

博士論文

論文題目 Studies on invariants of strictly
pseudoconvex CR manifolds
(強擬凸CR多様体の不変量に関する研究)

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Preface

A *CR manifold* is an odd-dimensional analog of a complex manifold. A typical example of a CR manifold is a real hypersurface in a complex manifold. In this thesis, we consider *strictly pseudoconvex* CR manifolds. In the seminal paper [Fef74], Fefferman has proved that two bounded strictly pseudoconvex domains in the complex Euclidean space are biholomorphic if and only if their boundaries, which are strictly pseudoconvex CR manifolds, are isomorphic as a CR manifold. Since then, there have been extensive researches on strictly pseudoconvex CR manifolds. The purpose of this thesis is to study geometric properties of both local and global invariants of strictly pseudoconvex CR manifolds.

In Chapter 1, we will consider Chern classes of closed strictly pseudoconvex CR manifolds. It is one of the most fundamental problems in CR geometry whether a given CR manifold can be realized as a real hypersurface in a complex manifold. There are many strictly pseudoconvex CR manifolds of dimension three with no such realizations; see [Ros65, Nir74, BE90b] for example. On the other hand, it is known that any closed connected strictly pseudoconvex CR manifold of dimension at least five can be realized as the boundary of a strictly pseudoconvex domain in a complex projective manifold [BdM75, HL75, Lem95]. This fact gives some restrictions to the topology of closed strictly pseudoconvex CR manifolds. For example, Bungart [Bun92] and Popescu-Pampu [PP08] have proved some vanishing results for the cup product on the cohomology with rational coefficients. By a similar method, we will obtain a constraint on Chern classes (Theorem 1.1.1). Through some examples, we will also show that our result is “sharp” in general (Propositions 1.3.1 and 1.3.2). This chapter is based on the paper [Tak18b].

In Chapter 2, we will deal with the existence of a pseudo-Einstein contact form. Recently, two global CR invariants have been introduced: the total Q -prime curvature and the Burns-Epstein invariant, which will be explained later. A *pseudo-Einstein contact form*, first introduced by Lee [Lee88], is a contact form satisfying a weak Einstein condition, and is necessary for defining those invariants. Therefore, the following problem arises when we consider the variation of these invariants: “is the existence of a pseudo-Einstein contact form preserved under deformations of CR structures?” We will solve this problem affirmatively for deformations as a real hypersurface in a fixed complex manifold (Corollary 2.1.2). This result follows from the fact that there exists a flat Hermitian metric on the canonical bundle of a sufficiently small tubular neighborhood of a strictly pseudoconvex real hypersurface if it admits a pseudo-Einstein contact form (Theorem 2.1.1). This chapter corresponds to the paper [Tak18c].

In Chapter 3, we will study relations between CR invariants constructed via the ambient space and Sasakian η -Einstein manifolds. Before stating our results, we recall the ambient space in even-dimensional conformal geometry, which is closely related to CR geometry.

Let N be a smooth manifold of dimension $2m \geq 2$, and C be a conformal class of pseudo-Riemannian metrics of signature (p, q) with $p + q = 2m$. The metric

bundle \mathcal{G} is the principal \mathbb{R}_+ -bundle over N whose fiber \mathcal{G}_x at $x \in N$ is given by

$$\mathcal{G}_x = \{ g_x \in S^2 T_x^* N \mid g \in C \}.$$

This \mathcal{G} has the tautological symmetric two-tensor g_0 . The *ambient space* is the space $\tilde{\mathcal{G}} = \mathcal{G} \times (-\epsilon, \epsilon)$ with a homogeneous pseudo-Riemannian metric \tilde{g} of signature $(p+1, q+1)$ such that $\tilde{g}|_{\mathcal{G} \times \{0\}} = g_0$ and \tilde{g} is asymptotically Ricci-flat on $\mathcal{G} \times \{0\}$. This space was first introduced by Fefferman and Graham [FG85]; see also the book [FG12] for details. By using the Laplacian of \tilde{g} , we can obtain some conformal invariants of (N, C) . For each integer $1 \leq k \leq m$, Graham, Jenne, Mason, and Sparling [GJMS92] have defined a conformally invariant differential operator P_k whose principal part is the k -th power of the Laplacian, called the *k -th GJMS operator*. It is known that P_1 agrees with the conformal Laplacian; in other words, the GJMS operators are higher order analogs of the conformal Laplacian. The *Q -curvature* is a smooth function on N determined for each representative of C , first introduced by Branson [Bra95]; see also [FG02, FH03]. This is not a conformal invariant, but its integral, the *total Q -curvature*, defines a global conformal invariant of (N, C) if N is closed. In the case of $m = 1$, the Q -curvature coincides with the Gauss curvature; that is, the Q -curvature is a generalization of the Gauss curvature in higher dimensional conformal geometry.

Assume that C contains an Einstein metric g . Gover [Gov06], and Fefferman and Graham [FG12] have proved that the k -th GJMS operator is decomposed into k factors, and each factor is the sum of the Laplacian and a constant determined only by m and the Einstein constant. They also have shown that the Q -curvature with respect to g is a constant depending only on m and the Einstein constant. Moreover, the variation of the total Q -curvature under deformations of conformal structures is well-understood: the first variation of the total Q -curvature at C is always zero [GH05], and the second variation is written in terms of the Lichnerowicz Laplacian [Mat13]; see also [GMS16].

Now we return the CR case. Let M be a strictly pseudoconvex CR manifold of dimension $2n+1$. Then we can construct a principal S^1 -bundle N over M with a conformal class C of Lorentzian metrics on N ; these are determined only by the CR structure of M [BDS77, Lee86]. The space $(\tilde{\mathcal{G}}, \tilde{g})$ with respect to (N, C) is called the *ambient space* of M . In this thesis, however, we take a complex-geometric approach to the ambient space following [HMM17]; see Section 0.3. Similar to the conformal case, the ambient space gives some CR invariants. For $(w, w') \in \mathbb{R}^2$ with $k = w + w' + n + 1 \in \{1, \dots, n+1\}$, Gover and Graham [GG05] have defined a *CR invariant powers of the sub-Laplacian* $P_{w, w'}$, a CR counterpart of the GJMS operators. A CR analog of the Q -curvature is the *Q -prime curvature*, introduced by Case and Yang [CY13], and Hirachi [Hir14]; it is a smooth function on M determined for each pseudo-Einstein contact form. Marugame [Mar18] has proved that its integral, the *total Q -prime curvature*, defines a global CR invariant for a closed M .

The results in Chapter 3 are CR counterparts of those in the paragraph before the previous one. A *Sasakian η -Einstein manifold* is a strictly pseudoconvex CR manifold $(S, T^{1,0}S)$ of dimension $2n+1$ with a contact form η satisfying a strong Einstein condition; see Definition 0.6.3. For such $(S, T^{1,0}S)$ and η , we will prove that $P_{w, w'}$ is factored into k components, and each component is written in terms of the sub-Laplacian, the Reeb vector field, n and the Einstein constant (Theorem 3.1.1). We will also show that the Q -prime curvature with respect to η is a constant determined only by n and the Einstein constant (Theorem 3.1.3). Note that the same results have been obtained independently by Case and Gover [CG17] in a different way. We will also consider the variation of the total Q -prime curvature at S under deformations as a real hypersurface. It will be proved that the first variation

of the total Q -prime curvature at S must be zero (Proposition 3.1.4). By applying some spectral results, we will also show that, if $n = 1$ or the Einstein constant is non-negative, then the second variation is non-positive, and equal to zero if and only if the deformation is “infinitesimally trivial” (Theorem 3.1.5). Through some examples, it will be seen that these results do not hold if $n \geq 2$ and the Einstein constant is negative (Theorem 3.1.6). This chapter is based on the paper [Tak18a]

In Chapter 4, we will compute another global CR invariant, the Burns-Epstein invariant, for the tubes associated with polarized Kähler-Einstein manifolds, which are typical examples of Sasakian η -Einstein manifolds. Burns and Epstein [BE90a] have introduced a global CR invariant for the boundaries of bounded strictly pseudoconvex domains in \mathbb{C}^{n+1} as the boundary term of the renormalized Gauss-Bonnet-Chern formula. Marugame [Mar16] has simplified their construction and generalized it to a global CR invariant for closed strictly pseudoconvex CR manifolds admitting a pseudo-Einstein contact form, which we call the *Burns-Epstein invariant* in this thesis. We will give a formula of this invariant for the tubes associated with polarized Kähler-Einstein manifolds in terms of characteristic numbers (Theorem 4.1.1). As an application, we will show that there is no proportional relationship between the Burns-Epstein invariant and the total Q -prime curvature in dimension at least five (Theorem 4.1.2). Note that, in dimension three, the Burns-Epstein invariant coincides with the total Q -prime curvature up to a universal constant.

Notation. Throughout this thesis, we assume that n is a positive integer. We write \mathbb{N}_+ (resp. \mathbb{R}_+) for the set of positive integers (resp. positive real numbers). We use Einstein’s summation convention and assume that

- uppercase Latin indices A, B, C, \dots run from 0 to $n + 1$;
- lowercase Latin indices a, b, c, \dots run from 1 to $n + 1$;
- lowercase Greek indices $\alpha, \beta, \gamma, \dots$ run from 1 to n .

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Basic materials

0.1. Strictly pseudoconvex CR manifolds

In this section, we recall some fundamental results on strictly pseudoconvex CR manifolds and the Tanaka-Webster connection.

Let M be an oriented smooth $(2n+1)$ -dimensional manifold without boundary. A *CR structure* is a complex n -dimensional subbundle $T^{1,0}M$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ such that

$$T^{1,0}M \cap T^{0,1}M = 0, \quad [\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M),$$

where $T^{0,1}M$ is the complex conjugate of $T^{1,0}M$ in $TM \otimes \mathbb{C}$. Set $HM = \operatorname{Re} T^{1,0}M$ and let $J: HM \rightarrow HM$ be a unique complex structure on HM such that

$$T^{1,0}M = \ker(J - \sqrt{-1}: HM \otimes \mathbb{C} \rightarrow HM \otimes \mathbb{C});$$

in particular, HM has a canonical orientation. Let HM° be the annihilator of HM ; that is,

$$HM^\circ = \{ \theta_p \in T_p^*M \mid \theta_p|_{H_pM} = 0 \}.$$

Since TM is oriented and HM is an oriented subbundle of TM of codimension one, HM° is an oriented real line bundle. An element $\theta_p \in H_pM^\circ \setminus \{0\}$ is said to be *positive* if the orientation of H_pM° determined by θ is compatible with the given orientation of H_pM° . We denote by HM_+° the set of positive elements in HM° , which is a trivial principal \mathbb{R}_+ -bundle over M . Let θ be a smooth section of HM_+° . The *Levi form* \mathcal{L}_θ with respect to θ is the Hermitian form on $T^{1,0}M$ defined by

$$\mathcal{L}_\theta(Z, W) = -\sqrt{-1}d\theta(Z, \bar{W}), \quad Z, W \in T^{1,0}M.$$

A CR structure $T^{1,0}M$ is said to be *pseudoconvex* (resp. *strictly pseudoconvex*) if \mathcal{L}_θ is non-negative (resp. positive definite) for some θ . This condition is independent of the choice of θ and determined only by $T^{1,0}M$ and the orientation of M . If $T^{1,0}M$ is strictly pseudoconvex, we call a section of HM_+° a *contact form*.

In the remainder of this section, we assume that the CR structure $T^{1,0}M$ is strictly pseudoconvex. Denote by T the *Reeb vector field* with respect to θ ; that is, a unique vector field satisfying

$$\theta(T) = 1, \quad \iota_T d\theta = 0.$$

Then the tangent bundle TM has the decomposition $TM = \mathbb{R}T \oplus HM$. One can define a Riemannian metric g_θ on M by

$$g_\theta(X, Y) = \frac{1}{2}d\theta(X, JY) + \theta(X)\theta(Y), \quad X, Y \in TM.$$

Here we extend J to an endomorphism on TM by $JT = 0$. Let (Z_α) be a local frame of $T^{1,0}M$, and set $Z_{\bar{\alpha}} = \bar{Z}_\alpha$. Then $(T, Z_\alpha, Z_{\bar{\alpha}})$ gives a local frame of $TM \otimes \mathbb{C}$, called an *admissible frame*. Its dual frame $(\theta, \theta^\alpha, \theta^{\bar{\alpha}})$ is called an *admissible coframe*. The two-form $d\theta$ is written as

$$d\theta = \sqrt{-1}l_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}},$$

where $(l_{\alpha\bar{\beta}})$ is a positive definite Hermitian matrix. We use $l_{\alpha\bar{\beta}}$ and its inverse $l^{\alpha\bar{\beta}}$ to raise and lower indices.

A contact form θ induces a canonical connection ∇ , called the *Tanaka-Webster connection* with respect to θ . It is defined by

$$\nabla T = 0, \quad \nabla Z_\alpha = \omega_\alpha^\beta Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}^{\bar{\beta}} Z_{\bar{\beta}} \quad (\omega_{\bar{\alpha}}^{\bar{\beta}} = \overline{\omega_\alpha^\beta})$$

with the following structure equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \omega_\alpha^\beta + A_{\bar{\alpha}}^\beta \theta \wedge \theta^{\bar{\alpha}}, \\ dl_{\alpha\bar{\beta}} &= \omega_\alpha^\gamma l_{\gamma\bar{\beta}} + l_{\alpha\bar{\gamma}} \omega_{\bar{\beta}}^{\bar{\gamma}}. \end{aligned}$$

The tensor $A_{\alpha\bar{\beta}} = \overline{A_{\bar{\alpha}\beta}}$ is shown to be symmetric and is called the *Tanaka-Webster torsion*. We denote the components of a successive covariant derivative of a tensor by subscripts preceded by a comma, for example, $K_{\alpha\bar{\beta},\gamma}$; we omit the comma if the derivatives are applied to a function. With this notation, introduce an operator $\bar{\partial}_b: C^\infty(M) \rightarrow \Gamma((T^{0,1}M)^*)$ by

$$\bar{\partial}_b f = f_{\bar{\alpha}} \theta^{\bar{\alpha}}.$$

A smooth function f is called a *CR holomorphic function* if $\bar{\partial}_b f = 0$. A *CR pluriharmonic function* is a real-valued smooth function that is locally the real part of a CR holomorphic function. We denote by \mathcal{P} the space of CR pluriharmonic functions.

The curvature form $\Omega_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$ of the Tanaka-Webster connection is of the form

$$(0.1.1) \quad \begin{aligned} \Omega_\alpha^\beta &= R_{\alpha\bar{\gamma}\delta}^\beta \theta^\gamma \wedge \theta^{\bar{\delta}} + A_{\alpha\gamma}^\beta \theta^\gamma \wedge \theta - A_{\bar{\gamma}\alpha}^\beta \theta^{\bar{\gamma}} \wedge \theta \\ &\quad - \sqrt{-1} A_{\alpha\gamma} \theta^\gamma \wedge \theta^\beta + \sqrt{-1} l_{\alpha\bar{\gamma}} A_{\bar{\delta}}^\beta \theta^{\bar{\gamma}} \wedge \theta^{\bar{\delta}}. \end{aligned}$$

We call the tensor $R_{\alpha\bar{\gamma}\delta}^\beta$ the *Tanaka-Webster curvature*. This tensor has the symmetry

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}} = R_{\alpha\bar{\delta}\gamma\bar{\beta}}.$$

Contracting the Tanaka-Webster curvature, we obtain the *Tanaka-Webster Ricci curvature* $\text{Ric}_{\gamma\bar{\delta}} = R_{\alpha\bar{\gamma}\delta}^\alpha$ and the *Tanaka-Webster scalar curvature* $\text{Scal} = \text{Ric}_{\gamma\bar{\gamma}}$.

Now consider commutators of covariant derivatives. We will use the index 0 for the component T or θ in our index notation. The commutators of the second derivatives for $f \in C^\infty(M)$ are given by

$$2f_{[\alpha\beta]} = 0, \quad 2f_{[\alpha\bar{\beta}]} = \sqrt{-1} l_{\alpha\bar{\beta}} f_0, \quad 2f_{[0\alpha]} = A_{\alpha\beta} f^\beta,$$

where $[\dots]$ means the anti-symmetrization over the enclosed indices. Define the *Kohn Laplacian* \square_b and the *sub-Laplacian* Δ_b by

$$\square_b f = -f_{\bar{\alpha}}^{\bar{\alpha}}, \quad \Delta_b f = -f_{\bar{\alpha}}^{\bar{\alpha}} - f_\alpha^\alpha = \square_b f + \bar{\square}_b f,$$

respectively. From the above commutation relations, we have

$$\square_b - \bar{\square}_b = \sqrt{-1} n T.$$

In particular,

$$(0.1.2) \quad 2\square_b = \Delta_b + \sqrt{-1} n T, \quad 2\bar{\square}_b = \Delta_b - \sqrt{-1} n T.$$

The second derivatives of a $(1, 0)$ -form $K = K_\alpha \theta^\alpha$ satisfy the following commutation relations:

$$\begin{aligned} 2K_{\alpha, [\beta\gamma]} &= \sqrt{-1}A_{\alpha\gamma}K_\beta - \sqrt{-1}A_{\alpha\beta}K_\gamma, \\ 2K_{\alpha, [\bar{\beta}\bar{\gamma}]} &= \sqrt{-1}l_{\alpha\bar{\beta}}A_{\bar{\gamma}}^\delta K_\delta - \sqrt{-1}l_{\alpha\bar{\gamma}}A_{\bar{\beta}}^\delta K_\delta, \\ 2K_{\alpha, [\beta\bar{\gamma}]} &= \sqrt{-1}l_{\beta\bar{\gamma}}K_{\alpha, 0} + R_\alpha^\delta{}_{\beta\bar{\gamma}}K_\delta, \\ 2K_{\alpha, [0\beta]} &= A_{\beta\gamma}K_{\alpha, \gamma} - A_{\alpha\beta, \gamma}K_\gamma, \\ 2K_{\alpha, [0\bar{\beta}]} &= A_{\bar{\beta}}^\gamma K_{\alpha, \gamma} + A_{\bar{\beta}, \alpha}^\gamma K_\gamma. \end{aligned}$$

For proofs of these facts, see [Lee88, Lemma 2.3].

A contact form θ is said to be *pseudo-Einstein* if the following two equalities hold:

$$\text{Ric}_{\alpha\bar{\beta}} = \frac{1}{n}\text{Scal}_{\alpha\bar{\beta}}, \quad \text{Scal}_\alpha = \sqrt{-1}nA_{\alpha\beta, \beta}.$$

From Bianchi identities for the Tanaka-Webster connection, we obtain

$$\left(\text{Ric}_{\alpha\bar{\beta}} - \frac{1}{n}\text{Scal}_{\alpha\bar{\beta}} \right)^{\bar{\beta}} = \frac{n-1}{n}(\text{Scal}_\alpha - \sqrt{-1}nA_{\alpha\beta, \beta});$$

see [Hir14, Lemma 5.7(iii)] for example. Therefore, the latter equality follows from the former one if $n \geq 2$; on the other hand, the former equality automatically holds if $n = 1$, and the latter one is a non-trivial condition. Another contact form $\hat{\theta} = e^\Upsilon \theta$ is pseudo-Einstein if and only if $\Upsilon \in \mathcal{P}$.

Now we introduce the Burns-Epstein invariant, a global CR invariant for strictly pseudoconvex CR manifolds admitting a pseudo-Einstein contact form. It is defined as the boundary term of the renormalized Gauss-Bonnet-Chern formula [Mar16, Theorem 1], but we do not use this definition but an explicit expression in terms of the Tanaka-Webster curvature and torsion. Assume that M is closed and admits a pseudo-Einstein contact form θ . To simplify the notation, set $\varsigma = \text{Scal}/n(n+1)$. Let

$$\begin{aligned} \theta_0^\beta &= \varsigma\theta^\beta - \sqrt{-1}A_{\bar{\gamma}}^\beta\theta^{\bar{\gamma}} + \sqrt{-1}\varsigma^\beta\theta, \\ \theta_\alpha^0 &= -l_{\alpha\bar{\gamma}}\theta^{\bar{\gamma}}, \\ \theta_0^0 &= \sqrt{-1}\varsigma\theta, \\ \Theta_\alpha^\beta &= \Omega_\alpha^\beta + \sqrt{-1}\varsigma d\theta\delta_\alpha^\beta - \varsigma l_{\alpha\bar{\gamma}}\theta^\beta \wedge \theta^{\bar{\gamma}} \\ &\quad + \sqrt{-1}\varsigma_\gamma\delta_\alpha^\beta\theta^\gamma \wedge \theta + \sqrt{-1}\varsigma_{\bar{\gamma}}\delta_\alpha^\beta\theta^{\bar{\gamma}} \wedge \theta + \sqrt{-1}\varsigma^\beta l_{\alpha\bar{\gamma}}\theta^{\bar{\gamma}} \wedge \theta \\ &\quad - \sqrt{-1}A_{\alpha\gamma}\theta^\gamma \wedge \theta^\beta + A_{\alpha\gamma, \beta}\theta^\gamma \wedge \theta - A_{\bar{\gamma}, \alpha}^\beta\theta^{\bar{\gamma}} \wedge \theta, \\ \Theta_\alpha^0 &= A_{\alpha\gamma}\theta^\gamma \wedge \theta. \end{aligned}$$

The *Burns-Epstein invariant* $\mu(M)$ is defined by

$$\mu(M) = \frac{1}{n!} \left(\frac{\sqrt{-1}}{2\pi} \right)^{n+1} \sum_{k=0}^n \binom{n}{k} \int_M (\Phi_k^{(0)} - \Phi_k^{(1)}),$$

where

$$\begin{aligned} \Phi_0^{(0)} &= \sum_{\sigma, \tau \in S_n} \text{sgn}(\sigma\tau) \theta_0^0 \wedge \theta_{\sigma(1)}^0 \wedge \theta_0^{\tau(1)} \wedge \cdots \wedge \theta_{\sigma(n)}^0 \wedge \theta_0^{\tau(n)}, \\ \Phi_k^{(0)} &= \sum_{\sigma, \tau \in S_n} \text{sgn}(\sigma\tau) \theta_0^0 \wedge \Theta_{\sigma(1)}^{\tau(1)} \wedge \cdots \wedge \Theta_{\sigma(k)}^{\tau(k)} \\ &\quad \wedge \theta_{\sigma(k+1)}^0 \wedge \theta_0^{\tau(k+1)} \wedge \cdots \wedge \theta_{\sigma(n)}^0 \wedge \theta_0^{\tau(n)} \quad (1 \leq k \leq n), \end{aligned}$$

$$\begin{aligned}\Phi_0^{(1)} &= \sum_{\sigma, \tau \in S_n} \operatorname{sgn}(\sigma\tau) \Theta_{\sigma(1)}^0 \wedge \theta_0^{\tau(1)} \wedge \Theta_{\sigma(2)}^0 \wedge \theta_0^{\tau(2)} \wedge \cdots \wedge \Theta_{\sigma(n)}^0 \wedge \theta_0^{\tau(n)}, \\ \Phi_k^{(1)} &= \sum_{\sigma, \tau \in S_n} \operatorname{sgn}(\sigma\tau) \Theta_{\sigma(1)}^0 \wedge \theta_0^{\tau(1)} \wedge \Theta_{\sigma(2)}^{\tau(2)} \wedge \cdots \wedge \Theta_{\sigma(k+1)}^{\tau(k+1)} \\ &\quad \wedge \Theta_{\sigma(k+2)}^0 \wedge \theta_0^{\tau(k+2)} \cdots \wedge \Theta_{\sigma(n)}^0 \wedge \theta_0^{\tau(n)} \quad (1 \leq k \leq n-1), \\ \Phi_n^{(1)} &= 0.\end{aligned}$$

This $\mu(M)$ is independent of the choice of a pseudo-Einstein contact form, and gives a global CR invariant of M [Mar16, Theorem 4.6].

0.2. Strictly pseudoconvex real hypersurfaces and domains

This section gives some basic facts on strictly pseudoconvex real hypersurfaces and domains.

Let M be an oriented real hypersurface in an $(n+1)$ -dimensional complex manifold X . Then M has the natural CR structure

$$T^{1,0}M = T^{1,0}X \cap (TM \otimes \mathbb{C}).$$

If $T^{1,0}M$ is pseudoconvex (resp. strictly pseudoconvex), we call M a *pseudoconvex* (resp. *strictly pseudoconvex*) *real hypersurface*. For any $\theta \in \Gamma(HM_+^0)$, there exists a smooth function ρ on a neighborhood of M such that $M = \rho^{-1}(0)$, $d\rho \neq 0$ on M , and $\theta = d^c\rho|_M$, where $d^c = (\sqrt{-1}/2)(\bar{\partial} - \partial)$; such a ρ is called a *defining function of M normalized by θ* . If $\hat{\rho}$ is another defining function, then there exists a smooth function Υ on a neighborhood of M such that $\hat{\rho} = e^\Upsilon\rho$. Take a sufficiently small tubular neighborhood U of M and a defining function ρ of M . Then $U \setminus M$ is divided into two parts: $U_- = U \cap \rho^{-1}((-\infty, 0))$ and $U_+ = U \cap \rho^{-1}((0, \infty))$. Moreover, U_- and U_+ are independent of the choice of a defining function ρ of M . If M is a pseudoconvex real hypersurface, we call U_- (resp. U_+) the *pseudoconvex side* (resp. *pseudoconcave side*) of U . It is known that any CR holomorphic function (resp. CR pluriharmonic function) on a strictly pseudoconvex real hypersurface can be extended to a holomorphic function (resp. pluriharmonic function) on the pseudoconvex side of a sufficiently small tubular neighborhood.

We next consider strictly pseudoconvex domains. Let Ω be a domain in X with smooth boundary $M = \partial\Omega$. (Throughout this thesis, we always assume a domain to be connected and relatively compact.) Then there exists a smooth function ρ on X such that

$$\Omega = \rho^{-1}((-\infty, 0)), \quad M = \rho^{-1}(0), \quad d\rho \neq 0 \text{ on } M;$$

such a ρ is called a *defining function* of Ω . A domain Ω is said to be *strictly pseudoconvex* if we can take a defining function ρ of Ω that is strictly plurisubharmonic near M . We call Ω a *Stein domain* if Ω admits a defining function ρ that is strictly plurisubharmonic on a neighborhood of the closure of Ω . Note that a Stein domain is a Stein manifold; this is because $-\rho^{-1}$ defines a strictly plurisubharmonic exhaustion function on Ω . It is known that any strictly pseudoconvex domain Ω is holomorphically convex, and consequently, there exist a Stein space Z and a proper surjective holomorphic map $\varphi: \Omega \rightarrow Z$ having some good properties, called the *Riemann reduction* of Ω ; see [GPR94, Chapter V] and references therein for details. In our setting, φ is described as follows. A compact analytic subset E of positive dimension at every point in Ω is called a *maximal compact analytic subset* of Ω if it is maximal among such subsets with respect to the inclusion relation; this E is determined uniquely by Ω . The map φ contracts each connected component of E to a point, and induces a biholomorphism $\Omega \setminus E \rightarrow Z \setminus \varphi(E)$. In particular, Z has at most finite normal isolated singularities.

The boundary of a strictly pseudoconvex domain is a closed connected strictly pseudoconvex real hypersurface [KR65, Corollary 7.3]. Conversely, it is known that any closed connected strictly pseudoconvex CR manifold of dimension at least five can be realized as the boundary of a strictly pseudoconvex domain in a complex projective manifold [BdM75, HL75, Lem95].

0.3. The ambient space

In this section, we give a brief introduction to the ambient space, which is a powerful tool for defining CR invariants.

Let X be an $(n+1)$ -dimensional complex manifold and $\pi_{\mathcal{X}}: \mathcal{X} = K_X^\times \rightarrow X$ be the total space of the canonical bundle of X with the zero section removed. For $\mu \in \mathbb{C}^\times$, define the dilation $\delta_\mu: \mathcal{X} \rightarrow \mathcal{X}$ by the scalar multiplication by μ^{n+2} . Denote by Z_0 the holomorphic vector field generating δ_μ ; that is, $Z_0 = (d/d\mu)|_{\mu=1} \delta_\mu^*$. A smooth function \mathbf{f} on an open set of \mathcal{X} is said to be *homogeneous of degree* $(w, w') \in \mathbb{R}^2$ if $Z_0 \mathbf{f} = w \mathbf{f}$ and $\bar{Z}_0 \mathbf{f} = w' \mathbf{f}$ hold. We write $\tilde{\mathcal{E}}(w, w')$ for the sheaf of smooth homogeneous functions of degree (w, w') , and sometimes abbreviate $\tilde{\mathcal{E}}(w, w)$ to $\tilde{\mathcal{E}}(w)$. By abuse of notation, we will write $\mathbf{f} \in \tilde{\mathcal{E}}(w, w')$. Note that a Hermitian metric of K_X can be identified with a positive function in $\tilde{\mathcal{E}}(n+2)$.

Let $z = (z^1, \dots, z^{n+1})$ be a local coordinate of X . Then the fiber coordinate ζ of \mathcal{X} is defined by $\zeta dz^1 \wedge \dots \wedge dz^{n+1}$. Choose a branch $z^0 = \zeta^{1/(n+2)}$, which is called a *branched fiber coordinate* in this paper. The holomorphic vector field Z_0 is equal to $z^0 \partial / \partial z^0$, and for each homogeneous function \mathbf{f} of degree (w, w') , there exists a smooth function f locally on X such that $\mathbf{f} = (z^0)^w (\bar{z}^0)^{w'} f$.

Let M be a strictly pseudoconvex real hypersurface in X and set $\mathcal{M} = \pi_{\mathcal{X}}^{-1}(M)$. We give an orientation on \mathcal{M} so that $\pi_{\mathcal{X}}^* \rho$ is a defining function of \mathcal{M} for any defining function ρ of M . Then \mathcal{M} is a \mathbb{C}^\times -invariant pseudoconvex real hypersurface in \mathcal{X} . Denote by $\mathcal{E}(w, w')$ the sheaf of homogeneous functions of degree (w, w') on \mathcal{M} .

For a defining function $\rho \in \tilde{\mathcal{E}}(1)$ of \mathcal{M} , the $(1, 1)$ -form $dd^c \rho$ defines a Lorentz-Kähler metric on a neighborhood of \mathcal{M} , denoted by $\mathbf{g}[\rho]$. We normalize ρ by using a complex Monge-Ampère equation. Take the tautological $(n+1, 0)$ -form ζ on K_X . Then

$$\text{vol}_{\mathcal{X}} = (\sqrt{-1})^{(n+2)^2} d\zeta \wedge \bar{d}\bar{\zeta}$$

gives a volume form on \mathcal{X} .

Proposition 0.3.1 ([HMM17, Proposition 2.2]). *There exists a defining function $\rho \in \tilde{\mathcal{E}}(1)$ of \mathcal{M} such that*

$$(dd^c \rho)^{n+2} = k_n (1 + \mathcal{O}\rho^{n+2}) \text{vol}_{\mathcal{X}},$$

where $\mathcal{O} \in \tilde{\mathcal{E}}(-n-2)$ and $k_n = -(n+1)!/(n+2)$. Moreover, such a ρ is unique modulo $O(\rho^{n+3})$, and $\mathcal{O}|_{\mathcal{M}}$ is independent of the choice of ρ .

We call such a ρ a *Fefferman defining function* and the Lorentz-Kähler metric $\mathbf{g}[\rho]$ with respect to ρ an *ambient metric*. The function \mathcal{O} is called the *obstruction function*. Note that the Ricci form of $\mathbf{g}[\rho]$ is given by $-dd^c \log(1 + \mathcal{O}\rho^{n+2})$.

A Fefferman defining function gives a necessary and sufficient condition for a contact form to be pseudo-Einstein in terms of a Hermitian metric on K_X . For a Hermitian metric \mathbf{h} of K_X , the function $\rho \cdot \mathbf{h}^{-1/(n+2)} \in \tilde{\mathcal{E}}(0)$ gives a defining function of M . Conversely, let ρ be a defining function of M . Then $\mathbf{h}_\rho = (\rho/\rho)^{n+2} \in \tilde{\mathcal{E}}(n+2)$ defines a Hermitian metric of K_X near M .

Proposition 0.3.2 ([HMM17, Proposition 2.6]). *A contact form θ on M is pseudo-Einstein if and only if there exists a defining function ρ of M normalized by θ such that \mathbf{h}_ρ is flat on the pseudoconvex side.*

In particular, any strictly pseudoconvex real hypersurface in a complex manifold X admits a pseudo-Einstein contact form if K_X admits a flat Hermitian metric.

Before the end of this section, we note some identifications for $\mathcal{E}(w)$. For a defining function ρ of M , the function $\mathbf{h}_\rho|_{\mathcal{M}} \in \mathcal{E}(n+2)$ depends only on the contact form θ normalizing ρ , denoted by \mathbf{h}_θ . The multiplication by $\mathbf{h}_\theta^{w/(n+2)}$ defines a linear isomorphism $C^\infty(M) \rightarrow \mathcal{E}(w)$. There exists also a canonical identification between $\mathcal{E}(-n-1)$ and the space $A^{2n+1}(M)$ of smooth $(2n+1)$ -forms on M . For $\varphi \in \mathcal{E}(-n-1)$, the $(2n+1)$ -form $\varphi d^c \rho \wedge (dd^c \rho)^n|_{\mathcal{M}}$ descends to a smooth $(2n+1)$ -form on M . This induces an isomorphism $\mathcal{E}(-n-1) \rightarrow A^{2n+1}(M)$. We denote by $\int_M \varphi$ the integral of the $(2n+1)$ -form corresponding to $\varphi \in \mathcal{E}(-n-1)$ for a closed M . Note that the composition

$$C^\infty(M) \rightarrow \mathcal{E}(-n-1) \rightarrow A^{2n+1}(M)$$

is given by $\varphi \mapsto \varphi \theta \wedge (d\theta)^n$.

0.4. Deformations of CR structures

In this section, we treat deformations of real hypersurfaces in a fixed complex manifold and corresponding deformations of CR structures; we follow the argument in [HMM17, Section 4].

Let M be a closed strictly pseudoconvex real hypersurface in an $(n+1)$ -dimensional complex manifold X , and $(M_t)_{t \in (-1,1)}$ be a smooth family of closed strictly pseudoconvex real hypersurfaces in X with $M_0 = M$. Take a Fefferman defining function ρ_t of $M_t = \pi_X^{-1}(M_t)$ that is smooth in t . Then $(d/dt)|_{t=0} \rho_t|_{\mathcal{M}} \in \mathcal{E}(1)$ is independent of the choice of ρ_t . Conversely, for any real-valued function $\varphi \in \mathcal{E}(1)$, there exists a smooth family $(M_t)_{t \in (-1,1)}$ such that $\varphi = (d/dt)|_{t=0} \rho_t|_{\mathcal{M}}$. Thus the space of infinitesimal deformations of M as a real hypersurface in X is naturally parametrized by $\mathcal{E}(1)_{\mathbb{R}}$, the space of real-valued functions in $\mathcal{E}(1)$. On the other hand, the space of infinitesimal deformations of CR structures on M , denoted by $\mathcal{D}(M, T^{1,0}M)$, is a linear subspace of $\Gamma(\text{Hom}(T^{0,1}M, T^{1,0}M))$. Each infinitesimal deformation as a real hypersurface induces that of a CR structure on M . This correspondence is represented by a differential operator

$$D: \mathcal{E}(1) \rightarrow \mathcal{D}(M, T^{1,0}M),$$

first introduced by Buchweitz and Millson [BM97, Chapter 8]; see also [AGL02, Section 4]. Remark that this operator also appears in a subcomplex of the BGG sequence [ČSS01]. If $\mathbf{F} \in \mathcal{E}(1)$ is pure imaginary, $D\mathbf{F}$ corresponds to the infinitesimal deformation induced from an infinitesimal contact diffeomorphism on M , which has been studied by Cheng and Lee [CL90, Lemma 3.4]. From this observation, we define a “trivial deformation” from CR point of view.

Definition 0.4.1. Let $(M_t)_{t \in (-1,1)}$ be a smooth family of closed strictly pseudoconvex real hypersurfaces in X with $M_0 = M$. It is said to be *infinitesimally trivial as a deformation of CR structures* if

$$\left. \frac{d}{dt} \right|_{t=0} \rho_t|_{\mathcal{M}} \in \text{Re ker } D.$$

As stated in the last paragraph in Section 0.3, a contact form θ on M gives an identification between $\mathcal{E}(1)$ and $C^\infty(M)$. Thus we obtain a differential operator

$$D_\theta: C^\infty(M) \rightarrow \mathcal{D}(M, T^{1,0}M),$$

written in terms of the Tanaka-Webster connection as follows:

$$2(D_\theta F)_\alpha^\beta = F_\alpha^\beta - \sqrt{-1}A_\alpha^\beta F.$$

0.5. Ambient constructions

In this section, we recall constructions of CR invariant powers of the sub-Laplacian, the P -prime operator, and the Q -prime curvature via the ambient space.

We use the notation in Sections 0.3 and 0.4. Fix a Fefferman defining function ρ and consider the corresponding Lorentz-Kähler metric $\mathbf{g} = \mathbf{g}[\rho]$. Let ∇ be the Levi-Civita connection of \mathbf{g} , and Δ be the (pseudo-Riemannian) Laplacian with respect to \mathbf{g} ; that is,

$$\Delta = -2\nabla^A \nabla_A.$$

This operator maps $\tilde{\mathcal{E}}(w, w')$ to $\tilde{\mathcal{E}}(w-1, w'-1)$.

Lemma 0.5.1 ([GG05, Theorem 1.1]). *Let $(w, w') \in \mathbb{R}^2$ such that $k = w + w' + n + 1$ is a positive integer. Then, for any $\tilde{\mathbf{f}} \in \tilde{\mathcal{E}}(w, w')$,*

$$(\Delta^k \tilde{\mathbf{f}})|_{\mathcal{M}} \in \mathcal{E}(w-k, w'-k)$$

depends only on $\mathbf{f} = \tilde{\mathbf{f}}|_{\mathcal{M}}$ and defines a differential operator

$$\mathbf{P}_{w,w'}: \mathcal{E}(w, w') \rightarrow \mathcal{E}(w-k, w'-k).$$

Moreover, the operator $\mathbf{P}_{w,w'}$ is independent of the choice of a Fefferman defining function if $k \leq n+1$.

If $k \leq n+1$, we call $\mathbf{P}_{w,w'}$ a CR invariant power of the sub-Laplacian. Note that $\mathbf{P}_{0,0}$ annihilates the constant functions, and satisfies

$$\int_M f_1 \cdot \overline{\mathbf{P}_{0,0} f_2} = \int_M \overline{f_2} \cdot \mathbf{P}_{0,0} f_1$$

for $f_1, f_2 \in C^\infty(M)$ if M is closed; see [GG05, Proposition 5.1]. In particular,

$$\int_M \mathbf{P}_{0,0} f = \int_M f \cdot \overline{\mathbf{P}_{0,0} 1} = 0$$

for any $f \in C^\infty(M)$.

When $w = w' = 1$, the operator $\mathbf{P}_{1,1}$ depends on the choice of a Fefferman defining function. However, a slight modification gives a new CR invariant differential operator, which is closely related to the variation of the total Q -prime curvature.

Lemma 0.5.2 ([HMM17, Lemma A.2]). *For $\tilde{\mathbf{f}} \in \tilde{\mathcal{E}}(1)$,*

$$[4\operatorname{Re}(\Delta^{n+1} \nabla^A \nabla^B \nabla_A \nabla_B \tilde{\mathbf{f}})]|_{\mathcal{M}} \in \mathcal{E}(-n-2)$$

depends only on $\mathbf{f} = \tilde{\mathbf{f}}|_{\mathcal{M}}$ and defines a differential operator

$$\mathbf{R}: \mathcal{E}(1) \rightarrow \mathcal{E}(-n-2).$$

Moreover, \mathbf{R} is independent of the choice of a Fefferman defining function. The operator \mathbf{R} coincides with $\mathbf{P}_{1,1}$ if the obstruction function \mathcal{O} vanishes along \mathcal{M} .

In the remainder of this section, we assume that θ is pseudo-Einstein. Take a defining function ρ of M normalized by θ such that \mathbf{h}_ρ is flat on the pseudoconvex side.

The P -prime operator is a differential operator acting on CR pluriharmonic functions, which appears in the transformation law of the Q -prime curvature.

Definition 0.5.3 ([Hir14, Definition 4.2]). *The P -prime operator*

$$\mathbf{P}'_\theta: \mathcal{P} \rightarrow \mathcal{E}(-n-1)$$

is defined by

$$\mathbf{P}'_\theta \Upsilon = -\frac{1}{n+2} [\Delta^{n+1} (\Upsilon \log \mathbf{h}_\rho)]|_{\mathcal{M}},$$

where $\Upsilon \in \mathcal{P}$ and $\tilde{\Upsilon}$ is a smooth extension of Υ that is pluriharmonic on the pseudoconvex side. The function $\mathbf{P}'_\theta \Upsilon$ is determined only by θ and Υ .

Since $\log \mathbf{h}_\rho$ is a pluriharmonic function on the pseudoconvex side, we have $\mathbf{P}'_\theta 1 = 0$. Moreover, Marugame [Mar18, Theorem 1.2] has proved that

$$\int_M \Upsilon_2 \cdot \mathbf{P}'_\theta \Upsilon_1 = \int_M \Upsilon_1 \cdot \mathbf{P}'_\theta \Upsilon_2$$

for $\Upsilon_1, \Upsilon_2 \in \mathcal{P}$ if M is closed. In particular,

$$\int_M \mathbf{P}'_\theta \Upsilon = \int_M \Upsilon \cdot \mathbf{P}'_\theta 1 = 0$$

for any $\Upsilon \in \mathcal{P}$.

Now we give the definition of the Q -prime curvature.

Definition 0.5.4 ([Hir14, Definition 5.4]). The Q -prime curvature \mathbf{Q}'_θ is defined by

$$\mathbf{Q}'_\theta = \frac{1}{2(n+2)^2} [\Delta^{n+1} (\log \mathbf{h}_\rho)^2] |_{\mathcal{M}} \in \mathcal{E}(-n-1).$$

This \mathbf{Q}'_θ is independent of the choice of a Fefferman defining function. We will also use the unbold $Q'_\theta \in C^\infty(M)$ defined by

$$Q'_\theta = \mathbf{Q}'_\theta \cdot \mathbf{h}_\theta^{(n+1)/(n+2)},$$

which is also called the Q -prime curvature.

If we take another pseudo-Einstein contact form $\hat{\theta} = e^\Upsilon \theta$ for $\Upsilon \in \mathcal{P}$, the Q -prime curvature transforms as follows [Hir14, Proposition 5.5]:

$$\mathbf{Q}'_{\hat{\theta}} = \mathbf{Q}'_\theta + \mathbf{P}'_\theta \Upsilon + \frac{1}{2} \mathbf{P}_{0,0}(\Upsilon^2).$$

Hence

$$\int_M \mathbf{Q}'_{\hat{\theta}} = \int_M \mathbf{Q}'_\theta + \int_M \mathbf{P}'_\theta \Upsilon + \frac{1}{2} \int_M \mathbf{P}_{0,0}(\Upsilon^2) = \int_M \mathbf{Q}'_\theta$$

for a closed M . Thus we see that the integral of the Q -prime curvature, the *total Q -prime curvature* $\overline{Q}'(M)$, is independent of the choice of a pseudo-Einstein contact form, and defines a global CR invariant.

We next recall variational formulas for the total Q -prime curvature under deformations of real hypersurfaces. Let $(M_t)_{t \in (-1,1)}$ be a smooth family of closed strictly pseudoconvex real hypersurfaces in a complex manifold X satisfying $M_0 = M$. Take a Fefferman defining function ρ_t of $\mathcal{M}_t = \pi_X^{-1}(M_t)$ such that it is smooth in the parameter $t \in (-1,1)$. Assume that there exists a flat Hermitian metric \mathbf{h} of K_X near M . The function $\rho_t = \rho_t \cdot \mathbf{h}^{-1/(n+2)}$ is a defining function of M_t , and the corresponding contact form $\theta_t = d^c \rho_t |_{M_t}$ is pseudo-Einstein.

Theorem 0.5.5 ([HMM17, Theorem 1.2]). *Under the above assumptions, the total Q -prime curvature satisfies*

$$(0.5.1) \quad \left. \frac{d}{dt} \right|_{t=0} \overline{Q}'(M_t) = c_n \int_M \varphi \mathcal{O} |_{\mathcal{M}},$$

where \mathcal{O} is the obstruction function of M , $\varphi = (d/dt)|_{t=0} \rho_t |_{\mathcal{M}} \in \mathcal{E}(1)$, and $c_n = 2^{n+1} n!(n+2)!$. Moreover if the obstruction function of M vanishes, then

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \overline{Q}'(M_t) = c'_n \int_M \varphi(\mathbf{R}\varphi),$$

where \mathbf{R} is as in Lemma 0.5.2 and $c'_n = -[4(n+1)(n+2)]^{-1}$.

0.6. Sasakian manifolds

This section contains a brief summary of Sasakian manifolds from CR point of view. See [BG08] and [Spa11] for a comprehensive introduction to Sasakian manifolds.

Let $(S, T^{1,0}S)$ be a strictly pseudoconvex CR manifold, η be a contact form on S , and ξ be the Reeb vector field with respect to η . Then an almost complex structure I on the cone $C(S) = \mathbb{R}_+ \times S$ of S is defined by

$$I(a(r\partial/\partial r) + b\xi + V) = -b(r\partial/\partial r) + a\xi + JV,$$

where r is the coordinate of \mathbb{R}_+ , $a, b \in \mathbb{R}$ and $V \in HS$. The bundle $T^{1,0}C(S)$ of $(1, 0)$ -vectors with respect to I is given by

$$T^{1,0}C(S) = \mathbb{C}(r\partial/\partial r - \sqrt{-1}\xi) \oplus T^{1,0}S.$$

Lemma 0.6.1. *The following three conditions are equivalent:*

- (1) *the Tanaka-Webster torsion with respect to η vanishes;*
- (2) $[\xi, \Gamma(T^{1,0}S)] \subset \Gamma(T^{1,0}S)$;
- (3) *the almost complex structure I is integrable.*

Definition 0.6.2. The triple $(S, T^{1,0}S, \eta)$ is called a *Sasakian manifold* if one of the equivalent conditions in the above lemma holds.

PROOF OF LEMMA 0.6.1. Take an admissible frame $(\xi, Z_\alpha, Z_{\bar{\alpha}})$, and the corresponding admissible coframe $(\eta, \theta^\alpha, \theta^{\bar{\alpha}})$. Then we obtain

$$\eta([\xi, Z_\alpha]) = -d\eta(\xi, Z_\alpha) = 0, \quad \theta^{\bar{\beta}}([\xi, Z_\alpha]) = -d\theta^{\bar{\beta}}(\xi, Z_\alpha) = -A^{\bar{\beta}}_\alpha$$

from the definition of the Reeb vector field and the structure equations of the Tanaka-Webster connection. This gives the equivalence between (1) and (2). On the other hand, from the integrability of $T^{1,0}S$, we can derive that (3) is equivalent to

$$[r\partial/\partial r - \sqrt{-1}\xi, \Gamma(T^{1,0}S)] \subset \Gamma(T^{1,0}S).$$

This yields the equivalence between (2) and (3). \square

Let $(S, T^{1,0}S, \eta)$ be a Sasakian manifold. The Riemannian metric $\bar{g} = dr \otimes dr + r^2 g_\eta$ on $C(S)$ is a Kähler metric on $C(S)$, and its Kähler form is equal to $dd^c r^2/2$. Moreover, the level set $\{r = 1\}$ is isomorphic to S as a CR manifold, and the one-form η is equal to $d^c \log r^2|_S$.

Consider the sub-Laplacian Δ_b with respect to η . Since the Tanaka-Webster torsion vanishes, one has

$$f_{0\alpha\bar{\beta}} = f_{\alpha 0\bar{\beta}} = f_{0\alpha\bar{\beta}}, \quad f_{0\bar{\beta}\alpha} = f_{\bar{\beta}\alpha 0}$$

for any $f \in C^\infty(M)$. Hence Δ_b commutes with ξ . We also note that the space \mathcal{P} is annihilated by $\Delta_b^2 + n^2\xi^2$. Let Υ be a CR pluriharmonic function. Locally, Υ is the real part of a CR holomorphic function f . On the other hand, we obtain

$$\Delta_b^2 + n^2\xi^2 = 4\bar{\square}_b \square_b = 4\square_b \bar{\square}_b$$

from $[\Delta_b, \xi] = 0$ and (0.1.2). Therefore,

$$(\Delta_b^2 + n^2\xi^2)\Upsilon = 2\bar{\square}_b \square_b f + 2\square_b \bar{\square}_b \bar{f} = 0.$$

We next consider an Einstein condition for Sasakian manifolds.

Definition 0.6.3. Let $(S, T^{1,0}S, \eta)$ be a $(2n + 1)$ -dimensional Sasakian manifold. It is called a *Sasakian η -Einstein manifold* if there exists a real constant λ such that the Tanaka-Webster Ricci curvature $\text{Ric}_{\alpha\bar{\beta}}$ of η satisfies

$$\text{Ric}_{\alpha\bar{\beta}} = (n + 1)\lambda l_{\alpha\bar{\beta}}.$$

In particular if $\lambda = 1$, it is called a *Sasaki-Einstein manifold*. In this thesis, we call the constant $(n+1)\lambda$ the *Einstein constant* of $(S, T^{1,0}S, \eta)$.

There exist characterizations of Sasakian η -Einstein manifolds in terms of g_η or \bar{g} .

Proposition 0.6.4. *Let $(S, T^{1,0}S, \eta)$ be a $(2n+1)$ -dimensional Sasakian manifold and λ be a real constant. Then the following are equivalent:*

- (1) $(S, T^{1,0}S, \eta)$ is a Sasakian η -Einstein manifold with Einstein constant $(n+1)\lambda$;
- (2) the Ricci curvature Ric_{g_η} of g_η satisfies

$$\text{Ric}_{g_\eta} = 2((n+1)\lambda - 1)g_\eta + 2(n+1)(1-\lambda)\eta \otimes \eta;$$

- (3) the Ricci curvature $\text{Ric}_{\bar{g}}$ of \bar{g} satisfies

$$\text{Ric}_{\bar{g}} = 2(n+1)(\lambda - 1)(g_\eta - \eta \otimes \eta).$$

PROOF. First, we show the equivalence between (1) and (2). We denote by ∇^{g_η} the Levi-Civita connection with respect to g_η . Then, for $U, V \in \Gamma(TS)$,

$$\nabla_U^{g_\eta} V = \nabla_U V - g_\eta(JU, V)\xi + \eta(U)JV + \eta(V)JU.$$

This follows from the fact that the Tanaka-Webster connection preserves the metric g_η and the torsion Tor of ∇ satisfies $\text{Tor}(U, V) = 2g_\eta(JU, V)\xi$. Hence the curvature R_{g_η} of ∇^{g_η} is related with the curvature R of ∇ as follows:

$$(0.6.1) \quad \begin{aligned} R_{g_\eta}(U, V)W &= R(U, V)W - g_\eta(JV, W)JU + g_\eta(JU, W)JV \\ &\quad - 2g_\eta(U, JV)JW - \eta(V)g_\eta(JU, JW)\xi \\ &\quad + \eta(U)g_\eta(JV, JW)\xi - \eta(U)\eta(W)V + \eta(V)\eta(W)U \end{aligned}$$

for $U, V, W \in \Gamma(TS)$. Taking the trace of (0.6.1) gives

$$\text{Ric}_{g_\eta} = \text{Ric} - 2g_\eta + 2(n+1)\eta \otimes \eta,$$

where Ric is the Ricci curvature of ∇ . On the other hand, Ric is given by

$$\text{Ric} = \text{Ric}_{\alpha\bar{\beta}}(\theta^\alpha \otimes \theta^{\bar{\beta}} + \theta^{\bar{\beta}} \otimes \theta^\alpha),$$

which follows from (0.1.1). This proves the equivalence between (1) and (2).

Next, we show the equivalence between (2) and (3). Let $\nabla^{\bar{g}}$ be the Levi-Civita connection with respect to \bar{g} . Then for $U, V \in \Gamma(TS)$,

$$\begin{aligned} \nabla_{r\partial/\partial r}^{\bar{g}}(r\partial/\partial r) &= r\partial/\partial r, \quad \nabla_{r\partial/\partial r}^{\bar{g}}U = \nabla_U^{\bar{g}}(r\partial/\partial r) = U, \\ \nabla_U^{\bar{g}}V &= \nabla_U^{g_\eta}V - g_\eta(U, V)r\partial/\partial r. \end{aligned}$$

Hence the curvature $R_{\bar{g}}$ of $\nabla^{\bar{g}}$ satisfies

$$\begin{aligned} R_{\bar{g}}(\cdot, r\partial/\partial r) &= 0, \quad R_{\bar{g}}(\cdot, \cdot)r\partial/\partial r = 0, \\ R_{\bar{g}}(U, V)W &= R_{g_\eta}(U, V)W - g_\eta(V, W)U + g_\eta(U, W)V, \end{aligned}$$

where $U, V, W \in \Gamma(TS)$. Taking the trace, we obtain

$$\text{Ric}_{\bar{g}} = \text{Ric}_{g_\eta} - 2ng_\eta.$$

Therefore, (2) is equivalent to (3). \square

Note that (3) is equivalent to

$$(0.6.2) \quad -dd^c \log \det \left(\frac{\partial^2 r^2}{\partial z^a \partial \bar{z}^b} \right) = (n+1)(\lambda - 1)dd^c \log r^2$$

for any holomorphic local coordinate (z^1, \dots, z^{n+1}) of $C(S)$ since $r^2/2$ is a Kähler potential of \bar{g} .

Example 0.6.5. Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere centered at the origin with the natural CR structure $T^{1,0}S^{2n+1}$, and η_{std} be the contact form on S^{2n+1} defined by

$$\eta_{\text{std}} = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n+1} (z^i d\bar{z}^i - \bar{z}^i dz^i)|_{S^{2n+1}}.$$

The cone $(C(S^{2n+1}), \bar{g})$ is isomorphic to $(\mathbb{C}^{n+1} \setminus \{0\}, g_{\text{Euc}})$ as an almost Kähler manifold by the map

$$C(S^{2n+1}) \rightarrow \mathbb{C}^{n+1} \setminus \{0\}; (r, p) \mapsto r^2 p.$$

Here g_{Euc} is the Euclidean metric on \mathbb{C}^{n+1} . Since g_{Euc} is a Ricci-flat Kähler metric, $(S^{2n+1}, T^{1,0}S^{2n+1}, \eta_{\text{std}})$ is a Sasaki-Einstein manifold.

A typical example of a Sasakian manifold is the *tube associated with a polarized Kähler manifold*. Let Y be an n -dimensional complex manifold, and (L, h_L) be a Hermitian holomorphic line bundle over Y such that $\omega = \sqrt{-1}\Theta_{h_L} = -dd^c \log h_L$ defines a Kähler metric on Y . We call such a triple (Y, L, h_L) a *polarized Kähler manifold*. Denote by $h_{L^{-1}}$ the dual Hermitian metric on L^{-1} . Now we consider the tube

$$S = \{ v \in L^{-1} \mid h_{L^{-1}}(v, v) = 1 \},$$

which is a real hypersurface in L^{-1} . This S has a canonical orientation as the boundary of the open set

$$\Omega = \{ v \in L^{-1} \mid h_{L^{-1}}(v, v) < 1 \}.$$

The one-form $\eta = d^c \log h_{L^{-1}}|_S$ is a connection one-form of the principal S^1 -bundle $p: S \rightarrow Y$, and satisfies $d\eta = p^*\omega$. Moreover, η gives a section of HS^1_+ , and $T^{1,0}S$ coincides with the horizontal lift of $T^{1,0}Y$ with respect to η . Since ω defines a Kähler metric, we have

$$-\sqrt{-1}d\eta(Z, \bar{Z}) = -\sqrt{-1}\omega(p_*Z, p_*\bar{Z}) > 0$$

for $0 \neq Z \in T^{1,0}S$; this implies that $(S, T^{1,0}S)$ is a strictly pseudoconvex CR manifold and η is a contact form on S . Note that the Reeb vector field ξ with respect to η is a generator of the S^1 -action on S .

Next, consider the Tanaka-Webster connection with respect to η . Take a local coordinate (z^1, \dots, z^n) of Y . The Kähler form ω is written as

$$\omega = \sqrt{-1}g_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta,$$

where $(g_{\alpha\bar{\beta}})$ is a positive definite Hermitian matrix. Let Z_α be the horizontal lift of $\partial/\partial z^\alpha$. Then $(\xi, Z_\alpha, \bar{Z}_\alpha = \overline{Z_\alpha})$ is an admissible frame on S . The corresponding admissible coframe is given by $(\eta, \theta^\alpha = p^*(dz^\alpha), \theta^{\bar{\alpha}} = p^*(d\bar{z}^\alpha))$. Since $d\eta = p^*\omega$, we have

$$d\eta = \sqrt{-1}(p^*g_{\alpha\bar{\beta}})\theta^\alpha \wedge \theta^{\bar{\beta}},$$

which implies $l_{\alpha\bar{\beta}} = p^*g_{\alpha\bar{\beta}}$. The connection form ϕ_α^β of the Kähler metric with respect to the frame $(\partial/\partial z^\alpha)$ satisfies

$$(0.6.3) \quad 0 = d(dz^\beta) = dz^\alpha \wedge \phi_\alpha^\beta, \quad dg_{\alpha\bar{\beta}} = \phi_\alpha^\gamma g_{\gamma\bar{\beta}} + g_{\alpha\bar{\gamma}} \phi_\beta^{\bar{\gamma}}.$$

We write as Φ_α^β the curvature form of the Kähler metric. Pulling back (0.6.3) by p gives

$$d\theta^\beta = \theta^\alpha \wedge (p^*\phi_\alpha^\beta), \quad dl_{\alpha\bar{\beta}} = (p^*\phi_\alpha^\gamma)l_{\gamma\bar{\beta}} + l_{\alpha\bar{\gamma}}(p^*\phi_\beta^{\bar{\gamma}}).$$

This yields that $\omega_\alpha^\beta = p^*\phi_\alpha^\beta$, and the Tanaka-Webster torsion vanishes identically; that is, $(S, T^{1,0}S, \eta)$ is a Sasakian manifold. Moreover, the curvature form Ω_α^β of the Tanaka-Webster connection is given by $\Omega_\alpha^\beta = p^*\Phi_\alpha^\beta$. Therefore, $(S, T^{1,0}S, \eta)$

is a Sasakian η -Einstein manifold with Einstein constant $(n+1)\lambda$ if and only if ω defines a Kähler-Einstein metric on Y with Einstein constant $(n+1)\lambda$.

Example 0.6.6 (S^{2n+1} revisited). Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere centered at the origin with the natural CR structure. We can identify S^{2n+1} with the tube of a polarized Kähler-Einstein manifold as follows. Consider the hyperplane bundle $L = \mathcal{O}(1)$ over $Y = \mathbb{C}\mathbb{P}^n$. Its dual $L^{-1} = \mathcal{O}(-1)$ can be realized as a subbundle of the trivial vector bundle of rank $n+1$:

$$\mathcal{O}(-1) = \{ (l, v) \in \mathbb{C}\mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in l \};$$

here we consider $\mathbb{C}\mathbb{P}^n$ as the set of all one-dimensional linear subspaces in \mathbb{C}^{n+1} . The standard Hermitian inner product on \mathbb{C}^{n+1} induces a Hermitian metric $h_{\mathcal{O}(-1)}$ on $\mathcal{O}(-1)$, and the unit sphere S^{2n+1} can be identified with the tube with respect to $h_{\mathcal{O}(-1)}$. Moreover, the curvature of the dual Hermitian metric $h_{\mathcal{O}(1)}$ on $\mathcal{O}(1)$ defines a Kähler-Einstein metric on $\mathbb{C}\mathbb{P}^n$ with Einstein constant $n+1$; this means that the triple $(\mathbb{C}\mathbb{P}^n, \mathcal{O}(1), h_{\mathcal{O}(1)})$ is a polarized Kähler-Einstein manifold with Einstein constant $n+1$. Therefore, S^{2n+1} is isomorphic as a CR manifold to the tube associated with a polarized Kähler-Einstein manifold $(\mathbb{C}\mathbb{P}^n, \mathcal{O}(1), h_{\mathcal{O}(1)})$ with Einstein constant $n+1$.

Example 0.6.7. Let Y_d be the Fermat hypersurface of degree d in $\mathbb{C}\mathbb{P}^{n+1}$; that is,

$$Y_d = \left\{ [X^0 : \dots : X^{n+1}] \in \mathbb{C}\mathbb{P}^{n+1} \mid \sum_{i=0}^{n+1} (X^i)^d = 0 \right\},$$

and L_d be the restriction to Y_d of the hyperplane bundle $\mathcal{O}(1)$ over $\mathbb{C}\mathbb{P}^{n+1}$. To simplify the notation, we denote by κ_d the cohomology class $c_1(L_d) = c_1(\mathcal{O}(1))|_{Y_d}$. The total Chern class $c(T^{1,0}Y_d)$ of $T^{1,0}Y_d$ is given by

$$c(T^{1,0}Y_d) = (1 + \kappa_d)^{n+2} (1 + d\kappa_d)^{-1},$$

or equivalently,

$$(0.6.4) \quad c_k(T^{1,0}Y_d) = \left[\sum_{l=0}^k \binom{n+2}{l} (-d)^{k-l} \right] \kappa_d^k.$$

It is known that L_d admits a Hermitian metric h_{L_d} whose curvature defines a Kähler-Einstein metric on Y_d [Tia00, Chapter 6.3]. Set $\lambda_d = (n+2-d)/(n+1)$. Since $c_1(T^{1,0}Y_d) = (n+2-d)\kappa_d$, the Einstein constant of this metric is $n+2-d = (n+1)\lambda_d$. Therefore, the tube S_d associated with (Y_d, L_d, h_{L_d}) is a Sasakian η -Einstein manifold with Einstein constant $(n+1)\lambda_d$.

A constraint on Chern classes of strictly pseudoconvex CR manifolds

1.1. Introduction

Let $(M, T^{1,0}M)$ be a closed connected strictly pseudoconvex CR manifold of dimension $2n+1 \geq 5$. As we noted in Section 0.2, M can be realized as the boundary of a strictly pseudoconvex domain in a complex projective manifold. This fact gives some restrictions to the topology of M . For example, Bungart [Bun92] has proved that the cup product

$$H^{i_1}(M, \mathbb{C}) \otimes \cdots \otimes H^{i_m}(M, \mathbb{C}) \rightarrow H^{|I|}(M, \mathbb{C})$$

vanishes for any multi-index $I = (i_1, \dots, i_m) \in \mathbb{N}^m$ satisfying $i_l \leq n-1$ and $|I| = i_1 + \cdots + i_m \geq n+2$; see also [PP08]. As remarked in the last paragraph of [Bun92], this result also follows from a result of Ohsawa via analytic Hodge decomposition [Ohs82].

In the next section, we will see that Ohsawa's result gives also a constraint on Chern classes of closed strictly pseudoconvex CR manifolds. For a complex vector bundle E and a multi-index $K = (k_1, \dots, k_m) \in \mathbb{N}^m$, we denote by $c_K(E)$ the cohomology class $c_{k_1}(E) \cdots c_{k_m}(E)$.

Theorem 1.1.1. *Let $(M, T^{1,0}M)$ be a closed strictly pseudoconvex CR manifold of dimension $2n+1 \geq 5$. Then $c_K(T^{1,0}M)$ vanishes in $H^{2|K|}(M, \mathbb{C})$ for any multi-index K with $2|K| \geq n+2$.*

This theorem implies that $c_2(T^{1,0}M) = 0$ in $H^4(M, \mathbb{C})$ for any closed strictly pseudoconvex CR manifold $(M, T^{1,0}M)$ of dimension five. Therefore, the assumption of the second Chern class in [CG17, Proposition 8.8] automatically holds for the strictly pseudoconvex case, which is compatible with [Mar16, Theorem 4.6].

Note that sharper results than Theorem 1.1.1 hold for some particular cases. For instance, Theorem 1.1.1 holds for \mathbb{Z} -coefficients if M can be realized as a real hypersurface in a Stein manifold. On the other hand, we can relax the degree condition to $2|K| \geq n+1$ if M is the link of an isolated singularity [Kol13]. However, we will see in Section 1.3 that Theorem 1.1.1 is "optimal" for a general closed strictly pseudoconvex CR manifold.

We also remark a relation of this result to contact topology. Let (M, H) be a closed $(2n+1)$ -dimensional contact manifold. Then we can define the k -th Chern class $c_k(H) \in H^{2k}(M, \mathbb{Z})$ of H by using an adapted almost complex structure on H . A contact structure H is said to be *holomorphically fillable* if M can be realized as the boundary of a strictly pseudoconvex domain and $H = \operatorname{Re} T^{1,0}M$ holds. Theorem 1.1.1 and the fact that $c_k(H) = c_k(T^{1,0}M)$ imply the following

Corollary 1.1.2. *Let H be a holomorphically fillable contact structure on a closed manifold M of dimension $2n+1 \geq 5$. Then $c_K(H)$ is torsion in $H^{2|K|}(M, \mathbb{Z})$ for any multi-index K satisfying $2|K| \geq n+2$.*

This result is no longer true for a general contact structure. As far as the author knows, such a constraint on Chern classes has not been obtained yet for a contact structure satisfying other fillability conditions.

1.2. Proof of Theorem 1.1.1

Let $(M, T^{1,0}M)$ be a closed strictly pseudoconvex CR manifold of dimension $2n + 1 \geq 5$. Without loss of generality, we may assume that M is connected. It can be realized as the boundary of a strictly pseudoconvex domain Ω in an $(n + 1)$ -dimensional complex projective manifold X [Lem95, Theorem 8.1]. The complex vector bundle $T^{1,0}X|_M$ is decomposed into the direct sum of $T^{1,0}M$ and a trivial complex line bundle, and consequently $c_K(T^{1,0}M) = c_K(T^{1,0}X|_M)$ is in the image of the restriction morphism $H^{2|K|}(\Omega, \mathbb{C}) \rightarrow H^{2|K|}(M, \mathbb{C})$. On the other hand, the natural morphism $H_c^i(\Omega, \mathbb{C}) \rightarrow H^i(\Omega, \mathbb{C})$ is surjective for $i \geq n + 2$ according to [Ohs82, Corollary 4]; see [Bun92, Lemma] for another proof. From the exact sequence

$$H_c^i(\Omega, \mathbb{C}) \rightarrow H^i(\Omega, \mathbb{C}) \rightarrow H^i(M, \mathbb{C}),$$

it follows that $H^i(\Omega, \mathbb{C}) \rightarrow H^i(M, \mathbb{C})$ is identically zero for $i \geq n + 2$. This completes the proof of Theorem 1.1.1.

1.3. Examples

In this section, we deal with some examples related to Theorem 1.1.1. We first show that Theorem 1.1.1 does not hold for \mathbb{Z} -coefficients in general.

Proposition 1.3.1. *For each $n \in \mathbb{N}_+$, there exists a closed strictly pseudoconvex CR manifold S of dimension $2n + 1$ such that $c_K(T^{1,0}S) \neq 0$ in $H^{2|K|}(S, \mathbb{Z})$ for every multi-index K with $0 \leq |K| \leq n$.*

PROOF. Let $\mathcal{O}(d)$ be the holomorphic line bundle over $\mathbb{C}\mathbb{P}^n$ of degree $d > 0$. There exists a Hermitian metric $h_{\mathcal{O}(d)}$ on $\mathcal{O}(d)$ whose curvature defines a Kähler metric on $\mathbb{C}\mathbb{P}^n$. Consider the tube S associated with $(\mathbb{C}\mathbb{P}^n, \mathcal{O}(d), h_{\mathcal{O}(d)})$, which is a principal S^1 -bundle over $\mathbb{C}\mathbb{P}^n$. Consider the Gysin exact sequence

$$H^{i-2}(\mathbb{C}\mathbb{P}^n, \mathbb{Z}) \xrightarrow{-d\cdot\kappa} H^i(\mathbb{C}\mathbb{P}^n, \mathbb{Z}) \xrightarrow{p^*} H^i(S, \mathbb{Z}) \rightarrow H^{i-1}(\mathbb{C}\mathbb{P}^n, \mathbb{Z}),$$

where $\kappa = c_1(\mathcal{O}(1))$ is a generator of $H^2(\mathbb{C}\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$. This gives that

$$H^{2k}(S, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}/d\mathbb{Z}, & 1 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

and $p^*\kappa^k$ is a generator of $H^{2k}(S, \mathbb{Z})$. On the other hand, since $T^{1,0}S$ is isomorphic to $p^*T^{1,0}\mathbb{C}\mathbb{P}^n$ as a complex vector bundle,

$$c_K(T^{1,0}S) = p^*c_K(T^{1,0}\mathbb{C}\mathbb{P}^n) = \left[\prod_{l=1}^m \binom{n+1}{k_l} \right] p^*\kappa^{|K|}.$$

Hence, for a multi-index K with $0 \leq |K| \leq n$, the K -th Chern class $c_K(T^{1,0}S)$ of $T^{1,0}S$ is equal to zero in $H^{2|K|}(S, \mathbb{Z})$ only if $\prod_{l=1}^m \binom{n+1}{k_l} \in d\mathbb{Z}$. In particular, if we choose d as a prime integer greater than $n + 1$, then $c_K(T^{1,0}S)$ does not vanish for any K with $0 \leq |K| \leq n$. \square

We next see that the degree condition in Theorem 1.1.1 is optimal.

Proposition 1.3.2. *For each $n \in \mathbb{N}_+$, there exists an $(n + 1)$ -dimensional Stein domain Ω such that its boundary $M = \partial\Omega$ satisfies $c_K(T^{1,0}M) \neq 0$ in $H^{2|K|}(M, \mathbb{C})$ for any multi-index K with $0 \leq 2|K| \leq n + 1$.*

PROOF. Let Ω_0 be a Stein domain in a two-dimensional complex manifold X_0 such that its boundary $M_0 = \partial\Omega_0$ satisfies $c_1(T^{1,0}M_0) \neq 0$ in $H^2(M_0, \mathbb{C})$; see [EO08, Theorem 6.2] for an example of Ω_0 . Take a defining function ρ of Ω_0 that is strictly plurisubharmonic on a neighborhood of the closure of Ω_0 . Without loss of generality, we may assume that ρ is an exhaustion function on X_0 . Then, for sufficiently small ϵ , there exists a diffeomorphism $\chi: (-\epsilon, \epsilon) \times M_0 \rightarrow \rho^{-1}((-\epsilon, \epsilon))$ satisfying $\chi(0, p) = p$ and $\rho(\chi(t, p)) = t$. The function $\psi_0 = -\rho^{-1}$ gives a strictly plurisubharmonic exhaustion function on Ω_0 .

We first show the statement for the case of odd n . For $k \in \mathbb{N}_+$, consider the domain

$$\Omega = \{ (p_1, \dots, p_k) \in (\Omega_0)^k \mid \psi_0(p_1) + \dots + \psi_0(p_k) < 2k/\epsilon \}.$$

The function $\psi(p_1, \dots, p_k) = \psi_0(p_1) + \dots + \psi_0(p_k)$ is a strictly plurisubharmonic exhaustion function on $(\Omega_0)^k$, and $d\psi \neq 0$ on $M = \partial\Omega$. Hence Ω is a Stein domain in $(\Omega_0)^k \subset (X_0)^k$. As noted in the proof of Theorem 1.1.1, the cohomology class $c_K(T^{1,0}M)$ coincides with $c_K(T^{1,0}(X_0)^k|_M)$. Consider the map

$$\iota: (M_0)^k \rightarrow M; (p_1, \dots, p_k) \mapsto (\chi(-\epsilon/2, p_1), \dots, \chi(-\epsilon/2, p_k)).$$

Since this map is homotopic to the natural embedding $(M_0)^k \hookrightarrow (X_0)^k$,

$$\iota^* c_K(T^{1,0}M) = c_K(\iota^* T^{1,0}(X_0)^k) = c_K((T^{1,0}X_0)^k|_{(M_0)^k}) = c_K((T^{1,0}M_0)^k).$$

From the assumption of Ω_0 , it follows that $c_K((T^{1,0}M_0)^k) \neq 0$ in $H^{2|K|}((M_0)^k, \mathbb{C})$ for any multi-index K with $0 \leq 2|K| \leq 2k$, and consequently, $c_K(T^{1,0}M) \neq 0$ in $H^{2|K|}(M, \mathbb{C})$ for such a K .

We next treat the case of even n . For $k \in \mathbb{N}_+$, consider the domain

$$\Omega = \{ (p_1, \dots, p_k, z) \in (\Omega_0)^k \times \mathbb{C} \mid \psi_0(p_1) + \dots + \psi_0(p_k) + |z|^2 < 2k/\epsilon \}.$$

This Ω is a Stein domain in $(\Omega_0)^k \times \mathbb{C} \subset (X_0)^k \times \mathbb{C}$. Consider the map

$$\iota: (M_0)^k \rightarrow M = \partial\Omega; (p_1, \dots, p_k) \mapsto (\chi(-\epsilon/2, p_1), \dots, \chi(-\epsilon/2, p_k), 0).$$

Then we obtain

$$\iota^* c_K(T^{1,0}M) = c_K((T^{1,0}M_0)^k) \neq 0$$

in $H^{2|K|}((M_0)^k, \mathbb{C})$ for any K satisfying $0 \leq 2|K| \leq 2k + 1$ in a similar way to the case of odd n . This proves the statement. \square

Stability of the existence of a pseudo-Einstein contact form

2.1. Introduction

A pseudo-Einstein contact form is necessary for defining the total Q -prime curvature and the Burns-Epstein invariant. When we consider the variation of such invariants, the question arises whether the existence of a pseudo-Einstein contact form is preserved under deformations of CR structures. In this chapter, we will show this stability for deformations as a real hypersurface in a fixed complex manifold of complex dimension at least three. More precisely, we will prove the following

Theorem 2.1.1. *Let Ω be a strictly pseudoconvex domain in a complex manifold X of complex dimension at least three. Assume that its boundary $M = \partial\Omega$ admits a pseudo-Einstein contact form. Then there exists a neighborhood U of M in X such that K_U has a flat Hermitian metric.*

The stability stated above follows from this theorem and Proposition 0.3.2.

Corollary 2.1.2. *Let Ω , X , M , and U be as in Theorem 2.1.1. Then any strictly pseudoconvex real hypersurface M' in U admits a pseudo-Einstein contact form.*

Note that this stability may have been already known when an ambient complex manifold is a Stein manifold of dimension at least three; see Remark 2.3.2.

Here we give an outline of the proof of Theorem 2.1.1. Take a tubular neighborhood U of M in X . The existence of a pseudo-Einstein contact form on M implies that there is a flat Hermitian metric on $K_{U \cap \Omega}$ if we take U sufficiently small. By using the Bott-Chern class, we will show that K_U admits a flat Hermitian metric if the morphism

$$(2.1.1) \quad H^1(U, \mathcal{O}) \rightarrow H^1(U \cap \Omega, \mathcal{O})$$

induced by the inclusion is injective (Lemma 2.2.1). On the other hand, a result of Andreotti and Grauert [AG62] yields that (2.1.1) is an isomorphism; here we use the assumption that the complex dimension of X is at least three. A proof of this fact will be given in Section 2.3.

Before the end of the introduction, we remark a relation between our result and the Lee conjecture. Lee [Lee88, Proposition D] has proved that the first Chern class $c_1(T^{1,0}M)$ of $T^{1,0}M$ is equal to zero in $H^2(M, \mathbb{R})$ if M admits a pseudo-Einstein contact form, and conjectured that the converse also holds if M is closed; this is called the *Lee conjecture*. There are some affirmative results on this conjecture [Lee88, Dra94, CC07, CSW12, CCT14], but it is still open. (Remark that we need an extra assumption on the Tanaka-Webster torsion in [CCT14, Theorem 1.1], which has been pointed out in the erratum [CCT16].) The stability of the existence of a pseudo-Einstein contact form follows from the Lee conjecture since the first Chern class of a CR structure is invariant under deformations of a CR structure. In other words, Corollary 2.1.2 can be considered as one of affirmative results on the Lee conjecture.

2.2. Bott-Chern class and the existence of a flat Hermitian metric

Let X be a complex manifold. The *real Bott-Chern cohomology* $H_{BC}^{1,1}(X, \mathbb{R})$ of bi-degree $(1, 1)$ is defined by

$$H_{BC}^{1,1}(X, \mathbb{R}) = \{ d\text{-closed real } (1, 1)\text{-forms on } X \} / \{ dd^c \psi \mid \psi \in C^\infty(X, \mathbb{R}) \}.$$

Let $f: Y \rightarrow X$ be a holomorphic map between complex manifolds. The pull-back by f induces a natural morphism

$$f^*: H_{BC}^{1,1}(X, \mathbb{R}) \rightarrow H_{BC}^{1,1}(Y, \mathbb{R}).$$

The cohomology $H_{BC}^{1,1}(X, \mathbb{R})$ has also a sheaf-theoretic interpretation. Let $\mathcal{A}^{p,q}$ be the sheaf of smooth (p, q) -forms and \mathcal{P} be that of pluriharmonic functions. There exists the following exact sequence of sheaves [Big69, Teorema (2.1)]:

$$0 \rightarrow \mathcal{P} \rightarrow \mathcal{A}_{\mathbb{R}}^{0,0} \xrightarrow{dd^c} \mathcal{A}_{\mathbb{R}}^{1,1} \xrightarrow{d} (\mathcal{A}^{2,1} \oplus \mathcal{A}^{1,2})_{\mathbb{R}}.$$

Here the subscript \mathbb{R} means the subsheaf consisting of real forms. Since $\mathcal{A}^{p,q}$ is a fine sheaf, this exact sequence implies that $H^1(X, \mathcal{P})$ is isomorphic to $H_{BC}^{1,1}(X, \mathbb{R})$. Note that a holomorphic map $f: Y \rightarrow X$ induces a natural morphism $f^*: H^1(X, \mathcal{P}) \rightarrow H^1(Y, \mathcal{P})$, which is compatible with $f^*: H_{BC}^{1,1}(X, \mathbb{R}) \rightarrow H_{BC}^{1,1}(Y, \mathbb{R})$ defined above.

For a holomorphic line bundle L over X , the *first Bott-Chern class* $c_1^{BC}(L) \in H_{BC}^{1,1}(X, \mathbb{R})$ is defined as follows. Take a Hermitian metric h of L . Then the curvature $(\sqrt{-1}/2\pi)\Theta_h = -(1/2\pi)dd^c \log h$ is a d -closed real $(1, 1)$ -form on X , and defines an element of $H_{BC}^{1,1}(X, \mathbb{R})$. This cohomology class is independent of the choice of h , denoted by $c_1^{BC}(L)$. From the definition, $c_1^{BC}(L) = 0$ if and only if L admits a flat Hermitian metric. Note that c_1^{BC} is natural; that is, $f^*c_1^{BC}(L) = c_1^{BC}(f^*L)$ for any holomorphic map $f: Y \rightarrow X$.

Now we give a sufficient condition to the existence of a flat Hermitian metric.

Lemma 2.2.1. *Let X and Y be complex manifolds and $f: Y \rightarrow X$ be a holomorphic map. Assume that f induces injective morphisms $H^1(X, \mathcal{O}) \hookrightarrow H^1(Y, \mathcal{O})$ and $H^2(X, \mathbb{R}) \hookrightarrow H^2(Y, \mathbb{R})$, and a surjective morphism $H^1(X, \mathbb{R}) \twoheadrightarrow H^1(Y, \mathbb{R})$. Then, for any holomorphic line bundle L over X , it admits a flat Hermitian metric if so does f^*L .*

PROOF. Assume that f^*L has a flat Hermitian metric. As we noted above, this is equivalent to $f^*c_1^{BC}(L) = c_1^{BC}(f^*L) = 0$. Hence it is enough to prove the injectivity of $f^*: H^1(X, \mathcal{P}) \rightarrow H^1(Y, \mathcal{P})$. Consider the following exact sequence of sheaves:

$$0 \rightarrow \mathbb{R} \xrightarrow{\sqrt{-1}} \mathcal{O} \xrightarrow{\text{Re}} \mathcal{P} \rightarrow 0.$$

This induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^1(X, \mathbb{R}) & \longrightarrow & H^1(X, \mathcal{O}) & \longrightarrow & H^1(X, \mathcal{P}) & \longrightarrow & H^2(X, \mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(Y, \mathbb{R}) & \longrightarrow & H^1(Y, \mathcal{O}) & \longrightarrow & H^1(Y, \mathcal{P}) & \longrightarrow & H^2(Y, \mathbb{R}). \end{array}$$

The injectivity of $H^1(X, \mathcal{P}) \rightarrow H^1(Y, \mathcal{P})$ follows from an easy diagram chasing. \square

2.3. Proof of Theorem 2.1.1

Let X , Ω , and M be as in Theorem 2.1.1. We first reduce the problem on X to that on a Stein space. Take a defining function ρ of Ω that is strictly plurisubharmonic near the boundary. Without loss of generality, we may assume that

$\rho: X \rightarrow \mathbb{R}$ is proper. Then, for sufficiently small $\delta > 0$, there exists a diffeomorphism

$$\chi: (-2\delta, 2\delta) \times M \rightarrow \rho^{-1}((-2\delta, 2\delta))$$

such that $\chi(0, p) = p$ and $\rho(\chi(t, p)) = t$. Replacing δ to a smaller one if necessary, we may assume that ρ is strictly plurisubharmonic on $\rho^{-1}((-2\delta, 2\delta))$. In particular, $\Omega' = \rho^{-1}((-\infty, \delta))$ is a strictly pseudoconvex domain in X containing Ω . Consider the Remmert reduction $\varphi: \Omega' \rightarrow Z$. From the strict plurisubharmonicity of ρ , it follows that the maximal compact analytic subset of Ω' cannot intersect with $\rho^{-1}((-\delta, \delta))$; in particular, φ is a biholomorphism on $\rho^{-1}((-\delta, \delta))$. Without loss of generality, we may assume that ρ descends to a smooth function on Z ; use the same letter ρ for abbreviation. It is sufficient to show the existence of a neighborhood $U \subset \rho^{-1}((-\delta, \delta))$ of $M = \rho^{-1}(0)$ such that K_U has a flat Hermitian metric. To this end, we construct a “good” exhaustion function on Z .

Lemma 2.3.1. *Fix $0 < \alpha < \delta$. There exists a smooth non-negative strictly plurisubharmonic exhaustion function ϕ on Z satisfying the following conditions:*

- $\phi^{-1}(0)$ coincides with the singular set A of Z ;
- ϕ is of the form

$$\phi(p) = \frac{\rho(p)}{\delta(\delta - \rho(p))} + K$$

- on $\rho^{-1}((-\alpha, \delta))$ for a constant $K > 0$;
- $\phi < K$ on $\rho^{-1}((-\infty, -\alpha])$.

The proof of this lemma is slightly complicated, and so will be given later. Now we complete the proof of Theorem 2.1.1 using Lemmas 2.2.1 and 2.3.1. Note that our proof is similar in spirit to the proof of [Yau81, Theorem B].

PROOF OF THEOREM 2.1.1. Set

$$\Omega(a, b) = \{ K + a < \phi < K + b \}$$

for $-\infty \leq a < b$. Note that $\phi^{-1}(K) = M$ and $\Omega(-K, b) = \Omega(-\infty, b) \setminus A$. It is enough to prove that the canonical bundle of $\Omega(-\epsilon, \epsilon)$ admits a flat Hermitian metric for some $\epsilon > 0$ if M has a pseudo-Einstein contact form. The existence of a pseudo-Einstein contact form on M implies that the canonical bundle of $\Omega(-\epsilon, 0)$ has a flat Hermitian metric for sufficiently small $\epsilon > 0$ by Proposition 0.3.2. We may also assume, by making ϵ small if necessary, that the inclusion $\Omega(-\epsilon, 0) \hookrightarrow \Omega(-\epsilon, \epsilon)$ induces isomorphisms

$$H^1(\Omega(-\epsilon, \epsilon), \mathbb{R}) \xrightarrow{\cong} H^1(\Omega(-\epsilon, 0), \mathbb{R}),$$

$$H^2(\Omega(-\epsilon, \epsilon), \mathbb{R}) \xrightarrow{\cong} H^2(\Omega(-\epsilon, 0), \mathbb{R}).$$

According to Lemma 2.2.1, it suffices to prove that

$$H^1(\Omega(-\epsilon, \epsilon), \mathcal{O}) \rightarrow H^1(\Omega(-\epsilon, 0), \mathcal{O})$$

is also an isomorphism. Consider the following commutative diagram induced by inclusions:

$$\begin{array}{ccc} H^1(\Omega(-K, \epsilon), \mathcal{O}) & \longrightarrow & H^1(\Omega(-\epsilon, \epsilon), \mathcal{O}) \\ \downarrow & & \downarrow \\ H^1(\Omega(-K, 0), \mathcal{O}) & \longrightarrow & H^1(\Omega(-\epsilon, 0), \mathcal{O}). \end{array}$$

From [AG62, Théorème 15], it follows that each row is an isomorphism; here we use the assumption that the complex dimension of X is at least three. Hence it is sufficient to show the left column is an isomorphism. Since $\Omega(-\infty, \epsilon)$ and

$\Omega(-\infty, 0)$ are Stein spaces, we obtain the following commutative diagram whose rows are isomorphisms:

$$\begin{array}{ccc} H^1(\Omega(-K, \epsilon), \mathcal{O}) & \xrightarrow{\cong} & H_A^2(\Omega(-\infty, \epsilon), \mathcal{O}) \\ \downarrow & & \downarrow \\ H^1(\Omega(-K, 0), \mathcal{O}) & \xrightarrow{\cong} & H_A^2(\Omega(-\infty, 0), \mathcal{O}). \end{array}$$

On the other hand, the right column of the above diagram is also an isomorphism by the excision property of the local cohomology. This completes the proof. \square

What is left is to show Lemma 2.3.1, the existence of a “good” exhaustion function ϕ on Z .

PROOF OF LEMMA 2.3.1. As noted in Section 0.2, the singular set A of Z is finite, given by $A = \{p_1, \dots, p_k\} \subset Z$. We first construct a smooth non-negative strictly plurisubharmonic exhaustion function ψ on Z with $\psi^{-1}(0) = A$. There exists a proper holomorphic regular embedding $f: Z \rightarrow \mathbb{C}^N$ for sufficiently large N [Nar60, Theorem 6]; in what follows, we identify Z with the image of f . Then $\psi_0 = |z|^2 + \sum_{j=1}^k \log |z - p_j|^2$ is a strictly plurisubharmonic exhaustion function on \mathbb{C}^N with $\psi_0^{-1}(-\infty) = A$. Hence $\psi = \exp \psi_0 = \exp(|z|^2) \prod_{j=1}^k |z - p_j|^2$ is a smooth non-negative strictly plurisubharmonic exhaustion function on \mathbb{C}^N with $\psi^{-1}(0) = A$.

Choose $\beta \in \mathbb{R}$ with $\alpha < \beta < \delta$, and take a smooth function $\lambda: \mathbb{R} \rightarrow [0, 1]$ such that $\lambda \equiv 1$ on $(-\infty, -\beta)$ and $\lambda \equiv 0$ on $(-\alpha, \infty)$. Then the function

$$\phi_1(p) = \lambda(\rho(p))\psi(p)$$

is strictly plurisubharmonic on $\rho^{-1}((-\infty, -\beta))$ and identically zero on $\rho^{-1}((-\alpha, \delta))$.

Next, take a non-negative smooth function g_1 on \mathbb{R} with

$$\text{supp } g_1 \subset ((2\delta)^{-1}, (\beta + \delta)^{-1}), \quad \int_{\mathbb{R}} g_1(t) dt = 1,$$

and set

$$g_2(t) = \int_0^t \int_0^s g_1(r) dr ds.$$

This g_2 is a non-negative and non-decreasing convex smooth function on \mathbb{R} , vanishes identically on $(-\infty, (2\delta)^{-1}]$, and

$$g_2(t) = t - \delta^{-1} + g_2(\delta^{-1}) > 0$$

on a neighborhood of $[(\beta + \delta)^{-1}, \infty)$. The function

$$\phi_2(p) = g_2\left(\frac{1}{\delta - \rho(p)}\right)$$

vanishes identically on $\rho^{-1}((-\infty, -\delta])$, is plurisubharmonic on $\rho^{-1}((-\delta, \delta))$, and

$$\phi_2(p) = \frac{\rho(p)}{\delta(\delta - \rho(p))} + g_2(\delta^{-1}) > 0.$$

on a neighborhood of $\rho^{-1}([-\beta, \delta])$. Therefore, for any $\epsilon > 0$, the sum $\phi = \epsilon\phi_1 + \phi_2$ is a non-negative smooth exhaustion function on Z such that it is strictly plurisubharmonic on $\rho^{-1}((-\infty, -\beta) \cup (-\alpha, \delta))$, and satisfies $\phi^{-1}(0) = A$. Since ϕ_2 is strictly plurisubharmonic on the compact set $\rho^{-1}([-\beta, -\alpha])$, the function ϕ is also strictly plurisubharmonic there for sufficiently small ϵ . Replacing ϵ by a smaller one, we also have $\phi < g_2(\delta^{-1})$ on $\rho^{-1}((-\infty, -\alpha])$. \square

Remark 2.3.2. Cao and Chang [CC07, Main Theorem (2)] state that if M is the boundary of a Stein domain of complex dimension at least three, then M admits a pseudo-Einstein contact form. However, as we saw in Proposition 1.3.2, there exists such an M satisfying $c_1(T^{1,0}M) \neq 0$ in $H^2(M, \mathbb{R})$; in particular, M has no pseudo-Einstein contact form. Here we give a short proof of a corrected statement: “if M is the boundary of a Stein domain of complex dimension at least three and satisfies $c_1(T^{1,0}M) = 0$ in $H^2(M, \mathbb{R})$, then M admits a pseudo-Einstein contact form”. A discussion in [Lee88, Section 6] gives that a closed strictly pseudoconvex CR manifold $(M, T^{1,0}M)$ of dimension at least five admits a pseudo-Einstein contact form if $c_1(T^{1,0}M) = 0$ in $H^2(M, \mathbb{R})$ and the Kohn-Rossi cohomology $H^{0,1}(M)$ of bi-degree $(0, 1)$ vanishes. On the other hand, a result of Yau [Yau81, Theorem B] yields that $H^{0,1}(M) = 0$ if M is as in the statement. Hence M admits a pseudo-Einstein contact form.

Ambient constructions for Sasakian η -Einstein manifolds

3.1. Introduction

In this chapter, we will give explicit formulas of CR invariants defined in Section 0.5 for Sasakian η -Einstein manifolds. Throughout this chapter, we assume that $(S, T^{1,0}S, \eta)$ is a $(2n+1)$ -dimensional Sasakian η -Einstein manifold with Einstein constant $(n+1)\lambda$, and consider S as a real hypersurface in the $(n+1)$ -dimensional complex manifold $X = C(S)$.

In Section 3.2, we will construct a Fefferman defining function ρ_S explicitly. From Lemma 0.5.1, we obtain the operator $\mathbf{P}_{w,w'}$ for any $(w, w') \in \mathbb{R}^2$ with $k = w + w' + n + 1 \in \mathbb{N}_+$; it may not be a CR invariant differential operator if $k \geq n + 2$. To simplify the formulas, we use unbold differential operators $P_{w,w'}$ and P'_η acting on functions on S instead of $\mathbf{P}_{w,w'}$ and \mathbf{P}'_η ; see Definition 3.3.1. The operators $P_{w,w'}$ and P'_η have expressions in terms of the sub-Laplacian Δ_b and the Reeb vector field ξ .

Theorem 3.1.1. *For $(w, w') \in \mathbb{R}^2$ with $k = w + w' + n + 1 \in \mathbb{N}_+$, the operator $P_{w,w'}$ has the formula*

$$(3.1.1) \quad P_{w,w'} = \prod_{j=0}^{k-1} L_{w'-w+k-2j-1}.$$

Here L_μ is the differential operator acting on $C^\infty(S)$ defined by

$$L_\mu = \Delta_b + \sqrt{-1}\mu\xi + \frac{1}{2}(n-\mu)(n+\mu)\lambda.$$

Theorem 3.1.2. *For any CR pluriharmonic function Υ on S ,*

$$P'_\eta \Upsilon = \frac{2^{n+1}(n-1)!}{n^n} \prod_{j=0}^n (\Delta_b + nj\lambda)\Upsilon.$$

In the case of the sphere or the Heisenberg group, Theorems 3.1.1 and 3.1.2 have been already obtained by Graham [Gra84] and Branson, Fontana, and Morpurgo [BFM13], respectively.

We also compute the Q -prime curvature of S . Similar to the above, we use unbold Q'_η instead of \mathbf{Q}'_η .

Theorem 3.1.3. *The Q -prime curvature Q'_η with respect to η is given by*

$$Q'_\eta = 2^{n+1}(n!)^2 \lambda^{n+1}.$$

As an application of this formula, we will compute the total Q -prime curvature for some Sasakian η -Einstein manifolds in Section 3.5.

Note that Case and Gover [CG17] have also obtained the same results as in Theorems 3.1.1 to 3.1.3 by using tractor calculus.

Finally, we consider the variation of the total Q -prime curvature at S under deformations as a real hypersurface. In the remainder of this section, we assume that S is closed.

Proposition 3.1.4. *Let $(M_t)_{t \in (-1,1)}$ be a smooth family of closed strictly pseudoconvex real hypersurfaces in X such that $M_0 = S$. Then the first variation of the total Q -prime curvature vanishes; that is,*

$$\left. \frac{d}{dt} \right|_{t=0} \bar{Q}'(M_t) = 0.$$

This proposition follows from the fact that the obstruction function of S vanishes; see Proposition 3.2.5. Moreover, spectral properties of Δ_b and ξ give the following

Theorem 3.1.5. *Let $(M_t)_{t \in (-1,1)}$ be as in Proposition 3.1.4. Assume that $n = 1$ or the Einstein constant is non-negative. Then the second variation of the total Q -prime curvature is non-positive; that is,*

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \bar{Q}'(M_t) \leq 0.$$

Moreover, the equality holds if and only if $(M_t)_{t \in (-1,1)}$ is infinitesimally trivial as a deformation of CR structures.

On the other hand, the conclusion of Theorem 3.1.5 does not hold for Sasakian η -Einstein manifolds of dimension greater than three and with negative Einstein constant.

Theorem 3.1.6. *For each integer $n \geq 2$, there exist a closed Sasakian η -Einstein manifold of dimension $2n+1$ with negative Einstein constant and an infinitesimally non-trivial smooth deformation such that the second variation of the total Q -prime curvature along this deformation is equal to zero. If n is even, one can find also a Sasakian η -Einstein manifold S and a smooth deformation of S such that the second variation of the total Q -prime curvature along this deformation is positive.*

This chapter is organized as follows. Section 3.2 is devoted to the construction of a Fefferman defining function for Sasakian η -Einstein manifolds. In Section 3.3, we provide a proof of Theorems 3.1.1 and 3.1.2. Section 3.4 deals with the variation of the total Q -prime curvature. In Section 3.5, we compute the total Q -prime curvature for some Sasakian η -Einstein manifolds.

3.2. Construction of Fefferman defining function

In this section, we construct a Fefferman defining function for Sasakian η -Einstein manifolds. To this end, we first construct a “good” defining function ρ_S of S in X . From this defining function, we obtain a flat Hermitian metric \mathbf{h}_S of K_X , and the desired Fefferman defining function ρ_S is given as the product $\rho_S \cdot \mathbf{h}_S^{1/(n+2)}$.

Define a smooth function ψ_λ on \mathbb{R} by

$$\psi_\lambda(x) = \begin{cases} \lambda^{-1} (\exp(\lambda x) - 1) & \text{if } \lambda \neq 0, \\ x & \text{if } \lambda = 0. \end{cases}$$

It can be seen that

$$\psi'_\lambda = 1 + \lambda \psi_\lambda, \quad \psi''_\lambda = \lambda \psi'_\lambda,$$

and

$$\rho_S = \psi_\lambda(\log r^2) \in C^\infty(X)$$

is a defining function of S normalized by η .

Proposition 3.2.1. *The defining function ρ_S satisfies the equation*

$$dd^c \log \mathcal{J}_z[\rho_S] = 0,$$

where $z = (z^1, \dots, z^{n+1})$ is a local coordinate of X and

$$\mathcal{J}_z[\phi] = -\det \begin{pmatrix} \phi & \partial\phi/\partial z^a \\ \partial\phi/\partial \bar{z}^b & \partial^2\phi/\partial z^a\partial \bar{z}^b \end{pmatrix}.$$

PROOF. To simplify notation, we write $\partial_a = \partial/\partial z^a$ and $\partial_{\bar{a}} = \partial^2/\partial z^a\partial \bar{z}^b$.

$$\begin{aligned} \mathcal{J}_z[\rho_S] &= -\det \begin{pmatrix} \rho_S & \partial_a \rho_S \\ \partial_{\bar{b}} \rho_S & \partial_{\bar{a}\bar{b}} \rho_S \end{pmatrix} \\ &= -\det \begin{pmatrix} \rho_S & (1 + \lambda \rho_S) \partial_a \log r^2 \\ \partial_{\bar{b}} \log r^2 & (1 + \lambda \rho_S) \partial_{\bar{a}\bar{b}} \log r^2 \end{pmatrix} \\ &= -(1 + \lambda \rho_S)^{n+1} \det \begin{pmatrix} \rho_S & \partial_a \log r^2 \\ \partial_{\bar{b}} \log r^2 & \partial_{\bar{a}\bar{b}} \log r^2 \end{pmatrix} \\ &= -(1 + \lambda \rho_S)^{n+1} r^{-2(n+1)} \det \begin{pmatrix} \rho_S & \partial_a r^2 \\ (1 + \rho_S) r^{-2} \partial_{\bar{b}} r^2 & \partial_{\bar{a}\bar{b}} r^2 \end{pmatrix}. \end{aligned}$$

Since $r^2/2$ is a Kähler potential of \bar{g} , we have $\partial_{\bar{a}\bar{b}} r^2 = 2\bar{g}(\partial_a, \partial_{\bar{b}})$. Hence

$$\begin{aligned} \mathcal{J}_z[\rho_S] &= -(1 + \lambda \rho_S)^{n+1} r^{-2(n+1)} [\rho_S - (1 + \rho_S)(2r^2)^{-1} \|\partial r^2\|_{\bar{g}}^2] \det(\partial_{\bar{a}\bar{b}} r^2) \\ &= (1 + \lambda \rho_S)^{n+1} r^{-2(n+1)} \det(\partial_{\bar{a}\bar{b}} r^2), \end{aligned}$$

where the last equality follows from $\|\partial r^2\|_{\bar{g}}^2 = 2r^2$. From (0.6.2), it follows that

$$\begin{aligned} -dd^c \log \mathcal{J}_z[\rho_S] &= -(n+1)(dd^c \log(1 + \lambda \rho_S) - dd^c \log r^2) - dd^c \log \det(\partial_{\bar{a}\bar{b}} r^2) \\ &= -(n+1)d(\lambda d^c \log r^2) + (n+1)\lambda dd^c \log r^2 \\ &= 0. \end{aligned}$$

This proves the statement. \square

Next, we construct a flat Hermitian metric of K_X by using ρ_S .

Lemma 3.2.2. *For each point $p \in X$, there exists a local coordinate z near p such that $\mathcal{J}_z[\rho_S] = 1$. Moreover, if $w = F(z)$ is also such a local coordinate, then $\det F'$ is a locally constant function whose absolute value is equal to one, where F' is the holomorphic Jacobi matrix of F .*

Definition 3.2.3. A local coordinate z of X is called a *flat local coordinate* if $\mathcal{J}_z[\rho_S] = 1$ holds.

PROOF OF LEMMA 3.2.2. Take a local coordinate $w = (w^1, \dots, w^{n+1})$ near p . From Proposition 3.2.1, $\log \mathcal{J}_w[\rho_S]$ is a pluriharmonic function. We may assume that this is the real part of a holomorphic function f ; that is,

$$\mathcal{J}_w[\rho_S] = e^{\operatorname{Re} f},$$

if we take a sufficiently small neighborhood of p . Take a holomorphic function g such that $\partial g/\partial w^1 = e^{f/2}$. In general, for another local coordinate $w' = G(w)$ of X ,

$$(3.2.1) \quad \mathcal{J}_{w'}[\phi] = |\det G'|^{-2} \mathcal{J}_w[\phi]$$

holds. The new local coordinate $z = (z^1 = g(w), z^2 = w^2, \dots, z^{n+1} = w^{n+1})$ satisfies $\mathcal{J}_z[\rho_S] = 1$. Thus we have a flat local coordinate at p . The second statement follows from (3.2.1) and the fact that a holomorphic function whose absolute value is locally constant is locally constant. \square

Corollary 3.2.4. *There exists a unique flat Hermitian metric \mathbf{h}_S on K_X such that $dz^1 \wedge \cdots \wedge dz^{n+1}$ is a local orthonormal frame of K_X for any flat local coordinate z , or equivalently, \mathbf{h}_S is written as $|z^0|^{2(n+2)}$, where z^0 is a branched fiber coordinate with respect to z .*

Proposition 3.2.5. *The defining function $\rho_S = \rho_S \cdot \mathbf{h}_S^{1/(n+2)} \in \tilde{\mathcal{E}}(1)$ of $\mathcal{S} = \pi_{\mathcal{X}}^{-1}(S)$ satisfies*

$$(dd^c \rho_S)^{n+2} = k_n \text{vol}_{\mathcal{X}}.$$

In particular, the obstruction function \mathcal{O} vanishes on \mathcal{X} .

PROOF. Take a flat local coordinate (z^1, \dots, z^{n+1}) and a branched fiber coordinate z^0 with respect to z . Then the volume form $\text{vol}_{\mathcal{X}}$ is written as

$$\text{vol}_{\mathcal{X}} = (\sqrt{-1})^{n+2} (n+2)^2 |z^0|^{2(n+1)} dz^0 \wedge d\bar{z}^0 \wedge \cdots \wedge dz^{n+1} \wedge d\bar{z}^{n+1}.$$

On the other hand, the $(n+2, n+2)$ -form $(dd^c \rho_S)^{n+2}$ is of the form

$$(\sqrt{-1})^{n+2} (n+2)! \det(\partial^2 \rho_S / \partial z^A \partial \bar{z}^B) dz^0 \wedge d\bar{z}^0 \wedge \cdots \wedge dz^{n+1} \wedge d\bar{z}^{n+1}.$$

Hence it suffices to show that

$$\det(\partial^2 \rho_S / \partial z^A \partial \bar{z}^B) = -|z^0|^{2(n+1)},$$

which follows from the computation below:

$$\begin{aligned} \det(\partial^2 \rho_S / \partial z^A \partial \bar{z}^B) &= \det \begin{pmatrix} \rho_S & z^0 \partial_a \rho_S \\ \bar{z}^0 \partial_{\bar{b}} \rho_S & |z^0|^2 \partial_{\bar{a}\bar{b}} \rho_S \end{pmatrix} \\ &= |z^0|^{2(n+1)} \det \begin{pmatrix} \rho_S & \partial_a \rho_S \\ \partial_{\bar{b}} \rho_S & \partial_{\bar{a}\bar{b}} \rho_S \end{pmatrix} \\ &= -|z^0|^{2(n+1)} \mathcal{J}_z[\rho_S] \\ &= -|z^0|^{2(n+1)}. \end{aligned}$$

Note that the last equality is a consequence of the definition of a flat local coordinate. \square

3.3. Proofs of factorization theorems

This section is devoted to the proofs of Theorems 3.1.1 and 3.1.2, product formulas for CR invariant powers of the sub-Laplacian and the P -prime operator. Throughout this section, we use the ambient metric $\mathbf{g} = \mathbf{g}[\rho_S]$ for ρ_S defined in Proposition 3.2.5.

Take a flat local coordinate z of X and let z^0 be a branched fiber coordinate with respect to z . For $(v, v') \in \mathbb{R}^2$, the multiplication by $(z^0)^v (\bar{z}^0)^{v'}$ defines an operator

$$\mathbf{M}_{v, v'} : \tilde{\mathcal{E}}(w, w') \rightarrow \tilde{\mathcal{E}}(w+v, w'+v').$$

This operator depends on the choice of z and z^0 in general, but the difference between two such z and z^0 is given by the multiplication by a locally constant function with its absolute value one. Note that $\mathbf{M}_{v, v}$ coincides with the multiplication by $\mathbf{h}_S^{v/(n+2)}$, and so independent of the choice of z and z^0 .

Now we introduce $P_{w, w'}$ and P'_η that appear in the statements of Theorems 3.1.1 and 3.1.2.

Definition 3.3.1. For $(w, w') \in \mathbb{R}^2$ with $k = w + w' + n + 1 \in \mathbb{N}_+$, a differential operator $P_{w, w'} : C^\infty(S) \rightarrow C^\infty(S)$ is defined by

$$P_{w, w'} = \mathbf{M}_{k-w, k-w'} \mathbf{P}_{w, w'} \mathbf{M}_{w, w'}.$$

Similarly, $P'_\eta : \mathcal{P} \rightarrow C^\infty(S)$ is defined by

$$P'_\eta = \mathbf{M}_{n+1, n+1} \mathbf{P}'_\eta.$$

These are independent of the choice of a flat local coordinate z and a branched fiber coordinate z^0 .

We need to study properties of Δ for the proofs of Theorems 3.1.1 and 3.1.2. To this end, we introduce the differential operator

$$C = [\Delta, M_{1,0}].$$

The operators Δ , $M_{v,v'}$, and C satisfy the following commutation relations.

Lemma 3.3.2.

$$\begin{aligned} [\Delta, M_{v,v'}] &= vM_{v-1,v'}C + v'M_{v,v'-1}\bar{C} + 2vv'\lambda M_{v-1,v'-1}, \\ [C, M_{v,v'}] &= 2v'\lambda M_{v,v'-1}, \quad [\bar{C}, M_{v,v'}] = 2v\lambda M_{v-1,v'}, \\ [\Delta, C] &= [\Delta, \bar{C}] = 0. \end{aligned}$$

PROOF. We first note that

$$[\bar{C}, M_{v,v'}] = \overline{[C, M_{v',v}]}, \quad [\Delta, \bar{C}] = \overline{[\Delta, C]},$$

and so it suffices to compute $[\Delta, M_{v,v'}]$, $[C, M_{v,v'}]$ and $[\Delta, C]$. Define a $(1,0)$ -vector field Z_{n+1} on X by

$$Z_{n+1} = \frac{1}{2}(r\partial_r - \sqrt{-1}\xi).$$

A direct computation shows that $[Z_{n+1}, \bar{W}] \in \Gamma(T^{0,1}X)$ for any $\bar{W} \in \Gamma(T^{0,1}X)$, which implies that Z_{n+1} is a holomorphic vector field. A local frame (Z_α) of $T^{1,0}S$ induces the frame (Z_0, Z_α, Z_{n+1}) of $T^{1,0}\mathcal{X}$. With this frame, the matrix representations of g and its inverse are given by

$$g_{A\bar{B}} = |z^0|^2 \begin{pmatrix} \rho_S & 0 & 1 + \lambda\rho_S \\ 0 & (1 + \lambda\rho_S)l_{\alpha\bar{\beta}} & 0 \\ 1 + \lambda\rho_S & 0 & \lambda(1 + \lambda\rho_S) \end{pmatrix},$$

and

$$(3.3.1) \quad g^{A\bar{B}} = |z^0|^{-2} \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & (1 + \lambda\rho_S)^{-1}l^{\alpha\bar{\beta}} & 0 \\ 1 & 0 & -\rho_S(1 + \lambda\rho_S)^{-1} \end{pmatrix}.$$

Denote by φ the holomorphic function z^0 for simplicity. Since the Laplacian Δ is of the form $-2\nabla^A\nabla_A$,

$$C = -2\varphi_A\nabla^A, \quad \bar{C} = -2\bar{\varphi}^A\nabla_A$$

Hence

$$[C, M_{v,v'}] = -2v'\varphi^v\bar{\varphi}^{v'-1}\varphi_A\bar{\varphi}^A = 2v'\lambda M_{v,v'-1}.$$

Here we use the fact that $\varphi_A\bar{\varphi}^A = -\lambda$, which follows from (3.3.1). Similarly, we have

$$\begin{aligned} [\Delta, M_{v,v'}] &= -2v\varphi^{v-1}\bar{\varphi}^{v'}\varphi_A\nabla^A - 2v'\varphi^v\bar{\varphi}^{v'-1}\bar{\varphi}^A\nabla_A - 2vv'\varphi^{v-1}\bar{\varphi}^{v'-1}\varphi_A\bar{\varphi}^A \\ &= vM_{v-1,v'}C + v'M_{v,v'-1}\bar{C} + 2vv'\lambda M_{v-1,v'-1}. \end{aligned}$$

It remains to show $[\Delta, C] = 0$.

$$\begin{aligned} \Delta C &= 4\nabla_B(\varphi_A\nabla^B\nabla^A) \\ &= 4[\varphi_{AB}\nabla^A\nabla^B + \varphi_A\nabla_B\nabla^B\nabla^A] \\ &= 4\varphi_{AB}\nabla^A\nabla^B + C\Delta; \end{aligned}$$

the last equality holds since \mathbf{g} is Ricci-flat. Therefore, it is sufficient to show that $\varphi_{AB} = 0$. From definition,

$$\varphi_{AB} = Z_B Z_A \varphi - (\nabla_{Z_B} Z_A) \varphi,$$

and

$$\mathbf{g}(\nabla_{Z_B} Z_A, Z_{\bar{C}}) = Z_B(\mathbf{g}(Z_A, Z_{\bar{C}})) - \mathbf{g}(Z_A, [Z_B, Z_{\bar{C}}]).$$

Since $\varphi = z^0$, we need only to consider the Z_0 -component of $\nabla_{Z_B} Z_A$. Hence it is enough to compute the value $\mathbf{g}(\nabla_{Z_B} Z_A, Z_{\bar{C}})$ for $C = 0$ or $n + 1$ from the matrix representation of \mathbf{g} . In this case,

$$\mathbf{g}(\nabla_{Z_B} Z_A, Z_{\bar{C}}) = Z_B(\mathbf{g}(Z_A, Z_{\bar{C}})),$$

since the $(0, 1)$ -vector fields $Z_{\bar{0}}$ and $Z_{\overline{n+1}}$ are anti-holomorphic. Under these observations, a direct calculation shows $Z_B Z_A \varphi = (\nabla_{Z_B} Z_A) \varphi$. \square

These commutation relations yield some non-trivial equalities for powers of the Laplacian Δ .

Proposition 3.3.3. *For $k \in \mathbb{N}_+$,*

$$(3.3.2) \quad \Delta^k = M_{-k-1,0}(M_{2,0}\Delta)^k M_{-k+1,0}.$$

Moreover if $k \geq 2$, then

$$(3.3.3) \quad \Delta^k = M_{-1,-1}\Delta^{k-2}M_{0,k-1}\Delta M_{k,-k+2}\Delta M_{-k+1,0}.$$

PROOF. We first prove (3.3.2) by induction in k . The case $k = 1$ is trivial. Assume that (3.3.2) holds for k . Then

$$\begin{aligned} & M_{-k-2,0}(M_{2,0}\Delta)^{k+1}M_{-k,0} \\ &= M_{-1,0}[M_{-k-1,0}(M_{2,0}\Delta)^k M_{-k+1,0}][M_{k+1,0}\Delta M_{-k,0}] \\ &= M_{-1,0}\Delta^k(M_{1,0}\Delta - kC) \\ &= \left(\Delta^k + \sum_{j=0}^{k-1} M_{-1,0}\Delta^{k-j-1}C\Delta^j \right) \Delta - kM_{-1,0}\Delta^k C \\ &= (\Delta^k + kM_{-1,0}\Delta^{k-1}C)\Delta - kM_{-1,0}\Delta^k C \\ &= \Delta^{k+1}. \end{aligned}$$

This proves (3.3.2) for $k + 1$.

Similarly, we show (3.3.3) by induction in k . If $k = 2$,

$$\begin{aligned} M_{-1,-1}M_{0,1}\Delta M_{2,0}\Delta M_{-1,0} &= M_{-1,0}\Delta M_{1,0}\Delta - M_{-1,0}\Delta C \\ &= \Delta^2 + M_{-1,0}C\Delta - M_{-1,0}\Delta C \\ &= \Delta^2. \end{aligned}$$

Assume that (3.3.3) holds for k . From Lemma 3.3.2, we obtain

$$\begin{aligned} & [\Delta, M_{0,k-1}]M_{0,1}\Delta M_{0,-k+1} + M_{0,k-1}\Delta M_{0,1}[\Delta, M_{0,-k+1}] \\ &= (k-1)M_{0,k-2}\bar{C}M_{0,1}\Delta M_{0,-k+1} + (-k+1)M_{0,k-1}\Delta M_{0,-k+1}\bar{C} \\ &= 0, \\ & [\Delta, M_{k,0}]M_{1,0}\Delta M_{-k,0} + M_{k,0}\Delta M_{1,0}[\Delta, M_{-k,0}] \\ &= kM_{k-1,0}CM_{1,0}\Delta M_{-k,0} - kM_{k,0}\Delta M_{-k,0}C \\ &= 0. \end{aligned}$$

Hence

$$\begin{aligned}
& M_{-1,-1} \Delta^{k-1} M_{0,k} \Delta M_{k+1,-k+1} \Delta M_{-k,0} \\
&= M_{-1,-1} \Delta^{k-2} ([\Delta, M_{0,k-1}] M_{0,1} \Delta M_{0,-k+1}) M_{k+1,0} \Delta M_{-k,0} \\
&\quad + M_{-1,-1} \Delta^{k-2} (M_{0,k-1} \Delta M_{0,1} [\Delta, M_{0,-k+1}]) M_{k+1,0} \Delta M_{-k,0} \\
&\quad + M_{-1,-1} \Delta^{k-2} M_{0,k-1} \Delta M_{0,-k+2} ([\Delta, M_{k,0}] M_{1,0} \Delta M_{-k,0}) \\
&\quad + M_{-1,-1} \Delta^{k-2} M_{0,k-1} \Delta M_{0,-k+2} (M_{k,0} \Delta M_{1,0} [\Delta, M_{-k,0}]) \\
&\quad + M_{-1,-1} \Delta^{k-2} M_{0,k-1} \Delta M_{k,-k+2} \Delta M_{-k+1,0} \Delta \\
&= \Delta^{k+1}.
\end{aligned}$$

This proves (3.3.3) for $k+1$. \square

PROOF OF THEOREM 3.1.1. Take an arbitrary $\tilde{\mathbf{f}} \in \tilde{\mathcal{E}}(w, w')$. For computation of $\Delta \tilde{\mathbf{f}}$, we need only to consider $\partial \bar{\partial} \tilde{\mathbf{f}}(Z_A, Z_{\bar{B}})$ and (A, \bar{B}) with $\mathbf{g}^{A\bar{B}} \neq 0$. Since Z_0 and Z_{n+1} are holomorphic vector fields,

$$\begin{aligned}
\partial \bar{\partial} \tilde{\mathbf{f}}(Z_0, Z_{\bar{0}}) &= w w' \tilde{\mathbf{f}}, & \partial \bar{\partial} \tilde{\mathbf{f}}(Z_0, Z_{\bar{n+1}}) &= w Z_{\bar{n+1}} \tilde{\mathbf{f}}, \\
\partial \bar{\partial} \tilde{\mathbf{f}}(Z_{n+1}, Z_{\bar{0}}) &= w' Z_{n+1} \tilde{\mathbf{f}}, & \partial \bar{\partial} \tilde{\mathbf{f}}(Z_{n+1}, Z_{\bar{n+1}}) &= \frac{1}{2} (Z_{n+1} Z_{\bar{n+1}} + Z_{\bar{n+1}} Z_{n+1}) \tilde{\mathbf{f}}.
\end{aligned}$$

On the other hand, the commutator $[Z_\alpha, Z_{\bar{\beta}}]$ is equal to

$$\nabla_{Z_\alpha} Z_{\bar{\beta}} - \nabla_{Z_{\bar{\beta}}} Z_\alpha + l_{\alpha\bar{\beta}} (Z_{n+1} - Z_{\bar{n+1}}).$$

Thus we have

$$\begin{aligned}
\partial \bar{\partial} \tilde{\mathbf{f}}(Z_\alpha, Z_{\bar{\beta}}) &= Z_\alpha Z_{\bar{\beta}} \tilde{\mathbf{f}} - \bar{\partial} \tilde{\mathbf{f}}([Z_\alpha, Z_{\bar{\beta}}]) \\
&= \frac{1}{2} (Z_\alpha Z_{\bar{\beta}} + Z_{\bar{\beta}} Z_\alpha) \tilde{\mathbf{f}} + \frac{1}{2} \partial \tilde{\mathbf{f}}([Z_\alpha, Z_{\bar{\beta}}]) - \frac{1}{2} \bar{\partial} \tilde{\mathbf{f}}([Z_\alpha, Z_{\bar{\beta}}]) \\
&= \frac{1}{2} (Z_\alpha Z_{\bar{\beta}} - \nabla_{Z_\alpha} Z_{\bar{\beta}} + Z_{\bar{\beta}} Z_\alpha - \nabla_{Z_{\bar{\beta}}} Z_\alpha) \tilde{\mathbf{f}} + \frac{1}{2} l_{\alpha\bar{\beta}} (Z_{n+1} + Z_{\bar{n+1}}) \tilde{\mathbf{f}}.
\end{aligned}$$

Let $f \in C^\infty(S)$ and \tilde{f} be a smooth extension of f to X . Then it follows that

$$\begin{aligned}
(3.3.4) \quad (M_{1-w, 1-w'} \Delta M_{w, w'} \tilde{f})|_S &= (\Delta_b + 2w w' \lambda) \tilde{f}|_S + (-n - 2w') (Z_{n+1} \tilde{f})|_S \\
&\quad + (-n - 2w) (Z_{\bar{n+1}} \tilde{f})|_S.
\end{aligned}$$

We first show (3.1.1) for the case of $k=1$. In this case, there exists a unique real number μ such that

$$(w, w') = (-(\mu + n)/2, (\mu - n)/2).$$

To simplify the notation, set

$$\Delta_\mu = M_{1-w, 1-w'} \Delta M_{w, w'}.$$

From (3.3.4), we obtain

$$P_{w, w'} f = (\Delta_\mu \tilde{f})|_S = \left[\Delta_b + \sqrt{-1} \mu \xi + \frac{1}{2} (n - \mu)(n + \mu) \lambda \right] f = L_\mu f,$$

which proves (3.1.1) for $k=1$.

Next, we consider the case of general $k \in \mathbb{N}_+$. By (3.3.2),

$$M_{k-w, k-w'} \Delta^k M_{w, w'} = \Delta_{w'-w+k-2(k-1)-1} \cdots \Delta_{w'-w+k-2j-1} \cdots \Delta_{w'-w+k-0-1}.$$

Thus we have

$$\begin{aligned}
P_{w, w'} f &= (\Delta_{w'-w+k-2(k-1)-1} \cdots \Delta_{w'-w+k-2j-1} \cdots \Delta_{w'-w+k-0-1} \tilde{f})|_S \\
&= L_{w'-w+k-2(k-1)-1} \cdots L_{w'-w+k-2j-1} \cdots L_{w'-w+k-0-1} f,
\end{aligned}$$

which completes the proof. \square

The proof of Theorem 3.1.2 is more complicated than Theorem 3.1.1. We need a new operator acting on CR pluriharmonic functions.

Proposition 3.3.4. *Let Υ be a CR pluriharmonic function on S and $\tilde{\Upsilon}$ be a smooth extension of Υ that is pluriharmonic on the pseudoconvex side. Then the function $\Lambda\tilde{\Upsilon}$ defined by*

$$\Lambda\tilde{\Upsilon} = M_{1,n+1}\Delta M_{n+1,-n+1}\Delta M_{-n,0}(\tilde{\Upsilon} \log |z^0|^2)$$

is an element of $\tilde{\mathcal{E}}(0)$ modulo a term that vanishes to infinite order at \mathcal{S} . Moreover, $(\Lambda\tilde{\Upsilon})|_{\mathcal{S}} \in \mathcal{E}(0)$ is determined only by Υ and given by

$$(\Lambda\tilde{\Upsilon})|_{\mathcal{S}} = -\frac{4}{n}\Delta_b(\Delta_b + n^2\lambda)\Upsilon.$$

PROOF. First, note that $Z_{n+1}Z_{n+1}^{-1}\tilde{\Upsilon}$ and $\Delta\tilde{\Upsilon}$ vanish to infinite order at S since $\tilde{\Upsilon}$ is pluriharmonic on the pseudoconvex side. In particular, (3.3.4) for $(w, w') = (0, 0)$ gives that

$$[(Z_{n+1} + Z_{n+1}^{-1})\tilde{\Upsilon}]|_{\mathcal{S}} = n^{-1}\Delta_b\tilde{\Upsilon}.$$

In the following, we compute modulo functions that vanish to infinite order at \mathcal{S} .

$$\begin{aligned} & \Delta M_{-n,0}(\tilde{\Upsilon} \log |z^0|^2) \\ &= -2\langle d\tilde{\Upsilon}, d((z^0)^{-n} \log |z^0|^2) \rangle_{\mathbf{g}} + \tilde{\Upsilon}\Delta((z^0)^{-n} \log |z^0|^2) \\ &= 2M_{-n-1,-1}[n(Z_{n+1}^{-1}\tilde{\Upsilon}) \log |z^0|^2 - (Z_{n+1} + Z_{n+1}^{-1})\tilde{\Upsilon} - n\lambda\tilde{\Upsilon}]. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ is the inner product on $T^*\mathcal{X}$ induced from \mathbf{g} . Hence

$$\begin{aligned} & M_{1,n+1}\Delta M_{n+1,-n+1}\Delta M_{-n,0}(\tilde{\Upsilon} \log |z^0|^2) \\ &= 2M_{1,n+1}\Delta M_{0,-n}[n(Z_{n+1}^{-1}\tilde{\Upsilon}) \log |z^0|^2 - (Z_{n+1} + Z_{n+1}^{-1})\tilde{\Upsilon} - n\lambda\tilde{\Upsilon}] \\ &= -4n(Z_{n+1}^2 + Z_{n+1}^2)\tilde{\Upsilon} - 4n^2\lambda(Z_{n+1} + Z_{n+1}^{-1})\tilde{\Upsilon} \\ &= -4n(Z_{n+1} - Z_{n+1}^{-1})^2\tilde{\Upsilon} - 4n^2\lambda(Z_{n+1} + Z_{n+1}^{-1})\tilde{\Upsilon} \\ &= 4n\xi^2\tilde{\Upsilon} - 4n^2\lambda(Z_{n+1} + Z_{n+1}^{-1})\tilde{\Upsilon}, \end{aligned}$$

which is an element of $\tilde{\mathcal{E}}(0)$. Moreover, on \mathcal{S} ,

$$(\Lambda\tilde{\Upsilon})|_{\mathcal{S}} = -\frac{4}{n}\Delta_b(\Delta_b + n^2\lambda)\Upsilon;$$

here we use the fact that $\Delta_b^2 + n^2\xi^2$ annihilates CR pluriharmonic functions on Sasakian manifolds. \square

PROOF OF THEOREM 3.1.2. From (3.3.3), it follows that

$$M_{n+1,n+1}\Delta^{n+1} = (M_{n,n}\Delta^{n-1}M_{-1,-1})(M_{1,n+1}\Delta M_{n+1,-n+1}\Delta M_{-n,0}).$$

Therefore, we have

$$\begin{aligned} P'_\eta\Upsilon &= -[(M_{n,n}\Delta^{n-1}M_{-1,-1})(M_{1,n+1}\Delta M_{n+1,-n+1}\Delta M_{-n,0})(\tilde{\Upsilon} \log |z^0|^2)]|_{\mathcal{S}} \\ &= -[(M_{n,n}\Delta^{n-1}M_{-1,-1})(\Lambda\tilde{\Upsilon})]|_{\mathcal{S}} \\ &= -P_{-1,-1}(\Lambda\tilde{\Upsilon})|_{\mathcal{S}} \\ &= \frac{4}{n}P_{-1,-1}\Delta_b(\Delta_b + n^2\lambda)\Upsilon \\ &= \frac{4}{n}\Delta_b(\Delta_b + n^2\lambda)P_{-1,-1}\Upsilon. \end{aligned}$$

It suffices to show that

$$P_{-1,-1}\Upsilon = \frac{2^{n-1}(n-1)!}{n^{n-1}} \prod_{j=1}^{n-1} (\Delta_b + nj\lambda)\Upsilon.$$

From Theorem 3.1.1, we obtain

$$P_{-1,-1} = \prod_{j=1}^{n-1} L_{n-2j}.$$

Since Υ is annihilated by $\Delta_b^2 + n^2\xi^2$,

$$\begin{aligned} L_{n-2j}L_{n-2(n-j)}\Upsilon &= (\Delta_b + 2j(n-j)\lambda)^2\Upsilon + (n-2j)^2\xi^2\Upsilon \\ &= (\Delta_b + 2j(n-j)\lambda)^2\Upsilon - \frac{(n-2j)^2}{n^2}\Delta_b^2\Upsilon \\ &= \left[\frac{2(n-j)}{n}(\Delta_b + nj\lambda) \right] \left[\frac{2(n-(n-j))}{n}(\Delta_b + n(n-j)\lambda) \right] \Upsilon. \end{aligned}$$

Thus we have

$$\begin{aligned} P_{-1,-1}\Upsilon &= \left[\prod_{j=1}^{n-1} \frac{2(n-j)}{n}(\Delta_b + nj\lambda) \right] \Upsilon \\ &= \frac{2^{n-1}(n-1)!}{n^{n-1}} \prod_{j=1}^{n-1} (\Delta_b + nj\lambda)\Upsilon, \end{aligned}$$

which completes the proof. \square

3.4. Variation of total Q -prime curvature

In this section, we consider the first and the second variation of the total Q -prime curvature at Sasakian η -Einstein manifolds. Throughout this section, we assume that S is closed. Let $(M_t)_{t \in (-1,1)}$ be a smooth family of closed strictly pseudoconvex real hypersurfaces in X with $M_0 = S$.

The first variation of the total Q -prime curvature can be computed via (0.5.1).

PROOF OF PROPOSITION 3.1.4. Since K_X has a flat Hermitian metric \mathbf{h}_S , we can apply (0.5.1). Proposition 3.1.4 follows from the vanishing of the obstruction function (Proposition 3.2.5). \square

Next, consider the second variation of the total Q -prime curvature. Take a Fefferman defining function ρ_t of $\mathcal{M}_t = \pi_X^{-1}(M_t)$ that is smooth in t and coincides with ρ_S constructed in Proposition 3.2.5 at $t = 0$. Then the second variation of $\overline{Q}'(M_t)$ satisfies

$$\frac{d^2}{dt^2} \Big|_{t=0} \overline{Q}'(M_t) = c'_n \int_M \varphi(P_{1,1}\varphi) \eta \wedge (d\eta)^n,$$

where $\varphi = (d/dt)|_{t=0}(\rho_t \cdot \mathbf{h}_S^{-1/(n+2)})|_S \in C^\infty(S)$. Therefore, it is enough to study spectral properties of $P_{1,1}$ for proofs of Theorems 3.1.5 and 3.1.6.

Before studying $P_{1,1}$, we consider a relation between D_η introduced in Section 0.4 and L_μ .

Lemma 3.4.1. *The operator $16D_\eta^*D_\eta$ coincides with $L_{n+2}L_n$. In particular, the operator $L_{n+2}L_n$ is a non-negative operator and its kernel coincides with $\ker D_\eta$. Similarly, the operator $L_{-n-2}L_{-n}$ is non-negative and its kernel is equal to that of \overline{D}_η .*

PROOF. Since $L_{-n-2}L_{-n}$ is the complex conjugate of $L_{n+2}L_n$, it suffices to prove the lemma for D_η and $L_{n+2}L_n$. Since the Tanaka-Webster torsion for η vanishes, the operator D_η is given by $2(D_\eta F)_\alpha^\beta = F_\alpha^\beta$. From this expression, it follows that

$$16D_\eta^*D_\eta F = 4F_\alpha^{\beta\bar{\alpha}}{}_\beta.$$

On the other hand,

$$\begin{aligned} F_\alpha^{\beta\bar{\alpha}} &= F_\alpha^{\bar{\alpha}\beta} - \sqrt{-1}F_0^\beta + R_\alpha^{\bar{\delta}\beta\bar{\alpha}}F_\delta \\ &= F_\alpha^{\bar{\alpha}\beta} - \sqrt{-1}F_0^\beta + (n+1)\lambda F^\beta. \end{aligned}$$

Hence

$$\begin{aligned} 4F_\alpha^{\beta\bar{\alpha}}{}_\beta &= 4(F_\alpha^{\bar{\alpha}\beta}{}_\beta - \sqrt{-1}F_{0\beta}^\beta + (n+1)\lambda F_\beta^\beta) \\ &= 4(\square_b^2 F + \sqrt{-1}\xi\square_b F - (n+1)\lambda\square_b F) \\ &= L_{n+2}L_n F. \end{aligned}$$

Thus we have $16D_\eta^*D_\eta = L_{n+2}L_n$. In particular, $L_{n+2}L_n$ is non-negative, and its kernel coincides with $\ker D_\eta$. \square

We rewrite Lemma 3.4.1 by using some spectral results on the sub-Laplacian and the Reeb vector field. Since the sub-Laplacian Δ_b is a non-negative subelliptic self-adjoint operator, its spectrum $\sigma(\Delta_b)$ is a discrete subset of $[0, \infty)$, consists only of eigenvalues, and the eigenspace \mathcal{H}_p with eigenvalue $p \in \sigma(\Delta_b)$ is a finite-dimensional subspace in $C^\infty(S)$. Note that $\mathcal{H}_0 = \mathbb{C}$. Moreover, the vector field $\sqrt{-1}\xi$ is formally self-adjoint and commutes with the sub-Laplacian. Hence each eigenspace \mathcal{H}_p is decomposed into the orthogonal direct sum of $\mathcal{H}_{p,q}$, where q is a real number and

$$\mathcal{H}_{p,q} = \{f \in \mathcal{H}_p \mid \sqrt{-1}\xi f = qf\}.$$

In what follows, let $(p, q) \in \mathbb{R}^2$ with $\mathcal{H}_{p,q} \neq 0$. Note that $p \geq n|q|$ since the operators $2\square_b = \Delta_b + \sqrt{-1}n\xi$ and $2\bar{\square}_b = \Delta_b - \sqrt{-1}n\xi$ are non-negative operators. On $\mathcal{H}_{p,q}$, the operator L_μ coincides with the multiplication by

$$p + \mu q + \frac{1}{2}(n - \mu)(n + \mu)\lambda.$$

From this point of view, Lemma 3.4.1 states that

$$(3.4.1) \quad (p + nq)(p + (n+2)q - 2(n+1)\lambda) \geq 0$$

and

$$(3.4.2) \quad (p - nq)(p - (n+2)q - 2(n+1)\lambda) \geq 0,$$

and the first (resp. second) equality holds if and only if $\mathcal{H}_{p,q}$ is contained in $\ker D_\eta$ (resp. $\ker \bar{D}_\eta$).

Now we return the study of $P_{1,1}$. From Theorem 3.1.1, $P_{1,1}$ coincides with the multiplication by

$$\prod_{j=0}^{n+2} (p + (n+2-2j)q + 2(j-1)(n+1-j)\lambda)$$

on $\mathcal{H}_{p,q}$. Equations (3.4.1) and (3.4.2) give that the quantity

$$\prod_{j=0,1,n+1,n+2} (p + (n+2-2j)q + 2(j-1)(n+1-j)\lambda)$$

is non-negative for $\mathcal{H}_{p,q} \neq 0$, and equal to zero if and only if $\mathcal{H}_{p,q}$ is contained in $\ker D_\eta + \ker \overline{D}_\eta$. Therefore, if $n \geq 2$, the sign of the second variation depends on that of

$$\prod_{j=2}^n (p + (n+2-2j)q + 2(j-1)(n+1-j)\lambda).$$

PROOF OF THEOREM 3.1.5. If $n = 1$, the theorem follows from the above argument. In the following, we consider the case $n \geq 2$. Let $2 \leq j \leq n$. If $\lambda \geq 0$, then

$$\begin{aligned} p + (n+2-2j)q + 2(j-1)(n+1-j)\lambda &\geq p \left(1 - \frac{|n+2-2j|}{n} \right) \\ &\geq 0; \end{aligned}$$

in the first inequality, we use the fact that $p \geq n|q|$. Moreover, the equality holds if and only if $\lambda = 0$ and $p = q = 0$. Therefore, the second variation is always non-positive, and equal to zero if and only if φ is an element of $\ker D_\eta + \ker \overline{D}_\eta$, or equivalently, in $\text{Re ker } D_\eta$, since φ is real-valued. \square

We next show Theorem 3.1.6. Let Σ be a closed Riemann surface of genus two, and g_Σ be a hyperbolic metric on Σ . Then there exists a complex structure on Σ such that g_Σ is Kähler. Consider the n -dimensional complex manifold $Y = \Sigma^n$. The product metric g_Y on Y satisfies $\text{Ric}_{g_Y} = -g_Y$. Moreover, its canonical line bundle $L = K_Y$ has the Hermitian metric h_L induced from g_Y , and $\omega = -dd^c \log h_L$ coincides with the Kähler form of g_Y . Hence the tube S associated with a polarized Kähler-Einstein manifold (Y, L, h_L) is a Sasakian η -Einstein manifold with Einstein constant -1 . Denote by π_1 the composition of the projection $S \rightarrow Y$ and the projection from Y to the first factor Σ . Let $f \in C^\infty(\Sigma)$ be an eigenfunction of Δ_{g_Σ} with eigenvalue p . Then $\pi_1^* f$ is an element of $\mathcal{H}_{p,0}$, and

$$L_{n+2-2j}(\pi_1^* f) = \left(p - \frac{2(j-1)(n+1-j)}{n+1} \right) \pi_1^* f.$$

Proposition 3.4.2. *For any $0 < p < 2$, there exists a hyperbolic metric g_Σ on Σ such that the first positive eigenvalue $\lambda_1(\Delta_{g_\Sigma})$ of Δ_{g_Σ} is equal to p .*

PROOF. Consider the space \mathcal{M}_{-1} of hyperbolic metrics on Σ with C^∞ topology. This space is known to be contractible [Tro92, Section 3.4]; in particular, it is connected. Moreover, the map $g_\Sigma \mapsto \lambda_1(\Delta_{g_\Sigma})$ defines a continuous function on \mathcal{M}_{-1} . Hence it is sufficient to show that

$$\inf_{g_\Sigma \in \mathcal{M}_{-1}} \lambda_1(\Delta_{g_\Sigma}) = 0, \quad \sup_{g_\Sigma \in \mathcal{M}_{-1}} \lambda_1(\Delta_{g_\Sigma}) \geq 2$$

from the intermediate value theorem. Buser [Bus77, Satz 1] has proved the first equality. On the other hand, it is known that there exists a hyperbolic metric on Σ such that its first eigenvalue is greater than 3.83 [Jen84, Section 1.2]; see [SU13, Section 5.3] for a more precise estimate of its value. This completes the proof. \square

PROOF OF THEOREM 3.1.6. Since $0 < 2(n-1)/(n+1) < 2$, we can take a hyperbolic metric g_Σ on Σ such that $\lambda_1(\Delta_{g_\Sigma}) = 2(n-1)/(n+1)$, and a real-valued eigenfunction $0 \neq f$ on Σ with eigenvalue $2(n-1)/(n+1)$. Then $\pi_1^* f$ is not contained in $\text{Re ker } D_\eta$, but $P_{1,1}(\pi_1^* f) = 0$ since $L_{n-2}(\pi_1^* f) = 0$. Therefore, if we take a smooth deformation of S such that $\varphi = f$, the second variation of the total Q -prime curvature along this deformation is equal to zero though this deformation is infinitesimally non-trivial as a deformation of CR structures.

Assume that $n = 2m$ for $m \in \mathbb{N}_+$. Let $0 \neq f \in C^\infty(\Sigma)$ be a real-valued eigenfunction of Δ_{g_Σ} with eigenvalue p . Then

$$\begin{aligned} P_{1,1}(\pi_1^* f) &= \prod_{j=0}^{2m+2} \left(p - \frac{2(j-1)(2m+1-j)}{2m+1} \right) \pi_1^* f \\ &= \left(p - \frac{2m^2}{2m+1} \right) \prod_{j=0}^m \left(p - \frac{2(j-1)(2m+1-j)}{2m+1} \right)^2 \pi_1^* f. \end{aligned}$$

Therefore, if we choose g_Σ and f such that p is sufficiently small, $\pi_1^* f$ is an eigenfunction of $P_{1,1}$ with negative eigenvalue. Thus we obtain a smooth deformation of S with positive second variation. \square

3.5. Computation of total Q -prime curvature

In this section, we compute the total Q -prime curvature for some examples. To this end, we first compute the Q -prime curvature for Sasakian η -Einstein manifolds.

PROOF OF THEOREM 3.1.3. It can be seen that

$$\Delta(\log |z^0|^2)^2 = -2\langle d \log |z^0|^2, d \log |z^0|^2 \rangle_{\mathbf{g}} = 4\lambda |z^0|^{-2}$$

and

$$\Delta |z^0|^{-2l} = -2\langle d(z^0)^{-l}, d(\bar{z}^0)^{-l} \rangle_{\mathbf{g}} = 2l^2 \lambda |z^0|^{-2(l+1)}.$$

Hence

$$Q'_\eta = \frac{1}{2} \Delta^{n+1} (\log |z^0|^2)^2 = 2^{n+1} (n!)^2 \lambda^{n+1} |z^0|^{-2(n+1)},$$

or equivalently, $Q'_\eta = 2^{n+1} (n!)^2 \lambda^{n+1}$. \square

Assume that S is closed. Then the total Q -prime curvature $\bar{Q}'(S)$ of S has the formula

$$\bar{Q}'(S) = 2^{n+1} (n!)^2 \lambda^{n+1} \int_S \eta \wedge (d\eta)^n = 2^{2n+1} (n!)^3 \lambda^{n+1} \text{Vol}(S, g_\eta),$$

where $\text{Vol}(S, g_\eta)$ is the volume of the Riemannian manifold (S, g_η) . We apply this formula to some examples of Sasakian η -Einstein manifolds.

3.5.1. Sasaki-Einstein manifolds. If $(S, T^{1,0}S, \eta)$ is a closed Sasaki-Einstein manifold, the total Q -prime curvature $\bar{Q}'(S)$ is equal to

$$\bar{Q}'(S) = 2^{2n+1} (n!)^3 \text{Vol}(S, g_\eta).$$

Hence it is enough to compute the volume of (S, g_η) .

Example 3.5.1 (S^{2n+1}). Consider the unit sphere S^{2n+1} as in Example 0.6.5. Then its total Q -prime curvature is equal to

$$\bar{Q}'(S^{2n+1}) = 2^{2n+1} (n!)^3 \text{Vol}(S^{2n+1}, g_{\eta_{\text{std}}}) = (4\pi)^{n+1} (n!)^2.$$

Here we use the fact that the metric $g_{\eta_{\text{std}}}$ is equal to that induced from the Euclidean metric on \mathbb{C}^{n+1} .

Example 3.5.2 ($Y^{p,q}$). In the study of the AdS/CFT correspondence, Gauntlett, Martelli, Sparks, and Waldram [GMSW04] have constructed five-dimensional Sasaki-Einstein manifolds $Y^{p,q}$ for coprime positive integers $q < p$, which are diffeomorphic to $S^2 \times S^3$. The total Q -prime curvature of $Y^{p,q}$ is given by

$$\bar{Q}'(Y^{p,q}) = 2^5 (2!)^3 \text{Vol}(Y^{p,q}) = \frac{2^8 q^2 (2p + (4p^2 - 3q^2)^{1/2})}{3p^2 (3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2})} \pi^3.$$

3.5.2. Tubes associated with polarized Kähler-Einstein manifolds.

Let (Y, L, h_L) be a closed n -dimensional polarized Kähler-Einstein manifold with Einstein constant $(n+1)\lambda$, and S be the tube associated with (Y, L, h_L) . We follow the notation in Section 0.6. The total Q -prime curvature $\overline{Q}'(S)$ of this S is given by

$$\begin{aligned}\overline{Q}'(S) &= 2^{n+1}(n!)^2\lambda^{n+1} \int_S \eta \wedge (d\eta)^n \\ &= (4\pi)^{n+1}(n!)^2\lambda^{n+1} \int_Y \left(\frac{\omega}{2\pi}\right)^n.\end{aligned}$$

Since $\omega/2\pi$ is a representative of the first Chern class $c_1(L)$ of L , it follows that

$$\overline{Q}'(S) = (4\pi)^{n+1}(n!)^2\lambda^{n+1} \int_Y c_1(L)^n.$$

Example 3.5.3 (S_d). Let Y_d , S_d , κ_d , and λ_d be as in Example 0.6.7. Then the total Q -prime curvature $\overline{Q}'(S_d)$ of S_d is given by

$$\overline{Q}'(S_d) = (4\pi)^{n+1}(n!)^2\lambda_d^{n+1} \int_{Y_d} \kappa_d^n.$$

Since Y_d is a smooth complex hypersurface in $\mathbb{C}\mathbb{P}^{n+1}$ of degree d , we have

$$\int_{Y_d} \kappa_d^n = d \int_{\mathbb{C}\mathbb{P}^{n+1}} c_1(\mathcal{O}(1))^{n+1} = d.$$

Hence

$$\overline{Q}'(S_d) = (4\pi)^{n+1}(n!)^2d \left(\frac{n+2-d}{n+1}\right)^{n+1}.$$

Burns-Epstein invariant for the tubes associated with polarized Kähler-Einstein manifolds

4.1. Introduction

In Section 3.5, we obtained an explicit formula of the total Q -prime curvature for Sasakian η -Einstein manifolds. In this chapter, we compute another CR invariant, the Burns-Epstein invariant, for the tubes associated with polarized Kähler-Einstein manifolds in terms of characteristic numbers.

Theorem 4.1.1. *Let S be the tube associated with a closed n -dimensional polarized Kähler-Einstein manifold (Y, L, h_L) with Einstein constant $(n+1)\lambda$. The Burns-Epstein invariant $\mu(S)$ of S is given by*

$$(4.1.1) \quad \mu(S) = \sum_{k=0}^n (-\lambda)^{n+1-k} \int_Y c_k(T^{1,0}Y) c_1(L)^{n-k}.$$

The proof of Theorem 4.1.1 given in the next section depends heavily on an expression of μ in terms of the Tanaka-Webster curvature and torsion, and is rather different from the proof of Theorem 3.1.3 in Section 3.5.

Next, consider the “difference” between the Burns-Epstein invariant μ and the total Q -prime curvature \overline{Q}' . We choose the unit sphere S^{2n+1} in \mathbb{C}^{n+1} as a normalization of these invariants. The Burns-Epstein invariant of S^{2n+1} is equal to -1 ; this may be a known fact, but we will derive it from Theorem 4.1.1 in Example 4.2.2. On the other hand, the total Q -prime curvature of S^{2n+1} is given by

$$\overline{Q}'(S^{2n+1}) = (4\pi)^{n+1} (n!)^2;$$

see Example 3.5.1. Therefore, the quantity

$$\mathcal{D}(M) = \mu(M) + \frac{1}{(4\pi)^{n+1} (n!)^2} \overline{Q}'(M)$$

is a global invariant for closed strictly pseudoconvex CR manifolds admitting a pseudo-Einstein contact form, and satisfies $\mathcal{D}(S^{2n+1}) = 0$.

It is non-trivial whether \mathcal{D} is identically zero or not. Actually, it is known that \mathcal{D} vanishes for three-dimensional CR manifolds; see [CY13, Theorem 1.2] and [Hir14, Theorem 6.6]. On the other hand, Hirachi, Marugame, and Matsumoto [HMM17, Theorem 1.3] have proved that \mathcal{D} is non-trivial for the five-dimensional case. As an application of Theorem 4.1.1, we will prove that \mathcal{D} is not identically zero if $n \geq 2$. To this end, we consider S_d as in Example 0.6.7. In Section 4.3, we will show the following

Theorem 4.1.2. *For $n \geq 2$, the invariant $\mathcal{D}(S_d)$ of S_d satisfies*

$$\lim_{d \rightarrow +\infty} \mathcal{D}(S_d) = (-1)^n \infty.$$

In particular, $\mathcal{D}(S_d) \neq 0$ for sufficiently large d .

4.2. Proof of Theorem 4.1.1

Let (Y, L, h_L) be an n -dimensional polarized Kähler-Einstein manifold with Einstein constant $(n+1)\lambda$, and consider the tube S associated with (Y, L, h_L) . We follow the notation in Section 0.6.

PROOF OF THEOREM 4.1.1. In our setting, $\varsigma = \text{Scal}/n(n+1) = \lambda$ and

$$\begin{aligned}\theta_0^\beta &= \lambda\theta^\beta, \quad \theta_\alpha^0 = -l_{\alpha\bar{\gamma}}\theta^{\bar{\gamma}}, \quad \theta_0^0 = \sqrt{-1}\lambda\eta, \\ \Theta_\alpha^\beta &= \Omega_\alpha^\beta + \sqrt{-1}\lambda d\eta\delta_\alpha^\beta - \lambda l_{\alpha\bar{\gamma}}\theta^\beta \wedge \theta^{\bar{\gamma}}, \quad \Theta_\alpha^0 = 0.\end{aligned}$$

Since $\Theta_\alpha^0 = 0$, the $(2n+1)$ -form $\Phi_k^{(1)}$ vanishes identically for any $0 \leq k \leq n$. On the other hand,

$$\begin{aligned}\frac{1}{n!} \left(\frac{\sqrt{-1}}{2\pi} \right)^{n+1} \sum_{k=0}^n \binom{n}{k} \Phi_k^{(0)} &= \left(\frac{\sqrt{-1}}{2\pi} \right)^{n+1} \theta_0^0 \wedge \det \left(\Theta_\alpha^\beta + \theta_\alpha^0 \wedge \theta_0^\beta \right) \\ &= \left(-\frac{\lambda}{2\pi} \eta \right) \wedge \det \left(-\frac{\lambda}{2\pi} d\eta\delta_\alpha^\beta + \frac{\sqrt{-1}}{2\pi} \Omega_\alpha^\beta \right) \\ &= \left(-\frac{\lambda}{2\pi} \eta \right) \wedge p^* \det \left(-\frac{\lambda}{2\pi} \omega\delta_\alpha^\beta + \frac{\sqrt{-1}}{2\pi} \Phi_\alpha^\beta \right).\end{aligned}$$

Hence

$$\begin{aligned}\mu(S) &= \int_S \left(-\frac{\lambda}{2\pi} \eta \right) \wedge p^* \det \left(-\frac{\lambda}{2\pi} \omega\delta_\alpha^\beta + \frac{\sqrt{-1}}{2\pi} \Phi_\alpha^\beta \right) \\ &= -\lambda \int_Y \det \left(-\frac{\lambda}{2\pi} \omega\delta_\alpha^\beta + \frac{\sqrt{-1}}{2\pi} \Phi_\alpha^\beta \right).\end{aligned}$$

Consider the expansion

$$\det \left(t\delta_\alpha^\beta + \frac{\sqrt{-1}}{2\pi} \Phi_\alpha^\beta \right) = \sum_{k=0}^n c_k(\Phi) t^{n-k}.$$

The coefficient $c_k(\Phi)$ is a closed $2k$ -form on Y , and gives a representative of $c_k(T^{1,0}Y) \in H^{2k}(Y, \mathbb{R})$. From this expansion, we obtain

$$\det \left(-\frac{\lambda}{2\pi} \omega\delta_\alpha^\beta + \frac{\sqrt{-1}}{2\pi} \Phi_\alpha^\beta \right) = \sum_{k=0}^n c_k(\Phi) \wedge \left(-\frac{\lambda}{2\pi} \omega \right)^{n-k}.$$

Moreover, the two-form $\omega/2\pi = (\sqrt{-1}/2\pi)\Theta_{h_L}$ is a representative of $c_1(L) \in H^2(Y, \mathbb{R})$. Thus we have

$$\mu(S) = \sum_{k=0}^n (-\lambda)^{n+1-k} \int_Y c_k(T^{1,0}Y) c_1(L)^{n-k},$$

which completes the proof. \square

Remark 4.2.1. From the Einstein condition, it follows that

$$c_1(T^{1,0}Y) = (n+1)\lambda c_1(L).$$

This gives that the formula (4.1.1) is rewritten as follows:

$$(4.2.1) \quad \mu(S) = -\lambda \sum_{k=0}^n \left(-\frac{1}{n+1} \right)^{n-k} \int_Y c_k(T^{1,0}Y) c_1(T^{1,0}Y)^{n-k}.$$

In particular, for the case of $n = 1$ or 2 , we can observe from (4.2.1) that $\mu(S)$ is determined only by λ and the topology of Y . Actually, if $n = 1$,

$$\mu(S) = -\lambda \left[-\frac{1}{2} \int_Y c_1(T^{1,0}Y) + \int_Y c_1(T^{1,0}Y) \right] = -\frac{\lambda}{2} \chi(Y),$$

where $\chi(Y)$ is the Euler characteristic of Y . If $n = 2$,

$$\begin{aligned}\mu(S) &= -\lambda \sum_{k=0}^2 \left(-\frac{1}{3}\right)^{2-k} \int_Y c_k(T^{1,0}Y) c_1(T^{1,0}Y)^{2-k} \\ &= -\lambda \left[\int_Y c_2(T^{1,0}Y) - \frac{2}{9} \int_Y c_1(T^{1,0}Y)^2 \right].\end{aligned}$$

From the Gauss-Bonnet-Chern theorem and the Hirzebruch signature theorem, it follows that

$$\int_Y c_1(T^{1,0}Y)^2 = 2\chi(Y) + 3\tau(Y), \quad \int_Y c_2(T^{1,0}Y) = \chi(Y),$$

where $\tau(Y)$ is the signature of Y , a topological invariant. This implies that

$$\mu(M) = -\lambda \left[\frac{5}{9}\chi(Y) - \frac{2}{3}\tau(Y) \right].$$

Example 4.2.2 (S^{2n+1}). Let S^{2n+1} be the unit sphere in \mathbb{C}^{n+1} with the natural CR structure. As we saw in Example 0.6.6, it is realized as the tube associated with $(\mathbb{C}\mathbb{P}^n, \mathcal{O}(1), h_{\mathcal{O}(1)})$. Therefore, we have

$$\begin{aligned}\mu(S^{2n+1}) &= \sum_{k=0}^n (-1)^{n+1-k} \int_{\mathbb{C}\mathbb{P}^n} c_k(T^{1,0}\mathbb{C}\mathbb{P}^n) c_1(\mathcal{O}(1))^{n-k} \\ &= \sum_{k=0}^n \binom{n+1}{k} (-1)^{n+1-k} \int_{\mathbb{C}\mathbb{P}^n} c_1(\mathcal{O}(1))^n \\ &= \sum_{k=0}^n \binom{n+1}{k} (-1)^{n+1-k} \\ &= (1-1)^{n+1} - 1 \\ &= -1.\end{aligned}$$

4.3. Application

In this section, we give a proof of Theorem 4.1.2. Let Y_d , S_d , and κ_d be as in Example 0.6.7. Applying Theorem 4.1.1 to S_d , we obtain the following

Proposition 4.3.1. *The Burns-Epstein invariant $\mu(S_d)$ of S_d is given by*

$$(4.3.1) \quad \mu(S_d) = -1 + \frac{n+2-d}{n+2} (1-d)^{n+1} + \frac{d}{(n+1)^{n+1}(n+2)} (d-1)^{n+1}.$$

PROOF. From Theorem 4.1.1 and (0.6.4), it follows that

$$\begin{aligned}\mu(S_d) &= \sum_{k=0}^n (-\lambda_d)^{n+1-k} \int_{Y_d} \left[\sum_{l=0}^k \binom{n+2}{l} (-d)^{k-l} \right] \kappa_d^k \cdot \kappa_d^{n-k} \\ &= \sum_{k=0}^n (-\lambda_d)^{n+1-k} \left[\sum_{l=0}^k \binom{n+2}{l} (-d)^{k-l} \right] \int_{Y_d} \kappa_d^n.\end{aligned}$$

Since $\int_{Y_d} \kappa_d^n = d$, it is enough to compute

$$N(d) = d \sum_{k=0}^n (-\lambda_d)^{n+1-k} \left[\sum_{l=0}^k \binom{n+2}{l} (-d)^{k-l} \right].$$

Note that this is a polynomial in d of degree at most $n + 2$. We first assume that $d \neq 1$. Then $d \neq \lambda_d$, and

$$\begin{aligned}
N(d) &= d \sum_{k=0}^n (-\lambda_d)^{k+1} \left[\sum_{l=0}^{n-k} \binom{n+2}{l} (-d)^{n-k-l} \right] \\
&= \lambda_d \sum_{l=0}^n \binom{n+2}{l} (-d)^{n+1-l} \sum_{k=0}^{n-l} \left(\frac{\lambda_d}{d} \right)^k \\
&= \lambda_d \sum_{l=0}^n \binom{n+2}{l} (-d)^{n+1-l} \frac{1 - (\lambda_d/d)^{n+1-l}}{1 - \lambda_d/d} \\
&= \frac{1}{d - \lambda_d} \left[-\lambda_d \sum_{l=0}^n \binom{n+2}{n+2-l} (-d)^{n+2-l} + d \sum_{l=0}^n \binom{n+2}{n+2-l} (-\lambda_d)^{n+2-l} \right] \\
&= \frac{1}{d - \lambda_d} \{ -\lambda_d [(1-d)^{n+2} + (n+2)d - 1] + d [(1-\lambda_d)^{n+2} + (n+2)\lambda_d - 1] \} \\
&= \frac{1}{d - \lambda_d} [\lambda_d - d - \lambda_d(1-d)^{n+2} + d(1-\lambda_d)^{n+2}] \\
&= -1 + \frac{n+2-d}{n+2} (1-d)^{n+1} + \frac{d}{(n+1)^{n+1}(n+2)} (d-1)^{n+1},
\end{aligned}$$

which proves (4.3.1) for $d \neq 1$. Since the both sides of the above equality are polynomials in d , it holds also for $d = 1$. This finishes the proof. \square

Now, we give a proof of Theorem 4.1.2 by using Proposition 4.3.1.

PROOF OF THEOREM 4.1.2. We already computed the total Q -prime curvature of S_d in Example 3.5.3; it is given by

$$\overline{Q}'(S_d) = (4\pi)^{n+1} (n!)^2 d \left(\frac{n+2-d}{n+1} \right)^{n+1}.$$

This and Proposition 4.3.1 yield that

$$\begin{aligned}
(4.3.2) \quad \mathcal{D}(S_d) &= -1 + \frac{n+2-d}{n+2} (1-d)^{n+1} \\
&\quad + \frac{d}{(n+1)^{n+1}(n+2)} (d-1)^{n+1} + d \left(\frac{n+2-d}{n+1} \right)^{n+1}.
\end{aligned}$$

We consider this as a polynomial in d . The coefficient of its leading term is

$$(-1)^n \left[\frac{1}{n+2} + \frac{(-1)^n}{(n+1)^{n+1}(n+2)} - \frac{1}{(n+1)^{n+1}} \right].$$

If $n \geq 2$, we can estimate the second and third terms as follows:

$$\left| \frac{(-1)^n}{(n+1)^{n+1}(n+2)} \right| + \left| \frac{1}{(n+1)^{n+1}} \right| \leq \frac{2}{(n+1)^2} < \frac{2}{2(n+2)} = \frac{1}{n+2};$$

here we use the fact that $(n+1)^2 > 2(n+2)$ for $n \geq 2$. Hence the coefficient of the leading term of (4.3.2) is $(-1)^n$ times a positive number. This implies $\mathcal{D}(S_d) \rightarrow (-1)^n \infty$ as $d \rightarrow +\infty$. \square

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