博士論文

論文題目　A combinatorial study on K-theoretic k-Schur functions and stable Grothendieck polynomials

(K-理論的 k-シューア関数と安定グロタンディーク多項式に関する組合せ論的研究)

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Preface

This thesis consists of four chapters. In Chapter 1 and 2 we study a family of symmetric functions $\{g_\lambda^{(k)}\}$ called *K-theoretic k-Schur functions* (or *K-k-Schur functions* for short), which is a K-theoretic and affine deformation of another family of symmetric functions called *Schur functions*. Although the K-k-Schur functions were introduced in the context of the geometry of affine Grassmannians, our focus is their combinatorial treatment and the proofs of the results are based on purely combinatorial arguments related to some fine structures on the affine symmetric groups.

In Chapter 3 and 4 of the thesis we return to the non-affine case; the *dual stable Grothendieck polynomials* g_λ are a K-theoretic deformation of Schur functions and can be considered as a certain limit of K-k-Schur functions. We give some properties of these functions that are absent in the affine case, related to the Hopf algebra structure of the ring of symmetric functions.

History

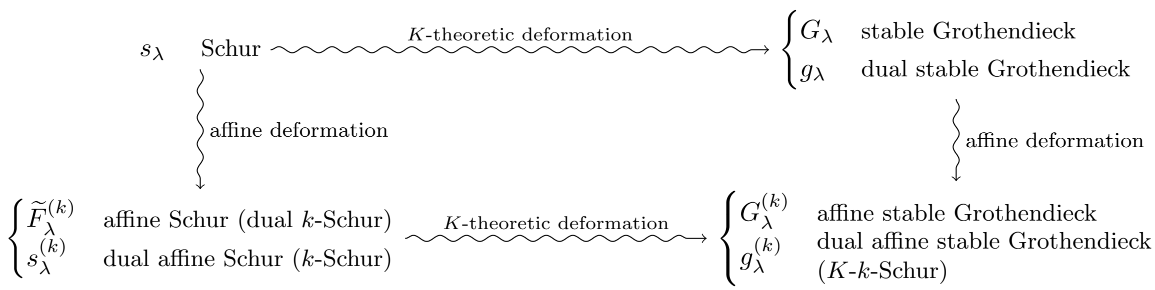
The Schur functions s_λ are a family of symmetric functions parametrized by integer partitions and among the most important objects in algebraic combinatorics; while written as weight generating functions of semi-standard tableaux and possessing many combinatorial properties such as the Pieri formula and the Littlewood–Richardson rule, they simultaneously represent the irreducible characters of the symmetric groups, the irreducible characters of the general linear groups, and the cohomology Schubert class in the Grassmannians (of type A). There has been a trend to generalize this classical setting, and we focus on two (independent) directions generalizing the connection between the Schur functions and geometry of Grassmannians: namely, (1) to consider K-theoretic analogue, i.e. to consider K-theory instead of cohomology, and (2) to consider affine analogue, i.e. to consider affine Grassmannians instead of Grassmannians.

The K-theoretic analogue of the Schur functions is the *stable Grothendieck polynomials* G_λ introduced by Fomin–Kirillov [FK96], which are symmetric power series obtained as the stable limit of the Grothendieck polynomials of Lascoux–Schützenberger [LS82]. These G_λ represent the K-cohomology classes of structure sheaves of Schubert varieties of the Grassmannians, and combinatorially they are an inhomogeneous deformation of s_λ, in that G_λ is a symmetric power series whose lowest degree part is s_λ. In [Buc02], Buch gave a formula for G_λ as signed weight generating functions of *set-valued tableaux*. Unlike the Schur functions, these G_λ are not self-dual under the Hall inner product (that is, the inner product $(\,,\,)$ on the ring of symmetric functions that satisfies $(h_\lambda, m_\mu) = \delta_{\lambda\mu}$ where h_λ and m_μ are the complete and monomial symmetric functions); their dual basis g_λ, first explicitly introduced in Lam–Pylyavskyy [LP07] and called the *dual stable Grothendieck polynomials*, are another inhomogeneous deformation of the Schur functions s_λ whose *highest* degree term are s_λ, and they represents K-homology classes of ideal sheaves of boundaries of Schubert varieties. Via duality, the generating function formula for G_λ using set-valued tableaux translates to the Pieri formula for g_λ using set-valued strips, giving a characterization of g_λ.

The affine analogue of the classical settings is another topic that has recently been studied intensively. Combinatorially the role of Schur functions is played by *k-Schur functions* $s_\lambda^{(k)}$ and their dual, *dual k-Schur functions* (or *affine Schur functions*) $\widetilde{F}_\lambda^{(k)}$. Historically k-Schur functions were first introduced by Lascoux, Lapointe and Morse [LLM03] in their study on Macdonald polynomial positivity which was seemingly unrelated to Schubert calculus, and subsequent studies (see [Lam06, LM05, LM07] for example) revealed that the combinatorial backbone of k-Schur theory lies in the setting of type A affine Weyl groups (i.e. affine symmetric groups) and led to several (conjecturally equivalent) characterizations of these functions: notably, $\widetilde{F}_\lambda^{(k)}$ are generating functions of *affine tableaux* (also known as *weak tableaux* or *k-tableaux* in literature), which is an affine-type deformation of semistandard tableaux, and dually, $s_\lambda^{(k)}$ can be characterized by the Pieri rule using *affine strips* (or *weak strips*). The role of these functions in affine Schubert calculus is established in a paper of Lam [Lam08] in which he proved that k-Schur (resp. dual k-Schur) functions correspond to the Schubert basis of homology (resp. cohomology) of the affine Grassmannian of type $A_k^{(1)}$. Moreover, Lam and Shimozono [LS12] showed that k-Schur functions play a central role in the explicit description of the Peterson isomorphism.

These developments have analogue in K-theory. Lam, Schilling and Shimozono [LSS10] established the identification between the K-cohomology classes of the structure sheaves of the Schubert varieties in the affine Grassmannian and a family of symmetric power series called *affine stable Grothendieck polynomials* $G_\lambda^{(k)}$. They also defined the *K-k-Schur functions* (or *dual affine stable Grothendieck polynomials*) $g_\lambda^{(k)}$ as the dual basis to $G_\lambda^{(k)}$ via the Hall inner product. The K-k-Schur functions form a basis of K-homology of the affine Grassmannians, representing the class of ideal sheaves of boundaries of the Schubert varieties. Combinatorially $G_\lambda^{(k)}$ are written as generating functions of *affine set-valued tableaux* [Mor12], which in some sense unifies the notions of affine tableaux and set-valued tableaux. We remark that generating function formulas for g_λ and $s_\lambda^{(k)}$ are given in [LP07] and [LLMS10], using *reverse plane partitions* and *strong marked tableaux*, respectively. The K-k-Schur functions $g_\lambda^{(k)}$, however, are currently only defined via duality (i.e. by Pieri rules using affine set-valued strips) and no combinatorial formula (i.e. as generating functions) is known.

These situations can be depicted as below:



Results in this thesis

First, in Chapter 1 and 2, we consider the K-theoretic affine setting. In Chapter 1, we propose to consider a new basis consisting of the sums of K-k-Schur functions $\sum_{\mu \leq \lambda} g_\mu^{(k)}$ and denote these sums by $\widetilde{g}_\lambda^{(k)}$, where $\leq$ comes from the strong (Bruhat) order on the affine symmetric groups, and prove properties of this basis such as

- the Pieri rule, i.e. a formula for the product $\widetilde{g}_{(a)}^{(k)}\widetilde{g}_\lambda^{(k)}$, and
- a k-rectangle factorization formula: $\widetilde{g}_{R_t\cup\lambda}^{(k)} = \widetilde{g}_{R_t}^{(k)}\widetilde{g}_\lambda^{(k)}$ where $R_t = (t^{k+1-t})$.

Notably, the k-rectangle factorization formula for $\widetilde{g}_\lambda^{(k)}$ holds in the same simple form as that for k-Schur functions $s_{R_t\cup\lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)}$, while that for the single K-k-Schur functions $g_\lambda^{(k)}$ does not (see (1.1.3)).

The Pieri rule for $\widetilde{g}_\lambda^{(k)}$ also has a simple form compared to that for the original K-k-Schur functions $g_\lambda^{(k)}$, in that the product $\widetilde{g}_{(a)}^{(k)}\widetilde{g}_\lambda^{(k)}$ is written as both a multiplicity-free sum over the order ideal generated by the leading terms (that is, the terms appearing in the highest-degree part), when expanded with the basis $g_\mu^{(k)}$; and an alternating sum of meets of the leading terms, when expanded with the basis $\widetilde{g}_\mu^{(k)}$.

The factorization formula for $\widetilde{g}_\lambda^{(k)}$ is easily deduced from the Pieri formula, as discussed in Section 1.6. The proof of the Pieri formula (in Section 1.5) is based on some properties on the strong and weak orderings of (arbitrary) Coxeter groups and arguments on some fine structures of the affine symmetric groups, developed in the first half (Section 1.3 and 1.4) of Chapter 1. Notably, in Section 1.3, for any Coxeter group W and any $x, y \in W$, we prove the existence of the elements

$$\min_{\leq}\{z \in W \mid x \leq z \geq_L y\} \qquad \text{and} \qquad \max_{\leq}\{z \in W \mid x \geq_L z \leq y\},$$

analogous to the join and meet. Here $\leq$ and $\leq_L$ denote the strong order and (left) weak order of W. Besides these elements are explicitly written by using the anti-Demazure action explained in the preliminary section (Section 1.2). In Section 1.4 we focus on the affine symmetric groups and prove some properties on the poset structure of the weak strips (over a fixed partition). A technical feature of this section is the use of *k-codes*, a powerful tool to treat the affine symmetric groups.

In Chapter 2 we provide alternative (and shorter) proofs of the Pieri rule and the k-rectangle factorization formula for $\widetilde{g}_\lambda^{(k)}$, by showing corresponding formulas for *non-commutative K-k-Schur functions*, which are the correspondents of K-k-Schur functions realized in a certain commutative subring (called the *K-affine Fomin–Stanley algebra*) of a certain non-commutative ring called the 0-*Hecke algebra*. The arguments in Chapter 2 uses the results in the first half (Section 1.3 and 1.4) of Chapter 1, and can replace the arguments in the latter half (Section 1.5 and 1.6) of Chapter 1.

Next, in Chapter 3 and 4, we consider the non-affine case. Since for any λ it holds $g_\lambda^{(k)} = g_\lambda$ and $G_\lambda^{(k)} = G_\lambda$ for sufficiently large k, by letting $k \to \infty$ some results from the affine case reduce to results in the non-affine case; for example, with similar notation $\widetilde{g}_\lambda = \sum_{\mu \subset \lambda} g_\mu$, the Pieri rule for $\widetilde{g}_\lambda$ has the same expression as that for $\widetilde{g}_\lambda^{(k)}$, in which the non-leading terms (i.e. terms with non-highest degree) are obtained by taking an alternating sum of meets of the leading terms (i.e. terms with the highest degree). On the other hand, in the non-affine case there are properties that are absent in the affine case. Notably,

(A) The bases g_λ and $\widetilde{g}_\lambda$ have the same product structure constants, i.e. the linear map I defined by $g_\lambda \mapsto \widetilde{g}_\lambda$ is an algebra automorphism on the ring of symmetric functions that sends the complete symmetric function h_i to $h_i + h_{i-1} + \cdots + h_1 + h_0$.

(B) The Pieri rule for G_λ has a quite similar expression to that for g_λ, in that the non-leading terms (i.e. terms with non-*lowest* degree) are obtained by taking an alternating sum of *joins* of the leading terms (i.e. terms with the *lowest* degree).

In Chapter 3 we explain that the ring automorphism in (A) is written as both

(a) the substitution $f(x) \mapsto f(1, x)$, (that is, $f(x_1, x_2, \cdots) \mapsto f(1, x_1, x_2, \cdots)$), and

(b) the map $H(1)^\perp$, where $H(1) = \sum_i h_i$,

where the linear map $F^\perp$ is the adjoint of the multiplication map $(F\cdot)$. The equivalence of two maps in (a) and (b) is previously known (more generally, $H(t)^\perp(f(x)) = f(t, x)$ where $H(t) = \sum_i t^i h_i$). The key observation to show $I(f(x)) = f(1, x)$ is that the substitution $f \mapsto f(1, 0, 0, \cdots)$ maps $g_{\lambda/\mu}$ to 1 for any skew shape λ/μ; then since I is a certain composition of this map and the coproduct on Λ it follows that $I = (f(x) \mapsto f(1, x))$. We also give:

- formulas for the image of $g_{\lambda/\mu}$ under I (and more generally $H(t)^\perp$), which generalizes $I(g_\lambda) = \sum_{\nu \subset \lambda} g_\nu$.
- similar formulas for the inverse automorphism $E(-t)^\perp$, where $E(-t) = \sum_i (-t)^i e_i = H(t)^{-1}$.
- presentations of the maps $(H(t)\cdot)$ and $(E(-t)\cdot)$ with respect to the basis $\{G_\lambda\}$, by the adjointness of $(F\cdot)$ and $F^\perp$ and the duality between G_λ and g_λ.

In Chapter 4 we give a short proof of (B), by showing that the coefficients appearing in the Pieri rules for G_λ and g_λ are the values of the Möbius functions of certain posets of horizontal strips (over a fixed partition).

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Contents

Chapter 1

A Pieri formula and a factorization formula for sums of K-theoretic k-Schur functions

Abstract

We consider the sum $\sum_{\mu\le\lambda} g_\mu^{(k)}$, which we denote by $\widetilde{g}_\lambda^{(k)}$, of Lam–Schilling–Shimozono's K-k-Schur (K-theoretic k-Schur) functions over a principal order ideal of the poset of k-bounded partitions under the strong (Bruhat) order, and give Pieri-type formulas for $\widetilde{g}_\lambda^{(k)}$ (Theorem 1.1.3 and 1.1.4) and a k-rectangle factorization formula (Theorem 1.1.5), mainly using combinatorial properties of the strong (Bruhat) and weak orderings on the affine symmetric groups.

1.1 Introduction

Let k be a positive integer. *K-k-Schur functions* $g_\lambda^{(k)}$ are inhomogeneous symmetric functions parametrized by k-bounded partitions λ, namely by the weakly decreasing strictly positive integer sequences $\lambda = (\lambda_1, \dots, \lambda_l)$, $l \in \mathbb{Z}_{\ge 0}$, whose terms are all bounded by k. They are K-theoretic analogues, in the sense mentioned in the history section of Preface, of another family of symmetric functions called *k-Schur functions* $s_\lambda^{(k)}$, which are homogeneous and also parametrized by k-bounded partitions. The K-k-Schur functions are introduced in [LSS10] and characterized by a Pieri-type formula (Definition 1.2.19).

Among the k-bounded partitions, those of the form $(t^{k+1-t}) = (\underbrace{t,\dots,t}_{k+1-t})$ for $1 \le t \le k$, called *k-rectangles* and denoted by R_t, play a special role. A notable property is the *k-rectangle factorization* for k-Schur functions [LM07, Theorem 40]: if a k-bounded partition has the form $R_t \cup \lambda$, where the symbol $\cup$ denotes the operation of concatenating the two sequences and reordering the terms in the weakly decreasing order, then the corresponding k-Schur function factorizes as follows:

$$s_{R_t\cup\lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)}. \tag{1.1.1}$$

It is natural to consider K-theoretic version of this formula. For several reasons listed below, it seems to make more sense to consider, in this regard, the sum of K-k-Schur functions $\sum_{\mu\le\lambda} g_\mu^{(k)}$ rather than the K-k-Schur function $g_\lambda^{(k)}$ (here $\le$ denotes the *strong order*, also known as the *Bruhat order*, which is transferred from that of the affine symmetric group $\tilde{S}_{k+1}$ through the bijection $\mathcal{P}_k \simeq \tilde{S}_{k+1}/S_{k+1}$ where $\mathcal{P}_k$ denotes the set of k-bounded partitions. See Section 1.2.1.1 and 1.2.2.3 for the details):

- *Connection to the K-Peterson isomorphism.*

  The (original) Peterson isomorphism, first presented by Peterson in his lectures at MIT and then published by Lam and Shimozono [LS10], states that the homology of the affine Grassmannian is isomorphic to the quantum cohomology of the flag variety after appropriate localization. As its K-theoretic version, an isomorphism between the K-homology of the affine Grassmannian and the quantum K-theory of the flag manifold, up to appropriate localization, is conjectured and called *K-Peterson isomorphism*:

  – In their attempt in [LLMS18] to verify the coincidence of the Schubert structure constants in the K-homology of the affine Grassmannian and the quantum K-theory of the flag manifold in torus-equivariant settings, Lam, Li, Mihalcea and Shimozono proved a special case of Theorem 1.1.5 for SL_2 (i.e. the case $k = 1$) with explicit calculations, in the context of geometry:

  $$\mathcal{O}_x \mathcal{O}_{t_{-\alpha^\vee}} = \mathcal{O}_{x t_{-\alpha^\vee}}, \qquad (1.1.2)$$

  where x is any affine Grassmannian element in the affine Weyl group, $\mathcal{O}_x$ is the Schubert class of structure sheaves on the affine Grassmannian and $t_{-\alpha^\vee}$ is the translation by the negative of the simple coroot of SL_2. (See also Remark 1.2.14.)

  – In [IIM18], Ikeda, Iwao and Maeno gave an explicit ring isomorphism, after appropriate localization, between the K-homology of the affine Grassmannian and the presentation of the quantum K-theory of the flag manifold that is conjectured by Kirillov and Maeno and proved by Anderson, Chen, and Tseng [ACT], as well as a conjectural description of the image of the quantum Grothendieck polynomials, which is conjectured to be the quantum Schubert classes. These presentations notably involve the dual stable Grothendieck polynomials g_{R_t} and their sum $\sum_{\mu \subset R_t} g_\mu$ corresponding to the k-rectangles R_t. Note that $\mu \subset R_t \iff \mu \leq R_t$, and that it is conjectured that $g^{(k)}_\lambda = g_\lambda$ for $\lambda \subset R_t$.

  Remark 1.1.1. After this article was submitted, there appeared a preprint [Kat] by Syu Kato in which he claims to have proved conjectures in [LLMS18] and in particular the factorization property for the structure sheaves in general type.

- *Natural appearances of $\sum_{\mu \leq \lambda} g^{(k)}_\mu$ in k-rectangle factorization formulas of $g^{(k)}_\lambda$.*

  It is suggested in [LSS10, Remark 7.4] that the K-k-Schur functions should also possess similar properties to (1.1.1), including the divisibility of $g^{(k)}_{R_t \cup \lambda}$ by $g^{(k)}_{R_t}$, for which the author's preceding work [Taka, Takb] gives an affirmative answer.

  Let us review the results of [Taka, Takb]. It is proved that $g^{(k)}_{R_t}$ divides $g^{(k)}_{R_t \cup \lambda}$ in the ring $\Lambda_{(k)} = \mathbb{Z}[h_1, \dots, h_k]$, of which the K-k-Schur functions $\{g^{(k)}_\mu\}_{\mu \in \mathcal{P}_k}$ form a basis. However, unlike (1.1.1), the quotient $g^{(k)}_{R_t \cup \lambda}/g^{(k)}_{R_t}$ is not a single term $g^{(k)}_\lambda$ but in general a linear combination of K-k-Schur functions with leading term $g^{(k)}_\lambda$: for any $\lambda \in \mathcal{P}_k$,

  $$g^{(k)}_{R_t \cup \lambda} = g^{(k)}_{R_t} \left(g^{(k)}_\lambda + \sum_\mu a_{\lambda\mu} g^{(k)}_\mu \right), \qquad (1.1.3)$$

  where the summation is taken with all $\mu \in \mathcal{P}_k$ such that $|\mu| < |\lambda|$, with some coefficients $a_{\lambda\mu}$ (which also depends on t). A special yet important case is the factorization of multiple k-rectangles: for $1 \leq t \leq k$ and $a > 1$,

  $$g^{(k)}_{R_t^a} = g^{(k)}_{R_t} \left(\sum_{\mu \subset R_t} g^{(k)}_\mu \right)^{a-1},$$

  where $R_t^a = R_t \cup \cdots \cup R_t$ (a times). Note that $\mu \subset R_t \iff \mu \leq R_t$. Furthermore, it is conjectured that the set of μ appearing in (1.1.3) forms an interval under the strong order: namely, for any $\lambda \in \mathcal{P}_k$

and $1 \le t \le k$, we expect there to exist $\nu \in \mathcal{P}_k$ such that

$$g^{(k)}_{R_t \cup \lambda} = g^{(k)}_{R_t} \sum_{\nu \le \mu \le \lambda} g^{(k)}_{\mu}.$$

These observations suggest the usefulness of Definition 1.1.2 below.

1.1.1 Main results

Let $\le$, $\le_L$, and $\le_R$ be the strong, left weak, and right weak order on the affine symmetric group $\tilde{S}_{k+1}$ (see Section 1.2.1.1 for the details).

From the observation above, we introduce the notation $\widetilde{g}^{(k)}_{\lambda}$ for the sum of K-k-Schur functions over the order ideal generated by λ under the strong order $\le$, which are the main objects of study in this chapter.

Definition 1.1.2. For any $\lambda \in \mathcal{P}_k$, we write $\widetilde{g}^{(k)}_{\lambda} = \sum_{\mu \le \lambda} g^{(k)}_{\mu}$.

Our first main theorem is a Pieri-type formula for $\widetilde{g}^{(k)}_{\lambda}$. We start with the Pieri rule for $g^{(k)}_{\lambda}$ [LSS10, Mor12]: for $\lambda \in \mathcal{P}_k$ and $1 \le r \le k$,

$$g^{(k)}_{\lambda} h_r = \sum_{(A,\mu)} (-1)^{|\lambda|+r-|\mu|} g^{(k)}_{\mu}, \tag{1.1.4}$$

summed over all pairs (A, μ) such that $(\mu/\lambda, A)$ are *affine set-valued strips* of size r (see Definition 1.2.7 and 1.2.19 for more details). Note that this is an inhomogeneous version of the Pieri rule for the k-Schur functions $s^{(k)}_{\lambda}$, that is, $s^{(k)}_{\lambda} h_r = \sum s^{(k)}_{\mu}$ where the summation is taken over all μ such that μ/λ are *weak strips* of size r (see Definition 1.2.6 and 1.2.17). In terms of $\widetilde{g}^{(k)}_{\lambda}$, this rule (1.1.4) becomes relatively simple:

Theorem 1.1.3. *Let $\lambda \in \mathcal{P}_k$ and $1 \le r \le k$, and define $\widetilde{h}_r = h_0 + h_1 + \cdots + h_r$. Then*

$$\widetilde{g}^{(k)}_{\lambda} \widetilde{h}_r = \sum_{\mu} g^{(k)}_{\mu},$$

summed over $\mu \in \mathcal{P}_k$ such that $\mu \le \kappa$ for some $\kappa \in \mathcal{P}_k$ such that κ/λ is a weak strip of size r.

To express its right-hand side as a linear combination of $\{\widetilde{g}^{(k)}_{\mu}\}_{\mu}$, we recall that the weak strips over λ correspond to certain proper subsets of $I = \{0, 1, \ldots, k\}$: for $\kappa \in \mathcal{P}_k$, κ/λ is a weak strip if and only if there exists $A \subsetneq I$ such that $\kappa = d_A\lambda \ge_L \lambda$, where d_A is the cyclically decreasing permutation corresponding to A (see Section 1.2.2.2, 1.2.2.3, and 1.2.2.4 for the details).

Theorem 1.1.4. *With the setting in Theorem 1.1.3, we let $d_{A_1}\lambda, d_{A_2}\lambda, \ldots$ be all weak strips of size r over λ. Then*

$$\widetilde{g}^{(k)}_{\lambda} \widetilde{h}_r = \sum_{m \ge 1} (-1)^{m-1} \sum_{a_1 < \cdots < a_m} \widetilde{g}^{(k)}_{d_{A_{a_1} \cap \cdots \cap A_{a_m}} \lambda}.$$

$$\left(= \sum_{a} \widetilde{g}^{(k)}_{d_{A_a}\lambda} - \sum_{a<b} \widetilde{g}^{(k)}_{d_{A_a \cap A_b}\lambda} + \sum_{a<b<c} \widetilde{g}^{(k)}_{d_{A_a \cap A_b \cap A_c}\lambda} - \cdots \right)$$

Moreover $d_{A_a \cap A_b \cap \ldots}\lambda = (d_{A_a}\lambda) \wedge (d_{A_b}\lambda) \wedge \ldots$, where $\wedge$ denotes the meet in the poset $\mathcal{P}_k$ with the strong order. See also Proposition 1.1.6.

Our second main theorem is the validity of the k-rectangle factorization formula for $\widetilde{g}^{(k)}_{\lambda}$, which takes the same simple form as that for k-Schur functions (1.1.1):

Theorem 1.1.5. *For any $\lambda \in \mathcal{P}_k$ and $1 \le t \le k$, we have*

$$\widetilde{g}^{(k)}_{R_t \cup \lambda} = \widetilde{g}^{(k)}_{R_t} \widetilde{g}^{(k)}_{\lambda}.$$

It is easy to deduce Theorem 1.1.5 from Theorem 1.1.4, as discussed in Section 1.6. The proof of Theorem 1.1.3 and 1.1.4, on the other hand, is the technical heart of this chapter and requires auxiliary work on the strong and weak orderings on the set of affine permutations as well as the structure of the set of weak strips, which are discussed in Section 1.3 and 1.4.

This chapter is organized as follows.

In Section 1.2, we review some notations and facts from the combinatorial background. In Section 1.2.1 we treat arbitrary Coxeter groups and their strong and weak orderings. It also contains quick reviews on the generalized quotients [BW88] and the Demazure products. Section 1.2.2 contains notations specific to the affine symmetric groups and a review on their Young-diagrammatic treatment. In Section 1.2.3 we briefly review the Pieri-type formulas for k-Schur and K-k-Schur functions.

Section 1.3 contains technical lemmas on the strong and weak orders on arbitrary Coxeter groups. In Section 1.3.1 the lattice property of the weak order is reviewed. Although it is known that the quotient of an affine Weyl group by its corresponding finite Weyl group forms a lattice under the weak order [Wau99], we include another proof for the type affine A using the k-Schur functions. Section 1.3.2 contains basic properties of the Demazure and anti-Demazure actions. In Section 1.3.3 we show the existence, for any elements x, y of any Coxeter group W, of $\min_{\leq}\{z \in W \mid x \leq z \geq_L y\}$ and $\max_{\leq}\{z \in W \mid x \geq_L z \leq y\}$, analogous to the join and meet. In Section 1.3.4 we consider an "interval-flipping" map $\Phi_z \colon [e, z]_L \longrightarrow [e, z]_R;\ x \mapsto zx^{-1}$ for any z in an arbitrary Coxeter group W, and show that Φ_z is anti-isomorphic under the strong order and that if $x, y \in [e, z]$ have a meet $x \wedge y$ in the strong order then Φ_z maps $x \wedge y$ to the join $\Phi_z(x) \vee \Phi_z(y)$ of $\Phi_z(x)$ and $\Phi_z(y)$ in the strong order taken in W. In Section 1.3.5 we show the Chain Property of lower weak intervals, analogous to the Chain Property of the generalized quotients (see Section 1.2.1.2).

In Section 1.4, we focus on the affine symmetric groups and give results on the structure of the set of weak strips, which includes:

Proposition 1.1.6 ($\subset$ Proposition 1.4.2)**.** *For any $\lambda \in \mathcal{P}_k$ and $A, B \subsetneq I$ with $d_A\lambda/\lambda$ and $d_B\lambda/\lambda$ are weak strips,*

(1) *$d_{A\cap B}\lambda/\lambda$ and $d_{A\cup B}\lambda/\lambda$ are weak strips.*

(2) *$d_{A\cap B}\lambda = d_A\lambda \wedge d_B\lambda$ under the strong order.*

Proposition 1.1.7 ($\subset$ Proposition 1.4.12)**.** *For any $\lambda \in \mathcal{P}_k$, there exists $i_\lambda \in I$ $(= \{0, 1, \ldots, k\})$ such that $i_\lambda \notin A$ for any weak strip $d_A\lambda/\lambda$.*

Section 1.5 and 1.6 are devoted to proving the Pieri-type formula for $\widetilde{g}_\lambda^{(k)}$ (Theorem 1.1.3 and 1.1.4) and the k-rectangle factorization formula for $\widetilde{g}_\lambda^{(k)}$ (Theorem 1.1.5), respectively.

1.2 Preliminaries

In this section we review some requisite combinatorial background.

1.2.1 Coxeter groups

For basic definitions for the Coxeter groups we refer the reader to [BB05] or [Hum90].

1.2.1.1 Strong and weak orderings

Let (W, S) be a Coxeter group, $T = \{wsw^{-1} \mid w \in W\}$ its set of reflections, and $l \colon W \longrightarrow \mathbb{Z}_{\geq 0}$ its length function. The *left weak order* (or simply *left order*) $\leq_L$, *right weak order* (or *right order*) $\leq_R$, and *strong order* (or *Bruhat order*) $\leq$ on W are generated by the covering relations:

$$\begin{aligned} u <_L v &\iff l(v) = l(u) + 1,\ v = su \text{ for some } s \in S, \\ u <_R v &\iff l(v) = l(u) + 1,\ v = us \text{ for some } s \in S, \\ u < v &\iff l(v) = l(u) + 1,\ v = tu \text{ for some } t \in T. \end{aligned}$$

Note that the definition of the strong order looks different from but coincides with the classical one.

A few observations follow immediately: for $u, v \in W$,

$$u \le_L v \iff l(vu^{-1}) + l(u) = l(v), \tag{1.2.1}$$

$$u \le_R v \iff l(u) + l(u^{-1}v) = l(v), \tag{1.2.2}$$

$$u \le_R uv \iff l(u) + l(v) = l(uv) \iff v \le_L uv. \tag{1.2.3}$$

We often use these equivalences without any mention. Using this translation from the weak order to length conditions, we can easily prove the following lemma:

Lemma 1.2.1. *For $x, y, z \in W$, we have*

(1) $z \le_L yz \le_L xyz \iff y \le_L xy$ *and* $z \le_L xyz$.

(2) $z \ge_L yz \ge_L xyz \iff y \le_L xy$ *and* $z \ge_L xyz$.

We often use the following notation taken from [BW88]: for $w \in W$ we let $\langle w \rangle$ denote any reduced expression for w, and $\langle u \rangle \langle v \rangle$ the concatenation of reduced expressions for u and v. Hence, saying that $\langle u \rangle \langle v \rangle$ is reduced means $l(u) + l(v) = l(uv)$.

For $u, v \in W$ with $u \le_L v$ the set $\{w \in W \mid u \le_L w \le_L v\}$ is called a *left interval* and denoted by $[u, v]_L$. We define a *right interval* $[u, v]_R$ and a *strong (or Bruhat) interval* $[u, v]$ similarly. We shall use the notation $[u, \infty)_L$ to denote the set $\{w \in W \mid u \le_L w\}$, and define $[u, \infty)_R$ and $[u, \infty)$ similarly.

In this chapter we heavily use some well-known results on the strong and weak orderings on Coxeter groups described below. See, for example, [BB05] for details. Let $v, w \in W$.

Strong Exchange Property. Suppose $w = s_1 s_2 \dots s_k$ ($s_i \in S$) and $t \in T$. If $l(tw) < l(w)$, then $tw = s_1 \dots \widehat{s_i} \dots s_k$ for some $i \in \{1, \cdots, k\}$. Furthermore, if $s_1 s_2 \dots s_k$ is a reduced expression then i is uniquely determined.

Subword Property. Let $w = s_1 s_2 \dots s_k$ be a reduced expression. Then $v \le w$ if and only if there exists a reduced expression $v = s_{i_1} s_{i_2} \dots s_{i_l}$ with $1 \le i_1 < i_2 < \cdots < i_l \le k$.

Chain Property.[1] If $v \le w$, then there exists a chain $v = x_0 \lessdot x_1 \lessdot \dots \lessdot x_k = w$.

Lifting Property (also known as *Z-property*). Let $s \in S$. If $sv > v$ and $sw > w$, then the following are equivalent: (1) $v \le w$, (2) $v \le sw$, and (3) $sv \le sw$.

1.2.1.2 Generalized quotients

For any subset $V \subset W$, let

$$W/V = \{w \in W \mid l(wv) = l(w) + l(v) \text{ for all } v \in V\}.$$

The subsets of the form W/V are called *(right) generalized quotients* [BW88]. Similarly the sets of the form

$$V \backslash W = \{w \in W \mid l(vw) = l(v) + l(w) \text{ for all } v \in V\}$$

are called *left generalized quotients*. Note that, when $V = W_J$, the parabolic subgroup corresponding to $J \subset I$, the generalized quotient W/W_J is just the parabolic quotient W^J, namely the set of shortest representatives collected from all cosets wW_J.

It is shown in [BW88, Lemma 2.2] that if $a, b, v \in W$ satisfy $l(av) = l(a) + l(v)$ and $l(bv) = l(b) + l(v)$, then $av < bv \iff a < b$. An immediate consequence is

$$W/\{v\} \simeq [v, \infty)_L; w \mapsto wv \tag{1.2.4}$$

under the strong order as well as the left weak order.

Chain Property for generalized quotients ([BW88, Corollary 3.5]). If $u, w \in W/V$ and $u < w$, then there exists a chain $u = x_0 \lessdot x_1 \lessdot \dots \lessdot x_k = w$ with $x_i \in W/V$ for all i.

[1] With the definition of $\le$ we employed here, this is trivial.

1.2.1.3 The 0-Hecke algebra and the Demazure product

The 0-*Hecke algebra* H associated to (W,S) is the associative algebra generated by $\{v_s \mid s \in S\}$ subject to the quadratic relations $v_s^2 = -v_s$ and the braid relations of (W,S), that is, $\underbrace{v_s v_t v_s \dots}_{m} = \underbrace{v_t v_s v_t \dots}_{m}$ for $s,t \in S$ where $m = m_{s,t}$ is the order of st in W. For $w \in W$ we can define without ambiguity $v_w \in H$ to be $v_{s_1} \dots v_{s_n}$ where $s_1 \dots s_n$ is any reduced expression for w. Furthermore, the elements $\{v_w \mid w \in W\}$ form a basis of H. The *Demazure product* (or *Hecke product*) $*$ on W describes the multiplication of basis elements in H: for $x,y \in W$, $x * y \in W$ is such that $v_x v_y = \pm v_{x*y}$. Some properties on the Demazure product can be found in [KM04, BM15].

We explicitly prepare the notation to denote the left multiplication in the Demazure product: for $s \in S$, we define the Demazure action $\phi_s \colon W \longrightarrow W$ by

$$\phi_s(x) = s * x = \begin{cases} x & (\text{if } x > sx) \\ sx & (\text{if } x < sx) \end{cases}.$$

Similarly we define the anti-Demazure action $\psi_s \colon W \longrightarrow W$ by

$$\psi_s(x) = \begin{cases} sx & (\text{if } x > sx) \\ x & (\text{if } x < sx) \end{cases}.$$

These maps $\{\phi_s\}_s$ and $\{\psi_s\}_s$ satisfy the quadratic relations $\phi_s^2 = \phi_s$, $\psi_s^2 = \psi_s$ and the braid relations of (W,S); a direct proof (found on [Ste07, Proposition 2.1]) of this (for ψ) is that both $\psi_s\psi_t\psi_s\dots$ and $\psi_t\psi_s\psi_t\dots$ (products of $m_{s,t}$ factors) send $x \in W$ to the shortest (resp. longest, if we consider ϕ instead of ψ) element of the parabolic coset $W_{\{s,t\}}x$. Therefore we can define without ambiguity $\phi_x, \psi_x : W \longrightarrow W$ for $x \in W$ by $\phi_x = \phi_{s_1} \dots \phi_{s_n}$ and $\psi_x = \psi_{s_1} \dots \psi_{s_n}$ where $x = s_1 \dots s_n$ is any reduced expression. Similarly we define the right Demazure and anti-Demazure actions $\phi_s^R, \psi_s^R \colon W \longrightarrow W$ for $s \in S$ by $\phi_s^R(x) = \phi_s(x^{-1})^{-1}$ and $\psi_s^R(x) = \psi_s(x^{-1})^{-1}$, that is, $\phi_s^R(x) = xs$ if $x < xs$, etc. We also define ϕ_x^R and ψ_x^R to be $\phi_{s_n}^R \dots \phi_{s_1}^R$ and $\psi_{s_n}^R \dots \psi_{s_1}^R$ (mind the order of composition) where $x = s_1 \dots s_n$ is any reduced expression. Note that $\phi_x(y) = x * y = \phi_y^R(x)$. When S is indexed by a set I, i.e. $S = \{s_i \mid i \in I\}$, we often write $\phi_i = \phi_{s_i}$ and $\psi_i = \psi_{s_i}$.

The following lemma is essentially given in [BW88, Theorem 4.2], and explicitly in [BM15, Proposition 3.1(e)]:

Lemma 1.2.2. *Let $x,y,z \in W$ with $x * y = z$, that is, $\phi_x(y) = z = \phi_y^R(x)$. Let $x' = zy^{-1}$ and $y' = x^{-1}z$, that is, $z = xy' = x'y$. Then we have*

(1) $x, x' \leq_R z$.

(2) $y, y' \leq_L z$.

(3) $l(z) = l(x) + l(y') = l(x') + l(y)$.

(4) $x' \leq x$.

(5) $y' \leq y$.

Proof. It follows easily from the definition of $*$ and the Subword Property. □

The proof of the following lemma is easy and similar to that of Lemma 1.2.2.

Lemma 1.2.3. *Let $x,y,z \in W$ with $\psi_x(y) = z$. Let $x' = zy^{-1}$, that is, $z = x'y$. Then we have*

(1) $x' \leq x$.

(2) $z \leq_L y$.

(3) $x'^{-1} \leq_R y$.

We see more properties of ϕ_x, ψ_x in Section 1.3.2.

1.2.2 Affine symmetric groups

In this section we briefly review the connection between affine permutations, bounded partitions and core partitions. We refer the reader to [LLM+14, Chapter 2] and [Den12] for the details.

Hereafter we fix a positive integer k.

1.2.2.1 Affine symmetric group

Let $I = \mathbb{Z}_{k+1} = \{0, 1, \dots, k\}$, where each of $0, 1, \dots, k$ should be understood to represent its coset modulo $k+1$. Let $[p, q] = \{p, p+1, \dots, q-1, q\} \subsetneq I$ for $p \neq q-1$. For example, $[4, 2] = \{4, 5, 0, 1, 2\}$ if $k = 5$. A subset $A \subset I$ is called *connected* if $A = [p, q]$ for some p, q. A *connected component* of $A \subsetneq I$ is a maximal connected subset of A.

The *affine symmetric group* $\tilde{S}_{k+1}$ is a group generated by the generators $\{s_i \mid i \in I\}$ subject to the relations $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ for $i - j \neq 0, \pm 1$, and its elements are called *affine permutations*. We sometimes write $s_{ij\dots}$ instead of $s_i s_j \cdots$. The parabolic quotient $\tilde{S}_{k+1}/S_{k+1}$, where S_{k+1} is the symmetric group $\langle s_1, \dots, s_k \rangle$ as a subgroup of $\tilde{S}_{k+1}$, is denoted by $\tilde{S}^{\circ}_{k+1}$ and its elements are called *affine Grassmannian elements*.

For $x \in \tilde{S}_{k+1}$, the set of *right descents* $D_R(x)$ is $\{i \in I \mid x > x s_i\}$ ($\subsetneq I$). The set of *left descents* $D_L(x)$ is defined similarly. For $i \in I$, an element $w \in \tilde{S}_{k+1}$ is called *i-dominant* if $D_R(w) \subset \{i\}$. Note that an affine permutation is 0-dominant if and only if it is affine Grassmannian.

1.2.2.2 Cyclically decreasing elements

A word $a = a_1 a_2 \dots a_m$ with letters from I is called *cyclically decreasing* (resp. *cyclically increasing*) if $a_1, a_2, \dots, a_m$ are distinct and any $j \in I$ does not precede $j+1$ (resp. $j-1$) in a. For $A \subsetneq I$, the *cyclically decreasing element* d_A is defined to be $s_{i_1} s_{i_2} \dots s_{i_m}$ where $A = \{i_1, i_2, \dots, i_m\}$ and the word $i_1 i_2 \dots i_m$ is cyclically decreasing. The *cyclically increasing element* $u_A = s_{i_m} s_{i_{m-1}} \dots s_{i_1}$ is defined similarly. Note that these definitions are independent of the choice of the word.

Example 1.2.4. Let $k = 5$ and $A = \{0, 1, 3, 5\} \subsetneq \mathbb{Z}_6$. The possible cyclically decreasing words for A are 1053, 1035, 1305 and 3105, and hence $d_A = s_1 s_0 s_5 s_3 = s_1 s_0 s_3 s_5 = s_1 s_3 s_0 s_5 = s_3 s_1 s_0 s_5$.

1.2.2.3 Connection to bounded partitions and core partitions

In this section we review the bijection between the set of k-bounded partitions, $k+1$-core partitions and affine Grassmannian elements in $\tilde{S}_{k+1}$. For the details see [LLM+14, Chapter 2] and references given there.

A *partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ is a weakly decreasing sequence in $\mathbb{Z}_{\geq 0}$ with $\sum_i \lambda_i < \infty$, and is often identified with its Young diagram $\{(i, j) \in (\mathbb{Z}_{>0})^2 \mid 1 \leq j \leq \lambda_i\}$. A partition λ is called *k-bounded* if $\lambda_1 \leq k$. Let $\mathcal{P}_k$ be the set of all k-bounded partitions. An *r-core* (or simply a *core* if no confusion can arise) is a partition none of whose cells have a hook length equal to r. We denote by $\mathcal{C}_r$ the set of all r-core partitions.

Now we recall the bijection

$$\mathcal{C}_{k+1} \simeq \mathcal{P}_k \simeq \tilde{S}^{\circ}_{k+1}. \tag{1.2.5}$$

The map $\mathfrak{p} \colon \mathcal{C}_{k+1} \longrightarrow \mathcal{P}_k; \kappa \mapsto \lambda$ is defined by $\lambda_i = \#\{j \mid (i, j) \in \kappa,\ \mathrm{hook}_{(i,j)}(\kappa) \leq k\}$. In fact $\mathfrak{p}$ is bijective and the inverse map $\mathfrak{c} = \mathfrak{p}^{-1} \colon \mathcal{P}_k \longrightarrow \mathcal{C}_{k+1}$ is algorithmically described as a “sliding cells” procedure.

The map $\mathfrak{s} \colon \tilde{S}^{\circ}_{k+1} \longrightarrow \mathcal{C}_{k+1}$ is constructed via an action of $\tilde{S}_{k+1}$ on $\mathcal{C}_{k+1}$: for $\kappa \in \mathcal{C}_{k+1}$ and $i \in I$, we define $s_i \cdot \kappa$ to be κ with all its addable (resp. removable) corners with residue i added (resp. removed), where the *residue* of a cell (i, j) is $j - i \mod k+1$. In fact this gives a well-defined $\tilde{S}_{k+1}$-action on $\mathcal{C}_{k+1}$, which induces the bijection $\mathfrak{s} : \tilde{S}^{\circ}_{k+1} \longrightarrow \mathcal{C}_{k+1}; w \mapsto w \cdot \varnothing$.

The map $\mathcal{P}_k \longrightarrow \tilde{S}^{\circ}_{k+1}; \lambda \mapsto w_\lambda$ is given by $w_\lambda = s_{i_1} s_{i_2} \dots s_{i_l}$, where $(i_1, i_2, \dots, i_l)$ is the sequence obtained by reading the residues of the cells in λ, from the shortest row to the largest, and within each row from right to left. See [LM05, Corollary 48] for the proof.

For $\lambda \in \mathcal{P}_k$, the *$k$-transpose* of λ is $\mathfrak{p}(\mathfrak{c}(\lambda)')$ and denoted by λ^{ω_k}. (Here μ' denotes the transpose of a partition μ.)

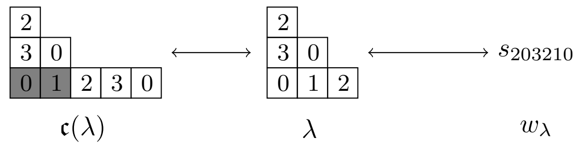


Figure 1.1: $k = 3$, $\lambda = (3, 2, 1) \in \mathcal{P}_3$, $\mathfrak{c}(\lambda) = (5, 2, 1) \in \mathcal{C}_4$, and $w_\lambda = s_{203210} \in \tilde{S}_4^\circ$.

Example 1.2.5. Let $k = 3$ and $\lambda = (3, 2, 1) \in \mathcal{P}_3$. The corresponding 4-core partition and affine permutation are $\mathfrak{c}(\lambda) = (5, 2, 1) \in \mathcal{C}_4$ and $w_\lambda = s_{203210} \in \tilde{S}_4^\circ$. (See Figure 1.1.)

1.2.2.4 Weak strips

Definition 1.2.6. For $v, w \in \tilde{S}_{k+1}^\circ$, we say v/w is a *weak strip* (or *affine strip*) of size r if $v = d_A w \geq_L w$ for some $A \subsetneq I$ with $|A| = r$. We also say v is a weak strip of size r over w.

Definition 1.2.7. For $v, w \in \tilde{S}_{k+1}^\circ$ and $A \subsetneq I$, we say $(v/w, A)$ is an *affine set-valued strip* of size r if $v = d_A * w \, (= \phi_{d_A}(w))$ and $|A| = r$. We also say (v, A) is a affine set-valued strip of size r over w.

Note that if $(v/w, A)$ is an affine set-valued strip of size r then v/w is an affine strip of size $\leq r$.

Remark 1.2.8. Identifying λ, $\mathfrak{c}(\lambda)$ and w_λ through the bijection $\mathcal{P}_k \simeq \mathcal{C}_{k+1} \simeq \tilde{S}_{k+1}^\circ$, we often say μ/λ (resp. κ/γ) is a weak strip for $\lambda, \mu \in \mathcal{P}_k$ (resp. $\kappa, \gamma \in \mathcal{C}_{k+1}$), etc.

Remark 1.2.9. Regarding $v, w \in \tilde{S}_{k+1}^\circ$ as bounded (or core) partitions as above, we see these notions are variants of the horizontal strip. For example, w_μ/w_λ is a weak strip if and only if the corresponding cores $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ form a horizontal strip and $w_\mu \geq_L w_\lambda$, and the term "affine set-valued" originates in affine set-valued tableaux. See, for example, [LLM+14, Mor12] for more details.

Example 1.2.10. Let $k = 3$ and $\lambda = (3, 2, 1) \in \mathcal{P}_3$, and thus $w_\lambda = s_{203210}$ and $\mathfrak{c}(\lambda) = (5, 2, 1)$. Figure 1.2 lists all v such that v/w_λ is a weak strip (the corresponding core partitions are displayed).

1.2.2.5 k-codes

The notion of k-codes was studied by T. Denton in [Den12]. We follow his definition and summarize the results relevant to our present work here.

A *k-code* is a function $\alpha : I \longrightarrow \mathbb{Z}_{\geq 0}$ such that there exists at least one $i \in I$ with $\alpha(i) = 0$. We often write $\alpha_i = \alpha(i)$. The *diagram* of a k-code α is the Ferrers diagram on a cylinder with $k+1$ columns indexed by I, where the i-th column contains α_i boxes. A k-code α may be identified with its *filling*, which is the diagram of α with all its boxes marked with their residues, that is, $i - j$ $(\in I)$ for one in the i-th column and j-th row. A *flattening* of the diagram of a k-code α is what is obtained by cutting out a column with no boxes (that is, column j with $\alpha_j = 0$). A *reading word* of α is obtained by reading the rows of a flattening of α from right to left, beginning with the last row. Note that, though a k-code may have multiple columns with no boxes, the affine permutation given by the reading word of α is independent of the choice of a flattening. Indeed, for a k-code α with m rows, letting A_i be the set of the residues of the boxes in the i-th row in the diagram of α, we have that $d_{A_m} \cdots d_{A_2} d_{A_1}$ is the affine permutation corresponding to α. In fact this correspondence is bijective (Theorem 1.2.11); an algorithm to obtain a k-code from an affine permutation is explained below.

Maximizing moves.

For a cyclically decreasing decomposition $w = d_{A_m} \cdots d_{A_1}$, there corresponds a "skew k-code diagram", that is, a set of boxes in the cylinder with $k+1$ columns indexed by I in which A_i is the set of the residues of the boxes in the i-th row. To justify it to the bottom, we consider the following "two-row moves": pick any consecutive two rows A_a and A_{a+1}, and let $i, j \in I$ with $j \neq i - 1$. Then,

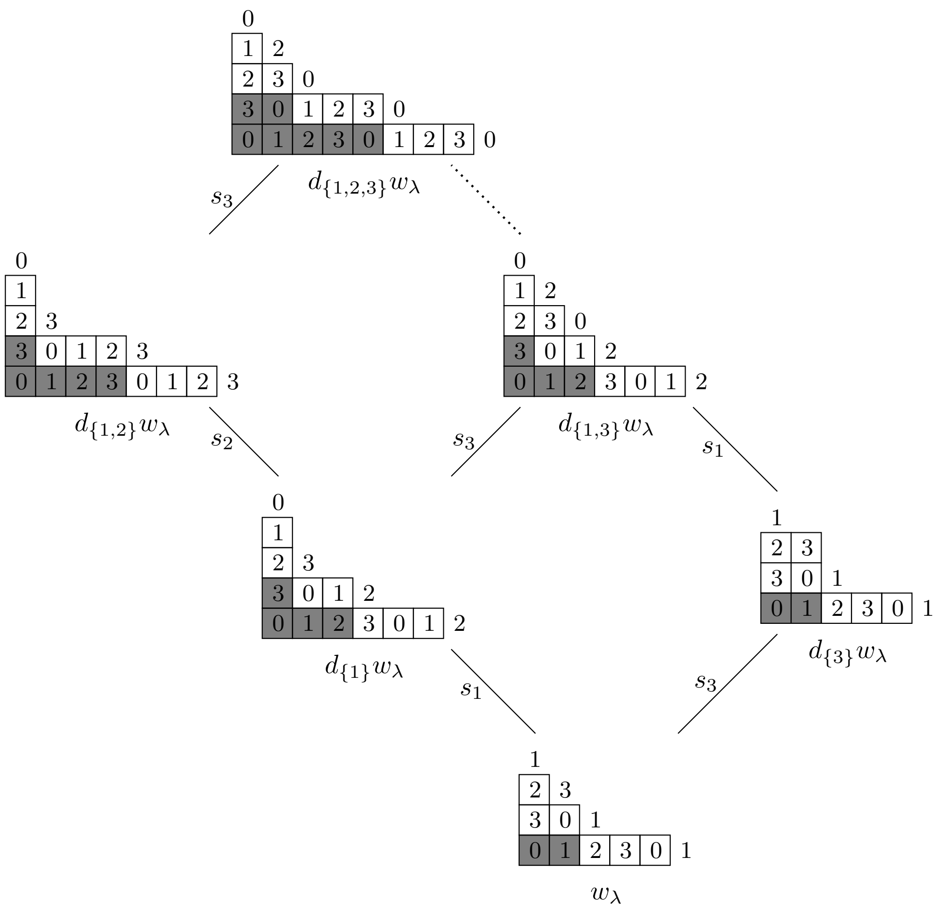


Figure 1.2: The weak strips over w_λ where $\lambda = (3, 2, 1)$. Left weak covers are represented as solid lines, and strong covers are solid or dotted lines. A solid edge between v and w is labelled with s_i if $v = s_i w$.

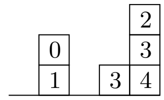
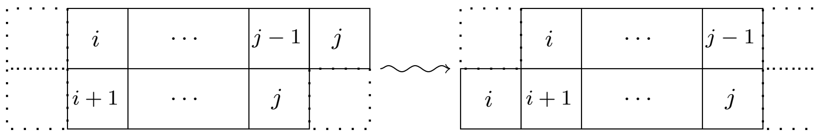


Figure 1.3: $\mathrm{RD}(w)$ where $k = 3$ and $w = s_2 s_{30} s_{431}$

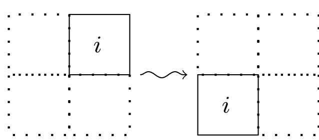
(1) if $i-1 \notin A_{a+1}$, $[i,j] \subset A_{a+1}$, $[i+1,j] \subset A_a$, and $i, j+1 \notin A_a$, then we replace A_a and A_{a+1} with $A_a \cup \{i\}$ and $A_{a+1} \setminus \{j\}$, reflecting the equation $(s_j s_{j-1} \dots s_i)(s_j \dots s_{i+1}) = (s_{j-1} \dots s_i)(s_j \dots s_{i+1} s_i)$.

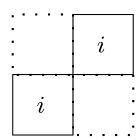


(2) if $i-1 \notin A_{a+1}$, $[i,j] \subset A_{a+1}$, $[i,j] \subset A_a$, and $j+1 \notin A_a$, then we conclude this decomposition does not give a reduced expression, reflecting the fact that $(s_j s_{j-1} \dots s_i)(s_j \dots s_{i+1} s_i)$ is not a reduced expression.

: not reduced

Note that these moves look simpler when $i = j$:

(1) 

, (2) 

: not reduced .

It is shown in [Den12, Section 3] that, for any decomposition $w = d_{A_m} \cdots d_{A_1}$ that gives a reduced expression, we can apply a finite series of moves of type (1) to justify its diagram to the bottom and obtain a k-code, which is in fact uniquely determined from w and denoted by $\mathrm{RD}(w)$, and gives the *maximal decreasing decomposition* $w = d_{B_n} \cdots d_{B_1}$, that is, the vector $(|B_1|, \cdots, |B_n|)$ is maximal in the lexicographical order among such decompositions for w. Furthermore, this procedure bijectively maps affine permutations to k-codes:

Theorem 1.2.11 ([Den12, Theorem 38]). *The map $w \mapsto \mathrm{RD}(w)$ gives a bijection between $\tilde{S}_{k+1}$ and the set of k-codes.*

Example 1.2.12. Let $k = 3$ and $w = s_2 s_{30} s_{431}$ (this expression gives the maximal decreasing decomposition). Then $\mathrm{RD}(w) = (0, 2, 0, 1, 3)$. (See Figure 1.3)

Note that this construction also works if maximal decreasing decomposition is replaced with *maximal increasing decompositions*, that is, the unique decomposition $w = u_{B_n} \cdots u_{B_1}$ into cyclically increasing elements with $l(w) = l(u_{B_n}) + \cdots + l(u_{B_1})$ and the vector $(|B_1|, \dots, |B_n|)$ being maximal in the lexicographical order, by modifying the notion of the filling of a k-code so that the box in the i-th column and j-th row is marked with $j - i$ instead of $i - j$. The resulting k-code is denoted by $\mathrm{RI}(w)$. The map $w \mapsto \mathrm{RI}(w)$ also gives a bijection between $\tilde{S}_{k+1}$ and the set of k-codes.

It is proved [Den12, Corollary 39] that $w \in \tilde{S}_{k+1}$ is i-dominant if and only if the flattening of the corresponding k-code $\mathrm{RD}(w)$ forms a k-bounded partition with residue i in its lower left box, that is, $\mathrm{RD}(w)_i \geq$

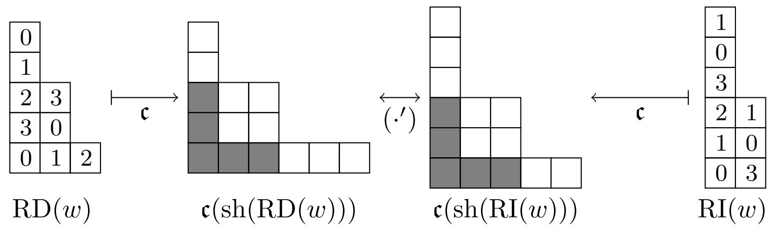


Figure 1.4:

$\mathrm{RD}(w)_{i+1} \geq \cdots \geq \mathrm{RD}(w)_{i-2} \geq \mathrm{RD}(w)_{i-1} = 0$. When $i = 0$, this mapping from 0-dominant permutations to k-bounded partitions coincides with the one described earlier in Section 1.2.2.3. Moreover, it is proved [Den12, Proposition 51] that, for $w \in \tilde{S}^{\circ}_{k+1}$ the two corresponding k-codes $\mathrm{RD}(w)$ and $\mathrm{RI}(w)$, regarded as k-bounded partitions, are transformed into each other by taking the k-transpose: $\mathrm{sh}(\mathrm{RI}(w)) = (\mathrm{sh}(\mathrm{RD}(w)))^{\omega_k}$ where $\mathrm{sh}(\alpha) \in \mathcal{P}_k$ is defined by $\mathrm{sh}(\alpha)_j = |\{i \mid \alpha_i \geq j\}|$.

It is also proved in [Den12, Proposition 56] that if $x \leq_L y$ then $\mathrm{RD}(x) \subset \mathrm{RD}(y)$ and $\mathrm{RI}(x) \subset \mathrm{RI}(y)$. Here the inclusion of k-codes should be understood as that of their (Ferrers) diagrams.

Example 1.2.13. Let $k = 3$ and $w = s_0 s_1 s_{32} s_{03} s_{210} = s_1 s_0 s_3 s_{12} s_{01} s_{30}$ (these presentations give the maximal decreasing and increasing decompositions). Then $\mathrm{RD}(w) = (5, 3, 1, 0)$ and $\mathrm{RI}(w) = (6, 3, 0, 0)$, and thus $\mathrm{sh}(\mathrm{RD}(w)) = (3, 2, 2, 1, 1) = (2, 2, 2, 1, 1, 1)^{\omega_3} = \mathrm{sh}(\mathrm{RI}(w))^{\omega_3}$. (See Figure 1.4)

1.2.2.6 k-rectangles

The partition $(t^{k+1-t}) = (t, t, \ldots, t) \in \mathcal{P}_k$, for $1 \leq t \leq k$, is denoted by R_t and called a *k-rectangle*.

Remark 1.2.14. Consider the affine permutation w_{R_i} corresponding to the k-rectangle R_i under the bijection (1.2.5). In fact w_{R_i} is congruent, in the extended affine Weyl group, to the translation $t_{-\varpi_i^\vee}$ by the negative of a fundamental coweight, modulo left multiplication by the length zero elements.

The next lemma describes the mapping $\lambda \mapsto R_t \cup \lambda$ in terms of affine permutations. For $A \subset I$ and $0 \leq t \leq k$, we write $A + t = \{a + t \mid a \in A\}$ $(\subset I)$.

Lemma 1.2.15. *Let $1 \leq t \leq k$. Define a group isomorphism*

$$f_t \colon \tilde{S}_{k+1} \longrightarrow \tilde{S}_{k+1} \ ; \ s_i \mapsto s_{i+t} \quad \text{for } i \in I.$$

For any $\lambda \in \mathcal{P}_k$, we have

$$w_{R_t \cup \lambda} = f_t(w_\lambda) w_{R_t}.$$

Proof. Let $d_{A_m} \cdots d_{A_1}$ and $d_{B_{k+1-t}} \cdots d_{B_1}$ be the maximal decreasing decompositions of w_λ and w_{R_t}. Then $d_{A_m+t} \cdots d_{A_1+t}$ is the maximal decomposition of $f_t(w_\lambda)$. Stacking the k-code diagram of $f_t(w_\lambda)$ on top of that of w_{R_t}, we obtain the diagram (not necessarily justified to the bottom) corresponding to the (not necessarily maximal) decreasing decomposition $f_t(w_\lambda) w_{R_t} = d_{A_m+t} \cdots d_{A_1+t} d_{B_{k+1-t}} \cdots d_{B_1}$ (See Figure 1.5). With maximizing moves, we can justify the diagram to obtain one with shape $R_t \cup \lambda$, which corresponds to the maximal decomposition of $w_{R_t \cup \lambda}$. □

The next lemma explains the correspondence between weak strips over λ and weak strips over $R_t \cup \lambda$.

Lemma 1.2.16. *Let $\lambda \in \mathcal{P}_k$.*
(1) *For $A \subsetneq I$, if $d_A \lambda / \lambda$ is a weak strip then $R_t \cup (d_A \lambda) = d_{A+t}(R_t \cup \lambda)$.*
Moreover, let $d_{A_1}\lambda, d_{A_2}\lambda, \ldots$ be the list of all weak strips over λ (of size r).
(2) *$R_t \cup (d_{A_1}\lambda), R_t \cup (d_{A_2}\lambda), \ldots$ is the list of all weak strips over $R_t \cup \lambda$ (of size r).*
(3) *$d_{A_1+t}(R_t \cup \lambda), d_{A_2+t}(R_t \cup \lambda), \ldots$ is the list of all weak strips over $R_t \cup \lambda$ (of size r).*

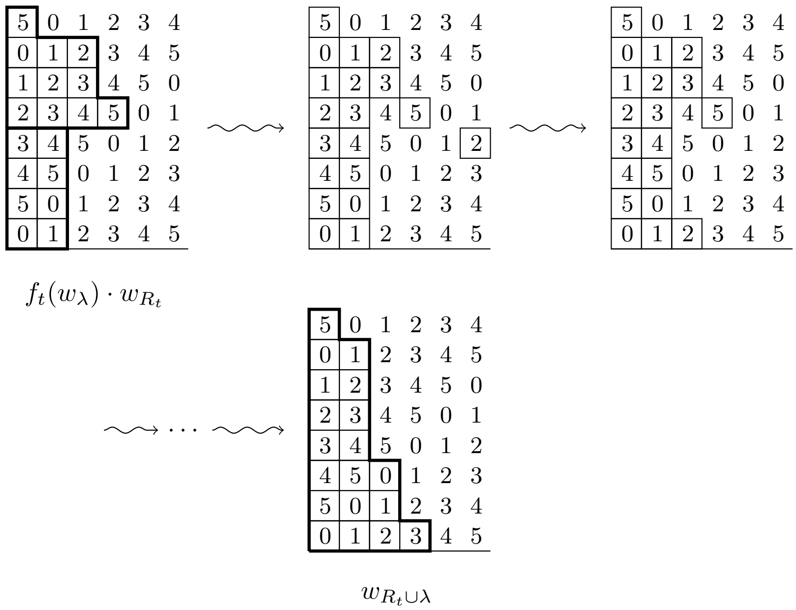


Figure 1.5: Justifying process with maximizing moves, where $k = 5$, $t = 2$, $R_2 = (2^4)$, and $\lambda = (4, 3, 3, 1)$.

Proof. (2) is [LM04, Theorem 20]. (3) follows from (1) and (2).

To prove (1), it suffices to show the case $|A| = 1$, that is, $R_t \cup (s_i\lambda) = s_{i+t}(R_t \cup \lambda)$ if $s_i\lambda \geq_L \lambda$. This is essentially shown in the process of proving [LM04, Theorem 20] by seeing correspondence between addable corners of $\mathfrak{c}(\lambda)$ with residue i and addable corners of $\mathfrak{c}(R_t \cup \lambda)$ with residue $i+t$, yet we here give another proof: by Lemma 1.2.15, it follows $w_{R_t \cup (s_i\lambda)} = f_t(w_{s_i\lambda})w_{R_t} = f_t(s_i w_\lambda)w_{R_t} = s_{i+t} f_t(w_\lambda) w_{R_t} = s_{i+t} w_{R_t \cup \lambda}$. □

1.2.3 Symmetric functions

For basic definitions for symmetric functions, see for instance [Mac95, Chapter I].

1.2.3.1 Symmetric functions

Let $\Lambda = \mathbb{Z}[h_1, h_2, \dots]$ be the ring of symmetric functions, generated by the *complete symmetric functions* $h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} \dots x_{i_r}$. For a partition λ we set $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_{l(\lambda)}}$. The set $\{h_\lambda\}_{\lambda \in \mathcal{P}}$ forms a $\mathbb{Z}$-basis of Λ.

1.2.3.2 Schur functions

The *Schur functions* $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ are the family of symmetric functions satisfying the *Pieri rule*:

$$h_r s_\lambda = \sum_{\mu/\lambda:\text{horizontal strip of size } r} s_\mu.$$

1.2.3.3 k-Schur functions

We recall a characterization of k-Schur functions given in [LM07], since it is a model for and has a relationship with K-k-Schur functions.

Definition 1.2.17 (the k-Schur function via the k-Pieri rule)**.** k-Schur functions $\{s_w^{(k)}\}_{w\in\tilde{S}_{k+1}^\circ}$ are the family of symmetric functions such that

$$s_e^{(k)} = 1,$$
$$h_r s_w^{(k)} = \sum_v s_v^{(k)} \quad \text{for } 0 \le r \le k \text{ and } w \in \tilde{S}_{k+1}^\circ,$$

summed over $v \in \tilde{S}_{k+1}^\circ$ such that v/w is a weak strip of size r.

It is known that $\{s_w^{(k)}\}_{w\in\tilde{S}_{k+1}^\circ}$ forms a basis of $\Lambda_{(k)} = \mathbb{Z}[h_1,\dots,h_k] \subset \Lambda$, and $s_w^{(k)}$ is homogeneous of degree $l(w)$. We regard $s_\lambda^{(k)}$ as $s_{w_\lambda}^{(k)}$ for $\lambda \in \mathcal{P}_k$. It is proved in [LM07, Theorem 40] that

Proposition 1.2.18 (the k-rectangle property)**.** *For $1 \le t \le k$ and $\lambda \in \mathcal{P}_k$, we have $s_{R_t\cup\lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)} (= s_{R_t} s_\lambda^{(k)})$.*

1.2.3.4 K-k-Schur functions

In this chapter we employ the following characterization with the Pieri rule ([LSS10, Corollary 7.6], [Mor12, Corollary 50]) of the K-k-Schur function as its definition.

Definition 1.2.19 (the K-k-Schur function via the K-k-Pieri rule)**.** K-k-Schur functions $\{g_w^{(k)}\}_{w\in\tilde{S}_{k+1}^\circ}$ are the family of symmetric functions such that $g_e^{(k)} = 1$ and

$$h_r \cdot g_w^{(k)} = \sum_{(A,v)} (-1)^{r+l(w)-l(v)} g_v^{(k)},$$

for $w \in \tilde{S}_{k+1}^\circ$ and $0 \le r \le k$, summed over $v \in \tilde{S}_{k+1}^\circ$ and $A \subsetneq I$ such that $(v/w, A)$ is an affine set-valued strip of size r.

It is known that $\{g_w^{(k)}\}_{w\in\tilde{S}_{k+1}^\circ}$ forms a basis of $\Lambda_{(k)}$. Besides, though $g_w^{(k)}$ is an inhomogeneous symmetric function in general, the degree of $g_w^{(k)}$ is $l(w)$ and its homogeneous part of highest degree is equal to $s_w^{(k)}$. In this chapter, for $f = \sum_w c_w g_w^{(k)} \in \Lambda_{(k)}$ we write $[g_v^{(k)}](f) = c_v$.

1.3 Properties of the strong and weak orderings on Coxeter groups

In this section we let (W,S) be an arbitrary Coxeter group.

Recall that for a poset $(P,\le)$ and a subset $A \subset P$, if the set $\{z \in P \mid z \le y \text{ for any } y \in A\}$ has the maximum element z_0 then z_0 is called the *meet* of A and is denoted by $\bigwedge A$, and if $\{z \in P \mid z \ge y \text{ for any } y \in A\}$ has the minimum element then it is called the *join* of A and denoted by $\bigvee A$. When $A = \{x,y\}$, its meet and join are simply called the meet and join of x and y, and denoted by $x \wedge y$ and $x \vee y$. A poset for which any nonempty subset has the meet is called a *complete meet-semilattice*. A poset for which any two elements have the meet and join is called a *lattice*. A subset of a complete meet-semilattice has the join if it has a common upper bound, since the join is the meet of all its common upper bounds then.

In this chapter we denote the meet of $x, y \in W$ under the strong (resp. left, right) order by $x \wedge y$ (resp. $x \wedge_L y$, $x \wedge_R y$) and call it the *strong meet* (resp. *left meet*, *right meet*) of $\{x,y\}$. We define $x \vee y$, $x \vee_L y$ and $x \vee_R y$ similarly.

1.3.1 Lattice property of the weak order

It is known that the weak order on any Coxeter group or its parabolic quotient forms complete meet-semilattices (see, for example, [BB05, Theorem 3.2.1]). The join of two elements in them, however, does not always exist, but it is known that the quotient of an affine Weyl group by its corresponding finite Weyl group forms a lattice under the weak order [Wau99]. We here include another proof for the type affine A case for the sake of completeness.

Lemma 1.3.1. *For any $v, w \in \tilde{S}^{\circ}_{k+1}$, their join $v \vee_L w$ under the left weak order exists.*

Proof. Since $\tilde{S}^{\circ}_{k+1}$ is a complete meet-semilattice, it remains to show the existence of a common upper bound of v and w under the left order. Let $s^{(k)}_v$ and $s^{(k)}_w$ denote the k-Schur functions corresponding to v and w, respectively. In the expansion of their product in the k-Schur function basis $s^{(k)}_v s^{(k)}_w = \sum_u c^u_{vw} s^{(k)}_u$, every u appearing in the right-hand side satisfies $w \leq_L u$ because $s^{(k)}_v$ can be written as a polynomial in $h_1, \dots, h_k$ and by the Pieri rule $h_i s^{(k)}_x$ is in general a linear combination of $s^{(k)}_y$ with $y \geq_L x$. By the same reason we have $v \leq_L u$. □

We proved the following corollary in the proof of the lemma above:

Corollary 1.3.2. *For any $v, w \in \tilde{S}^{\circ}_{k+1}$, every u appearing with a nonzero coefficient in the right-hand side of $s^{(k)}_v s^{(k)}_w = \sum_u c^u_{vw} s^{(k)}_u$ satisfies $u \geq_L v \vee_L w$.*

With the K-k-Pieri rule instead of the k-Pieri in hand, the same holds for the K-k-Schur functions:

Corollary 1.3.3. *For any $v, w \in \tilde{S}^{\circ}_{k+1}$, every u appearing with a nonzero coefficient in the right-hand side of $g^{(k)}_v g^{(k)}_w = \sum_u d^u_{vw} g^{(k)}_u$ satisfies $u \geq_L v \vee_L w$.*

1.3.2 Properties of Demazure and anti-Demazure actions

Lemma 1.3.4. *Let $x \in W$ and ϕ_x, ψ_x be the Demazure and anti-Demazure actions defined in Section 1.2.1.3.*

(1) *$\phi_x(w) \geq_L w$ and $\psi_x(w) \leq_L w$ for any $w \in W$.*

(2) *ϕ_x and ψ_x are order-preserving under $\leq$. Namely, if $v \leq w$ then $\phi_x(v) \leq \phi_x(w)$ and $\psi_x(v) \leq \psi_x(w)$.*

(3) *For any $y \in W$, the map $(x \mapsto \phi_x(y))$ is order-preserving and the map $(x \mapsto \psi_x(y))$ is order-reversing under $\leq$.*

(4) *$\phi_x \psi_{x^{-1}}(y) \geq y$ and $\psi_{x^{-1}} \phi_x(y) \leq y$ for any $y \in W$.*

(5) *ϕ_x preserves strong meets and ψ_x preserves strong joins. Namely, for $v, w \in W$,*

   (a) *if $v \wedge w$ exists then $\phi_x(v) \wedge \phi_x(w)$ exists and equals to $\phi_x(v \wedge w)$.*

   (b) *if $v \vee w$ exists then $\psi_x(v) \vee \psi_x(w)$ exists and equals to $\psi_x(v \vee w)$.*

Remark 1.3.5. This lemma also works for ϕ^R_x and ψ^R_x instead of ϕ_x and ψ_x.

Remark 1.3.6. For the statements on ϕ_x, (1) of this lemma is done in [BM15, Proposition 3.1(d)]; (2) and (3) in [BM15, Proposition 3.1(c)].

Proof. (1) is clear from the definition of ϕ_s and ψ_s. (2) is from the Lifting Property. (3) is clear from (1) and the Subword Property. For (4), the case $x = s \in S$ is clear from the definition of ϕ_s, ψ_s, and the general case follows from this and (2).

For (5)(a), it suffices to prove it when $x = s \in S$. Write simply $\phi = \phi_s$ and $\psi = \psi_s$. Assume $v \wedge w$ exists. We have $\phi(v \wedge w) \leq \phi(v), \phi(w)$ by (2). To show that $\phi(v \wedge w)$ is the meet of $\phi(v)$ and $\phi(w)$, take arbitrary u with $u \leq \phi(v), \phi(w)$. Then $\psi(u) \leq \psi(v), \psi(w)$ from the Lifting Property, and hence $\psi(u) \leq v, w$, which implies $\psi(u) \leq v \wedge w$. Applying ϕ, we have $\phi(u) = \phi(\psi(u)) \leq \phi(v \wedge w)$, and hence $u \leq \phi(v \wedge w)$. (5)(b) is essentially the same as (5)(a). □

Remark 1.3.7. The map ϕ_x (resp. ψ_x) does not preserve strong joins (resp. meets) in general. For example, letting $W = S_4$, we have $s_{212} \wedge s_{232} = s_2$ but $\psi_2(s_{212}) \wedge \psi_2(s_{232}) = s_{12} \wedge s_{32} = s_2 \neq \psi_2(s_2)$, where we write $s_{ij\ldots}$ instead of $s_i s_j \cdots$. Mapping everything above via $x \mapsto xw_0$ where w_0 is the longest element of W, we obtain a counterexample for ϕ_x preserving joins.

Corollary 1.3.8. *Let $u, v, x, y \in W$ with $\langle u\rangle\langle x\rangle$ and $\langle v\rangle\langle y\rangle$ reduced and $ux = vy$ (or namely, $u \leq_L ux = vy \geq_L v$). Then $u \geq v \iff x \leq y$.*

Proof. By Lemma 1.3.4(3) we have $u \geq v \iff u^{-1} \geq v^{-1} \implies (x =)\, \psi_{u^{-1}}(ux) \leq \psi_{v^{-1}}(vy)\, (= y)$. The other direction is similar. □

1.3.3 Half-strong, half-weak meets and joins

Analogous to the meets and joins under the weak order, we show the existence of the minimum element (under $\leq$) of the set

$$\{z \in W \mid x \leq z \geq_L y\},$$

and the maximum of

$$\{z \in W \mid x \geq_L z \leq y\}.$$

Remark 1.3.9. The existence of such elements may have been known, but we have not been able to find an appropriate reference. M. Shimozono explicitly used part (1) of the following key proposition, in his Sage implementation to compute the Deodhar lift [Deo87]. Upon our request for information, he sent us a proof of (1) [Shi], which we include here since we find it better than our original proof. We thank him for his permission for us to use it in our paper.

Proposition 1.3.10. *Let $x, y \in W$.*

(1) *The set $\{u \in W \mid x \leq \phi_u(y)\}$ has the minimum element $\psi^R_{y^{-1}}(x)$ under the strong order.*

(2) *The set $\{u \in W \mid \psi_{u^{-1}}(x) \leq y\}$ has the minimum element $\psi^R_{y^{-1}}(x)$ under the strong order.*

Proof. (1): We prove it by induction on $l(y)$. The base case $l(y) = 0$ being clear, we assume $l(y) > 0$. Take $s \in S$ such that $y > ys$. Let $x' = \psi^R_s(x)\, (= \min(x, xs))$ and $y' = ys$. Since $y = y' * s$, for any u we see $u * y = u * y' * s$, whence by the Lifting Property $x \leq u * y \iff x' \leq u * y'$. Hence $D(x, y) = D(x', y')$ where

$$D(x, y) = \{u \in W \mid x \leq \phi_u(y)\, (= u * y)\}.$$

By the induction hypothesis it follows that $D(x, y) = D(x', y')$ has the minimum element $\psi^R_{y'^{-1}}(x')$, which equals to $\psi^R_{y^{-1}}(x)$.
(2): Let $E(x, y) = \{u \in W \mid \psi_{u^{-1}}(x) \leq y\}$. It suffices to show $D(x, y) = E(x, y)$. By Lemma 1.3.4(2),(4) we have $x \leq \phi_u(y) \implies \psi_{u^{-1}}(x) \leq \psi_{u^{-1}}\phi_u(y) \leq y$ and $\psi_{u^{-1}}(x) \leq y \implies x \leq \phi_u\psi_{u^{-1}}(x) \leq \phi_u(y)$. □

Proposition 1.3.11. *Let $x, y \in W$.*

(1) *The set $\{z \in W \mid x \leq z \geq_L y\}$ has the minimum element $\psi^R_{y^{-1}}(x)y$ under the strong order.*

(2) *The set $\{z \in W \mid x \geq_L z \leq y\}$ has the maximum element $\left(\psi^R_{y^{-1}}(x)\right)^{-1}x$ under the strong order.*

Proof. (1): By (1.2.4), we have $D(x, y) \supset \{u \mid x \leq uy \geq_L y\} \simeq \{z \mid x \leq z \geq_L y\}$; $u \mapsto uy$, where the isomorphism is under $\leq$. The minimum element u of $D(x, y)$ satisfies $u * y = uy$ i.e. $uy \geq_L y$, since otherwise $(u * y)y^{-1}$ is a smaller element of $D(x, y)$ by Lemma 1.2.2. Hence by Proposition 1.3.10(1) we have $\psi^R_{y^{-1}}(x)y = \min_{\leq}\{z \mid x \leq z \geq_L y\}$.
(2): By Corollary 1.3.8 we have $E(x, y) \supset \{u \mid x \geq_L u^{-1}x \leq y\} \underset{\text{anti}}{\simeq} \{z \mid x \geq_L z \leq y\}$; $u \mapsto u^{-1}x$, where the anti-isomorphism is under $\leq$. For a similar reason to (1) we have $\max_{\leq}\{z \mid x \geq_L z \leq y\} = (\min_{\leq} E(x, y))^{-1}x = (\psi^R_{y^{-1}}(x))^{-1}x$. □

From the proposition above, we define

$$x_S\!\vee_L y = y_L\!\vee_S x := \min_{\leq}\{z \in W \mid x \leq z \geq_L y\} = \psi^R_{y^{-1}}(x)y,$$
$$x_L\!\wedge_S y = y_S\!\wedge_L x := \max_{\leq}\{z \in W \mid x \geq_L z \leq y\} = \left(\psi^R_{y^{-1}}(x)\right)^{-1}x.$$

We define $x_S\!\vee_R y$ and $x_S\!\wedge_R y$ similarly.

1.3.4 Flipping lower weak intervals

For any $z \in W$, define the map

$$\Phi_z : [e, z]_L \longrightarrow [e, z]_R; x \mapsto zx^{-1}$$

and its inverse

$$\Psi_z : [e, z]_R \longrightarrow [e, z]_L; y \mapsto y^{-1}z.$$

Proposition 1.3.12 below demonstrates that these maps behave well along with the strong order on W and its meet/join operations.

Proposition 1.3.12. *Let $z \in W$.*

(1) *Φ_z and Ψ_z are anti-isomorphisms under the strong order.*

(2) *$l(\Phi_z(x)) = l(z) - l(x)$ for any $x \in [e, z]_L$ and $l(\Psi_z(y)) = l(z) - l(y)$ for any $y \in [e, z]_R$.*

(3) *Φ_z and Ψ_z send strong meets to strong joins. Namely,*

  (a) *for $x, y \in [e, z]_L$ such that $x \wedge y$ exists and $x \wedge y \in [e, z]_L$, we have $\Phi_z(x \wedge y) = \Phi_z(x) \vee \Phi_z(y)$.*

  (b) *for $x, y \in [e, z]_R$ such that $x \wedge y$ exists and $x \wedge y \in [e, z]_R$, we have $\Psi_z(x \wedge y) = \Psi_z(x) \vee \Psi_z(y)$.*

  (*Note that the meets and joins are not taken in $[e, z]_L$ or $[e, z]_R$ but in W.*)

Proof. (1) is done in Corollary 1.3.8, and (2) is obvious.

For (3), we only prove (a) since (b) is shown similarly. Let $x, y, x \wedge y \in [e, z]_L$. From (1) it follows that $\Phi_z(x \wedge y) \geq \Phi_z(x), \Phi_z(y)$. To show the minimality of $\Phi_z(x \wedge y)$, let us take arbitrary $w \in W$ such that $w \geq \Phi_z(x), \Phi_z(y)$. From Proposition 1.3.11, we can let $w' = z_R\!\wedge_S w$. Since $\Phi_z(x), \Phi_z(y) \in [e, z]_R \cap [e, w]$, we have $\Phi_z(x), \Phi_z(y) \leq w'$. Since $w' \leq_R z$, applying Ψ_z $(= \Phi_z^{-1})$, we have $x, y \geq \Psi_z(w')$. Hence $x \wedge y \geq \Psi_z(w')$. Applying Φ_z, we have $\Phi_z(x \wedge y) \leq w'$, and hence $\Phi_z(x \wedge y) \leq w$. Therefore $\Phi_z(x \wedge y)$ is the join of $\{\Phi_z(x), \Phi_z(y)\}$. □

Remark 1.3.13. It seems to be true that Φ_z and Ψ_z send strong joins to strong meets. Its proof would require that there be the strong-minimum element of $\{z \mid x \leq z \leq_L y\}$ and the strong-maximum of $\{z \mid x \leq_L z \leq y\}$ for any $x, y \in W$, analogous to Proposition 1.3.11.

1.3.5 Chain Property for lower weak intervals

In this section we prove the Chain Property for the lower weak intervals $[e, u]_L$ and $[e, u]_R$ for an arbitrary Coxeter group W and its element $u \in W$. This can be regarded as a dual statement to the Chain Property for certain generalized quotients, since $[e, u]_L = \{x \mid x \leq_L u\}$ whereas the poset $W/\{u\}$, formed with the strong order, is isomorphic to $\{x \mid x \geq_L u\}$ with the same order. Besides it is shown in [BW88, Corollary 4.5] that the class of right generalized quotients and lower left intervals coincide for finite W, so that the validity of the claim is known for finite Coxeter groups. When W is infinite, however, these do not coincide, as we give a counterexample below. Beforehand we recall [BW88, Theorem 4.10]: for any Coxeter group W, the left generalized quotients and the right generalized quotients are in bijection by $U \mapsto W/U$ and $V\backslash W \leftarrow V$, and a subset $U \subset W$ is a right generalized quotient if and only if $U = W/(U\backslash W)$.

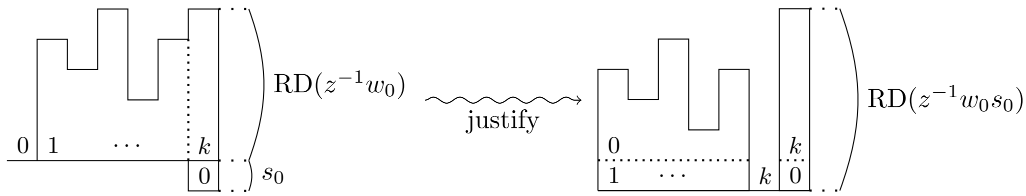


Figure 1.6: Inserting s_0 into $\mathrm{RD}(z^{-1}w_0)$ and justifying it to obtain a k-code for $z^{-1}w_0s_0$

Example 1.3.14. Let $W = \tilde{S}_{k+1} = \langle s_0, s_1, \ldots, s_k\rangle$. Let w_0 be the longest element of $S_{k+1} = \langle s_1, \ldots, s_k\rangle$. From the following claim it follows that any right generalized quotient of $\tilde{S}_{k+1}$ containing w_0 also contains s_0w_0, so that $S_{k+1} = [e, w_0]_L$ is not a right generalized quotient of $\tilde{S}_{k+1}$.

Claim. For any $z \in \tilde{S}_{k+1}$, the product $\langle w_0\rangle\langle z\rangle$ is reduced if and only if $\langle s_0w_0\rangle\langle z\rangle$ is reduced.

Proof of Claim. The "if" direction is clear. Toward the "only if" direction, assume $\langle w_0\rangle\langle z\rangle$ is reduced, that is, $\langle z^{-1}\rangle\langle w_0\rangle$ is reduced. Since $z^{-1}w_0 \geq_L w_0$, we have $\mathrm{RD}(z^{-1}w_0) \supset \mathrm{RD}(w_0)$. Hence, since the first row of $\mathrm{RD}(w_0)$ is $\{1, \ldots, k\}$ and the rows of a k-code are proper subsets of $\{0, 1, \ldots, k\}$, the first row of $\mathrm{RD}(z^{-1}w_0)$ is also $\{1, \ldots, k\}$. Thus, inserting s_0 into $\mathrm{RD}(z^{-1}w_0)$ from the bottom (see Figure 1.6) and justifying it to the bottom with maximizing moves, we successfully obtain $\mathrm{RD}(z^{-1}w_0s_0)$, the i-th column of which is

- the k-th column of $\mathrm{RD}(z^{-1}w_0)$ with an s_0 added, when $i = 0$,
- the i-th column of $\mathrm{RD}(z^{-1}w_0)$ when $i = 1, \ldots, k-1$,
- empty when $i = k$.

In particular $\langle z^{-1}w_0\rangle\langle s_0\rangle$ is reduced. Since $\langle z^{-1}\rangle\langle w_0\rangle$ is also reduced, we have $\langle z^{-1}\rangle\langle w_0\rangle\langle s_0\rangle$ is reduced, and hence so is $\langle s_0\rangle\langle w_0\rangle\langle z\rangle$, as desired. □

The proof of the following proposition is parallel to that of [BW88, Theorem 3.4]. Beforehand we recall that, for $x, y \in W$ with $x \geq y$ and any fixed reduced expression $x = s_1 \ldots s_m$, there exists $1 \leq j_1 < j_2 < \cdots < j_l \leq m$ such that $x = y^{(0)} \gtrdot y^{(1)} \gtrdot \cdots \gtrdot y^{(l)} = y$ where

$$y^{(a)} = s_1 \ldots \widehat{s_{j_1}} \ldots \widehat{s_{j_a}} \ldots s_m.$$

See, for example, [BW88, Section 3] or [BW82] for the details.

Proposition 1.3.15. *Let $u, x, y \in W$ with $xu, yu \leq_L u$ and $xu \leq yu$. Note that $xu \leq yu \iff x^{-1} \geq y^{-1} \iff x \geq y$ for $xu, yu \leq_L u$. Fix a reduced expression for $x = s_1 \ldots s_m$ and take $y^{(0)}, \ldots, y^{(l)}$ as in the paragraph immediately above. Then $y^{(a)}u \leq_L u$ for any a.*

Proof. Suppose to the contrary that there exists a such that $y^{(a)}u \not\leq_L u$. Since $y^{(l)}u = yu \leq_L u$, we can take such a that $y^{(a)}u \not\leq_L u$ and $y^{(a+1)}u \leq_L u$.

Since $xu \leq_L u$, we have $s_{j_a+1} \ldots s_m u \leq_L u$. Hence there exists $p < j_a$ such that

$$s_p zu \not\leq_L u \quad \text{and} \quad zu \leq_L u, \tag{1.3.1}$$

where we put

$$z = s_{p+1} \ldots \widehat{s_{j_a}} \ldots s_{j_{a+1}} \ldots s_m,$$

where there may be more indices omitted between s_{p+1} and $\widehat{s_{j_a}}$ (including s_{p+1}), according to the omissions in $y^{(a)} = s_1 \ldots \widehat{s_{j_1}} \ldots \widehat{s_{j_a}} \ldots s_m$. Since $y^{(a+1)}u \leq_L u$, we have

$$s_p \widehat{z} u \leq_L u \quad \text{and} \quad \widehat{z} u \leq_L u, \tag{1.3.2}$$

where we put

$$\widehat{z} = s_{p+1} \dots \widehat{s_{j_a}} \dots \widehat{s_{j_{a+1}}} \dots s_m.$$

We have $zu \lessdot s_p zu$ by (1.3.1) and $\widehat{z}u \gtrdot s_p\widehat{z}u$ by (1.3.2). Besides, since $y^{(a)} > y^{(a+1)}$ it follows $z > \widehat{z}$, and thereby $zu \lessdot \widehat{z}u$. Hence we have $s_p zu = \widehat{z}u$ by the Lifting Property and length arguments. Therefore $s_p z = \widehat{z} \lessdot z$, which contradicts the fact that $s_p z$ is a consecutive subword of a reduced expression for $y^{(a)}$. ☐

As a corollary, we have the Chain Property for weak lower intervals:

Theorem 1.3.16. *For any $u \in W$, the interval $[e,u]_L$ (resp. $[e,u]_R$) under the left (resp. right) weak order has the Chain Property.*

Proof. The statement for left lower intervals follows from Proposition 1.3.15 and that $\{x \in W \mid xu \leq_L u\} = [e,u^{-1}]_L$ for $u \in W$, which follows from $xu \leq_L u \iff x^{-1} \leq_R u \iff x \leq_L u^{-1}$. The statement for right intervals is proved by a parallel argument. ☐

1.4 Properties of the weak strips

Hereafter we restrict our attention to $\tilde{S}_{k+1}$ rather than general Coxeter groups and let $W = \tilde{S}_{k+1}$ and $W^\circ = \tilde{S}^\circ_{k+1}$. In Section 1.2.2 we put $I = \mathbb{Z}_{k+1} = \{0,1,\dots,k\}$ and let d_A denote the cyclically decreasing element corresponding to $A \subsetneq I$.

In this section we prove some properties on weak strips. First we define for any $u \in W$,

$$\begin{aligned}
Z_{u,+} &= \{v \in W \mid v = d_A u \geq_L u \text{ for } \exists A \subsetneq I\},\\
Z'_{u,+} &= \{A \subsetneq I \mid d_A u \geq_L u\} = \{A \subsetneq I \mid d_A u \in Z_{u,+}\},\\
Z_{u,-} &= \{v \in W \mid v = d_A^{-1} u \leq_L u \text{ for } \exists A \subsetneq I\},\\
Z'_{u,-} &= \{A \subsetneq I \mid d_A^{-1} u \leq_L u\} = \{A \subsetneq I \mid d_A^{-1} u \in Z_{u,-}\}.
\end{aligned}$$

It is an immediate observation from the Subword Property that

- The map $(Z'_{u,+}, \subset) \longrightarrow (Z_{u,+}, \leq)\,;\, A \mapsto d_A u$ is an isomorphism of posets.
- The map $(Z'_{u,-}, \subset) \longrightarrow (Z_{u,-}, \leq)\,;\, A \mapsto d_A^{-1} u$ is an anti-isomorphism of posets.

Since if $u \in W^\circ$ and $v \leq_L u$ then $v \in W^\circ$, for $u \in W^\circ$ we have

$$Z_{u,-} = \{v \mid u/v \text{ is a weak strip}\}.$$

On the other hand, the set $Z_{u,+}$ does not coincide with the set of v such that v/u is a weak strip. More precisely, for $u \in W^\circ$ we have by definition

$$v/u \text{ is a weak strip} \iff v \in Z_{u,+} \text{ and } v \in W^\circ.$$

Recalling that $v \in W^\circ \iff vw_0^J \geq_L w_0^J$ where $J = \{1,\dots,k\}$ and w_0^J is the longest element of $W_J = S_{k+1}$, by Lemma 1.2.1 we have

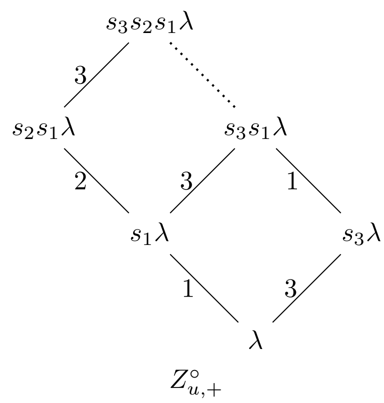
$$\begin{aligned}
v/u \text{ is a weak strip} &\iff vw_0^J \in Z_{uw_0^J,+}\\
&\iff v = d_A u \text{ with } A \in Z'_{uw_0^J,+}.
\end{aligned}$$

In other words, defining

$$\begin{aligned}
Z^\circ_{u,+} &= \{v \mid v/u \text{ is a weak strip}\},\\
Z'^\circ_{u,+} &= \{A \subsetneq I \mid d_A u/u \text{ is a weak strip}\} = \{A \subsetneq I \mid d_A u \in Z^\circ_{u,+}\},
\end{aligned}$$

we have

$$\begin{aligned}
Z^\circ_{u,+} &\simeq Z_{uw_0^J,+}\ ;\ v \mapsto vw_0^J,\\
Z'^\circ_{u,+} &= Z'_{uw_0^J,+}.
\end{aligned}$$



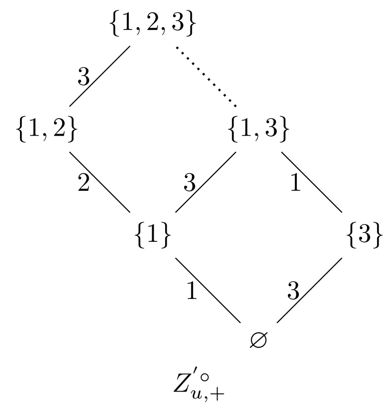


Figure 1.7: The posets $Z^{\circ}_{u,+}$ ($\simeq Z_{uw_0^J,+}$) and $Z'^{\circ}_{u,+}$ ($= Z'_{uw_0^J,+}$) for $u = w_\lambda$ where $k = 3$ and $\lambda = (3,2,1) \in \mathcal{P}_3$ (and w_0^J is the longest element of S_4). Left weak covers are represented as solid lines, and strong covers are solid or dotted lines. A solid edge between v and w is labelled with i if $v = s_i w$.

Example 1.4.1. Figure 1.7 illustrates the same example as Example 1.2.10.

From the example above, we would expect these properties:

(1) $Z'_{u,\pm}$ is closed under intersection and union.

(2) $Z'_{u,\pm}$ has the maximum element.

(3) $Z_{u,\pm}$ and $Z'_{u,\pm}$ have the Chain Property. (See Section 1.4.3 for the details.)

(1), (2), (3) are proved in Section 1.4.1, 1.4.2, 1.4.3, respectively.

1.4.1 Intersection and union

In this section we prove the following proposition as the compilation of Lemma 1.4.5, 1.4.9 and 1.4.10.

Proposition 1.4.2. *For $u \in W$, we have*

(1) $A, B \in Z'_{u,\pm}$ *and* $A \cup B \neq I \implies A \cup B \in Z'_{u,\pm}$.

(2) $A, B \in Z'_{u,\pm} \implies A \cap B \in Z'_{u,\pm}$.

(3) $A, B \in Z'_{u,+} \implies d_{A\cap B}u = (d_A u) \wedge (d_B u)$.

(4) $A, B \in Z'_{u,-} \implies d^{-1}_{A\cap B}u = (d^{-1}_A u) \vee (d^{-1}_B u)$.

In this section we say $A, B \subset I$ are *strongly disjoint* if for any $i \in A$ and $j \in B$ it holds that $i - j \not\equiv 0, \pm 1$, and $x, y \in W$ are *strongly commutative* if any Coxeter generator s_i appearing in a reduced expression of x and any s_j appearing in that of y satisfy $i - j \not\equiv 0, \pm 1$. The next lemma is straightforward.

Lemma 1.4.3. *Let $A, B \subsetneq I$ and $x, y \in W$.*

(1) *If A, B are strongly disjoint, then d_A, d_B are strongly commutative.*

(2) *For the decomposition $A = A_1 \sqcup \cdots \sqcup A_m$ into connected components, $A_1, \ldots, A_m$ are pairwise strongly disjoint and $d_{A_1}, \ldots, d_{A_m}$ are pairwise strongly commutative.*

(3) *For $x' \le x$ and $y' \le y$, if x, y are strongly commutative then so are x', y'.*

(4) *If x, y are strongly commutative, then x, y are commutative and $l(xy) = l(x) + l(y)$.*

Lemma 1.4.4. *Let $x, y, z \in W$ with x, y strongly commutative. Then*

(1) $z \leq_L xyz \iff z \leq_L xz, yz$.

(2) $z \geq_L xyz \iff z \geq_L xz, yz$.

Proof. We only prove (1) since (2) is shown similarly.

The "only if" direction immediately follows by the definition of the weak order and commutativity of x, y. We prove the "if" direction by induction on $l(x) + l(y)$. It is clear when $l(x) = 0$ or $l(y) = 0$. In particular the case $l(x) + l(y) \leq 1$ is done and we may assume $l(x) + l(y) \geq 2$ and $l(x), l(y) > 0$.

Step A: the case $l(x) + l(y) = 2$, i.e. $l(x) = l(y) = 1$.

We can write $x = s_i$ and $y = s_j$ with $s_i \neq s_j$, $s_i s_j = s_j s_i$ from the strong commutativity. We have $s_i z, s_j z \geq_L z$ by the assumption. Hence $z \in W/W_{\{i,j\}}$, where $W_{\{i,j\}} = \langle s_i, s_j \rangle = \{e, s_i, s_j, s_i s_j\}$. Therefore $s_i s_j z \geq_L z$.

Step B: the case $l(x) + l(y) > 2$.

From the commutativity of x, y we may assume $l(y) \geq l(x)$; in particular $l(y) > 1$. Take a reduced expression of $y = s_{i_1} \dots s_{i_l}$ and put $y' = s_{i_1} \dots s_{i_{l-1}}$, $z' = s_{i_l} z$. Since $z \leq_L yz$ and $s_{i_l} \leq_L y$, we have $z \leq_L z'$. Now we can obtain $z' \leq_L xy'z'$, which implies $z \leq_L z' \leq_L xy'z' = xyz$ as desired, by applying the induction hypothesis for $(x, y, z) := (x, y', z')$, having its assumption satisfied as follows:

- x, y' are strongly commutative.
  Proof. From Lemma 1.4.3 (3).
- $z' \leq_L y'z'$.
  Proof. Since $z \leq_L yz$ and $s_{i_l} \leq_L y$, by Lemma 1.2.1(1) we have $z' = s_{i_l} z \leq_L yz = y'z'$.
- $z' \leq_L xz'$.
  Proof. Since $l(x) + l(y) > l(x) + l(s_{i_l})$, we can obtain $z \leq_L xz'$ by applying the induction hypothesis for $(x, y, z) := (x, s_{i_l}, z)$, having that its assumption described below is clearly satisfied:
  - x and s_{i_l} are strongly commutative.
  - $z \leq_L xz$.
  - $z \leq_L s_{i_l} z$.

  Besides $s_{i_l} \leq_L x s_{i_l}$, hence we have $z' \leq_L xz'$ by Lemma 1.2.1(1).

□

Lemma 1.4.5. *Let $w \in W$ and $A, B \subsetneq I$ with $w \leq_L d_A w, d_B w$.*

(1) $w \leq_L d_{A \cap B} w$.

(2) *The element $d_{A \cap B} w$ is the strong meet of $d_A w$ and $d_B w$.*

Remark 1.4.6. The same statement with all d_X replaced with u_X is proved similarly.

Remark 1.4.7. It does not generally hold that if $w \leq_L xw, yw$ and $x \wedge y$ exists then $w \leq_L (x \wedge y)w$; a counterexample is $W = S_4$, $x = s_{21}$, $y = s_{23}$, $w = s_2$.

Proof. (1): Within this proof we say $x \in W$ satisfies $(*)$ if $w \leq_L xw$.

Decomposing A, B into connected components $A = A_1 \sqcup \dots \sqcup A_m$ and $B = B_1 \sqcup \dots \sqcup B_n$, we have $A \cap B = \bigsqcup_{i,j} (A_i \cap B_j)$. Each nonempty $A_i \cap B_j$ has at most two connected components, each component C of which satisfies $d_{A_i} = x d_C$ for some $x \in W$ or $d_{B_j} = y d_C$ for some $y \in W$ as easily seen. Having that both d_A $(\geq_L d_{A_i})$ and d_B $(\geq_L d_{B_j})$ satisfy $(*)$ and that lower bounds in $\leq_L$ inherit $(*)$, we see each d_C satisfies $(*)$. Besides $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = (A_i \cap A_{i'}) \cap (B_j \cap B_{j'})$ is empty unless $(i, j) = (i', j')$, we thus have

$A \cap B$ decomposes as $A \cap B = C_1 \sqcup \cdots \sqcup C_l$ into connected components, where each d_{C_i} satisfies ($*$). Now it follows from Lemma 1.4.4 (1) that $d_{A\cap B} = d_{C_1} \dots d_{C_l}$ satisfies ($*$), as desired.

(2): By the Subword Property we have $d_{A\cap B} = d_A \wedge d_B$. From the assumption and (1), we have $\phi_w^R(d_X) = d_X w$ for $X = A, B, A\cap B$. Hence by Lemma 1.3.4 (5) we have $d_{A\cap B}w = d_A w \wedge d_B w$. □

Corollary 1.4.8. *Let $\lambda \in \mathcal{P}_k$, and $\kappa^{(1)}, \kappa^{(2)}$ be weak strips over λ. Write $\kappa^{(i)} = d_{A_i}\lambda$ for each i with $A_i \subsetneq I$. Then $d_{A_1\cap A_2}\lambda$ is a weak strip over λ and is the meet of $\kappa^{(1)}, \kappa^{(2)}$ in the poset $\mathcal{P}_k$ with the strong order: $\kappa^{(1)} \wedge \kappa^{(2)} = d_{A_1\cap A_2}\lambda$.*

Proof. Let $w_\lambda \in W^\circ$ be the affine Grassmannian permutation corresponding to λ, and w_0 the longest element of S_{k+1}. By Lemma 1.2.1, the condition $d_A\lambda/\lambda$ is a weak strip is equivalent to $d_A w_\lambda w_0 \geq_L w_\lambda w_0$. From this and Lemma 1.4.5(1) we see $d_{A_1\cap A_2}\lambda/\lambda$ is a weak strip. From Lemma 1.4.5(2) we have $d_{A_1\cap A_2}w_\lambda = (d_{A_1}w_\lambda) \wedge (d_{A_2}w_\lambda)$ in W. Since $W^\circ \subset W$ is a subposet, this is also the meet in $W^\circ \simeq \mathcal{P}_k$. □

Lemma 1.4.9. *Let $w \in W$ and $A, B \subsetneq I$ with $d_A^{-1}w, d_B^{-1}w \leq_L w$.*

(1) *$d_{A\cap B}^{-1}w \leq_L w$.*

(2) *The element $d_{A\cap B}^{-1}w$ is the strong join of $d_A^{-1}w$ and $d_B^{-1}w$.*

Proof. (1) is proved parallelly to Lemma 1.4.5 (1), making use of Lemma 1.4.4(2) instead of Lemma 1.4.4(1).

Next we show (2). We have $d_A^{-1}w, d_B^{-1}w, d_{A\cap B}^{-1}w \in [e, w]_L$ by (1). The map Φ_w in Lemma 1.3.12 sends $d_A^{-1}w, d_B^{-1}w, d_{A\cap B}^{-1}w$ to $d_A, d_B, d_{A\cap B}$ respectively. Since $d_{A\cap B} = d_A \wedge d_B$, sending them back via Ψ_w, we have $d_{A\cap B}^{-1}w = (d_A^{-1}w) \vee (d_B^{-1}w)$ by Lemma 1.3.12(3). □

Lemma 1.4.10. *Let $u \in W$ and $A, B \subsetneq I$ with $A \cup B \neq I$.*

(1) *If $d_A u, d_B u \geq_L u$, then $d_{A\cup B}u \geq_L u$.*

(2) *If $d_A^{-1}u, d_B^{-1}u \leq_L u$, then $d_{A\cup B}^{-1}u \leq_L u$.*

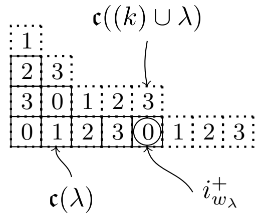
Proof. We only give a proof of (1) since that of (2) is quite similar.

Assume $d_A u, d_B u \geq_L u$. Take the decomposition $A = A_1 \sqcup \cdots \sqcup A_m$ and $B = B_1 \sqcup \cdots \sqcup B_n$ into connected components. Since $d_{A_i} \leq_L d_A$, we have $d_{A_i}u \geq_L u$ for any i, and similarly $d_{B_j}u \geq_L u$ for any j. Since $A \cup B = (\dots(A\cup B_1)\cup\dots)\cup B_n$, we only need to prove it when B is connected. Assume B is connected. It is also easy to see, from Lemma 1.4.3 and Lemma 1.4.4(1), that it suffices to prove it when A, B and $A\cup B$ are connected. We therefore assume A, B and $A\cup B$ are connected. The case $A \subset B$ or $B \subset A$ being clear, we assume $A \not\subset B$ and $B \not\subset A$; namely we let $A = [i, j]$ and $B = [p, q]$ with $p \leq i \leq q+1 \leq j+1$ without loss of generality, where we employ an ordering $r+1 < \cdots < k < 0 < \cdots < r-1$ of $I \setminus \{r\}$ with an arbitrarily fixed element $r \in I \setminus (A \cup B)$. Since $d_B = s_q \dots s_p \geq_L s_{i-1}\dots s_p = d_{B\setminus A}$ and $d_B u \geq_L u$, we have $d_{B\setminus A}u \geq_L u$. Hence we may replace B by $B \setminus A$ $(= [p, i-1])$.

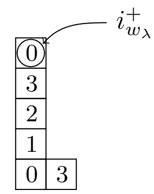
Let $B' = B \setminus \{i-1\} = [p, i-2]$ and $u' = d_{B'}u$. Since $d_{B'} \leq_L d_B$ and $d_B u \geq_L u$, it follows that $u' \geq_L u$. Since $s_{i-1}u' = d_B u \geq_L u$ and $d_A u' = d_A d_{B'} u \geq_L u$, the latter of which is from Lemma 1.4.4 (1), it easily follows that $s_{i-1}u' \geq_L u'$ and $d_A u' \geq_L u'$ from Lemma 1.2.1.

Toward a contradiction, suppose $d_{A\cup B}u \not\geq_L u$. Then we have $d_A s_{i-1}u' \not\geq_L u'$ since $d_{A\cup B}u = d_A s_{i-1}u'$ and $u \leq_L u'$. Since $s_{i-1}u' \geq_L u'$, there exists $a \in [i, j]$ such that $xs_{i-1}u' \geq_L u'$ and $s_a x s_{i-1}u' \not\geq_L u'$, which implies $s_a x s_{i-1}u' \lessdot x s_{i-1}u'$, where we write $x = s_{a-1}s_{a-2}\dots s_{i+1}s_i$. On the other hand, since $d_A u' \geq_L u'$ we have $s_a x u' \geq_L u'$ and $xu' \geq_L u'$. Besides we have $xs_{i-1}u' \gtrdot xu'$ from the Subword Property. Hence the Lifting Property implies that $xu' \leq s_a x s_{i-1}u'$, which is actually an equality since both sides have the same length. Therefore we have $(s_{a-1}s_{a-2}\dots s_{i+1}s_i =)\ x = s_a x s_{i-1}$ $(= s_a s_{a-1}\dots s_i s_{i-1})$, which is absurd. □

Remark 1.4.11. Unlike the "cap" case, it does not always hold that $d_{A\cup B}u = (d_A u) \vee (d_B u)$ in (1), or $d_{A\cup B}^{-1}u = (d_A^{-1}u) \wedge (d_B^{-1}u)$ in (2). A counterexample for (1) is given by $W = S_3$, $u = e$, $A = \{1\}$ and $B = \{2\}$.



$\mathfrak{c}(\lambda)$ and $\mathfrak{c}((k)\cup\lambda)$



$\mathrm{RI}(w_\lambda) = \lambda^{\omega_3}$

Figure 1.8: An example where $k = 3$, $\lambda = (3, 2, 1)$ and $\mathfrak{c}(\lambda) = (5, 2, 1)$. The dotted shape on the left figure represents $\mathfrak{c}((k) \cup \lambda)$, and the solid one does $\mathfrak{c}(\lambda)$. In this case $w_{(k)\cup\lambda} = s_3 s_2 s_1 w_\lambda = d_{\{1,2,3\}} w_\lambda$ and therefore $i^+_{w_\lambda} = 0$.

1.4.2 Non-appearing indices

Proposition 1.4.12. (1) *For any $w \in W$, there exists $i^-_w \in I$ such that $i^-_w \notin A$ for any $A \subsetneq I$ with $d_A^{-1} w \leq_L w$.*
(2) *For any $w \in W^\circ$, there exists $i^+_w \in I$ such that $i^+_w \notin A$ for any $A \subsetneq I$ with $d_A w \geq_L w$ and $d_A w \in W^\circ$.*

Proof. (1): For any $A \subsetneq I$, we have

$$\begin{aligned} d_A^{-1} w \leq_L w &\iff d_A \leq_R w \\ &\iff u_A \leq_L w^{-1} \\ &\implies \mathrm{RI}(u_A) \subset \mathrm{RI}(w^{-1}), \end{aligned}$$

and the last condition is equivalent to A being included by the first row of $\mathrm{RI}(w^{-1})$. Hence we can take i^-_w from the complement of the first row of $\mathrm{RI}(w^{-1})$.
(2): By Lemma 1.3.1 we may take $z := \bigvee_L \{d_A w \mid A \subsetneq I \text{ s.t. } d_A w \geq_L w,\, d_A w \in W^\circ\}$, the left join of all weak strips over w. Take any $A \subsetneq I$ such that $d_A w \geq_L w$ and $d_A w \in W^\circ$. Since $w, d_A w \leq_L z$, we have $zw^{-1} \geq_R z(d_A w)^{-1} = zw^{-1} u_A$, which is equivalent to $wz^{-1} \geq_L d_A w z^{-1}$. Hence, similarly to the proof of (1) we have A is a subset of the first row of $\mathrm{RD}((wz^{-1})^{-1}) = \mathrm{RD}(zw^{-1})$, which is a proper subset of I and independent of A, and therefore we can take i^+_w from its complement. □

Remark 1.4.13. The index i^+_w in (2) above is in fact uniquely determined as follows: a bounded partition $\lambda \in \mathcal{P}_k$, corresponding to a 0-dominant affine permutation $w_\lambda \in W^\circ$, has the unique weak strip of size k, namely $(k) \cup \lambda$. Since the corresponding core $\mathfrak{c}((k) \cup \lambda)$ has k more boxes in the first row than $\mathfrak{c}(\lambda)$ does, the only possibility for $i^+_{w_\lambda}$ is what is determined by the following equivalent descriptions:

- The residue of the rightmost box in the first row of $\mathfrak{c}(\lambda)$.
- The negative of the residue written in the leftmost box in the last row of $\mathrm{RI}(w_\lambda) = \lambda^{\omega_k}$.
- $m - 1$, where $w_\lambda = u_{A_m} \dots u_{A_1}$ is the maximal increasing decomposition for w_λ. (Note that $A_m = \{i, i+1, \dots, m-2, m-1\}$ for some i.)

Remark 1.4.14. We cannot drop the assumption on 0-dominantness of $d_A w$ in (2) of the proposition. For example, let $k = 3$ and $w = s_3 s_0$. Then $w = u_{\{3,0\}}$ is the maximal increasing decomposition and hence i^+_w should be 0, but $d_{\{0\}} w = s_0 s_3 s_0 \geq_L w$.

Corollary 1.4.15. *Let $u \in W$.*

(1) *$Z'_{u,+}$ has the maximum element under $\subset$. Hence, $Z_{u,+}$ has the maximum element under $\leq$.*

(2) *$Z'_{u,-}$ has the maximum element under $\subset$. Hence, $Z_{u,-}$ has the minimum element under $\leq$.*

Proof. By Proposition 1.4.2 (1) and Proposition 1.4.12. □

1.4.3 Chain Property

Recall that an *order ideal* of a poset P is a subset $X \subset P$ such that if $x \in X$ and $y \leq x$ then $y \in X$, and an *order filter* of P is a subset $X \subset P$ such that if $x \in X$ and $y \geq x$ then $y \in X$.

Proposition 1.4.16. *The sets $Z_{u,+}$ and $Z_{u,-}$ have the Chain Property. Namely, for any $x, y \in Z_{u,\pm}$ such that $x \leq y$, there exists a sequence $x = z^{(0)} \lessdot z^{(1)} \lessdot \ldots \lessdot z^{(l)} = y$ such that $z^{(i)} \in Z_{u,\pm}$ for any i.*

Proof. First we note a few immediate observations:

- For a poset P and a subposet $Q \subset P$, if $A \subset P$ is an order ideal then $A \cap Q$ is an order ideal of Q.
- If a subset X of a Coxeter group W has the Chain Property and $Y \subset X$ is an order ideal, then Y also has the Chain Property.

Let $D = \{d_A \mid A \subsetneq I\}$. Since $D \subset W$ is an order ideal, the set $\{d_A \mid d_A \leq_R u\} = D \cap [e, u]_R$ is an order ideal of $[e, u]_R$ and hence has the Chain Property since so does $[e, u]_R$ as proved in Theorem 1.3.16. Hence $Z_{u,-}$ also has the Chain Property since it is the image of $\{d_A \mid d_A \leq_R u\}$ under the the anti-isomorphism $\Psi_u \colon [e, u]_R \longrightarrow [e, u]_L; x \mapsto x^{-1}u$.

Similarly, the set $\{d_A \mid d_A u \geq_L u\} = D \cap (W/\{u\})$ has the Chain Property since it is an order ideal of $W/\{u\}$, which has the Chain Property [BW88, Corollary 3.5]. Hence, since $Z_{u,+}$ is the image of $\{d_A \mid d_A u \geq_L u\}$ under the isomorphism $(\cdot u) \colon W/\{u\} \longrightarrow [u, \infty)_L$, we conclude that $Z_{u,+}$ has the Chain Property. $\square$

From the isomorphism $(Z_{u,+}, \leq) \simeq (Z'_{u,+}, \subset)$ and the anti-isomorphism $(Z_{u,-}, \leq) \underset{\text{anti}}{\simeq} (Z'_{u,-}, \subset)$, we have the Chain Property for $Z'_{u,\pm}$:

Corollary 1.4.17. *The sets $Z'_{u,+}$ and $Z'_{u,-}$ have the Chain Property. Namely, for any $A, B \in Z'_{u,\pm}$ with $A \subset B$, there exists a sequence $A = C^{(0)} \subset\!\!\cdot\; C^{(1)} \subset\!\!\cdot \ldots \subset\!\!\cdot\; C^{(l)} = B$ such that $C^{(i)} \in Z'_{u,\pm}$ for any i.*

1.5 Proof of the Pieri rule for $\widetilde{g}_\lambda^{(k)}$

This section is devoted for the proof of Theorem 1.1.3 and 1.1.4.

1.5.1 Outline

Recall the K-k-Pieri rule (Definition 1.2.19): for $v \in W^\circ$ and $0 \leq i \leq k$,

$$g_v^{(k)} h_i = \sum_{\substack{A \subset I, |A| = i \\ d_A * v \in W^\circ}} (-1)^{i - (l(d_A * v) - l(v))} g_{d_A * v}^{(k)}. \tag{1.5.1}$$

Let $w = w_\lambda \in W^\circ$ be the affine Grassmannian element corresponding to λ. Summing (1.5.1) up over $v \in W^\circ \cap [e, w]$ and $i \in \{0, 1, \ldots, r\}$, we have

$$\widetilde{g}_w^{(k)} \widetilde{h}_r = \sum_{\substack{v \leq w \\ v \in W^\circ}} \sum_{\substack{A \subset I, |A| \leq r \\ d_A * v \in W^\circ}} (-1)^{|A| - (l(d_A * v) - l(v))} g_{d_A * v}^{(k)},$$

and its coefficient of $g_u^{(k)}$ (for $u \in W^\circ$) is

$$[g_u^{(k)}](\widetilde{g}_w^{(k)} \widetilde{h}_r) = \sum_{\substack{v \leq w \\ v \in W^\circ}} \sum_{\substack{A \subset I, |A| \leq r \\ u = d_A * v}} (-1)^{|A| - (l(u) - l(v))}. \tag{1.5.2}$$

We shall illustrate, in the example below, that if the summation above is not empty then there are exactly one larger number of pairs (v, A) with $(-1)^{|A| - (l(u) - l(v))} = +1$ than those with $(-1)^{|A| - (l(u) - l(v))} = -1$, and consequently the value of the summation in (1.5.2) is equal to 1.

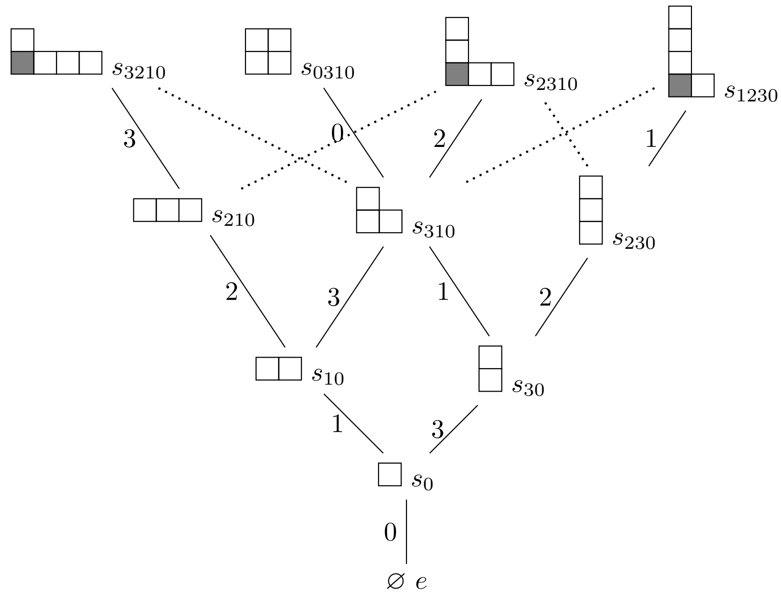


Figure 1.9: The poset of 4-cores (and corresponding elements in $\tilde{S}_4^\circ$) up to those of size 4. The left weak covers are represented by solid lines, and the strong covers are dotted or solid lines. A solid edge labelled with i corresponds to the left multiplication by s_i.

Example 1.5.1. Let $k = 3$ and $u = s_{310} = w_\lambda \in \tilde{S}_4^\circ$ where $\lambda = (2,1) \in \mathcal{P}_3$. Table 1.1 lists the pairs (v, A) such that $d_A * v = u$, organized according to the size of A. Apparently there are the same number of pairs (v, A) with $|A| = r'$ and $(-1)^{|A|-(l(u)-l(v))} = +1$, and those with $|A| = r'$ and $(-1)^{|A|-(l(u)-l(v))} = -1$, for each fixed $r' > 0$. Furthermore, introducing the condition $v \le w$ for $w = s_{210}$, say, we see that the set of the pairs (v, A) with $d_A * v = u$ and $v \le w$ is $\{(s_{10}, \{3\}), (s_0, \{1,3\}), (s_{10}, \{1,3\})\}$, and that the number of such pairs (v, A) with $|A| = r'$ and $(-1)^{|A|-(l(u)-l(v))} = +1$ and those with $|A| = r'$ and $(-1)^{|A|-(l(u)-l(v))} = -1$ coincide whenever $r' \neq 1$, and differ by 1 when $r' = 1$.

	(v, A)	$(-1)^{\lvert A\rvert-(l(u)-l(v))}$
$\lvert A\rvert = 0$	$(s_{310}, \varnothing)$	$+1$
$\lvert A\rvert = 1$	$(s_{30}, \{1\})$	$+1$
	$(s_{310}, \{1\})$	-1
	$(s_{10}, \{3\})$	$+1$
	$(s_{310}, \{3\})$	-1
$\lvert A\rvert = 2$	$(s_0, \{1,3\})$	$+1$
	$(s_{10}, \{1,3\})$	-1
	$(s_{30}, \{1,3\})$	-1
	$(s_{310}, \{1,3\})$	$+1$

Table 1.1: The list of (v, A) such that $d_A * v = u$, where $u = s_{310}$.

According to the observation above, for $u \in W^\circ$ and $A \subsetneq I$ we let

$$X_{A,u} = \{v \in W \mid d_A * v = u\} = \{v \in W \mid \phi_{d_A}(v) = u\},$$
$$Y_{A,u} = X_{A,u} \cap [e, w],$$

and

$$X'_{A,u} = \{B \subsetneq I \mid d_B^{-1}u \in X_{A,u}\},$$
$$Y'_{A,u} = \{B \subsetneq I \mid d_B^{-1}u \in Y_{A,u}\}.$$

Note that, for any $v \in X_{A,u}$, Lemma 1.3.4(1) implies $v \leq_L u$, and hence it follows from $u \in W^\circ$ that $v \in W^\circ$. Hence

$$[g_u^{(k)}](\widetilde{g}_w^{(k)}\widetilde{h}_r) = \sum_{|A|\leq r} \sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))}. \tag{1.5.3}$$

The flow of the proof is as follows:

Step 1. Every element of $X_{A,u}$ has the form $d_B^{-1}u$ with $B \subset A$, and thereby $X_{A,u}$ is anti-isomorphic to the subposet $X'_{A,u}$ of $[\varnothing, A]$ by $d_B^{-1}u \mapsto B$.

Step 2. The poset $X'_{A,u} \subset [\varnothing, A]$ has the minimum element B and is a boolean poset; $X'_{A,u} = [B, A]$.

Step 3. The subset $Y_{A,u}$ of $X_{A,u}$ being an order ideal, its image $Y'_{A,u}$ under $X_{A,u} \simeq X'_{A,u}$ is an order filter of $X'_{A,u}$. Moreover $Y'_{A,u}$ is closed under intersection, reflecting join-closedness of $Y_{A,u}$. Hence $Y'_{A,u}$ is also a boolean poset. Therefore, the value of the summation over $v \in Y_{A,u}$ in (1.5.3) is 0 unless $|Y_{A,u}| = 1$ since its summands cancel out, and 1 if $|Y_{A,u}| = 1$.

Step 4. If $u \leq d_B w$ for some $B \subsetneq I$ with $|B| = r$ and $d_B w \geq_L w$, then there uniquely exists A such that $|Y_{A,u}| = 1$, and hence the value of the right-hand side in (1.5.3) is 1. If there does not exist such B, then neither does such A, and hence (1.5.3) is 0.

Remark 1.5.2. The set $X_{A,u}$ is a fiber of the Demazure action ϕ_{d_A}. In Step 2 (Corollary 1.5.11) this fiber is shown to be a boolean poset. Meanwhile, for the longest element w_J of a finite parabolic subgroup W_J, any fiber of its Demazure action ϕ_{w_J} is a parabolic coset $W_J x$, whence isomorphic to W_J. More generally it might be interesting to find fibers of the Demazure action ϕ_w of an arbitrary element w.

1.5.2 Proof of Theorem 1.1.3 and 1.1.4

We fix $u \in W^\circ$.

1.5.2.1 Step 1

We fix $A \subsetneq I$. Since $Y_{A,u} \subset X_{A,u}$, the summation over v in (1.5.3) is 0 when $X_{A,u} = \varnothing$. We hence assume $X_{A,u} \neq \varnothing$, since otherwise such A does not contribute to the value of the right-hand side of (1.5.3). Take arbitrary $v \in X_{A,u}$. From Lemma 1.2.2 and the definition of $X_{A,u}$ we have

(1) $v, d_A^{-1}u \leq_L u$,

(2) $d_A^{-1}u \leq v$.

From Proposition 1.3.12 (1) and (1) above, (2) is equivalent to

(3) $uv^{-1} \leq d_A$.

The Subword Property and (3) imply $uv^{-1} = d_B$, or equivalently $v = d_B^{-1}u$, for some $B \subset A$. We have $A, B \in Z'_{u,-}$ from (1).

The argument above is restated as follows (see also Figure 1.10):

Lemma 1.5.3. (1) *$X_{A,u} \neq \varnothing \implies A \in Z'_{u,-}$.*

(2) *$X_{A,u} \subset [d_A^{-1}u, u]$.*

(3) *$(X'_{A,u}, \subset)$ and $(X_{A,u}, \leq)$ are anti-isomorphic by $B \mapsto d_B^{-1}u$.*

$$\begin{array}{ccccccc}
 & & B & \longmapsto & d_B^{-1}u & & \\
 & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\
[\varnothing, I] & \supset & Z'_{u,-} & \underset{\text{anti}}{\simeq} & Z_{u,-} & \subset & [e,u]_L \\
\cup & & \cup & & \cup & & \cup \\
[\varnothing, A] & \supset & X'_{A,u} & \underset{\text{anti}}{\simeq} & X_{A,u} & \subset & [e,u]_L \cap [d_A^{-1}u, u] \\
 & & \cup & & \cup & & \\
 & & Y'_{A,u} & \underset{\text{anti}}{\simeq} & Y_{A,u} & = & X_{A,u} \cap [e,w]
\end{array}$$

Figure 1.10: Relation between $Z_{u,-}, Z'_{u,-}, X_{A,u}, X'_{A,u}, Y_{A,u}, Y'_{A,u}$.

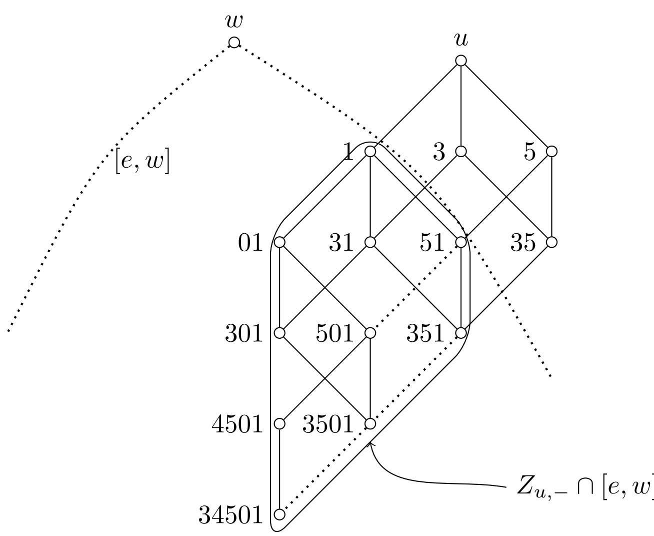


Figure 1.11: Each vertex labelled with $i_1 \dots i_m$ represents $s_{i_1} \dots s_{i_m} u \in Z_{u,-}$. Left covers are represented by solid edges, and strong covers are dotted or solid edges.

(4) $X'_{A,u} \subset [\varnothing, A]$.

(5) $X'_{A,u} \subset Z'_{u,-}$.

Proof. It remains to show that the mapping $B \mapsto d_B^{-1}u$ in (3) is order-reversing, which follows from Proposition 1.3.12 (1) and the Subword Property. □

1.5.2.2 Step 2 and 3

Let us start with an example to describe the situation.

Example 1.5.4. Let $k = 5$, $\lambda = (5, 3, 2, 1)$, $\mu = (5, 2, 2, 2)$, $u = w_\lambda$ and $w = w_\mu$ (see Figure 1.11). When $A = \{5, 0, 1\}$[2], for example, $X_{A,u} = Y_{A,u} = \{s_1u, s_{01}u, s_{51}u, s_{501}u\}$ and $X'_{A,u} = Y'_{A,u} = [\{1\}, \{5, 0, 1\}]$. Similarly, when $A = \{3, 5, 1\}$ we see $X'_{A,u} = [\varnothing, \{3, 5, 1\}]$ and $Y'_{A,u} = [\{1\}, \{3, 5, 1\}]$.

Lemma 1.5.5. *X_A and Y_A are convex under the strong order. Namely, if $v \le v' \le v''$ and $v, v'' \in X_A$ (resp. Y_A) then $v' \in X_A$ (resp. Y_A).*

[2] In this example we follow the cyclic ordering $3 < 4 < 5 < 0 < 1$ on $I \setminus \{2\}$, as we see $i_u^- = 2$, i.e. every element of $Z'_{u,-}$ is a subset of $I \setminus \{2\}$.

Proof. It follows from Lemma 1.3.4(2). ☐

Remark 1.5.6. It is not a very immediate consequence of Lemma 1.5.5 that $X'_{A,u}$ and $Y'_{A,u}$ are convex in the boolean poset $[\varnothing, I]$, yet it is shown to be true in Corollary 1.5.11.

In this section we write $\{i_1, \ldots, i_m\}_<$ to denote the set $\{i_1, \ldots, i_m\}$ for which the condition that $(i_1, \ldots, i_m)$ is cyclically increasing is imposed.

Lemma 1.5.7. (1) $B, C \in X'_{A,u} \implies B \cap C \in X'_{A,u}$.

(2) $B, C \in Y'_{A,u} \implies B \cap C \in Y'_{A,u}$.

Proof. We prove (1) by induction on $|A|$. The base case $A = \varnothing$ is clear. Assume $|A| = m > 0$. Write $A = \{i_1, \ldots, i_m\}_<$. We need to show $\phi_{d_A}(d^{-1}_{B\cap C}u) = u$ if $\phi_{d_A}(d^{-1}_B u) = u$ and $\phi_{d_A}(d^{-1}_B u) = u$ for $B, C \subset A$. Note that $B \cap C \in Z'_{u,-}$ by Lemma 1.5.3(5). Let $A' = A \setminus \{i_1\}$, $B' = B \setminus \{i_1\}$, $C' = C \setminus \{i_1\}$, $B'' = B \cup \{i_1\}$ and $C'' = C \cup \{i_1\}$. Note that $\phi_{d_A} = \phi_{i_m} \ldots \phi_{i_1} = \phi_{d_{A'}}\phi_{i_1}$.

<u>*Claim 1.*</u> (a) $\phi_{i_1}(d^{-1}_B u) = d^{-1}_{B'}u$ and $\phi_{i_1}(d^{-1}_C u) = d^{-1}_{C'}u$. (b) $B'', C'' \in Z'_{u,-}$.

Proof of Claim 1. We only give a proof of the statement for B since that for C is the same.

(Case 1) When $i_1 \in B$, we see $d^{-1}_{B''}u = d^{-1}_B u = s_{i_1} d^{-1}_{B'}u \lessdot d^{-1}_{B'}u$, and hence both (a) and (b) are clear.

(Case 2) When $i_1 \notin B$, we claim that $s_{i_1} d^{-1}_B u < d^{-1}_B u$; suppose, on the contrary, $s_{i_1} d^{-1}_B u > d^{-1}_B u$. Then we have $s_{i_1} d^{-1}_B u \not\leq_L u$ since $l(s_{i_1} d^{-1}_B u) > l(u) - l(s_{i_1} d^{-1}_B)$. On the other hand, $u = \phi_{d_A}(d^{-1}_B u) = \phi_{d_{A'}}(s_{i_1} d^{-1}_B u)$ since $s_{i_1} d^{-1}_B u > d^{-1}_B u$, and therefore $s_{i_1} d^{-1}_B u \leq_L u$ by Lemma 1.2.2, which is in contradiction.

Therefore $s_{i_1} d^{-1}_B u < d^{-1}_B u$. Now (a) is clear since $d^{-1}_B u = d^{-1}_{B'}u$, and (b) follows from $d^{-1}_{B''}u = s_{i_1} d^{-1}_B u$. *Claim 1 is proved.*

<u>*Claim 2.*</u> $\phi_{i_1}(d^{-1}_{B\cap C}u) = d^{-1}_{B'\cap C'}u$.

Proof of Claim 2. By Claim 1(b) and Proposition 1.4.2 (2), we have $B'' \cap C'' \in Z'_{u,-}$, that is, $u \geq_L d^{-1}_{B''\cap C''}u$. Since $B'' \cap C'' = (B' \cap C') \cup \{i_1\}$, we have $d^{-1}_{B''\cap C''} = s_{i_1} d^{-1}_{B'\cap C'} > d^{-1}_{B'\cap C'}$, and hence $d^{-1}_{B''\cap C''}u = s_{i_1} d^{-1}_{B'\cap C'}u \lessdot d^{-1}_{B'\cap C'}u$ by Lemma 1.2.1. Noting that $B \cap C = B' \cap C'$ or $B'' \cap C''$, in either case $\phi_{i_1}(d^{-1}_{B\cap C}u) = d^{-1}_{B'\cap C'}u$. *Claim 2 is proved.*

<u>*Claim 3.*</u> $B' \cap C' \in X'_{A',u}$.

Proof of Claim 3. By Claim 1(a) and that $B \in X'_{A,u}$, we have $u = \phi_{d_A}(d^{-1}_B u) = \phi_{d_{A'}}\phi_{i_1}(d^{-1}_B u) = \phi_{d_{A'}}(d^{-1}_{B'}u)$, and hence $B' \in X'_{A',u}$. Similarly $C' \in X'_{A',u}$. Hence $B' \cap C' \in X'_{A',u}$ by the induction hypothesis. *Claim 3 is proved.*

Now we have

$$
\begin{aligned}
\phi_{d_A}(d^{-1}_{B\cap C}u) &= \phi_{d_{A'}}\phi_{i_1}(d^{-1}_{B\cap C}u) \\
&= \phi_{d_{A'}}(d^{-1}_{B'\cap C'}u) && \text{(by Claim 2)} \\
&= u. && \text{(by Claim 3)}
\end{aligned}
$$

Hence (1) is proved. (2) follows from (1) and the definition of join and $Y_{A,u}$. ☐

Lemma 1.5.8. *Let* $A, A' \in Z'_{u,-}$ *with* $A' \subset A$ *and* $|A \setminus A'| = 1$. *Then* $A' \in X'_{A,u}$.

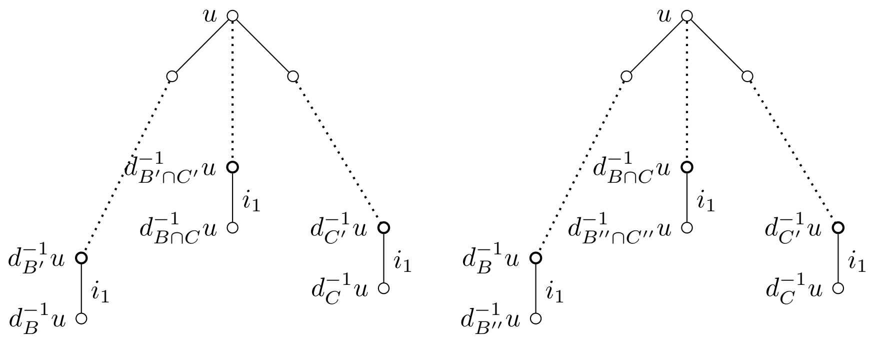
Proof. Let $A = \{i_1, \ldots, i_m\}_<$ and $A' = \{i_1, \ldots, \widehat{i_k}, \ldots, i_m\}_<$.

Since $u \geq_L d^{-1}_{A'}u = s_{i_1} \ldots \widehat{s_{i_k}} \ldots s_{i_m} u$,

- $\phi_{i_j}(s_{i_j} \ldots s_{i_{k-1}} s_{i_{k+1}} \ldots s_{i_m} u) = s_{i_{j+1}} \ldots s_{i_{k-1}} s_{i_{k+1}} \ldots s_{i_m} u$ for $1 \leq j < k$,
- $\phi_{i_j}(s_{i_j} \ldots s_{i_m} u) = s_{i_{j+1}} \ldots s_{i_m} u$ for $k < j \leq m$.

Since $u \geq_L d^{-1}_A u = s_{i_1} \ldots s_{i_m} u$,

- $\phi_{i_k}(s_{i_{k+1}} \ldots s_{i_m} u) = s_{i_{k+1}} \ldots s_{i_m} u$.



(When $i_1 \in B \cap C$) (When $i_1 \notin B$ and $i_1 \in C$)

Figure 1.12: For Lemma 1.5.7

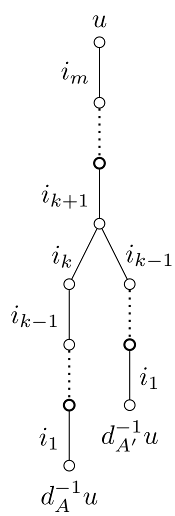


Figure 1.13: For Lemma 1.5.8

Hence

$$\begin{aligned}
\phi_{d_A}(d^{-1}_{A'}u) &= \phi_{i_m}\dots\phi_{i_{k+1}}\phi_{i_k}\phi_{i_{k-1}}\dots\phi_{i_1}(s_{i_1}\dots s_{i_{k-1}}s_{i_{k+1}}\dots s_{i_m}u)\\
&= \phi_{i_m}\dots\phi_{i_{k+1}}\phi_{i_k}(s_{i_{k+1}}\dots s_{i_m}u)\\
&= \phi_{i_m}\dots\phi_{i_{k+1}}(s_{i_{k+1}}\dots s_{i_m}u)\\
&= u.
\end{aligned}$$

□

Lemma 1.5.9. *Let $A = \{i_1, \dots, i_m\}_< \in Z'_{u,-}$ and $B \in X'_{A,u}$. By Lemma 1.5.3(4) we can write $B = \{i_1, \dots, \widehat{i_{j_1}}, \dots, \widehat{i_{j_l}}, \dots, i_m\}$ for some $1 \le j_1 < \cdots < j_l \le m$. Let $A^{(a)} = \{i_{j_a+1}, i_{j_a+2}, \dots, i_{m-1}, i_m\}$ and $B^{(a)} = B \cap A^{(a)} = \{i_{j_a+1}, \dots, \widehat{i_{j_{a+1}}}, \dots, \widehat{i_{j_l}}, \dots, i_m\}$ for each $a \in \{1, \dots, l\}$. Then, for each $1 \le a \le l$,*

$$s_{i_{j_a}} d^{-1}_{B^{(a)}} u < d^{-1}_{B^{(a)}} u.$$

Proof. We carry out induction on $l = |A \setminus B|$, with trivial base case $l = 0$. Assume $l > 0$. From Lemma 1.5.3(5), we have $u \ge_L d^{-1}_B u = s_{i_1} \dots s_{i_{j_1-1}} d^{-1}_{B^{(1)}} u$, and hence $d^{-1}_{B^{(1)}} u \ge_L s_{i_1} \dots s_{i_{j_1-1}} d^{-1}_{B^{(1)}} u$ by Lemma 1.2.1.

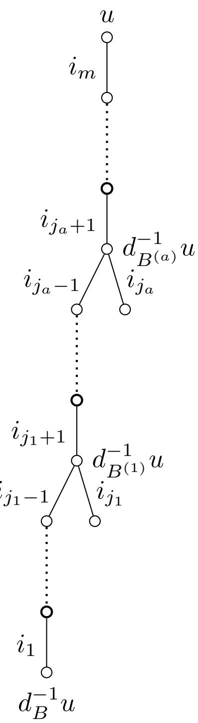


Figure 1.14: For Lemma 1.5.9

Hence

$$\begin{aligned} u &= \phi_{d_A}(d_B^{-1}u) \\ &= \phi_{d_{A^{(1)}}}\phi_{i_{j_1}}\phi_{i_{j_1-1}}\dots\phi_{i_1}(s_{i_1}\dots s_{i_{j_1-1}}d^{-1}_{B^{(1)}}u) \\ &= \phi_{d_{A^{(1)}}}\phi_{i_{j_1}}(d^{-1}_{B^{(1)}}u). \end{aligned} \tag{1.5.4}$$

We now claim $s_{i_{j_1}}d^{-1}_{B^{(1)}}u < d^{-1}_{B^{(1)}}u$; suppose to the contrary that $s_{i_{j_1}}d^{-1}_{B^{(1)}}u > d^{-1}_{B^{(1)}}u$. Then we have $s_{i_{j_1}}d^{-1}_{B^{(1)}}u \not\leq_L u$ since $l(s_{i_{j_1}}d^{-1}_{B^{(1)}}u) > l(u) - l(s_{i_{j_1}}d^{-1}_{B^{(1)}})$. On the other hand, $s_{i_{j_1}}d^{-1}_{B^{(1)}}u > d^{-1}_{B^{(1)}}u$ implies $\phi_{i_{j_1}}(d^{-1}_{B^{(1)}}u) = s_{i_{j_1}}d^{-1}_{B^{(1)}}u$, which implies $\phi_{d_{A^{(1)}}}(s_{i_{j_1}}d^{-1}_{B^{(1)}}u) = u$ by (1.5.4), which implies $s_{i_{j_1}}d^{-1}_{B^{(1)}}u \leq_L u$ by Lemma 1.2.2, which is in contradiction.

Therefore $s_{i_{j_1}}d^{-1}_{B^{(1)}}u < d^{-1}_{B^{(1)}}u$, that is, $\phi_{i_{j_1}}(d^{-1}_{B^{(1)}}u) = d^{-1}_{B^{(1)}}u$, and hence $\phi_{d_{A^{(1)}}}(d^{-1}_{B^{(1)}}u) = u$ by (1.5.4). Hence, since $|A^{(1)} \setminus B^{(1)}| = |A \setminus B| - 1$, we obtain $s_{i_{j_a}}d^{-1}_{B^{(a)}}u < d^{-1}_{B^{(a)}}u$ for $a = 2, \dots, l$ by the induction hypothesis applied for $(A, B) := (A^{(1)}, B^{(1)})$. □

Lemma 1.5.10. *Let $A, B \in Z'_{u,-}$ with $B \subset A$. The following are equivalent:*

(1) $B \in X'_{A,u}$.

(2) $B \cup \{i\} \in Z'_{u,-}$ *for any* $i \in A \setminus B$.

(3) $B \cup \{i\} \in X'_{A,u}$ *for any* $i \in A \setminus B$.

(4) $A \setminus \{i\} \in Z'_{u,-}$ *for any* $i \in A \setminus B$.

(5) $A \setminus \{i\} \in X'_{A,u}$ *for any* $i \in A \setminus B$.

(6) $[B, A] \subset Z'_{u,-}$.

(7) $[B, A] \subset X'_{A,u}$.

Proof. (2) $\iff$ (4) $\iff$ (6): (6) $\implies$ (4) and (6) $\implies$ (2) are obvious. (2) $\implies$ (4) $\implies$ (6) is from Lemma 1.4.2(1).
(1) $\implies$ (2): We use the notations $A^{(a)}$ and $B^{(a)}$ in Lemma 1.5.9. From Lemma 1.5.9 we have $\{i_{j_a}\} \cup B^{(a)} \in Z'_{u,-}$ for any a, and hence $B \cup \{i_{j_a}\} = (\{i_{j_a}\} \cup B^{(a)}) \cup B \in Z'_{u,-}$ by Proposition 1.4.2(1).
(1) $\implies$ (7): We already proved (1) $\implies$ (2) $\iff$ (6). Hence, since $A, B \in X'_{A,u}$ and $[B, A] \subset Z'_{u,-}$, by Lemma 1.5.5 we have $[B, A] \subset X'_{A,u}$.
(1) $\iff$ (3) $\iff$ (5) $\iff$ (7): It is obvious that (7) $\implies$ (3), (5). From Lemma 1.5.7(1) we have (3) $\implies$ (1) and (5) $\implies$ (1). Besides we already proved (1) $\implies$ (7).
(4) $\implies$ (5): By Lemma 1.5.8. $\square$

We write $\bigcap X = \bigcap_{x \in X} x$ for a set X of sets.

Corollary 1.5.11. *We have $X'_{A,u} = [\bigcap X'_{A,u}, A]$ if $X'_{A,u} \neq \varnothing$, and $Y'_{A,u} = [\bigcap Y'_{A,u}, A]$ if $Y'_{A,u} \neq \varnothing$. In particular, $X'_{A,u}$ and $Y'_{A,u}$ are isomorphic to boolean posets, and therefore so are $X_{A,u}$ and $Y_{A,u}$.*

Proof. Assume $Y'_{A,u}$ is nonempty. Then $Y'_{A,u}$ has the minimum element $C = \bigcap Y'_{A,u}$ by Lemma 1.5.7(2). By Lemma 1.5.10(1) $\implies$ (7) we have $[C, A] \subset X'_{A,u}$. Moreover, since $Y_{A,u}$ is an order ideal of $X_{A,u}$ we have $Y'_{A,u}$ is an order filter of $X'_{A,u}$, and therefore $[C, A] \subset Y'_{A,u}$. The opposite inclusion $Y'_{A,u} \subset [C, A]$ is implied by minimality of C. Therefore $Y'_{A,u} = [C, A]$.

It is proved similarly that $X'_{A,u} = [\bigcap X'_{A,u}, A]$ whenever $X'_{A,u} \neq \varnothing$. $\square$

Therefore we have

$$\begin{aligned}\sum_{v \in Y_{A,u}} (-1)^{|A|-(l(u)-l(v))} &= \sum_{B \in Y'_{A,u}} (-1)^{|A|-(l(u)-l(d_B^{-1}u))} \\ &= \sum_{B \in Y'_{A,u}} (-1)^{|A|-|B|} \\ &= \begin{cases} 1 & \text{if } |Y'_{A,u}| = 1, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } |Y_{A,u}| = 1, \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \tag{1.5.5}$$

1.5.2.3 Step 4

Next we discuss which A satisfies the condition $|Y_{A,u}| = 1$.

Since $Z_{u,-} \subset [e, u]_L$ is an order filter, so is $Z_{u,-} \cap [e, w] \subset [e, u]_L \cap [e, w]$. Hence, if $(w_S \wedge_L u =)$ $\max([e, u]_L \cap [e, w]) \notin Z_{u,-}$, then $Z_{u,-} \cap [e, w] = \varnothing$, and hence $Y_{A,u} = \varnothing$ for any A since $Y_{A,u} = X_{A,u} \cap [e, w] \subset Z_{u,-} \cap [e, w]$. We hence assume $w_S \wedge_L u \in Z_{u,-}$ and write $w_S \wedge_L u = d_{A_0}^{-1} u$ with $A_0 \in Z'_{u,-}$. Write $Z_{u,-}^{\leq w} = Z_{u,-} \cap [e, w]$. Note that $w_S \wedge_L u = \max Z_{u,-}^{\leq w}$.

Example 1.5.12. Recall Example 1.5.4. In that case $\max(Z_{u,-} \cap [e, w]) = s_1 u$ and hence $A_0 = \{1\}$. It is easily checked that $X_{\{1\},u} = \{u, s_1 u\}$ and $Y_{\{1\},u} = \{s_1 u\}$.

Lemma 1.5.13. $|Y_{A,u}| = 1 \iff A = A_0$.

Proof. ($\implies$): Clearly $d_{A_0}^{-1} u \in Y_{A_0,u}$. On the contrary, take any $v \in Y_{A_0,u}$. Then $v = d_B^{-1} u$ for some $B \in Y'_{A_0,u}$. Since $Y'_{A_0,u} \subset X'_{A_0,u} \subset [\varnothing, A_0]$, we have $B \subset A_0$. On the other hand, since $v \in Y_{A_0,u} = X_{A_0,u} \cap [e, w] \subset Z_{u,-}^{\leq w}$, we have $v \leq \max Z_{u,-}^{\leq w} = d_{A_0}^{-1} u$, and hence $B \supset A_0$. Therefore $B = A_0$.
($\impliedby$): If $A \notin Z'_{u,-}$, then $|Y_{A,u}| \leq |X_{A,u}| = 0$ from Lemma 1.5.3(1). We hence assume $A \in Z'_{u,-}$. Then $d_A^{-1} u \in Z_{u,-}$.

If $d_A^{-1}u \not\leq w$, then $Y_{A,u} = \varnothing$ since $d_A^{-1}u$ is the minimum element of $X_{A,u}$ and $Y_{A,u} = X_{A,u} \cap [e, w]$ is an order ideal of $X_{A,u}$.

Hence we assume $d_A^{-1}u \leq w$. Since $d_{A_0}^{-1}u = \max Z_{u,-}^{\leq w}$, we have $d_A^{-1}u \leq d_{A_0}^{-1}u$, and hence $A_0 \subset A$. Suppose $A_0 \subsetneq A$. By Corollary 1.4.17 there exists an $A' \in Z'_{u,-}$ such that $A_0 \subset A' \subset A$ and $|A \setminus A'| = 1$. By Lemma 1.5.8 and that $d_{A'}^{-1}u \leq d_{A_0}^{-1}u \leq w$ we have $d_{A'}^{-1}u \in Y_{A,u}$. Hence $Y_{A,u} \supset \{d_A^{-1}u, d_{A'}^{-1}u\}$. □

Therefore, substituting (1.5.5) and the result of Lemma 1.5.13 into the right-hand side of (1.5.3) and noting that $|A_0| = l(u) - l(w_S \wedge_L u)$, we have

$$[g_u^{(k)}](\widetilde{g}_w^{(k)} \widetilde{h}_r) = \begin{cases} 1 & \text{if } w_S \wedge_L u \in Z_{u,-} \text{ and } l(u) - l(w_S \wedge_L u) \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we show the following:

Lemma 1.5.14. *The following are equivalent*:

1. $w_S \wedge_L u \in Z_{u,-}$ *and* $l(u) - l(w_S \wedge_L u) \leq r$.
2. *There exists* A *such that* $|A| \leq r$ *and* $u \geq_L d_A^{-1}u \leq w$.
3. *There exists* A *such that* $|A| \leq r$ *and* $u \leq d_A w \geq_L w$.
4. *There exists* A *such that* $|A| = r$ *and* $u \leq d_A w \geq_L w$.

Proof. (1) $\iff$ (2): Clear.
(3) $\iff$ (4): (4) $\implies$ (3) is obvious. (3) $\implies$ (4) follows from the fact $Z'_{u,+}$ has the Chain Property and the maximum element of size k, which corresponds to the maximum element of $Z_{u,+}$.
(2) $\implies$ (3): Assume $u \geq_L d_A^{-1}u \leq w$. Then $u = \phi_{d_A}(d_A^{-1}u) \leq \phi_{d_A}(w)$ by Lemma 1.3.4(2). Besides, we have $\phi_{d_A}(w) = d_B w \geq_L w$ for some $B \subset A$ by Lemma 1.2.2, and $|B| \leq |A| \leq r$.
(3) $\implies$ (2): Proved similarly to (2) $\implies$ (3), with Lemma 1.2.3 instead of Lemma 1.2.2. □

Now we finished, from Lemma 1.5.14 (1) $\iff$ (4), the proof of Theorem 1.1.3:

$$\widetilde{g}_w^{(k)} \widetilde{h}_r = \sum_u g_u^{(k)},$$

summed over $u \in W^\circ$ such that $u \leq d_A w$ for some $A \subsetneq I$ with $|A| = r$ and $d_A w \geq_L w$.

Theorem 1.1.4 follows from Theorem 1.1.3, Corollary 1.4.8, and the Inclusion-Exclusion Principle.

1.6 Proof of the k-rectangle factorization formula

This section is devoted for the proof of Theorem 1.1.5.

The idea of the proof is similar to that of Proposition 1.2.18; we consider a linear map $\Theta : \Lambda_{(k)} \longrightarrow \Lambda_{(k)}$ extending $\widetilde{g}_\lambda^{(k)} \mapsto \widetilde{g}_{R_t \cup \lambda}^{(k)}$, having that $\{\widetilde{g}_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ forms a basis of $\Lambda_{(k)}$. It suffices to show Θ is a $\Lambda_{(k)}$-homomorphism, since it implies $\widetilde{g}_{R_t \cup \lambda}^{(k)} = \Theta(\widetilde{g}_\lambda^{(k)}) = \widetilde{g}_\lambda^{(k)} \Theta(1) = \widetilde{g}_\lambda^{(k)} \Theta(\widetilde{g}_\varnothing^{(k)}) = \widetilde{g}_\lambda^{(k)} \widetilde{g}_{R_t}^{(k)}$. Since $\{\widetilde{h}_i\}_{1 \leq i \leq k}$ generate $\Lambda_{(k)}$, we only need to show

$$\Theta(\widetilde{h}_r \widetilde{g}_\lambda^{(k)}) = \widetilde{h}_r \Theta(\widetilde{g}_\lambda^{(k)}). \tag{1.6.1}$$

Let $d_{A_1}\lambda, d_{A_2}\lambda, \ldots$ be the list of all weak strips over λ of size r. Applying Theorem 1.1.4 to both sides of (1.6.1), we have

$$\begin{aligned} \text{(LHS)} &= \Theta\left(\sum_a \widetilde{g}_{d_{A_a}\lambda}^{(k)} - \sum_{a<b} \widetilde{g}_{d_{A_a \cap A_b}\lambda}^{(k)} + \cdots \right) \\ &= \sum_a \widetilde{g}_{R_t \cup (d_{A_a}\lambda)}^{(k)} - \sum_{a<b} \widetilde{g}_{R_t \cup (d_{A_a \cap A_b}\lambda)}^{(k)} + \cdots, \end{aligned} \tag{1.6.2}$$

and by Lemma 1.2.16 (3) we have

$$\begin{aligned}
(\text{RHS}) &= \widetilde{h}_r \widetilde{g}^{(k)}_{R_t \cup \lambda} \\
&= \sum_a \widetilde{g}^{(k)}_{d_{A_a+t}(R_t \cup \lambda)} - \sum_{a<b} \widetilde{g}^{(k)}_{d_{(A_a+t)\cap(A_b+t)}(R_t \cup \lambda)} + \cdots . \qquad (1.6.3)
\end{aligned}$$

Since $(A_a + t) \cap (A_b + t) \cap \cdots = (A_a \cap A_b \cap \cdots) + t$, by Lemma 1.2.16 (1) we have (1.6.2) = (1.6.3).

Now Theorem 1.1.5 is proved.

Chapter 2

Alternative proofs using non-commutative K-k-Schur functions

2.1 Introduction

The contents of this chapter give an alternative for the latter half (Section 1.5 and 1.6) of Chapter 1, giving another proof for the Pieri formula (Theorem 1.1.4) and the k-rectangle factorization formula (Theorem 1.1.5) for K-k-Schur functions, using the *non-commutative K-k-Schur functions* which is also introduced in [LSS10].

2.2 Coxeter groups and 0-Hecke algebras

We follow the notations in Chapter 1 and recall some here; see Section 1.2 in Chapter 1 for undefined notations.

Let (W, S) be an arbitrary Coxeter group and H the associated 0-Hecke algebra; the associative algebra H is generated by $\{T_s \mid s \in S\}$ subject to the quadratic relation $T_s^2 = -T_s$ and the braid relations of (W, S). For $w \in W$ the element $T_w \in H$ is defined without ambiguity to be $T_{s_1} \cdots T_{s_l}$ where $w = s_1 \cdots s_l$ is a reduced expression. Then $\{T_w \mid w \in W\}$ form a basis: $H = \bigoplus_{w \in W} \mathbb{Z} T_w$.

Let $\widetilde{T}_s = 1 + T_s$. Then these $\widetilde{T}_s$ satisfy $\widetilde{T}_s^2 = \widetilde{T}_s$ and the braid relations of W. Hence we can define without ambiguity $\widetilde{T}_w = \widetilde{T}_{s_1} \dots \widetilde{T}_{s_l}$ for $w \in W$ where $w = s_1 \dots s_l$ is a reduced expression. By the definition of the Demazure product $*$ we have $\widetilde{T}_v \widetilde{T}_w = \widetilde{T}_{v*w}$, and in particular $\widetilde{T}_v \widetilde{T}_w = \widetilde{T}_{vw}$ when $\langle v \rangle \langle w \rangle$ is reduced. The following fact is standard (for the proof, see for example [Ste07]):

$$\widetilde{T}_w = \sum_{v \leq w} T_v. \tag{2.2.1}$$

Lemma 2.2.3 below plays a crucial role in the arguments in the rest of this chapter.

Lemma 2.2.1. *Let $v, w \in W$ and $s \in S$. If $\langle v \rangle \langle s \rangle$, $\langle s \rangle \langle w \rangle$ and $\langle v \rangle \langle w \rangle$ are reduced then so is $\langle v \rangle \langle s \rangle \langle w \rangle$.*

Proof. Suppose to the contrary that $\langle v \rangle \langle s \rangle$, $\langle s \rangle \langle w \rangle$ and $\langle v \rangle \langle w \rangle$ are reduced but $\langle v \rangle \langle s \rangle \langle w \rangle$ is not. Take a reduced expression $v = s_1 \dots s_m$. Then there exists $l \in \{1, \dots, m\}$ such that $\langle s_{l+1} \dots s_m \rangle \langle s \rangle \langle w \rangle$ is reduced and $\langle s_l \dots s_m \rangle \langle s \rangle \langle w \rangle$ is not. Write $v' = s_{l+1} \dots s_m$ and $t = s_l$. Since $v' \leq_L tv' \leq_L v$, that $\langle v \rangle \langle s \rangle$ and $\langle v \rangle \langle w \rangle$ are reduced implies that so are $\langle t \rangle \langle v' \rangle \langle s \rangle$ and $\langle t \rangle \langle v' \rangle \langle w \rangle$. Since $\langle v' \rangle \langle w \rangle$ and $\langle v' \rangle \langle s \rangle \langle w \rangle$ are reduced, we have $v'w \lessdot v'sw$ by the Subword Property. Since $\langle v' \rangle \langle s \rangle \langle w \rangle$ is reduced and $\langle tv' \rangle \langle s \rangle \langle w \rangle$ is not, we have $tv'sw \lessdot v'sw$. Since $\langle t \rangle \langle v' \rangle \langle w \rangle$ is reduced, we have $v'w \lessdot tv'w$. Hence by the Lifting Property we have $tv'sw \geq v'w$. Since $v'w$ and $tv'sw$ have the same length, we have $v'w = tv'sw$, which implies $v' = tv's$, which contradicts the fact that $\langle t \rangle \langle v' \rangle \langle s \rangle$ is reduced. ∎

Remark 2.2.2. It does not always hold that for $u, v, w \in W$ if $\langle v\rangle\langle u\rangle$, $\langle u\rangle\langle w\rangle$ and $\langle v\rangle\langle w\rangle$ are reduced then so is $\langle v\rangle\langle u\rangle\langle w\rangle$; a counterexample is $v = s_2$, $u = s_{12}$ and $w = s_1$ where $W = S_3$.

Lemma 2.2.3. *For $v, w \in W$, if $\langle v\rangle\langle w\rangle$ is not reduced then $T_v\widetilde{T}_w = 0$.*

Proof. We induct on $l(w)$. The case $l(w) = 0$ is clear. When $l(w) = 1$, we can write $w = s \in S$. By the assumption we have $vs < v$ and hence $T_v\widetilde{T}_s = T_v(1 + T_s) = T_v + T_vT_s = T_v - T_v = 0$.

Next we assume $l(w) > 1$. Take an $s \in S$ such that $sw < w$, and write $w' = sw$. When $vs < v$, we have $T_v\widetilde{T}_s = 0$ by the induction hypothesis, and hence $T_v\widetilde{T}_w = T_v\widetilde{T}_s\widetilde{T}_{w'} = 0$. When $vs > v$, by the assumption and Lemma 2.2.1 we have $\langle v\rangle\langle w'\rangle$ is not reduced, whence by the induction hypothesis $T_v\widetilde{T}_{w'} = 0$. Besides, since $vw = vsw'$ clearly $\langle vs\rangle\langle w'\rangle$ is not reduced, hence $T_{vs}\widetilde{T}_{w'} = 0$ by the induction hypothesis. Hence we have

$$T_v\widetilde{T}_w = T_v\widetilde{T}_s\widetilde{T}_{w'} = T_v(1 + T_s)\widetilde{T}_{w'} = T_v\widetilde{T}_{w'} + T_{vs}\widetilde{T}_{w'} = 0,$$

completing the induction. □

2.3 K-affine Fomin–Stanley algebras and non-commutative K-k-Schur functions

Hereafter we let $W = \tilde{S}_{k+1} = \langle s_0, s_1, \cdots, s_k\rangle$ and $W^\circ = \tilde{S}^\circ_{k+1}$.

Denote by $\mathbb{K}$ the associated 0-Hecke algebra: $\mathbb{K} = \langle T_0, T_1, \dots, T_k\rangle = \bigoplus_{w\in W}\mathbb{Z}T_w$. Here we simply wrote $T_i = T_{s_i}$. Let $\mathbf{k}_m = \sum_{|A|=m} T_{d_A}$ $(\in \mathbb{K})$ for $m = 1, \dots, k$, where A runs over subsets of $\mathbb{Z}_{k+1} = \{0, 1, \dots, k\}$ with size m, and d_A $(\in W)$ denotes the corresponding cyclically decreasing element (see Section 1.2.2.2). Then these $\mathbf{k}_1, \dots, \mathbf{k}_k$ commute with each other (see [Lam06, Section 18.4] or [LSS10, Section 7]). The *K-affine Fomin–Stanley subalgebra* $\mathbb{L}$ $(\subset \mathbb{K})$ is defined to be $\langle \mathbf{k}_1, \dots, \mathbf{k}_k\rangle$. Note that these $\mathbb{K}$ and $\mathbb{L}$ are denoted by $\mathbb{K}_0$ and $\mathbb{L}_0$ in [LSS10].

Theorem 2.3.1 ([LSS10, Theorem 7.17])**.** *There is a Hopf isomorphism*

$$\phi\colon \Lambda_{(k)} = \mathbb{Z}[h_1, \dots, h_k] \longrightarrow \mathbb{L} = \langle \mathbf{k}_1, \dots, \mathbf{k}_k\rangle\,;\ h_i \mapsto \mathbf{k}_i. \tag{2.3.1}$$

We do not review their coalgebra structure here as we need not. The *non-commutative K-k-Schur function* $\mathbf{g}^{(k)}_\lambda$ are defined to be $\phi(g^{(k)}_\lambda) \in \mathbb{L}$. Let J be the $\mathbb{Z}$-submodule of $\mathbb{K}$ spanned by non-affine-Grassmannian elements $\{T_w \mid w \notin W^\circ\}$:

$$J = \bigoplus_{w\in W\setminus W^\circ} \mathbb{Z}T_w \quad (\subset \mathbb{K}). \tag{2.3.2}$$

Since J is a left ideal of $\mathbb{K}$, we have

$$y \equiv y' \mod J \implies xy \equiv xy' \mod J \quad (\text{for } x, y, y' \in \mathbb{K}). \tag{2.3.3}$$

With the identification $\mathcal{P}_k \simeq \tilde{S}^\circ_{k+1}\,;\ \lambda \mapsto w_\lambda$ in (1.2.5), for $\lambda \in \mathcal{P}_k$ we denote T_{w_λ} by T_λ, and $\widetilde{T}_{w_\lambda}$ by $\widetilde{T}_\lambda$. The following fact is crucial:

Theorem 2.3.2 (a corollary of [LSS10, Proposition 7.16])**.** $\mathbf{g}^{(k)}_\lambda \equiv T_\lambda \mod J$ *for any* $\lambda \in \mathcal{P}_k$.

This theorem states the non-commutative K-k-Schur functions have the unique term T_w with w affine Grassmannian, and moreover, an element of $\mathbb{L}$ is determined by its congruence class modulo J: since $\{g^{(k)}_\lambda\}_{\lambda\in\mathcal{P}_k}$ is a basis of $\Lambda_{(k)}$, by Theorem 2.3.1 it follows that $\{\mathbf{g}^{(k)}_\lambda\}_{\lambda\in\mathcal{P}_k}$ is a basis of $\mathbb{L}$. Hence, for $f \in \mathbb{L}$,

$$f \equiv \sum_{\lambda\in\mathcal{P}_k} a_\lambda T_\lambda \mod J \implies f = \sum_{\lambda\in\mathcal{P}_k} a_\lambda \mathbf{g}^{(k)}_\lambda. \tag{2.3.4}$$

In particular, for $f, f' \in \mathbb{L}$,

$$f \equiv f' \mod J \iff f = f'. \tag{2.3.5}$$

Recall the definition $\widetilde{g}^{(k)}_\lambda = \sum_{\mu\le\lambda} g^{(k)}_\mu$. We define $\widetilde{\mathbf{g}}^{(k)}_\lambda := \phi(\widetilde{g}^{(k)}_\lambda) = \sum_{\mu\le\lambda}\mathbf{g}^{(k)}_\mu$. By (2.2.1) and Theorem 2.3.2 we have

$$\widetilde{\mathbf{g}}^{(k)}_\lambda \equiv \widetilde{T}_\lambda \mod J. \tag{2.3.6}$$

Note that $\widetilde{T}_\lambda$ $(=\sum_{v\le w_\lambda} T_v)$ is not equal to $\sum_{\mu\le\lambda} T_\mu$ $(=\sum_{v\le w_\lambda, v\in W^\circ} T_v)$, but equal modulo J.

Let w_0 be the longest element of S_{k+1}. Recall that $w\in W^\circ$ if and only if $\langle w\rangle\langle w_0\rangle$ is reduced.

Lemma 2.3.3. *Let $x\in\mathbb{K}$. Then $x\in J$ if and only if $x\widetilde{T}_{w_0}=0$.*

Proof. Write $x=\sum_{w\in W} a_w T_w$. Then $x\widetilde{T}_{w_0} = \sum_{w\in W} a_w T_w \widetilde{T}_{w_0} = \sum_{w\in W^\circ} a_w T_w\widetilde{T}_{w_0}$, where the last equation is by Lemma 2.2.3. Since for each $w\in W^\circ$ we can write $T_w\widetilde{T}_{w_0} = T_{ww_0} + \sum_{v<ww_0} c_{w,v}T_v$ with some coefficients $c_{w,v}$, from the linear independence of $\{T_{ww_0}\}_{w\in W^\circ}$ we conclude that $\{T_w\widetilde{T}_{w_0}\}_{w\in W^\circ}$ are linearly independent. Hence $x\in J \iff a_w = 0\ (\forall w\in W^\circ) \iff x\widetilde{T}_{w_0}=0$. □

Lemma 2.3.4. *For $v\in W$ and $w\in W^\circ$, if $vw\notin W^\circ$ then $T_v\widetilde{T}_w\in J$.*

Proof. If $\langle v\rangle\langle w\rangle$ is not reduced then $T_v\widetilde{T}_w=0$ by Lemma 2.2.3. Assume $\langle v\rangle\langle w\rangle$ is reduced and $vw\notin W^\circ$, i.e. $\langle v\rangle\langle w\rangle\langle w_0\rangle$ is not reduced. By the assumption $\langle w\rangle\langle w_0\rangle$ is reduced. Hence $\langle v\rangle\langle ww_0\rangle$ is not reduced, which implies $T_v\widetilde{T}_{ww_0}=0$ by Lemma 2.2.3, and hence $T_v\widetilde{T}_w\in J$ by Lemma 2.3.3. □

2.4 k-rectangle factorization formula

In this section we give an alternative proof of the k-rectangle factorization formula (Theorem 1.1.5). Via the isomorphism (2.3.1), it suffices to show $\widetilde{\mathbf{g}}^{(k)}_{R_t\cup\lambda} = \widetilde{\mathbf{g}}^{(k)}_{R_t}\widetilde{\mathbf{g}}^{(k)}_\lambda$ for $\lambda\in\mathcal{P}_k$ and $1\le t\le k$.

The lemma below is standard; see for example the proof of [Lam08, Proposition 4.5]. A more intuitive proof is one using the k-code, explained below.

Lemma 2.4.1. *For $v\in W$ and $1\le t\le k$, if $vw_{R_t}\ge_L w_{R_t}$ and $vw_{R_t}\in W^\circ$ then v is t-dominant.*

Proof. It suffices to show that if $i\ne t$ then $s_iw_{R_t}<w_{R_t}$ or $s_iw_{R_t}\notin W^\circ$, which is straightforward to check by stacking the k-code diagram of s_i on that of w_{R_t} and justifying them with maximizing moves, similarly to Lemma 1.2.15. □

Define an group automorphism $\gamma\colon W\longrightarrow W;\, s_i\mapsto s_{i+1}$. By abusing the notation we also define an algebra automorphism $\gamma\colon\mathbb{K}\longrightarrow\mathbb{K}$ by $T_i\mapsto T_{i+1}$, so that $\gamma(T_w)=T_{\gamma(w)}$. Note that $\gamma^{k+1}=\mathrm{id}$ and γ maps an i-dominant element to an $(i+1)$-dominant element. Since the only 0-dominant element appearing in the expansion of $\mathbf{g}^{(k)}_\lambda$ with the basis $\{T_w\}_{w\in W}$ is T_λ (Theorem 2.3.2) and elements of $\mathbb{L}$ are invariant under γ,

$$\text{the only } t\text{-dominant element in the expansion of } \mathbf{g}^{(k)}_\lambda \text{ with the basis } \{T_w\}_{w\in W} \text{ is } \gamma^t(T_\lambda). \tag{2.4.1}$$

Now we have the following equalities in which the modulus of congruences is J in (2.3.2):

$$\widetilde{\mathbf{g}}^{(k)}_\lambda\widetilde{\mathbf{g}}^{(k)}_{R_t} \equiv \widetilde{\mathbf{g}}^{(k)}_\lambda\widetilde{T}_{R_t} \qquad \text{(by (2.3.3) and (2.3.6))} \tag{2.4.2}$$

$$= \sum_{\mu\le\lambda}\mathbf{g}^{(k)}_\mu\widetilde{T}_{R_t} \qquad \text{(by definition)} \tag{2.4.3}$$

$$\equiv \sum_{\mu\le\lambda}\gamma^t(T_\mu)\widetilde{T}_{R_t} \qquad \text{(by (2.4.1) and Lemma 2.3.4 and 2.4.1)} \tag{2.4.4}$$

$$= \gamma^t\Big(\sum_{\mu\le\lambda}T_\mu\Big)\widetilde{T}_{R_t}, \tag{2.4.5}$$

and

$$\widetilde{\mathbf{g}}^{(k)}_{R_t\cup\lambda} \equiv \widetilde{T}_{R_t\cup\lambda} \qquad \text{(by (2.3.6))} \tag{2.4.6}$$
$$= \widetilde{T}_{\gamma^t(w_\lambda)}\widetilde{T}_{R_t} \quad \text{(by Proposition 1.2.15)} \tag{2.4.7}$$
$$= \gamma^t(\widetilde{T}_\lambda)\widetilde{T}_{R_t}. \tag{2.4.8}$$

Since $\widetilde{T}_\lambda = \sum_{v\le w_\lambda} T_v \equiv \sum_{\mu\le\lambda} T_\mu \mod J$, the elements $\gamma^t(\widetilde{T}_\lambda)$ and $\gamma^t(\sum_{\mu\le\lambda} T_\mu)$ are congruent modulo $\gamma^t(J)$, the $\mathbb{Z}$-span of non-t-dominant elements. Hence, by Lemma 2.3.4 and 2.4.1 we have (2.4.5) $\equiv$ (2.4.8) mod J, and therefore $\widetilde{\mathbf{g}}^{(k)}_\lambda\widetilde{\mathbf{g}}^{(k)}_{R_t} = \widetilde{\mathbf{g}}^{(k)}_{R_t\cup\lambda}$ by (2.3.5), reproving Theorem 1.1.5.

2.5 Pieri formula for $\widetilde{\mathbf{g}}^{(k)}_\lambda$

In this section we give an alternative proof for the Pieri formula for $\widetilde{g}^{(k)}_\lambda$ (Theorem 1.1.4). Again, it suffices to show it in the non-commutative version.

Fix $\lambda\in\mathcal{P}_k$. Let us recall some results from Section 1.4 in Chapter 1. Let

$$Z = \{A \subsetneq \mathbb{Z}_{k+1} \mid d_A\lambda/\lambda \text{ is a weak strip}\} = \{A \subsetneq \mathbb{Z}_{k+1} \mid \langle d_A\rangle\langle w_\lambda w_0\rangle \text{ is reduced}\}. \tag{2.5.1}$$

This Z is denoted by $Z'^{\circ}_{w_\lambda,+} = Z'_{w_\lambda w_0,+}$ in Section 1.4 in Chapter 1. Then this Z is closed under intersection (Proposition 1.4.2), i.e.

$$A, B\in Z \implies A\cap B \in Z. \tag{2.5.2}$$

Note that for any $A, B\subsetneq\mathbb{Z}_{k+1}$ it holds $d_A\le d_B$ if and only if $A\subset B$ by the Subword Property, and hence $d_A\wedge d_B = d_{A\cap B}$ where $\wedge$ is the meet under the strong order $\le$.

Next we fix $a\in\{1,\cdots,k\}$ and let $d_{A_1}\lambda, d_{A_2}\lambda,\cdots$ be the list of all weak strips of size a over λ, i.e. $A_1, A_2,\cdots$ be the elements of Z with size a. Let us briefly write $A_{i,j,\dots} = A_i\cap A_j\cap\cdots$.

In what follows the modulus of all congruences is J, defined in (2.3.2). For $A\in Z$ we let

$$\widehat{T}_{d_A} = \sum_{\substack{B\in Z\\ B\subset A}} T_{d_B}. \tag{2.5.3}$$

Since $\widetilde{T}_{d_A} = \sum_{B\subset A} T_{d_B}$, by Lemma 2.3.4 and the definition of Z we have

$$\widehat{T}_{d_A}\widetilde{T}_\lambda \equiv \widetilde{T}_{d_A}\widetilde{T}_\lambda. \tag{2.5.4}$$

Moreover, by (2.5.2) and the Inclusion-Exclusion Principle we have

$$\sum_{\substack{A\in Z\\ |A|\le a}} T_{d_A} = \sum_i \widehat{T}_{d_{A_i}} - \sum_{i<j}\widehat{T}_{d_{A_{i,j}}} + \sum_{i<j<l}\widehat{T}_{d_{A_{i,j,l}}} - \cdots. \tag{2.5.5}$$

Note that $\mathbf{g}^{(k)}_{(a)} = \mathbf{k}_a$ and hence $\widetilde{\mathbf{g}}^{(k)}_{(a)} = \sum_{0\le i\le a} \mathbf{k}_i$. Now we have

$$\begin{aligned}
\widetilde{\mathbf{g}}^{(k)}_{(a)}\widetilde{\mathbf{g}}^{(k)}_{\lambda} &\equiv \sum_{0\le i\le a} \mathbf{k}_i \widetilde{T}_\lambda && \text{(by (2.3.3) and (2.3.6))} && (2.5.6)\\
&= \sum_{|A|\le a} T_{d_A}\widetilde{T}_\lambda && \text{(by the definition of } \mathbf{k}_a\text{)} && (2.5.7)\\
&\equiv \sum_{\substack{A\in Z\\ |A|\le a}} T_{d_A}\widetilde{T}_\lambda. && \text{(by Lemma 2.3.4)} && (2.5.8)\\
&= \Big(\sum_i \widehat{T}_{d_{A_i}} - \sum_{i<j}\widehat{T}_{d_{A_{i,j}}} + \sum_{i<j<l}\widehat{T}_{d_{A_{i,j,l}}} - \cdots\Big)\widetilde{T}_\lambda && \text{(by (2.5.5))} && (2.5.9)\\
&= \sum_i \widehat{T}_{d_{A_i}}\widetilde{T}_\lambda - \sum_{i<j}\widehat{T}_{d_{A_{i,j}}}\widetilde{T}_\lambda + \sum_{i<j<l}\widehat{T}_{d_{A_{i,j,l}}}\widetilde{T}_\lambda - \cdots && && (2.5.10)\\
&\equiv \sum_i \widetilde{T}_{d_{A_i}}\widetilde{T}_\lambda - \sum_{i<j}\widetilde{T}_{d_{A_{i,j}}}\widetilde{T}_\lambda + \sum_{i<j<l}\widetilde{T}_{d_{A_{i,j,l}}}\widetilde{T}_\lambda - \cdots && \text{(by (2.5.4))} && (2.5.11)\\
&= \sum_i \widetilde{T}_{d_{A_i}\lambda} - \sum_{i<j}\widetilde{T}_{d_{A_{i,j}}\lambda} + \sum_{i<j<l}\widetilde{T}_{d_{A_{i,j,l}}\lambda} - \cdots && && (2.5.12)\\
&\equiv \sum_i \widetilde{\mathbf{g}}^{(k)}_{d_{A_i}\lambda} - \sum_{i<j}\widetilde{\mathbf{g}}^{(k)}_{d_{A_{i,j}}\lambda} + \sum_{i<j<l}\widetilde{\mathbf{g}}^{(k)}_{d_{A_{i,j,l}}\lambda} - \cdots. && \text{(by (2.3.6))} && (2.5.13)
\end{aligned}$$

Hence by (2.3.5) we have

$$\widetilde{\mathbf{g}}^{(k)}_{(a)}\widetilde{\mathbf{g}}^{(k)}_{\lambda} = \sum_i \widetilde{\mathbf{g}}^{(k)}_{d_{A_i}\lambda} - \sum_{i<j}\widetilde{\mathbf{g}}^{(k)}_{d_{A_{i,j}}\lambda} + \sum_{i<j<l}\widetilde{\mathbf{g}}^{(k)}_{d_{A_{i,j,l}}\lambda} - \cdots,$$

reproving the Pieri rule (Theorem 1.1.4).

Chapter 3

Automorphisms on the ring of symmetric functions and stable and dual stable Grothendieck polynomials

3.1 Introduction

The stable Grothendieck polynomials G_λ and the dual stable Grothendieck polynomials g_λ are certain families of inhomogeneous symmetric functions parametrized by interger partitions λ. They are certain K-theoretic deformations of the Schur functions and dual to each other via the Hall inner product.

A notable property of the dual stable Grothendieck polynomials is that g_λ and their sums $\sum_{\mu\subset\lambda} g_\mu$ have the same product structure constants, i.e. the linear map I defined by

$$I\colon \Lambda \longrightarrow \Lambda\,;\, g_\lambda \mapsto \sum_{\mu\subset\lambda} g_\mu \tag{3.1.1}$$

is a ring automorphism on the ring of symmetric functions Λ. (We note that they represent K-homology classes of boundary ideal sheaves and structure sheaves of Schubert varieties in the Grassmannians. See Remark 3.2.7.)

In this chapter we explain that the map I above is written as both

(a) the substitution $f(x) \mapsto f(1,x)$, (that is, $f(x_1,x_2,\cdots) \mapsto f(1,x_1,x_2,\cdots)$), and

(b) the map $H(1)^\perp$, where $H(1) = \sum_i h_i$,

where the linear map $F^\perp\colon \Lambda \longrightarrow \Lambda$ is the adjoint of the multiplication map $(F\cdot)\colon \widehat{\Lambda} \longrightarrow \widehat{\Lambda}$, where $\widehat{\Lambda}$ is the completion of Λ. The equivalence of two maps in (a) and (b) is previously known in a more general form ((3.3.1) and (3.3.2)): $H(t)^\perp(f(x)) = f(t,x)$ where $H(t) = \sum_i t^i h_i$. The key observation to show $I(f(x)) = f(1,x)$ is that the substitution $f \mapsto f(1,0,0,\cdots)$ maps $g_{\lambda/\mu}$ to 1 for any skew shape λ/μ (Proposition 3.4.1); then since I is a certain composition of this map and the coproduct on Λ it follows that $I = (f(x) \mapsto f(1,x))$.

We also give:

- formulas for the image of $g_{\lambda/\mu}$ under I (and more generally $H(t)^\perp$), which generalizes $I(g_\lambda) = \sum_{\nu\subset\lambda} g_\nu$. ((3.4.10) and Proposition 3.4.3)
- similar formulas for the inverse automorphism $E(-t)^\perp$, where $E(-t) = \sum_i (-t)^i e_i = H(t)^{-1}$. (Proposition 3.5.3)
- presentations of the maps $(H(t)\cdot)$ and $(E(-t)\cdot)$ with respect to the basis $\{G_\lambda\}$, by the adjointness of $(F\cdot)$ and $F^\perp$ and the duality between G_λ and g_λ. ((3.4.11) and (3.5.3))

Organization

In Section 3.2 we recall Hopf algebras, symmetric functions and stable Grothendieck and dual stable Grothendieck polynomials. In Section 3.3 we recall algebra automorphisms $H(t)^{\perp}$ and $E(t)^{\perp}$, with some relevant arguments included in Section 3.7 as an appendix. In Section 3.4 we give descriptions for the maps $H(t)^{\perp}$ and $(H(t)\cdot)$ and prove $I = (f(x) \mapsto f(1,x)) = H(1)^{\perp}$. In Section 3.5 we treat $E(t)^{\perp}$ and $(E(t)\cdot)$ and give similar presentations. In Section 3.6 we give some examples.

3.2 Preliminaries

Throughout this chapter, we fix a commutative ring K and assume that any modules, algebras, morphisms etc. are over K.

3.2.1 Hopf algebra

We recall some generalities on the Hopf algebra. For more details we refer the reader to [Swe69, Abe80, GR] for example.

An *algebra* A is a K-module equipped with a *product* (or *multiplication*) $m = m_A\colon A\otimes A \longrightarrow A$ and a *unit* $u = u_A\colon K \longrightarrow A$ satisfying $m\circ(m\otimes \mathrm{id}) = m\circ(\mathrm{id}\otimes m)$ and $m\circ(\mathrm{id}\otimes u) = \mathrm{id} = m\circ(u\otimes \mathrm{id})$. A *coalgebra* C is a K-module equipped with a *coproduct* (or *comultiplication*) $\Delta = \Delta_C\colon C \longrightarrow C\otimes C$ and a *counit* $\epsilon = \epsilon_C\colon C \longrightarrow K$ satisfying $(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta$ and $(\mathrm{id}\otimes\epsilon)\circ\Delta = \mathrm{id} = (\epsilon\otimes\mathrm{id})\circ\Delta$. A K-linear map $\varphi\colon A \longrightarrow B$ between algebras is an *algebra morphism* if $\varphi\circ m_A = m_B\circ(\varphi\otimes\varphi)$ and $\varphi\circ u_A = u_B$. A K-linear map $\varphi\colon C \longrightarrow D$ between coalgebras is a *coalgebra morphism* if $(\varphi\otimes\varphi)\circ\Delta_C = \Delta_D\circ\varphi$ and $\epsilon_C = \epsilon_D\circ\varphi$. A K-module A equipped with m, u, Δ, ϵ is a *bialgebra* if (A, m, u) is an algebra, (A, Δ, ϵ) is a coalgebra, and the following equivalent conditions hold: (a) Δ, ϵ are algebra morphisms; (b) m, u are coalgebra morphisms. A bialgebra A equipped with an *antipode* map $S\colon A \longrightarrow A$ satisfying $m\circ(S\otimes\mathrm{id})\circ\Delta = u\circ\epsilon$ is called a *Hopf algebra*.

3.2.1.1 duals

For a K-module A, let $A^* = \mathrm{Hom}(A, K) = \{f\colon A \longrightarrow K\colon K\text{-linear}\}$ and $(\ ,\ ) = (\ ,\ )_A\colon A^*\times A \longrightarrow K\,;\,(f,a) = f(a)$. For a graded K-module $A = \bigoplus_{n\geq 0} A_n$, we denote by A^o the *graded dual* $\bigoplus_n A_n^*$, and A is called *of finite type* if every A_n is a finite free K-module. For any coalgebra C, its dual C^* is an algebra by

$$(m_{C^*}(f\otimes g), a)_C = (f\otimes g, \Delta_C(a))_{C\otimes C} \tag{3.2.1}$$

for $f, g \in C^*$ and $a \in C$. If an algebra A is a finite free K-module (resp. A is a graded algebra of finite type), then its dual A^* (resp. graded dual A^o) is a coalgebra by

$$(\Delta_{A^*}(f), a\otimes b)_{A\otimes A} = (f, ab)_A \tag{3.2.2}$$

for $f \in A^*$ (resp. A^o) and $a, b \in A$.

For a coalgebra C and an algebra A, the space of linear maps $\mathrm{Hom}(C, A)$ becomes an associative algebra by the *convolution product* $*$ defined by $f * g = m_A\circ(f\otimes g)\circ\Delta_C$. Then $u_A\circ\epsilon_C$ is the identity for $*$, and the convolution product on $C^* = \mathrm{Hom}(C, K)$ coincides with the product given in (3.2.1).

3.2.1.2 Module and comodule morphisms

For a coalgebra C, a linear map $\phi\colon C \longrightarrow C$ is *C-comodule morphism* if $\Delta\circ\phi = (\phi\otimes\mathrm{id})\circ\Delta$. For an algebra A, a linear map $\psi\colon A \longrightarrow A$ is *A-module morphism* if $\psi\circ m = m\circ(\psi\otimes\mathrm{id})$.

Lemma 3.2.1. *Let C be a coalgebra and C^* its dual algebra. For a linear map $\phi\colon C \longrightarrow C$, the following are equivalent: (1) $\phi\colon C \longrightarrow C$ is a C-comodule morphism. (2) $\phi^*\colon C^* \longrightarrow C^*$ is a C^*-module morphism.*

Proof. It easily follows from $(f, \phi(a)) = (\phi^*(f), a)$ and $(m(f \otimes g), a) = (f \otimes g, \Delta(a))$ (for $f, g \in C^*$ and $a \in C$). $\square$

For a coalgebra C and $f \in C^*$, the map $f^\perp \colon C \longrightarrow C$ is defined by $f^\perp(c) = \sum_{(c)}(f, c_1)c_2$ where we write $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$ by the Sweedler notation. In other words $f^\perp = (f \otimes \mathrm{id}) \circ \Delta$. By (3.2.1), $f^\perp$ is the adjoint of $(f\cdot) \colon C^* \longrightarrow C^*\,;\, g \mapsto fg$, i.e. $(fg, c) = (g, f^\perp(c))$. Since the multiplication map $(f\cdot)$ is a C^*-module morphism, by Lemma 3.2.1 we see that $f^\perp$ is a C-comodule morphism. Conversely, any C-comodule endomorphism on C has the form $f^\perp$:

Lemma 3.2.2. (1) *For an algebra A, if $\psi \colon A \longrightarrow A$ is an A-module morphism then ψ is the multiplication by $\psi(1_A)$.*

(2) *For a coalgebra C, if $\phi \colon C \longrightarrow C$ is a C-comodule morphism then $\phi = (\phi^*(1_{C^*}))^\perp$.*

Proof. (1) is clear. (2) follows from (1), Lemma 3.2.1 and the adjointness of $(\phi^*(1)\cdot)$ and $\phi^*(1)^\perp$. $\square$

3.2.2 Symmetric functions

For basic definitions for symmetric functions, see for instance [Mac95, Chapter I].

Let Λ $(= \Lambda(x) = \Lambda_K = \Lambda_K(x))$ be the ring of symmetric functions, namely the set of all symmetric formal power series of bounded degree in variable $x = (x_1, x_2, \dots)$ with coefficients in K. We omit the variable x when no confusion arise. Let $\widehat{\Lambda}$ be its completion, consisting of all symmetric formal power series (with possibly unbounded degree). The Schur functions s_λ $(\lambda \in \mathcal{P})$ are a family of homogeneous symmetric functions satisfying $\Lambda = \bigoplus_{\lambda \in \mathcal{P}} K s_\lambda$ and $\widehat{\Lambda} = \prod_{\lambda \in \mathcal{P}} K s_\lambda$.

The *Hall inner product* $(\,,\,)$ is a bilinear form on Λ for which the Schur functions form an orthonormal basis, i.e. $(s_\lambda, s_\mu) = \delta_{\lambda\mu}$. This is naturally extended to $(\,,\,) \colon \widehat{\Lambda} \times \Lambda \longrightarrow K$, whence we can identify $\widehat{\Lambda}$ with Λ^* and Λ with $\Lambda^o = \bigoplus_{n \geq 0} \Lambda_n^*$. Here Λ_n denotes the homogeneous component of Λ with degree n.

The ring Λ is a Hopf algebra with a product $m \colon \Lambda \otimes \Lambda \longrightarrow \Lambda\,;\, f \otimes g \mapsto fg$, a unit $u \colon K \longrightarrow \Lambda\,;\, 1 \mapsto 1$, a coproduct $\Delta \colon \Lambda = \Lambda(x) \longrightarrow \Lambda(x, y) \hookrightarrow \Lambda(x) \otimes \Lambda(y)\,;\, f(x) \mapsto f(x, y)$, a counit $\epsilon \colon \Lambda \longrightarrow K\,;\, f \mapsto f(0, 0, \dots)$, i.e. $\epsilon(s_\lambda) = \delta_{\lambda\varnothing}$, and an antipode $S \colon \Lambda \longrightarrow \Lambda\,;\, s_\lambda \mapsto (-1)^{|\lambda|} s_{\lambda'}$. Here λ' denotes the transpose of $\lambda \in \mathcal{P}$. The coincidence between the coefficients in the Littlewood-Richardson rules $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$ and $\Delta(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu$ implies that Λ is self-dual, i.e. the Hopf structure on Λ^o via (3.2.1) and (3.2.2) coincides with one coming from the identification $\Lambda \simeq \Lambda^o$. Note that $\widehat{\Lambda} \simeq \Lambda^*$ is an algebra but not a coalgebra, since if $f \in \widehat{\Lambda}$ has unbounded degree then $f(x, y)$ may be unable to be written as a finite sum of $f_1(x)f_2(y)$ for $f_1, f_2 \in \widehat{\Lambda}$.

For $F \in \widehat{\Lambda}$, we have linear maps

- $(F, -) \colon \Lambda \longrightarrow K\,;\, f \mapsto (F, f)$, and
- $F^\perp \colon \Lambda \longrightarrow \Lambda\,;\, f \mapsto \sum (F, f_1) f_2$

where we put $\Delta(f) = \sum f_1 \otimes f_2$ for $f \in \Lambda$ by the Sweedler notation. By the identification $\widehat{\Lambda} \simeq \Lambda^*$ this notation is the same as given in Section 3.2.1. Note that

$$F^\perp = ((F, -) \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes (F, -)) \circ \Delta \tag{3.2.3}$$

where the second equality is by cocommutativity. We also have

$$(F, -) = \epsilon \circ F^\perp \tag{3.2.4}$$

since $\epsilon \circ F^\perp = \epsilon \circ ((F, -) \otimes \mathrm{id}) \circ \Delta = ((F, -) \otimes \epsilon) \circ \Delta = (F, -) * \epsilon = (F, -)$. The following lemma is standard:

Lemma 3.2.3. *For $F, G \in \widehat{\Lambda}$,*

(1) $(FG, -) = (F, -) * (G, -)$ *where $*$ denotes the convolution product on* $\mathrm{Hom}(\Lambda, K)$.

(2) $(FG)^\perp = G^\perp \circ F^\perp$ $(= F^\perp \circ G^\perp)$.

By arguments in Section 3.2.1.2 we have the following lemmas.

Lemma 3.2.4. *For $F \in \widehat{\Lambda}$,*

(1) $F^{\perp}\colon \Lambda \longrightarrow \Lambda$ *is a* Λ*-comodule morphism.*
(2) $(F^{\perp})^* = (F\cdot)$*. Namely* $(FG, f) = (G, F^{\perp}(f))$ *for* $G \in \widehat{\Lambda}$ *and* $f \in \Lambda$*.*

Lemma 3.2.5. *Let* $\phi\colon \Lambda \longrightarrow \Lambda$ *be a* Λ*-comodule morphism. Then the dual map* $\phi^*\colon \widehat{\Lambda} \longrightarrow \widehat{\Lambda}$ *is the multiplication by* $\phi^*(1)$*. Moreover* $\phi = (\phi^*(1))^{\perp}$*.*

3.2.3 Stable and dual stable Grothendieck polynomials

The *stable Grothendieck polynomials* (parametrized by permutations) were introduced by Fomin and Kirillov [FK96] as a stable limit of the Grothendieck polynomials of Lascoux–Schützenberger [LS82]. In [Buc02, Theorem 3.1] Buch gave a combinatorial description of the stable Grothendieck polynomials G_λ for partitions as (signed) generating functions of so-called *set-valued tableaux.* We do not review the detail here and just recall some of its properties: $G_\lambda \in \widehat{\Lambda}$ (although $G_\lambda \notin \Lambda$ if $\lambda \neq \varnothing$), G_λ is an infinite linear combination of $\{s_\mu\}_{\mu\in\mathcal{P}}$ whose lowest degree component is s_λ. Hence $\widehat{\Lambda} = \prod_{\lambda\in\mathcal{P}} KG_\lambda$, i.e. every element in $\widehat{\Lambda}$ is uniquely written as an infinite linear combination of G_λ. Moreover the span $\bigoplus_\lambda KG_\lambda$ ($\subset \widehat{\Lambda}$) is a bialgebra, in particular the expansion of the product $G_\mu G_\nu = \sum_\lambda c^\lambda_{\mu\nu} G_\lambda$ and the coproduct $\Delta(G_\lambda) = \sum_{\mu,\nu} d^\lambda_{\mu\nu} G_\mu \otimes G_\nu$ are finite.

Next we recall the dual stable Grothendieck polynomial $g_{\lambda/\mu}$. For a skew shape λ/μ, a *reverse plane partition* of shape λ/μ is a filling of the boxes in λ/μ with positive integers such that the numbers are weakly increasing in every row and column.

Definition 3.2.6 ([LP07])**.** For a skew shape λ/μ, the *dual stable Grothendieck polynomial* $g_{\lambda/\mu}$ is defined by

$$g_{\lambda/\mu} = \sum_T x^T, \tag{3.2.5}$$

summed over reverse plane partitions T of shape λ/μ, where $x^T = \prod_i x_i^{T(i)}$ where $T(i)$ is the number of columns of T that contain i.

When $\mu = \varnothing$ we write $g_\lambda = g_{\lambda/\varnothing}$. It is shown in [LP07] that $g_{\lambda/\mu} \in \Lambda$ and g_λ has the highest degree component s_λ and forms a basis of Λ that is dual to G_λ via the Hall inner product:

$$(G_\lambda, g_\mu) = \delta_{\lambda\mu}. \tag{3.2.6}$$

Hence the product (resp. coproduct) structure constants for $\{G_\lambda\}$ coincide with the coproduct (resp. product) structure constants for $\{g_\lambda\}$: $g_\mu g_\nu = \sum_\lambda d^\lambda_{\mu\nu} g_\lambda$ and $\Delta(g_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} g_\mu \otimes g_\nu$.

As stated in (3.1.1), we denote by I the linear map $\Lambda \longrightarrow \Lambda$ defined by $I(g_\lambda) = \sum_{\mu\subset\lambda} g_\mu$. Note that the inverse map is given by $I^{-1}(g_\lambda) = \sum_{\lambda/\mu:\ \text{rook strip}} (-1)^{|\lambda/\mu|} g_\mu$. Here λ/μ is called a *rook strip* if any cell of λ/μ is removable corner of λ.

Remark 3.2.7. We recall geometric interpretations of G_λ and g_λ. Let $\mathrm{Gr}(k,n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$, $R = (n-k)^k$ the rectangle of shape $(n-k)\times k$, and $\mathcal{O}_\lambda$ ($\lambda \subset R$) the structure sheaves of Schubert varieties of $\mathrm{Gr}(k,n)$. The K-theory $K^*(\mathrm{Gr}(k,n))$, the Grothendieck group of algebraic vector bundles on $\mathrm{Gr}(k,n)$, has a basis $\{[\mathcal{O}_\lambda]\}_{\lambda\subset R}$, and the surjection $\bigoplus_{\lambda\in\mathcal{P}} \mathbb{Z}G_\lambda \longrightarrow K^*(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda\subset R} \mathbb{Z}[\mathcal{O}_\lambda]$ that maps G_λ to $[\mathcal{O}_\lambda]$ (which is considered as 0 if $\lambda \not\subset R$) is an algebra homomorphism [Buc02].

There is another basis of $K^*(\mathrm{Gr}(k,n))$ consisting of the classes $[\mathcal{I}_\lambda]$ of ideal sheaves of boundaries of Schubert varieties. In [Buc02, Section 8] it is shown that the bases $\{[\mathcal{O}_\lambda]\}_{\lambda\subset R}$ and $\{[\mathcal{I}_\lambda]\}_{\lambda\subset R}$ relates to each other by $[\mathcal{O}_\lambda] = \sum_{\lambda\subset\mu\subset R} [\mathcal{I}_\mu]$ and that they are dual: more precisely $([\mathcal{O}_\lambda], [\mathcal{I}_{\tilde{\mu}}]) = \delta_{\lambda\mu}$ where $\tilde{\mu} = (n-k-\mu_k, \cdots, n-k-\mu_1)$ is the rotated complement of $\mu \subset R$ and the pairing (,) is defined by $(\alpha,\beta) = \rho_*(\alpha\otimes\beta)$ where ρ_* is the pushforward to a point.

The K-homology $K_*(\mathrm{Gr}(k,n))$, the Grothendieck group of coherent sheaves, is naturally isomorphic to $K^*(\mathrm{Gr}(k,n))$. Lam and Pylyavskyy proved in [LP07, Theorem 9.16] that the surjection $\Lambda = \bigoplus_{\lambda\in\mathcal{P}} \mathbb{Z}g_\lambda \longrightarrow K_*(\mathrm{Gr}(k,n)) = \bigoplus_{\mu\subset R} \mathbb{Z}[\mathcal{I}_\mu]$ that maps g_λ to $[\mathcal{I}_{\tilde{\lambda}}]$ (which is considered as 0 if $\lambda \subset R$) identifies the coproduct and product on Λ with the pushforwards of the diagonal embedding map and the direct sum map.

Since $\mu \subset \lambda \iff \tilde{\mu} \supset \tilde{\lambda}$, under this identification we see that $\sum_{\mu\subset\lambda} g_\mu \in \Lambda$ corresponds to $[\mathcal{O}_{\tilde{\lambda}}] \in K_*(\mathrm{Gr}(k,n))$.

3.3 Automorphisms $H^\perp(t)$ and $E^\perp(t)$

There are well-known generating functions

$$H(t) = \sum_{i\geq 0} t^i h_i, \qquad E(t) = \sum_{i\geq 0} t^i e_i$$

where $t \in K$ (hence $H(t), E(t) \in \widehat{\Lambda}$). Let

$$H^\perp(t) := H(t)^\perp = \sum_{i\geq 0} t^i h_i^\perp, \qquad E^\perp(t) := E(t)^\perp = \sum_{i\geq 0} t^i e_i^\perp.$$

It is known (see [Mac95, Chapter 1.5, Example 29]) that

$$H^\perp(t),\ E^\perp(t)\colon \Lambda \longrightarrow \Lambda \text{ are ring automorphisms,} \tag{3.3.1}$$

$$H^\perp(t)(f(x_1,x_2,\cdots)) = f(t,x_1,x_2,\cdots) \qquad \text{for } f \in \Lambda. \tag{3.3.2}$$

(The proof of (3.3.1) was as follows: for $F \in \widehat{\Lambda}$, the map $F^\perp\colon \Lambda \longrightarrow \Lambda$ is an algebra automorphism if and only if $F(x,y) = F(x)F(y)$ and $F(0) = 1$, and it is easy to see that $H(t)$ and $E(t)$ satisfy them. See Section 3.7 for more details. To show (3.3.2), it then suffices to consider the case where $f = h_n$, which is straightforward.)

From (3.3.1), (3.3.2) and (3.2.4) we have

$$(H(t),-),\ (E(t),-)\colon \Lambda \longrightarrow K \text{ are ring homomorphisms,} \tag{3.3.3}$$

$$(H(t),f) = f(t,0,0,\cdots). \tag{3.3.4}$$

Since $H(t)E(-t) = 1$, by Lemma 3.2.3 and the fact that the counit is the identity with respect to the convolution product we have

Lemma 3.3.1. (1) $(H(t),-) * (E(-t),-) = \epsilon$, *where* $\epsilon\colon \Lambda \longrightarrow K$ *is the counit.*
(2) $H(t)^\perp \circ E(-t)^\perp = \mathrm{id}_\Lambda$.

Remark 3.3.2. We give here a note on the relation between $H^\perp(t)$ and $E^\perp(t)$.

Let $\phi\colon \Lambda \longrightarrow \Lambda$ be a graded self-adjoint algebra endomorphism. Then ϕ naturally extends to an algebra endomorphism on $\widehat{\Lambda}$, and for $F \in \widehat{\Lambda}$ we have

$$(\phi(F),-) = (F,-)\circ\phi, \qquad \phi\circ(\phi(F))^\perp = F^\perp\circ\phi.$$

(*Proof.* For $F, G \in \widehat{\Lambda}$ and $f \in \Lambda$ we have $(\phi(F),f) = (F,\phi(f))$ and $(G, \phi\circ(\phi(F))^\perp(f)) = (\phi(G), (\phi(F))^\perp(f)) = (\phi(F)\phi(G), f) = (\phi(FG), f) = (FG, \phi(f)) = (G, F^\perp\circ\phi(f))$.)

Define an algebra homomorphism $\phi_t\colon \Lambda \longrightarrow \Lambda\,;\ f(x) \mapsto f(tx)$, where $x = (x_1,x_2,\ldots)$ and $tx = (tx_1,tx_2,\ldots)$. Clearly ϕ_t is graded, and since $(\phi_t(s_\lambda), s_\mu) = t^{|\lambda|}\delta_{\lambda\mu} = (s_\lambda, \phi_t(s_\mu))$ it is self-adjoint. Since $H(t) = \phi_t(H(1))$ and $E(t) = \phi_t(E(1))$, we have

$$(H(t),-) = (H(1),-)\circ\phi_t, \qquad \phi_t\circ H(t)^\perp = H(1)^\perp\circ\phi_t,$$
$$(E(t),-) = (E(1),-)\circ\phi_t, \qquad \phi_t\circ E(t)^\perp = E(1)^\perp\circ\phi_t.$$

Since $\omega\colon \Lambda \longrightarrow \Lambda\,;\ h_i \mapsto e_i$ is a graded self-adjoint algebra involution which sends $H(t)$ to $E(t)$, it follows

$$(H(t),-)\circ\omega = (E(t),-), \qquad H^\perp(t)\circ\omega = \omega\circ E^\perp(t). \tag{3.3.5}$$

Remark 3.3.3. By (3.3.4) it follows that $(H(t),h_i) = t^i$ for $i \geq 0$, $(H(t),e_i) = 0$ for $i \geq 2$, and $(H(t),p_i) = t^i$ for $i \geq 1$.

3.4 Descriptions of $H(t)$, $(H(t), -)$ and $H(t)^\perp$

We start with the following observation. Let $c(\lambda/\mu)$ denote the number of columns in the skew shape λ/μ.

Proposition 3.4.1. *The algebra homomorphism $(H(t), -)$ satisfies $(H(t), g_{\lambda/\mu}) = t^{c(\lambda/\mu)}$ for any skew shape λ/μ, and in particular $(H(t), g_\lambda) = t^{c(\lambda)}$ for any $\lambda \in \mathcal{P}$.*

Proof. By (3.3.4) we have $(H(t), g_{\lambda/\mu}) = g_{\lambda/\mu}(t, 0, 0, \cdots)$. By (3.2.5), it is the generating function of reverse plane partitions on λ/μ filled with one alphabet 1. Clearly there is exactly one such filling, whose weight is $x_1^{c(\lambda/\mu)}$. Hence $g_{\lambda/\mu}(t, 0, 0, \cdots) = t^{c(\lambda/\mu)}$. □

Since $\{g_\lambda\}_\lambda$ is a basis of Λ and $\{G_\lambda\}_\lambda$ is their dual (3.2.6), from Proposition 3.4.1 we have

$$(H(t), -) = \Big(\sum_{\lambda \in \mathcal{P}} t^{c(\lambda)} G_\lambda, -\Big), \tag{3.4.1}$$

whence

$$H(t) = \sum_{\lambda \in \mathcal{P}} t^{c(\lambda)} G_\lambda. \tag{3.4.2}$$

Another consequence of the proposition above is formulas on the structure constants in $g_\mu g_\nu = \sum_\lambda d_{\mu\nu}^\lambda g_\lambda$ and $g_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda g_\nu$:

Corollary 3.4.2. (1) *For any $\mu, \nu \in \mathcal{P}$, we have $t^{c(\mu)+c(\nu)} = \sum_\lambda d_{\mu\nu}^\lambda t^{c(\lambda)}$.*
(2) *For any $\lambda, \mu \in \mathcal{P}$, we have $t^{c(\lambda/\mu)} = \sum_\nu c_{\mu\nu}^\lambda t^{c(\nu)}$.*

3.4.1 Proof of $I = H(1)^\perp$

Next we give another description of the map $I\colon g_\lambda \mapsto \sum_{\mu \subset \lambda} g_\mu$.

For a skew shape λ/μ and a totally ordered set X called *alphabets* (most commonly $\{1, 2, 3, \dots\}$), we shall denote by $\mathrm{RPP}(\lambda/\mu, X)$ the set of reverse plane partition of shape λ/μ where each box is filled with an element of X. The expression (3.2.5) of $g_{\lambda/\mu}$ as a generating function of reverse plane partitions implies

$$\Delta(g_{\lambda/\mu}) = \sum_{\mu \subset \nu \subset \lambda} g_{\lambda/\nu} \otimes g_{\nu/\mu}, \tag{3.4.3}$$

since we have a natural bijection between $\mathrm{RPP}(\lambda/\mu, \{1, 2, \cdots, 1', 2', \dots\})$ and $\bigsqcup_{\mu \subset \nu \subset \lambda} \mathrm{RPP}(\nu/\mu, \{1, 2, \cdots\}) \times \mathrm{RPP}(\lambda/\nu, \{1', 2', \cdots\})$ where $1 < 2 < \cdots < 1' < 2' < \cdots$.

Proposition 3.4.3. *The algebra automorphism $H(t)^\perp\colon \Lambda \longrightarrow \Lambda$ satisfies*

$$H(t)^\perp(g_{\lambda/\mu}) = \sum_{\mu \subset \nu \subset \lambda} t^{c(\lambda/\nu)} g_{\nu/\mu} = \sum_{\mu \subset \nu \subset \lambda} t^{c(\nu/\mu)} g_{\lambda/\nu} \tag{3.4.4}$$

for any $\mu \subset \lambda$. In particular,

$$H(t)^\perp(g_\lambda) = \sum_{\nu \subset \lambda} t^{c(\lambda/\nu)} g_\nu = \sum_{\nu \subset \lambda} t^{c(\nu)} g_{\lambda/\nu} \tag{3.4.5}$$

for any $\lambda \in \mathcal{P}$.

Proof. By (3.2.3), we have by applying $(H(t), -) \otimes \mathrm{id}$ to (3.4.3) that

$$H^\perp(t)(g_{\lambda/\mu}) = \sum_{\mu \subset \nu \subset \lambda} (H(t), g_{\lambda/\nu}) \cdot g_{\nu/\mu} = \sum_{\mu \subset \nu \subset \lambda} t^{c(\lambda/\nu)} g_{\nu/\mu}, \tag{3.4.6}$$

where the last equation is by Proposition 3.4.1. Similarly, by applying $(\mathrm{id} \otimes (H(t), -))$ to (3.4.3) we have

$$H^{\perp}(t)(g_{\lambda/\mu}) = \sum_{\mu \subset \nu \subset \lambda} g_{\lambda/\nu} \cdot (H(t), g_{\nu/\mu}) = \sum_{\mu \subset \nu \subset \lambda} t^{c(\nu/\mu)} g_{\lambda/\nu}. \tag{3.4.7}$$

□

Setting $\mu = \varnothing$ and $t = 1$ in (3.4.6), for any $\lambda \in \mathcal{P}$ we have

$$H^{\perp}(1)(g_\lambda) = \sum_{\nu \subset \lambda} g_\nu \ \Big(= I(g_\lambda)\Big).$$

Since $\{g_\lambda\}_\lambda$ form a basis of Λ, this implies

Proposition 3.4.4. *We have*

$$I = H^{\perp}(1) = \big(f(x) \mapsto f(1, x)\big). \tag{3.4.8}$$

Besides, since $H(1) = \sum_\lambda G_\lambda$ *by* (3.4.2), *we have*

$$I = \sum_{\lambda \in \mathcal{P}} G_\lambda^{\perp}. \tag{3.4.9}$$

In particular (3.4.8) shows that $I\colon \Lambda \longrightarrow \Lambda$ is a ring automorphism. Moreover, (3.4.6), (3.4.7) and (3.4.8) imply that for any skew shape λ/μ

$$I(g_{\lambda/\mu}) = \sum_{\mu \subset \nu \subset \lambda} g_{\nu/\mu} = \sum_{\mu \subset \nu \subset \lambda} g_{\lambda/\nu}. \tag{3.4.10}$$

In Example 3.6.1 we see an example for the identity (3.4.10).

Remark 3.4.5. (1) By Lemma 3.2.4, I is a Λ-comodule automorphism. With the second equality of (3.4.8), we can see each side of the following equality (the condition for being comodule morphisms)

$$\Delta \circ I = (I \otimes \mathrm{id}_\Lambda) \circ \Delta = (\mathrm{id}_\Lambda \otimes I) \circ \Delta$$

maps $f(x) \in \Lambda(x)$ to $f(1, x, y) \in \Lambda(x) \otimes \Lambda(y)$.
(2) By (3.2.6) and $\Delta(g_\lambda) = \sum_\nu g_\nu \otimes g_{\lambda/\nu}$ (by (3.4.3)), we have $G_\mu^{\perp}(g_\lambda) = g_{\lambda/\mu}$. Here we consider $g_{\lambda/\mu} = 0$ if $\mu \not\subset \lambda$. Since $F^{\perp}$ (for $F \in \widehat{\Lambda}$) commute each other, I commutes with $G_\mu^{\perp}$. Hence, by applying $G_\mu^{\perp}$ to the equation $I(g_\lambda) = \sum_{\nu \subset \lambda} g_\nu$ we have

$$I(g_{\lambda/\mu}) = I(G_\mu^{\perp}(g_\lambda)) = G_\mu^{\perp}(I(g_\lambda)) = G_\mu^{\perp}\Big(\sum_{\nu \subset \lambda} g_\nu\Big) = \sum_{\nu \subset \lambda} G_\mu^{\perp}(g_\nu) = \sum_{\nu \subset \lambda} g_{\nu/\mu},$$

re-proving the first equation in (3.4.10). Similarly, by applying $I = \sum_\nu G_\nu^{\perp}$ to g_λ we get a special case of the second equation of (3.4.10):

$$I(g_\lambda) = \sum_\nu G_\nu^{\perp}(g_\lambda) = \sum_\nu g_{\lambda/\nu}.$$

3.4.2 Dual map

Recall that $H^{\perp}(t)\colon \Lambda \longrightarrow \Lambda$ and $(H(t)\cdot)\colon \widehat{\Lambda} \longrightarrow \widehat{\Lambda}$ are adjoint (Lemma 3.2.4 (2)). By (3.2.6) and $H(t)^{\perp}(g_\mu) = \sum_{\lambda \subset \mu} t^{c(\mu/\lambda)} g_\lambda$ shown in (3.4.5) we have

$$H(t) G_\lambda = \sum_{\lambda \subset \mu} t^{c(\mu/\lambda)} G_\mu. \tag{3.4.11}$$

Note that (3.4.2) is obtained by setting $\lambda = \varnothing$ in (3.4.11). By (3.4.2) and (3.4.11) we have for any $\lambda \in \mathcal{P}$

$$\Big(\sum_{\mu \in \mathcal{P}} t^{c(\mu)} G_\mu\Big) G_\lambda = \sum_{\lambda \subset \mu} t^{c(\mu/\lambda)} G_\mu. \tag{3.4.12}$$

Remark 3.4.6. Since $I = H^{\perp}(1)$ it follows that $I^* = (H(1)\cdot) = \big((\sum_\lambda G_\lambda)\cdot\big)$, and (3.4.12) specializes to

$$\Big(I^*(G_\lambda) = \Big) \quad \big(\sum_{\mu\in\mathcal{P}} G_\mu\big)G_\lambda = \sum_{\lambda\subset\mu} G_\mu \tag{3.4.13}$$

which appeared in [Buc02, Section 8].

3.5 Description of $E(t)$, $(E(t), -)$ and $E(t)^{\perp}$

In this section we give descriptions using G_λ and g_λ for the element $E(t)$ and maps $(E(t), -)$ and $E^{\perp}(t)$. Note that by $I = H^{\perp}(1)$ and $I^* = (H(1)\cdot)$ it follows that $I^{-1} = E^{\perp}(-1)$ and $(I^*)^{-1} = (E(-1)\cdot)$.

We postpone the proof of the following proposition until Section 3.5.1.

Proposition 3.5.1. *The ring homomorphism $(E(t), -)\colon \Lambda \longrightarrow K$ satisfies*

$$(E(t), g_{\lambda/\mu}) = \begin{cases} t^{c(\lambda/\mu)}(t+1)^{|\lambda/\mu|-c(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a vertical strip,} \\ 0 & \text{otherwise} \end{cases}$$

for any skew shape λ/μ. In particular, for any $\lambda \in \mathcal{P}$,

$$(E(t), g_\lambda) = \begin{cases} 1 & \text{if } \lambda = \varnothing, \\ t(t+1)^{n-1} & \text{if } \lambda = (1^n)\ (n \geq 1), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.5.2. (1) By (3.3.5) and Remark 3.3.3 it follows that $(E(t), e_i) = t^i$ for $i \geq 0$, $(E(t), h_i) = 0$ for $i \geq 2$, and $(E(t), p_i) = (-1)^{i-1}t^i$ for $i \geq 1$.
(2) Setting $t = -1$ in Proposition 3.5.1, we have $(E(-1), g_{\lambda/\mu}) = (-1)^{|\lambda/\mu|}$ if λ/μ is a rook strip, and $(E(-1), g_{\lambda/\mu}) = 0$ otherwise. In particular $(E(-1), g_\varnothing) = 1$, $(E(-1), g_{(1)}) = -1$, and $(E(-1), g_\lambda) = 0$ for any $\lambda \in \mathcal{P}$ with $|\lambda| > 1$.
(3) Unlike (3.3.4), there is no $a_1, a_2, \dots \in \mathbb{R}$ such that $(E(-1), f) = f(a_1, a_2, \dots)$, since such numbers should satisfy $-1 = (E(-1), p_2) = a_1^2 + a_2^2 + \cdots$.

Before proving Proposition 3.5.1, we give as its corollaries descriptions for $E(t)$ and $E(t)^{\perp}$.

Proposition 3.5.3. *The ring automorphism $E(t)^{\perp}\colon \Lambda \longrightarrow \Lambda$ satisfies*

$$\begin{aligned} E(t)^{\perp}(g_{\lambda/\mu}) &= \sum_{\substack{\mu\subset\nu\subset\lambda \\ \lambda/\nu:\ \text{vertical strip}}} t^{c(\lambda/\nu)}(t+1)^{|\lambda/\nu|-c(\lambda/\nu)} g_{\nu/\mu} \\ &= \sum_{\substack{\mu\subset\nu\subset\lambda \\ \nu/\mu:\ \text{vertical strip}}} t^{c(\nu/\mu)}(t+1)^{|\nu/\mu|-c(\nu/\mu)} g_{\lambda/\nu} \end{aligned}$$

for any skew shape λ/μ. In particular, for any $\lambda \in \mathcal{P}$,

$$\begin{aligned} E(t)^{\perp}(g_\lambda) &= \sum_{\substack{\nu\subset\lambda \\ \lambda/\nu:\ \text{vertical strip}}} t^{c(\lambda/\nu)}(t+1)^{|\lambda/\nu|-c(\lambda/\nu)} g_\nu \\ &= \begin{cases} g_\lambda + \sum_{k=1}^{l(\lambda)} t(t+1)^{k-1} g_{\lambda/(1^k)} & \text{if } \lambda \neq \varnothing, \\ g_\varnothing & \text{if } \lambda = \varnothing. \end{cases} \end{aligned} \tag{3.5.1}$$

Proof. Proved similarly to Proposition 3.4.3, with Proposition 3.5.1 in hand. □

Now we have a description of $E(-1)^{\perp} = I^{-1}$ by setting $t = -1$ in the proposition above.

Corollary 3.5.4. *The ring automorphism $E(-1)^\perp = I^{-1}\colon \Lambda \longrightarrow \Lambda$ satisfies*

$$I^{-1}(g_{\lambda/\mu}) = \sum_{\substack{\mu\subset\nu\subset\lambda \\ \lambda/\nu:\ \text{rook strip}}} (-1)^{|\lambda/\nu|} g_{\nu/\mu} = \sum_{\substack{\mu\subset\nu\subset\lambda \\ \nu/\mu:\ \text{rook strip}}} (-1)^{|\nu/\mu|} g_{\lambda/\mu}.$$

In particular, when $\mu = \varnothing$ we have

$$I^{-1}(g_\lambda) = \sum_{\lambda/\nu:\ \text{rook strip}} (-1)^{|\lambda/\nu|} g_\nu = \begin{cases} g_\lambda - g_{\lambda/(1)} & \text{if } \lambda \neq \varnothing, \\ 1 & \text{if } \lambda = \varnothing. \end{cases} \tag{3.5.2}$$

Since $E^\perp(t)$ and $(E(t)\cdot)$ are adjoint, by (3.5.1) and (3.2.6) we have the following:

Proposition 3.5.5. *The element $E(t) = \sum_{i\geq 0} t^i e_i \in \widehat{\Lambda}$ satisfies*

$$E(t)G_\lambda = \sum_{\mu/\lambda:\ \text{vertical strip}} t^{c(\mu/\lambda)}(t+1)^{|\mu/\lambda|-c(\mu/\lambda)} G_\mu. \tag{3.5.3}$$

In particular, setting $\lambda = \varnothing$ we have

$$E(t) = 1 + \sum_{n\geq 1} t(t+1)^{n-1} G_{(1^n)},$$

and hence

$$\Big(1 + \sum_{n\geq 1} t(t+1)^{n-1} G_{(1^n)}\Big) G_\lambda = \sum_{\mu/\lambda:\ \text{vertical strip}} t^{c(\mu/\lambda)}(t+1)^{|\mu/\lambda|-c(\mu/\lambda)} G_\mu. \tag{3.5.4}$$

Remark 3.5.6. Setting $t = -1$, Proposition 3.5.5 specializes to $E(-1) = 1 - G_1$ and

$$E(-1)G_\lambda = (1 - G_1)G_\lambda = \sum_{\mu/\lambda:\ \text{rook strip}} (-1)^{|\mu/\lambda|} G_\mu, \tag{3.5.5}$$

which appeared in [Buc02, Section 8].

3.5.1 Proof of Proposition 3.5.1

We recall the *incidence algebras* (see [Sta12, Chapter 3.6] for details). Let $\mathrm{Int}(\mathcal{P}) = \{(\mu,\lambda) \in \mathcal{P}\times\mathcal{P} \mid \mu \subset \lambda\}$, consisting of all comparable (ordered) pairs in $\mathcal{P}$ (or equivalently all skew shapes, by identifying (μ,λ) with λ/μ). The *incidence algebra* $I(\mathcal{P}) = I(\mathcal{P}, K)$ is the algebra of all functions $f\colon \mathrm{Int}(\mathcal{P}) \longrightarrow K$ where multiplication is defined by the convolution

$$(fg)(\mu,\lambda) = \sum_{\mu\subset\nu\subset\lambda} f(\mu,\nu)g(\nu,\lambda). \tag{3.5.6}$$

Then $I(\mathcal{P}, K)$ is an associative algebra with two-sided identity $\delta := ((\mu,\lambda) \mapsto \delta_{\mu\lambda})$.

A linear function $f\colon \Lambda \longrightarrow K$ can be considered as an element of $I(\mathcal{P}, K)$ by setting $f(\mu,\lambda) = f(g_{\lambda/\mu})$. Then the convolution product $*$ on $\mathrm{Hom}(\Lambda, K)$ coincides with the multiplication on $I(\mathcal{P})$ due to (3.4.3), i.e. this inclusion $\mathrm{Hom}(\Lambda, K) \longrightarrow I(\mathcal{P})$ is as algebras.[1] Note that the counit $\epsilon \in \mathrm{Hom}(\Lambda, K)$ is mapped to $\delta \in I(\mathcal{P})$.

[1] Since $\Delta(s_{\lambda/\mu}) = \sum_\nu s_{\lambda/\nu} \otimes s_{\nu/\mu}$, by setting $f(\mu,\lambda) = f(s_{\lambda/\mu})$ we can obtain another algebra inclusion, although we do not use it.

Proof of Proposition 3.5.1. Define $i_t, j_t \in I(\mathcal{P})$ by

$$i_t(\mu, \lambda) = t^{c(\lambda/\mu)}$$

and

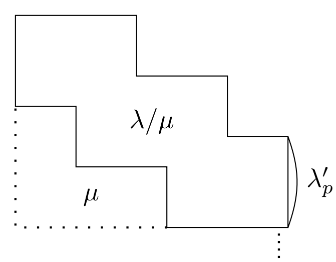
$$j_t(\mu, \lambda) = \begin{cases} (-1)^{|\lambda/\mu|} t^{c(\lambda/\mu)} (t-1)^{|\lambda/\mu| - c(\lambda/\mu)} & \text{if } \lambda/\mu \text{ is a vertical strip,} \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 3.4.1 $(H(t), -) \in \mathrm{Hom}(\Lambda, K)$ corresponds to $i_t \in I(\mathcal{P})$. Since $(H(t), -) * (E(-t), -) = \epsilon$, it suffices to show that $i_t j_t = \delta$ in order to prove that $(E(-t), -)$ corresponds to j_t, whence Proposition 3.5.1 follows by replacing t with $-t$.

By the definitions of i_t and j_t and (3.5.6)

$$(i_t j_t)(\mu, \lambda) = \sum_{\substack{\mu \subset \nu \subset \lambda \\ \lambda/\nu:\ \text{vertical strip}}} t^{c(\nu/\mu)} (-1)^{|\lambda/\nu|} t^{c(\lambda/\nu)} (t-1)^{|\lambda/\nu| - c(\lambda/\nu)}. \tag{3.5.7}$$

We need to show that the value of the right-hand side of (3.5.7) is $\delta_{\mu\lambda}$. It is clear that if $\mu = \lambda$ then the value of (3.5.7) is 1. Assume $\mu \subsetneq \lambda$. Since the value of (3.5.7) is invariant under removal of empty rows in the skew shape λ/μ, we can assume there is a box in the first row of λ/μ, i.e. $\lambda_1 > \mu_1$. Let p be the index of the rightmost column of λ, i.e. $\lambda_1 = p$. Note that $\lambda'_p > 0 = \mu'_p$. Let $\tilde{\lambda}$ be the partition obtained by removing the p-th (rightmost) column of λ, i.e. $\tilde{\lambda}_i = \min(\lambda_i, p-1)$. (Note: in the figure below and hereafter we display Young diagrams in the French notation.)



For a vertical strip λ/ν with $\mu \subset \nu$, by removing the p-th column of ν (let $\tilde{\nu}$ denote the resulting shape) we get a vertical strip $\tilde{\lambda}/\tilde{\nu}$ that satisfies $\mu \subset \tilde{\nu}$ and $\tilde{\nu}'_{p-1} \geq \lambda'_p$. Conversely, for any vertical strip $\tilde{\lambda}/\kappa$ with $\mu \subset \kappa$ and $\kappa'_{p-1} \geq \lambda'_p$ and any integer $0 \leq i \leq \lambda'_p$, by adding i boxes in the p-th column of κ we get the shape $\kappa + (1^i)$, for which $\lambda/(\kappa + (1^i))$ is a vertical strip. Therefore we have a bijection

$$\{\nu \mid \mu \subset \nu \subset \lambda,\ \lambda/\nu:\ \text{vertical strip}\} \simeq \{\kappa \mid \mu \subset \kappa \subset \tilde{\lambda},\ \tilde{\lambda}/\kappa:\ \text{vertical strip},\ \kappa'_{p-1} \geq \lambda'_p\} \times \{0, 1, \ldots, \lambda'_p\} \tag{3.5.8}$$

in which ν corresponds to $(\tilde{\nu}, \nu'_p)$, where $\tilde{\nu}$ is ν with its p-th column removed. For ν in the left-hand side of (3.5.8), it is easy to see that

$$c(\nu/\mu) = c(\tilde{\nu}/\mu) + \delta\left[\nu'_p > 0\right],$$
$$|\lambda/\nu| = |\tilde{\lambda}/\tilde{\nu}| + \lambda'_p - \nu'_p,$$
$$c(\lambda/\nu) = c(\tilde{\lambda}/\tilde{\nu}) + \delta\left[\nu'_p < \lambda'_p\right],$$

where we use the notation $\delta[P] = 1$ if P is true and $\delta[P] = 0$ if P is false for a condition P.

Write simply $A = \{\kappa \mid \mu \subset \kappa \subset \tilde{\lambda},\ \tilde{\lambda}/\kappa\text{: vertical strip},\ \kappa'_{p-1} \geq \lambda'_p\}$. Substituting these to (3.5.7), we have

$$(\text{RHS of (3.5.7)}) = \sum_{\kappa \in A} \sum_{i=0}^{\lambda'_p} t^{c(\kappa/\mu)+\delta[i>0]}(-1)^{|\tilde{\lambda}/\kappa|+\lambda'_p-i} t^{c(\tilde{\lambda}/\kappa)+\delta[i<\lambda'_p]}(t-1)^{|\tilde{\lambda}/\kappa|+\lambda'_p-i-(c(\tilde{\lambda}/\kappa)+\delta[i<\lambda'_p])}$$

$$= \sum_{\kappa \in A}(-1)^{|\tilde{\lambda}/\kappa|} t^{c(\kappa/\mu)+c(\tilde{\lambda}/\kappa)}(t-1)^{|\tilde{\lambda}/\kappa|-c(\tilde{\lambda}/\kappa)} \underbrace{\sum_{i=0}^{\lambda'_p}(-1)^{\lambda'_p-i} t^{\delta[i>0]+\delta[i<\lambda'_p]}(t-1)^{\lambda'_p-i-\delta[i<\lambda'_p]}}_{(X)}.$$

We shall show $(X) = 0$. Letting $q = \lambda'_p\ (>0)$ and $j = q - i$ we rewrite (X) as

$$(X) = \sum_{j=0}^{q}(-1)^j t^{\delta[j>0]+\delta[j<q]}(t-1)^{j-\delta[j>0]}. \tag{3.5.9}$$

It is easy to check (3.5.9) $= 0$ when $q = 1$. When $q \geq 2$, by checking

$$(-1)^q t(t-1)^{q-1} + (-1)^{q-1}t^2(t-1)^{q-2} = (-1)^{q-1}t(t-1)^{q-2},$$

we can carry induction on q to obtain (3.5.9) $= 0$.

Therefore we conclude (3.5.7) $= 0$ if $\lambda/\mu \neq \varnothing$, finishing the proof of Proposition 3.5.1. □

3.6 Example

We display Young diagrams in the French notation.

Example 3.6.1. Let $\lambda/\mu = (3,2,1)/(1) = (3,2,1)/(1)$. We shall verify (3.4.10) for this λ/μ by expanding each term into a linear combination of $\{g_\nu\}$. We can check

$$g_{(3,2,1)/(1)} = g_{(3,2)} + g_{(3,1,1)} + g_{(2,2,1)} - g_{(3,1)} - g_{(2,2)} - g_{(2,1,1)} + g_{(2,1)}, \tag{3.6.1}$$

using recursively the Pieri formula for skew dual stable Grothendieck polynomials [Yel, Theorem 7.1]

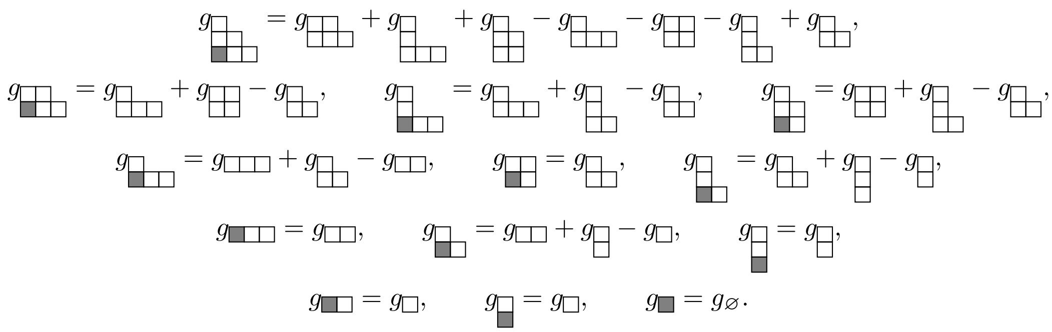
$$h_k g_{\mu/\nu} = \sum_{\substack{\lambda/\mu\text{: horizontal strip} \\ \nu/\eta\text{: vertical strip}}} (-1)^{k-|\lambda/\mu|} \binom{a(\lambda/\!\!/\mu) - a(\nu'/\!\!/\eta') - |\nu/\eta|}{k - |\lambda/\mu| - |\nu/\eta|} g_{\lambda/\eta}, \tag{3.6.2}$$

where $a(\alpha/\!\!/\beta)$ is the number of $i \geq 1$ satisfying $\beta_i > \alpha_{i+1}$ and $\beta_i > \beta_{i+1}$, and the binomial coefficient $\binom{m}{n}$ is considered as 0 when $n < 0$. (Note: another way to check (3.6.1) is to use (3.5.2).)

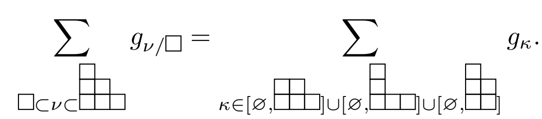
Applying I to (3.6.1) and using $I(g_\kappa) = \sum_{\alpha \subset \kappa} g_\alpha$, we compute the first term of (3.4.10) as

$$\begin{aligned} I(g_{(3,2,1)/(1)}) &= I(g_{(3,2)} + g_{(3,1,1)} + g_{(2,2,1)} - g_{(3,1)} - g_{(2,2)} - g_{(2,1,1)} + g_{(2,1)}) \\ &= \sum_{\kappa \subset (3,2)} g_\kappa + \sum_{\kappa \subset (3,1,1)} g_\kappa + \sum_{\kappa \subset (2,2,1)} g_\kappa - \sum_{\kappa \subset (3,1)} g_\kappa - \sum_{\kappa \subset (2,2)} g_\kappa - \sum_{\kappa \subset (2,1,1)} g_\kappa + \sum_{\kappa \subset (2,1)} g_\kappa \\ &= \sum_{\kappa \in [\varnothing,(3,2)] \cup [\varnothing,(3,1,1)] \cup [\varnothing,(2,2,1)]} g_\kappa. \end{aligned} \tag{3.6.3}$$

Next we compute the second term of (3.4.10), $\sum_{(1)\subset\nu\subset(3,2,1)} g_{\nu/(1)}$. Again using (3.6.2) we have

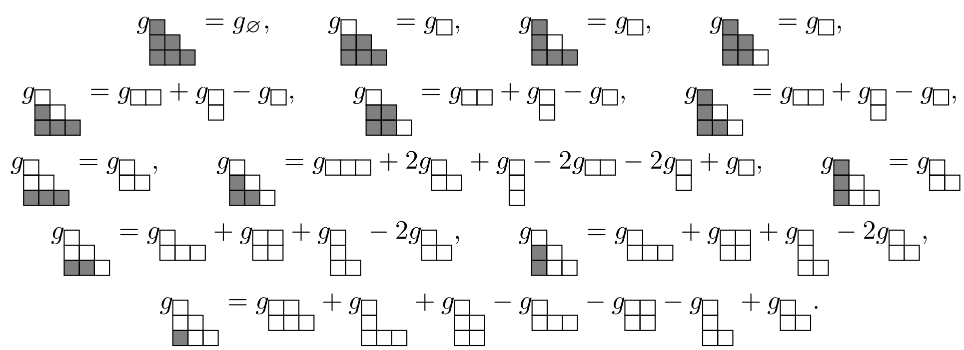


Summing them up, we have

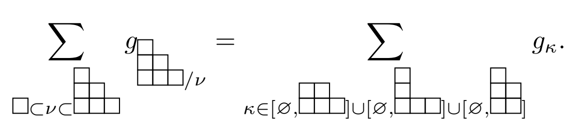


(3.6.4)

Finally we compute the last term of (3.4.10), $\sum_{(1)\subset\nu\subset(3,2,1)} g_{(3,2,1)/\nu}$. We can check



Summing them up, we have



(3.6.5)

Hence we see (3.6.3) = (3.6.4) = (3.6.5), verifying (3.4.10).

Remark 3.6.2. From (3.6.3) in the example above and a Pieri-type formula given later in Proposition 4.3.2 ($I(g_\lambda)I(g_{(k)}) = \sum_\mu g_\mu$, summed over μ satisfying the set difference $\mu \setminus \lambda$ is a horizontal strip of size $\leq k$), one may expect positivity in the expansions $I(g_{\lambda/\mu}) = \sum_\nu \tilde{c}^\lambda_{\mu\nu} g_\nu$ and $I(g_\mu)I(g_\nu) = \sum_\lambda \tilde{d}^\lambda_{\mu\nu} g_\lambda$ (note that $\tilde{c}^\lambda_{\mu\nu} = \sum_{\mu\subset\kappa\subset\lambda} c^\lambda_{\kappa,\nu} = \sum_{\mu\subset\kappa\subset\lambda} c^\kappa_{\mu,\nu}$ and $\tilde{d}^\lambda_{\mu\nu} = \sum_{\substack{\alpha\subset\mu \\ \beta\subset\nu}} d^\lambda_{\alpha,\beta}$). However, neither hold in general; a counterexample for the former is $\tilde{c}^{(53221)}_{(321),(321)} = -1$, and one for the latter is $\tilde{d}^{(5321)}_{(321),(321)} = -1$.

3.7 Appendix: Group-like elements in $\widehat{\Lambda}$ and automorphisms on Λ

An element a of a coalgebra (A, Δ, ϵ) is called *group-like* if $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$. The set of group-like elements in a Hopf algebra forms a group; if a and b are group-like then so are ab and $S(a) = a^{-1}$. If A is a bialgebra and $a \in A$ is group-like then the multiplication map $L_a\colon A \longrightarrow A\,;\, b \mapsto ab$ is a coalgebra morphism since $\Delta \circ L_a(b) = \Delta(ab) = \Delta(a)\Delta(b) = (a \otimes a)\Delta(b) = (L_a \otimes L_a) \circ \Delta(b)$, and hence the dual map $L_a^*\colon A^* \longrightarrow A^*$ is an algebra morphism.

Although $\widehat{\Lambda}$ is not a coalgebra, we shall also say $F \in \widehat{\Lambda}$ is *group-like* if $F(x,y) = F(x)F(y)$ and its constant term $F(0)$ is 1. Again, here we mean $F(x) = F(x_1, x_2, \cdots)$, $F(y) = F(y_1, y_2, \cdots)$, $F(x,y) = F(x_1, x_2, \cdots, y_1, y_2, \cdots)$ and $F(0) = F(0, 0, \cdots)$. Then these elements satisfy expected properties seen above:

Lemma 3.7.1. *For group-like elements $F, F' \in \widehat{\Lambda}$,*

(1) *FF' is group-like.*
(2) *$S(F) = F^{-1}$ is group-like. Here we extend the antipode $S\colon \Lambda \longrightarrow \Lambda\,;\ s_\lambda \mapsto (-1)^{|\lambda|} s_{\lambda'}$ to $\widehat{\Lambda} \longrightarrow \widehat{\Lambda}$.*

Proof. (1) By $FF'(x,y) = F(x,y)F'(x,y) = F(x)F(y)F'(x)F'(y) = FF'(x) \cdot FF'(y)$ and $FF'(0) = F(0)F'(0) = 1$.
(2) Write $F = \sum_\lambda A_\lambda s_\lambda$ with $A_\lambda \in K$ (possibly an infinite sum).

Since $F(x,y) = \sum_\lambda A_\lambda s_\lambda(x,y) = \sum_\lambda A_\lambda \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x)s_\nu(y) = \sum_{\mu,\nu} \big(\sum_\lambda A_\lambda c^\lambda_{\mu\nu}\big) s_\mu(x)s_\nu(y)$ and $F(x)F(y) = \big(\sum_\mu A_\mu s_\mu(x)\big)\big(\sum_\nu A_\nu s_\nu(y)\big) = \sum_{\mu,\nu} A_\mu A_\nu s_\mu(x)s_\nu(y)$, it follows that

$$F = \sum_\lambda A_\lambda s_\lambda \text{ is group-like} \iff A_\varnothing = 1 \text{ and } A_\mu A_\nu = \sum_\lambda A_\lambda c^\lambda_{\mu\nu} \text{ for } \forall \mu, \nu. \tag{3.7.1}$$

Let $F' := S(F) = \sum A_\lambda (-1)^{|\lambda|} s_{\lambda'}$. Similarly we can see that F' is group-like if and only if $A_\varnothing = 1$ and $A_\mu A_\nu = \sum_\lambda A_\lambda c^{\lambda'}_{\mu'\nu'}$ for any μ, ν. Since $c^\lambda_{\mu\nu} = c^{\lambda'}_{\mu'\nu'}$ it follows that F' is group-like.

Since $\Delta(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu$, by applying $m \circ (\mathrm{id} \otimes S)$ we have $\sum_{\mu,\nu} (-1)^{|\nu|} c^\lambda_{\mu\nu} s_\mu s_{\nu'} = m \circ (\mathrm{id} \otimes S) \circ \Delta(s_\lambda) = u \circ \epsilon(s_\lambda) = \delta_{\lambda,\varnothing}$. From this and (3.7.1) we have $FF' = \big(\sum_\mu A_\mu s_\mu\big)\big(\sum_\nu (-1)^{|\nu|} A_\nu s_{\nu'}\big) = \sum_{\mu,\nu} (-1)^{|\nu|} A_\mu A_\nu s_\mu s_{\nu'} = \sum_{\mu,\nu}\sum_\lambda (-1)^{|\nu|} c^\lambda_{\mu\nu} A_\lambda s_\mu s_{\nu'} = \sum_\lambda A_\lambda \big(\sum_{\mu,\nu} (-1)^{|\nu|} c^\lambda_{\mu\nu} s_\mu s_{\nu'}\big) = \sum_\lambda A_\lambda \delta_{\lambda,\varnothing} = A_\varnothing = 1$. Hence $F' = F^{-1}$. □

Lemma 3.7.2. *For $F \in \widehat{\Lambda}$, the followings are equivalent.*

(1) *$F \in \widehat{\Lambda}$ is group-like.*
(2) *$(F, -)\colon \Lambda \longrightarrow K$ is an algebra homomorphism.*
(3) *$F^\perp\colon \Lambda \longrightarrow \Lambda$ is an algebra automorphism.*

Proof. (1) $\iff$ (2) Again we write $F = \sum_\lambda A_\lambda s_\lambda$ with $A_\lambda \in K$. (2) is equivalent to $(F, 1) = 1$ and $(F, s_\mu)(F, s_\nu) = (F, s_\mu s_\nu)$ for any μ, ν, which is equivalent to (3.7.1) since $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$.
(2) $\implies$ (3) Since $(F, -)\colon \Lambda \longrightarrow K$ and $\mathrm{id}_\Lambda\colon \Lambda \longrightarrow \Lambda$ are algebra morphisms, $(F, -) \otimes \mathrm{id}_\Lambda\colon \Lambda \otimes \Lambda \longrightarrow \Lambda$ is an algebra morphism. Since $\Delta\colon \Lambda \longrightarrow \Lambda \otimes \Lambda$ is an algebra morphism by the axiom of bialgebras, it follows that $F^\perp = ((F, -) \otimes \mathrm{id}) \circ \Delta$ is an algebra morphism.

By $S(F) = F^{-1}$ and Lemma 3.2.3 (2) we have $S(F)^\perp = (F^\perp)^{-1}$. Hence $F^\perp$ is invertible.
(3) $\implies$ (2) By (3.2.4) and the axiom of bialgebras that $\epsilon\colon \Lambda \longrightarrow K$ is an algebra morphism, it follows that $(F, -) = \epsilon \circ F^\perp$ is an algebra morphism. □

Remark 3.7.3. There is no group-like element in Λ except 1 since $f(x,y) = f(x)f(y)$ implies $\deg(f) = \deg(f) + \deg(f)$.

Remark 3.7.4. By Lemma 3.2.4 and 3.2.5, an algebra automorphism on Λ is of the form $F^\perp$ for some $F \in \widehat{\Lambda}$ if and only if it is a Λ-comodule morphism.

By Lemma 3.7.2, the following lemma implies that $H^\perp(t)$ and $E^\perp(t)$ are algebra automorphisms on Λ.

Lemma 3.7.5. *The elements $H(t), E(t) \in \widehat{\Lambda}$ are group-like.*

Proof. By $\Delta(h_k) = \sum_{i+j=k} h_i \otimes h_j$, we have $H(t)(x,y) = \sum_{k \geq 0} t^k h_k(x,y) = \sum_{k \geq 0} t^k \sum_{i+j=k} h_i(x)h_j(y) = \big(\sum_{i \geq 0} t^i h_i(x)\big)\big(\sum_{j \geq 0} t^j h_j(y)\big) = \big(H(t)(x)\big)\big(H(t)(y)\big)$. The proof for $E(t)$ is similar. □

Chapter 4

On the Pieri rules of stable and dual stable Grothendieck polynomials

4.1 Introduction

The stable Grothendieck polynomials G_λ and the dual stable Grothendieck polynomials g_λ are certain families of inhomogeneous symmetric functions parametrized by interger partitions λ. They are certain K-theoretic deformations of the Schur functions and dual to each other via the Hall inner product.

The Pieri formulas for G_λ and g_λ ((4.2.1) and (4.2.2)) involve certain binomials coefficients, and we show in this chapter that these coefficients are the values of the Möbius functions of certain posets of horizontal strips (Lemma 4.3.1), and hence the Pieri formulas can be rewritten as certain multiplicity-free sums ((4.3.6) and (4.3.9)) by using certain sums $\widetilde{g}_\lambda = \sum_{\mu\subset\lambda} g_\mu$ and $\widetilde{G}_\lambda = \sum_{\mu\supset\lambda} G_\mu$; or, as alternating sums of meets/joins of the leading terms (Proposition 4.3.2 and 4.3.3).

4.2 Stable and dual stable Grothendieck polynomials

We follow the settings in Chapter 3; see Section 3.2 for undefined notations. In particular, see Section 3.2.3 for the stable Grothendieck polynomials G_λ and the dual stable Grothendieck polynomials g_λ.

Let $\mathcal{P}$ be the set of integer partitions. For partitions $\lambda,\mu\in\mathcal{P}$, the inclusion $\lambda\subset\mu$ means $\lambda_i\le\mu_i$ for all i, and $\lambda\cap\mu$ and $\lambda\cup\mu$ ($\in\mathcal{P}$) are given by $(\lambda\cap\mu)_i=\min(\lambda_i,\mu_i)$ and $(\lambda\cup\mu)_i=\max(\lambda_i,\mu_i)$ for all i. In other words, $\cap$ and $\cup$ are the meet and join of the poset $(\mathcal{P},\subset)$.

4.2.1 Pieri rules

The (row) Pieri formula for G_λ was given by Lenart [Len00, Theorem 3.2]: for any partition $\lambda\in\mathcal{P}$ and integer $a\ge 0$,

$$G_{(a)}G_\lambda=\sum_{\mu/\lambda:\ \text{horizontal strip}}(-1)^{|\mu/\lambda|-a}\binom{r(\mu/\lambda)-1}{|\mu/\lambda|-a}G_\mu, \tag{4.2.1}$$

where $r(\mu/\lambda)$ denotes the number of the rows in the skew shape μ/λ. Subsequently, the (row) Pieri formula for g_λ is given in [Buc02, Corollary 7.1] (as a formula for $d^\mu_{\lambda,(a)}$, the coproduct structure constants for G_λ):

$$g_{(a)}g_\lambda=\sum_{\mu/\lambda:\ \text{horizontal strip}}(-1)^{a-|\mu/\lambda|}\binom{r(\lambda/\bar{\mu})}{a-|\mu/\lambda|}g_\mu, \tag{4.2.2}$$

where $\bar{\mu}=(\mu_2,\mu_3,\dots)$.

4.2.2 Their sums

For $\lambda \in \mathcal{P}$ we let

$$\widetilde{g}_\lambda = \sum_{\mu \subset \lambda} g_\mu \ (\in \Lambda), \qquad \widetilde{G}_\lambda = \sum_{\mu \supset \lambda} G_\mu \ (\in \widehat{\Lambda}).$$

In Chapter 3 it is shown (see Proposition 3.4.4 and (3.4.13)) that

$$H(1)^\perp(g_\lambda) = \widetilde{g}_\lambda, \tag{4.2.3}$$

$$H(1)G_\lambda = \widetilde{G}_\lambda. \tag{4.2.4}$$

Here $H(1) = \sum_{i \geq 0} h_i = \sum_{\lambda \in \mathcal{P}} G_\lambda$. Recall that $H(1)^\perp \colon \Lambda \longrightarrow \Lambda$ is an algebra automorphism. We remark that the equality (4.2.4) appeared in [Buc02, Section 8].

4.3 Description for the Pieri coefficients

In this section we give an explanation for the Pieri coefficients for G_λ (4.2.1) and g_λ (4.2.2); their non-leading terms (higher-degree terms for the case of G_λ; lower-degree terms for the case of g_λ) are obtained by taking an alternating sum of meets/joins of the leading terms ((4.3.8) and (4.3.11)). Another equivalent description is that the product $\widetilde{G}_\lambda G_{(a)}$ (resp. $\widetilde{g}_\lambda \widetilde{g}_{(a)}$) is expanded into a certain multiplicity-free sum of G_μ (resp. g_μ) ((4.3.6) and (4.3.9)).

The key fact is that the coefficients in the Pieri rule (4.2.1) and (4.2.2) are values of the Möbius functions of certain posets of horizontal strips over λ: for $\lambda \in \mathcal{P}$ and $a \in \mathbb{Z}_{>0}$, let

$$\mathrm{HS}(\lambda) = \{\mu \in \mathcal{P} \mid \mu/\lambda \text{ is a horizontal strip}\}, \tag{4.3.1}$$

$$\mathrm{HS}_{\leq a}(\lambda) = \{\mu \in \mathrm{HS}(\lambda) \mid |\mu/\lambda| \leq a\}, \qquad \widehat{\mathrm{HS}}_{\leq a}(\lambda) = \mathrm{HS}_{\leq a}(\lambda) \sqcup \{\hat{1}\}, \tag{4.3.2}$$

$$\mathrm{HS}_{\geq a}(\lambda) = \{\mu \in \mathrm{HS}(\lambda) \mid |\mu/\lambda| \geq a\}, \qquad \widehat{\mathrm{HS}}_{\geq a}(\lambda) = \mathrm{HS}_{\geq a}(\lambda) \sqcup \{\hat{0}\}. \tag{4.3.3}$$

Here $\hat{0}$ and $\hat{1}$ are the minimum and maximal elements. For a poset P, let μ_P denote its Möbius function (see Section 4.4). Then we have

Lemma 4.3.1. (1) *For any $\mu \in \mathrm{HS}_{\geq a}(\lambda)$, we have $c^\mu_{\lambda,(a)} = -\mu_{\widehat{\mathrm{HS}}_{\geq a}(\lambda)}(\hat{0}, \mu)$. That is,*

$$\sum_{\mu \supset \nu \in \mathrm{HS}_{\geq a}(\lambda)} c^\nu_{\lambda,(a)} = 1. \tag{4.3.4}$$

(2) *For any $\mu \in \mathrm{HS}_{\leq a}(\lambda)$, we have $d^\mu_{\lambda,(a)} = -\mu_{\widehat{\mathrm{HS}}_{\leq a}(\lambda)}(\mu, \hat{1})$. That is,*

$$\sum_{\mu \subset \nu \in \mathrm{HS}_{\leq a}(\lambda)} d^\nu_{\lambda,(a)} = 1. \tag{4.3.5}$$

Before proving Lemma 4.3.1 we show the following propositions. Let $\lambda^{(1)}, \lambda^{(2)}, \cdots$ be the list of all horizontal strips over λ of size a. Then

Proposition 4.3.2. *We have*

$$\widetilde{g}_{(a)} \widetilde{g}_\lambda = \sum_{\mu \subset \lambda^{(i)} \text{ for } \exists i} g_\mu \tag{4.3.6}$$

$$= \sum_i \widetilde{g}_{\mu^{(i)}} - \sum_{i<j} \widetilde{g}_{\mu^{(i)} \cap \mu^{(j)}} + \sum_{i<j<k} \widetilde{g}_{\mu^{(i)} \cap \mu^{(j)} \cap \mu^{(k)}} - \cdots, \tag{4.3.7}$$

and

$$g_{(a)} g_\lambda = \sum_i g_{\lambda^{(i)}} - \sum_{i<j} g_{\lambda^{(i)} \cap \lambda^{(j)}} + \sum_{i<j<k} g_{\lambda^{(i)} \cap \lambda^{(j)} \cap \lambda^{(k)}} - \cdots. \tag{4.3.8}$$

Proposition 4.3.3. *We have*

$$G_{(a)}\widetilde{G}_\lambda = \sum_{\mu\supset\lambda^{(i)}\ for\ \exists i} G_\mu \tag{4.3.9}$$

$$= \sum_i \widetilde{G}_{\lambda^{(i)}} - \sum_{i<j} \widetilde{G}_{\lambda^{(i)}\cup\lambda^{(j)}} + \sum_{i<j<k} \widetilde{G}_{\lambda^{(i)}\cup\lambda^{(j)}\cup\lambda^{(k)}} - \cdots, \tag{4.3.10}$$

and

$$G_{(a)}G_\lambda = \sum_i G_{\lambda^{(i)}} - \sum_{i<j} G_{\lambda^{(i)}\cup\lambda^{(j)}} + \sum_{i<j<k} G_{\lambda^{(i)}\cup\lambda^{(j)}\cup\lambda^{(k)}} - \cdots \tag{4.3.11}$$

Note that the left-hand side of (4.3.9) is not $\widetilde{G}_{(a)}\widetilde{G}_\lambda$ but $G_{(a)}\widetilde{G}_\lambda$ while that of (4.3.6) is $\widetilde{g}_{(a)}\widetilde{g}_\lambda$, reflecting the fact that the map $G_\lambda \mapsto \widetilde{G}_\lambda$ is a module morphism while $g_\lambda \mapsto \widetilde{g}_\lambda$ is a ring morphism.

Remark 4.3.4. The equations (4.3.6) and (4.3.7) are mere specializations of corresponding results (Theorem 1.1.3 and 1.1.4) for affine dual stable Grothendieck polynomials (also known as K-k-Schur functions) $g_\lambda^{(k)}$ shown in Chapter 1, but here we give another proof since it is easier and also applicable to G_λ. It is also notable that in the affine case (that is, for $g_\lambda^{(k)}$), equations of the form (4.3.6) and (4.3.7) hold but (4.3.8) does not.

Proof of Proposition 4.3.2. The right-hand sides of (4.3.6) and (4.3.7) are equal by the Inclusion-Exclusion Principle, and (4.3.7) and (4.3.8) are equivalent by (4.2.3).

Let P be the order ideal of $\mathcal{P}$ generated by $\{\lambda^{(1)}, \lambda^{(2)}, \cdots\}$ (i.e. the set of $\mu \in \mathcal{P}$ satisfying $\mu \subset \lambda^{(i)}$ for some i) and $\widehat{P} = P \sqcup \{\hat{1}\}$ where $\hat{1}$ is the maximum element. Note that $\{\lambda^{(1)}, \lambda^{(2)}, \cdots\}$ is the set of coatoms in $\widehat{P}$ and $\widehat{\mathrm{HS}}_{\le a}(\lambda)$ $(\subset \widehat{P})$ is closed under meet. Then

$$\widetilde{g}_\lambda\widetilde{g}_{(a)} = \sum_\nu d^\nu_{\lambda,(a)}\widetilde{g}_\nu \qquad ((4.2.2) \text{ and } (4.2.3)) \tag{4.3.12}$$

$$= -\sum_\nu \mu_{\widehat{\mathrm{HS}}_{\le a}(\lambda)}(\nu,\hat{1})\widetilde{g}_\nu \qquad (\text{Lemma } 4.3.1\ (2)) \tag{4.3.13}$$

$$= -\sum_\nu \mu_{\widehat{P}}(\nu,\hat{1})\widetilde{g}_\nu \qquad (\text{Lemma } 4.4.1\ (3)) \tag{4.3.14}$$

$$= \sum_{\mu\in P} g_\mu. \qquad (\text{Lemma } 4.4.1\ (1)) \tag{4.3.15}$$

Hence (4.3.6) follows. □

Proof of Proposition 4.3.3. Similarly to Proposition 4.3.2, the equivalence of (4.3.9), (4.3.10) and (4.3.11) follows and we have by (4.2.1), (4.2.4), Lemma 4.3.1 (1) and Lemma 4.4.1 (with all ordering reversed)

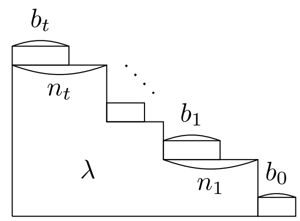
$$\widetilde{G}_\lambda G_{(a)} = \sum_\nu c^\nu_{\lambda,(a)}\widetilde{G}_\nu = \sum_{\mu\in Q} G_\mu, \tag{4.3.16}$$

where Q is the order filter of $\mathcal{P}$ generated by $\{\lambda^{(1)}, \lambda^{(2)}, \cdots\}$, i.e. the set of $\mu \in \mathcal{P}$ satisfying $\mu \supset \lambda^{(i)}$ for some i. Hence (4.3.9) follows. □

Proof of Lemma 4.3.1. Fix $\lambda \in \mathcal{P}$. Let $r_0 < r_1 < \cdots < r_t$ be the row indices for which rows there are addable corners of λ, i.e. $\lambda_{r_i-1} > \lambda_{r_i}$ (we consider $\lambda_0 = \infty$, whence $r_0 = 1$). Let $n_i = \lambda_{r_i-1} - \lambda_{r_i}$, i.e. the number of boxes that can be added to λ in the r_i-th row (we consider $n_0 = \infty$). Then

$$\mathrm{HS}(\lambda) \simeq \{(b_0, \ldots, b_t) \in \mathbb{Z}^{t+1} \mid 0 \le b_i \le n_i \ (\text{for } 0 \le i \le t)\},$$

where $(b_0, \dots, b_t)$ in the right-hand side corresponds to the partition obtained by adding b_i boxes to λ in the r_i-th row.



Under this correspondence $\mu \mapsto (b_0, \dots, b_t)$ and $\nu \mapsto (c_0, \dots, c_t)$, we have $\nu \subset \mu \iff c_i \le b_i$ (for all i) and

$$|\nu/\lambda| = \sum_{i=0}^{t} c_i, \qquad r(\nu/\lambda) = \sum_{i=0}^{t} \delta[c_i > 0], \qquad r(\lambda/\bar{\nu}) = \sum_{i=1}^{t} \delta[c_i < n_i], \tag{4.3.17}$$

where we use the notation $\delta[P] = 1$ if P is true and $\delta[P] = 0$ if P is false for a condition P.

Now we prove (4.3.4). For $\mu \in \mathrm{HS}_{\ge a}(\lambda)$ by (4.3.17) we have

$$(\text{LHS of (4.3.4)}) = \sum_{\substack{\nu \in \mathrm{HS}_{\ge a}(\lambda) \\ \nu \subset \mu}} (-1)^{|\nu/\lambda| - a} \binom{r(\nu/\lambda) - 1}{|\nu/\lambda| - a} \tag{4.3.18}$$

$$= \sum_{0 \le c_0 \le b_0} \sum_{0 \le c_1 \le b_1} \cdots \sum_{0 \le c_t \le b_t} \delta\left[\sum_{i=0}^{t} c_i \ge a\right] (-1)^{\sum_{i=0}^{t} c_i - a} \binom{\sum_{i=0}^{t} \delta[c_i > 0] - 1}{\sum_{i=0}^{t} c_i - a}. \tag{4.3.19}$$

Applying Lemma 4.3.5 below to simplify the summation on c_t, we have

$$= \sum_{0 \le c_0 \le b_0} \cdots \sum_{0 \le c_{t-1} \le b_{t-1}} \delta\left[b_t + \sum_{i=0}^{t-1} c_i \ge a\right] (-1)^{b_t + \sum_{i=0}^{t-1} c_i - a} \binom{\sum_{i=0}^{t-1} \delta[c_i > 0] - 1}{b_t + \sum_{i=0}^{t-1} c_i - a}. \tag{4.3.20}$$

Repeating this to simplify the summations on $c_0, \dots, c_{t-1}$, we have

$$= \dots \tag{4.3.21}$$

$$= \delta\left[\sum_{i=0}^{t} b_i \ge a\right] (-1)^{\sum_{i=0}^{t} b_i - a} \binom{-1}{\sum_{i=0}^{t} b_i - a} \tag{4.3.22}$$

$$= \delta\left[\sum_{i=0}^{t} b_i \ge a\right] = \delta[|\mu/\lambda| \ge a] = 1. \tag{4.3.23}$$

Hence (4.3.4) is proved.

Next we prove (4.3.5). By similar arguments we have

$$(\text{LHS of (4.3.5)}) = \sum_{b_0 \le c_0 \le n_0} \sum_{b_1 \le c_1 \le n_1} \cdots \sum_{b_t \le c_t \le n_t} \delta\left[\sum_{i=0}^{t} c_i \le a\right] (-1)^{a - \sum_{i=0}^{t} c_i} \binom{\sum_{i=1}^{t} \delta[c_i < n_i]}{a - \sum_{i=0}^{t} c_i}. \tag{4.3.24}$$

Note that this is actually a finite sum despite $n_0 = \infty$, and we can replace n_0 with a sufficiently large positive integer without changing the value of (4.3.24). Noticing $\delta[c_0 < n_0] = 1$ for any c_0 that contributes to the summation (4.3.24), and letting $b'_i = n_i - b_i$, $c'_i = n_i - c_i$ and $a' = (\sum_{i=0}^{t} n_i) - a$, we have

$$(4.3.24) = \sum_{0 \le c'_0 \le b'_0} \sum_{0 \le c'_1 \le b'_1} \cdots \sum_{0 \le c'_t \le b'_t} \delta\left[\sum_{i=0}^{t} c'_i \ge a'\right] (-1)^{\sum_{i=0}^{t} c'_i - a'} \binom{\sum_{i=0}^{t} \delta[c'_i > 0] - 1}{\sum_{i=0}^{t} c'_i - a'}. \tag{4.3.25}$$

Since this summation is of the same form as (4.3.19), by the same arguments we have

$$= \delta\left[\sum_{i=0}^{t} b_i' \geq a'\right] = \delta\left[\sum_{i=0}^{t} b_i \leq a\right] = \delta\left[|\mu/\lambda| \leq a\right] = 1. \tag{4.3.26}$$

Hence (4.3.5) is proved. □

Lemma 4.3.5. *For $R, q, b, b' \in \mathbb{Z}$ with $b' \leq b$, we have*

$$\sum_{b' \leq x \leq b} \delta[x \geq R]\,(-1)^{x-R}\binom{q+\delta[x > b']}{x-R} = \delta[b \geq R]\,(-1)^{b-R}\binom{q}{b-R},$$

where we use the notation $\delta[P] = 1$ if P is true and $\delta[P] = 0$ if P is false for a condition P.

Proof. We carry induction on $b - b'$. The lemma is clear when $b' = b$. When $b' < b$, it is easy to check

$$-\delta[b' \geq R]\binom{q}{b'-R} + \delta[b'+1 \geq R]\binom{q+1}{b'+1-R} = \delta[b'+1 \geq R]\binom{q}{b'+1-R}.$$

Hence we can replace b' with $b'+1$, completing the proof. □

4.4 Appendix: Möbius function of a poset

For basic definitions for posets we refer the reader to [Sta12, Chapter 3].

For a locally finite (i.e. every interval is finite) poset P, the *Möbius function* $\mu_P(x, y)$ (for $x, y \in P$ with $x \leq y$) is characterized by

$$\sum_{x \leq z \leq y} \mu_P(x, z) = \delta_{xy} \quad \text{for any } x \leq y,$$

or equivalently

$$\sum_{x \leq z \leq y} \mu_P(z, y) = \delta_{xy} \quad \text{for any } x \leq y. \tag{4.4.1}$$

Lemma 4.4.1. *Let $\widehat{P}$ be a locally finite poset with the maximum element $\hat{1}$. Let $P = \widehat{P} \backslash \{\hat{1}\}$ and $\{x_1, \cdots, x_n\}$ be the maximal elements in P, i.e. the coatoms in $\widehat{P}$. Consider formal variables $\{g(s) \mid s \in \widehat{P}\}$ and let $\widetilde{g}(t) = \sum_{s \leq t} g(s)$ for $t \in \widehat{P}$.*
(1) *We have*

$$\sum_{s \in P} g(s) = -\sum_{s \in P} \mu_{\widehat{P}}(s, \hat{1})\widetilde{g}(s). \tag{4.4.2}$$

(2) *Assume that P admits the meet operation $\wedge$. Then*

$$\sum_{s \in P} g(s) = \sum_{m \geq 1} (-1)^{m-1} \sum_{i_1 < \cdots < i_m} \widetilde{g}(x_{i_1} \wedge \cdots \wedge x_{i_m}) \tag{4.4.3}$$

$$\left(= \sum_{i} \widetilde{g}(x_i) - \sum_{i<j} \widetilde{g}(x_i \wedge x_j) + \sum_{i<j<k} \widetilde{g}(x_i \wedge x_j \wedge x_k) - \dots, \right) \tag{4.4.4}$$

(3) *In the same situation as (2), $\mu_{\widehat{P}}(s, \hat{1}) = 0$ unless s is of the form $s = x_{i_1} \wedge \cdots \wedge x_{i_l}$, and*

$$\mu_{\widehat{P}}(s, \hat{1}) = \mu_{\widehat{P}'}(s, \hat{1}) \tag{4.4.5}$$

for any subposet $\widehat{P}'$ of $\widehat{P}$ that contains all elements of the form $x_{i_1} \wedge \cdots \wedge x_{i_l}$ (including $\hat{1}$ as the meet of an empty set).

Proof. It is known (see [Sta12, Proposition 3.7.1] for example) that

$$g(t) = \sum_{s \leq t} \mu_{\widehat{P}}(s,t)\widetilde{g}(s) \quad (\text{for } \forall t \in \widehat{P}). \tag{4.4.6}$$

Hence we have

$$\sum_{s \in P} g(s) = \widetilde{g}(\hat{1}) - g(\hat{1}) = \widetilde{g}(\hat{1}) - \sum_{s \in \widehat{P}} \mu_{\widehat{P}}(s,\hat{1})\widetilde{g}(s) = -\sum_{s \in P} \mu_{\widehat{P}}(s,\hat{1})\widetilde{g}(s), \tag{4.4.7}$$

proving (1). (2) is by the Inclusion-Exclusion Principle. (3) follows from (1) and (2). □

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