} A &= P_{av}^T A_i (A_i + A_e)^{-1} A_e P_{av}, \\ a(u,v) &= \langle Au, v \rangle \end{aligned}$$

for all $(u,v) \in V \times V$. We have the following lemma.

Lemma 3.1 ([5, Theorem 6]). *The bidomain bilinear form $a(\cdot,\cdot)$ is symmetric, continuous and coercive on V,*

$$\begin{aligned} \alpha\|u\|_V^2 &\le a(u,u) + \alpha\|u\|_H^2, && \text{for all } u \in V, \\ |a(u,v)| &\le M\|u\|_V\|v\|_V, && \text{for all } u, v \in V, \end{aligned}$$

for some constants α, $M > 0$. Furthermore, there exists an increasing sequence $0 = \lambda_0 < \cdots \le \lambda_i \le \cdots$ in $\mathbb{R}$ and an orthonormal Hilbert basis of H of eigenvectors $(\psi_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $\psi_i \in V$ and $v \in V$ it is $a(\psi_i, v) = \lambda_i(\psi_i, v)$.

We next define the strong bidomain operator in the L^q-setting for $1 < q < \infty$. We will use the same notation as for the weak setting since it will be clear from the context whether we consider the weak or strong formulation. To this end, let $L^q_{av}(\Omega) := \{u \in L^q(\Omega) : \int_\Omega u \, \mathrm{d}x = 0\}$ and let P_{av} be the orthogonal projection from $L^q(\Omega)$ to $L^q_{av}(\Omega)$, i.e., $P_{av}u := u - \frac{1}{|\Omega|}\int_\Omega u \, \mathrm{d}x$. Then, we define the elliptic operators A_i and A_e by

$$\begin{aligned} A_{i,e}u &:= -\operatorname{div}(\sigma_{i,e}\nabla u), \\ D(A_{i,e}) &:= \big\{u \in W^{2,q}(\Omega) \cap L^q_{av}(\Omega) : \sigma_{i,e}\nabla u \cdot \nu = 0 \text{ a.e. on } \partial\Omega\big\} \subset L^q_{av}(\Omega). \end{aligned}$$

Here $A_{i,e}$ and $\sigma_{i,e}$ mean that either A_i and σ_i or A_e and σ_e are considered. Condition (3.2.2) implies that $D(A_i) = D(A_e)$. Hence, it is possible to define the sum $A_i + A_e$ with the domain $D(A_i) = D(A_e)$. Note that the inverse operator $(A_i + A_e)^{-1}$ on $L^q_{av}(\Omega)$ is a bounded linear operator.

Following [16] we define the bidomain operator as follows. Let σ_i and σ_e satisfy Assumption C. Then the bidomain operator A is defined as

$$A = A_i(A_i + A_e)^{-1}A_eP_{av} \tag{3.2.3}$$

with domain

$$D(A) := \{u \in W^{2,q}(\Omega) : \nabla u \cdot \nu = 0 \text{ a.e. on } \partial\Omega\}.$$

If we assume conservation of currents, i.e.,

$$\int_\Omega (s_i(t) + s_e(t)) \, \mathrm{d}x = 0, \quad t \geq 0 \tag{3.2.4}$$

and moreover $\int_\Omega u_e \, \mathrm{d}x = 0$, the bidomain equations (BDE) may be equivalently rewritten as an evolution equation [5, 16] of the form

$$\text{(ABDE)} \qquad \begin{cases} \partial_t u + Au + f(u, w) = s, & \text{in } (0, \infty), \\ \partial_t w + g(u, w) = 0, & \text{in } (0, \infty), \\ u(0) = u_0, \ \ w(0) = w_0, \end{cases}$$

where

$$s := s_i - A_i(A_i + A_e)^{-1}(s_i + s_e) \tag{3.2.5}$$

is the modified source term. The functions u_e and u_i can be recovered from u by virtue of the following relations

$$\begin{aligned} u_e &= (A_i + A_e)^{-1}\{(s_i + s_e) - A_iP_{av}u\}, \\ u_i &= u + u_e. \end{aligned}$$

3.3 Weak time-periodic solutions

In this section, we show the existence of weak time-periodic solutions by using a Galerkin approximation. We consider the bidomain equations under some growth conditions on

f and g which contain the nonlinearities introduced in Section 3.1.

We use the abstract form

$$\text{(PABDE)} \qquad \begin{cases} u' + Au + f(u,w) = s, & \text{in } \mathbb{T}\times\Omega, \\ w' + g(u,w) = 0, & \text{in } \mathbb{T}\times\Omega, \\ u(t+T,x) = u(t,x), \;\; w(t+T,x) = w(t,x), \end{cases}$$

where s is a T-periodic function for some $T > 0$ and $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$ denotes the time torus. We assume that the nonlinear terms f and g satisfy the following conditions.

Assumption N. Let $p > 1$ be a number so that the Sobolev embedding $V \subset L^p(\Omega)$ holds. In other words, $2 \le p$ if $d = 2$; or $2 \le p \le 6$ if $d = 3$. The nonlinear terms $f, g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are of the form

$$f(u,w) = f_1(u) + f_2(u)w,$$
$$g(u,w) = g_1(u) + g_2 w,$$

where $g_2 \in \mathbb{R}$ and $f_1, f_2, g_1 : \mathbb{R} \to \mathbb{R}$ are continuous functions. The functions are assumed to satisfy that there exist constants $C_0 \in \mathbb{R}$, $C_i > 0$ $(i = 1,\dots,5)$ and $r > 0$ such that

$$C_0 + C_1|u|^p + C_2|w|^2 \le rf(u,w)u + g(u,w)w \tag{3.3.1}$$
$$|f_1(u)| \le C_3(1 + |u|^{p-1}) \tag{3.3.2}$$
$$|f_2(u)| \le C_4(1 + |u|^{p/2-1}) \tag{3.3.3}$$
$$|g_1(u)| \le C_5(1 + |u|^{p/2}) \tag{3.3.4}$$

for all $u, w \in \mathbb{R}$.

This assumption is a modified version of the assumption used in [5]. Note that Assumption N, with $p = 4$, holds in the three models introduced in Section 3.1. We shall check it in the appendix. We see that the following inequality holds: for any $(u,w) \in L^p(\Omega) \times H$, we have

$$\|f(u,w)\|_{p'}^{p'} \le C_6(1 + \|u\|_p^p + \|w\|_H^2)$$
$$\|g(u,w)\|_H^2 \le C_7(1 + \|u\|_p^p + \|w\|_H^2)$$

for some $C_i > 0$ $(i = 6,7)$ depending on p and $C_3,\dots,C_5$, where p' is the Hölder conjugate exponent, i.e., $1/p + 1/p' = 1$. In particular, $f(u,w) \in L^{p'}(Q)$ and $g(u,w) \in L^2(Q)$ for all $u \in L^p(Q), w \in L^2(Q)$. See in [5, Lemma 25].

Under this assumption, weak time-periodic solutions for (PABDE) are defined as follows.

Definition 3.2. Let $T > 0$, $s \in L^2(\mathbb{T}; V')$. Suppose that the Assumption N holds. Then a pair of (u,w) of $u : \mathbb{T}\times\Omega \to \mathbb{R}$, $w : \mathbb{T}\times\Omega \to \mathbb{R}$ is called a weak T-periodic solution to (PABDE) if

(i) $u \in C_w(\mathbb{T}; H) \cap L^2(\mathbb{T}; V) \cap L^p(Q)$, $w \in C_w(\mathbb{T}; H)$,

(ii) For all $\varphi_1 \in W^{1,2}(0,T;H) \cap L^2(0,T;V) \cap L^p(Q)$ and all $\varphi_2 \in W^{1,2}(0,T;H)$,

$$\int_0^t \{(u, \partial_t \varphi_1) - a(u, \varphi_1) - {}_{p'}\langle f(u,w), \varphi_1 \rangle_p\} \, \mathrm{d}\tau = -\int_0^t {}_{V'}\langle s, \varphi_1 \rangle_V \, \mathrm{d}\tau + (u(t), \varphi_1(t)) - (u(0), \varphi_1(0)),$$

$$\int_0^t \{(w, \partial_t \varphi_2) - (g(u,w), \varphi_2)\} \, \mathrm{d}\tau = (w(t), \varphi_2(t)) - (w(0), \varphi_2(0)),$$

for all $t \in (0,T)$. Here $(\cdot,\cdot)$ denotes the L^2-inner product and $C_w(\mathbb{T}, H)$ denotes the space of all weakly continuous functions u on $\mathbb{T}$ with values in H, i.e., $u : \mathbb{T} \to H$ such that $(u(t), \psi)$ is continuous in t for all $\psi \in H$.

A weak T-periodic solution (u,w) is called strong if, in addition to above, it holds

$$u \in W^{1,2}(\mathbb{T};H) \cap L^2(\mathbb{T};H^2(\Omega)), w \in W^{1,2}(\mathbb{T};H).$$

The result on existence of weak time-periodic solutions reads as follows.

Theorem 3.3. *Let $T > 0$. For every T-periodic function $s \in L^2(\mathbb{T};V')$ there exists at least one weak T-periodic solution (u,w) to* (PABDE).

Proof. Let $\{\psi_i\}_{i=0}^\infty \subset V$ be the orthonormal basis of eigenvectors of the bidomain bilinear form a in H and let $\{\lambda_i\}_{i=0}^\infty \subset \mathbb{R}_{\geq 0}$ be the corresponding eigenvalues as in Lemma 3.1. Let

$$u_k(t,x) := \sum_{i=0}^k \alpha_{ki}(t)\psi_i(x), \tag{3.3.5}$$

$$w_k(t,x) := \sum_{i=0}^k \beta_{ki}(t)\psi_i(x), \tag{3.3.6}$$

with $\alpha_k(t) = \{\alpha_{kj}(t)\}_{j=0}^k$, $\beta_k(t) = \{\beta_{kj}(t)\}_{j=0}^k$, which are the solutions of the system of the ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{kj} = -\alpha_{kj}\lambda_j - \displaystyle\int_\Omega f(u_k, w_k)\psi_j \, \mathrm{d}x + {}_{V'}\langle s(t), \psi_j \rangle_V, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\beta_{kj} = -\displaystyle\int_\Omega g(u_k, w_k)\psi_j \, \mathrm{d}x, \\ \alpha_{kj}(0) = a_j, \\ \beta_{kj}(0) = b_j, \end{cases} \tag{3.3.7}$$

for $j = 0, 1, \ldots, k$. The initial data $\mathbf{a}_k = \{a_j\}_{j=0}^k$ and $\mathbf{b}_k = \{b_j\}_{j=0}^k$ are fixed later. By the standard theory of ordinary differential equations, this system admits a unique solution $(\alpha_k, \beta_k) \subset (W^{1,2}(0,T_k))^{2(k+1)}$ on some interval $(0,T_k)$. It is either $|\alpha_k(t)| + |\beta_k(t)| \to \infty$ as $t \nearrow T_k$ or we can take any finite time T_k. In the following, it is shown that $|\alpha_k(t)| + |\beta_k(t)| \to \infty$ as $t \nearrow T_k$ does not occur by using a priori estimates. To this end, multiplying the first equation of (3.3.7) with $r \cdot \alpha_{kj}$, where r is the constant defined in Assumption N, the second equation with β_{kj}, and summing over j yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(r\|u_k(t)\|_H^2 + \|w_k(t)\|_H^2\right) + ra(u_k(t), u_k(t)) + \int_\Omega rf(u_k(t), w_k(t))u_k(t) + g(u_k(t), w_k(t))w_k(t) \, \mathrm{d}x$$

$$= r_{V'}\langle s(t), u_k(t)\rangle_V.$$

We recall that the bidomain bilinear form a has the coercivity of the form

$$\alpha\|U\|_V^2 \leq a(U,U) + \alpha\|U\|_H^2$$

for all $U \in V$ and for some constant $\alpha > 0$, see [5]. By the coercivity of a, the Assumption N, and Young's inequality, it is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r\|u_k(t)\|_H^2 + \|w_k(t)\|_H^2\right) + C_{11}\|u_k(t)\|_V^2 + C_{12}\|u_k(t)\|_p^p - C_{13}\|u_k(t)\|_H^2 + C_{14}\|w_k(t)\|_H^2$$
$$\leq C_{15}\|s(t)\|_{V'}^2 + C_{16},$$

for some constants $C_{1i} = C_{1i}(r, \alpha, C_j) > 0$ $(i = 1, \dots, 6,\ j = 0, \dots, 2)$; we emphasize that all constants C_{1i} are independent of k. We use the estimate

$$C_{17}\|u_k(t)\|_p^p - C_{18} \leq C_{12}\|u_k(t)\|_p^p - C_{13}\|u_k(t)\|_2^2$$

for some $C_{17}, C_{18} > 0$ since $2 < p < \infty$. Therefore, we have the following estimate

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(r\|u_k(t)\|_H^2 + \|w_k(t)\|_H^2\right) + C_{21}\left(r\|u_k(t)\|_V^2 + \|u_k(t)\|_p^p + \|w_k(t)\|_H^2\right) \\ &\leq C_{22}\|s(t)\|_{V'}^2 + C_{23}, \end{aligned} \tag{3.3.8}$$

for some constants $C_{2i} > 0$ $(i = 1, 2, 3)$.

Then, we apply Gronwall's inequality for the inequality (3.3.8), then

$$\begin{aligned} &r\|u_k(t)\|_H^2 + \|w_k(t)\|_H^2 \\ &\leq \mathrm{e}^{-C_{21}t}(r\|\mathbf{a}_k\|_H^2 + \|\mathbf{b}_k\|_H^2) + \int_0^t \mathrm{e}^{-C_{21}(t-\tau)}(C_{22}\|s(\tau)\|_{V'}^2 + C_{23})\,\mathrm{d}\tau. \end{aligned} \tag{3.3.9}$$

Since $\|u_k(t)\|_H^2 = |\alpha_k(t)|^2$ and $\|w_k(t)\|_H^2 = |\beta_k(t)|^2$, this implies T_k does not blow up at any finite time. We consider the Poincaré map

$$\begin{aligned} \mathcal{S} &: \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} \to \mathbb{R}^{k+1} \times \mathbb{R}^{k+1}, \\ &\mathcal{S}(\mathbf{a}_k, \mathbf{b}_k) := (\alpha_k(T), \beta_k(T)). \end{aligned}$$

We define

$$\mathbb{B}_R := \left\{ (\mathbf{a}_k, \mathbf{b}_k) = (\{a_j\}_{j=0}^k, \{b_j\}_{j=0}^k) \in \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} \,\middle|\, r\left(\sum_{j=0}^k |a_j|^2\right)^{1/2} + \left(\sum_{j=0}^k |b_j|^2\right)^{1/2} \leq R \right\}$$

with

$$R^2 = \frac{\int_0^T \mathrm{e}^{-C_{21}(T-\tau)}(C_{22}\|s(\tau)\|_{V'}^2 + C_{23})\,\mathrm{d}\tau}{1 - \mathrm{e}^{-C_{21}T}}. \tag{3.3.10}$$

Then, it follows that $\mathcal{S}$ maps $\mathbb{B}_R$ into itself from (3.3.9). Since $\mathcal{S}$ is also continuous, by Brouwer's fixed point theorem we conclude that $\mathcal{S}$ admits a fixed point $(\bar{\mathbf{a}}_k, \bar{\mathbf{b}}_k) =$

$\mathcal{S}(\bar{\mathbf{a}}_k, \bar{\mathbf{b}}_k)$ in $\mathbb{B}_R$ for all $k \in \mathbb{N}$.

In the following, we denote by u_k and w_k the functions defined in (3.3.5) and (3.3.6) respectively, corresponding to the solutions α_k, β_k of (3.3.7) with initial values $\bar{\mathbf{a}}_k$, $\bar{\mathbf{b}}_k$. Then, $u_k(0,x) = u_k(T,x)$ and $w_k(0,x) = w_k(T,x)$. Moreover, we see $u_k(t+T,x) = u_k(t,x)$ and $w_k(t+T,x) = w_k(t,x)$ for all $t \in \mathbb{R}$ by periodically expansion. In the next step, we would like to pass to the limit $k \to \infty$ and show the existence of a weak solution to the original problem (ABDE). To do so, we consider the uniform boundedness. For the inequality (3.3.9), we take the supremum from $t = 0$ to $t = T$, then by using (3.3.10)

$$\|u_k\|_{L^\infty(0,T;H)} + \|w_k\|_{L^\infty(0,T;H)} \le C_{31}\|s\|_{L^2(0,T;V')} + C_{32},$$

for some $C_{3i} > 0$ $(i = 1,2)$. Moreover, for the inequality (3.3.8), integrate from $t = 0$ to $t = T$ to get

$$\|u_k\|^2_{L^2(0,T;V)} + \|u_k\|^p_{L^p(Q)} + \|w_k\|^2_{L^2(0,T;H)} \le C_{41}\|s\|^2_{L^2(0,T;V')} + C_{42}. \tag{3.3.11}$$

for some $C_{4i} > 0$ $(i = 1,2)$. This implies that there are sub-sequences of $\{u_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$, for convenience still denoted by $\{u_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$, that converges to u weakly in $L^2(0,T;V) \cap L^p(Q)$ and converges to w weakly in $L^2(0,T;H)$.

By construction of the function u_k and w_k,

$$(\partial_t u_k(t), \psi_\ell) + a(u_k(t), \psi_\ell) + {}_{p'}\langle f(u_k, w_k), \psi_\ell\rangle_p = {}_{V'}\langle s(t), \psi_\ell\rangle_V$$
$$(\partial_t w_k(t), \psi_\ell) + (g(u_k, w_k), \psi_\ell) = 0$$

for all $\ell = 0, \dots, k$. Integrate from t_0 to t_1 $(0 \le t_0 \le t_1 \le T)$, then we get

$$\begin{aligned}
&|(u_k(t_1), \psi_\ell) - (u_k(t_0), \psi_\ell)| \\
=&\Big|\int_{t_0}^{t_1} -a(u_k, \psi_\ell) - {}_{p'}\langle f(u_k, w_k), \psi_\ell\rangle_p + {}_{V'}\langle s, \psi_\ell\rangle_V \,\mathrm{d}\tau\Big| \\
\le& C_{M,\psi_\ell}\left(\int_{t_0}^{t_1} (\|u_k\|_V + \|f(u_k, w_k)\|_{L^{p'}(\Omega)}) \,\mathrm{d}\tau + \|s\|_{L^2(t_0,t_1;V')}\right) \\
\le& C_{M,\psi_\ell}\left(|t_1 - t_0|^{1/2} + |t_1 - t_0|^{1/p} + \|s\|_{L^2(t_0,t_1;V')}\right), \\
&|(w_k(t_1), \psi_\ell) - (w_k(t_0), \psi_\ell)| \\
=&\Big|\int_{t_0}^{t_1} -(g(u_k, w_k), \psi_\ell) \,\mathrm{d}\tau\Big| \\
\le& \|\psi_\ell\|_{L^2(\Omega)} \int_{t_0}^{t_1} \|g(u_k, w_k)\|_{L^2(\Omega)} \,\mathrm{d}\tau \\
\le& C_{\psi_\ell}|t_1 - t_0|^{1/2}
\end{aligned}$$

for some C_{M,ψ_ℓ} independent of t_0, t_1 and k, where we use the embedding assumption $(\psi_\ell \in) V \subset L^p(\Omega)$, the Schwarz inequality and the inequality 3.3.11. Therefore, it follows that for any $\varepsilon > 0$ there is a $\delta > 0$ with

$$|(u_k(t_1), \psi_\ell) - (u_k(t_0), \psi_\ell)| + |(w_k(t_1), \psi_\ell) - (w_k(t_0), \psi_\ell)| < \varepsilon \quad \text{if } |t_1 - t_0| \le \delta,\ k = 1, 2, \dots.$$

This means the families $\{(u_k(t),\psi_\ell)\}_{k=1}^\infty$ and $\{(w_k(t),\psi_\ell)\}_{k=1}^\infty$ are equicontinuous. Since $\{(u_k,\psi_\ell)\}_{k=1}^\infty$ and $\{(w_k,\psi_\ell)\}_{k=1}^\infty$ are uniform bounded in k, from Ascoli-Arzela's theorem, it follows that the subsequences $\{(u_k(t),\psi_\ell)\}_{k=1}^\infty$ and $\{(w_k(t),\psi_\ell)\}_{k=1}^\infty$ converge uniformly to continuous functions $(u(t),\psi_\ell)$ and $(w(t),\psi_\ell)$ for each fixed ℓ. By the Cantor diagonalization argument and a density argument, this convergence can be generalized that for each $\psi \in H$, $\{(u_k(t),\psi)\}_{k=1}^\infty$ and $\{(w_k(t),\psi)\}_{k=1}^\infty$ converge uniformly to continuous functions $(u(t),\psi)$ and $(w(t),\psi)$. Therefore, we have $u \in C_w(\mathbb{T};H)$ and $w \in C_w(\mathbb{T};H)$.

It remains to show the weak convergence of the nonlinear terms $f(u_k,w_k), g(u_k,w_k)$. We first prove $u_k \to u$ in $L^2(Q)$. To do so, we use Friedrich's inequality, which states that for any $\varepsilon > 0$, there exists $J \in \mathbb{N}$ and $\phi_1,\phi_2,\dots,\phi_J \in H$ such that for all $U \in V$, the following inequality holds

$$\|U\|_H^2 \le \sum_{j=1}^J \left| \int_\Omega U\phi_j \,\mathrm{d}x \right|^2 + \varepsilon\|\nabla U\|_H^2.$$

For Friedrich's inequality, see e.g. [10]. This inequality with $U = u_k - u$, the uniform boundedness of $\{u_k\}_{k=1}^\infty \subset L^2(0,T;V)$, and $u_k \to u$ in $C_w(\mathbb{T};H)$ implies that $u_k \to u$ in $L^2(Q)$. Since we have $u_k \to u$ a.e. in Q and f_1, f_2, g_1 are continuous, $f_1(u_k) \to f_1(u)$, $f_2(u_k) \to f_2(u)$, $g_1(u_k) \to g_1(u)$ a.e. in Q are satisfied. We shall show uniform boundedness in $L^{p'}(Q)$ for $f(u_k,w_k)$ and uniform boundedness in $L^2(Q)$ for $g(u_k,w_k)$, which implies $f(u_k,w_k) \to f(u,w)$ weakly in $L^{p'}(Q)$ and $g(u_k,w_k) \to g(u,w)$ weakly in $L^2(Q)$. Fortunately, under the Assumption N, it has already been proved in [5, p.477]. Since the functions u_k, w_k satisfy that for all $\varphi_1 \in H^1(0,T;H) \cap L^2(0,T;V) \cap L^p(Q)$ and all $\varphi_2 \in W^{1,2}(0,T;H)$,

$$\int_0^t \{(u_k,\partial_t\varphi_1) - a(u_k,\varphi_1) - {}_{p'}\langle f(u_k,w_k),\varphi_1\rangle_p\} \,\mathrm{d}\tau = -\int_0^t {}_{V'}\langle s,\varphi_1\rangle_V \,\mathrm{d}\tau + (u_k(t),\varphi_1(t)) - (u_k(0),\varphi_1(0)),$$

$$\int_0^t \{(w_k,\partial_t\varphi_2) - (g(u_k,w_k),\varphi_2)\} \,\mathrm{d}\tau = (w_k(t),\varphi_2(t)) - (w_k(0),\varphi_2(0)),$$

for all $t \in (0,T)$, combining above discussions about the weak convergence, we show the existence of a weak T-periodic solution. □

3.4 Regularity of weak periodic solution

In this section, we shall show that for the FitzHugh–Nagumo nonlinearities introduced in Section 3.1 the weak time-periodic solution constructed in the previous section is actually a strong solution. In order to do so, we first review the global strong well-posedness result by Colli Franzone and Savaré [11]. After that, we use a weak-strong uniqueness argument to show the existence of a strong time-periodic solution for (PABDE) with FitzHugh–Nagumo nonlinearities. In [11], they considered the initial boundary value problem for the bidomain equations of the form

$$\text{(BDE II)} \qquad \begin{cases} \partial_t u - \operatorname{div}(\sigma_i \nabla u_i) + F(u) + \theta w = s_i, & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + \operatorname{div}(\sigma_e \nabla u_e) + F(u) + \theta w = -s_e, & \text{in } (0,\infty)\times\Omega, \\ u = u_i - u_e, & \text{in } (0,\infty)\times\Omega, \\ \partial_t w + \gamma w - \eta u = 0, & \text{in } (0,\infty)\times\Omega, \\ \sigma_i \nabla u_i \cdot \nu = g_i,\ \sigma_e \nabla u_e \cdot \nu = g_e, & \text{in } (0,\infty)\times\partial\Omega, \\ u(0) = u_0,\ \ w(0) = w_0, & \text{in } \Omega, \end{cases}$$

with $\theta, \gamma, \eta > 0$.

Let $\mathcal{T}$ be a sufficiently large time such that $T < \mathcal{T}$. They regarded the bidomain equation as the *degenerate* variational formulation of the form

$$\begin{cases} (Bu)' + Au + \mathcal{F}u = L & t \in (0,\mathcal{T}) \\ (Bu)(0) = \ell^0, \end{cases}$$

and constructed the global weak formulation and their regularity.

Let Ω be a Lipschitz domain of $\mathbb{R}^d$, $\Gamma := \partial\Omega$, and the measurable function $\sigma_{i,e} : \overline{\Omega} \to \mathbb{R}^{d\times d}$ satisfy the uniform ellipticity condition. Assume the nonlinear term F is a continuous function with

$$\text{(3.4.1)} \qquad F(0) = 0,\ \exists \lambda_F \geq 0 : \frac{F(x) - F(y)}{x - y} \geq -\lambda_F,\ \ \forall x, y \in \mathbb{R},\ \text{with } x \neq y.$$

Their result is as follows.

Theorem 3.4 (Franzone-Savaré '02 [11]). *Assume $s_{i,e} \in L^2(0,\mathcal{T};H)$, $g_{i,e} \in W^{1,1}(0,\mathcal{T};H^{-1/2}(\Gamma))$ satisfy $s_i + s_e \in W^{1,1}(0,\mathcal{T};H)$ and the compatibility condition*

$$\int_\Omega (s_i + s_e)\,\mathrm{d}x + {}_{H^{-1/2}(\Gamma)}\langle g_i + g_e, 1\rangle_{H^{1/2}(\Gamma)} = 0.$$

Then for any initial data $u_0, w_0 \in H$, there uniquely exist a couple

$$u_{i,e} \in L^2(0,\mathcal{T};V),\ \int_\Omega u_e = 0 \text{ a.e } t$$

and

$$\begin{aligned} & u \in C([0,\mathcal{T}];H) \cap L^2(0,\mathcal{T};V),\ \partial_t u \in L^2_{loc}(0,\mathcal{T};H), \\ & F(u(t)) \in L^1(\Omega) \cap V' \ a.e.\ t \in (0,\mathcal{T}), \\ & w, \partial_t w \in C([0,\mathcal{T}];H), \end{aligned}$$

which solves the bidomain equation in the sense of

$$\begin{aligned} \int_\Omega (\partial_t u \hat{u} + \frac{\theta}{\eta}\partial_t w \hat{w})\,\mathrm{d}x + \int_\Omega F(u)\hat{u}\,\mathrm{d}x + \sum_{i,e} \int_\Omega \sigma_{i,e}\nabla u_{i,e} \cdot \nabla \hat{u}_{i,e}\,\mathrm{d}x \\ + \frac{\theta\gamma}{\eta}\int_\Omega w\hat{w}\,\mathrm{d}x + \theta\int_\Omega (w\hat{u} - u\hat{w})\,\mathrm{d}x \end{aligned}$$

$$= \sum_{i,e} \int_\Omega s_{i,e} \hat{u}_{i,e} \, \mathrm{d}x + \sum_{i,e} {}_{H^{-1/2}(\Gamma)} \langle g_{i,e}, \hat{u}_{i,e} \rangle_{H^{1/2}(\Gamma)},$$

$$\int_\Omega (u(0)\hat{u} + \frac{\theta}{\eta} w(0)\hat{w}) \, \mathrm{d}x = \int_\Omega (u_0 \hat{u} + \frac{\theta}{\eta} w_0 \hat{w}) \, \mathrm{d}x,$$

for a.e. $t \in (0, \mathcal{T})$ *and all* $\hat{u}_{i,e} \in V \times V$ *with* $\int_\Omega \hat{u}_e \, \mathrm{d}x = 0$ *and* $\hat{u} = \hat{u}_i - \hat{u}_e$ *and* $\hat{w} \in H$.

Moreover if $u_0 \in V, u_0 F(u_0) \in L^1(\Omega)$, *then*

$$u_{i,e} \in C([0, \mathcal{T}]; V), \; \partial_t u \in L^2(0, \mathcal{T}; H), \; w \in C([0, \mathcal{T}]; V).$$

Furthermore they derived the regularity results.

Proposition 3.5. *In addition to the assumption in the theorem, suppose that* $d = 3$, *and the nonlinear term* F *has a cubic growth at infinity, i.e.,*

$$0 < \liminf_{|r| \to \infty} \frac{F(r)}{r^3} \leq \limsup_{|r| \to \infty} \frac{F(r)}{r^3} < +\infty.$$

Then the bidomain equation admits a unique strong solution $u_{i,e}, u, w$. *Moreover, it satisfies*

$$-\operatorname{div}(\sigma_{i,e} \nabla u_{i,e}) \in L^2(0, \mathcal{T}; H)$$

Remark 3.6. Let Ω be of class $C^{1,1}$, $\sigma_{i,e}$ be Lipschitz in Ω and $g_{i,e} \in L^2(0, \mathcal{T}; H^{1/2}(\Gamma))$. Then by the standard regularity theorem, we see

$$u_{i,e} \in L^2(0, \mathcal{T}; H^2(\Omega)).$$

Remark 3.7. If we look at the function f of the FitzHugh–Nagumo nonlinearity introduced in Section 3.1 as $f(u, w) = F(u) + w = u(u - a)(u - 1) + w$, then the function $F(u)$ satisfies the assumptions for the nonlinearity in Proposition 3.5 as well as Assumption (3.4.1).

Now, we combine the results from the previous sections to obtain a strong time-periodic solution for the bidomain equations with FitzHugh–Nagumo type nonlinearities subject to arbitrary large forces. We would like to identify our *weak* time-periodic solution (v, z) constructed in Section 3.3 with a *strong* solution (u, w) to the initial value problem with initial data $v(t_0), z(t_0)$ for some $t_0 > 0$ satisfying $v(t_0) \in V$ and $f(v(t_0))v(t_0) \in L^1(\Omega)$. Since $\|f(v)v\|_{L^1(Q)} \leq \|f(v)\|_{L^{p'}(Q)} \|v\|_{L^p(Q)}$ with $p = 4$, this guarantees the existence of a $t_0 > 0$ such that $f(v(t_0))(v(t_0)) \in L^1(\Omega)$. So we can use the theorem by Colli-Franzone and Savaré for the global strong solution with the initial values $v(t_0)$, $z(t_0)$. Finally, we show that the weak solution (v, z) coincides with the strong solution (u, w) and therefore obtain the existence of a strong time-periodic solution. We follow the approach given in [12].

To be more precise, for given T-time-periodic functions $s_{i,e} \in L^2(0, \mathcal{T}; H)$ with $s_i + s_e \in W^{1,1}(0, \mathcal{T}; H)$ and $\int_\Omega (s_i + s_e) \, \mathrm{d}x = 0$ for a.e. t, let (v, z) be a weak T-time-periodic solution of (PABDE) for $s = s_i - A_i(A_i + A_e)^{-1}(s_i + s_e)$ $(\in L^2(\mathbb{T}; H))$ corresponding to Theorem 3.3. We take t_0 such that $v(t_0) \in V$ and $v(t_0)f(v(t_0)) \in L^1(\Omega)$. Since (v, z) is a weak T-time-periodic solution, it satisfies that for all $\varphi_1 \in W^{1,2}(t_0, \mathcal{T}; H) \cap L^2(t_0, \mathcal{T}; V) \cap$

$L^4(Q)$ and all $\varphi_2 \in W^{1,2}(t_0, \mathcal{T}; H)$,

$$\int_{t_0}^{t} \{(v, \partial_t \varphi_1) - a(v, \varphi_1) - (f(v, z), \varphi_1)\} \, \mathrm{d}\tau = -\int_{t_0}^{t} (s(\tau), \varphi_1(\tau)) \, \mathrm{d}\tau + (v(t), \varphi_1(t)) - (v(t_0), \varphi_1(t_0)), \tag{3.4.2}$$

$$\int_{t_0}^{t} \{(z, \partial_t \varphi_2) - (g(v, z), \varphi_2)\} \, \mathrm{d}\tau = (z(t), \varphi_2(t)) - (z(t_0), \varphi_2(t_0)), \tag{3.4.3}$$

for all $t \in (t_0, \mathcal{T})$, and (v, z) satisfies the following strong energy inequality:

$$\begin{aligned} &\left(\|v(t)\|_H^2 + \|z(t)\|_H^2\right) + 2\int_{t_0}^{t} a(v(\tau), v(\tau)) \, \mathrm{d}\tau \\ &\qquad + 2\int_{t_0}^{t}\int_{\Omega} f(v(\tau), z(\tau))v(\tau) + g(v(\tau), z(\tau))z(\tau) \, \mathrm{d}x \, \mathrm{d}\tau \\ \le &\|v(t_0)\|_H^2 + \|z(t_0)\|_H^2 + 2\int_{t_0}^{t} (s(\tau), v(\tau)) \, \mathrm{d}\tau, \end{aligned} \tag{3.4.4}$$

for all $t \in [t_0, \mathcal{T}]$.

We next consider the unique global strong solution

$$(u, w) \in \left(W^{1,2}(t_0, \mathcal{T}; H) \cap L^2(t_0, \mathcal{T}; H^2(\Omega))\right) \times C^1([t_0, \mathcal{T}]; H)$$

corresponding to the initial-boundary value problem for the bidomain equation with initial value $(v(t_0), z(t_0))$ and T-periodic right-hand side $s_{i,e}$ and $g_{i,e} = 0$. In the following, we show that the weak solution (v, z) agrees with the strong solution (u, w).

Since (u, w) is a strong solution, it satisfies that for all $\mathcal{T} > t_0$ and all $\phi_1 \in W^{1,2}(t_0, \mathcal{T}; H) \cap L^2(t_0, \mathcal{T}; V) \cap L^4(Q)$ and all $\phi_2 \in W^{1,2}(t_0, \mathcal{T}; H)$

$$\int_{t_0}^{t} \{(u, \partial_t \phi_1) - a(u, \phi_1) - (f(u, w), \phi_1)\} \, \mathrm{d}\tau = -\int_{t_0}^{t} (s(\tau), \phi_1(\tau)) \, \mathrm{d}\tau + (u(t), \phi_1(t)) - (v(t_0), \phi_1(t_0)), \tag{3.4.5}$$

$$\int_{t_0}^{t} \{(w, \partial_t \phi_2) - (g(u, w), \phi_2)\} \, \mathrm{d}\tau = (w(t), \phi_2(t)) - (z(t_0), \phi_2(t_0)), \tag{3.4.6}$$

for all $t \in (t_0, \mathcal{T})$, and (u, w) satisfies the following strong energy identity:

$$\begin{aligned} &\left(\|u(t)\|_H^2 + \|w(t)\|_H^2\right) + 2\int_{t_0}^{t} a(u(\tau), u(\tau)) \, \mathrm{d}\tau \\ &\qquad + 2\int_{t_0}^{t}\int_{\Omega} f(u(\tau), w(\tau))u(\tau) + g(u(\tau), w(\tau))w(\tau) \, \mathrm{d}x \, \mathrm{d}\tau \\ = &\|v(t_0)\|_H^2 + \|z(t_0)\|_H^2 + 2\int_{t_0}^{t} (s(\tau), u(\tau)) \, \mathrm{d}\tau. \end{aligned} \tag{3.4.7}$$

Next, denote by

$$v_h(t) := \int_0^{\mathcal{T}} j_h(t-\tilde{t})v(\tilde{t})\,\mathrm{d}\tilde{t}, \quad z_h(t) := \int_0^{\mathcal{T}} j_h(t-\tilde{t})z(\tilde{t})\,\mathrm{d}\tilde{t},$$
$$u_h(t) := \int_0^{\mathcal{T}} j_h(t-\tilde{t})u(\tilde{t})\,\mathrm{d}\tilde{t}, \quad w_h(t) := \int_0^{\mathcal{T}} j_h(t-\tilde{t})w(\tilde{t})\,\mathrm{d}\tilde{t}$$

the (Friedrichs) time-mollifier of v, z, u, and w, respectively, where $j_h \in \mathrm{C}_c^\infty(-h,h)$, $0 < h < \mathcal{T}$, is even and positive with $\int_{\mathbb{R}} j_h(\tilde{t})\,\mathrm{d}\tilde{t} = 1$. Then, as is well known,

(3.4.8)
$$\lim_{h\to 0}\int_0^{\mathcal{T}} \|v_h(\tau)-v(\tau)\|_V^2\,\mathrm{d}\tau = 0, \quad \operatorname{esssup}_{t\in[0,\mathcal{T}]}\|v_h(t)\|_2 \le \operatorname{esssup}_{t\in[0,\mathcal{T}]}\|v(t)\|_2,$$
(3.4.9)
$$\lim_{h\to 0}\int_0^{\mathcal{T}} \|u_h(\tau)-u(\tau)\|_{H^2}^2\,\mathrm{d}\tau = 0, \quad \operatorname{esssup}_{t\in[0,\mathcal{T}]}\|u_h(t)\|_V \le \operatorname{esssup}_{t\in[0,\mathcal{T}]}\|u(t)\|_V,$$
(3.4.10)
$$\lim_{h\to 0}\int_0^{\mathcal{T}} \|z_h(\tau)-z(\tau)\|_H^2\,\mathrm{d}\tau = 0, \quad \lim_{h\to 0}\int_0^{\mathcal{T}} \|w_h(\tau)-w(\tau)\|_H^2\,\mathrm{d}\tau = 0.$$

The weak continuity of v and u implies

$$\lim_{h\to 0}(u(t),v_h(t)) = \lim_{h\to 0}(u_h(t),v(t)) = (u(t),v(t)),\ t\ge t_0, \tag{3.4.11}$$
$$\lim_{h\to 0}(w(t),z_h(t)) = \lim_{h\to 0}(w_h(t),z(t)) = (w(t),z(t)),\ t\ge t_0. \tag{3.4.12}$$

Furthermore since

$$\int_{t_0}^t (u,\partial_t v_h)\,\mathrm{d}\tau = -\int_{t_0}^t (\partial_t u, v_h)\,\mathrm{d}\tau + (u(t),v_h(t)) - (u(t_0),v_h(t_0)),$$
$$\int_{t_0}^t (w,\partial_t z_h)\,\mathrm{d}\tau = -\int_{t_0}^t (\partial_t w, z_h)\,\mathrm{d}\tau + (w(t),z_h(t)) - (w(t_0),z_h(t_0)),$$

by taking the limit,

$$\lim_{h\to 0}\left\{\int_{t_0}^t (u,\partial_t v_h) + (\partial_t u, v_h)\,\mathrm{d}\tau\right\} = (u(t),v(t)) - \|v(t_0)\|_H^2$$
$$\lim_{h\to 0}\left\{\int_{t_0}^t (w,\partial_t z_h) + (\partial_t w, z_h)\,\mathrm{d}\tau\right\} = (w(t),z(t)) - \|z(t_0)\|_H^2.$$

We now replace φ_1 by u_h in (3.4.2), φ_2 by w_h in (3.4.3), ϕ_1 by v_h in (3.4.5), and ϕ_2 by z_h in (3.4.6). Then, we sum up the resulting equations, pass to the limit $h\to 0$, and use the properties of the time-mollifier mentioned above to obtain

$$\int_{t_0}^t \{-2a(u,v) - (f(v,z),u) - (f(u,w),v) - (g(v,z),w) - (g(u,w),z)\}\,\mathrm{d}\tau$$
(3.4.13)
$$= -\int_{t_0}^t (s(\tau), u(\tau)+v(\tau))\,\mathrm{d}\tau + (u(t),v(t)) - \|v(t_0)\|_H^2 + (w(t),z(t)) - \|z(t_0)\|_H^2.$$

To prove $(u,w)=(v,z)$, we calculate

$$\begin{aligned}
&\|u(t)-v(t)\|_H^2+\|w(t)-z(t)\|_H^2+2\int_{t_0}^t a(u(\tau)-v(\tau),u(\tau)-v(\tau))\,\mathrm{d}\tau\\
=&\left(\|u(t)\|_H^2+\|w(t)\|_H^2+2\int_{t_0}^t a(u(\tau),u(\tau))\,\mathrm{d}\tau\right)+\left(\|v(t)\|_H^2+\|z(t)\|_H^2+2\int_{t_0}^t a(v(\tau),v(\tau))\,\mathrm{d}\tau\right)\\
&-2(u(t),v(t))-2(w(t),z(t))-4\int_{t_0}^t a(u(\tau),v(\tau))\,\mathrm{d}\tau.
\end{aligned}$$

For the first two parts, we use the strong energy equality (3.4.7) and the strong energy inequality (3.4.4), and for the last term, we use the relation (3.4.13). Then, we have

$$\begin{aligned}
&\|u(t)-v(t)\|_H^2+\|w(t)-z(t)\|_H^2+2\int_{t_0}^t a(u(\tau)-v(\tau),u(\tau)-v(\tau))\,\mathrm{d}\tau\\
&\le 2\int_{t_0}^t\{(f(v,z),u)+(f(u,w),v)-(f(u,w),u)-(f(v,z),v)\\
&\qquad+(g(v,z),w)+(g(u,w),z)-(g(u,w),w)-(g(v,z),z)\}\,\mathrm{d}\tau\\
&\le -2\int_{t_0}^t(f(u,w)-f(v,z),u-v)+(g(u,w)-g(v,z),w-z)\,\mathrm{d}\tau
\end{aligned}$$

Here, for the first term we use the Assumption (3.4.1) and Young's inequality to get

$$\begin{aligned}
&-2\int_{t_0}^t(f(u,w)-f(v,z),u-v)\,\mathrm{d}\tau\\
\le&2\lambda_f\int_{t_0}^t\|u(\tau)-v(\tau)\|_H^2\,\mathrm{d}\tau-2\int_{t_0}^t(w-z,u-v)\,\mathrm{d}\tau\\
\le&2\lambda_f\int_{t_0}^t\|u(\tau)-v(\tau)\|_H^2\,\mathrm{d}\tau+\int_{t_0}^t\varepsilon_1\|w-z\|_H^2+C(\varepsilon_1)\|u-v\|_H^2\,\mathrm{d}\tau
\end{aligned}$$

for some constants ε_1, $C(\varepsilon_1)>0$. On the other hand since the function $g(u,w)=-\varepsilon(ku-w)$ is linear,

$$|(g(u,w)-g(v,z),w-z)|\le C(\|u-v\|_H^2+\|w-z\|_H^2).$$

for some $C>0$. Therefore, we have

$$\begin{aligned}
&\|u(t)-v(t)\|_H^2+\|w(t)-z(t)\|_H^2+2\int_{t_0}^t a(u(\tau)-v(\tau),u(\tau)-v(\tau))\,\mathrm{d}\tau\\
\le&C\int_{t_0}^t(\|u(\tau)-v(\tau)\|_H^2+\|w(\tau)-z(\tau)\|_H^2\,\mathrm{d}\tau,
\end{aligned}$$

for some $C>0$, which is different from the previous constant.

Hence, we are able to apply Gronwall's lemma to conclude that

$$u-v\equiv 0, w-z\equiv 0 \text{ a.e. in } \Omega\times[t_0,\mathcal{T}].$$

This implies the existence of a strong T-time-periodic solution (u,w) when the source term $s_{i,e}$ is a T-time-periodic function.

We write down the main theorem of the existence of strong periodic solutions without assuming smallness conditions for the external forces.

Theorem 3.8. *Let $d = 3$, $T > 0$, and $s_{i,e} \in L^2(\mathbb{T}; H)$ with $s_i + s_e \in W^{1,1}(\mathbb{T}; H)$ and $\int_\Omega (s_i + s_e)\, \mathrm{d}x = 0$ for a.e. t. Let the conductivity matrices $\sigma_{i,e}$ satisfy Assumption C and the nonlinear term F satisfy Assumption (3.4.1) and assume that there exist constants $C_0 \in \mathbb{R}$ and $C_1 > 0$ such that*

$$C_0 + C_1|u|^4 \leq F(u)u \tag{3.4.14}$$

for all $u \in \mathbb{R}$. Then for the bidomain equations with FitzHugh–Nagumo type

$$\begin{cases} \partial_t u - \operatorname{div}(\sigma_i \nabla u_i) + F(u) + w = s_i, & \text{in } (0,\infty)\times\Omega, \\ \partial_t u + \operatorname{div}(\sigma_e \nabla u_e) + F(u) + w = -s_e, & \text{in } (0,\infty)\times\Omega, \\ u = u_i - u_e, & \text{in } (0,\infty)\times\Omega, \\ \partial_t w - \varepsilon(ku - w) = 0, & \text{in } (0,\infty)\times\Omega, \\ \sigma_i \nabla u_i \cdot \nu = 0,\ \sigma_e \nabla u_e \cdot \nu = 0, & \text{in } (0,\infty)\times\partial\Omega, \\ u(0) = u_0,\ \ w(0) = w_0, & \text{in } \Omega, \end{cases}$$

there exists a strong T-periodic solution

$$(u_i, u_e) \in (W^{1,2}(\mathbb{T}; H) \cap L^2(\mathbb{T}; H^2(\Omega)))^2 \text{ with } \int_\Omega u_e \, \mathrm{d}x = 0 \text{ a.e. } t$$
$$(u, w) \in (W^{1,2}(\mathbb{T}; H) \cap L^2(\mathbb{T}; H^2(\Omega)) \cap L^4(\mathbb{T}\times\Omega)) \times C^1(\mathbb{T}; H).$$

Remark 3.9. The Assumption N of the existence of the weak periodic solutions is replaced by (3.4.14).

Remark 3.10. We do not treat the ionic models by Rogers–McCulloch and Aliev-Panfilov due to the lack of a suitable global well-posedness result for the initial value problem in the L^2 setting.

3.5 Appendix

In this appendix, we check that the three models introduced in Section 3.1 satisfy the Assumption N. Since the growth conditions (3.3.2)-(3.3.4) are trivial as $p = 4$, we confirm the condition (5.2.5).

3.5.1 FitzHugh–Nagumo model

The FitzHugh–Nagumo type is

$$f(u,w) = u(u-a)(u-1) + w$$
$$g(u,w) = -\varepsilon(ku - w)$$

with $0 < a < 1$ and $k, \varepsilon > 0$. Then, we are able to calculate as follows ($r = 1$):

$$f(u,w)u + g(u,w)w = u^4 - (a+1)u^3 + au^2 + uw - \varepsilon kuw + \varepsilon w^2$$

and by

$$\begin{aligned}|(a+1)u^3| &\le \frac{1}{8}u^4 + c_{11},\\ |au^2| &\le \frac{1}{8}u^4 + c_{12},\\ |uw| &\le \frac{1}{8}u^4 + \frac{\varepsilon}{4}w^2 + c_{13},\\ |\varepsilon uw| &\le \frac{1}{8}u^4 + \frac{\varepsilon}{4}w^2 + c_{14},\end{aligned}$$

for some $c_{1i} > 0$ $(i = 1, \dots, 4)$, we have

$$f(u,w)u + g(u,w)w \ge \frac{1}{2}u^4 + \frac{\varepsilon}{2}w^2 + c_1$$

for some $c_1 \in \mathbb{R}$. Therefore, the FitzHugh–Nagumo model satisfies the Assumption N.

3.5.2 Rogers–McCulloch model

The Rogers–McCulloch type is

$$\begin{aligned}f(u,w) &= bu(u-a)(u-1) + uw\\ g(u,w) &= -\varepsilon(ku - w)\end{aligned}$$

with $0 < a < 1$ and $b, k, \varepsilon > 0$. Then, we are able to calculate as follows:

$$rf(u,w)u + g(u,w)w = rbu^4 - rb(a+1)u^3 + rbau^2 + ru^2w - \varepsilon kuw + \varepsilon w^2$$

and, based on the calculation

$$|ru^2w| \le \frac{C^2}{2}u^4 + \frac{r^2}{2C^2}w^2,$$

we choose $r, C > 0$ depending on b, ε, such that

$$\begin{cases} c_{21} := rb - \frac{C^2}{2} > 0,\\ c_{22} := \varepsilon - \frac{r^2}{2C^2} > 0.\end{cases}$$

By

$$\begin{aligned}|rb(a+1)u^3| &\le \frac{c_{21}}{6}u^4 + c_{23},\\ |rbau^2| &\le \frac{c_{21}}{6}u^4 + c_{24},\\ |\varepsilon kuw| &\le \frac{c_{21}}{6}u^4 + \frac{c_{22}}{2}w^2 + c_{25},\end{aligned}$$

for some $c_{2i} > 0$ $(i = 3, \dots, 5)$, we have

$$rf(u,w)u + g(u,w)w \ge \frac{c_{21}}{2}u^4 + \frac{c_{22}}{2}w^2 + c_2$$

for some $c_2 \in \mathbb{R}$. Therefore, the Rogers–McCulloch model satisfies the Assumption N.

3.5.3 Aliev–Panfilov model

The modified Aliev–Panfilov type is

$$\begin{aligned} f(u,w) &= bu(u-a)(u-1) + uw \\ g(u,w) &= \varepsilon(ku(u-1-d) + w) \end{aligned}$$

with $0 < a, d < 1$, $b, k, \varepsilon > 0$, and $b > k$. Then, we are able to calculate as follows:

$$rf(u,w)u + g(u,w)w = rbu^4 + rb(a+1)u^3 + rbau^2 + ru^2w + \varepsilon ku^2w - \varepsilon k(1+d)uw + \varepsilon w^2$$

and, based on the calculation

$$|(r+\varepsilon k)u^2w| \le \frac{C^2}{2}u^4 + \frac{(r+\varepsilon k)^2}{2C^2}w^2,$$

we choose $r, C > 0$ depending on b, k, ε, such that

$$\begin{cases} c_{31} := rb - \frac{C^2}{2} > 0, \\ c_{32} := \varepsilon - \frac{(r+\varepsilon k)^2}{2C^2} > 0. \end{cases}$$

Here, the assumption $b > k$ is essential. By

$$\begin{aligned} |rb(a+1)u^3| &\le \frac{c_{31}}{6}u^4 + c_{33}, \\ |rbau^2| &\le \frac{c_{31}}{6}u^4 + c_{34}, \\ |\varepsilon k(1+d)uw| &\le \frac{c_{31}}{6}u^4 + \frac{c_{32}}{2}w^2 + c_{35}, \end{aligned}$$

for some $c_{3i} > 0$ $(i = 3, \dots, 5)$, we have

$$rf(u,w)u + g(u,w)w \ge \frac{c_{31}}{2}u^4 + \frac{c_{32}}{2}w^2 + c_3$$

for some $c_3 \in \mathbb{R}$. Note that we are not able to take c_{3i} $(i = 1, 2)$ in the case $b = k$. Therefore, the modified Aliev-Panfilov model satisfies the Assumption N.

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Chapter 4

Strong well-posedness for the phase-field Navier–Stokes equations in the maximal regularity class

In this chapter we study the dynamics of vesicle membranes in incompressible viscous fluids. We prove existence and uniqueness of the local strong solution for this model coupling of the Navier–Stokes equations with a phase field equation in an L_p-L_q setting. We transform the equation into a quasi-linear parabolic evolution equation and use the general theory proved by Prüss et al. [16, 23, 25]. Since the operator and the nonlinear term are analytic, we have that the solution is real analytic in time and space. At last it is shown that the variational strict stable solution is exponentially stable, provided the product of the viscosity coefficient and the mobility constant is large.

Keywords: phase-field Navier–Stokes equations; well-posedness; stability

4.1 Introduction

It is important to understand the deformation of vesicle membranes in a liquid in many biological and physiological applications. The vesicle contains a liquid and is surrounded by another liquid. The function of the vesicle is to store and/or transport substances. They are not only essential to the function of cells but also interesting since they change their shapes such as spheres, discocytes, stomatocytes, tori and double tori. These are concrete examples of minimizers of different surface energies, such as the bending elasticity (Willmore, mean curvature square) energy in the calculus of variations and its different variations like the general Helfrich energy [2, 14, 27, 34]. We consider that the equilibrium configurations of vesicle membranes can be characterized the minimizer of the following Helfrich bending elastic energy of the surface:

$$E_{\text{elastic}} = \int_\Gamma \frac{k}{2}(H - c_0)^2 dS,$$

where Γ is the surface of the vesicle membrane, H is the mean curvature of Γ, c_0 is the spontaneous curvature that describes certain physical/chemical difference between the inside and the outside of the membrane, and k is the bending modulus (bending rigidity) that depends on the local heterogeneous concentration of the species. Here we assumed the evolution of the vesicle membrane does not change its topology so that the energy is simplified. For details, see [9, 27, 34].

The model which represent the deformation of the vesicle in a liquid was first constructed in [8]. They derived the system of the equations via an energetic variational approach. In this phase field Navier–Stokes equations, the description of the membrane is given the terms of a phase field function φ. The labeling function φ takes value $+1$ inside the vesicle membrane and -1 outside, and the thin transition layer of width is characterized by a small parameter ε. The zero level set of φ ($\{x \mid \varphi(x) = 0\}$) represents the surface of the vesicle membrane. The fluid is modeled by the incompressible Navier–Stokes equations in the whole domain containing the inside and outside of the vesicle.

The phase field approximation of the Helfrich bending elasticity energy is given by a modified Willmore energy [10]:

$$E_\varepsilon(\varphi) = \int_\Omega \frac{k}{2\varepsilon}\left(\varepsilon\Delta\varphi + (\frac{1}{\varepsilon}\varphi + c_0\sqrt{2})(1-\varphi^2)\right)^2 dx.$$

The convergence from $E_\varepsilon(\varphi)$ to E_{elastic} was studied in [7, 30]. In this chapter, for the sake of simplicity, we assume that k is a positive constant and $c_0 = 0$. Since the vesicle preserves its volume and surface area, we use the penalty formulation about its energy [27]. These two constraint functionals for the vesicle volume and surface area are given by

$$A(\varphi) = \int_\Omega \varphi dx, \qquad B(\varphi) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{4\varepsilon}(\varphi^2-1)^2 dx,$$

respectively. The modified energy is formulated in the following form:

$$E(\varphi) = E_\varepsilon(\varphi) + \frac{1}{2}M_1[A(\varphi) - A(\varphi_0)]^2 + \frac{1}{2}M_2[B(\varphi) - B(\varphi_0)]^2,$$

where M_1 and M_2 are two penalty constants and φ_0 is the initial phase function.

We consider the phase field Navier–Stokes equations derived from above energy [8, 32]. Let $\Omega \subset \mathbb{R}^n$ be a bounded C^4-domain, the function u be the unknown velocity field, the function p be the pressure and φ be the phase field function. We denote by ν the fluid viscosity and γ the mobility coefficient, which are positive constants. The model is

$$\text{(PFNS)} \quad \begin{cases} \partial_t u + (u\cdot\nabla)u = \nu\Delta u - \nabla p + \frac{\delta E(\varphi)}{\delta\varphi}\nabla\varphi & \text{in } [0,T]\times\Omega, \\ \operatorname{div} u = 0 & \text{in } [0,T]\times\Omega, \\ \partial_t\varphi + (u\cdot\nabla)\varphi = -\gamma\frac{\delta E(\varphi)}{\delta\varphi} & \text{in } [0,T]\times\Omega, \\ u = 0 & \text{on } [0,T]\times\partial\Omega, \\ \varphi = -1, \ \ \Delta\varphi = 0 & \text{on } [0,T]\times\partial\Omega, \\ u(0) = u_0, \ \ \varphi(0) = \varphi_0 & \text{in } \Omega, \end{cases}$$

where

$$\frac{\delta E(\varphi)}{\delta\varphi} = kg(\varphi) + M_1[A(\varphi) - A(\varphi_0)] + M_2[B(\varphi) - B(\varphi_0)]f(\varphi)$$

$$
\begin{aligned}
&= k\left(\varepsilon\Delta^2\varphi - \frac{1}{\varepsilon}\Delta(\varphi^3) - \frac{3}{\varepsilon}\varphi^2\Delta\varphi + \frac{2}{\varepsilon}\Delta\varphi + \frac{1}{\varepsilon^3}(3\varphi^5 - 4\varphi^3 + \varphi)\right)\\
&\qquad + M_1[A(\varphi) - A(\varphi_0)] + M_2[B(\varphi) - B(\varphi_0)]f(\varphi)\\
&=: W(\varphi)\\
&=: k\varepsilon\Delta^2\varphi + L(\varphi)
\end{aligned}
$$

and

$$
\begin{aligned}
f(\varphi) &= -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}(\varphi^2 - 1)\varphi,\\
g(\varphi) &= -\Delta f(\varphi) + \frac{1}{\varepsilon^2}(3\varphi^2 - 1)f(\varphi).
\end{aligned}
$$

We note that the term $\frac{\delta E(\varphi)}{\delta\varphi} = W(\varphi)$ is the so-called chemical potential and $E_\varepsilon(\varphi) = \frac{k}{2\varepsilon}\int_\Omega |f(\varphi)|^2 dx$. In this chapter as usual we consider the Dirichlet type for the phase field function φ and the no-slip boundary condition for the velocity field u.

The well-posedness of the system has been studied in [6, 17, 31]. In [6], they proved the existence of the global weak solution by using the Galerkin method. With a better regularity assumption on the weak solutions, as in the case of the conventional Navier–Stokes equations [28], they proved the uniqueness result. In [17], they proved the local in time existence and uniqueness of the strong solution in an L_2 framework. The idea was to rewrite (PFNS) as a semi-linear equation for the Stokes equation coupled with a parabolic equation whose operator is bi-Laplace operator. However, to estimate the nonlinear term $\Delta^2\varphi\nabla\varphi$ they needed the higher regularity class for the function φ compared with the usual parabolic equation. More precisely, they proved that if u_0, φ_0 satisfy $u_0 \in H_0^1(\Omega)$, $\operatorname{div} u_0 = 0$, $\varphi_0 + 1 \in H^{2+\frac{3}{8}}(\Omega) \cap H_0^1(\Omega)$ then (PFNS) has a unique strong solution

$$
\begin{cases}
u \in L_2(0,T; H^2(\Omega)) \cap H^1(0,T; L_2(\Omega))\\
\varphi \in L_2(0,T; H^{4+\frac{3}{8}}(\Omega)) \cap H^1(0,T; H^{\frac{3}{8}}(\Omega))
\end{cases}
$$

for some $T = T(\|u_0\|_{H^1}, \|\varphi_0\|_{H^{2+\frac{3}{8}}}) > 0$. In [31] they considered the (PFNS) in a periodic box and proved existence/uniqueness of strong solutions and some regularity criteria. Moreover they investigated the stability of the system near local minimizers of the elastic bending energy by using Lojasiewicz–Simon-type inequality.

The main purpose of this chapter is to study the existence, uniqueness and regularity of the strong solution to (PFNS) in an L_p-L_q framework as well as their exponential stability in the n dimension, while papers [6, 17, 31] are L_2 setting and $n = 3$. The main idea is to consider (PFNS) as a quasi-linear equation not as a semi-linear equation. We consider the quasi-linear operator $A(z)(z = {}^T(u, \phi))$ given by

$$
A(z) = \begin{pmatrix} \nu\mathcal{A} & -k\varepsilon\mathbb{P}\mathcal{B}(\phi) \\ 0 & \gamma k\varepsilon\mathcal{D} \end{pmatrix},
$$

where $\mathcal{A}$ denotes the Stokes operator, $\mathcal{D}$ the bi-Laplace operator, $\mathbb{P}$ the Helmholtz projection, and $\mathcal{B}$ is given by $\mathcal{B}(\phi)h := \Delta^2 h\nabla\phi$. For the quasi-linear parabolic equation, we use maximal regularity in a weighted L_p spaces and well-posed result proved by Prüss et al. [16,23,25]. This quasi-linear approach has already used to analyze nematic liquid crystal flows [15] and viscoelastic Poiseuille-type flows [12] as pioneering works. We employ time weight L_p spaces:

$$L_{p,\mu}(0,T;X) := \{z : (0,T) \to X \mid t^{1-\mu}z \in L_p(0,T;X)\},$$
$$H^1_{p,\mu}(0,T;X) := \{z \in L_{p,\mu}(0,T;X) \cap W^1_1(0,T;X) \mid \dot{z} \in L_{p,\mu}(0,T;X)\},$$

for $p \in (1,\infty)$, $\mu \in (1/p,1]$ and a Banach space X. The merit of the time weight is to observe that the class of initial data can be taken larger and the solution regularizes instantly in time. Furthermore, we prove the stationary solution $(0,\varphi^*)$ is exponentially stable even under including fluid effect if the product of the coefficients $\nu\gamma$ is sufficiently large and φ^* is the variational strict stable solution, i.e. φ^* satisfies $W(\varphi^*) = 0$ and $(\frac{\delta^2 E(\varphi^*)}{\delta\varphi^2}\psi,\psi) \geq c\|\psi\|^2_{L^2}$ for some $c > 0$ and for any $\psi \in H^4_2(\Omega)$ satisfying $\psi|_{\partial\Omega} = -1, \Delta\psi|_{\partial\Omega} = 0$.

Let us state the main results.

Theorem 4.1 (Local existence and uniqueness of strong solutions)**.** *Let $p,q \in (1,\infty)$, $\mu \in (1/p,1]$ be $\frac{1}{2}+\frac{1}{p}+\frac{n}{2q} < \mu \leq 1$ and assume that*

$$\begin{cases} (u_0,\varphi_0) \in B^{2(\mu-1/p)}_{q,p}(\Omega) \times B^{4(\mu-1/p)}_{q,p}(\Omega) & \\ \operatorname{div} u_0 = 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \\ \varphi_0 = -1, \ \ \Delta\varphi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists $T = T\left(\|u_0\|_{B^{2(\mu-1/p)}_{q,p}}, \|\varphi_0\|_{B^{4(\mu-1/p)}_{q,p}}\right) > 0$ such that the equations (PFNS) *have a unique strong solution*

$$\begin{cases} u \in H^1_{p,\mu}(0,T;L_{q,\sigma}(\Omega)) \cap L_{p,\mu}(0,T;(H^2_q(\Omega))^n), \\ \varphi \in H^1_{p,\mu}(0,T;L_q(\Omega)) \cap L_{p,\mu}(0,T;H^4_q(\Omega)), \\ \nabla p \in L_{p,\mu}(0,T;(L_q(\Omega))^n), \end{cases}$$

where the interval $[0,T)$ is a maximal time interval of existence. Moreover the solution depends continuously on u_0 and φ_0.

Remark 4.2. The condition $\frac{1}{2}+\frac{1}{p}+\frac{n}{2q} < \mu \leq 1$ comes from the embedding exponent such that $B^{2(\mu-1/p)}_{q,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$. Note that this condition implies that the embedding $B^{4(\mu-1/p)}_{q,p}(\Omega) \hookrightarrow C^2(\overline{\Omega})$.

Theorem 4.3. *The solution ${}^T(u,\varphi)$ in Theorem 4.1 satisfies for each $j \in \mathbb{N}$,*

$$t^j \left[\frac{d}{dt}\right]^j \begin{pmatrix} u \\ \varphi \end{pmatrix} \in H^1_{p,\mu}(0,T;L_{q,\sigma}(\Omega) \times L_q(\Omega)) \cap L_{p,\mu}(0,T;(H^2_q(\Omega))^n \times H^4_q(\Omega)).$$

Moreover, the solution ${}^T(u,\varphi)$ is real analytic from $(0,T)$ to $(H^2_q(\Omega))^n \times H^4_q(\Omega)$.

Remark 4.4. By the scaling techniques in time and space, the maximal regularity and the implicit function theorem, it is proved that ${}^T(u,\varphi)$ is real analytic in $(0,T)\times\Omega$. See [22] for parabolic equations, and see [24] for Navier–Stokes equations.

At last we study the stability of the solution near the local minimizers of the elastic bending energy.

Theorem 4.5. *Let* $p, q \in (1, \infty), \mu \in (1/p, 1]$ *satisfy the assumption in Theorem 4.1 and* ${}^T(0, \varphi^*) \in \{0\} \times H_q^4(\Omega)$ *be the variational strictly stable solution of (PFNS) i.e.* φ^* *satisfies* $W(\varphi^*) = 0$ *and* $(\frac{\delta^2 E(\varphi^*)}{\delta \varphi^2}\psi, \psi) \geq c\|\psi\|_{L^2}^2$ *for some* $c > 0$ *and for any* $\psi \in H_2^4(\Omega)$ *satisfying* $\psi|_{\partial\Omega} = -1, \Delta\psi|_{\partial\Omega} = 0$*. Then there are* $\varepsilon > 0, C > 0$ *such that for each* ${}^T(u_0, \varphi_0) \in B_{q,p}^{2(\mu-1/p)}(\Omega) \times B_{q,p}^{4(\mu-1/p)}(\Omega)$ *satisfying* $\|u_0\|_{B_{q,p}^{2(\mu-1/p)}} + \|\varphi_0 - \varphi^*\|_{B_{q,p}^{4(\mu-1/p)}} < \varepsilon$ *and for any* ν *and* γ *satisfying* $\nu\gamma > C$*, there exists a unique global solution*

$$\begin{cases} u \in H_{p,\mu,loc}^1(\mathbb{R}_+; L_{q,\sigma}(\Omega)) \cap L_{p,\mu,loc}(\mathbb{R}_+; (H_q^2(\Omega))^n), \\ \varphi \in H_{p,\mu,loc}^1(\mathbb{R}_+; L_q(\Omega)) \cap L_{p,\mu,loc}(\mathbb{R}_+; H_q^4(\Omega)). \end{cases}$$

Furthermore, there is a $\beta > 0$ *such that*

$$\begin{cases} e^{\beta t}u \in H_{p,\mu}^1(\mathbb{R}_+; L_{q,\sigma}(\Omega)) \cap L_{p,\mu}(\mathbb{R}_+; (H_q^2(\Omega))^n) \cap C_0(\mathbb{R}_+; (B_{q,p}^{2(\mu-1/p)}(\Omega))^n), \\ e^{\beta t}(\varphi - \varphi^*) \in H_{p,\mu}^1(\mathbb{R}_+; L_q(\Omega)) \cap L_{p,\mu}(\mathbb{R}_+; H_q^4(\Omega)) \cap C_0(\mathbb{R}_+; B_{q,p}^{4(\mu-1/p)}(\Omega)). \end{cases}$$

In particular, the equilibrium ${}^T(0, \varphi^*)$ *is exponentially stable in* $B_{q,p}^{2(\mu-1/p)}(\Omega) \times B_{q,p}^{4(\mu-1/p)}(\Omega)$*.*

In this chapter we are not able to guarantee the existence of the variational strict stable solution because of the two penalty terms. In [31] they dealt with the case of $c = 0$ and other stability result.

4.2 General theory for quasi-linear evolution equations

We explain the theory of the quasi-linear parabolic equation via the maximal regularity. For details we refer the papers [16,23]. See also [25].

Let X_0 and X_1 be Banach spaces such that $X_1 \overset{d}{\hookrightarrow} X_0$, i.e. X_1 is continuously and densely embedded in X_0. Let $T > 0$ or $T = \infty$. For a closed linear operator A in X_0, we say A has the property of the maximal $L_{p,\mu}$-regularity, for short $A \in \mathcal{MR}_{p,\mu}(X_1, X_0)$, if for each $f \in L_{p,\mu}(0, T; X_0)$ there exists a unique solution $u \in H_{p,\mu}^1(0, T; X_0) \cap L_{p,\mu}(0, T; X_1)$ of the linear problem $\dot{u} + Au = f$ $(t \in (0, T))$ with initial value $u(0) = 0$. For classical case $\mu = 1$, denote $A \in \mathcal{MR}_p(X_1, X_0)$. In [16,23] it was proved that

$$A \in \mathcal{MR}_{p,\mu}(X_1, X_0) \Leftrightarrow A \in \mathcal{MR}_p(X_1, X_0) \qquad \forall p \in (1, \infty), \mu \in (1/p, 1],$$

and, concerning nontrivial initial data, if $A \in \mathcal{MR}_p(X_1, X_0)$ then

$$\begin{aligned} &A \in \mathcal{MR}_{p,\mu}(X_1, X_0) \Leftrightarrow \forall f \in L_{p,\mu}(0, T; X_0)\, \forall u_0 \in X_{\gamma,\mu}, \\ &\exists! u \in H_{p,\mu}^1(0, T; X_0) \cap L_{p,\mu}(0, T; X_1) \text{ s.t. } \dot{u} + Au = f \ (t \in (0, T)), u(0) = u_0, \end{aligned}$$

where $X_{\gamma,\mu} := (X_0, X_1)_{\mu-1/p,p}$ is the trace space for $p \in (1, \infty)$ and $\mu \in (1/p, 1]$. For trace spaces, see also [21]. The case $\mu = 1$, denote $X_\gamma := X_{\gamma,1}$.

We consider the following quasi-linear parabolic equation (QL):

$$\text{(QL)} \qquad \begin{cases} \dot{z}(t) + A(z(t))z(t) = F(z(t)) & t \in (0,T), \\ z(0) = z_0. \end{cases}$$

Here we impose regularity assumptions

$$(A_-)\ A \in \mathrm{Lip}_{loc}(X_{\gamma,\mu}; \mathcal{L}(X_1, X_0)), \qquad (F_-)\ F \in \mathrm{Lip}_{loc}(X_{\gamma,\mu}; X_0)$$

and $z_0 \in X_{\gamma,\mu}$ for $p \in (1,\infty)$, $\mu \in (1/p, 1]$. By $\mathcal{L}(X_1, X_0)$ we denote the set of all bounded linear operator from X_1 to X_0. We now give existence and uniqueness results for (QL). Local in time existence and uniqueness of (QL) was shown by Clément and Li [3] in the case $\mu = 1$ and by Köhne, Prüss and Wilke [16] for the case $\mu \in (1/p, 1]$.

Proposition 4.6. *Let $1 < p < \infty$, $\mu \in (1/p, 1]$, $z_0 \in X_{\gamma,\mu}$, and suppose that the assumption $(A_-), (F_-)$ and $A(z_0) \in \mathcal{MR}_p(X_1, X_0)$ are satisfied. Then, there exists $a > 0$, such that* (QL) *admits a unique solution z on $J = [0, a]$ in the regularity class*

$$z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1) \hookrightarrow C(J; X_{\gamma,\mu}) \cap C((0,a]; X_\gamma).$$

The solution depends continuously on z_0, and can be extended to a maximal interval of existence $J(z_0) = [0, t^+(z_0))$.

Smoothing effects often appear in parabolic problems. The parameter trick method by Angenent [1] is well known. A similar method has already been used in the study of Navier–Stokes equation in [19, 20]. We state the regularity of the solution of (QL) in terms of the regularity of A and F.

We use the following notation to state two propositions:

$$(A_k)\ A \in C^k(X_{\gamma,\mu}; \mathcal{L}(X_1, X_0)), \qquad (F_k)\ F \in C^k(X_{\gamma,\mu}; X_0)$$

for $k \in \mathbb{N} \cup \{\infty, \omega\}$, where the index ω refers to real analyticity.

Let us recall the definition of the real analytic between Banach spaces [33]. Suppose X, Y are two Banach spaces. We say the operator $T : X \to Y$ is analytic if for any $x_0 \in X$ there exists a small neighborhood of x_0 such that

$$T(x_0 + h) - T(x_0) = \sum_{n \geq 1} T_n(x_0)(h, \cdots, h) \qquad \forall h \in X,\ \|h\|_X < r \ll 1.$$

Here $T_n(x_0)$ is a continuous symmetrical n-linear operator on $X^n \to Y$ and satisfies

$$\sum_{n \geq 1} \|T_n(x_0)\|_{\mathcal{L}(X^n, Y)} \|h\|_X^n < \infty.$$

Proposition 4.7. *Let $1 < p < \infty$, $\mu \in (1/p, 1]$, $z_0 \in X_{\gamma,\mu}$, $k \in \mathbb{N} \cup \{\infty, \omega\}$ and suppose that the assumption $(A_k), (F_k)$ and $A(z_0) \in \mathcal{MR}_p(X_1, X_0)$ are satisfied. Let z be the solution in Proposition 4.6 and assume $A(z(t)) \in \mathcal{MR}_p(X_1, X_0)$ for all $t \in J$. Then*

$$t^j \left[\frac{d}{dt}\right]^j z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1), \qquad j \leq k$$

Furthermore, if $k = \infty$ then $z \in C^\infty(J; X_1)$, and if $k = \omega$ then $z \in C^\omega(J; X_1)$.

If we impose the Fréchet differentiability of A and F, then the solution exists globally provided the initial data is close to an equilibrium point, see [22] for the case $\mu = 1$, and [16] for the case $\mu \in (1/p, 1]$. Let $\mathcal{E} := \{z \in X_1 \mid A(z)z = F(z)\}$ be the equilibria of (QL) and A_0 be the linearization of (QL) at $z^* \in \mathcal{E}$, i.e.

$$A_0 w = A(z^*)w + (A'(z^*)w)z^* - F'(z^*)w, \qquad w \in X_1.$$

We denote the spectrum of the operator A by $\sigma(A)$ and denote the resolvent set of the operator A by $\rho(A)$.

Proposition 4.8. *Let $1 < p < \infty$, $\mu \in (1/p, 1]$ and $z^* \in \mathcal{E}$ be $A(z^*) \in \mathcal{MR}_p(X_1, X_0)$ on $\mathbb{R}_+$ and the assumptions (A_1) and (F_1) are satisfied. Suppose that $\sigma(A_0) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > 0\}$. Then there is $\varepsilon > 0$ such that for each $z_0 \in B_\varepsilon(z^*) \subset X_{\gamma,\mu}$ there exists a unique global solution $z \in H^1_{p,\mu,loc}(\mathbb{R}_+; X_0) \cap L_{p,\mu,loc}(\mathbb{R}_+; X_1)$ of (QL). Furthermore, there is a $\beta > 0$ such that*

$$e^{\beta t}(z - z^*) \in H^1_{p,\mu}(\mathbb{R}_+; X_0) \cap L_{p,\mu}(\mathbb{R}_+; X_1) \cap C_0(\mathbb{R}_+; X_{\gamma,\mu}).$$

In particular, the equilibrium z^* is exponentially stable in $X_{\gamma,\mu}$.

In this proposition, the constant $\varepsilon > 0$ depends only on the maximal regularity constant of A_0 and the local Lipschitz constants of A and F. Here $C_0(\mathbb{R}_+; X_{\gamma,\mu})$ is the space of $X_{\gamma,\mu}$-valued continuous function vanishing at the time-infinity.

4.3 Quasilinear Approach for the phase-field Navier–Stokes equations

4.3.1 Quasilinear formulation

In this section we transform (PFNS) into quasi-linear evolution equations for the unknown $z = {}^T(u, \phi)$. Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded C^4-domain. We choose the Banach space

$$X_0 := L_{q,\sigma}(\Omega) \times L_q(\Omega).$$

As usual, $L_{q,\sigma}(\Omega)$ is the subspace of $(L_q(\Omega))^n$ consisting of solenoidal vector fields. We denote by $\mathbb{P} : (L_q(\Omega))^n \to L_{q,\sigma}(\Omega)$ the Helmholtz projection and the Stokes operator $\mathcal{A}_q : D(\mathcal{A}_q) \to L_{q,\sigma}(\Omega)$, where $D(\mathcal{A}_q) = \{u \in (H^2_q(\Omega))^n \cap L_{q,\sigma}(\Omega) \mid u = 0 \text{ a.e. on } \partial\Omega\}$, $\mathcal{A}_q = -\mathbb{P}\Delta$. The maximal L_q-regularity result for the Stokes operator $\mathcal{A}_q$ is well-known; see e.g. [11, 13]. The bi-Laplace operator $\mathcal{D}_q$ in $L_q(\Omega)$ is defined by $\mathcal{D}_q = \Delta^2$ with the domain $D(\mathcal{D}_q) = \{\phi \in H^4_q(\Omega) \mid \phi = \Delta\phi = 0 \text{ a.e. on } \partial\Omega\}$. The maximal L_q-regularity result for bi-Laplace operator $\mathcal{D}_q$ is also well-known; see e.g. [13]. We choose the Banach space

$$X_1 := D(\mathcal{A}_q) \times D(\mathcal{D}_q),$$

equipped with its canonical norms. Then $X_1 \overset{d}{\hookrightarrow} X_0$.

For we treat (PFNS) as the quasi-linear equation, we define the operator

$$A(z) := \begin{pmatrix} \nu \mathcal{A}_q & -k\varepsilon \mathbb{P}\mathcal{B}_q(\phi) \\ 0 & \gamma k \varepsilon \mathcal{D}_q \end{pmatrix},$$

where the operator $\mathcal{B}_q$ is given by $\mathcal{B}_q(\phi)h := \Delta^2 h \nabla\phi$. Let $\phi = \varphi + 1$ and apply the Helmholtz projection $\mathbb{P}$ to the first equation in (PFNS), then we are able to rewrite (PFNS) of the form:

$$(*) \begin{cases} \dfrac{d}{dt} z + A(z)z = F(z) := \begin{pmatrix} -\mathbb{P}((u\cdot\nabla)u) + \mathbb{P}(L(\phi-1)\nabla\phi) \\ -(u\cdot\nabla)\phi - \gamma L(\phi - 1) \end{pmatrix} & t \in (0,T), \\ z(0) = z_0 := \begin{pmatrix} u_0 \\ \phi_0 \end{pmatrix} := \begin{pmatrix} u_0 \\ \varphi_0 + 1 \end{pmatrix}. \end{cases}$$

We show the $A(z)$ has the property of maximal regularity for each $z \in X_{\gamma,\mu}$ and assumptions (A_ω) and (F_ω). We have that $\mathcal{B}_q(\phi) : D(\mathcal{D}_q) \to (L_q(\Omega))^n$ is bounded for each $\phi \in C^1(\overline{\Omega})$ and the map $\phi \to \mathbb{P}\mathcal{B}_q(\phi)$ is real analytic. So $A(z) \in C^\omega(L_{q,\sigma}(\Omega) \times C^1(\overline{\Omega}), \mathcal{L}(X_1, X_0))$. By the tri-diagonal structure of $A(z)$ and by the regularity of $\mathcal{B}_q$ one can easily see that $A(z) \in \mathcal{MR}_p(X_1, X_0)$ for each $z = {}^T(u,\phi) \in L_{q,\sigma}(\Omega) \times C^1(\overline{\Omega})$. Indeed, from $\begin{pmatrix} \nu\mathcal{A}_q & 0 \\ 0 & \gamma k \varepsilon \mathcal{D}_q \end{pmatrix} \in \mathcal{MR}_p(X_1, X_0)$, for any ${}^T(f,g) \in L_p(0,T;X_0)$, we can take ${}^T(\tilde{u},\tilde{\phi}) \in H^1_p(0,T;X_0) \cap L_p(0,T;X_1)$ which is the solution of

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \tilde{u} \\ \tilde{\phi} \end{pmatrix} + \begin{pmatrix} \nu\mathcal{A}_q & 0 \\ 0 & \gamma k\varepsilon\mathcal{D}_q \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} f + k\varepsilon\mathbb{P}\mathcal{B}_q(\phi)\overline{\phi} \\ g \end{pmatrix} & t \in (0,T), \\ \begin{pmatrix} \tilde{u} \\ \tilde{\phi} \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{cases}$$

where $\overline{\phi}$ is the solution of

$$\begin{cases} \dfrac{d}{dt}\overline{\phi} + k\gamma\varepsilon\mathcal{D}_q\overline{\phi} = g & t \in (0,T), \\ \overline{\phi}(0) = 0. \end{cases}$$

Note that $k\varepsilon\mathbb{P}\mathcal{B}_q(\phi)\overline{\phi} \in L_p(0,T;L_{q,\sigma}(\Omega))$ for each $\phi \in C^1(\overline{\Omega})$ and $\tilde{\phi} = \overline{\phi}$. This implies that for any ${}^T(f,g) \in L_p(0,T;X_0)$, we can take ${}^T(\tilde{u},\tilde{\phi}) \in H^1_p(0,T;X_0) \cap L_p(0,T;X_1)$ which is the solution of

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \tilde{u} \\ \tilde{\phi} \end{pmatrix} + \begin{pmatrix} \nu\mathcal{A}_q & -k\varepsilon\mathbb{P}\mathcal{B}_q(\phi) \\ 0 & \gamma k\varepsilon\mathcal{D}_q \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} & t \in (0,T), \\ \begin{pmatrix} \tilde{u} \\ \tilde{\phi} \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

So $A(z) \in \mathcal{MR}_p(X_1, X_0)$ for each $z = {}^T(u,\phi) \in L_{q,\sigma}(\Omega) \times C^1(\overline{\Omega})$.

The nonlinear term F is also real analytic from $(C^1(\overline{\Omega}))^n \times C^2(\overline{\Omega})$ into X_0. If we get $X_{\gamma,\mu} \hookrightarrow (C^1(\overline{\Omega}))^n \times C^2(\overline{\Omega})$, then (A_ω) and (F_ω) hold. The space $X_{\gamma,\mu}$ is given by

$$X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p} = D_{\mathcal{A}_q}(\mu - 1/p, p) \times D_{\mathcal{D}_q}(\mu - 1/p, p),$$

provided $p \in (1, \infty)$ and $\mu \in (1/p, 1]$; see [18, 21]. Here $D_{\mathcal{A}_q}(\theta, p)$ is the real interpolation $(L_{q,\sigma}(\Omega), D(\mathcal{A}_q))_{\theta,p}$ and $D_{\mathcal{D}_q}(\theta, p) = (L_q(\Omega), D(\mathcal{D}_q))_{\theta,p}$. We need to consider two embedding exponent, one is $D_{\mathcal{A}_q}(\mu - 1/p, p) \hookrightarrow (C^1(\overline{\Omega}))^n$ and the other is $D_{\mathcal{D}_q}(\mu - 1/p, p) \hookrightarrow C^2(\overline{\Omega})$:

$$\frac{1}{2} + \frac{1}{p} + \frac{n}{2q} < \mu \leq 1 \Rightarrow D_{\mathcal{A}_q}(\mu - 1/p, p) \hookrightarrow (C^1(\overline{\Omega}))^n.$$
$$\frac{1}{2} + \frac{1}{p} + \frac{n}{4q} < \mu \leq 1 \Rightarrow D_{\mathcal{D}_q}(\mu - 1/p, p) \hookrightarrow C^2(\overline{\Omega}).$$

Note that $\frac{1}{2} + \frac{1}{p} + \frac{n}{4q} < \frac{1}{2} + \frac{1}{p} + \frac{n}{2q}$.

Under the condition $\frac{1}{2} + \frac{1}{p} + \frac{n}{2q} < \mu \leq 1$, we can characterize the interpolation spaces by Besov spaces:

$$u \in D_{\mathcal{A}_q}(\mu - 1/p, p) \Leftrightarrow u \in (B^{2(\mu-1/p)}_{q,p}(\Omega))^n \cap L_{q,\sigma}(\Omega), \;\; u = 0, \text{ a.e. on } \partial\Omega,$$
$$\phi \in D_{\mathcal{D}_q}(\mu - 1/p, p) \Leftrightarrow \phi \in B^{4(\mu-1/p)}_{q,p}(\Omega), \;\; \phi = \Delta\phi = 0, \text{ a.e. on } \partial\Omega.$$

For Besov spaces, see [29].

We are ready to prove well-posedness results in Section 4.1. The proof is based on propositions in Section 4.2.

Proof of Theorem 4.1. We transformed (PFNS) into the quasi-linear parabolic equation $(*)$. The condition $z_0 \in X_{\gamma,\mu}$ is equivalent to the conditions in Theorem 4.1 and $\tilde{A}(z_0) \in \mathcal{MR}_p(X_1, X_0)$. So we are able to apply Proposition 4.6. □

Proof of Theorem 4.3. We have already checked the conditions $(A_\omega), (F_\omega)$. Since the solution $z(t) \in X_{\gamma,\mu}$ for all $t \in J$, assumptions in Proposition 4.7 are satisfied. □

4.3.2 Spectral analysis of the linearized operator

In order to prove the stability result of Theorem 4.5, we calculate the linearized operator near the local minimizers of the elastic bending energy. The equilibria $\mathcal{E}$ is the set

$$\mathcal{E} = \{z^* = {}^T(0, \phi^*) \in X_1 \mid W(\phi^* - 1) = 0\}.$$

The linearized operator A_0 at z^* is given by $A_0 w = A(z^*)z + (A'(z^*)w)z^* - F'(z^*)w$ for $w = {}^T(w_1, w_2) \in X_1$. By direct calculation

$$A(z^*)w = \begin{pmatrix} -\nu\mathbb{P}\Delta & -k\varepsilon\mathbb{P}(\Delta^2 \cdot \nabla\phi^*) \\ 0 & \gamma k\varepsilon\Delta^2 \end{pmatrix} w$$
$$(A'(z^*)w)z^* = \begin{pmatrix} 0 & -k\varepsilon\mathbb{P}(\Delta^2 \cdot \nabla w_2) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \phi^* \end{pmatrix}$$

$$
= \begin{pmatrix} 0 & -k\varepsilon\mathbb{P}(\Delta^2\phi^*\nabla\cdot) \\ 0 & 0 \end{pmatrix} w
$$

$$
F'(z^*)w = \begin{pmatrix} 0 & -k\varepsilon\mathbb{P}(\Delta^2\phi^*\nabla\cdot) + \mathbb{P}(G(\phi^*)\cdot\nabla\phi^*) \\ -C(\phi^*) & -\gamma G(\phi^*) \end{pmatrix} w,
$$

where the linear operator $C(\phi^*)$ and $G(\phi^*)$ is defined as follows:

$$
\begin{aligned}
&C(\phi^*)w_1 = (w_1\cdot\nabla)\phi^* \\
&G(\phi^*)w_2 = G_1(\phi^*)w_2 + M_1\int_\Omega w_2 dx \\
&\qquad\qquad + M_2\left\{ f(\phi^*-1)\int_\Omega f(\phi^*-1)w_2 dx + G_2(\phi^*)w_2 \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&G_1(\phi^*)w_2 \\
&= -\frac{k}{\varepsilon}\left\{ \left(6(\phi^*-1)^2-2\right)\Delta w_2 + 6\nabla((\phi^*-1)^2)\cdot\nabla w_2 \right. \\
&\quad \left. + \left(3\Delta((\phi^*-1)^2) + 6(\phi^*-1)\Delta\phi^* + (15(\phi^*-1)^4 - 12(\phi^*-1)^2+1)\right) w_2 \right\} \\
&G_2(\phi^*)w_2 = [B(\phi^*-1) - B(\phi_0-1)]\left\{ -\varepsilon\Delta w_2 + \frac{1}{\varepsilon}\left(3(\phi^*-1)^2-1\right) w_2 \right\}.
\end{aligned}
$$

Therefore the linearized operator A_0 is

$$
A_0 = \begin{pmatrix} -\nu\mathbb{P}\Delta & -\mathbb{P}\left((k\varepsilon\Delta^2 + G(\phi^*))\cdot\nabla\phi^*)\right) \\ C(\phi^*) & \gamma(k\varepsilon\Delta^2 + G(\phi^*)) \end{pmatrix}.
$$

Since $\frac{\delta^2 E(\phi^*)}{\delta\phi^2} = k\varepsilon\Delta^2 + G(\phi^*)$, the realization of this operator A_0 in L^q spaces can be rewritten as

$$
A_0 = \begin{pmatrix} \nu\mathcal{A}_q & -\mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\cdot\nabla\phi^*\right) \\ \mathcal{C}_q & \gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2} \end{pmatrix}, \quad D(A_0) = D(\mathcal{A}_q)\times D(\mathcal{D}_q),
$$

where $\mathcal{C}_q = C(\phi^*)$ with the domain $D(\mathcal{C}_q) = L_{q,\sigma}(\Omega)$.

From now we show that $\sigma(A_0)\subset\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda > 0\}$. Denote $\Sigma_{\eta,M} := \{\lambda = re^{i\theta}\in\mathbb{C}\setminus\{0\}\mid r\geq M, \eta < |\theta|\}$ for some $M\geq 0$ and $\eta\in(0,\pi/2)$, and denote $\mathbb{C}_+ := \{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda > 0\}$ and $\overline{\mathbb{C}}_- := \{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda\leq 0\}$.

Lemma 4.9. *Assume that there exists $c > 0$ such that for all $\psi\in D(\mathcal{D}_2)$, $\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\psi,\psi\right)\geq c\|\psi\|^2_{L^2}$. Then $\sigma\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)\subset\mathbb{C}_+$.*

Proof. We consider that $G(\phi^*)$ is a lower order perturbation of $k\varepsilon\Delta^2$. Then for any $0 < \eta < \pi/2$, there exists $M > 0$ such that $\Sigma_{\eta,M}\subset\rho(\frac{\delta^2 E(\phi^*)}{\delta\phi^2})$. Fix $\lambda_0\in\rho(\frac{\delta^2 E(\phi^*)}{\delta\phi^2})$. From compactness of the operator $(\lambda_0 - \frac{\delta^2 E(\phi^*)}{\delta\phi^2})^{-1}$ and Fredholm theory, we have the injection of the operator $\lambda - \frac{\delta^2 E(\phi^*)}{\delta\phi^2}$ implies $\lambda\in\rho(\frac{\delta^2 E(\phi^*)}{\delta\phi^2})$. Let $\psi\in D(\mathcal{D}_q)$ satisfy $(\lambda - \frac{\delta^2 E(\phi^*)}{\delta\phi^2})\psi = 0$. We may assume $\psi\in D(\mathcal{D}_2)$. In fact, the boundedness of Ω implies that

$D(\mathcal{D}_q) \subset D(\mathcal{D}_2)$, when $2 \leq q < \infty$. On the other hand, when $1 < q < 2$, by the Sobolev embedding theorem and bootstrap argument for the equation $(\lambda+\lambda_0-\frac{\delta^2 E(\phi^*)}{\delta\phi^2})\psi = \lambda_0\psi$, we see that $\psi \in D(\mathcal{D}_2)$. From $(\lambda - \frac{\delta^2 E(\phi^*)}{\delta\phi^2})\psi = 0$, we see

$$\mathrm{Re}\,\lambda\|\psi\|_2 - \left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\psi, \psi\right) = 0$$

and, from the assumption on $\frac{\delta^2 E(\phi^*)}{\delta\phi^2}$ we see $\psi = 0$ when $\lambda \in \overline{\mathbb{C}}_-$. □

Remark 4.10. We have the following resolvent estimate for the operator $\frac{\delta^2 E(\phi^*)}{\delta\phi^2}$:

$$\left\|\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda - \frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\right\|_{\mathcal{L}(L^q(\Omega))} \leq C_\eta \ \ (\forall \lambda \in \Sigma_{\eta,0}).$$

Theorem 4.11. *Assume that there exists $c > 0$ such that for all $\psi \in D(\mathcal{D}_2)$, $\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\psi, \psi\right) \geq c\|\psi\|_{L^2}^2$. Then there exists $C = C_{\phi^*} > 0$ such that if ν and γ satisfy $\nu\gamma > C$ then $\sigma(A_0) \subset \mathbb{C}_+$.*

Proof. By the similar method of Lemma 4.9, it suffices that $(\lambda - A_0)z = 0$ $(z = {}^T(u, \phi) \in X_1)$ implies $z = 0$ for $\lambda \in \overline{\mathbb{C}}_-$. The second equation of this resolvent equation

$$\begin{aligned}(\lambda - \nu\mathcal{A}_q)u + \mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\phi\nabla\phi^*\right) = 0\\ -\mathcal{C}_q u + \left(\lambda - \gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)\phi = 0\end{aligned}$$

derives

$$-\mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda - \gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q u\nabla\phi^*\right) + \mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\phi\nabla\phi^*\right) = 0.$$

By subtracting the first equation,

$$\left(\lambda - \nu\mathcal{A}_q + \mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda - \gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q\cdot\nabla\phi^*\right)\right)u = 0.$$

We use the perturbation theory of the generator of analytic semigroups. The calculation

$$\begin{aligned}&\left\|\mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda - \gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q\cdot\nabla\phi^*\right)u\right\|_{L_{q,\sigma}}\\ \leq&\|\nabla\phi^*\|_\infty\left\|\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda - \gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q u\right\|_q\\ \leq&\|\nabla\phi^*\|_\infty\sup_{\lambda\in\overline{\mathbb{C}}_-}\left\|\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda - \frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\right\|_{\mathcal{L}(L^q)}\frac{1}{\gamma}\|(u\cdot\nabla)\phi^*\|_q\\ \leq&\|\nabla\phi^*\|_\infty^2\sup_{\lambda\in\overline{\mathbb{C}}_-}\left\|\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda - \frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\right\|_{\mathcal{L}(L^q)}\frac{1}{\gamma}\|u\|_q\end{aligned}$$

$$
\begin{aligned}
&\leq C\|\nabla\phi^*\|_\infty^2 \sup_{\lambda\in\overline{\mathbb{C}}_-}\left\|\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda-\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\right\|_{\mathcal{L}(L^q)}\frac{1}{\nu\gamma}\|\nu\mathcal{A}_q u\|_q \\
&\leq C_{\phi^*}\frac{1}{\nu\gamma}\|\nu\mathcal{A}_q u\|_q
\end{aligned}
$$

implies that if $\nu\gamma$ is sufficiently large, then $\mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda-\gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q\cdot\nabla\phi^*\right)$ is small perturbation of $\nu\mathcal{A}_q$. So from $\lambda\in\overline{\mathbb{C}}_-\subset\rho(\nu\mathcal{A}_q)$, we have

$$
\lambda\in\rho\left(\nu\mathcal{A}_q-\mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\left(\lambda-\gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q\cdot\nabla\phi^*\right)\right)
$$

and then $u=0$. By lemma 4.9, $\phi=0$. It concludes that $\sigma(A_0)\subset\mathbb{C}_+$. □

Proof of Theorem 4.5. Since $\phi^*=\varphi^*-1$, the proof is straightly based on Proposition 4.8. □

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Chapter 5

Global well-posedness for a Cahn–Hilliard equation on bounded domains with permeable and non-permeable walls in maximal regularity spaces

We consider the strong solution of the Cahn–Hilliard equation on bounded domains with permeable and non-permeable walls in maximal L_p regularity spaces. From the maximal L_p regularity result of the linear equation with the dynamic boundary condition, the fixed point theorem and a priori estimate, we prove that the solution exists uniquely and globally in time

Keywords: Cahn–Hilliard equation, dynamic boundary condition, global well-posedness, maximal L_p regularity

5.1 Introduction

Let $0 < T < \infty$ be a some fixed time, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain whose boundary $\Gamma := \partial\Omega$ is smooth. Denote $J := (0, T)$, $Q := J \times \Omega$ and $\Sigma := J \times \Gamma$. We consider the following Cahn–Hilliard equation

$$\partial_t u - \Delta\mu = 0 \quad \text{in } Q, \tag{5.1.1}$$

$$\mu = -\Delta u + F'(u) \quad \text{in } Q. \tag{5.1.2}$$

Here u is the order parameter, μ and F are the chemical and physical potentials, respectively. In this chapter we consider one of following two boundary conditions:

$$\Delta\mu + b\partial_\nu\mu + c\mu = 0 \quad \text{on } \Sigma, \tag{5.1.3}$$

$$-\alpha\Delta_\Gamma u + \partial_\nu u + G'(u) = \mu/b \quad \text{on } \Sigma, \tag{5.1.4}$$

or

$$\Delta\mu + b\partial_\nu\mu - c\Delta_\Gamma\mu = 0 \quad \text{on } \Sigma, \tag{5.1.5}$$

$$-\alpha\Delta_\Gamma u + \partial_\nu u + G'(u) = \mu/b \quad \text{on } \Sigma. \tag{5.1.6}$$

The first one appears the case that the domain has porous (permeable) walls and the second one corresponds to non-permeable walls.

In the boundary conditions, α, b, c are positive constants, Δ_Γ is the Laplace–Beltrami operator on Γ, ν is the unit outward normal vector to Γ and G is the nonlinear term which comes from the surface energy. A typical example of F and G are $F(u) = (1/4)(u^2-1)^2$ and $G(u) = (g_s/2)u^2 - h_s u$ with $g_s > 0, h_s \neq 0$. We also treat the case $c = 0$ in subsection 5.2.4.

Our aim of this chapter is to prove existence and uniqueness of this Cahn–Hilliard equation with these boundary conditions in maximal L_p spaces for $1 < p < \infty$. So far, the study of the Cahn–Hilliard equation has been considered in L_2 frameworks. The L_p approach has been done by the papers [18, 19] but only for the classical dynamic boundary condition. In the last decades, other type of boundary conditions has been considered and discussed in L_2 frameworks. See the next paragraph for the previous works. However, as far as we know, the study of L_p frameworks has not been treated under our boundary conditions yet. In this chapter we fill this gap by a simple approach using the linear theory of abstract parabolic equations constructed in the paper [5]. The authors considered the equations called *relaxation type*, which contains our linearized Cahn–Hilliard equation with the boundary conditions we consider. So we obtain the maximal L_p regularity result on the linearized equations. For the nonlinear Cahn–Hilliard equation (5.1.1)–(5.1.2) on permeable walls (5.1.3)–(5.1.4) and on non-permeable walls (5.1.5)–(5.1.6), we prove local existence and uniqueness of solutions by fixed point argument. The key is to show the contraction property of non-linear term by restricting a small time interval and taking exponent p large, see Proposition 5.4 and Proposition 5.10. To extend global solutions, we use energy estimates from integration by parts. Combining with a priori estimates, we claim that the unique local solution does not blow up at any time, which means the solution is a global solution.

The Cahn–Hilliard equation is known as describing the spinodal decomposition of binary mixtures, which we can see in the cooling processes of alloys, glasses or polymer mixtures (see [1, 13, 16, 17]). For the study of the Cahn–Hilliard equation, various boundary conditions has been considered. At first, we would like to mention the following usual boundary conditions:

$$\partial_\nu \mu = 0 \qquad \text{on } \Sigma, \tag{5.1.7}$$

$$\partial_\nu u = 0 \qquad \text{on } \Sigma. \tag{5.1.8}$$

The condition (5.1.7) derives that the total mass $\int_\Omega u dx$ does not change for all time $t > 0$. The other condition (5.1.8) is called the variational boundary condition since it derives that the following bulk free energy

$$E_\Omega(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) \right) dx \tag{5.1.9}$$

does not increase with (5.1.7). For the Cahn–Hilliard equation (5.1.1)–(5.1.2) with (5.1.7)–(5.1.8), the global well-posedness result and large time behavior were constructed. See [6, 21, 22].

However, in [13] it was proposed by physicists that one should add the following surface free energy

$$E_\Gamma(u) := \int_\Gamma \Big(\frac{\alpha}{2}|\nabla_\Gamma u|^2 + G(u)\Big)\, dS \tag{5.1.10}$$

to the bulk free energy $E_\Omega(u)$, where ∇_Γ is the surface gradient. Together with the no-flux boundary condition (5.1.7), the total energy $E(u) = E_\Omega(u) + E_\Gamma(u)$ makes non-increasing when the dynamic boundary condition

$$\alpha\Delta_\Gamma u - \partial_\nu u + G'(u) = \frac{1}{\Gamma_s} u_t \qquad \text{on } \Sigma, \tag{5.1.11}$$

is posed, with some $\Gamma_s > 0$. For this problem, see e.g., [3,18,20,23]. We would like to mention the paper [18]. The authors of [18] obtained results on the maximal L_p regularity of the solution and asymptotic behavior of the solution of this problem. Moreover it has shown the existence of a global attractor. These results was extended to the non-isothermal setting by a similar maximal regularity result in [19].

The Wentzell boundary condition (5.1.3) we would like to study was proposed in the paper [8]. Thanks to the boundary condition (5.1.4), the total energy $E(u)$ is non-increasing:

$$\frac{d}{dt}E(u(t)) = -\int_\Omega |\nabla \mu|^2 dx - \frac{c}{b}\int_\Gamma \mu^2 dS \le 0 \ \ (t > 0). \tag{5.1.12}$$

Since $\frac{d}{dt}(\int_\Omega u dx + \int_\Gamma u \frac{dS}{b}) = -c\int_\Gamma \mu \frac{dS}{b}$, the case $c = 0$ corresponds to the case of the conservation of the total mass in the bulk and on the boundary. In the paper [8] the existence and uniqueness of a global solution were proved via the Caginalp type equation, which is the similar method in [20]. Later in [9], these results were extended under more general assumptions. In the papers [23]($c > 0$) and [10]($c = 0$), it was shown that each solution of this model converges to a steady state as time goes to infinity and their convergence rate by using Lojasiewicz–Simon inequality.

In contrast to permeable walls, recently, the Cahn–Hilliard equation (5.1.1)–(5.1.2) with (5.1.5)–(5.1.6) in the non-permeable walls was considered, e.g., [2,11,12]. The first boundary condition (5.1.5) represents the Cahn–Hilliard equation on the boundary Γ. The second boundary condition (5.1.6) called the variational boundary condition (5.1.4) leads non-increasing for $E(u)$. In this system, $\int_\Omega u dx + \int_\Gamma u \frac{dS}{b}$ is constant. The existence and uniqueness of weak solutions and their asymptotic behavior were shown in [12]. The well-posedness results for this equation with singular potentials in [4] and numerical results in [7] were also studied. More recently, another boundary condition was proposed in [14] via an energetic variational approach that combines the least action principle and Onsager's principle of maximum energy dissipation.

In this chapter we prove the global existence and uniqueness of the Cahn–Hilliard equation on permeable and non-permeable walls in maximal L_p regularity spaces. This article is organized as follows. In Section 5.2, we study the equation on permeable walls. In subsection 5.2.1, we give the linear theory. We use the general theory of maximal regularity of relaxation type proved by Denk–Prüss–Zacher [5]. We collect their result in Appendix A and apply it for the Cahn–Hilliard equation on permeable walls in Appendix B. In subsection 5.2.2, we give local well-popsedness of this equation by using usual fixed point argument. The estimate we use is essentially based on the paper in [19]. In subsection 5.2.3, we extend this local solution to the global solution by a energy estimate and a priori estimate. In subsection 5.2.4, we focus on the case $c = 0$ in the boundary condition (5.1.3). Since the estimates used in subsection 5.2.3 are different from the case $c > 0$, we calculate the case $c = 0$ again. We are able to get existence and uniqueness result as well. In Section 5.3, we study the equation on non-permeable walls. The strategy for non-permeable walls is almost same as Section 5.2, so we show a few estimates and give some comment, then we state our results.

Before we study the Cahn–Hilliard equation, we would like to mention about the equation on the boundary. In this chapter we distinguish u, μ in the domain and u_Γ, μ_Γ on the boundary, but $u|_\Gamma = u_\Gamma, \mu|_\Gamma = \mu_\Gamma$, where "$|_\Gamma$" is the trace operator on the boundary Γ. Moreover for the boundary condition (5.1.3) and (5.1.5), we replace $(\Delta\mu)|_\Gamma$ with $\partial_t u_\Gamma$ since $\partial_t u = \Delta\mu$ in the domain Ω. So the equations we analyze are as follows

$$\begin{cases} \partial_t u = \Delta\mu, \quad \mu = -\Delta u + F'(u) & \text{in} \quad Q, \\ \partial_t u_\Gamma + b\partial_\nu \mu + c\mu_\Gamma = 0, \quad -\alpha\Delta_\Gamma u_\Gamma + \partial_\nu u + G'(u_\Gamma) = \frac{\mu_\Gamma}{b} & \text{on} \quad \Sigma, \\ u|_\Gamma = u_\Gamma, \quad \mu|_\Gamma = \mu_\Gamma & \text{on} \quad \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

and

$$\begin{cases} \partial_t u = \Delta\mu, \quad \mu = -\Delta u + F'(u) & \text{in} \quad Q, \\ \partial_t u_\Gamma + b\partial_\nu \mu - c\Delta_\Gamma\mu_\Gamma = 0, \quad -\alpha\Delta_\Gamma u_\Gamma + \partial_\nu u + G'(u_\Gamma) = \frac{\mu_\Gamma}{b} & \text{on} \quad \Sigma, \\ u|_\Gamma = u_\Gamma, \quad \mu|_\Gamma = \mu_\Gamma & \text{on} \quad \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

Here note that the unknown functions are u and u_Γ. We do not use the functions μ and μ_Γ except for energy estimates.

Throughout this chapter, we use fractional Sobolev space $W_p^s(J, X)$ for a Banach space X, $s \in \mathbb{R}_{\geq 0} \setminus \mathbb{N}$ and $1 < p < \infty$, which is characterize as follows. Let $[s] \in \mathbb{N} \cup \{0\}$ and $\{s\} \in (0, 1)$ be $s = [s] + \{s\}$. Then by using real interpolation method, it is

$$W_p^s(J, X) := (W_p^{[s]}(J, X), W_p^{[s]+1}(J, X))_{\{s\},p}.$$

Similarly, Besov space is defined as follows.

$$B_{p,p}^s(\Omega) := (W_p^{[s]}(\Omega), W_p^{[s]+1}(\Omega))_{\{s\},p}.$$

To treat nonlinear term, let $C^{m-}(\mathbb{R})(m \in \mathbb{N})$ be the space of all functions $f \in C^{m-1}(\mathbb{R})$ such that $\partial^\alpha f$ is Lipschitz continuous for each $|\alpha| = m$.

5.2 A Cahn–Hilliard equation on permeable walls

5.2.1 The linear theory

In this section we study the following linearized equation of the form

$$
(*)\begin{cases}\partial_t v + \Delta^2 v = f & \text{in} \quad Q,\\ \partial_t v_\Gamma - b\partial_\nu \Delta v + bc\partial_\nu v - \alpha bc \Delta_\Gamma v_\Gamma = g & \text{on} \quad \Sigma,\\ v|_\Gamma = v_\Gamma, \quad -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b \Delta_\Gamma v_\Gamma = h & \text{on} \quad \Sigma,\\ v(0) = v_0 \quad \text{in}\ \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{on}\ \Gamma. & \end{cases}
$$

Here the functions $f, g, h, v_0, v_{0\Gamma}$ are given and v, v_Γ are unknown. Since this linearized equation is included in the general framework studied by [5], we collect and write down these results in Appendix A, and apply it in Appendix B. Then we get the following linear theory.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^4 and $1 < p < \infty$ be $p \neq 5/4, 5/2, 5$. Let $\kappa_0 = 1/4 - 1/(4p), \kappa_1 = 1/2 - 1/(4p)$. Then the linearized Cahn–Hilliard equation $(*)$ admits a unique solution*

$$
\begin{aligned}(v, v_\Gamma) \in Z \times Z_\Gamma := &\big(W_p^1(J, L_p(\Omega)) \cap L_p(J, W_p^4(\Omega))\big)\\ &\times \big(W_p^{1+\kappa_0}(J, L_p(\Gamma)) \cap W_p^1(J, W_p^{4\kappa_0}(\Gamma)) \cap L_p(J, W_p^{3+4\kappa_0}(\Gamma))\big)\end{aligned}
$$

if and only if

$$
\begin{aligned}(f, g, h) \in\ & X \times Y_0 \times Y_1\\ :=\ & L_p(J, L_p(\Omega)) \times (W_p^{\kappa_0}(J, L_p(\Gamma)) \cap L_p(J, W^{4\kappa_0}(\Gamma)))\\ & \times (W_p^{\kappa_1}(J, L_p(\Gamma)) \cap L_p(J, W^{4\kappa_1}(\Gamma))),\\ (v_0, v_{0\Gamma}) \in\ & \pi Z \times \pi Z_\Gamma := B_{p,p}^{4-4/p}(\Omega) \times B_{p,p}^{4-4/p}(\Gamma),\end{aligned}
$$

and the compatibility conditions

$$
\begin{aligned}& v_0|_\Gamma = v_{0\Gamma} && \text{on}\ \Gamma && \text{if} \quad p > 5/4,\\ & -(\Delta v_0)|_\Gamma - b\partial_\nu v_{0\Gamma} + \alpha b \Delta_\Gamma v_{0\Gamma} = h|_{t=0} && \text{on}\ \Gamma && \text{if} \quad p > 5/2,\\ & g|_{t=0} + b\partial_\nu \Delta v_0 - bc\partial_\nu v_0 + \alpha bc \Delta_\Gamma v_{0\Gamma} \in B_{p,p}^{1-5/p}(\Gamma) && && \text{if} \quad p > 5.\end{aligned}
$$

are satisfied.

Remark 5.2. If we use time weighted L_p maximal regularity result, then we are able to relax the compatibility conditions while the regularity class of the solution for $t > 0$ is same, see [15].

5.2.2 Local well-posedness

In this section we prove the local well-posedness for the Cahn–Hilliard equation on permeable walls

$$(\mathrm{CH})_{\mathrm{per.}}\begin{cases}\partial_t u + \Delta^2 u = \Delta F'(u) + f & \text{in} \quad Q,\\ \partial_t u_\Gamma - b\partial_\nu \Delta u + bc\partial_\nu u - \alpha bc\Delta_\Gamma u_\Gamma = -b\partial_\nu F'(u) - bcG'(u_\Gamma) + g & \text{on} \quad \Sigma,\\ u|_\Gamma = u_\Gamma, \quad -(\Delta u)|_\Gamma - b\partial_\nu u + \alpha b\Delta_\Gamma u_\Gamma = -F'(u)|_\Gamma + bG'(u_\Gamma) & \text{on} \quad \Sigma,\\ u(0) = u_0 \quad \text{in}\ \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on}\ \Gamma.\end{cases}$$

Here $F \in C^{4-}(\mathbb{R}), G \in C^{2-}(\mathbb{R})$. The original equation we explained in the introduction is the case $f = g = 0$, but we are able to add non-homogeneous terms f, g. We will prove existence and uniqueness of this solution. So first we need to consider the compatibility conditions for the boundary. Let $(g, u_0, u_{0\Gamma}) \in Y_0 \times \pi Z \times \pi Z_\Gamma$ satisfy the following compatibility conditions

$$u_0|_\Gamma = u_{0\Gamma} \qquad \text{on}\ \Gamma \quad \text{if} \quad p > 5/4, \tag{5.2.1}$$

$$-(\Delta u_0)|_\Gamma - b\partial_\nu u_{0\Gamma} + \alpha b\Delta_\Gamma u_{0\Gamma} = -F'(u_0)|_\Gamma + bG'(u_{0\Gamma}) \ \text{on}\ \Gamma \quad \text{if} \quad p > 5/2, \tag{5.2.2}$$

$$\begin{aligned} &g|_{t=0} + b\partial_\nu \Delta u_0 - bc\partial_\nu u_0 + \alpha bc\Delta_\Gamma u_{0\Gamma}\\ &\qquad - b\partial_\nu F'(u_0) - bcG'(u_{0\Gamma}) \in B^{1-5/p}_{p,p}(\Gamma) \qquad \text{if} \quad p > 5.\end{aligned} \tag{5.2.3}$$

We use the notation $J_a := (0, a) \subset J$, $X(a)$, $Y_i(a)(i = 0, 1)$ and $Z(a)$, $Z_\Gamma(a)$ to indicate the time interval under consideration.

We can state now the following main result of this section.

Theorem 5.3. *Let $1 < p < \infty$ be $p > (n + 4)/4$ and $p \neq 5/2, 5$, and let $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi Z_\Gamma$ satisfy the compatibility conditions* (5.2.1)–(5.2.3) *and $F \in C^{4-}(\mathbb{R}), G \in C^{2-}(\mathbb{R})$. Then there is an $a \in (0, T]$ and a unique solution $(u, u_\Gamma) \in Z(a) \times Z_\Gamma(a)$ of $(\mathrm{CH})_{\mathrm{per.}}$. Furthermore the solution depends continuously on the data, and if the data (f, g) are independent of t, the map $(u_0, u_{0\Gamma}) \mapsto (u(t), u_\Gamma(t))$ defines a local semiflow in the natural phase manifold $\mathcal{M}$ defined by $\pi Z \times \pi Z_\Gamma$ and the compatibility conditions* (5.2.1)–(5.2.3).

Proof. The proof is based on the contraction mapping theorem. At first we take the function $(u^*, u^*_\Gamma) \in Z(T) \times Z_\Gamma(T)$ that is the solution of the linearized equation

$$\begin{cases}\partial_t u^* + \Delta^2 u^* = f & \text{in} \quad Q,\\ \partial_t u^*_\Gamma - b\partial_\nu \Delta u^* + bc\partial_\nu u^* - \alpha bc\Delta_\Gamma u^*_\Gamma = g - \tilde{g} & \text{on} \quad \Sigma,\\ u^*|_\Gamma = u^*_\Gamma, \quad -(\Delta u^*)|_\Gamma - b\partial_\nu u^* + \alpha b\Delta_\Gamma u^*_\Gamma = -\tilde{h} & \text{on} \quad \Sigma,\\ u^*(0) = u_0 \quad \text{in}\ \Omega, \quad u^*_\Gamma(0) = u_{0\Gamma} \quad \text{on}\ \Gamma.\end{cases}$$

Here

$$\tilde{g} = \begin{cases} 0 & \text{if}\ p < 5,\\ e^{-t\Delta^2_\Gamma}(b\partial_\nu F'(u_0) + bcG'(u_{0\Gamma})) & \text{if}\ 5 < p,\end{cases}$$

$$\tilde{h} = \begin{cases} 0 & \text{if } p < 5/2, \\ e^{-t\Delta_\Gamma^2}(F'(u_0)|_\Gamma - bG'(u_{0\Gamma})) & \text{if } 5/2 < p, \end{cases}$$

are the modified terms, so that we are able to use linear theory. Note that $-\Delta_\Gamma^2$ is the generator of an analytic (C_0)-semigroup in $B^{1-5/p}_{p,p}(\Gamma)$ and $B^{2-5/p}_{p,p}(\Gamma)$.

For given $a \in (0, T]$ to be fixed later, we define

$$\mathbb{E} := \{(u, u_\Gamma) \in Z(a) \times Z_\Gamma(a) \mid u|_\Gamma = u_\Gamma\}, \ {}_0\mathbb{E} := \{(u, u_\Gamma) \in \mathbb{E} \mid (u, u_\Gamma)|_{t=0} = (0, 0)\}$$

with canonical norm $\|\cdot\|_\mathbb{E}$ and

$$\mathbb{F} := X(a) \times Y_0(a) \times Y_1(a), \ {}_0\mathbb{F} := \{(f, g, h) \in \mathbb{F} \mid h|_{t=0} = 0\}$$

with norm $\|\cdot\|_\mathbb{F}$. Define the linear operator $\mathbb{L} : \mathbb{E} \to \mathbb{F}$ by means of

$$\mathbb{L}(v, v_\Gamma) := \begin{bmatrix} \partial_t v + \Delta^2 v \\ \partial_t v_\Gamma - b\partial_\nu \Delta v + bc\partial_\nu v - \alpha bc\Delta_\Gamma v_\Gamma \\ -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b\Delta_\Gamma v_\Gamma \end{bmatrix}.$$

By theorem 5.1, $\mathbb{L} : {}_0\mathbb{E} \to {}_0\mathbb{F}$ is linear, bounded and bijective, hence an isomorphism. Next we define the nonlinear mapping $N : \mathbb{E} \times {}_0\mathbb{E} \to {}_0\mathbb{F}$ by

$$N((u^*, u^*_\Gamma), (v, v_\Gamma)) := \begin{bmatrix} \Delta F'(u^* + v) \\ -b\partial_\nu F'(u^* + v) - bcG'(u^*_\Gamma + v_\Gamma) + \tilde{g} \\ -F'(u^* + v)|_\Gamma + bG'(u^*_\Gamma + v_\Gamma) + \tilde{h} \end{bmatrix}.$$

We will show the key proposition, which needs to use contraction mapping theorem and to show the range of N is ${}_0\mathbb{F}$. Let $\mathbb{B}_R((0, 0)) \subset {}_0\mathbb{E}$ be a closed ball with center $(0, 0)$, radius $R > 0$, and set $\mathbb{B}_R((u^*, u^*_\Gamma)) := (u^*, u^*_\Gamma) + \mathbb{B}_R((0, 0))$.

Proposition 5.4. *Let $p > (n + 4)/4$, $F \in C^{4-}(\mathbb{R}), G \in C^{2-}(\mathbb{R})$, $J_a \subset J$ and $R > 0$. Then there exist functions $\lambda_j = \lambda_j(a)$ with $\lambda_j(a) \to 0$ as $a \to 0$, $j = 1, \cdots, 5$ such that for all $(u, u_\Gamma), (v, v_\Gamma) \in \mathbb{B}_R((u^*, u^*_\Gamma))$ the following statements hold:*

$$\begin{aligned} \|\Delta F'(u) - \Delta F'(v)\|_X &\le \lambda_1(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_\mathbb{E}, \\ \|\partial_\nu F'(u) - \partial_\nu F'(v)\|_{Y_0} &\le \lambda_2(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_\mathbb{E}, \\ \|G'(u_\Gamma) - G'(v_\Gamma)\|_{Y_0} &\le \lambda_3(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_\mathbb{E}, \\ \|F'(u)|_\Gamma - F'(v)|_\Gamma\|_{Y_1} &\le \lambda_4(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_\mathbb{E}, \\ \|G'(u_\Gamma) - G'(v_\Gamma)\|_{Y_1} &\le \lambda_5(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_\mathbb{E}. \end{aligned}$$

The first and second inequalities are the same in [18, Proposition 3.2] and the others are easily followed.

We see that $u = u^* + v$, $u_\Gamma = u^*_\Gamma + v_\Gamma$ is a unique solution of $(\mathrm{CH})_{\mathrm{per.}}$ if and only if

(5.2.4) $$\mathbb{L}(v, v_\Gamma) = N((u^*, u^*_\Gamma), (v, v_\Gamma)) \ \text{ i.e. } (v, v_\Gamma) = \mathbb{L}^{-1}N((u^*, u^*_\Gamma), (v, v_\Gamma))$$

since

$$\begin{aligned}
\mathbb{L}(u^*+v, u_\Gamma^*+v_\Gamma) &= \mathbb{L}(u^*, u_\Gamma^*) + \mathbb{L}(v, v_\Gamma)\\
&= \begin{bmatrix} f \\ g-\tilde{g} \\ -\tilde{h} \end{bmatrix} + \begin{bmatrix} \Delta F'(u^*+v) \\ -b\partial_\nu F'(u^*+v) - bcG'(u_\Gamma^*+v_\Gamma) + \tilde{g} \\ -F'(u^*+v)|_\Gamma + bG'(u_\Gamma^*+v_\Gamma) + \tilde{h} \end{bmatrix}\\
&= \begin{bmatrix} \Delta F'(u^*+v) + f \\ -b\partial_\nu F'(u^*+v) - bcG'(u_\Gamma^*+v_\Gamma) + g \\ -F'(u^*+v)|_\Gamma + bG'(u_\Gamma^*+v_\Gamma) \end{bmatrix},\\
(u^*+v, u_\Gamma^*+v_\Gamma)(0) &= (u^*, u_\Gamma^*)(0) + (v, v_\Gamma)(0) = (u_0, u_{0\Gamma}).
\end{aligned}$$

Define the operator $S : \mathbb{B}_R((0,0)) \to {}_0\mathbb{E}$ by means of $S(v, v_\Gamma) := \mathbb{L}^{-1}N((u^*, u_\Gamma^*), (v, v_\Gamma))$. We show that the operator S is a contraction map on $\mathbb{B}_R((0,0))$ with small time interval J_a.

First we prove that $S\mathbb{B}_R((0,0)) \subset \mathbb{B}_R((0,0))$ by the following calculation. Let $(w, w_\Gamma) \in \mathbb{B}_R((0,0))$.

$$\begin{aligned}
&\|S(w, w_\Gamma)\|_{\mathbb{E}} \le \|\mathbb{L}^{-1}\|_{\mathcal{L}(\mathbb{F},\mathbb{E})} \|N((u^*, u_\Gamma^*), (w, w_\Gamma))\|_{\mathbb{F}}\\
&\le C(\|N((u^*, u_\Gamma^*), (w, w_\Gamma)) - N((u^*, u_\Gamma^*), (0,0))\|_{\mathbb{F}} + \|N((u^*, u_\Gamma^*), (0,0))\|_{\mathbb{F}})\\
&\le C(\|\Delta F'(u^*+w) - \Delta F'(u^*)\|_X + \|\partial_\nu F'(u^*+w) - \partial_\nu F'(u^*)\|_{Y_0}\\
&\quad + \|G'(u_\Gamma^*+w_\Gamma) - G'(u_\Gamma^*)\|_{Y_0} + \|F'(u^*+w)|_\Gamma - F'(u^*)|_\Gamma\|_{Y_1} + \|G'(u_\Gamma^*+w_\Gamma) - G'(u_\Gamma^*)\|_{Y_1}\\
&\quad + \|\Delta F'(u^*)\|_X + \|\partial_\nu F'(u^*)\|_{Y_0} + \|G'(u_\Gamma^*)\|_{Y_0} + \|\tilde{g}\|_{Y_0}\\
&\quad + \|F'(u^*)|_\Gamma\|_{Y_1} + \|G'(u_\Gamma^*)\|_{Y_1} + \|\tilde{h}\|_{Y_1})\\
&\le C(\lambda(a)\|(w, w_\Gamma)\|_{\mathbb{E}} + \|\Delta F'(u^*)\|_X + \|\partial_\nu F'(u^*)\|_{Y_0} + \|G'(u_\Gamma^*)\|_{Y_0} + \|\tilde{g}\|_{Y_0}\\
&\quad + \|F'(u^*)|_\Gamma\|_{Y_1} + \|G'(u_\Gamma^*)\|_{Y_1} + \|\tilde{h}\|_{Y_1})
\end{aligned}$$

for some function $\lambda(a)$, which goes to 0 as $a \to 0$, since $(u^*, u_\Gamma^*), (u^*+w, u_\Gamma^*+w_\Gamma) \in \mathbb{B}_R((u^*, u_\Gamma^*))$ and Proposition 5.4. The remaining terms $\|\Delta F'(u^*)\|_{X(a)}$, $\|\partial_\nu F'(u^*)\|_{Y_0(a)}$, $\|G'(u_\Gamma^*)\|_{Y_0(a)}$, $\|\tilde{g}\|_{Y_0(a)}$, $\|F'(u^*)|_\Gamma\|_{Y_1(a)}$, $\|G'(u_\Gamma^*)\|_{Y_1(a)}$, $\|\tilde{h}\|_{Y_1(a)}$ also goes to 0 as $a \to 0$. So we have $\|S(w, w_\Gamma)\|_{\mathbb{E}} \le R$, i.e. $S\mathbb{B}_R((0,0)) \subset \mathbb{B}_R((0,0))$ when a is sufficiently small.

We next show the following contraction property. Let $(w_1, w_{1\Gamma}), (w_2, w_{2\Gamma}) \in \mathbb{B}_R((0,0))$.

$$\begin{aligned}
&\|S(w_1, w_{1\Gamma}) - S(w_2, w_{2\Gamma})\|_{\mathbb{E}}\\
\le& \|\mathbb{L}^{-1}\|_{\mathcal{L}(\mathbb{F},\mathbb{E})} \|N((u^*, u_\Gamma^*), (w_1, w_{1\Gamma})) - N((u^*, u_\Gamma^*), (w_2, w_{2\Gamma}))\|_{\mathbb{F}}\\
\le& C(\|\Delta F'(u^*+w_1) - \Delta F'(u^*+w_2)\|_X + \|\partial_\nu F'(u^*+w_1) - \partial_\nu F'(u^*+w_2)\|_{Y_0}\\
&\quad + \|G'(u_\Gamma^*+w_{1\Gamma}) - G'(u_\Gamma^*+w_{2\Gamma})\|_{Y_0} + \|F'(u_\Gamma^*+w_{1\Gamma})|_\Gamma - F'(u_\Gamma^*+w_{2\Gamma})|_\Gamma\|_{Y_1}\\
&\quad + \|G'(u_\Gamma^*+w_{1\Gamma}) - G'(u_\Gamma^*+w_{2\Gamma})\|_{Y_1})\\
\le& \frac{1}{2}\|(w_1, w_{1\Gamma}) - (w_2, w_{2\Gamma})\|_{\mathbb{E}},
\end{aligned}$$

provided a is sufficiently small by Proposition 5.4.

Therefore from the fixed point theorem, we get a unique solution $(v, v_\Gamma) \in \mathbb{B}_R((0,0))$ such that $(v, v_\Gamma) = \mathbb{L}^{-1}N((u^*, u_\Gamma^*), (v, v_\Gamma))$. The function (u^*, u_Γ^*) depends continuously on the data f, g and (v, v_Γ) depends continuously on (u^*, u_Γ^*). This implies that the unique solution $u = u^* + v$ and $u_\Gamma = u_\Gamma^* + v_\Gamma$ of $(\mathrm{CH})_{\mathrm{per.}}$ depends continuously on the data as well. If the data f, g are independent of the time, then translation is invariant. So the solution map $(u_0, u_{0\Gamma}) \mapsto (u(t), u_\Gamma(t))$ defines a local semiflow in the natural phase manifold $\pi Z \times \pi Z_\Gamma$ and the compatibility conditions (5.2.1)–(5.2.3). □

Remark 5.5. This proof also show that the existence of maximal time interval $J_{\max} = (0, a_{\max})$, which is characterized by

$$\begin{cases} \lim\limits_{t \to a_{\max}} u(t) & \text{does not exist in } \pi Z \\ \lim\limits_{t \to a_{\max}} u_\Gamma(t) & \text{does not exist in } \pi Z_\Gamma \end{cases} \text{and/or } \ \|(u, u_\Gamma)\|_{\mathbb{E}(a_{\max})} = \infty,$$

if $a_{\max} < T$.

5.2.3 Global well-posedness

In this section we consider the global solution for the equation with non-homogeneous terms f, g;

$$\begin{cases} \partial_t u = \Delta\mu + f, \quad \mu = -\Delta u + F'(u) & \text{in} \quad Q, \\ \partial_t u_\Gamma + b\partial_\nu \mu + c\mu_\Gamma = g, \quad -\alpha\Delta_\Gamma u_\Gamma + \partial_\nu u + G'(u_\Gamma) = \frac{\mu_\Gamma}{b} & \text{on} \quad \Sigma, \\ u|_\Gamma = u_\Gamma, \quad \mu|_\Gamma = \mu_\Gamma & \text{on} \quad \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

As we explained in introduction, the unknown functions are only u and u_Γ though we use μ and μ_Γ. By the subsection 5.2.2 there is a unique solution on some maximal time interval $J_{\max} = (0, a_{\max})$. We fix some arbitrary J_a for $0 < a \le a_{\max} (\le T)$ and show the boundedness near the point $t = a$ from a priori estimate derived from energy estimate. Multiplying the equation by u and μ, integration by parts and the boundary conditions lead

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u)\right) + |\nabla\mu|_2^2 \\
&= -\int_\Omega \nabla\mu \cdot \nabla u + \int_\Gamma u_\Gamma \partial_\nu \mu + \int_\Gamma \partial_t u_\Gamma \partial_\nu u + \int_\Gamma \mu_\Gamma \partial_\nu \mu + \int_\Omega f u + \int_\Omega f\mu, \\
\Rightarrow &\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right) + |\nabla\mu|_2^2 + \frac{c}{b}|\mu_\Gamma|_{2,\Gamma}^2 \\
&= -\int_\Omega \nabla\mu \cdot \nabla u - \frac{c}{b}\int_\Gamma u_\Gamma \mu_\Gamma + \int_\Gamma \partial_t u_\Gamma (\partial_\nu u - \frac{\mu_\Gamma}{b}) + \int_\Omega f u + \int_\Omega f\mu + \frac{1}{b}\int_\Gamma g\mu_\Gamma \\
&= -\int_\Omega \nabla\mu \cdot \nabla u - \frac{c}{b}\int_\Gamma u_\Gamma \mu_\Gamma + \int_\Gamma \partial_t u_\Gamma (\alpha\Delta_\Gamma u_\Gamma + G'(u_\Gamma)) + \int_\Omega f u + \int_\Omega f\mu + \frac{1}{b}\int_\Gamma g\mu_\Gamma \\
\Rightarrow &\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2 + \frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2 + \int_\Gamma G(u_\Gamma)\right) + |\nabla\mu|_2^2 + \frac{c}{b}|\mu_\Gamma|_{2,\Gamma}^2 \\
&= -\int_\Omega \nabla\mu \cdot \nabla u - \frac{c}{b}\int_\Gamma u_\Gamma \mu_\Gamma + \int_\Omega f u + \int_\Omega f\mu + \frac{1}{b}\int_\Gamma g\mu_\Gamma.
\end{aligned}$$

For simplicity, we set

$$E(u,u_\Gamma) := \frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2 + \frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2 + \int_\Gamma G(u_\Gamma).$$

By Poincaré's inequality $|\mu|_2 \le C(|\nabla\mu|_2 + |\mu_\Gamma|_{2,\Gamma})$ and Young's inequality with ε, we have

$$\frac{d}{dt}E(u,u_\Gamma) + C_1(|\nabla\mu|_2^2 + |\mu_\Gamma|_{2,\Gamma}^2) \le C_2\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right) + C_3(|f|_2^2 + |g|_{2,\Gamma}^2)$$

for some $C_i > 0$ $(i = 1,2,3)$.

To get energy estimate, we assume that F and G satisfy the following condition:

$$\text{(5.2.5)} \qquad \begin{cases} F(s) \ge -c_1, & c_1 > 0,\ s \in \mathbb{R}, \\ G(s) \ge -\frac{1}{2b}s^2 - c_2, & c_2 > 0,\ s \in \mathbb{R}. \end{cases}$$

Note that the typical example in the introduction satisfies this assumption. Under this condition, the function $E(u,u_\Gamma)$ is bounded from below. We get the inequality

$$\frac{d}{dt}E(u,u_\Gamma) + C_1(|\nabla\mu|_2^2 + |\mu_\Gamma|_{2,\Gamma}^2) \le C_2E(u,u_\Gamma) + C_3(|f|_2^2 + |g|_{2,\Gamma}^2 + 1).$$

We apply Gronwall's lemma, then we get energy estimate

$$E(u,u_\Gamma) \le C\left(E(u_0,u_{0\Gamma}) + \int_0^{a_{\max}} (|f|_2^2 + |g|_{2,\Gamma}^2 + 1)\right)$$

and

$$\text{(5.2.6)} \qquad (u,u_\Gamma) \in L_\infty(J_{a_{\max}}, W_2^1(\Omega) \times W_2^1(\Gamma))$$

when $(f,g,u_0,u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi Z_\Gamma$ as $p \ge 2$ and $p > (n+4)/4$. Here the constant C depends only on $T > 0$ and is independent of $a_{\max}$.

We use the following lemma, which is obtained in the paper [18, Lemma 4.1]. To do so, we have to assume that the dimension $n = 2,3$ and some growth condition on F and G:

$$\text{(5.2.7)} \qquad \begin{cases} |F'''(s)| \le C(1+|s|^\beta), & s \in \mathbb{R}, \\ |G'(s)| \le C(1+|s|^{\beta+2}), & s \in \mathbb{R}, \end{cases} \quad \text{with} \begin{cases} \beta < 3 \text{ in the case } n = 3, \\ \beta > 0 \text{ in the case } n = 2. \end{cases}$$

Lemma 5.6. *Suppose $2 \le p < \infty$, $n = 2,3$, the function F and G satisfy (5.2.5) and (5.2.7) and let $(u,u_\Gamma) \in \mathbb{E}(a)$ be the solution of $(\mathrm{CH})_{\mathrm{per.}}$. Then there exist constants $m, C > 0$ and $\delta \in (0,1)$, independent of $a > 0$, such that*

$$\begin{aligned}&\|\Delta F'(u)\|_{X(a)} + \|\partial_\nu F'(u)\|_{Y_0(a)} + \|G'(u_\Gamma)\|_{Y_0(a)} + \|F'(u)|_\Gamma\|_{Y_1(a)} + \|G'(u_\Gamma)\|_{Y_1(a)} \\ \le& C(1 + \|u\|_{Z(a)}^\delta \|u\|_{L_\infty(J_a, W_2^1(\Omega))}^m).\end{aligned}$$

Proof. The estimates of the first term $\|\Delta F'(u)\|_{X(a)}$ and the second term $\|\partial_\nu F'(u)\|_{Y_0(a)}$ is just in [18, Lemma 4.1]. Since the trace operator is bounded from $W_p^{1/2}(J_a, L_p(\Omega)) \cap$

$L_p(J_a, W_p^2(\Omega))$ to Y_1, $Y_1 \subset Y_0$ and $u|_\Gamma = u_\Gamma$, the other three terms are also estimated. See [19, Appendix (b)]. □

Combining maximal L_p regularity estimate,

$$
\begin{aligned}
&\|(u, u_\Gamma)\|_{\mathbb{E}(a)} \\
\le& C(\|\Delta F'(u)\|_{X(a)} + \|\partial_\nu F'(u)\|_{Y_0(a)} + \|G'(u_\Gamma)\|_{Y_0(a)} + \|F'(u)|_\Gamma\|_{Y_1(a)} \\
&\qquad + \|G'(u_\Gamma)\|_{Y_1(a)} + \|f\|_{X(T)} + \|g\|_{Y_0(T)} + \|(u_0, u_{0\Gamma})\|_{\pi Z\times\pi Z_\Gamma}) \\
\le& \widetilde{C}(1 + \|u\|_{Z(a)}^\delta),
\end{aligned}
\tag{5.2.8}
$$

where the constant $\widetilde{C}$ is independent of a. Hence $\|u\|_{Z(a)}$ is bounded and it derives the boundedness of $\|u_\Gamma\|_{Z_\Gamma(a)}$. Therefore the solution $(u, u_\Gamma) \in \mathbb{E}(a)$ is global solution, i.e. $a_{\max} = T$. We obtained the following first main theorem of this chapter.

Theorem 5.7. *Suppose $2 \le p < \infty$, $p \ne 5/2, 5$, $n = 2, 3$ and that the function F and G satisfy (5.2.5) and (5.2.7). Then for any $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi Z_\Gamma$ satisfying the compatibility conditions (5.2.1)–(5.2.3), there exists a unique global solution $(u, u_\Gamma) \in Z(T) \times Z_\Gamma(T)$ of $(\mathrm{CH})_{\mathrm{per.}}$. The solution depends continuously on the given data and if the data are independent of t, the map $(u_0, u_{0\Gamma}) \mapsto (u(t), u_\Gamma(t))$ defines a global semiflow on the natural phase manifold $\pi Z \times \pi Z_\Gamma$ and the compatibility conditions (5.2.1)–(5.2.3).*

5.2.4 The degenerate case: $c = 0$

In this section we focus on the case $c = 0$ in the boundary condition (5.1.3). Almost all results for now can be applied to this case. The linear theory and local well-posedness result is completely the same as the case $c > 0$. The point different from the case $c > 0$ is the energy estimate. Multiplying the equation by u and μ, integration by parts and Young's inequality lead

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2 + \frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2 + \int_\Gamma G(u_\Gamma)\right) + C_1|\nabla\mu|_2^2 \\
\le& C_2\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right) + C_3|f|_2^2 + \int_\Omega f\mu + \frac{1}{b}\int_\Gamma g\mu_\Gamma
\end{aligned}
$$

for some $C_i > 0$ $(i = 1, 2, 3)$. Here we assume $\int_\Omega f dx + \int_\Gamma g \frac{dS}{b} = 0$. Then we see

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u dx &= \int_\Omega (\Delta\mu + f) dx \\
&= \int_\Gamma \partial_\nu \mu dS + \int_\Omega f dx \\
\Rightarrow \frac{d}{dt}\left(\int_\Omega u dx + \int_\Gamma u_\Gamma \frac{dS}{b}\right) &= \int_\Omega f dx + \int_\Gamma g\frac{dS}{b} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega f\mu + \frac{1}{b}\int_\Gamma g\mu_\Gamma &= \int_\Omega f(\mu - \overline{\mu}) + \frac{1}{b}\int_\Gamma g(\mu_\Gamma - \overline{\mu}) \\
&\le \frac{C_1}{2}|\nabla\mu|_2^2 + C_4(|f|_2^2 + |g|_2^2),
\end{aligned}
$$

where $\overline{\mu} = \frac{1}{|\Omega|}\int_\Omega \mu dx$ and some $C_4 > 0$ by Poincaré's inequality. This implies that

$$\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2 + \frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2 + \int_\Gamma G(u_\Gamma)\right) + \tilde{C}_1|\nabla\mu|_2^2$$
$$\leq \tilde{C}_2\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right) + \tilde{C}_3(|f|_2^2 + |g|_2^2)$$

for some $\tilde{C}_i > 0$ $(i = 1, 2, 3)$. This inequality deduces a priori estimate (5.2.6) under the assumption (5.2.5). Thus we have the global well-posedness result for the case $c = 0$.

Theorem 5.8. *Suppose $2 \leq p < \infty$, $p \neq 5/2, 5$, $n = 2, 3$ and that the function F and G satisfy (5.2.5) and (5.2.7). Then for any $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi Z_\Gamma$ satisfying the compatibility conditions (5.2.1)–(5.2.3) with $c = 0$ and $\int_\Omega f dx + \int_\Gamma g \frac{dS}{b} = 0$, there exists a unique global solution $(u, u_\Gamma) \in Z(T) \times Z_\Gamma(T)$ of $(\mathrm{CH})_{\mathrm{per.}}$ with $c = 0$.*

5.3 A Cahn–Hilliard equation on non-permeable walls

5.3.1 The linear theory

In this section we study the linear theory of the Cahn–Hilliard equation on non-permeable walls. The linear equation is as follows:

$$(**)\begin{cases} \partial_t v + \Delta^2 v = f & \text{in} \quad Q, \\ \partial_t v_\Gamma - b\partial_\nu \Delta v - bc\Delta_\Gamma \partial_\nu v + \alpha bc\Delta_\Gamma^2 v_\Gamma = g & \text{on} \quad \Sigma, \\ v|_\Gamma = v_\Gamma, \quad -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b\Delta_\Gamma v_\Gamma = h & \text{on} \quad \Sigma, \\ v(0) = v_0 \quad \text{in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

We again use the general theory in [5] and the assumption of the theorem is checked in Appendix C. However we have to assume a condition on the coefficients α, b, c to get (LS) condition. The assumption is the following:

Assumption (A) The coefficients $\alpha, b, c > 0$ satisfy $\alpha bc < 2(\alpha b + c)$.
Let $\overline{Z}_\Gamma := W_p^{1+\kappa_0}(J, L_p(\Gamma)) \cap L_p(J, W_p^{4+4\kappa_0}(\Gamma))$ and $\pi\overline{Z}_\Gamma := B_{p,p}^{5-5/p}(\Gamma)$

Theorem 5.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^5 and $1 < p < \infty$ be $p \neq 5/4, 5/2, 5$. Suppose that the constants $\alpha, b, c > 0$ satisfy the Assumption (A). Then the linearized Cahn–Hilliard equation $(**)$ admits a unique solution $(v, v_\Gamma) \in Z \times \overline{Z}_\Gamma$ if and only if $(f, g, h) \in X \times Y_0 \times Y_1$ and $(v_0, v_{0\Gamma}) \in \pi Z \times \pi\overline{Z}_\Gamma$, and the compatibility conditions*

$$\begin{aligned} & v_0|_\Gamma = v_{0\Gamma} && \text{on } \Gamma && \text{if} \quad p > 5/4, \\ & -(\Delta v_0)|_\Gamma - b\partial_\nu v_{0\Gamma} + \alpha b\Delta_\Gamma v_{0\Gamma} = h|_{t=0} && \text{on } \Gamma && \text{if} \quad p > 5/2, \\ & g|_{t=0} + b\partial_\nu \Delta v_0 - bc\Delta_\Gamma \partial_\nu v_0 - \alpha bc\Delta_\Gamma^2 v_{0\Gamma} \in B_{p,p}^{1-5/p}(\Gamma) && && \text{if} \quad p > 5, \end{aligned}$$

are satisfied.

5.3.2 The nonlinear theory

In this subsection we state the nonlinear theory. We state the different point from the case of permeable walls. We need the estimate of the nonlinear term $\Delta_\Gamma G'(u_\Gamma)$ corresponding to Proposition 5.4 and Lemma 5.6. From now, we restrict the case that $G(u_\Gamma) = (g_s/2)u_\Gamma^2 - h_s u_\Gamma$ with $g_s > 0, h_s \neq 0$. Thus we study the Cahn–Hilliard equation on non-permeable walls.

$$(\mathrm{CH})_{\text{non-per.}} \begin{cases} \partial_t u = \Delta\mu + f, \quad \mu = -\Delta u + F'(u) & \text{in } \quad Q, \\ \partial_t u_\Gamma + b\partial_\nu \mu - c\Delta_\Gamma \mu_\Gamma = g, \quad -\alpha\Delta_\Gamma u_\Gamma + \partial_\nu u + g_s u_\Gamma^2 - h_s = \frac{\mu_\Gamma}{b} & \text{on } \quad \Sigma, \\ u|_\Gamma = u_\Gamma, \quad \mu|_\Gamma = \mu_\Gamma & \text{on } \quad \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

We see the following proposition.

Proposition 5.10. *Let $p > (n+4)/4$, $J_a \subset J$ and $R > 0$. Then there exist functions $\lambda_6 = \lambda_6(a)$ with $\lambda_6(a) \to 0$ as $a \to 0$ such that for all $(u, u_\Gamma), (v, v_\Gamma) \in \mathbb{B}_R((u^*, u_\Gamma^*))$ the following statements hold:*

$$\|\Delta_\Gamma u_\Gamma - \Delta_\Gamma v_\Gamma\|_{Y_0} \le \lambda_6(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_{\mathbb{E}}.$$

This proposition is enough to show the local well-posedness result. To extend the global solution, we show the energy estimate. Multiplying the equation by u and μ, integration by parts and the boundary conditions lead

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2 + \frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2 + \int_\Gamma G(u_\Gamma)\right) \\ &\quad + |\nabla\mu|_2^2 + \frac{c}{b}|\nabla_\Gamma \mu_\Gamma|_{2,\Gamma}^2 \\ &= -\int_\Omega \nabla\mu\cdot\nabla u - \frac{c}{b}\int_\Gamma \nabla_\Gamma u_\Gamma\cdot\nabla_\Gamma\mu_\Gamma + \int_\Omega fu + \int_\Omega f\mu + \frac{1}{b}\int_\Gamma g u_\Gamma + \frac{1}{b}\int_\Gamma g\mu_\Gamma. \end{aligned}$$

Here as the case $c = 0$, we assume that $\int_\Omega f\,dx + \int_\Gamma g\frac{dS}{b} = 0$. Then we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2 + \frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2 + \int_\Gamma G(u_\Gamma)\right) \\ &\quad + C_1(|\nabla\mu|_2^2 + |\nabla_\Gamma\mu_\Gamma|_{2,\Gamma}^2) \\ &\le C_2\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right) + C_3(|f|_2^2 + |g|_2^2) \end{aligned}$$

for some $C_i > 0$ $(i = 1, 2, 3)$.

Under the assumption (5.2.5) on F, we see $(u, u_\Gamma) \in L_\infty(J_{a_{\max}}, W_2^1(\Omega) \times W_2^1(\Gamma))$. We prepare the following lemma.

Lemma 5.11. *Suppose $2 \le p < \infty$, $n = 2, 3$, let $(u, u_\Gamma) \in \mathbb{E}(a)$ be the solution of $(\mathrm{CH})_{\text{non-per.}}$. Then there exist constants $C > 0$ and $\delta \in (0, 1)$, independent of $a > 0$, such that*

$$\|\Delta_\Gamma u_\Gamma\|_{Y_0(a)} \le C(1 + \|u\|_{Z(a)}^\delta \|u\|_{L_\infty(J_a, W_2^1(\Omega))}^{1-\delta}).$$

Proof. By the trace theory and the mixed derivative theorem, it is enough to see the existence of $0 < \delta < 1$

$$\|u\|_{W_p^{3/4}(J_a,L_p(\Omega))} \leq C\|u\|^{\delta}_{W_p^{7/8}(J_a,W_p^{1/2}(\Omega))}\|u\|^{1-\delta}_{L_\infty(J_a,W_2^1(\Omega))}.$$

By Gagliardo–Nirenberg's inequality, we check the existence of δ satisfying

$$\begin{cases} \frac{3}{4} - \frac{1}{p} \leq \delta(\frac{7}{8} - \frac{1}{p}) \\ -\frac{n}{p} \leq \delta(\frac{1}{2} - \frac{n}{p}) + (1-\delta)(1-\frac{n}{2}). \end{cases}$$

Since the second inequality is $\frac{n}{2} - \frac{n}{p} - 1 \leq \delta(\frac{n}{2} - \frac{n}{p} - \frac{1}{2})$, we choose δ is sufficiently close to 1, then the inequalities are satisfied. □

Combining the estimates in (5.6), we are able to prove the global well-posedness result.

Theorem 5.12. *Suppose $2 \leq p < \infty$, $p \neq 5/2, 5$, $n = 2, 3$ and that the function F satisfy (5.2.5) and (5.2.7). Suppose that the constants $\alpha, b, c > 0$ satisfy the Assumption (A). Then for any $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi \overline{Z}_\Gamma$ satisfying the compatibility conditions*

$$\begin{aligned} &u_0|_\Gamma = u_{0\Gamma} && \text{on } \Gamma \quad \text{if} \quad p > 5/4, \\ &-(\Delta u_0)|_\Gamma - b\partial_\nu u_{0\Gamma} + \alpha b \Delta_\Gamma u_{0\Gamma} = -F'(u_0)|_\Gamma + bg_s u_{0\Gamma} - bh_s && \text{on } \Gamma \quad \text{if} \quad p > 5/2, \\ &g|_{t=0} + b\partial_\nu \Delta u_0 + bc\partial_\nu u_0 - \alpha bc \Delta_\Gamma^2 u_{0\Gamma} \\ &\qquad - b\partial_\nu F'(u_0) + bcg_s \Delta_\Gamma u_{0\Gamma} \in B^{1-5/p}_{p,p}(\Gamma) && \quad \text{if} \quad p > 5, \end{aligned}$$

and $\int_\Omega f\,dx + \int_\Gamma g \frac{dS}{b} = 0$, there exists a unique global solution $(u, u_\Gamma) \in Z(T) \times \overline{Z}_\Gamma(T)$ of $(\mathrm{CH})_{\text{non-per.}}$.

5.4 Appendix A

We collect the linear theory of the dynamic boundary condition proved in the papers [5]. We represent the simplified their result to fit our equations. They studied the parabolic initial boundary value problems of the general form (so called *relaxation type*)

$$\begin{cases} \partial_t u + \mathcal{A}(t,x,D)u = f(t,x) & \text{in } Q, \\ \partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho = g_0(t,x) & \text{on } \Sigma, \\ \mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho = g_j(t,x) \ \ (j = 1,\cdots,m) & \text{on } \Sigma, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ \rho(0,x) = \rho_0(x) & \text{on } \Gamma, \end{cases}$$

where

$$\mathcal{A}(t,x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(t,x)D^\alpha,$$
$$\mathcal{B}_j(t,x,D) = \sum_{|\beta|\leq m_j} b_{j\beta}(t,x)D^\beta,$$

$$\mathcal{C}_j(t,x,D_\Gamma) = \sum_{|\gamma|\leq k_j} c_{j\gamma}(t,x)D_\Gamma^\gamma,$$

are differential operators of order $2m$, $0 \leq m_j < 2m$, $0 \leq k_j$ $(j = 0,1,\cdots,m)$, respectively, with $m \in \mathbb{N}$ and m_j, $k_j \in \mathbb{N}_0$. The symbols D, respectively D_Γ mean $-i\nabla$, respectively $-i\nabla_\Gamma$, where ∇ denotes the gradient in Ω and ∇_Γ the surface gradient on Γ. Assume that all boundary operators $\mathcal{B}_j$ and at least one $\mathcal{C}_j$ are nontrivial, and set $k_j = -\infty$ in case $\mathcal{C}_j = 0$. The initial values u_0, ρ_0 as well as the right-hand sides f and g_j are given functions.

Let $\kappa_j := 1 - m_j/(2m) - 1/(2mp)$, $l_j := k_j - m_j + m_0$ and $l := \max_{j=0,1,\cdots,m} l_j$. We state their results limited to the case $l \leq 2m$, the coefficients a_α, $b_{j\beta}$ and $c_{j\gamma}$ are smooth, Ω is a bounded domain and u and ρ are $\mathbb{C}$-valued functions, which adopt our case.

The essential assumptions are *the normally ellipticity condition* (E) and *the Lopatinskii–Shapiro condition* (LS), which are necessary for the maximal L_p regularity, hence are unavoidable. For the case $\ell < 2m$, which is just applied to the linearized Cahn–Hilliard equation on permeable walls, we need another necessary condition called *the asymptotic Lopatinskii–Shapiro condition* (LS_∞^-). Let the subscript $\#$ be the principal part of the corresponding differential operator. The assumptions are as follows.
(E) For all $t \in J$, $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$, $|\xi| = 1$, we have

$$\sigma\left(\mathcal{A}_\#(t,x,\xi)\right) \subset \mathbb{C}_+ := \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}.$$

(LS) For each fixed $t \in J$ and $x \in \Gamma$, and for all $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \overline{\mathbb{C}}_+$ with $|\xi'| + |\lambda| \neq 0$, the ordinary differential equation in $\mathbb{R}_+ = [0,\infty)$ given by

$$\begin{cases} (\lambda + \mathcal{A}_\#(t,x,\xi',D_y))\, v(y) = 0 \ \ (y > 0), \\ \mathcal{B}_{0\#}(t,x,\xi',D_y)v(0) + (\lambda + \mathcal{C}_{0\#}(t,x,\xi'))\,\sigma = 0, \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) + \mathcal{C}_{j\#}(t,x,\xi')\sigma = 0 \ \ (j = 1,\cdots,m) \end{cases}$$

has only the trivial solution $(v,\sigma) = (0,0)$.
($\mathbf{LS}_\infty^-$) Let $\ell < 2m$. For all fixed $t \in J$ and $x \in \Gamma$, and for all $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \overline{\mathbb{C}}_+$ with $|\xi'| + |\lambda| \neq 0$, the ordinary differential equation in $\mathbb{R}_+ = [0,\infty)$ given by

$$\begin{cases} (\lambda + \mathcal{A}_\#(t,x,\xi',D_y))\, v(y) = 0 \ \ (y > 0), \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) = 0 \ \ (j = 1,\cdots,m) \end{cases}$$

and for $|\xi'| = 1$ and $\lambda \in \overline{\mathbb{C}}_+$,

$$\begin{cases} \mathcal{A}_\#(t,x,\xi',D_y)v(y) = 0 \ \ (y > 0), \\ \mathcal{B}_{0\#}(t,x,\xi',D_y)v(0) + (\lambda + \mathcal{C}_{0\#}(t,x,\xi'))\,\sigma = 0, \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) + \mathcal{C}_{j\#}(t,x,\xi')\sigma = 0 \ \ (j = 1,\cdots,m) \end{cases}$$

admit the unique trivial solution $(v,\sigma) = (0,0)$.

The existence and uniqueness results of this boundary condition are as follows.

Theorem 5.13 (Denk–Prüss–Zacher). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^{2m+l-m_0}. Assume* (E), (LS) *and for $\ell < 2m$ the condition* (LS_∞^-) *and the coefficients a_α, $b_{j\beta}$, $c_{j\gamma}$ are* *smooth. Let $1 < p < \infty$ be such that $\kappa_j \neq 1/p$, $j = 0, 1, \cdots, m$. Then the linear equation admits a unique solution*

$$(u,\rho) \in Z \times Z_\rho := \left(W_p^1(J, L_p(\Omega)) \cap L_p(J, W_p^{2m}(\Omega))\right) \times \left(W_p^{1+\kappa_0}(J, L_p(\Gamma)) \cap W_p^1(J, W_p^{2m\kappa_0}(\Gamma)) \cap L_p(J, W_p^{\ell+2m\kappa_0}(\Gamma))\right)$$

if and only if

$$\begin{aligned}(f, g_0, g_1, \cdots, g_m) &\in X \times Y_0 \times Y_1 \times \cdots \times Y_m \\ &:= L_p(J, L_p(\Omega)) \times \otimes_{j=0}^m \left(W_p^{\kappa_j}(J, L_p(\Gamma)) \cap L_p(J, W_p^{2m\kappa_j}(\Gamma))\right) \\ (u_0, \rho_0) &\in \pi Z_u \times \pi Z_\rho := B_{p,p}^{2m(1-1/p)}(\Omega) \times B_{p,p}^{2m\kappa_0+\ell(1-1/p)}(\Gamma),\end{aligned}$$

and the compatibility conditions

$$\begin{aligned}&\mathcal{B}_j(0,x)u_0(x) + \mathcal{C}_j(0,x)\rho_0(x) = g_j(0,x), \;\; x \in \Gamma, \text{ if } \kappa_j > 1/p, \;\; j = 1, 2, \cdots, m, \\ &g_0(0,\cdot) - \mathcal{B}_0(0,\cdot)u_0 - \mathcal{C}_0(0,\cdot)\rho_0 \in \pi_1 Z_\rho := B_{p,p}^{2m(\kappa_0-1/p)}(\Gamma), \text{ if } \kappa_0 > 1/p,\end{aligned}$$

are satisfied.

In [5], they treated the case $l > 2m$, non-smooth coefficients case and u, ρ are $\mathcal{HT}$ Banach valued case. However it is sufficient to consider above statement. By the Newton polygon method, they characterized

$$Z_\rho = W_p^{1+\kappa_0}(J, L_p(\Gamma)) \cap L_p(J, W_p^{\ell+2m\kappa_0}(\Gamma))$$

when $\ell = 2m$, which is applied to the linearized Cahn–Hilliard equation on non-permeable walls.

5.5 Appendix B

We apply this general linear theory for the linearized Cahn–Hilliard equation on permeable walls:

$$(*)_{\text{per.}} \begin{cases} \partial_t v + \Delta^2 v = f & \text{in} \quad Q, \\ \partial_t v_\Gamma - b\partial_\nu \Delta v + bc\partial_\nu v - \alpha bc\Delta_\Gamma v_\Gamma = g & \text{on} \quad \Sigma, \\ v|_\Gamma = v_\Gamma, \quad -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b\Delta_\Gamma v_\Gamma = h & \text{on} \quad \Sigma, \\ v(0) = v_0 \quad \text{in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

This problem fits into the setting $\mathcal{A} = \Delta^2$, $\mathcal{B}_0 = -b\partial_\nu\Delta$, $\mathcal{C}_0 = -\alpha bc\Delta_\Gamma$, $\mathcal{B}_1 = -(\Delta\cdot)|_\Gamma$, $\mathcal{C}_1 = \alpha b\Delta_\Gamma$, $\mathcal{B}_2 = 1$, $\mathcal{C}_2 = -1$, $g_2 = 0$ and $m = 2$, $m_0 = 3$, $k_0 = 2$, $m_1 = 2$, $k_1 = 2$, $m_2 = 0$, $k_2 = 0$, $\ell_0 = 2$, $\ell_1 = 3$, $\ell_2 = 3$. Then $\ell = \ell_1 = \ell_2 = 3 < 2m$, $\kappa_0 = 1/4 - 1/(4p)$, $\kappa_1 = 1/2 - 1/(4p)$ and $\kappa_2 = 1 - 1/(4p)$.

We check the conditions (E) and (LS). Since $\sigma\left(\mathcal{A}_\#(t,x,\xi)\right) = \sigma(|\xi|^4) = \{1\} \subset \mathbb{C}_+$ for $\xi \in \mathbb{R}^n$, $|\xi| = 1$, the condition (E) is satisfied.

To see (LS) condition, we need to solve the ordinary differential equation

$$\begin{aligned}
&(5.5.1) \qquad && \big((\lambda + (-|\xi'|^2 + \partial_y^2)^2\big)\, v(y) = 0 \ \ (y > 0),\\
&(5.5.2) \qquad && -b(-\partial_y)(-|\xi'|^2 + \partial_y^2)v(0) + \big((\lambda - \alpha bc(-|\xi'|^2)\big)\,\sigma = 0,\\
&(5.5.3) \qquad && v(0) - \sigma = 0,\\
&(5.5.4) \qquad && -(-|\xi'|^2 + \partial_y^2)v(0) + \alpha b(-|\xi'|^2)\sigma = 0.
\end{aligned}$$

For the case $\lambda = 0$, from (5.5.1), $v(y) = (c_1 + c_2 y)e^{-|\xi'|y}$ for some $c_1, c_2 \in \mathbb{C}$. By the boundary conditions (5.5.2)–(5.5.4),

$$\begin{cases} -b|\xi'|^2(c_2 - |\xi'|c_1) + b(3|\xi'|^2c_2 - |\xi'|^3c_1) + \alpha bc|\xi'|^2c_1 = 0,\\ |\xi'|^2c_1 - (-2|\xi'|c_2 + |\xi'|^2c_1) - \alpha b|\xi'|^2c_1 = 0.\end{cases}$$

$$\Rightarrow \begin{cases} \alpha c c_1 + 2c_2 = 0,\\ \alpha b|\xi'|c_1 - 2c_2 = 0.\end{cases}$$

The determinant of the coefficient matrix is $-2(\alpha c + \alpha b|\xi'|) \neq 0$. So we have $(c_1, c_2) = (0, 0)$, which implies the unique trivial solution $(v, \sigma) = (0, 0)$.

For the case $\lambda \neq 0$, $v(y) = c_1e^{z_1y} + c_2e^{z_2y}$ with

$$z_k := -\sqrt{|\xi'|^2 + (-1)^{k-1}\sqrt{-\lambda}} \quad (k = 1, 2).$$

Here and hereafter we shall use the argument of the square root of complex numbers belongs $(-\pi/2, \pi/2]$, so that the real part of the square root of complex numbers is non-negative. By the boundary conditions (5.5.2)–(5.5.4),

$$\begin{cases} -b|\xi'|^2(c_1z_1 + c_2z_2) + b(c_1z_1^3 + c_2z_2^3) + (\lambda + \alpha bc|\xi'|^2)(c_1 + c_2) = 0,\\ |\xi'|^2(c_1 + c_2) - (c_1z_1^2 + c_2z_2^2) - \alpha b|\xi'|^2(c_1 + c_2) = 0.\end{cases}$$

$$\Rightarrow \begin{cases} bcz_1(z_1^2 - |\xi'|^2) + bcz_2(z_2^2 - |\xi'|^2) + (\lambda + \alpha bc|\xi'|^2)(c_1 + c_2) = 0,\\ -c_1(z_1^2 - |\xi'|^2) - c_2(z_2^2 - |\xi'|^2) - \alpha b|\xi'|^2(c_1 + c_2) = 0.\end{cases}$$

Since $z_k^2 - |\xi'|^2 = (-1)^{k-1}\sqrt{-\lambda}$ $(k = 1, 2)$, we see

$$\begin{cases} (\lambda + \alpha bc|\xi'|^2 + b\sqrt{-\lambda}z_1)c_1 + (\lambda + \alpha bc|\xi'|^2 - b\sqrt{-\lambda}z_2)c_2 = 0\\ (\alpha b|\xi'|^2 + \sqrt{-\lambda})c_1 + (\alpha b|\xi'|^2 - \sqrt{-\lambda})c_2 = 0.\end{cases}$$

We calculate the determinant of the coefficient matrix:

$$\begin{aligned}
&\begin{vmatrix} \lambda + \alpha bc|\xi'|^2 + b\sqrt{-\lambda}z_1 & \lambda + \alpha bc|\xi'|^2 - b\sqrt{-\lambda}z_2 \\ \alpha b|\xi'|^2 + \sqrt{-\lambda} & \alpha b|\xi'|^2 - \sqrt{-\lambda} \end{vmatrix}\\
=&\begin{vmatrix} \lambda + \alpha bc|\xi'|^2 + b\sqrt{-\lambda}z_1 & -b\sqrt{-\lambda}(z_1 + z_2) \\ \alpha b|\xi'|^2 + \sqrt{-\lambda} & -2\sqrt{-\lambda} \end{vmatrix}\\
=&-\sqrt{-\lambda}\left\{2(\lambda + \alpha bc|\xi'|^2) + 2b\sqrt{-\lambda}z_1 - b(z_1 + z_2)(\alpha b|\xi'|^2 + \sqrt{-\lambda})\right\}\\
=&-\sqrt{-\lambda}\left\{2(\lambda + \alpha bc|\xi'|^2) - \alpha b^2|\xi'|^2(z_1 + z_2) + b\sqrt{-\lambda}(z_1 - z_2)\right\}
\end{aligned}$$

$$=: -\sqrt{-\lambda}((\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})).$$

We claim that the real part of the last term (Ⅲ) is non-negative. Then the determinant never become zero since the real part of the first term (I) and the second term (Ⅱ) is positive. We focus on the sign of the term (Ⅲ). From the equality $\sqrt{-\lambda}(z_1 - z_2) = -2\lambda(z_1 + z_2)^{-1}$,

$$\mathrm{sign}\,(\mathrm{Re}\,(\mathrm{III})) = \mathrm{sign}\,(\mathrm{Re}\,(-\lambda)\mathrm{Re}\,(\mathrm{z}_1 + \mathrm{z}_2) + \mathrm{Im}\,(-\lambda)\mathrm{Im}\,(\mathrm{z}_1 + \mathrm{z}_2)).$$

Here $\mathrm{Re}\,(-\lambda), \mathrm{Re}\,(z_1 + z_2) \le 0$ and $\mathrm{sign}\,\mathrm{Im}\,(-\lambda) = \mathrm{sign}\,\mathrm{Im}\,(z_1 + z_2)$ since

$$z_1 + z_2 = -\sqrt{2|\xi'|^2 + 2\sqrt{|\xi'|^4 + \lambda}}.$$

This implies $\mathrm{sign}\,(\mathrm{Re}\,(\mathrm{III}))$ is non-negative. This means that $(v, \sigma) = (0, 0)$, which concludes that the (LS) condition is satisfied. The other condition (LS^-_∞) is easily checked, so we skip the calculation.

5.6 Appendix C

We apply this general linear theory for the linearized Cahn–Hilliard equation on non-permeable walls:

$$(*)_{\text{non-per.}} \begin{cases} \partial_t v + \Delta^2 v = f & \text{in} \quad Q, \\ \partial_t v_\Gamma - b\partial_\nu \Delta v - bc\Delta_\Gamma \partial_\nu v + \alpha bc \Delta_\Gamma^2 v_\Gamma = g & \text{on} \quad \Sigma, \\ v|_\Gamma = v_\Gamma, \quad -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b \Delta_\Gamma v_\Gamma = h & \text{on} \quad \Sigma, \\ v(0) = v_0 \quad \text{in}\ \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{on}\ \Gamma. \end{cases}$$

This problem fits into the setting $\mathcal{A} = \Delta^2$, $\mathcal{B}_0 = -b\partial_\nu\Delta - bc\Delta_\Gamma\partial_\nu$, $\mathcal{C}_0 = \alpha bc\Delta_\Gamma^2$, $\mathcal{B}_1 = -(\Delta\cdot)|_\Gamma$, $\mathcal{C}_1 = \alpha b\Delta_\Gamma$, $\mathcal{B}_2 = 1$, $\mathcal{C}_2 = -1$, $g_2 = 0$ and $m = 2$, $m_0 = 3$, $k_0 = 4$, $m_1 = 2$, $k_1 = 2$, $m_2 = 0$, $k_2 = 0$, $\ell_0 = 4$, $\ell_1 = 3$, $\ell_2 = 3$. And then $\ell = \ell_0 = 4 = 2m$, $\kappa_0 = 1/4 - 1/(4p)$, $\kappa_1 = 1/2 - 1/(4p)$ and $\kappa_2 = 1 - 1/(4p)$. The condition (E) is satisfied as before.

To see (LS) condition, we need to solve the ordinary differential equation

(5.6.1) $\big((\lambda + (-|\xi'|^2 + \partial_y^2)^2\big)\, v(y) = 0 \ \ (y > 0),$

(5.6.2) $-\, b(-\partial_y)(-|\xi'|^2 + \partial_y^2)v(0) - bc(-|\xi'|^2)(-\partial_y)v(0) + \big((\lambda + \alpha bc(-|\xi'|^2)^2\big)\, \sigma = 0,$

(5.6.3) $v(0) - \sigma = 0,$

(5.6.4) $-\, (-|\xi'|^2 + \partial_y^2)v(0) + \alpha b(-|\xi'|^2)\sigma = 0.$

For the case $\lambda = 0$, $v(y) = (c_1 + c_2 y)e^{-|\xi'|y}$ for some $c_1, c_2 \in \mathbb{C}$. By the boundary conditions (5.6.2)–(5.6.4),

$$\begin{cases} -b|\xi'|^2(c_2 - |\xi'|c_1) + b(3|\xi'|^2 c_2 - |\xi'|^3 c_1) - bc|\xi'|^2(c_2 - |\xi'|c_1) + \alpha bc|\xi'|^4 c_1 = 0, \\ |\xi'|^2 c_1 - (-2|\xi'|c_2 + |\xi'|^2 c_1) - \alpha b|\xi'|^2 c_1 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} c|\xi'|(\alpha|\xi'| + 1)c_1 + (2 - c)c_2 = 0, \\ \alpha b|\xi'|c_1 - 2c_2 = 0. \end{cases}$$

The determinant of the coefficient matrix is $-|\xi'|(2\alpha c|\xi'| + 2\alpha b + 2c - \alpha bc)$. So we assume $\alpha bc < 2(\alpha b + c)$ (Assumption A), then we have $(c_1, c_2) = (0, 0)$, which implies the unique trivial solution $(v, \sigma) = (0, 0)$.

For the case $\lambda \neq 0$, $v(y) = c_1 e^{z_1 y} + c_2 e^{z_2 y}$ with the same z_k as before. By the boundary conditions (5.6.2)–(5.6.4),

$$\begin{cases} -b|\xi'|^2(c_1 z_1 + c_2 z_2) + b(c_1 z_1^3 + c_2 z_2^3) \\ \qquad -bc|\xi'|^2(c_1 z_1 + c_2 z_2) + (\lambda + \alpha bc(-|\xi'|^2)^2)(c_1 + c_2) = 0, \\ |\xi'|^2(c_1 + c_2) - (c_1 z_1^2 + c_2 z_2^2) - \alpha b|\xi'|^2(c_1 + c_2) = 0. \end{cases}$$

$$\Rightarrow \begin{cases} (\lambda + \alpha bc|\xi'|^4 + b\sqrt{-\lambda} z_1 - bc|\xi'|^2 z_1)c_1 + (\lambda + \alpha bc|\xi'|^2 - b\sqrt{-\lambda} z_2 - bc|\xi'|^2)c_2 = 0, \\ (\alpha b|\xi'|^2 + \sqrt{-\lambda})c_1 + (\alpha b|\xi'|^2 - \sqrt{-\lambda})c_2 = 0. \end{cases}$$

We calculate the determinant of the coefficient matrix:

$$\begin{aligned} &\begin{vmatrix} \lambda + \alpha bc|\xi'|^4 + b\sqrt{-\lambda} z_1 - bc|\xi'|^2 z_1 & \lambda + \alpha bc|\xi'|^2 - b\sqrt{-\lambda} z_2 - bc|\xi'|^2 z_2 \\ \alpha b|\xi'|^2 + \sqrt{-\lambda} & \alpha b|\xi'|^2 - \sqrt{-\lambda} \end{vmatrix} \\ =&\begin{vmatrix} \lambda + \alpha bc|\xi'|^4 + b\sqrt{-\lambda} z_1 - bc|\xi'|^2 z_1 & -b\sqrt{-\lambda}(z_1 + z_2) + bc|\xi'|^2(z_1 - z_2) \\ \alpha b|\xi'|^2 + \sqrt{-\lambda} & -2\sqrt{-\lambda} \end{vmatrix} \\ =& -\sqrt{-\lambda}\Big\{2(\lambda + \alpha bc|\xi'|^4) + b\sqrt{-\lambda}(z_1 - z_2) \\ &\qquad -bc|\xi'|^2(z_1 + z_2) - \alpha b^2|\xi'|^2(z_1 + z_2) + \alpha b^2 c|\xi'|^4 \frac{2}{z_1 + z_2}\Big\}, \end{aligned}$$

where we used $z_1 - z_2 = 2\sqrt{-\lambda}(z_1 + z_2)^{-1}$. We see the real part of $2(\lambda + \alpha bc|\xi'|^4) + b\sqrt{-\lambda}(z_1 - z_2)$ is positive. We claim that the real part of the others is non-negative by using the Assumption (A). From the Assumption (A),

$$\begin{aligned} &\mathrm{Re}\left(-bc|\xi'|^2(z_1 + z_2) - \alpha b^2|\xi'|^2(z_1 + z_2) + \alpha b^2 c|\xi'|^4 \frac{2}{z_1 + z_2}\right) \\ \geq&\mathrm{Re}\left(-bc|\xi'|^2(z_1 + z_2) - \alpha b^2|\xi'|^2(z_1 + z_2) + 2(\alpha b + c)b|\xi'|^4 \frac{2}{z_1 + z_2}\right) \\ =&(bc|\xi'|^2 + \alpha b^2)\mathrm{Re}\left(\frac{4|\xi'|^2}{z_1 + z_2} - (z_1 + z_2)\right). \end{aligned}$$

Note that

$$\frac{4|\xi'|^2}{z_1 + z_2} - (z_1 + z_2) = 2(z_1 + z_2)^{-1}(|\xi'|^2 - z_1 z_2),$$

$z_1 z_2 = \sqrt{\lambda + |\xi'|^4}$, $\mathrm{Re}\,(|\xi'|^2 - z_1 z_2) \leq 0$ and $\mathrm{Im}\,(z_1 + z_2)\mathrm{Im}\,(|\xi'|^2 - z_1 z_2) \geq 0$. So we have

$$\begin{aligned} &\mathrm{Sign}\,\mathrm{Re}\,\left((z_1 + z_2)^{-1}(|\xi'|^2 - z_1 z_2)\right) \\ =&\mathrm{Sign}\left(\mathrm{Re}\,(z_1 + z_2)\mathrm{Re}\,(|\xi'|^2 - z_1 z_2) + \mathrm{Im}\,(z_1 + z_2)\mathrm{Im}\,(|\xi'|^2 - z_1 z_2)\right) \\ \geq&0. \end{aligned}$$

This implies that the determinant of the coefficients never 0 and $(v, \sigma) = (0, 0)$. So it was shown (LS) condition.

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Chapter 6

Well-posedness for the quasi-linear parabolic equations from the theory of maximal L_p regularity with critical time weights

We construct the local well-posedness for the quasi-linear parabolic evolution equations with the time-dependent operator and the non-linear terms in time weighted L_p spaces. Moreover we give a sufficient condition to extend the global solution for the semi-linear parabolic evolution equations in terms of a priori estimate. This result is a generalization from the time-independent case and the bi-linear non-linearities in a series of papers by J. Prüss to the time-independent case and the general non-linearities.

Keywords: maximal L_p regularity with time weights, well-posedness, quasi-linear parabolic evolution equations

6.1 Introduction

For the quasi-linear parabolic problem there are many papers on the well-posedness results. At first in the paper [1] they proved existence and uniqueness of the strong solution from maximal L_p regularity. Later their method using a contraction mapping theorem was applied to the various equations. Among them, J. Prüss and G. Simonett [10] introduced time weighted maximal L_p regularity and applied it to get well-posedness in [6, 7, 14]. The merit of the time weight is to reduce the initial regularity while keeping the regularity of the solution excepting for the behavior near $t = 0$. This theory of time weights is useful to consider the global solution and to gain the compactness properties of orbits. Moreover they proved that the initial regularity is critical by illustrating a counterexample in spaces strictly larger than the initial spaces they constructed. They clarified the relationship between the critical spaces, the scaling invariant spaces and the interpolation-extrapolation spaces. The theory of critical time weights was applied to a lot of equations, e.g. the vorticity equations for the Navier–Stokes problem, convection-diffusion equations, the Nernst–Planck–Poisson equations in electro-chemistry, chemotaxis equations, the MHD equations and some other well-known parabolic equations in [12], the Navier–Stokes equations with the Navier boundary conditions in [13] and the bidomain equations in [14]. For other time weighted theory of the Navier–Stokes equations, see [4].

In a series of papers by J. Prüss, they constructed a framework of the local well-posedness of the quasilinear parabolic equations of the form

$$\dot{u} + A(u)u = F(u),\ t > 0, \quad u(0) = u_0,$$

based on the maximal L_p regularity for the operator $A(u_0)$, locally Lipschitz assumptions on A and F and Banach's fixed point theorem with a technique of time weights. In this paper we generalize above time-independent form to time-dependent form of

$$\text{(QL)} \qquad \dot{u} + A(t,u)u = F(t,u),\ t > 0, \quad u(0) = u_0,$$

and prove the local well-posedness result. By this form we are able to contain a non-homogeneous term in the equations, i.e. we can consider a effect of a source term. The assumptions and the proof are almost same as before, but it is important to check well-posedness of this form. We are able to derive the continuity for the non-homogeneous terms. After we get the unique local strong solution, we give a sufficient condition to extend global solution in terms of a priori estimate. However we had to restrict that the equations are the semi-linear parabolic equations. An assumption on a priori estimate is also same as [12], but we generalize from the bi-linear (or multi-linear) non-linearities used in [12] to general non-linearities used in the local well-posedness results.

The outline of this paper is as follows. In section 6.2, we write down some assumptions and statements of the theorem of local well-posedness for (QL). In section 6.3, we prove the local well-posedness. The strategy is to use maximal L_p regularity estimates, Lipschitz assumptions on A and F and the contraction map as usual. In section 6.4, we consider the global well-posedness for the general semi-linear parabolic evolution equations after some preparations. In section 6.5, we apply the local well-posedness for a quasi-linear heat equation.

6.2 Local well-posedness

Let X_0 and X_1 be Banach spaces such that dense embedding $X_1 \hookrightarrow X_0$ and let $1 < p < \infty$ and $1/p < \mu \leq 1$. We consider the quasilinear parabolic problems

$$\text{(6.2.1)} \qquad \begin{aligned} \dot{u}(t) + A(t,u(t))u(t) = F_1(t,u(t)) + F_2(u(t)),\ t > 0, \\ u(0) = u_0 \end{aligned}$$

in weighted L_p-framework, i.e. we look for the solution in the class

$$u \in \mathbb{E}_{1,\mu}(J) := H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1) (\hookrightarrow C(\overline{J}; X_{\gamma,\mu})).$$

Here $J = (0, \mathcal{T})$ denotes the time interval and

$$\begin{aligned} L_{p,\mu}(J; X_1) &:= \{u \in L_{1,loc}(J; X_1) \mid t^{1-\mu}u \in L_p(J; X_1)\}, \\ H^1_{p,\mu}(J; X_0) &:= \{u \in L_{p,\mu}(J; X_0) \cap H^1_1(J; X_0) \mid t^{1-\mu}\dot{u} \in L_p(J; X_0)\}, \\ X_{\gamma,\mu} &:= (X_0, X_1)_{\mu-1/p,p} \quad \text{real interpolation space.} \end{aligned}$$

Let $V_\mu \subset X_{\gamma,\mu}$ be open. Let $X_\beta := [X_0, X_1]_\beta$ denote the complex interpolation spaces for $\beta \in (0,1)$ to treat the part F_2, which is the singular part in contrast to the regular part F_1. For a comprehensive interpolation theory, see [15]. We will impose the following assumptions.

(H0) $A \in C(\overline{J} \times V_\mu; \mathcal{B}(X_1, X_0))$ and $F_1 : J \times V_\mu \to X_0$ satisfies assumptions of Caratheodory type, i.e. $F_1(\cdot, u)$ is measurable for each $u \in V_\mu$ and $F_1(t, \cdot)$ is continuous for a.e. $t \in J$.

(H1) $A(t, \cdot) \in \mathrm{Lip}_{\mathrm{loc}}(V_\mu; \mathcal{B}(X_1, X_0))$ for all $t \in \overline{J}$, whose Lipschitz constant is uniform in t, and $F_1(t, \cdot) \in \mathrm{Lip}_{\mathrm{loc}}(V_\mu; X_0)$ for a.e. $t \in J$, whose Lipschitz constant is a function of t in $L_{p,\mu}(J)$. Namely for all $u^* \in V_\mu$, there exists $\varepsilon_0 > 0$ with $\overline{B}^{X_{\gamma,\mu}}(u^*, \varepsilon_0) \subset V_\mu$, $L > 0$ and $M \in L_{p,\mu}(J)$ such that for all $w_1, w_2 \in \overline{B}^{X_{\gamma,\mu}}(u^*, \varepsilon_0)$,

$$|A(t, w_1) - A(t, w_2)|_{\mathcal{B}(X_1, X_0)} \le L |w_1 - w_2|_{X_{\gamma,\mu}}$$

for all $t \in J$ as well as

$$|F_1(t, w_1) - F_1(t, w_2)|_{X_0} \le M(t) |w_1 - w_2|_{X_{\gamma,\mu}}$$

for a.e. $t \in J$.

(H2) $F_2 : V_\mu \cap X_\beta \to X_0$ satisfies the estimate

$$|F_2(u_1) - F_2(u_2)|_{X_0} \le C \sum_{j=1}^{m} \left(1 + |u_1|_{X_\beta}^{\rho_j} + |u_2|_{X_\beta}^{\rho_j}\right) |u_1 - u_2|_{X_{\beta_j}},$$

for some numbers $m \in \mathbb{N}$, $\rho_j \ge 0$, $\beta \in (\mu - 1/p, 1)$, $\beta_j \in [\mu - 1/p, \beta]$ and a constant $C > 0$, which may depend on $|u_i|_{X_{\gamma,\mu}}$.

(H3) For all $j = 1, \cdots, m$, we have

$$\rho_j(\beta - (\mu - 1/p)) + (\beta_j - (\mu - 1/p)) \le 1 - (\mu - 1/p).$$

When we use this equality for some j (it is called *critical* case) not strict inequality (it is called *sub-critical* case), we have to assume additionally the following structural condition on the Banach spaces X_0 and X_1.

(S) The space X_0 is of class UMD. The embedding

$$H_p^1(\mathbb{R}; X_0) \cap L_p(\mathbb{R}; X_1) \hookrightarrow H_p^{1-\beta}(\mathbb{R}; X_\beta),$$

is valid for each $\beta \in (0,1)$, $p \in (1, \infty)$.

As $\beta_j \le \beta < 1$, any j with $\rho_j = 0$ is subcritical. The assumption (H3) determines *the critical weight* μ_c defined by

$$\mu_c := \frac{1}{p} + \beta - \min_j \frac{1 - \beta_j}{\rho_j}.$$

Here we take minimum for j such that $\rho_j \ne 0$. For any $\mu \in [\mu_c, 1]$ the assumption (H3) is satisfied. Note that $X_{\gamma,\mu_c} = (X_0, X_1)_{\mu_c - 1/p, p}$ and the first real interpolation index $\mu_c - 1/p$ is independent of p.

The strategy to get well-posedness heavily depends on the maximal $L_{p,\mu}$-regularity for the operator. We say that a densely defined closed linear operator A on X_0 with

$D(A) = X_1$ has the property of maximal $L_{p,\mu}$-regularity, for short $A \in \mathcal{MR}_{p,\mu}(X_1, X_0)$, if for each $f \in \mathbb{E}_{0,\mu}(\mathbb{R}_+) := L_{p,\mu}(\mathbb{R}_+; X_0)$ there exists a unique solution

$$u \in \mathbb{E}_{1,\mu}(\mathbb{R}_+) = H^1_{p,\mu}(\mathbb{R}_+; X_0) \cap L_{p,\mu}(\mathbb{R}_+; X_1)$$

of the linear problem

$$\dot{u} + Au = f,\ t > 0, \quad u(0) = 0.$$

Thanks to [10], it was proved that the class $\mathcal{MR}_{p,\mu}(X_1, X_0)$ and $\mathcal{MR}_p(X_1, X_0)$ are equivalent, where $\mathcal{MR}_p(X_1, X_0) := \mathcal{MR}_{p,1}(X_1, X_0)$ is a operator space satisfying usual maximal L_p regularity property. For the nontrivial initial data, $u_0 \in X_{\gamma,\mu}$ is the natural space since for any $u \in \mathbb{E}_{1,\mu}(\mathbb{R}_+)$, we have $u|_{t=0} \in X_{\gamma,\mu}$ and vice versa, i.e. $x \in X_{\gamma,\mu}$ implies $e^{-tA}x \in \mathbb{E}_{1,\mu}(\mathbb{R}_+)$, see [10, 11]. In many applications the condition of maximal regularity on $\mathbb{R}_+$ is too strong since it means the semigroup e^{-tA} is exponential decay but we use this notation because it is satisfied in many cases when we add κu for both sides in the equations (6.2.1) for sufficiently large $\kappa > 0$, e.g. [9].

The local well-posedness result is as follows.

Theorem 6.1. *Suppose that the structural assumption* **(S)** *holds, and assume that hypothesis* **(H0)**-**(H3)** *are valid. Fix any $u_0 \in V_\mu$ such that $A_0 := A(0, u_0) \in \mathcal{MR}_{p,\mu}(X_1, X_0)$ and $F_1(\cdot, u_0) \in \mathbb{E}_{0,\mu}(J)$. Then there is $T = T(u_0) \in (0, \mathcal{T}]$ and $\varepsilon = \varepsilon(u_0) \in (0, \varepsilon_0]$ such that $\overline{B}^{X_{\gamma,\mu}}(u_0, \varepsilon) \subset V_\mu$ and such that problem (6.2.1) admits a unique solution*

$$u(\cdot, u_1) \in \mathbb{E}_{1,\mu}(0, T) \cap C([0, T], V_\mu),$$

for each initial value $u_1 \in \overline{B}^{X_{\gamma,\mu}}(u_0, \varepsilon)$. There is a constant $c = c(u_0) > 0$ such that

$$\|u(\cdot, u_1) - u(\cdot, u_2)\|_{\mathbb{E}_{1,\mu}(0,T)} \le c|u_1 - u_2|_{X_{\gamma,\mu}},$$

for all $u_1, u_2 \in \overline{B}^{X_{\gamma,\mu}}(u_0, \varepsilon)$. Moreover, for each $\delta \in (0, T)$ we have in addition

$$u \in \mathbb{E}_{1,1}(\delta, T)(\hookrightarrow C([\delta, T]; X_{\gamma,1}),$$

i.e. the solution regularizes instantaneously.

In particular, by dividing $F_1(t, u) = F_{11}(t) + F_{12}(u)$, we get the following corollary.

Corollary 6.2. *Suppose that the structural assumption* **(S)** *holds, and assume that $A \in C(\overline{J} \times V_\mu; \mathcal{B}(X_1, X_0))$, $(A(t, \cdot), F_{12}) \in \mathrm{Lip}_{\mathrm{loc}}(V_\mu; \mathcal{B}(X_1, X_0) \times X_0)$ for all $t \in J$, and hypothesis* **(H2)**-**(H3)** *are valid. Fix any $u_0 \in V_\mu$ such that $A_0 := A(0, u_0) \in \mathcal{MR}_{p,\mu}(X_1, X_0)$. Then for any $F_{11} \in L_{p,\mu}(J, X_0)$, the same unique existence statement of the theorem 6.1 for the quasi-linear parabolic problem*

$$\dot{u}(t) + A(t, u(t))u(t) = F_{11}(t) + F_{12}(u(t)) + F_2(u(t)),\ t > 0,$$

holds. Moreover there is a constant $c = c(u_0) > 0$ such that

$$\|u(\cdot, u_1, F^1_{11}) - u(\cdot, u_2, F^2_{11})\|_{\mathbb{E}_{1,\mu}(0,T)} \le c(|u_1 - u_2|_{X_{\gamma,\mu}} + |F^1_{11} - F^2_{11}|_{\mathbb{E}_{0,\mu}(J)}),$$

i.e. the continuity for the source terms F_{11} holds.

As another choice, the case that $F_1(t,u) = F(t) + B(t)u$ for $F \in L_{p,\mu}(J;X_0)$, $B \in L_{p,\mu}(J;\mathcal{B}(X_{\gamma,\mu},X_0))$ also satisfies the assumption **(H0)** and **(H1)**.

As we can see in the paper [7, 11], the continuation of local solutions holds.

Corollary 6.3. *Let the assumptions of theorem 6.1 be satisfied and assume that $A(t,v) \in \mathcal{MR}_{p,\mu}(X_1,X_0)$ for each $t \in J$ and all $v \in V_\mu$. Then the solution $u(t)$ has a maximal interval of existence $J(u_0) = [0, t_+(u_0))$, which is characterized by*
(i) *Global existence: $t_+(u_0) = \mathcal{T}$;*
(ii) $\liminf_{t\to t_+(u_0)} \operatorname{dist}_{X_{\gamma,\mu}}(u(t), \partial V_\mu) = 0$;
(iii) $\lim_{t\to t_+(u_0)} u(t)$ *does not exist in $X_{\gamma,\mu}$.*

6.3 Proof of Theorem, local-wellposedness

Proof of Theorem 6.1. The proof is almost same as [7, Theorem 2.1] and [14, Theorem 1.2]. Let $f := F(\cdot, u_0)$. We first introduce a reference solution $u_0^* \in \mathbb{E}_{1,\mu}(J)$ as the solution of the linear problem

$$\dot{w} + A_0 w = f,\ t \in J, \quad w(0) = u_0.$$

Here note that there exists a unique solution $w = u_0^*$ since $f \in \mathbb{E}_{0,\mu}(J)$, $u_0 \in V_\mu (\subset X_{\gamma,\mu})$ and the operator A_0 satisfies maximal $L_{p,\mu}$ regularity property. Let $u_1 \in \overline{B}^{X_{\gamma,\mu}}(u_0,\varepsilon) (\subset V_\mu)$ and a closed ball

$$\mathbb{B}_{r,T,u_1} := \{v \in \mathbb{E}_{1,\mu}(0,T) \mid v|_{t=0} = u_1, \|v - u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} \le r\}$$

for some $\varepsilon \in (0,\varepsilon_0]$, $r \in (0,1]$ and $T > 0$ to be fixed later. As in [7], we shall show that for all $v \in \mathbb{B}_{r,T,u_1}$, it holds that $v(t) \in \overline{B}^{X_{\gamma,\mu}}(u_0,\varepsilon_0) (\subset V_\mu)$ for all $t \in [0,T]$, provided that $r, T, \varepsilon > 0$ are sufficiently small. To do so, we replace the reference solution u_1^* in [7] by the solution of

$$\dot{w} + A_0 w = f,\ t \in J,\ w(0) = u_1.$$

Then we have

$$\begin{aligned}
&\|v - u_0\|_{\infty,X_{\gamma,\mu}} \\
\le& \|v - u_1^*\|_{\infty,X_{\gamma,\mu}} + \|u_1^* - u_0^*\|_{\infty,X_{\gamma,\mu}} + \|u_0^* - u_0\|_{\infty,X_{\gamma,\mu}} \\
\le& C_1\|v - u_1^*\|_{\mathbb{E}_{1,\mu}(0,T)} + \|u_1^* - u_0^*\|_{\infty,X_{\gamma,\mu}} + \left\| e^{-tA_0}u_0 + \int_0^t e^{-(t-s)A_0} f(s)ds - u_0 \right\|_{\infty,X_{\gamma,\mu}} \\
\le& C_1(\|v - u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} + \|u_0^* - u_1^*\|_{\mathbb{E}_{1,\mu}(0,T)}) + \|u_1^* - u_0^*\|_{\infty,X_{\gamma,\mu}} \\
&\qquad + \|e^{-tA_0}u_0 - u_0\|_{\infty,X_{\gamma,\mu}} + C_0\|f\|_{\mathbb{E}_{0,\mu}(0,T)} \\
\le& C_1 r + C_\gamma |u_0 - u_1|_{X_{\gamma,\mu}} + \|e^{-tA_0}u_0 - u_0\|_{\infty,X_{\gamma,\mu}} + C_0\|f\|_{\mathbb{E}_{0,\mu}(0,T)}.
\end{aligned}$$

Here we use that C_1 and C_γ are independent of T and the maximal $L_{p,\mu}$ regularity for A_0 with its constant $C_0 > 0$ which is also independent of $T > 0$. The last line can be estimated by ε_0 when ε, r, T are sufficiently small. In the rest of the proof, we keep this smallness condition. Note that $v(t) \in \overline{B}^{X_{\gamma,\mu}}(u_0, \varepsilon_0) \cap X_1 (\subset V_\mu \cap X_\beta)$ for a.e. t.

We will prove the existence and uniqueness of the problem (6.2.1) by Banach's fixed point theorem. We define a mapping $\mathcal{S}_{u_1} : \mathbb{B}_{r,T,u_1} \to \mathbb{E}_{1,\mu}(0,T)$ by $\mathcal{S}_{u_1} v = u$, where $v \in \mathbb{B}_{r,T,u_1}$ and u is the unique solution of *the linear problem*

$$\dot{u} + A_0 u = F_1(t,v) + F_2(v) + (A_0 - A(t,v))v, \; t \in (0,T), \quad u(0) = u_1.$$

To use Banach's fixed point theorem, we need to show the self-mapping property $\mathcal{S}_{u_1}\mathbb{B}_{r,T,u_1} \subset \mathbb{B}_{r,T,u_1}$ and the contraction property:

$$\|\mathcal{S}_{u_1} v - \mathcal{S}_{u_1} \overline{v}\|_{\mathbb{E}_{1,\mu}(0,T)} \le \kappa \|v - \overline{v}\|_{\mathbb{E}_{1,\mu}(0,T)},$$

is valid for some $\kappa \in (0,1)$ and for all $v, \overline{v} \in \mathbb{B}_{r,T,u_1}$. We will first prove the self-mapping property. Let $v \in \mathbb{B}_{r,T,u_1}$, then we have

$$\begin{aligned}
&\|\mathcal{S}_{u_1} v - u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} \\
\le &\|\mathcal{S}_{u_1} v - u_1^*\|_{\mathbb{E}_{1,\mu}(0,T)} + \|u_1^* - u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} \\
\le &C_0 \left(\|F_1(t,v) - F_1(t,u_0)\|_{\mathbb{E}_{0,\mu}(0,T)} + \|F_2(v)\|_{\mathbb{E}_{0,\mu}(0,T)} + \|(A_0 - A(t,v))v\|_{\mathbb{E}_{0,\mu}(0,T)} \right) \\
&\qquad + \|u_1^* - u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} \\
=: &I + II + III + IV.
\end{aligned}$$

The second term II and the last term IV can be estimated by $r/4$ under the assumptions of the theorem and ε and T are sufficiently small, see [6, 7, 14]. We next calculate the first term I:

$$\begin{aligned}
\|F_1(t,v) - F_1(t,u_0)\|_{\mathbb{E}_{0,\mu}(0,T)} &\le |M|_{L_{p,\mu}(0,T)} \|v - u_0\|_{L_\infty(0,T;X_{\gamma,\mu})} \\
&\le |M|_{L_{p,\mu}(0,T)} \varepsilon_0
\end{aligned}$$

Since $|M|_{L_{p,\mu}(0,T)} \to 0$ as $T \to 0$, we see that the first term I can be estimated by $r/4$. The third term III is estimated as follows. Let $\tilde{L}(T) := \sup_{t\in[0,T]} \|A_0 - A(t,u_0)\|_{\mathcal{B}(X_1,X_0)}$. We have

$$\begin{aligned}
&\|(A_0 - A(t,v))v\|_{\mathbb{E}_{0,\mu}(0,T)} \\
\le &\sup_{t\in[0,T]} \left(\|A_0 - A(t,u_0)\|_{\mathcal{B}(X_1,X_0)} + \|A(t,u_0) - A(t,v(t))\|_{\mathcal{B}(X_1,X_0)} \right) \|v\|_{L_{p,\mu}(0,T;X_1)} \\
\le &\left(\tilde{L}(T) + L\|v - u_0\|_{L_\infty(0,T;X_{\gamma,\mu})} \right) \left(\|v - u_0^*\|_{L_{p,\mu}(0,T;X_1)} + \|u_0^*\|_{L_{p,\mu}(0,T;X_1)} \right) \\
\le &\left(\tilde{L}(T) + L \left(C_1 r + C_\gamma \varepsilon + \|e^{-tA_0}u_0 - u_0\|_{\infty,X_{\gamma,\mu}} + C_0 \|f\|_{\mathbb{E}_{0,\mu}(0,T)} \right) \right) \\
&\qquad \times \left(\|v - u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} + \|u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} \right) \\
\le &\left(\tilde{L}(T) + L \left(C_1 r + C_\gamma \varepsilon + \|e^{-tA_0}u_0 - u_0\|_{\infty,X_{\gamma,\mu}} + C_0 \|f\|_{\mathbb{E}_{0,\mu}(0,T)} \right) \right) \left(r + \|u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} \right)
\end{aligned}$$

Since $\tilde{L}(T), \|u_0^*\|_{\mathbb{E}_{1,\mu}(0,T)} \to 0$ as $T \to 0$, we see that the third term III can be estimated by $r/4$ when ε, r, T are sufficiently small. Combining all estimates, we derived the self-mapping property when ε, r, T are sufficiently small.

It remains to prove the contraction property and the continuous dependence of the initial data. Let $u_2 \in \overline{B}^{X_{\gamma,\mu}}(u_0, \varepsilon)$, $\overline{v} \in \mathbb{B}_{r,T,u_2}$. We have

$$\begin{aligned}
&\|\mathcal{S}_{u_1} v - \mathcal{S}_{u_2}\overline{v}\|_{\mathbb{E}_{1,\mu}(0,T)} \\
\le &\|e^{-tA_0}(u_1 - u_2)\|_{\mathbb{E}_{1,\mu}(0,T)} + C_0(\|F_1(t,v) - F_1(t,\overline{v})\|_{\mathbb{E}_{0,\mu}(0,T)} + \|F_2(v) - F_2(\overline{v})\|_{\mathbb{E}_{0,\mu}(0,T)} \\
&\qquad + \|(A_0 - A(t,v))v - (A_0 - A(t,\overline{v}))\overline{v}\|_{\mathbb{E}_{0,\mu}(0,T)}) \\
\le &\|e^{-tA_0}(u_1 - u_2)\|_{\mathbb{E}_{1,\mu}(0,T)} + C_0(\|F_1(t,v) - F_1(t,\overline{v})\|_{\mathbb{E}_{0,\mu}(0,T)} + \|F_2(v) - F_2(\overline{v})\|_{\mathbb{E}_{0,\mu}(0,T)} \\
&\qquad + \|(A(t,v) - A_0)(v - \overline{v})\|_{\mathbb{E}_{0,\mu}(0,T)} + \|(A(t,v) - A(t,\overline{v}))\overline{v}\|_{\mathbb{E}_{0,\mu}(0,T)}).
\end{aligned}$$

Note that we can use the completely same argument in [7, 14] even we treat time-dependent operator $A(t,u)$ and non-linear term $F_1(t,u)$. Therefore it has already estimated as follows.

$$\|\mathcal{S}_{u_1} v - \mathcal{S}_{u_2}\overline{v}\|_{\mathbb{E}_{1,\mu}(0,T)} \le \frac{1}{2}\|v - \overline{v}\|_{\mathbb{E}_{1,\mu}(0,T)} + c\|u_1 - u_2\|_{X_{\gamma,\mu}} \tag{6.3.1}$$

for some $c = c(u_0) > 0$ when ε, r, T are sufficiently small. In particular, by $u_1 = u_2$, the inequality means that $\mathcal{S}_{u_1}$ is the contraction map in $\mathbb{B}_{r,T,u_1}$. So there exists a unique fixed point $\tilde{u} \in \mathbb{B}_{r,T,u_1}$ such that $\mathcal{S}_{u_1}\tilde{u} = \tilde{u}$, which is the unique solution of the quasi-linear parabolic problem (6.2.1) with initial value u_1. Furthermore, denoting $u(t,u_1)$ and $u(t,u_2)$ by the solutions of (6.2.1) with initial values $u_1, u_2 \in \overline{B}^{X_{\gamma,\mu}}(u_0, \varepsilon)$, respectively, the continuous dependence of the initial data follows from (6.3.1). □

Remark 6.4. The proof of Corollary 6.2 is straightforward since $\|F_1(t,u) - F_1(t,\overline{v})\|_{\mathbb{E}_{0,\mu}(0,T)}$ in the proof of the theorem 6.1 is replaced by

$$\|F_{11}^1 - F_{11}^2\|_{\mathbb{E}_{0,\mu}(0,T)} + \|F_{12}(v) - F_{12}(\overline{v})\|_{\mathbb{E}_{0,\mu}(0,T)}.$$

6.4 Global well-posedness

Let $t_+ \in (0, \mathcal{T}]$ be the maximal time interval of the solution. Let $a \in (0, t_+)$ be fixed. By the mixed derivative theorem and Sobolev embedding in weighted spaces, see for instance [8], we have

$$\mathbb{E}_{1,\mu}(0,a) = H^1_{p,\mu}(0,a;X_0) \cap L_{p,\mu}(0,a;X_1) \hookrightarrow H^{1-\mu}_{p,\mu}(0,a;X_\mu) \hookrightarrow L_p(0,a;X_\mu).$$

We remind that the space X_μ denote complex interpolation spaces $X_\mu = [X_0, X_1]_\mu$.

Conversely, we would like to prove that if $t_+ < \mathcal{T}$ then $u \notin L_p(0,t_+;X_\mu)$. In particular, the solution exists globally if $u \in L_p(0,t_+;X_\mu)$. This means that the global existence is equivalent to an integral a priori bound.

In this section we restrict the equations are the semi-linear parabolic evolution equations

$$\dot{u} + A(t)u = F_1(t,u) + F_2(u),\ t > 0, \quad u(0) = u_0, \tag{SL}$$

and $V_\mu = X_{\gamma,\mu}$. Here the assumption on F_1 and F_2 are same as section 6.2. As another restricted assumption, the operator A belongs to $C(\overline{J};\mathcal{B}(X_1,X_0))$ and $A(s) \in \mathcal{BIP}(X_0)$

with power angle $\theta_A < \pi/2$ for all $s \in \overline{J}$, which is stronger than the maximal L_p regularity property on some interval. For the class of $\mathcal{BIP}(X_0)$, see [2,11].

For the semi-linear equations (SL), we may assume $F_2(u^*) = 0$ for all $u^* \in X_\beta$ since $\tilde{F}_1(t,u) := F_1(t,u) + F_2(u^*)$ satisfies the assumption (H0) and (H1). Then the assumption (H2) is

$$|F_2(u)|_{X_0} \le C \sum_{j=1}^{m} (1 + |u|_{X_\beta}^{\rho_j}) |u|_{X_{\beta_j}}. \tag{6.4.1}$$

Without loss of generality, let $\rho_j \neq 0$. Let

$$\frac{1}{r} := \frac{\beta_j - (\mu - 1/p)}{1 - (\mu - 1/p)}, \quad \frac{1}{r'} := \rho_j \frac{\beta - (\mu - 1/p)}{1 - (\mu - 1/p)}, \quad \frac{1}{r''} := 1 - \frac{1}{r} - \frac{1}{r'}.$$

From the definition of μ, β_j, β and the assumption (H3), $1/r < 1$, $1/r' < \rho_j$ and $0 \le 1/r'' < 1$. The case $1/r'' = 0$ is the critical case. Multiply the inequality (6.4.1) by $t^{1-\mu}$, take $L_p(0,T)$ norm in time and Hölder's inequality, we have

$$\begin{aligned} |F_2(u)|_{\mathbb{E}_{0,\mu}(0,T)} &\le C \sum_{j=1}^{m} \left(\int_0^T t^{(1-\mu)p} (1 + |u(t)|_{X_\beta}^{\rho_j})^p |u(t)|_{X_{\beta_j}}^p \, dt \right)^{1/p} \\ &\le C \sum_{j=1}^{m} \kappa_{r''}(T) \left(\kappa_{r'}(T) + \|u\|_{L_{\rho_j p r', \sigma'}(0,T;X_\beta)}^{\rho_j} \right) \|u\|_{L_{pr,\sigma}(0,T;X_{\beta_j})}, \end{aligned}$$

where

$$\kappa_\ell(T) := (\int_0^T t^{(1-\mu)p} dt)^{1/\ell} = \left(\frac{T^{(1-\mu)p+1}}{(1-\mu)p+1} \right)^{1/\ell} \qquad (\ell = r', r'')$$
$$\sigma := 1 - \frac{1}{r} + \frac{\mu}{r}, \quad \sigma' := 1 - \frac{1}{\rho_j r'} + \frac{\mu}{\rho_j r'}.$$

Note that $\sigma > 1/(pr)$ and $\sigma' > 1/(\rho_j p r')$ is admissible and $\kappa_\ell(T) \to 0$ as $T \to 0$ if $1/r'' \neq 0$.

We would like to use the estimate of the form

$$\|u\|_{L_{\rho_j p r', \sigma'}(0,T;X_\beta)}^{\rho_j} \|u\|_{L_{pr,\sigma}(0,T;X_{\beta_j})} \le C \|u\|_{L_p(0,T;X_\mu)}^{\rho_j} \|u\|_{\mathbb{E}_{1,\mu}(0,T)}, \tag{6.4.2}$$

where $C > 0$ is independent of T and u. To do so, we use the following interpolation result.

Lemma 6.5 ([14, Appendix A.1])**.** *Suppose X_1 is densely embedded in X_0, $A : X_1 \to X_0$ is bounded abd $A \in \mathcal{BIP}(X_0)$. Let $\mathcal{F}_j$, $j = 0, 1$ be complete function spaces over an interval $J = (0, a)$ and let $\theta \in (0, 1)$. Then*

$$[\mathcal{F}_0(J, X_{\beta_0}), \mathcal{F}_1(J, X_{\beta_1})]_\theta \simeq \mathcal{F}_\theta(J, X_\beta), \quad \beta = (1-\theta)\beta_0 + \theta\beta_1,$$

where $[\cdot,\cdot]_\theta$ means complex interpolation, $\mathcal{F}_\theta = [\mathcal{F}_0, \mathcal{F}_1]_\theta$ and $X_\alpha = [X_0, X_1]_\alpha$ for $\alpha \in (0, 1)$.

Using this lemma, the problem is reduced to the existence of $\alpha_j^1, \alpha_j^2 \in (0,1)$ and $t_j, s_j \in (0,1)$ such that

$$\begin{aligned}
&[L_p(0,T;X_\mu), \mathbb{E}_{1,\mu}(0,T)]_{t_j} \\
&\hookrightarrow [L_p(0,T;X_\mu), H^{1-\alpha_j^1}_{p,\mu}(0,T;X_{\alpha_j^1})]_{t_j} = H^{(1-\alpha_j^1)t_j}_{p,(1-t_j)+\mu t_j}(0,T;X_{(1-t_j)\mu+t_j\alpha_j^1}) \\
&\hookrightarrow L_{\rho_j pr',\sigma'}(0,T;X_\beta)
\end{aligned}$$

and

$$\begin{aligned}
&[L_p(0,T;X_\mu), \mathbb{E}_{1,\mu}(0,T)]_{s_j} \\
&\hookrightarrow [L_p(0,T;X_\mu), H^{1-\alpha_j^2}_{p,\mu}(0,T;X_{\alpha_j^2})]_{s_j} = H^{(1-\alpha_j^2)s_j}_{p,(1-s_j)+\mu s_j}(0,T;X_{(1-s_j)\mu+s_j\alpha_j^2}) \\
&\hookrightarrow L_{pr,\sigma}(0,T;X_{\beta_j})
\end{aligned}$$

with $\rho_j t_j + s_j = 1$ since these two complex interpolations imply

$$\begin{aligned}
\|u\|_{L_{\rho_j pr',\sigma'}(0,T;X_\beta)} &\le C\|u\|^{1-t_j}_{L_p(0,T;X_\mu)}\|u\|^{t_j}_{\mathbb{E}_{1,\mu}(0,T)}, \\
\|u\|_{L_{pr,\sigma}(0,T;X_{\beta_j})} &\le C\|u\|^{1-s_j}_{L_p(0,T;X_\mu)}\|u\|^{s_j}_{\mathbb{E}_{1,\mu}(0,T)},
\end{aligned}$$

and therefore the inequality (6.4.2) holds from $\rho_j t_j + s_j = 1$. The constants C may differ from line to line, but they are independent of T and u. Two time-space embeddings are satisfied when

$$\alpha_j^1 := \frac{\beta - \mu + \mu t_j}{t_j} \in (0,1), \quad \alpha_j^2 := \frac{\beta_j - \mu\rho_j t_j}{1-\rho_j t_j} \in (0,1), \quad 0 < t_j < \min\{1/\rho_j, 1\}.$$

This is guaranteed if t_j such that

$$(0 <) \max\left\{\frac{\mu-\beta}{\mu}, \frac{\beta-\mu}{1-\mu}\right\} < t_j < \min\left\{\frac{\beta_j}{\mu\rho_j}, \frac{1-\beta_j}{(1-\mu)\rho_j}\right\} (< \frac{1}{\rho_j})$$

exists for all j with $\rho_j \neq 0$. Namely we assume

$$\max\left\{\frac{\mu-\beta}{\mu}, \frac{\beta-\mu}{1-\mu}\right\} < \min\left\{\frac{\beta_j}{\mu\rho_j}, \frac{1-\beta_j}{(1-\mu)\rho_j}\right\}. \tag{6.4.3}$$

This is a sufficient condition to show the inequality (6.4.2). The statement of global well-posedness is as follows.

Theorem 6.6. *Assume that (S), (H2)-(H3), (6.4.3) and $F_1 : J \times X_{\gamma,\mu} \to X_0$ satisfies assumptions Caratheodory type and $F_1(t,\cdot) \in \mathrm{Lip_{loc}}(X_{\gamma,\mu}; X_0)$ for a.e. $t \in J$, whose Lipschitz constant is a function of t in $L_{p,\mu}(J)$. Assume that $A \in C(\overline{J}; \mathcal{B}(X_1, X_0))$ and $A(s) \in \mathcal{BIP}(X_0)$ with power angle $\theta_A < \pi/2$. Let u be the unique solution of Theorem 6.1 with maximal time interval of existence $[0, t_+)$ for a semi-linear parabolic equation (SL). Then*
(i) *$u \in L_p(0,a;X_\mu)$ for each $a < t_+$.*
(ii) *If $t_+ < \mathcal{T}$ then $u \notin L_p(0,t_+;X_\mu)$.*
In particular, the solution exists globally ($t_+ = \mathcal{T}$) if $u \in L_p(0,t_+;X_\mu)$.

Proof. We have already mentioned (i) in the beginning of this section. We consider (ii) that the integral a priori bound implies the global solution. Suppose $t_+ < \mathcal{T}$ and let $a_0 \in (0, t_+)$ be fixed. Note that

$$|F_2(u)|_{\mathbb{E}_{0,\mu}(a_0,a)} \leq C \sum_{j=1}^m \kappa_{r''}(a_0,a) \left(\kappa_{r'}(a_0,a) + \|u\|^{\rho_j}_{L_{\rho_j pr',\sigma'}(a_0,a;X_\beta)} \right) \|u\|_{L_{pr,\sigma}(a_0,a;X_{\beta_j})},$$

with

$$\kappa_\ell(a_0,a) := (\int_{a_0}^a (t-a_0)^{(1-\mu)p} dt)^{1/\ell} = \left(\frac{(a-a_0)^{(1-\mu)p+1}}{(1-\mu)p+1} \right)^{1/\ell} \quad (\to 0 \text{ as } a_0 \to a) \quad (\ell = r', r'')$$

for all $a \in (a_0, t_+)$. The interpolation (6.4.2) implies

$$|F_2(u)|_{\mathbb{E}_{0,\mu}(a_0,a)} \leq C \sum_{j=1}^m \kappa_{r''}(a_0,a) \left(\kappa_{r'}(a_0,a) \|u\|_{L_{pr,\sigma}(a_0,a;X_{\beta_j})} + \|u\|^{\rho_j}_{L_p(a_0,a;X_\mu)} \|u\|_{\mathbb{E}_{1,\mu}(a_0,a)} \right),$$

where the constant C is independent of a_0 and $a \in (a_0, t_+)$. Let M be the supremum of the constant of maximal regularity of $A(s)$ for the interval $[0, t_+)$ which is larger than other maximal regularity constant for sub-interval of $[0, t_+)$. Let $t_0 \in (a_0, t_+)$ with $u|_{t_0} \in X_1$ to be fixed later. We use the estimate

$$\begin{aligned}
\|u\|_{L_{pr,\sigma}(t_0,a;X_{\beta_j})} &\leq \|u - u|_{t_0}\|_{L_{pr,\sigma}(t_0,a;X_{\beta_j})} + \|u|_{t_0}\|_{L_{pr,\sigma}(t_0,a;X_{\beta_j})} \\
&\leq C\|u - u|_{t_0}\|_{\mathbb{E}_{1,\mu}(t_0,a)} + \|u|_{t_0}\|_{L_{pr,\sigma}(t_0,a;X_{\beta_j})} \\
&\leq C\|u\|_{\mathbb{E}_{1,\mu}(t_0,a)} + C_{u|_{t_0}}(t_0,a)
\end{aligned}$$

where the constant C is independent of t_0, a and the constant $C_{u|_{t_0}}(t_0,a) \to 0$ as $t_0 \to a$, and the other estimate

$$\begin{aligned}
&\|F_1(t,u)\|_{\mathbb{E}_{0,\mu}(t_0,a)} \\
\leq &\|F_1(t,u) - F_1(t,u_0)\|_{\mathbb{E}_{0,\mu}(t_0,a)} + \|F_1(t,u_0)\|_{\mathbb{E}_{0,\mu}(t_0,a)} \\
\leq &\kappa_p(t_0,a)\|u - u_0\|_{L_\infty(t_0,a;X_{\gamma,\mu})} + \|F_1(t,u_0)\|_{\mathbb{E}_{0,\mu}(t_0,a)} \\
\leq &\kappa_p(t_0,a)(C\|u - u|_{t_0}\|_{\mathbb{E}_{1,\mu}(t_0,a)} + \|u|_{t_0} - u_0\|_{L_\infty(t_0,a;X_{\gamma,\mu})}) + \|F_1(t,u_0)\|_{\mathbb{E}_{0,\mu}(t_0,a)} \\
\leq &C\kappa_p(t_0,a)\|u\|_{\mathbb{E}_{1,\mu}(t_0,a)} + C_{u|_{t_0},u_0,F_1}(t_0,a)
\end{aligned}$$

where the constant C is independent of t_0, a and the constants $\kappa_p(t_0,a), C_{u|_{t_0},u_0,F_1}(t_0,a) \to 0$ as $t_0 \to a$. To treat $A(t)$ we estimate as follows:

$$\|(A(t_0) - A(t))u\|_{\mathbb{E}_{0,\mu}(t_0,a)} \leq \sup_{t\in[t_0,a]} \|A(t_0) - A(t)\|_{\mathcal{B}(X_1,X_0)} \|u\|_{\mathbb{E}_{1,\mu}(t_0,a)}$$

Here $\sup_{t\in[t_0,a]} \|A(t_0) - A(t)\|_{\mathcal{B}(X_1,X_0)} \to 0$ as $t_0 \to a$.

Combining all estimate, by the maximal regularity of $A(t_0)$,

$$\begin{aligned}
&\|u\|_{\mathbb{E}_{1,\mu}(t_0,a)} \\
&\leq M \left(|u|_{t_0}|_{X_{\gamma,\mu}} + \|F_1(t,u)\|_{\mathbb{E}_{0,\mu}(t_0,a)} + \|F_2(u)\|_{\mathbb{E}_{0,\mu}(t_0,a)} + \|(A(t_0) - A(t))u\|_{\mathbb{E}_{0,\mu}(t_0,a)} \right)
\end{aligned}$$

$$\leq M(|u|_{t_0}|_{X_{\gamma,\mu}} + K_1(t_0,a)\|u\|_{\mathbb{E}_{1,\mu}(t_0,a)} + K_2(t_0,a))$$

with $K_1(t_0,a), K_2(t_0,a) \to 0$ as $t_0 \to a$. Remark that the term $\|u\|_{L_p(t_0,a;X_\mu)}$ also goes to zero as $t_0 \to a$. Take t_0 sufficiently close to t_+ and $a \in (t_0,t_+)$. such that $K_1(t_0,a) < 1/(2M)$. Then we have

$$\begin{aligned}\|u\|_{\mathbb{E}_{1,\mu}(t_0,a)} &\leq 2M(|u|_{t_0}|_{X_{\gamma,\mu}} + K_2(t_0,a))\\ &\leq 2M(|u|_{t_0}|_{X_{\gamma,\mu}} + \widetilde{K_2}(t_0,\mathcal{T}))\ \text{(independent of } a)\end{aligned}$$

Therefore $u \in \mathbb{E}_{1,\mu}(t_0,t_+)$, which does not blow-up at $t = t_+$ This contradicts that t_+ is maximal time interval. □

Remark 6.7. The assumption (6.4.3) is satisfied when $\mu = \mu_c$, $\beta_j = \beta$ and $\max_j \rho_j < p$. This case with $m = 1$, $\rho_1 = 1$ is the bi-linear non-linearities $(F_2 =)G : X_\beta \times X_\beta \to X_0$ in [12].

Remark 6.8. We are able to generalize this global well-posedness for the quasilinear parabolic equations (QL) if

$$\|(A_0 - A(t,u))u\|_{\mathbb{E}_{0,\mu}(0,T)} \leq C(1 + \|u\|_{L_p(0,T;X_\mu)})\|u\|_{\mathbb{E}_{1,\mu}(0,T)}$$

holds for some $C > 0$ which is independent of T and u.

6.5 Example

In this section we apply the local well-posedness for the quasi-linear heat equation. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. Let $\kappa > 2$ and a function $F : [0,\mathcal{T}) \times \mathbb{R} \to \mathbb{R}$ of Caratheodory type. A continuous function $a : [0,\mathcal{T}) \times \mathbb{R} \to \mathbb{R}$ satisfies that there exists $a_0 > 0$ such that $a(t,s) \geq a_0 > 0$ for all t, s. We consider the following quasi-linear heat equation.

$$\text{(QH)}\begin{cases} u' - a(t,u)\Delta u = F(t,u) + |\nabla u|^\kappa & \text{in } (0,\mathcal{T}) \times \Omega,\\ \partial_\nu u = 0 & \text{on } (0,\mathcal{T}) \times \partial\Omega,\\ u(0) = u_0. \end{cases}$$

Let $1 < p, q < \infty$. We set $X_0 := L_q(\Omega), X_1 := \{u \in H_q^2(\Omega) \mid \partial_\nu u|_{\partial\Omega} = 0\}$ and $F_\kappa(u) := |\nabla u|^\kappa$. Then,

$$\begin{aligned} X_\beta := [X_0, X_1]_\beta &= \begin{cases} H_q^{2\beta}(\Omega) & \text{if } 2\beta < 1 + 1/q,\\ \{u \in H_q^{2\beta}(\Omega) \mid \partial_\nu u|_{\partial\Omega} = 0\} & \text{if } 2\beta > 1 + 1/q \end{cases}\\ &\hookrightarrow H^1_{\kappa q}(\Omega) \text{ if } \beta = \frac{1}{2} + \frac{d}{2q}\left(1 - \frac{1}{\kappa}\right). \end{aligned}$$

Moreover we have the assumption (H2) as follows:

$$\begin{gathered} |F_\kappa(u)|_{X_0} \leq C|\nabla u|^\kappa_{\kappa q} \leq C|u|^\kappa_{X_\beta},\\ |F_\kappa(u) - F_\kappa(v)|_{X_0} \leq C(|u|^{\kappa-1}_{X_\beta} + |v|^{\kappa-1}_{X_\beta})|u - v|_{X_\beta}. \end{gathered}$$

The critical weight μ_c and the critical space X_{γ,μ_c} are determined by

$$\mu_c := \frac{1}{p} + \frac{d}{2q} + \frac{\kappa-2}{2(\kappa-1)},$$
$$X_{\gamma,\mu_c} =: {}_\nu B_{q,p}^{\frac{d}{q}+\frac{\kappa-2}{\kappa-1}}(\Omega) = \begin{cases} B_{q,p}^{\frac{d}{q}+\frac{\kappa-2}{\kappa-1}}(\Omega) & \text{if } \frac{d}{q}+\frac{\kappa-2}{\kappa-1} < 1+\frac{1}{q}, \\ \{u \in B_{q,p}^{\frac{d}{q}+\frac{\kappa-2}{\kappa-1}}(\Omega) \mid \partial_\nu u|_{\partial\Omega} = 0\} & \text{if } \frac{d}{q}+\frac{\kappa-2}{\kappa-1} > 1+\frac{1}{q}. \end{cases}$$

The condition $1/p < \mu_c$ holds from $\kappa > 2$ and the conditions $\beta < 1$ and $\mu_c \le 1$ are

$$\frac{2}{p} + \frac{d}{q} \le \frac{\kappa}{\kappa-1}. \tag{6.5.1}$$

The critical space X_{γ,μ_c} is compactly embedded into $C(\overline{\Omega})$ for all $p, q \in (1,\infty)$, $\kappa > 2$. Therefore the operator A defined by $A(t,w)u := -a(t,w)\Delta u$ satisfies (H0) and the maximal L_p regularity on some interval, see [3]. Assume that $a(t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ for all $t \in [0,\mathcal{T})$, whose Lipschitz constant is uniform in t and $F(t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ for a.e. $t \in (0,\mathcal{T})$, whose Lipschitz constant is a $L_{p,\mu_c}(0,\mathcal{T})$ function. Then the function A and F satisfy (H0) and (H1) since

$$a(t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(L_\infty(\Omega); L_\infty(\Omega)),$$
$$F(t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(L_\infty(\Omega); L_\infty(\Omega)) \subset \mathrm{Lip}_{\mathrm{loc}}(X_{\gamma,\mu_c}; X_0).$$

The local well-posedness theorem is as follows.

Theorem 6.9. *Let $\kappa > 2$, $p, q \in (1,\infty)$ satisfy* (6.5.1). *Let a and F be a continuous function and $a(t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ for all $t \in [0,\mathcal{T})$, whose Lipschitz constant is uniform in t and $a(t,s) \ge a_0 > 0$ for some a_0, and $F(t,\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ for a.e. $t \in (0,\mathcal{T})$, whose Lipschitz constant is a $L_{p,\mu_c}(0,\mathcal{T})$ function. Then for any $u_0 \in {}_\nu B_{q,p}^{\frac{d}{q}+\frac{\kappa-2}{\kappa-1}}(\Omega)$, there exists $T \in (0,\mathcal{T}]$ and the unique solution $u \in \mathbb{E}_{1,\mu_c}(0,T)$ for the quasi-linear heat equation* (QH). *The solution has continuity for the initial data. Moreover the solution can be extended to the maximal time interval t_+ and belongs to the class of*

$$u \in C([0,t_+); {}_\nu B_{q,p}^{\frac{d}{q}+\frac{\kappa-2}{\kappa-1}}(\Omega)) \cap C((0,t_+); {}_\nu B_{q,p}^{2(1-1/p)}(\Omega)).$$

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