博士論文 （要約）

論文題目　Relationship between orbit decompositions on
flag varieties and multiplicities of induced representations
(旗多様体上の軌道分解と誘導表現の重複度の関係性について)

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1 Introduction

Let Q be a parabolic subgroup of a real reductive Lie group G and H a closed subgroup of G. The aim of this thesis is to study the relationship between the orbit decompositions on flag varieties G/Q with respect to H-action and the behavior of the multiplicities of the induced representations of G from Q-representations in the regular representations on G/H.

1.1 Three conditions (i_Q), (ii_Q) and (iii_Q)

In the field of global analysis on homogeneous spaces, a rich theory has been developed by the I. M. Gelfand school and Harish-Chandra for group manifolds, by S. Helgason for Riemannian symmetric spaces, and in the framework of semisimple symmetric spaces by Flensted-Jensen, T. Oshima and P. Delorme among others. In the late 80s, T. Kobayashi raised a problem on what is the "most general framework" in which we could expect reasonable and detailed analysis of function spaces on G/H. As a solution to this problem, Kobayashi and Oshima established a finiteness criterion for multiplicities of the regular representation on a homogeneous space G/H.

Fact 1.1.1 ([23, Theorem A])**.** Suppose that G and H are defined algebraically over $\mathbb{R}$. Then the following two conditions on the pair (G, H) are equivalent:

(i) $\dim \mathrm{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$ for any $(\pi, \tau) \in \hat{G}_{\mathrm{smooth}} \times \hat{H}_{\mathrm{alg}}$,

(ii) G/H is real spherical.

Here $\hat{G}_{\mathrm{smooth}}$ denotes the set of equivalence classes of irreducible smooth admissible Fréchet representations of G with moderate growth, and $\hat{H}_{\mathrm{alg}}$ that of algebraic irreducible finite-dimensional representations of H. Given $\tau \in \hat{H}_{\mathrm{alg}}$, we write $C^\infty(G/H, \tau)$ for the Fréchet space of smooth sections of the G-homogeneous vector bundle over G/H associated to τ. The terminology *real sphericity* was introduced by Kobayashi [19] in his study of a broader framework for global analysis on homogeneous spaces than the usual (e.g., semisimple symmetric spaces).

Definition 1.1.2. A homogeneous space G/H is *real spherical* if a minimal parabolic subgroup P of G has an open orbit on G/H.

As we have seen, real spherical homogeneous spaces are one important class of homogeneous spaces in the sense that the all multiplicities of the regular representation on them are finite. The theory of real spherical homogeneous spaces was actively studied in the last decade and applied to the theory of blanching laws of infinite-dimensional representations. For example, Kobayashi and T. Matsuki classified the pairs (G, H) such that $G \times H/diag(H)$ is real spherical in [22], and symmetry breaking operators were classified in [25, 26] for the first time by Kobayashi and B. Speh in the framework of real spherical homogeneous spaces.

The following is one of the characterizations of real spherical homogeneous spaces. This is a consequence of the rank one reduction of Matsuki [29] and the classification of real spherical homogeneous spaces of real rank one by B. Kimelfeld [17].

Fact 1.1.3 ([3])**.** For the pair (G, H), the following two conditions are equivalent:

(ii) G/H is real spherical,

(iii) $\#(H\backslash G/P) < \infty$.

Therefore, for a minimal parabolic subgroup P, the three conditions (i), (ii), and (iii) are equivalent by Facts 1.1.1 and 1.1.3 (see Figure 1.1 below). We recall that the regular representation $L^2(G/H)$ may be decomposed into irreducible tempered representations when H is "small", but may need more singular representations such as unitarily induced representations from general parabolic subgroups when H is "large", see [2] for the precise criterion. Thus we ask a question what will happen to the relationship among the three conditions, if we replace P by a general parabolic subgroup Q of G. This is one of the main interests of this thesis. There is an obvious extension of the conditions (ii) and (iii) to a general parabolic subgroup Q (see Definition 1.1.5 below). In order to formulate a variant of (i) for a parabolic subgroup Q of G, we review the notion of Q-series.

Definition 1.1.4 ([20, Def. 6.6])**.** Let $\pi \in \hat{G}_{\mathrm{smooth}}$. We say that π belongs to *Q-series* if π occurs as a subquotient of the degenerate principal series representation $C^\infty(G/Q, \tau)$ for some $\tau \in \hat{Q}_{\mathrm{f}}$.

Here $\hat{Q}_{\mathrm{f}}$ is the set of equivalence classes of irreducible finite dimensional representations of Q. Set $\hat{G}^Q_{\mathrm{smooth}} := \{\pi \in \hat{G}_{\mathrm{smooth}} \mid \pi \text{ belongs to } Q\text{-series}\}$. Obviously, $\hat{G}^Q_{\mathrm{smooth}} \supset \hat{G}^{Q'}_{\mathrm{smooth}}$ if $Q \subset Q'$. Moreover, $\hat{G}^Q_{\mathrm{smooth}}$ is equal to $\hat{G}_{\mathrm{smooth}}$ if $Q = P$ (minimal parabolic) by Harish-Chandra's subquotient theorem [9] and to $\hat{G}_{\mathrm{f}}$ if $Q = G$.

Definition 1.1.5. For a parabolic subgroup Q of G, we define three conditions (i_Q), (ii_Q), and (iii_Q) as follows:

(i_Q) $\dim \mathrm{Hom}_G(\pi, C^\infty(G/H, \tau)) < \infty$ for all $(\pi, \tau) \in \hat{G}^Q_{\mathrm{smooth}} \times \hat{H}_{\mathrm{alg}}$, (ii$_Q$) Q has an open orbit on G/H,

(iii$_Q$) $\#(H\backslash G/Q) < \infty$.

Then we consider the following problem.

Question. Determine the relationship among the three conditions (i$_Q$), (ii$_Q$) and (iii$_Q$).

The conditions (i$_Q$), (ii$_Q$), and (iii$_Q$) reduce to (i), (ii), and (iii), respectively, if $Q = P$ (minimal parabolic), and we have seen in Facts 1.1.1 and 1.1.3 that the following equivalences hold for $Q = P$,

$$(\mathrm{i}_P) \iff (\mathrm{ii}_P) \iff (\mathrm{iii}_P).$$

Furthermore, if $Q = G$, the condition (i$_Q$) automatically holds by the Frobenius reciprocity, while (ii$_Q$) and (iii$_Q$) are obvious. Hence

$$(\mathrm{i}_G) \iff (\mathrm{ii}_G) \iff (\mathrm{iii}_G).$$

For a general parabolic subgroup Q, clearly, (iii$_Q$) implies (ii$_Q$). However there is an easy counterexample for the converse statement.

Example 1.1.6. The projective space $\mathbb{RP}^2 = SL(3,\mathbb{R})/Q$ splits into an open orbit and continuously many fixed points of the unipotent radical H of Q.

On the other hand, the implication (i$_Q$) $\Rightarrow$ (ii$_Q$) holds. To see this, we define a subset $\hat{H}_{\mathrm{f}}(G)$ of $\hat{H}_{\mathrm{f}}$ by

$$\hat{H}_{\mathrm{f}}(G) := \{\tau \in \hat{H}_{\mathrm{f}} \mid \tau \text{ appears as a quotient of some element of } \hat{G}_{\mathrm{f}}\}.$$

The implication (i$_Q$) $\Rightarrow$ (ii$_Q$) is derived from the following stronger assertion.

Fact 1.1.7 ([20, Cor. 6.8])**.** If there exists $\tau \in \hat{H}_{\mathrm{f}}(G)$ such that for all $\pi \in \hat{G}^{Q}_{\mathrm{smooth}}$, $\dim \mathrm{Hom}_G(\pi, C^\infty(G/H,\tau)) < \infty$, then Q has an open orbit on G/Q, namely, (ii$_Q$) holds.

We summarize the known relationship among the three conditions as follows.

P : minimal parabolic

$$\begin{tikzcd}
 & (\mathrm{i}) \arrow[dl, Leftrightarrow, "\text{Fact 1.1.1}"'] \arrow[dr, Leftrightarrow] & \\
(\mathrm{ii}) \arrow[rr, Leftrightarrow, "\text{Fact 1.1.3}"'] & & (\mathrm{iii})
\end{tikzcd}$$

Figure 1.1

Q : general parabolic

$$\begin{tikzcd}
 & (\mathrm{i_Q}) \arrow[dl, Rightarrow, bend left, "\text{Fact 1.1.7}"] \\
(\mathrm{ii_Q}) \arrow[ur, Rightarrow, "\text{No}"] \arrow[r, Rightarrow, "\text{No}"'] & (\mathrm{iii_Q}) \arrow[l, Rightarrow, bend right]
\end{tikzcd}$$

Figure 1.2

1.2 Main Theorems on the relationship between the conditions (i_Q) and (iii_Q)

Figure 1.2 indicates that the relationship between the conditions (i_Q) and (iii_Q) is unsettled for a general parabolic Q of G. In this thesis, we give a partial answer for this, namely, we prove theorems below.

Theorem 1.2.1. Let Q be a maximal parabolic subgroup of $G = SL(2n, \mathbb{R})$ such that G/Q is isomorphic to the real projective space $\mathbb{RP}^{2n-1}$. Then if $n \geq 2$, there exists an algebraic subgroup H of G satisfying the following two conditions:

1) $\#(H\backslash G/Q) < \infty$,

2) $\dim \mathrm{Hom}_G(C^\infty(G/Q, \chi), C^\infty(G/H)) = \infty$ for some class-one character χ of Q.

Furthermore, if $n \geq 3$, H satisfies the following condition:

2') $\dim \mathrm{Hom}_G(C^\infty(G/Q, \chi), C^\infty(G/H)) = \infty$ for any class-one character χ of Q.

Here a one-dimensional representation χ of Q is referred to as a class-one character if χ is trivial on $Q \cap K$, where K is a maximal compact subgroup. We note that χ is of class one if and only if $C^\infty(G/Q, \chi)$ has a nonzero K-fixed vector.

Theorem 1.2.2. Let G be a real reductive algebraic group, H a real algebraic subgroup and Q a parabolic subgroup of G.

(1) If the number of *orientable* p-dimensional H-orbits on G/Q is infinite, then for $0 \leq p < \dim G/Q$,

$$\dim \mathrm{Hom}_G(C^\infty(G/Q, \wedge^p(\mathfrak{g}/\mathfrak{q})^\vee), C^\infty(G/H)) = \infty.$$

(2) If the number of *transverse orientable* p-dimensional H-orbits on G/Q is infinite, then for $0 \leq p < \dim G/Q$,

$$\dim \mathrm{Hom}_G(C^\infty(G/Q, \wedge^p(\mathfrak{g}/\mathfrak{q})^\vee \otimes or), C^\infty(G/H)) = \infty.$$

Theorem 1.2.1 implies that (iii_Q) $\Rightarrow$ (i_Q) does not hold. Theorem 1.2.2 gives an affirmative answer under certain mild condition on orientation for the implication (i_Q) $\Rightarrow$ (iii_Q). Figures below summarize the relationship among the three conditions. In Figure 1.3, the symbol Δ on the arrow means that the implication is proved under an additional assumption of orientation.

P : minimal parabolic

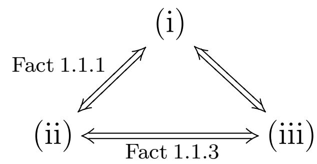

Figure 1.1

Q : general parabolic

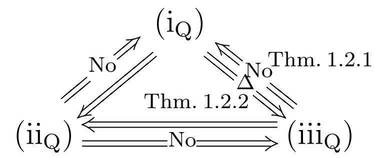

Figure 1.3

1.3 Uniform boundedness of the multiplicities for induced representations

Let $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$ be complexifications of G and H, respectively. By finding upper and lower estimates of the dimensions of $\mathrm{Hom}_G(\pi, C^\infty(G/H,\tau))$, Kobayashi and Oshima also established the criterion for the uniform boundedness of the multiplicities of induced representations.

Fact 1.3.1. Suppose that G and H are defined algebraically over $\mathbb{R}$. Then the following two conditions on the pair (G, H) are equivalent:

(i) $\displaystyle \sup_{\tau\in\hat{H}_{\mathrm{alg}}} \sup_{\pi\in\hat{G}_{\mathrm{smooth}}} \frac{1}{\dim\tau} \dim \mathrm{Hom}_G(\pi, C^\infty(G/H,\tau)) < \infty,$

(ii) $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.

Here a homogeneous space $G_{\mathbb{C}}/H_{\mathbb{C}}$ is called spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit on $G_{\mathbb{C}}/H_{\mathbb{C}}$.

Remark 1.3.2. $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if and only if $\#(H_{\mathbb{C}}\backslash G_{\mathbb{C}}/B) < \infty$ holds [6, 29, 33]. This is a special case of Fact 1.1.3.

1.4 Main Theorems on uniform boundedness of the multiplicities

Fact 1.3.1 implies that the finiteness of the number of $H_{\mathbb{C}}$-orbits on $G_{\mathbb{C}}/B$ characterizes the uniformly bounded multiplicity property of the regular representation on G/H. Motivated by this result, we prove the following uniformly bounded multiplicity property of Q-series representations in the regular representations on G/H.

Theorem 1.4.1. Let Q be a parabolic subgroup of a real reductive Lie group G and H a closed subgroup of G. Suppose $\#(H_{\mathbb{C}}\backslash G_{\mathbb{C}}/Q_{\mathbb{C}}) < \infty$. Then we have

$$\sup_{(\eta,\tau)\in\hat{Q}_{\mathrm{f}}\times\hat{H}_{\mathrm{f}}} \frac{1}{\dim\eta\cdot\dim\tau} \dim \mathrm{Hom}_G(C^\infty(G/Q,\eta), C^\infty(G/H,\tau)) < \infty.$$

For the proof of this theorem, we also prove the following uniform boundedness of the dimensions of the spaces of group invariant hyperfunctions. We write $\mathcal{B}_M$ for the sheaf of Sato's hyperfunctions on a real analytic manifold M.

Theorem 1.4.2. Let M be a real analytic manifold, X a complexification of M and U a relatively compact open semianalytic set of M. Suppose that a complex Lie group $H_{\mathbb{C}}$ acts on X with $\#(H_{\mathbb{C}}\backslash X) < \infty$. Then there exists $C > 0$ such that for any $\tau \in \hat{\mathfrak{h}}_{\mathrm{f}}$, the following inequality holds:

$$\dim(\mathcal{B}_M(U) \otimes \tau)^{\mathfrak{h}} < C \cdot \dim \tau.$$

Here we say that a complex manifold X is a complexification of a real analytic manifold M if X contains M and the tangent space T_xX at x is equal to $T_xM \oplus \sqrt{-1}T_xM$ for any $x \in M \subset X$. We give another proof of (b)⇒(a) in Fact 1.3.1 as a corollary of Theorem 1.4.2. A. Aizenbud, D. Gourevitch and A. Minchenko gave another proof of a weaker version of (b)⇒(a) in Fact 1.3.1 using the theory of $\mathcal{D}$-modules and the universality of the Weil representation in [1]. This is different from the proof of this thesis which uses the theory of the multiplicities of $\mathcal{D}$-modules.

As we have seen above, the orbit decomposition of H on G/P and its complexification have information of harmonic analysis on G/H. In particular, the finiteness of the number of H-orbits on G/P (resp. $H_{\mathbb{C}}$-orbits on $G_{\mathbb{C}}/B$) characterizes the finite (resp. bounded) multiplicity property of the regular representations on G/H. Moreover $\#(H_{\mathbb{C}}\backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$ also implies the bounded multiplicity property. Therefore we want to know how the multiplicities behave in the intermediate case, namely, the case that $\#(H\backslash G/P) < \infty$ although $\#(H_{\mathbb{C}}\backslash G_{\mathbb{C}}/P_{\mathbb{C}}) = \infty$. Thus we are interested in finding such subgroups H of G. The following proposition gives examples.

Proposition 1.4.3. *Let G be a real reductive Lie group without compact factors, Q a parabolic subgroup of G and $Q = MAN$ its Langlands decomposition. Then the following four conditions on Q are equivalent:*

(i) $\mathfrak{m}$ *is abelian,*

(ii) $\#(A_{\mathbb{C}}N_{\mathbb{C}}\backslash G_{\mathbb{C}}/Q_{\mathbb{C}}) < \infty$*,*

(iii) $G_{\mathbb{C}}/N_{\mathbb{C}}$ *is spherical,*

(iv) $G_{\mathbb{C}}/A_{\mathbb{C}}N_{\mathbb{C}}$ *is spherical.*

Although this is known by experts, we give two proofs of Proposition 1.4.3. One is a representation theoretic proof and the other is a geometric proof.

For any real reductive Lie group G, we have $\#(AN\backslash G/P) < \infty$ by the Bruhat decomposition, where P is a minimal parabolic subgroup of G (for example, see [18, Thm. 7.40]). Therefore Proposition 1.4.3 implies that the number of AN-orbits on G/P is finite but the number of $A_{\mathbb{C}}N_{\mathbb{C}}$-orbits on $G_{\mathbb{C}}/P_{\mathbb{C}}$ is infinite if G is not quasi-split, namely, $\mathfrak{m}$ is not abelian.

Remark 1.4.4. Let G be the special indefinite unitary group $SU(1,n)$. Then it is pointed out that $\#(N_{\mathbb{C}}\backslash G_{\mathbb{C}}/P_{\mathbb{C}}) = \infty$ holds if $n \geq 3$ by Matsuki [29, Remark 7].

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