博士論文

論文題目　Arithmetic degrees of self-maps of algebraic varieties
(代数多様体の自己写像の算術次数)

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Preface

In classical dynamics, one studies topological and analytic properties of orbits of points under self-maps of manifolds. In arithmetic dynamics, we study arithmetic properties of orbits of rational points under self-maps of algebraic varieties defined over number fields. Many of the motivating problems in arithmetic dynamics come via analogy with classical problems in arithmetic geometry, especially deep results on properties of abelian varieties. An Abelian variety X with multiplication by n map can be considered as a discrete dynamical system, and we can formulate dynamical analogues of classical notions and theorems on abelian varieties. For instance, the Néron-Tate height functions on abelian varieties are fundamental tool to study arithmetic of abelian varieties, and the dynamical analogues of them are called (dynamical) canonical height functions.

Let us consider a dominant rational map $f\colon X \dashrightarrow X$ defined over $\overline{\mathbb{Q}}$. It is important to understand the asymptotic behavior of Weil height functions on the orbit $\{f^n(P)\}$ where $P \in X(\overline{\mathbb{Q}})$ is a point whose f-orbit is well-defined. (For example, we need to know the growing rate of certain height function to define/study the canonical height function of a dynamical system.) A measure of the growing rate of height functions along an orbit is *arithmetic degree* which is introduced by Silverman in [50]. In [50], he expects that the arithmetic degrees of any Zariski dense orbits are equal to the dynamical degree of the self-map. A refined version of this conjecture was formulated by Kawaguchi and Silverman in [29]. Related topics are studied in [23, 24, 27, 28, 29, 35, 36, 39, 40, 41, 47, 50, 51].

In this paper, we prove several properties of arithmetic degree, and study arithmetic degrees of self-morphisms of surfaces and semi-abelian varieties. In these cases, we prove the conjecture. Problems over number fields often have natural analogues over function fields. We also discuss the function field analogue of arithmetic degree.

The content of this paper is as follows. In Chapter 1, we collect some definitions and basic properties of several notions that are used throughout this paper. We also introduce the Kawaguchi-Silverman conjecture, which is the central problem in this paper.

In Chapter 2, we prove several upper bounds of the sequence $\{h_X(f^n(P))\}$ where h_X is an arbitrary ample Weil height function on X, and as a consequence we get the inequality $\overline{\alpha}_f(P) \leq \delta_f$ where $\overline{\alpha}_f(P)$ is the upper arithmetic degree. As a corollary, we prove convergence of dynamical canonical heights under some assumptions. This chapter is based on [39].

In Chapter 3, we prove Kawaguchi-Silverman conjecture for endomorphism of smooth projective surfaces. We investigate endomorphisms of surfaces by using classification of surfaces. We also prove that there exists at least one point such that its arithmetic degree is equal to the dynamical degree when the self-map is a morphism. This chapter is based on [40].

In Chapter 4, we discuss the function filed analogue of arithmetic degrees. We give another proof of the inequality $\overline{\alpha}_f(P) \leq \delta_f$ by using a model over a curve. We also prove that the set of points whose arithmetic degree is equal to the dynamical degree is Zariski dense when the coefficient field of the function field is uncountable. The key is that we can translate height theoretic problems into a geometric problems on models over curves. This chapter is based on [41].

In Chapter 5, we prove Kawaguchi-Silverman conjecture for endomorphism (not necessarily a group homomorphism) of semi-abelian varieties. Moreover, we determine the set of arithmetic degrees of such endomorphisms and characterize preperiodic points in terms of its arithmetic degree (under an assumption that is naturally needed). We deduce the problem to the case where the self-morphism is a group homomorphism and its minimal polynomial is a power of an irreducible polynomial. For such a self-morphism, we calculate the arithmetic degrees by using Silverman's results on group endomorphisms of algebraic tori [50] and Kawaguchi-Silverman's results on group endomorphisms of abelian varieties [28]. This chapter is based on [42].

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Contents

Chapter 1

Preliminaries

We give definitions of some notions that are used throughout this paper.

1.1 Dynamical degrees

Let X be a smooth projective variety defined over an algebraically closed field of characteristic zero and $f\colon X \dashrightarrow X$ a (not necessarily dominant) rational map. We define pull-back $f^*\colon N^1(X) \longrightarrow N^1(X)$ as follows, where $N^1(X)$ is the group of divisors modulo numerical equivalence. Take a resolution of indeterminacy $\pi\colon X' \longrightarrow X$ of f with X' smooth projective. Write $f' = f \circ \pi$. Then we define $f^* = \pi_* \circ f'^*$. This is independent of the choice of resolution.

Definition 1.1.1. Let $f\colon X \dashrightarrow X$ be a dominant rational map. Fix an ample divisor H on X. Then the (first) dynamical degree of f is

$$\delta_f = \lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n}.$$

This is independent of the choice of H. We refer to [8, 9, 52], [13, §4] for basic properties of dynamical degrees.

Remark 1.1.2.

(1) There are other definitions of dynamical degree. Fix a norm $\|\cdot\|$ on $\mathrm{Hom}(N^1(X)_{\mathbb{R}}, N^1(X)_{\mathbb{R}})$, where $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\delta_f = \lim_{n\to\infty} \|(f^n)^*\|^{1/n}$. When f is a morphism, δ_f is the spectral radius of $f^*\colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$. If the ground field is $\mathbb{C}$, this is equal to the spectral radius of $f^*\colon H^{1,1}(X) \longrightarrow H^{1,1}(X)$ (cf. [13, §4]).

(2) If f is a morphism, we have $\delta_{f^n} = \delta_f^n$ for every $n \geq 1$ since $(f^n)^* = (f^*)^n$.

(3) Let $\rho((f^n)^*)$ be the spectral radius of the linear self-map $(f^n)^* \colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$. The dynamical degree δ_f is equal to the limit $\lim_{n\to\infty}(\rho((f^n)^*))^{1/n}$. ([12, Proposition 1.2 (iii)], [29, Remark 7])

(4) Dynamical degree is a birational invariant. That is, let $\pi \colon X \dashrightarrow X'$ be a birational map and $f \colon X \dashrightarrow X$ a dominant rational map and set $f' = \pi \circ f \circ \pi^{-1} \colon X' \dashrightarrow X'$. Then $\delta_f = \delta_{f'}$.

(5) Let X, Y be smooth projective varieties and $f \colon X \dashrightarrow X$, $g \colon Y \dashrightarrow Y$ dominant rational maps. Let $f \times g \colon X \times Y \dashrightarrow X \times Y$ be the product of f and g. Then, by definition, it is easy to see that $\delta_{f\times g} = \max\{\delta_f, \delta_g\}$.

1.2 Height functions

We briefly recall the definition of Weil height function. Standard references for Weil height functions are [5, 19, 32], for example.

The height function on a projective space $\mathbb{P}^N(\overline{\mathbb{Q}})$ is

$$\mathbb{P}^N(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R} \ ; \ (x_0 : \cdots : x_N) \mapsto \frac{1}{[K:\mathbb{Q}]} \sum_v \log\max\{|x_0|_v, \ldots, |x_N|_v\}$$

where K is a number field (finite extension of $\mathbb{Q}$ contained in the fixed algebraic closure $\overline{\mathbb{Q}}$) containing the coordinates $x_0, \ldots, x_N$, the sum runs over all places v of K, and $|\ |_v$ is the absolute value associated with v normalized as follows:

$$|x|_v = \begin{cases} \#\ (\mathcal{O}_K/\mathfrak{p}_v)^{-\operatorname{ord}_v(x)} & \text{if } v \text{ is non-archimedian} \\ |\sigma_v(x)|^{[K_v:\mathbb{R}]} & \text{if } v \text{ archimedian.} \end{cases}$$

Here $\mathcal{O}_K$ is the ring of integers of K. When v is non-archimedian, $\mathfrak{p}_v$ is the maximal ideal corresponding to v and ord_v is the valuation associated with v. When v is archimedian, σ_v is the embedding of K into $\mathbb{C}$ corresponding to v. This definition is independent of the choice of homogeneous coordinates and the number field K.

Let X be a projective variety over $\overline{\mathbb{Q}}$. A Cartier $\mathbb{R}$-divisor D on X determines a (logarithmic) Weil height function h_D up to bounded functions as follows. When D is a very ample integral divisor, h_D is the composite of the embedding by $|D|$ and the height on the projective space we have just defined. For a general D, we write

$$D = \sum_{i=1}^{m} a_i H_i \tag{1.2.1}$$

where a_i are real numbers and H_i are very ample divisors. Then we define

$$h_D = \sum_{i=1}^{m} a_i h_{H_i}.$$

The function h_D does not depend on the choice of the representation (1.2.1) up to bounded function. We call any representative of the class h_D modulo bounded functions a height function associated with D. We call a height function associated with an ample divisor an ample height function.

1.3 Arithmetic degrees

In this section, the ground field is $\overline{\mathbb{Q}}$.

Definition 1.3.1. Let $f\colon X \dashrightarrow X$ be a dominant rational self-map of a smooth projective variety.

(1) We write $X_f(\overline{\mathbb{Q}}) = \{P \in X(\overline{\mathbb{Q}}) \mid f^n(P) \notin I_f \text{ for every } n \geq 0\}$ where I_f is the indeterminacy locus of f.

(2) Let H be an ample divisor on X and take a Weil height function h_H associated with H. The arithmetic degree $\alpha_f(P)$ of f at $P \in X_f(\overline{\mathbb{Q}})$ is defined by

$$\alpha_f(P) = \lim_{n\to\infty} \max\{1, h_H(f^n(P))\}^{1/n}$$

if the limit exists. Since the convergence of this limit is not proved in general, we introduce the following:

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} \max\{1, h_H(f^n(P))\}^{1/n},$$
$$\underline{\alpha}_f(P) = \liminf_{n\to\infty} \max\{1, h_H(f^n(P))\}^{1/n}.$$

We call $\overline{\alpha}_f(P)$ the upper arithmetic degree and $\underline{\alpha}_f(P)$ the lower arithmetic degree. The definitions of the (upper, lower) arithmetic degrees do not depend on the choice of H and h_H ([29, Proposition 12] [40, Theorem 3.4]).

In [29], Kawaguchi and Silverman formulated the following conjecture.

Conjecture 1.3.2 (Kawaguchi-Silverman conjecture (KSC))**.** *Let X be a smooth projective variety and $f\colon X \dashrightarrow X$ a dominant rational map, both defined over $\overline{\mathbb{Q}}$. Let $P \in X_f(\overline{\mathbb{Q}})$.*

(1) *The limit defining $\alpha_f(P)$ exists.*

(2) *The arithmetic degree $\alpha_f(P)$ is an algebraic integer.*

(3) *The collections of arithmetic degrees $\{\alpha_f(Q) \mid Q \in X_f(\overline{\mathbb{Q}})\}$ is a finite set.*

(4) *If the orbit $\mathcal{O}_f(P) = \{f^n(P) \mid n = 0, 1, 2, \dots\}$ is Zariski dense in X, then $\alpha_f(P) = \delta_f$.*

This conjecture, especially the last part, is the central problem in this paper.

Remark 1.3.3. This conjecture is proved in the following situations:

(1) $N^1(X)_{\mathbb{R}} = \mathbb{R}$ and f is a morphism [27].

(2) $f\colon \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is a monomial map and $P \in \mathbb{G}_{\mathrm{m}}^N(\overline{\mathbb{Q}})$ [50].

(3) X is a surface and f is a morphism (Chapter 3, [25, 40]).

(4) $X = \mathbb{P}^N$ and f is a rational map extending a regular affine automorphism [27].

(5) X is an abelian variety [28, 51].

(6) X is a hyper-Kähler variety and f is an automorphism [35].

(7) X is a semi-abelian variety and f is a self-morphism (Chapter 5, [42]). (The conjecture actually makes sense when X is quasi-projective.)

(8) f is an endomorphism and X is the product $\prod_{i=1}^n X_i$ of smooth projective varieties, with the assumption that each variety X_i satisfies one of the following conditions [47] :

- the first Betti number of $(X_i)_{\mathbb{C}}$ is zero and the Néron–Severi group of X_i has rank one,
- X_i is an abelian variety,
- X_i is an Enriques surface, or

- X_i is a $K3$ surface.

(9) f is an endomorphism and X is the product $X_1 \times X_2$ of positive dimensional varieties such that one of X_1 or X_2 is of general type [47]. (In fact, there do not exist Zariski dense forward f-orbits on such $X_1 \times X_2$.)

When f is a morphism, the first three parts of Conjecture 1.3.2 are proved by Kawaguchi and Silverman in [28] (cf. Remark 2.1.4). See [27,35,40,42,50] for more details and results related to this conjecture.

Remark 1.3.4. Recently, Lesieutre and Satriano found a counter example to Conjecture 1.3.2 (3) [36]. That is, there exists a dominant rational self-map such that the set of arithmetic degrees is infinite. In chapter 5, however, we prove that the set of arithmetic degrees of a self-morphism of a semi-abelian variety is finite.

Remark 1.3.5. Let X be a smooth projective variety over an algebraically closed field of characteristic zero. Assume $\kappa(X) > 0$. Let $\Phi : X \dashrightarrow W$ be the Iitaka fibration of X, and $f\colon X \dashrightarrow X$ a dominant rational self-map on X. Then standard argument of pluricanonical system shows that f induces a birational map $g\colon W \dashrightarrow W$ such that $g \circ \Phi = \Phi \circ f$. By [46, Theorem A], this g is an automorphism of finite order. This implies that any dominant rational self-maps on a smooth projective varieties of positive Kodaira dimension have no Zariski dense orbits. So the last part of Conjecture 1.3.2 has meaning for smooth projective varieties of non-positive Kodaira dimension.

Notation

In this paper, a variety over a field k means an irreducible reduced separated scheme of finite type over k. The following is a list of the notation that we use throughout this paper.

- Let X, Y be projective varieties over an algebraically closed field k and $f\colon X \dashrightarrow Y$ be a rational map.
  (1) The group of Cartier divisors on X modulo numerical equivalence is denoted by $N^1(X)$. When X is smooth, the Neron-Severi group of X is denoted by $\mathrm{NS}(X)$.
  (2) Linear equivalence of divisors is denoted by $\sim$; $\mathbb{Q}$-linear equivalence and $\mathbb{R}$-linear equivalence are denoted by $\sim_{\mathbb{Q}}, \sim_{\mathbb{R}}$ respectively; numerical equivalence is denoted by $\equiv$.

(3) The indeterminacy locus of f is denoted by I_f.

(4) When $Y = X$ and $P \in X(k)$ is a point such that $f^n(P) \notin I_f$ for all $n \geq 0$, the (forward) orbit $\{P, f(P), f^2(P), \dots\}$ is denoted by $\mathcal{O}_f(P)$. We say P is preperiodic under f if the orbit $\mathcal{O}_f(P)$ is finite. This is equivalent to $f^n(P) = f^m(P)$ for some $n \neq m \geq 0$.

(5) We write $X_f(k) = \{P \in X(k) \mid f^n(P) \notin I_f$ for every $n \geq 0\}$.

- Let f, g and h be real-valued functions on a set S. The equality $f = g + O(h)$ means that there is a positive constant C such that $|f(x) - g(x)| \leq C|h(x)|$ for every $x \in S$. The equality $f = g + O(1)$ means that there is a positive constant C' such that $|f(x)-g(x)| \leq C'$ for every $x \in S$.

- Let M be a $\mathbb{Z}$-module. We write $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, and so on.

Chapter 2

Upper bound of arithmetic degrees

2.1 Summary

Let X be a smooth projective variety and $f\colon X \dashrightarrow X$ a dominant rational map, both defined over $\overline{\mathbb{Q}}$.

In this chapter, we give upper bounds of heights of $f^n(P)$ in terms of δ_f (see Theorem 2.1.1). Actually, this theorem is stated as Theorem 1 in [29]. However, the proof of Theorem 1 in [29] unfortunately contains a mistake (cf. Remark 2.1.2). In this chapter, we give a correct proof of Theorem 1 in [29].

Let h_X be the height function associated with an ample divisor on X. We write $h_X^+ = \max\{h_X, 1\}$.

The main theorem of this chapter is the following.

Theorem 2.1.1. *Let $f\colon X \dashrightarrow X$ be a dominant rational map defined over $\overline{\mathbb{Q}}$. For any $\epsilon > 0$, there exists $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C(\delta_f + \epsilon)^n h_X^+(P)$$

for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$. In particular, for any $P \in X_f(\overline{\mathbb{Q}})$, we have

$$\overline{\alpha}_f(P) \leq \delta_f.$$

Remark 2.1.2. This theorem is stated as Theorem 1 in [29], but unfortunately their proof is incorrect. Precisely, in the proof of Theorem 24 (Theorem 1) in [29], the constant C_1 and therefore C_8 depends on m. Thus one can not conclude the equality $\lim_{m\to\infty}(C_8 r m^r)^{1/ml} = 1$ which is a key in the argument of the proof in [29].

If f is a morphism, we have the following slightly stronger inequalities.

Theorem 2.1.3. *Let $f\colon X \longrightarrow X$ be a surjective morphism. Let $r = \dim N^1(X)_{\mathbb{R}}$ be the Picard number of X.*

(1) *When $\delta_f = 1$, there exists a constant $C > 0$ such that*
$$h_X^+(f^n(P)) \leq Cn^{2r} h_X^+(P)$$
for all $n \geq 1$ and $P \in X(\overline{\mathbb{Q}})$.

(2) *Assume that $\delta_f > 1$. Then there exists a constant $C > 0$ such that*
$$h_X^+(f^n(P)) \leq Cn^{r-1}\delta_f^n h_X^+(P)$$
for all $n \geq 1$ and $P \in X(\overline{\mathbb{Q}})$.

Remark 2.1.4. In [28], Kawaguchi and Silverman prove a similar inequality under the same assumption of Theorem 2.1.3. Moreover, they prove that the arithmetic degree $\alpha_f(P)$ exists and is equal to one of the eigenvalues of the linear map $f^*\colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$.

Remark 2.1.5. The exponent $2r$ in Theorem 2.1.3 (1) is the best possible. For example, let X be an elliptic curve with identity element $0 \in X$ and $P \in X$ a non-torsion point. Let $f = T_P\colon X \longrightarrow X$ be the translation by P. Then, $\delta_f = 1$ since $f^* = \mathrm{id}$. Let h be the Neron-Tate height on X. Then $h^+(f^n(0)) = h^+(nP) = \max\{1, n^2 h(P)\}$.

If the Picard number of X is one, we have the following stronger inequalities.

Theorem 2.1.6. *Let X be a smooth projective variety of Picard number one. Let $f\colon X \dashrightarrow X$ be a dominant rational map.*

(1) *For a positive integer $k > 0$, there exists a constant $C > 0$ such that*
$$h_X^+(f^n(P)) \leq Cn^2 \rho((f^k)^*)^{n/k} h_X^+(P)$$
for all $P \in X_f(\overline{\mathbb{Q}})$ and $n \geq 1$.

(2) *Let $k > 0$ be a positive integer. Assume that $\rho((f^k)^*) > 1$. Then there exists a constant $C > 0$ such that*
$$h_X^+(f^n(P)) \leq C\rho((f^k)^*)^{n/k} h_X^+(P)$$
for all $P \in X_f(\overline{\mathbb{Q}})$ and $n \geq 0$.

A dominant rational map f is said to be algebraically stable if $(f^n)^* = (f^*)^n \colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$ for all $n > 0$. In this case, $\delta_f = \rho(f^*)$. As a corollary of Theorem 2.1.1, we get the following.

Proposition 2.1.7. *Assume that the Picard number of X is one and let $f \colon X \dashrightarrow X$ be an algebraically stable dominant rational map with $\delta_f > 1$. Then the limit*

$$\hat{h}_{X,f}(P) = \lim_{n\to\infty} \frac{h_X(f^n(P))}{\delta_f^n}$$

exists for all $P \in X_f(\overline{\mathbb{Q}})$.

More generally,

Proposition 2.1.8. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. Let $f \colon X \dashrightarrow X$ be a dominant rational self-map defined over $\overline{\mathbb{Q}}$. Assume $\delta_f > 1$ and there exists a nef $\mathbb{R}$-divisor H on X such that $f^*H \equiv \delta_f H$. Fix a height function h_H associated with H. Then for any $P \in X_f(\overline{\mathbb{Q}})$, the limit*

$$\hat{h}_{X,f}(P) = \lim_{n\to\infty} \frac{h_H(f^n(P))}{\delta_f^n}$$

converges or diverges to $-\infty$.

Question. Are there any examples that the limits diverge to $-\infty$?

The function $\hat{h}_{X,f}$ is the function which is called the canonical height function in [50]. The canonical height functions of dynamical systems of self-morphisms are systematically studied in [7]. On the other hand, little is known about the canonical heights of rational maps. There are several recent studies on them. In [23, Theorem D], it is proved that any birational self-maps of surfaces with dynamical degree greater than one admit canonical heights up to birational conjugate. In [26], the canonical heights of regular affine automorphisms are studied in detail.

We prove Theorem 2.1.3 in §2.2, Theorem 2.1.1 in §2.3, Theorem 2.1.6 and Proposition 2.1.7, 2.1.8 in §2.4. In the proof of Theorem 2.1.6, we use the computation in the proof of Theorem 2.3.2 in §2.3.

In this chapter, we give a method to estimate $h_H(f^n(P))$ in terms of the behavior of f on the group $N^1(X)_{\mathbb{R}}$ by controlling error terms arising from divisors numerically equivalent to zero. We give an expression of error terms as a linear combinations of fixed height functions whose coefficients can be controlled easily.

Remark 2.1.9 (Other ground fields)**.** All of the results and arguments in this chapter remain valid without change for other ground fields $\overline{K}$ of characteristic 0 where K is a field with a set of non-trivial absolute values satisfying the product formula. The main theorems (Theorem 2.1.1, 2.1.3) also hold over a field of positive characteristic, see Appendix 2.5.2.

Notation.

$\|\| \ \|\|$	For a real vector $v \in \mathbb{R}^n$ or a real matrix $M \in M_{n\times m}(\mathbb{R})$, $\|\|v\|\|$ and $\|\|M\|\|$ are the maximum among the absolute values of the coordinates.
$\langle\ ,\ \rangle$	For two column vectors $v = {}^t(v_1, \dots, v_n), w = {}^t(w_1, \dots, w_n)$ of the same size, we write $\langle v, w\rangle = \sum v_i w_i$. We use this notation whenever the multiplication $v_i w_i$ is defined (e.g. v_i are real numbers, and w_i are $\mathbb{R}$-divisors or real valued functions). Similarly, for a real matrix M and a vector w whose entries are divisors or real valued functions, Mw is defined in the obvious manner.
$\mathbf{h} \circ f$	For a column vector valued function $\mathbf{h} = {}^t(h_1, \dots, h_n)$ on a set X and a map f to X, we write $\mathbf{h} \circ f = {}^t(h_1 \circ f, \dots, h_n \circ f)$.

2.2 Endomorphism case

We first treat the case where f is a morphism. The purpose of this section is to prove the following theorem.

Theorem 2.2.1 (Theorem 2.1.3)**.** *Let X be a projective variety over $\overline{\mathbb{Q}}$ and $f\colon X \longrightarrow X$ be a surjective morphism defined over $\overline{\mathbb{Q}}$. Let δ_f be the spectral radius of $f^*\colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$. (Actually, δ_f is equal to the dynamical degree of f which is defined by taking a resolution of singularities.) Let $r = \dim N^1(X)_{\mathbb{R}}$ be the Picard number of X. Fix an ample height function h_X on X.*

(1) *When $\delta_f = 1$, there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C n^{2r} h_X^+(P)$$

for all $n \geq 1$ and $P \in X(\overline{\mathbb{Q}})$.

(2) *Assume that* $\delta_f > 1$. *Then there exists a constant* $C > 0$ *such that*

$$h_X^+(f^n(P)) \le Cn^{r-1}\delta_f^n h_X^+(P)$$

for all $n \ge 1$ *and* $P \in X(\overline{\mathbb{Q}})$.

Proof. Let $D_1, \dots, D_r$ be $\mathbb{R}$-divisors which form a basis for $N^1(X)_{\mathbb{R}}$. Let H be an ample divisor on X such that $H + D_i,\ H - D_i$ $(i = 1, \dots, r)$ are ample. For $\mathbb{R}$-divisors α, β, $\alpha \equiv \beta$ means α and β are numerically equivalent. Let $f^*D_i \equiv \sum_{k=1}^r a_{ki}D_k$, and $A = (a_{ki})_{k,i}$. We can write $H \equiv \sum_{i=1}^r c_iD_i$. Then

$$f^*H \equiv \sum_{j=1}^r \sum_{k=1}^r c_j a_{kj} D_k = \left\langle A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix}, \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_r \end{pmatrix} \right\rangle = \left\langle A\vec{c}, \vec{D} \right\rangle.$$

Let

$$E = f^*H - \left\langle A\vec{c}, \vec{D} \right\rangle \tag{2.2.1}$$

$$E_i = f^*D_i - \sum_{k=1}^r a_{ki}D_k. \tag{2.2.2}$$

Then

$$\vec{E} = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_r \end{pmatrix} = f^*\vec{D} - {}^{\mathrm{t}}A\vec{D}.$$

Note that E, E_i are numerically zero.

The choice of Height functions.

First, we take and fix height functions $h_{D_1}, \dots h_{D_r}$ associated with $D_1, \dots, D_r$. Next, we take and fix a height function h_H associated with H so that $h_H \ge 1,\ h_H \ge |h_{D_i}|$ $(i = 1, \dots r)$. Then $h_{D_i} \circ f,\ h_H \circ f$ are height functions associated with f^*D_i and f^*H. We write

$$\mathbf{h}_{\vec{D}} = \begin{pmatrix} h_{D_1} \\ h_{D_2} \\ \vdots \\ h_{D_r} \end{pmatrix}.$$

We define

$$h_E = h_H \circ f - \left\langle A\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle \tag{2.2.3}$$

$$\mathbf{h}_{\vec{E}} = \begin{pmatrix} h_{E_1} \\ h_{E_2} \\ \vdots \\ h_{E_r} \end{pmatrix} = \mathbf{h}_{\vec{D}} \circ f - {}^{\mathrm{t}}A\mathbf{h}_{\vec{D}} \ . \tag{2.2.4}$$

Then, by (2.2.1)(2.2.2), h_E and h_{E_i} are height functions associated with E and E_i. Now, since E, E_i are numerically zero, there exists a constant $C > 0$ such that for all $Q \in X(\overline{\mathbb{Q}})$

$$|h_E(Q)| \leq C\sqrt{h_H(Q)} \tag{2.2.5}$$

$$|h_{E_i}(Q)| \leq C\sqrt{h_H(Q)} \quad i = 1, \dots, r. \tag{2.2.6}$$

See for example [19, Theorem B.5.9] and Proposition 2.5.5.

Let us begin the estimation of $h_H(f^n(P))$. Let $P \in X(\overline{\mathbb{Q}})$ be an arbitrary point. Then we have

$$h_H(f(P)) = h_E(P) + \left\langle A\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (P).$$

For $n \geq 2$, we have

$$\begin{aligned}
&h_H(f^n(P)) \\
&=(h_H \circ f)(f^{n-1}(P)) - \left\langle A\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-1}(P)) \\
&\quad + \left\langle A\vec{c}, \mathbf{h}_{\vec{D}} \circ f \right\rangle (f^{n-2}(P)) - \left\langle A^2\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-2}(P)) \\
&\quad + \cdots \\
&\quad + \left\langle A^{n-2}\vec{c}, \mathbf{h}_{\vec{D}} \circ f \right\rangle (f(P)) - \left\langle A^{n-1}\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f(P)) \\
&\quad + \left\langle A^{n-1}\vec{c}, \mathbf{h}_{\vec{D}} \circ f \right\rangle (P) \\
&=h_E(f^{n-1}(P)) \\
&\quad + \left\langle A\vec{c}, {}^{\mathrm{t}}A\mathbf{h}_{\vec{D}} + \mathbf{h}_{\vec{E}} \right\rangle (f^{n-2}(P)) - \left\langle A^2\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-2}(P)) \\
&\quad + \cdots \\
&\quad + \left\langle A^{n-2}\vec{c}, {}^{\mathrm{t}}A\mathbf{h}_{\vec{D}} + \mathbf{h}_{\vec{E}} \right\rangle (f(P)) - \left\langle A^{n-1}\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f(P)) \\
&\quad + \left\langle A^{n-1}\vec{c}, {}^{\mathrm{t}}A\mathbf{h}_{\vec{D}} + \mathbf{h}_{\vec{E}} \right\rangle (P) \qquad \text{by (2.2.3)(2.2.4)} \\
&=h_E(f^{n-1}(P)) \\
&\quad + \left\langle A\vec{c}, \mathbf{h}_{\vec{E}} \right\rangle (f^{n-2}(P)) \\
&\quad + \cdots \\
&\quad + \left\langle A^{n-2}\vec{c}, \mathbf{h}_{\vec{E}} \right\rangle (f(P)) \\
&\quad + \left\langle A^{n-1}\vec{c}, \mathbf{h}_{\vec{E}} \right\rangle (P) + \left\langle A^{n}\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (P).
\end{aligned}$$

By (2.2.5)(2.2.6)

$$|\left\langle A^m\vec{c},\mathbf{h}_{\vec{E}}\right\rangle(Q)|\le r^2\|\vec{c}\|\|A^m\|C\sqrt{h_H(Q)}\ \text{ for } Q\in X(\overline{\mathbb{Q}}).$$

Also, by the choice of h_H and h_{D_i}, we have

$$|\left\langle A^n\vec{c},\mathbf{h}_{\vec{D}}\right\rangle(P)|\le r^2\|\vec{c}\|\|A^n\|h_H(P).$$

Thus

$$\begin{aligned}h_H(f^n(P))\le C\Big(&\sqrt{h_H(f^{n-1}(P))}+r^2\|\vec{c}\|\|A\|\sqrt{h_H(f^{n-2}(P))}+\cdots \\ &+r^2\|\vec{c}\|\|A^{n-2}\|\sqrt{h_H(f(P))}+r^2\|\vec{c}\|\|A^{n-1}\|\sqrt{h_H(P)}\Big)+r^2\|\vec{c}\|\|A^n\|h_H(P).\end{aligned}\tag{2.2.7}$$

For simplicity, we write $\delta=\delta_f$. Let $\rho(f^*)$ be the spectral radius of the linear map $f^*\colon N^1(X)_{\mathbb{R}}\longrightarrow N^1(X)_{\mathbb{R}}$. Let $\rho(A)$ be the spectral radius of the matrix A. By definition, we have $\delta=\rho(f^*)=\rho(A)=\lim_{n\to\infty}\|A^n\|^{1/n}$. Note that

$$\frac{r^2\|\vec{c}\|\|A^k\|}{k^{r-1}\rho(A)^k}=\frac{r^2\|\vec{c}\|\|A^k\|}{k^{r-1}\delta^k}$$

is bounded with respect to $k>0$.

Let $C_1=\sup_{k>0}\left\{r^2\|\vec{c}\|\|A^k\|/k^{r-1}\delta^k\right\}$. Set $C_2=\max\{1,C_1,CC_1,C\}$. Then dividing inequality (2.2.7) by $n^{r-1}\delta^n$, we get

$$\begin{aligned}&\frac{h_H(f^n(P))}{n^{r-1}\delta^n} \\ &\le C\Bigg(\frac{r^2\|\vec{c}\|\|A^{n-1}\|}{n^{r-1}\delta^n}\sqrt{h_H(P)}+ \\ &\quad\sum_{k=1}^{n-2}\frac{r^2\|\vec{c}\|\|A^{n-1-k}\|}{(n-1-k)^r\delta^{n-1-k}}\sqrt{\frac{h_H(f^k(P))}{k^{r-1}\delta^k}}\frac{(n-1-k)^{r-1}k^{(r-1)/2}}{n^{r-1}\delta^{1+k/2}} \\ &\quad+\sqrt{\frac{h_H(f^{n-1}(P))}{(n-1)^{r-1}\delta^{n-1}}}\frac{(n-1)^{(r-1)/2}}{n^{r-1}\delta^{1+(n-1)/2}}\Bigg)+\frac{r^2\|\vec{c}\|\|A^n\|}{n^{r-1}\delta^n}h_H(P) \\ &\le C_2\Bigg(\sqrt{h_H(P)}+\sum_{k=1}^{n-2}\sqrt{\frac{h_H(f^k(P))}{k^{r-1}\delta^k}}\frac{(n-1-k)^{r-1}k^{(r-1)/2}}{n^{r-1}\delta^{1+k/2}} \\ &\quad+\sqrt{\frac{h_H(f^{n-1}(P))}{(n-1)^{r-1}\delta^{n-1}}}\frac{(n-1)^{(r-1)/2}}{n^{r-1}\delta^{1+(n-1)/2}}+h_H(P)\Bigg).\end{aligned}\tag{2.2.8}$$

First we assume that $\delta > 1$. Then $k^{(r-1)/2}/\delta^{1+k/2}$ is bounded with respect to k. Thus, there exists a constant $C_3 > 0$ which is independent of n, P so that

$$\frac{h_H(f^n(P))}{n^{r-1}\delta^n} \leq C_3 \left(\sqrt{h_H(P)} + \sum_{k=1}^{n-1} \sqrt{\frac{h_H(f^k(P))}{k^{r-1}\delta^k}} + h_H(P) \right).$$

Applying Lemma 2.5.2 to the sequence

$$\begin{aligned} a_0 &= h_H(P), \\ a_n &= h_H(f^n(P))/n^r\delta^n \ (n \geq 1), \end{aligned}$$

there exists a constant $C_4 > 0$ independent of n, P such that

$$\frac{h_H(f^n(P))}{n^{r-1}\delta^n} \leq C_4 n^2 h_H(P)$$

for all $n \geq 1$. Again from (2.2.8),

$$\frac{h_H(f^n(P))}{n^{r-1}\delta^n} \leq C_2 \left(\sqrt{h_H(P)} + \sum_{k=1}^{n-1} \sqrt{C_4 h_H(P)} \frac{k^{1+(r-1)/2}}{\delta^{1+k/2}} + h_H(P) \right).$$

Since $\sum_{k=1}^{\infty} k^{1+(r-1)/2}/\delta^{1+k/2}$ is convergent, there exists a constant $C_5 > 0$ independent of n, P such that

$$\frac{h_H(f^n(P))}{n^{r-1}\delta^n} \leq C_5 h_H(P).$$

Thus $h_H(f^n(P)) \leq C_5 n^{r-1}\delta^n h_H(P)$. Now, since h_H and h_X are ample height functions and we take $h_H \geq 1$, there exists an integer $m > 0$ such that

$$m h_H \geq h_X^+,\ m h_X^+ \geq h_H.$$

Thus

$$h_X^+(f^n(P)) \leq m h_H(f^n(P)) \leq m C_5 n^{r-1}\delta^n h_H(P) \leq m^2 C_5 n^{r-1}\delta^n h_X^+(P).$$

This completes the proof of Theorem 2.2.1(2).

Now assume that $\delta = 1$. Dividing both sides of (2.2.8) by n^{r-1}, we get

$$\begin{aligned}\frac{h_H(f^n(P))}{n^{2r-2}} &\le C_2 \left(\frac{\sqrt{h_H(P)}}{n^{r-1}} + \sum_{k=1}^{n-2} \sqrt{\frac{h_H(f^k(P))}{k^{2r-2}}} \frac{(n-1-k)^{r-1}k^{r-1}}{n^{2r-2}} \right. \\ &\qquad \left. + \sqrt{\frac{h_H(f^{n-1}(P))}{(n-1)^{2r-2}}} \frac{(n-1)^{r-1}}{n^{2r-2}} + \frac{h_H(P)}{n^{r-1}} \right) \\ &\le C_2 \left(\sqrt{h_H(P)} + \sum_{k=1}^{n-1} \sqrt{\frac{h_H(f^k(P))}{k^{2r-2}}} + h_H(P) \right).\end{aligned}$$

By Lemma 2.5.2, there exists a constant $C_6 > 0$ independent of n, P such that

$$h_H(f^n(P)) \le C_6 n^{2r} h_H(P) \quad \text{for all } n \ge 1.$$

By the same argument at the end of the proof of (2), this proves Theorem 2.2.1(1). □

2.3 Rational self-map case

Now we prove the main theorem of this chapter.

Theorem 2.3.1 (Theorem 2.1.1)**.** *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f\colon X \dashrightarrow X$ be a dominant rational map defined over $\overline{\mathbb{Q}}$. Let δ_f be the first dynamical degree of f. Fix an ample height function h_X on X. Then, for any $\epsilon > 0$, there exists $C > 0$ such that*

$$h_X^+(f^n(P)) \le C(\delta_f + \epsilon)^n h_X^+(P)$$

for all $n \ge 0$ and $P \in X_f(\overline{\mathbb{Q}})$. In particular, for any $P \in X_f(\overline{\mathbb{Q}})$, we have

$$\overline{\alpha}_f(P) \le \delta_f.$$

We deduce this theorem from the following theorem.

Theorem 2.3.2. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f\colon X \dashrightarrow X$ be a dominant rational map defined over $\overline{\mathbb{Q}}$ with first dynamical degree δ_f. Fix an ample height function h_X on X. Then, for any $\epsilon > 0$, there exist a positive integer k and a constant $C > 0$ such that*

$$h_X^+(f^{nk}(P)) \le C(\delta_f + \epsilon)^{nk} h_X^+(P)$$

for all $n \ge 0$ and $P \in X_f(\overline{\mathbb{Q}})$.

Lemma 2.3.3. *In the situation of Theorem 2.3.2, there exists a constant $C_0 \geq 1$ such that*

$$h_X^+(f^n(P)) \leq C_0^n h_X^+(P)$$

for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$.

Proof. Let H be an ample divisor on X. Take a height function h_H associated with H so that $h_H \geq 1$. Let h_{f^*H} be a height function associated with f^*H. Then, from [29, Proposition 21]

$$h_H(f(P)) \leq h_{f^*H}(P) + O(1)$$

for all $P \in X_f(\overline{\mathbb{Q}})$. Here $O(1)$ is a bounded function on $X_f(\overline{\mathbb{Q}})$ which depends on $f, H, f^*H, h_H, h_{f^*H}$ but is independent of P. Since H is ample and $h_H \geq 1$, for a sufficiently large $C_0 \geq 1$, we have

$$h_{f^*H}(P) + O(1) \leq C_0 h_H(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$. Thus, we get

$$h_H(f(P)) \leq C_0 h_H(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$. Therefore

$$h_H(f^n(P)) \leq C_0^n h_H(P).$$

By the same argument at the end of the proof of Theorem 2.2.1(2), this proves the statement. □

Proof of Theorem 2.3.2 $\Longrightarrow$ Theorem 2.3.1. From Theorem 2.3.2, for any $\epsilon > 0$, there exist a positive integer k and a positive constant $C > 0$ such that

$$h_X^+(f^{nk}(P)) \leq C(\delta_f + \epsilon)^{nk} h_X^+(P)$$

for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$. For any integer $m \geq 0$, we write $m = qk + t, q \geq 0, 0 \leq t < k$. Let C_0 be the constant in Lemma 2.3.3. Then for any $P \in X_f(\overline{\mathbb{Q}})$,

$$\begin{aligned} h_X^+(f^m(P)) &\leq C(\delta_f + \epsilon)^{qk} h_X^+(f^t(P)) \\ &\leq C C_0^t (\delta_f + \epsilon)^{qk} h_X^+(P) \\ &\leq C C_0^{k-1} (\delta_f + \epsilon)^m h_X^+(P). \end{aligned}$$

This proves the first statement in Theorem 2.3.1.

The second statement is an easy consequence of the first one. That is,

$$\begin{aligned}\overline{\alpha}_f(P) &= \limsup_{n\to\infty} h_X^+(f^n(P))^{1/n}\\ &\le \limsup_{n\to\infty} \left(Ch_X^+(P)\right)^{1/n}(\delta_f+\epsilon)\\ &= \delta_f+\epsilon.\end{aligned}$$

Since ϵ is arbitrary, we get $\overline{\alpha}_f(P) \le \delta_f$. □

Before starting the proof of Theorem 2.3.2, we prove an interesting corollary.

Corollary 2.3.4. *In the situation of Theorem 2.3.2,*

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} h_X^+(f^{nk}(P))^{1/nk} = \overline{\alpha}_{f^k}(P)^{1/k}$$

for any $k > 0$ and any point $P \in X_f(\overline{\mathbb{Q}})$.

Proof. We compute

$$\begin{aligned}\overline{\alpha}_f(P) &= \limsup_{m\to\infty} h_X^+(f^m(P))^{1/m}\\ &= \limsup_{n\to\infty} \max_{0\le i<k} h_X^+(f^{nk+i}(P))^{1/nk+i}\\ &\le \limsup_{n\to\infty} \max_{0\le i<k} (C_0^i h_X^+(f^{nk}(P)))^{1/nk+i} && \text{by Lemma 2.3.3}\\ &\le \limsup_{n\to\infty} (C_0^{k-1} h_X^+(f^{nk}(P)))^{1/nk}\\ &= \limsup_{n\to\infty} h_X^+(f^{nk}(P))^{1/nk}\\ &\le \overline{\alpha}_f(P).\end{aligned}$$

Then we have $\overline{\alpha}_f(P) = \limsup_{n\to\infty} h_X^+(f^{nk}(P))^{1/nk} = \overline{\alpha}_{f^k}(P)^{1/k}$. □

Now we turn to the proof of Theorem 2.3.2.

Proof of Theorem 2.3.2. Let $D_1, \ldots, D_r$ be very ample divisors on X which forms a basis for $N^1(X)_{\mathbb{R}}$. Take an ample divisor H on X so that $H \pm D_i$, $i = 1, \ldots, r$ are ample and if we write $H \equiv \sum_{i=1}^r c_i D_i$ then $c_i \ge 0$.

We take a resolution of indeterminacy $p\colon Y \longrightarrow X$ of f as follows. p is a sequence of blowing ups at smooth centers and the images of centers in X

are contained in the indeterminacy locus I_f of f. Let $g = f \circ p$.

$$\begin{array}{ccc} & Y & \\ {}^{p}\swarrow & & \searrow^{g} \\ X & \underset{f}{\dashrightarrow} & X \end{array}$$

Let $\mathrm{Exc}(p)$ be the exceptional locus of p. By the negativity lemma (see for example [31, Lemma 3.39]),

$$Z_i = p^* p_* g^* D_i - g^* D_i$$

is an effective divisor on Y whose support is contained in $\mathrm{Exc}(p)$. Let $F_i = g^* D_i$ for $i = 1, \dots, r$. Then,

$$p^* p_* F_i - F_i = Z_i. \tag{2.3.1}$$

Take divisors $F_{r+1}, \dots, F_s$ on Y so that $F_1, \dots, F_s$ forms a basis for $N^1(Y)_{\mathbb{R}}$. There exists an ample $\mathbb{Q}$-divisor H' on Y such that $p^*H - H'$ is an effective $\mathbb{Q}$-divisor whose support is contained in $\mathrm{Exc}(p)$. Indeed, take an effective p-exceptional divisor G such that $-G$ is p-ample. (For the existence of such a divisor, see for example [31, Lemma 2.62]). Then, for sufficiently large $N > 0$, $H' = -\frac{1}{N} G + p^* H$ satisfies desired properties. Let

$$g^* D_i \equiv \sum_{m=1}^{s} a_{mi} F_m \quad (i = 1, \dots r) \tag{2.3.2}$$

$$p_* F_j \equiv \sum_{l=1}^{r} b_{lj} D_l \quad (j = 1, \dots, s) \tag{2.3.3}$$

and

$$\begin{aligned} A &= (a_{mi})_{mi} \quad s \times r\text{-matrix} \\ B &= (b_{lj})_{lj} \quad r \times s\text{-matrix}. \end{aligned}$$

By the definition of F_j, A is the following form.

$$A = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & \\ & & \end{pmatrix}. \tag{2.3.4}$$

Note that BA is the representation matrix of f^* with respect to the basis $D_1, \ldots, D_r$. We write

$$\vec{D} = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_r \end{pmatrix}, \vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_s \end{pmatrix}, \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix}, \vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_r \end{pmatrix}.$$

Let

$$E = g^*H - \left\langle A\vec{c}, \vec{F} \right\rangle \tag{2.3.5}$$

$$\vec{E}' = \begin{pmatrix} E'_1 \\ E'_2 \\ \vdots \\ E'_s \end{pmatrix} = p_*\vec{F} - {}^{\mathrm{t}}B\vec{D}. \tag{2.3.6}$$

These are numerically zero divisors.

The choice of height functions.

Fix height functions $h_{D_1}, \ldots, h_{D_r}$ associated with $D_1, \ldots, D_r$. Fix a height function h_H associated with H so that $h_H \geq 1$ and $h_H \geq |h_{D_i}|$ for $i = 1, \ldots, r$. Note that $h_{D_1}, \ldots, h_{D_r}$ and h_H are independent of f.

We define $h_{F_j} = h_{D_j} \circ g,\ j = 1, \ldots, r$. These are height functions associated with F_j. For $j = r+1, \ldots, s$, fix any height functions h_{F_j} associated with F_j. Fix height functions $h_{p_*F_j}$ associated with p_*F_j for $j = 1, \ldots, s$. We write

$$\mathbf{h}_{\vec{D}} = \begin{pmatrix} h_{D_1} \\ h_{D_2} \\ \vdots \\ h_{D_r} \end{pmatrix},\ \mathbf{h}_{\vec{F}} = \begin{pmatrix} h_{F_1} \\ h_{F_2} \\ \vdots \\ h_{F_s} \end{pmatrix},\ \mathbf{h}_{p_*\vec{F}} = \begin{pmatrix} h_{p_*F_1} \\ h_{p_*F_2} \\ \vdots \\ h_{p_*F_s} \end{pmatrix}.$$

Define

$$\mathbf{h}_{\vec{E'}} = \begin{pmatrix} h_{E'_1} \\ h_{E'_2} \\ \vdots \\ h_{E'_s} \end{pmatrix} = \mathbf{h}_{p_*\vec{F}} - {}^{\mathrm{t}}B\mathbf{h}_{\vec{D}} \tag{2.3.7}$$

$$h_E = h_H \circ g - \langle A\vec{c}, \mathbf{h}_{\vec{F}} \rangle \tag{2.3.8}$$

$$\mathbf{h}_{\vec{Z}} = \begin{pmatrix} h_{Z_1} \\ h_{Z_2} \\ \vdots \\ h_{Z_r} \end{pmatrix} = \begin{pmatrix} h_{p_*F_1} \\ h_{p_*F_2} \\ \vdots \\ h_{p_*F_r} \end{pmatrix} \circ p - \begin{pmatrix} h_{F_1} \\ h_{F_2} \\ \vdots \\ h_{F_r} \end{pmatrix}. \tag{2.3.9}$$

By (2.3.6), (2.3.5) and (2.3.1), $h_{E'_j}$ is a height function associated with E'_j for $j = 1, \ldots, s$, h_E is the one with E and h_{Z_i} is the one with Z_i for $i = 1, \ldots, r$. By adding a bounded function to $h_{p_*F_i}$, we may assume that $h_{Z_i} \geq 0$ on $Y \setminus Z_i$ (see for example [19, Theorem B.3.2(e)]). Fix a height function $h_{H'} \geq 1$ associated with H'. Fix a height function $h_{p^*H-H'}$ associated with $p^*H - H'$ so that $h_{p^*H-H'} \geq 0$ on $Y \setminus \mathrm{Exc}(p)$. Note that there exists a constant $\gamma \geq 0$ such that

$$h_H \circ p \geq h_{p^*H-H'} + h_{H'} - \gamma \quad \text{on } Y(\overline{\mathbb{Q}}). \tag{2.3.10}$$

Since E, E'_j are numerically zero, there exists a constant $C > 0$ such that

$$|h_E| \leq C\sqrt{h_{H'}} \tag{2.3.11}$$

$$|h_{E'_j}| \leq C\sqrt{h_H}. \tag{2.3.12}$$

Let $M(f)$ be the representation matrix of the linear map $f^*\colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$ with respect to the basis $D_1, \ldots, D_r$.

Claim 2.3.5. *Let $R = \max\{1, r^2\|\vec{c}\|\|M(f)\|\}$. Then there exists $K > 0$ such that*

$$h_H(f^n(P)) \leq Kn^2R^n h_H(P)$$

for all $n \geq 1$ and $P \in X_f(\overline{\mathbb{Q}})$. Note that the constant K depends on f but $h_H, r, \vec{c}$ and $D_1, \ldots, D_r$ do not depend on f.

Proof of the claim. Let $P \in X_f(\overline{\mathbb{Q}})$. Note that p^{-1} is defined at $f^i(P)$ for every $i \geq 0$. For $n \geq 1$

$$\begin{aligned} &h_H(f^n(P)) \qquad (2.3.13)\\ =&(h_H \circ g)(p^{-1}f^{n-1}(P)) - \left\langle A\vec{c}, \mathbf{h}_{p_*\vec{F}} \circ p \right\rangle (p^{-1}f^{n-1}(P)) + \left\langle A\vec{c}, \mathbf{h}_{p_*\vec{F}} \right\rangle (f^{n-1}(P)) \end{aligned}$$

by (2.3.7)(2.3.8),

$$= \left\langle A\vec{c}, \mathbf{h}_{\vec{F}} - \mathbf{h}_{p_*\vec{F}} \circ p \right\rangle (p^{-1}f^{n-1}(P)) + h_E(p^{-1}f^{n-1}(P)) + \left\langle BA\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-1}(P)) + \left\langle A\vec{c}, \mathbf{h}_{\vec{E'}} \right\rangle (f^{n-1}(P))$$

by (2.3.9),

$$= \left\langle \vec{c}, -\mathbf{h}_{\vec{Z}} \right\rangle (p^{-1}f^{n-1}(P)) + h_E(p^{-1}f^{n-1}(P)) + \left\langle BA\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-1}(P)) + \left\langle \vec{c}, {}^{\mathrm{t}}A\mathbf{h}_{\vec{E'}} \right\rangle (f^{n-1}(P))$$

since $h_{Z_i} \geq 0$ on $Y \setminus \mathrm{Exc}(p)$,

$$\leq h_E(p^{-1}f^{n-1}(P)) + \left\langle BA\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-1}(P)) + \left\langle \vec{c}, {}^{\mathrm{t}}A\mathbf{h}_{\vec{E'}} \right\rangle (f^{n-1}(P))$$

by (2.3.4)(2.3.11)(2.3.12),

$$\leq r^2\|\vec{c}\|\|BA\|h_H(f^{n-1}(P)) + r\|\vec{c}\|C\sqrt{h_H(f^{n-1}(P))} + C\sqrt{h_{H'}(p^{-1}(f^{n-1}(P)))}$$

by (2.3.10) and $h_{p^*H-H'} \geq 0$ on $Y \setminus \mathrm{Exc}(p)$,

$$\leq r^2\|\vec{c}\|\|BA\|h_H(f^{n-1}(P)) + r\|\vec{c}\|C\sqrt{h_H(f^{n-1}(P))} + C\sqrt{h_H(f^{n-1}(P)) + \gamma}.$$

Note that C, γ depend on f. On the other hand, $r, H, D_1, \dots, D_r$, and h_H do not depend on f. Thus $\vec{c}$ also does not depend on f.

Since BA is the representation matrix of f^* with respect to $D_1, \dots, D_r$, $BA = M(f)$ and $R = \max\{1, r^2\|\vec{c}\|\|BA\|\}$. Then, dividing the both sides of (2.3.13) by R^n, we get

$$\frac{h_H(f^n(P))}{R^n} \leq \frac{h_H(f^{n-1}(P))}{R^{n-1}} + r\|\vec{c}\|C\sqrt{\frac{h_H(f^{n-1}(P))}{R^{n-1}}} + C\sqrt{\frac{h_H(f^{n-1}(P))}{R^{n-1}} + \gamma}\ .$$

Let

$$a_n = \frac{h_H(f^n(P))}{R^n} \quad \text{for } n \geq 0.$$

Then $a_n > 0$ and $a_0 = h_H(P)$ and the sequence $(a_n)_n$ satisfies the following inequality.

$$a_n \leq a_{n-1} + r\|\vec{c}\|C\sqrt{a_{n-1}} + C\sqrt{a_{n-1} + \gamma}$$

By Lemma 2.5.1, there exist a constant $K > 0$ independent of n, P such that

$$a_n \leq Kn^2 a_0 \quad \text{for all } n \geq 1.$$

Therefore

$$h_H(f^n(P)) \leq Kn^2 R^n h_H(P).$$

Thus we get the claim. □

Now, fix any positive real number $\epsilon > 0$. Let $\delta = \delta_f$. Let $M(f^k)$ be the representation matrix of $(f^k)^*\colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$ with respect to the basis $D_1, \ldots, D_r$. Since $\lim_{k\to\infty} \|M(f^k)\|^{1/k} = \delta$, there exists a positive integer $k > 0$ such that

$$\frac{\|M(f^k)\|}{(\delta+\epsilon)^k} r^2 \|\vec{c}\| < 1. \tag{2.3.14}$$

Fix such a k. We apply the claim to f^k in the place of f. Then,

$$h_H(f^{kn}(P)) \leq Kn^2 \left(\frac{R}{(\delta+\epsilon)^k}\right)^n (\delta+\epsilon)^{kn} h_H(P).$$

Recall $R = \max\{1, r^2\|\vec{c}\|\|M(f^k)\|\}$. Thus, by (2.3.14)

$$\frac{R}{(\delta+\epsilon)^k} < 1.$$

Thus there exists a constant K' such that

$$Kn^2 \left(\frac{R}{(\delta+\epsilon)^k}\right)^n \leq K'$$

for all n. Then we get

$$h_H(f^{kn}(P)) \leq K'(\delta+\epsilon)^{kn} h_H(P).$$

By the same argument at the end of the proof of Theorem 2.2.1(2), this proves Theorem 2.3.2(2).

□

Remark 2.3.6. One can prove Theorem 2.3.1 over any ground field K such that Weil height functions can be defined. If the characteristic of K is zero, the same proof works. For the case when the characteristic of K is positive, see Appendix 2.5.2.

2.4 Picard rank one case

When the Picard number of X is one, we can say much more about the behavior of the sequence $\{h_X(f^n(P))\}_n$.

Theorem 2.4.1 (Theorem 2.1.6). *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ of Picard number one. Let $f\colon X \dashrightarrow X$ be a dominant rational self-map defined over $\overline{\mathbb{Q}}$. Fix an ample height function h_X on X.*

(1) *For any positive integer $k > 0$, there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \le Cn^2\rho((f^k)^*)^{n/k}h_X^+(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$ and $n \ge 1$.

(2) *Let $k > 0$ be a positive integer. Assume that $\rho((f^k)^*) > 1$. Then there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \le C\rho((f^k)^*)^{n/k}h_X^+(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$ and $n \ge 0$.

Proof. We use the notation in the proof of Theorem 2.3.2. For simplicity, we write $\rho_k = \rho((f^k)^*)$ for $k > 0$. We apply (2.3.13) to f^k. By the assumption $r = 1$, thus $BA = \rho_k$ is a real number. By (2.3.13),

$$\begin{aligned} h_H(f^{nk}(P)) =& - c_1 h_{Z_1}(p^{-1}f^{k(n-1)}(P)) + h_E(p^{-1}f^{k(n-1)}(P)) \\ &+ \rho_k c_1 h_{D_1}(f^{k(n-1)}(P)) + c_1 h_{E_1'}(f^{k(n-1)}(P)) \\ \le& \rho_k c_1 h_{D_1}(f^{k(n-1)}(P)) + C\sqrt{h_H(f^{k(n-1)}(P)) + \gamma} \\ &+ c_1 C\sqrt{h_H(f^{k(n-1)}(P))} \end{aligned} \tag{2.4.1}$$

Let $N = c_1 D_1 - H$. By the definition of c_1, this is a numerically zero divisor. Define

$$h_N = c_1 h_{D_1} - h_H.$$

Then, this is a height function associated with N. Thus there exists a constant $\widetilde{C} > 0$ such that

$$|h_N| \le \widetilde{C}\sqrt{h_H}.$$

Then

$$\begin{aligned} h_H(f^{nk}(P)) \le& \rho_k h_H(f^{k(n-1)}(P)) + \widetilde{C}\sqrt{h_H(f^{k(n-1)}(P))} \\ &+ C\sqrt{h_H(f^{k(n-1)}(P)) + \gamma} + c_1 C\sqrt{h_H(f^{k(n-1)}(P))}. \end{aligned}$$

Divide both sides of this inequality by ρ_k^n. By Lemma 2.5.1, there exists a constant $\widetilde{K} > 0$ (which is independent of n, P, but depends on k) such that

$$h_H(f^{nk}(P)) \leq \widetilde{K} n^2 \rho_k^{nk/k} h_H(P) \quad \text{for all } n \geq 1. \tag{2.4.2}$$

By the same argument as in (Proof of Theorem 2.3.2 $\Longrightarrow$Theorem 2.3.1), we can prove the first statement.

Now assume $\rho_k > 1$. Then

$$\frac{h_H(f^{nk}(P))}{\rho_k^n} \leq \frac{h_H(f^{k(n-1)}(P))}{\rho_k^{n-1}} + \left(\widetilde{C} + C + c_1 C\right) \frac{\sqrt{h_H(f^{k(n-1)}(P))}}{\rho_k^n} + \frac{C\sqrt{\gamma}}{\rho_k^n}$$

By (2.4.2),

$$\sqrt{h_H(f^{k(n-1)}(P))} \leq \sqrt{\widetilde{K} h_H(P)}(n-1)\rho_k^{(n-1)/2}$$

and thus

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ \left(\widetilde{C} + C + c_1 C\right) \frac{\sqrt{h_H(f^{k(n-1)}(P))}}{\rho_k^n} + \frac{C\sqrt{\gamma}}{\rho_k^n} \right\} \\ \leq &\sum_{n=1}^{\infty} \left\{ \left(\widetilde{C} + C + c_1 C\right) \frac{\sqrt{\widetilde{K} h_H(P)}(n-1)\rho_k^{(n-1)/2}}{\rho_k^n} + \frac{C\sqrt{\gamma}}{\rho_k^n} \right\}. \end{aligned}$$

Since $\rho_k > 1$, there exists a constant $\widetilde{K}_1$ (independent of n, P) such that

$$\frac{h_H(f^{nk}(P))}{\rho_k^n} \leq \widetilde{K}_1 h_H(P).$$

Thus

$$h_H(f^{nk}(P)) \leq \widetilde{K}_1 \rho_k^{nk/k} h_H(P).$$

By the same argument as in (Proof of Theorem 2.3.2 $\Longrightarrow$Theorem 2.3.1), we can prove the second statement.

□

Now, we prove the convergence of canonical heights.

Proposition 2.4.2 (Proposition 2.1.7). *Let X and f be as in Theorem 2.4.1. Assume f is algebraically stable and $\delta_f > 1$. Fix an ample height function h_X on X. Then*

$$\hat{h}_{X,f}(P) = \lim_{n\to\infty} \frac{h_X(f^n(P))}{\delta_f^n}$$

exists for all $P \in X_f(\overline{\mathbb{Q}})$.

Proof. Since any ample heights are bounded below, this follows from the following more general statement. □

Proposition 2.4.3. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. Let $f\colon X \dashrightarrow X$ be a dominant rational self-map defined over $\overline{\mathbb{Q}}$. Assume $\delta_f > 1$ and there exists a nef $\mathbb{R}$-divisor H on X such that $f^*H \equiv \delta_f H$. Fix a height function h_H associated with H. Then for any $P \in X_f(\overline{\mathbb{Q}})$, the limit*

$$\lim_{n\to\infty} \frac{h_H(f^n(P))}{\delta_f^n}$$

converges or diverges to $-\infty$.

Proof. We take a resolution of indeterminacy $p\colon Y \longrightarrow X$ of f so that p is an isomorphism outside the indeterminacy locus I_f of f:

$$\begin{array}{ccc} & Y & \\ {}^{p}\swarrow & & \searrow^{g} \\ X & \overset{f}{\dashrightarrow} & X. \end{array}$$

Write $g = f \circ p$. By negativity lemma, $p^*p_*g^*H - g^*H$ is a p-exceptional effective divisor on Y. Then as in the proof of [**?**, Proposition 21], we have $h_H \circ f \le h_{f^*H} + O(1)$ on $X \setminus I_f$ where h_H and h_{f^*H} are height functions associated with H and f^*H. Fix an ample height h_X on X. Since $f^*H \equiv \delta_f H$, we have $h_{f^*H} - \delta_f h_H = O\left(\sqrt{h_X^+}\right)$. Thus, we have

$$h_H \circ f \le \delta_f h_H + O\left(\sqrt{h_X^+}\right) \quad \text{on } X \setminus I_f.$$

Write $B = h_H \circ f - \delta_f h_H$. Then, for any $P \in X_f$,

$$\begin{aligned} h_H(f^n(x)) &= \sum_{k=0}^{n-1} \delta_f^{n-1-k}\bigl(h_H(f^{k+1}(P)) - \delta_f h_H(f^k(P))\bigr) + \delta_f^n h_H(P) \\ &= \sum_{k=0}^{n-1} \delta_f^{n-1-k} B(f^k(P)) + \delta_f^n h_H(P). \end{aligned}$$

Take $\epsilon > 0$ so that $\sqrt{\delta_f + \epsilon} < \delta_f$. By Theorem 2.1.1, there exists $C > 0$ such that $B(f^k(P)) \le C\sqrt{\delta_f + \epsilon}^k$ for all $k \ge 0$. Set

$$a_k = \frac{B(f^k(P))}{\sqrt{\delta_f + \epsilon}^k}.$$

Note that a_k is bounded above. Then

$$\begin{aligned}\frac{h_H(f^n(P))}{\delta_f^n} &= h_H(P) + \sum_{k=0}^{n-1} \frac{B(f^k(P))}{\delta_f^{k+1}} \\ &= h_H(P) + \frac{1}{\delta_f} \sum_{k=0}^{n-1} a_k \left(\frac{\sqrt{\delta_f + \epsilon}}{\delta_f} \right)^k \\ &= h_H(P) + \frac{1}{\delta_f} \left\{ \sum_{\substack{0 \le k \le n-1 \\ a_k \ge 0}} a_k \left(\frac{\sqrt{\delta_f + \epsilon}}{\delta_f} \right)^k - \sum_{\substack{0 \le k \le n-1 \\ a_k < 0}} (-a_k) \left(\frac{\sqrt{\delta_f + \epsilon}}{\delta_f} \right)^k \right\}.\end{aligned}$$

The first summation in the bracket is convergent since a_k is bounded above and the second summation is monotonically increasing. Hence, the claim follows.

□

2.5 Appendix

2.5.1 Lemmas

Lemma 2.5.1. *Let $(a_n)_{n \ge 0}$ be a sequence of positive real numbers with $a_0 \ge 1$ which satisfies*

$$a_n \le a_{n-1} + C_1 \left(\sqrt{a_{n-1}} + \sqrt{a_{n-1} + C_2} \right)$$

for all $n \ge 1$. Here C_1, C_2 are non-negative constants. Then there exists a positive constant $\widetilde{C}$ depending only on C_1, C_2 such that

$$a_n \le \widetilde{C} n^2 a_0$$

for all $n \ge 1$.

Proof. Let $C > 0$ be a large positive constant which we will see how large it should be later. Let $b_n = a_n / C n^2$ for $n \ge 1$. Then

$$b_n \le \left(1 - \frac{1}{n}\right)^2 b_{n-1} + C_1 \frac{n-1}{\sqrt{C} n^2} \left(\sqrt{b_{n-1}} + \sqrt{b_{n-1} + \frac{C_2}{C(n-1)^2}} \right)$$

for $n \ge 2$.

If, say, $C \geq \max\{9C_1^2, C_2\}$, then we can easily show that $b_n \leq \max\{b_{n-1}, 1\}$. Thus we get $a_n = Cn^2 b_n \leq Cn^2 \max\{1, b_1\} = n^2 \max\{C, a_1\}$. Since $a_1 \leq a_0 \left(1 + C_1(1 + \sqrt{1 + C_2})\right)$, we have $a_n \leq n^2 \max\{C, Ca_0\} = Cn^2 a_0$ if $C \geq \max\{9C_1^2, C_2, 1 + C_1(1 + \sqrt{1 + C_2})\}$.

□

Lemma 2.5.2. *Let $(a_n)_{n \geq 0}$ be a positive real sequence with $a_0 \geq 1$ which satisfies*

$$a_n \leq C(a_0 + \sqrt{a_0} + \sqrt{a_1} + \cdots + \sqrt{a_{n-1}}) \quad \text{for all } n \geq 1$$

where C is a positive constant. For any $\widetilde{C} \geq 1$ such that $\widetilde{C} \geq \max\{\frac{C^2}{4}, 1 + C\}$, we have

$$a_n \leq \widetilde{C} n^2 a_0 \qquad \text{for all } n \geq 1.$$

Proof. Let $(b_n)_{n \geq 0}$ be a sequence such that

$$\begin{aligned} &b_0 = a_0 \\ &b_n = C\left(b_0 + \sqrt{b_0} + \cdots + \sqrt{b_{n-1}}\right) \qquad \text{for all } n \geq 1. \end{aligned}$$

Then clearly $a_n \leq b_n$ for all $n \geq 0$. By the definition of b_n, we have $b_{n+1} = b_n + C\sqrt{b_n}$. Thus the statement follows from Lemma 2.5.1 and its proof. □

2.5.2 Positive characteristic

In this section, we briefly remark how to modify the proof of Theorem 2.3.2 when the ground field has positive characteristic. Let K be an algebraically closed field with height function (e.g. $\overline{\mathbb{F}_q(t)}$ the algebraic closure of the function field over a finite field).

Proposition 2.5.3. *Let $f \colon X \dashrightarrow Z$ be a dominant rational map of smooth projective varieties over K.*

(1) *Let Y be a projective variety with a birational morphism $p \colon Y \longrightarrow X$ and a morphism $g \colon Y \longrightarrow Z$ such that $f \circ p = g$. For a Cartier divisor D on Z, we define $f^*D = p_*[g^*D]$. Here $[g^*D]$ is the codimension one cycle associated with the Cartier divisor g^*D. Then, the divisor f^*D is independent of the choice of Y.*

(2) *Let $\Gamma \subset X \times Z$ be the graph of f. For a Cartier divisor D on Z, we have $f^*D = \mathrm{pr}_{1*}(\mathrm{pr}_2^* D \cdot \Gamma)$.*

(3) *The map f^* induces a homomorphism $f^*\colon N^1(Z) \longrightarrow N^1(X)$. This definition of pull-back coincides the definition in [8, 53].*

For a dominant rational self-map $f\colon X \dashrightarrow X$, let $p\colon Y \longrightarrow X$ be a blow-up of X with a suitable ideal sheaf $\mathcal{I}$ whose support is the indeterminacy locus I_f. More precisely, take an embedding $i\colon X \longrightarrow \mathbb{P}^N$. Then the linear system defining the morphism $i \circ f\colon X \setminus I_f \longrightarrow \mathbb{P}^N$ is uniquely extended to a linear system on X. Then we can take $\mathcal{I}$ to be the base ideal of this linear system. Then there exists a surjective morphism $g\colon Y \longrightarrow X$ such that $g = f \circ p$. Using this setting, we can argue as in the proof of Theorem 2.3.2.

The only non-trivial point is the following. In the proof, we need to bound height functions associated with numerically zero divisors. Precisely, we need the inequality (2.3.11). On a smooth projective variety, this is well-known (see for example [19]). Now we need this inequality on Y, which is possibly singular. Actually, this inequality holds on any projective variety.

Lemma 2.5.4 (see for example [30, Theorem 9.5.4])**.** *Let Y be a normal projective variety over an algebraically closed field. Then there exists a morphism $\alpha\colon Y \longrightarrow A$ with A is an Abelian variety with the following property. For any line bundle L on Y which is algebraically equivalent to zero, there exists a line bundle M on A which is algebraically equivalent to zero such that $L \simeq \alpha^* M$.*

By this lemma and the argument in the proof of [19, Theorem B.5.9], we can easily prove the following.

Proposition 2.5.5. *Let Y be a projective variety over K and E, H divisors on Y with E numerically equivalent to zero and H ample. Fix height functions h_E, h_H associated with these divisors with $h_H \geq 1$. Then there exists a positive constant $C > 0$ such that*

$$|h_E| \leq C\sqrt{h_H}$$

on $Y(K)$.

Chapter 3

Endomorphisms on smooth projective surfaces

(Joint work with Kaoru Sano and Takahiro Shibata.)

3.1 Summary

Let k be a number field, X a smooth projective variety over $\overline{k}$, and $f\colon X \dashrightarrow X$ a dominant rational self-map on X over $\overline{k}$.

Let H be an ample divisor on X defined over $\overline{k}$. Recall that the (*first*) *dynamical degree* of f is defined by

$$\delta_f := \lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n}.$$

The *arithmetic degree* of f at a $\overline{k}$-rational point $P \in X_f(\overline{k})$ is defined by

$$\alpha_f(P) := \lim_{n\to\infty} h_H^+(f^n(P))^{1/n}$$

if the limit on the right hand side exists. Here, $h_H\colon X(\overline{k}) \longrightarrow [0,\infty)$ is the (absolute logarithmic) Weil height function associated with H, and we put $h_H^+ := \max\{h_H, 1\}$.

In this chapter, we consider the following part of Kawaguchi-Silverman conjecture.

Conjecture 3.1.1. *For every $\overline{k}$-rational point $P \in X_f(\overline{k})$, the arithmetic degree $\alpha_f(P)$ exists. Moreover, if the forward f-orbit $\mathcal{O}_f(P)$ is Zariski dense in X, the arithmetic degree $\alpha_f(P)$ is equal to the dynamical degree δ_f, i.e., we have*

$$\alpha_f(P) = \delta_f.$$

Remark 3.1.2. We have in general $\overline{\alpha}_f(P) \leq \delta_f$ (see Chapter 2, [29, Theorem 4], [39, Theorem 1.4]). Hence, in order to prove Conjecture 3.1.1, it is enough to prove the opposite inequality $\underline{\alpha}_f(P) \geq \delta_f$.

In this chapter, we prove Conjecture 3.1.1 for any endomorphism on any smooth projective surface:

Theorem 3.1.3. *Let k be a number field, X a smooth projective surface over $\overline{k}$, and $f\colon X \longrightarrow X$ a surjective endomorphism on X. Then Conjecture 3.1.1 holds for f.*

Remark 3.1.4. Kawaguchi proved Conjecture 3.1.1 for automorphism of smooth projective surfaces [25].

As by-products of our arguments, we also obtain the following two cases for which Conjecture 3.1.1 holds:

Theorem 3.1.5 (Theorem 3.2.6)**.** *Let k be a number field, X a smooth projective irrational surface over $\overline{k}$, and $f\colon X \dashrightarrow X$ a birational automorphism on X. Then Conjecture 3.1.1 holds for f.*

Theorem 3.1.6 (Theorem 3.2.7)**.** *Let k be a number field, X a smooth projective toric variety over $\overline{k}$, and $f\colon X \longrightarrow X$ a toric surjective endomorphism on X. Then Conjecture 3.1.1 holds for f.*

In [37], Lin gives a precise description of the arithmetic degrees of toric self-maps on toric varieties.

As we will see in the proof of Theorem 3.1.3, there does not always exist a Zariski dense orbit for a given self-map. For instance, a self-map cannot have a Zariski dense orbit if it is a self-map over a variety of positive Kodaira dimension. So it is also important to consider whether a self-map has a $\overline{k}$-rational point whose orbit has full arithmetic complexity, that is, whose arithmetic degree coincides with the dynamical degree. We prove that such a point always exists for any surjective endomorphism on any smooth projective variety.

Theorem 3.1.7. *Let k be a number field, X a smooth projective variety over $\overline{k}$, and $f\colon X \longrightarrow X$ a surjective endomorphism on X. Then there exists a $\overline{k}$-rational point $P \in X(\overline{k})$ such that $\alpha_f(P) = \delta_f$.*

If f is an automorphism, we can construct a "large" collection of points whose orbits have full arithmetic complexity.

Theorem 3.1.8. *Let k be a number field, X a smooth projective variety over $\overline{k}$, and $f\colon X \longrightarrow X$ an automorphism. Then there exists a subset $S \subset X(\overline{k})$ which satisfies all of the following conditions.*

(1) *For every $P \in S$, $\alpha_f(P) = \delta_f$.*

(2) *For $P, Q \in S$ with $P \neq Q$, $\mathcal{O}_f(P) \cap \mathcal{O}_f(Q) = \emptyset$.*

(3) *S is Zariski dense in X.*

Notation

- Throughout this chapter, we fix a number field k.
- A *variety* always means an integral separated scheme of finite type over $\overline{k}$ in this chapter.
- An *endomorphism* on a variety X means a morphism from X to itself defined over $\overline{k}$. A *non-invertible endomorphism* is a surjective endomorphism which is not an automorphism.
- A *curve* (resp. *surface*) simply means a smooth projective variety of dimension 1 (resp. dimension 2) unless otherwise stated.
- For any curve C, the genus of C is denoted by $g(C)$.
- When we say that P is a point of X or write as $P \in X$, it means that P is a $\overline{k}$-valued point of X.

Outline of this chapter

In Section 3.2, at first we recall some lemmata about reduction for Conjecture 3.1.1, which were proved in [47] and [51]. Then, we prove the birational invariance of arithmetic degree. As its corollary, we prove Theorem 3.1.5 by reducing to the automorphism case, using minimal models. And we also prove Theorem 3.1.6. In Section 3.3, by using the Enriques classification of smooth projective surfaces, we reduce Theorem 3.1.3 to three cases, i.e. the case of $\mathbb{P}^1$-bundles, hyperelliptic surfaces, and surfaces of Kodaira dimension one. In Section 3.4 we recall fundamental properties of $\mathbb{P}^1$-bundles over curves. In Section 3.5, Section 3.6, and Section 3.7, we prove Theorem 3.1.3 in each case explained in Section 3.3. Finally, in Section 3.8, we prove Theorem 3.1.7 and Theorem 3.1.8. In the proof of Theorem 3.1.7, we use a nef $\mathbb{R}$-divisor D that satisfies$f^*D \equiv \delta_f D$.

3.2 Some reductions for Conjecture 3.1.1

3.2.1 Reductions

We recall some lemmata which are useful to reduce the proof of some cases of Conjecture 3.1.1 to easier cases.

Lemma 3.2.1. *Let X be a smooth projective variety and $f\colon X \longrightarrow X$ a surjective endomorphism. Then Conjecture 3.1.1 holds for f if and only if Conjecture 3.1.1 holds for f^t for some $t \geq 1$.*

Proof. See [47, Lemma 3.3]. □

Lemma 3.2.2 ([51, Lemma 6]). *Let $\psi\colon X \longrightarrow Y$ be a finite morphism between smooth projective varieties. Let $f_X\colon X \longrightarrow X$ and $f_Y\colon Y \longrightarrow Y$ be surjective endomorphisms on X and Y, respectively. Assume that $\psi \circ f_X = f_Y \circ \psi$.*

(i) *For any $P \in X(\overline{k})$, we have $\alpha_{f_X}(P) = \alpha_{f_Y}(\psi(P))$.*

(ii) *Assume that ψ is surjective. Then Conjecture 3.1.1 holds for f_X if and only if Conjecture 3.1.1 holds for f_Y.*

Proof. (i) Take any point $P \in X(\overline{k})$. Let H be an ample divisor on Y. Then ψ^*H is an ample divisor on X. Hence we have

$$\begin{aligned}\alpha_{f_X}(P) &= \lim_{n\to\infty} h^+_{\psi^*H}(f_X^n(P))^{1/n} \\ &= \lim_{n\to\infty} h^+_{H}(\psi \circ f_X^n(P))^{1/n} \\ &= \lim_{n\to\infty} h^+_{H}(f_Y^n \circ \psi(P))^{1/n} \\ &= \alpha_{f_Y}(\psi(P)).\end{aligned}$$

(ii) For a point $P \in X(\overline{k})$, the forward f_X-orbit $\mathcal{O}_{f_X}(P)$ is Zariski dense in X if and only if the forward f_Y-orbit $\mathcal{O}_{f_Y}(\psi(P))$ is Zariski dense in Y since ψ is a finite surjective morphism. Moreover we have $\dim X = \dim Y$. So we obtain

$$\begin{aligned}\delta_{f_X} &= \lim_{n\to\infty} ((f_X^n)^*\psi^*H \cdot (\psi^*H)^{\dim X-1})^{1/n} \\ &= \lim_{n\to\infty} (\psi^*(f_Y^n)^*H \cdot (\psi^*H)^{\dim Y-1})^{1/n} \\ &= \lim_{n\to\infty} (\deg(\psi)((f_Y^n)^*H \cdot H^{\dim Y-1}))^{1/n} \\ &= \delta_{f_Y}.\end{aligned}$$

Therefore the assertion follows. □

3.2.2 Birational invariance of the arithmetic degree

We show that the arithmetic degree is invariant under birational conjugacy.

Lemma 3.2.3. *Let $\mu\colon X \dashrightarrow Y$ be a birational map of smooth projective varieties. Take Weil height functions h_X, h_Y associated with ample divisors H_X, H_Y on X, Y, respectively. Then there are constants $M \in \mathbb{R}_{>0}$ and $M' \in \mathbb{R}$ such that*

$$h_X(P) \geq M h_Y(\mu(P)) + M'$$

for any $P \in X(\overline{k}) \setminus I_\mu(\overline{k})$.

Proof. Replacing H_Y by a positive multiple, we may assume that H_Y is very ample. Take a smooth projective variety Z and a birational morphism $p\colon Z \longrightarrow X$ such that p is isomorphic over $X \setminus I_\mu$ and $q = \mu \circ p\colon Z \longrightarrow Y$ is a morphism. Let $\{F_i\}_{i=1}^r$ be the collection of prime p-exceptional divisors. We take H_Y as not containing $q(F_i)$ for any i, so q^*H_Y does not contain F_i for any i. Then $E = p^*p_*q^*H_Y - q^*H_Y$ is an effective divisor contained in the exceptional locus of p. Take a sufficiently large integer N such that $NH_X - p_*q^*H_Y$ is very ample. Then, for $P \in X(\overline{k}) \setminus I_\mu$, we have

$$\begin{aligned} h_X(P) &= \frac{1}{N}(h_{NH_X - p_*q^*H_Y}(P) + h_{p_*q^*H_Y}(P)) + O(1) \\ &\geq \frac{1}{N} h_{p_*q^*H_Y}(P) + O(1) \\ &= \frac{1}{N} h_{p^*p_*q^*H_Y}(p^{-1}(P)) + O(1) \\ &= \frac{1}{N} h_{q^*H_Y}(p^{-1}(P)) + h_E(p^{-1}(P)) + O(1) \\ &= \frac{1}{N} h_Y(\mu(P)) + h_E(p^{-1}(P)) + O(1). \end{aligned}$$

We know that $h_E \geq O(1)$ on $Z(\overline{k}) \setminus \operatorname{Supp} E$ (cf. [19, Theorem B.3.2(e)]). Since $\operatorname{Supp} E \subset p^{-1}(I_\mu)$, $h_E(p^{-1}(P)) \geq O(1)$ for $P \in X(\overline{k}) \setminus I_\mu$. Finally, we obtain that $h_X(P) \geq (1/N)h_Y(\mu(P)) + O(1)$ for $P \in X(\overline{k}) \setminus I_\mu$. □

Theorem 3.2.4. *Let $f\colon X \dashrightarrow X$ and $g\colon Y \dashrightarrow Y$ be dominant rational self-maps on smooth projective varieties and $\mu\colon X \dashrightarrow Y$ a birational map such that $g \circ \mu = \mu \circ f$.*

(i) *Let* $U \subset X$ *be a Zariski open subset such that* $\mu|_U \colon U \longrightarrow \mu(U)$ *is an isomorphism. Then* $\overline{\alpha}_f(P) = \overline{\alpha}_g(\mu(P))$ *and* $\underline{\alpha}_f(P) = \underline{\alpha}_g(\mu(P))$ *for* $P \in X_f(\overline{k}) \cap \mu^{-1}(Y_g(\overline{k}))$ *such that* $\mathcal{O}_f(P) \subset U(\overline{k})$.

(ii) *Take* $P \in X_f(\overline{k}) \cap \mu^{-1}(Y_g(\overline{k}))$. *Assume that* $\mathcal{O}_f(P)$ *is Zariski dense in* X *and both* $\alpha_f(P)$ *and* $\alpha_g(\mu(P))$ *exist. Then* $\alpha_f(P) = \alpha_g(\mu(P))$.

Proof. (i) Using Lemma 3.2.3 for both μ and μ^{-1}, there are constants $M_1, L_1 \in \mathbb{R}_{>0}$ and $M_2, L_2 \in \mathbb{R}$ such that

$$M_1 h_Y(\mu(P)) + M_2 \leq h_X(P) \leq L_1 h_Y(\mu(P)) + L_2 \qquad (*)$$

for $P \in U(\overline{k})$. The claimed equalities follow from $(*)$.

(ii) Since $\mathcal{O}_f(P)$ is Zariski dense in X, we can take a subsequence $\{f^{n_k}(P)\}_k$ of $\{f^n(P)\}_n$ contained in U. Using $(*)$ again, it follows that

$$\alpha_f(P) = \lim_{k\to\infty} h_X^+(f^{n_k}(P))^{1/n_k} = \lim_{k\to\infty} h_Y^+(g^{n_k}(\mu(P)))^{1/n_k} = \alpha_g(\mu(P)).$$

□

Remark 3.2.5. In [50], Silverman dealt with a height function on $\mathbb{G}_m^n$ induced by an open immersion $\mathbb{G}_m^n \hookrightarrow \mathbb{P}^n$ and proved Conjecture 3.1.1 for monomial maps on $\mathbb{G}_m^n$. It seems that it had not be checked in the literature that the arithmetic degrees of endomorphisms on quasi-projective varieties do not depend on the choice of open immersions to projective varieties. Now by Theorem 3.2.4, the arithmetic degree of a rational self-map on a quasi-projective variety at a point does not depend on the choice of an open immersion of the quasi-projective variety to a projective variety. Furthermore, by the birational invariance of dynamical degrees, we can state Conjecture 3.1.1 for rational self-maps on quasi-projective varieties, such as semi-abelian varieties.

3.2.3 Applications of the birational invariance

In this subsection, we prove Theorem 3.1.5 and Theorem 3.1.6 as applications of Theorem 3.2.4.

Theorem 3.2.6 (Theorem 3.1.5)**.** *Let* X *be an irrational surface and* $f \colon X \dashrightarrow X$ *a birational automorphism on* X. *Then Conjecture 3.1.1 holds for* f.

Proof. Take a point $P \in X_f(\overline{k})$. If $\mathcal{O}_f(P)$ is finite, the limit $\alpha_f(P)$ exists and is equal to 1. Next, assume that the closure $\overline{\mathcal{O}_f(P)}$ of $\mathcal{O}_f(P)$ has dimension

1. Let Z be the normalization of $\overline{\mathcal{O}_f(P)}$ and $\nu\colon Z \longrightarrow X$ the induced morphism. Then an endomorphism $g\colon Z \longrightarrow Z$ satisfying $\nu \circ g = f \circ \nu$ is induced. Take a point $P' \in Z$ such that $\nu(P') = P$. Then $\alpha_g(P') = \alpha_f(P)$ since ν is finite by Lemma 3.2.2 (i). It follows from [28, Theorem 2] that $\alpha_g(P')$ exists (note that [28, Theorem 2] holds for possibly non-surjective endomorphisms on possibly reducible normal varieties). Therefore $\alpha_f(P)$ exists.

Finally, assume that $\mathcal{O}_f(P)$ is Zariski dense. If $\delta_f = 1$, then $1 \leq \underline{\alpha}_f(P) \leq \overline{\alpha}_f(P) \leq \delta_f = 1$ by Remark 3.1.2, so $\alpha_f(P)$ exists and $\alpha_f(P) = \delta_f = 1$. So we may assume that $\delta_f > 1$. Since X is irrational and $\delta_f > 1$, $\kappa(X)$ must be non-negative (cf. [9, Theorem 0.4, Proposition 7.1 and Theorem 7.2]). Take a birational morphism $\mu\colon X \longrightarrow Y$ to the minimal model Y of X and let $g\colon Y \dashrightarrow Y$ be the birational automorphism on Y defined as $g = \mu \circ f \circ \mu^{-1}$. Then g is in fact an automorphism since, if g has indeterminacy, Y must have a K_Y-negative curve. It is obvious that $\mathcal{O}_g(\mu(P))$ is also Zariski dense in Y. Since $\mu(\mathrm{Exc}(\mu))$ is a finite set, there is a positive integer n_0 such that $\mu(f^n(P)) = g^n(\mu(P)) \notin \mu(\mathrm{Exc}(\mu))$ for $n \geq n_0$. So we have $f^n(P) \notin \mathrm{Exc}(\mu)$ for $n \geq n_0$. Replacing P by $f^{n_0}(P)$, we may assume that $\mathcal{O}_f(P) \subset X \setminus \mathrm{Exc}(\mu)$. Applying Theorem 3.2.4 (i) to P, it follows that $\alpha_f(P) = \alpha_g(\mu(P))$. We know that $\alpha_g(\mu(P))$ exists since g is a morphism. So $\alpha_f(P)$ also exists. The equality $\alpha_g(\mu(P)) = \delta_g$ holds as a consequence of Conjecture 3.1.1 for automorphisms on surfaces (cf. Remark 3.1.4). Since the dynamical degree is invariant under birational conjugacy, it follows that $\delta_g = \delta_f$. So we obtain the equality $\alpha_f(P) = \delta_f$. □

Theorem 3.2.7 (Theorem 3.1.6)**.** *Let X be a smooth projective toric variety and $f\colon X \longrightarrow X$ a toric surjective endomorphism on X. Then Conjecture 3.1.1 holds for f.*

Proof. Let $\mathbb{G}_m^d \subset X$ be the torus embedded as an open dense subset in X. Then $f|_{\mathbb{G}_m^d} : \mathbb{G}_m^d \longrightarrow \mathbb{G}_m^d$ is a homomorphism of algebraic groups by assumtion. Let $\mathbb{G}_m^d \subset \mathbb{P}^d$ be the natural embedding of $\mathbb{G}_m^d$ to the projective space $\mathbb{P}^d$ and $g\colon \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ be the induced rational self-map. Then g is a monomial map.

Take $P \in X(\overline{k})$ such that $\mathcal{O}_f(P)$ is Zariski dense. Note that $\alpha_f(P)$ exists since f is a morphism. Since $\mathcal{O}_f(P)$ is Zariski dense and $f(\mathbb{G}_m^d) \subset \mathbb{G}_m^d$, there is a positive integer n_0 such that $f^n(P) \in \mathbb{G}_m^d$ for $n \geq n_0$. By replacing P by $f^{n_0}(P)$, we may assume that $\mathcal{O}_f(P) \subset \mathbb{G}_m^d$. Applying Theorem 3.2.4 (i) to P, it follows that $\alpha_f(P) = \alpha_g(P)$.

The equality $\alpha_g(P) = \delta_g$ holds as a consequence of Conjecture 3.1.1

for monomial maps (cf. Remark 1.3.3 (2)). Since the dynamical degree is invariant under birational conjugacy, it follows that $\delta_g = \delta_f$. So we obtain the equality $\alpha_f(P) = \delta_f$. □

3.3 Endomorphisms on surfaces

We start to prove Theorem 3.1.3. Since Conjecture 3.1.1 for automorphisms on surfaces is already proved by Kawaguchi (see Remark 3.1.4), it is sufficient to prove Theorem 3.1.3 for *non-invertible* endomorphisms, that is, surjective endomorphisms which are not automorphisms.

Let $f\colon X \longrightarrow X$ be a non-invertible endomorphism on a surface. We divide the proof of Theorem 3.1.3 according to the Kodaira dimension of X.

(I) $\kappa(X) = -\infty$; we need the following result due to Nakayama.

Lemma 3.3.1 (cf. [45, Proposition 10])**.** *Let $f\colon X \longrightarrow X$ be a non-invertible endomorphism on a surface X with $\kappa(X) = -\infty$. Then there is a positive integer m such that $f^m(E) = E$ for any irreducible curve E on X with negative self-intersection.*

Proof. See [45, Proposition 10]. □

Let $\mu\colon X \longrightarrow X'$ be the contraction of a (-1)-curve E on X. By Lemma 3.3.1, there is a positive integer m such that $f^m(E) = E$. Then f^m induces an endomorphism $f'\colon X' \longrightarrow X'$ such that $\mu \circ f^m = f' \circ \mu$. Using Lemma 3.2.1 and Theorem 3.2.4, the assertion of Theorem 3.1.3 for f follows from that for f'. Continuing this process, we may assume that X is relatively minimal.

When X is irrational and relatively minimal, X is a $\mathbb{P}^1$-bundle over a curve C with $g(C) \geq 1$.

When X is rational and relatively minimal, X is isomorphic to $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ for some $n \geq 0$ with $n \neq 1$. Note that Conjecture 3.1.1 holds for surjective endomorphisms on projective spaces (see Remark 1.3.3 (1)).

(II) $\kappa(X) = 0$; for surfaces with non-negative Kodaira dimension, we use the following result due to Fujimoto.

Lemma 3.3.2 (cf. [14, Lemma 2.3 and Proposition 3.1])**.** *Let $f\colon X \longrightarrow X$ be a non-invertible endomorphism on a surface X with $\kappa(X) \geq 0$. Then X is minimal and f is étale.*

Proof. See [14, Lemma 2.3 and Proposition 3.1] □

So X is either an abelian surface, a hyperelliptic surface, a K3 surface, or an Enriques surface. Since f is étale, we have $\chi(X, \mathcal{O}_X) = \deg(f)\chi(X, \mathcal{O}_X)$. Now $\deg(f) \geq 2$ by assumption, so $\chi(X, \mathcal{O}_X) = 0$ (cf. [14, Corollary 2.4]). Hence X must be either an abelian surface or a hyperelliptic surface because K3 surfaces and Enriques surfaces have non-zero Euler characteristics. Note that Conjecture 3.1.1 is valid for endomorphisms on abelian varieties (see Remark 1.3.3 (5)).

(III) $\kappa(X) = 1$; this case will be treated in Section 3.7.

(IV) $\kappa(X) = 2$; the following fact is well-known.

Lemma 3.3.3. *Let X be a smooth projective variety of general type. Then any surjective endomorphism on X is an automorphism. Furthermore, the group of automorphisms* $\mathrm{Aut}(X)$ *on X has finite order.*

Proof. See [14, Proposition 2.6], [22, Theorem 11.12], or [38, Corollary 2]. □

So there is no non-invertible endomorphism on X. As a summary, the remaining cases for the proof of Theorem 3.1.3 are the following:

- Non-invertible endomorphisms on $\mathbb{P}^1$-bundles over a curve.
- Non-invertible endomorphisms on hyperelliptic surfaces.
- Non-invertible endomorphisms on surfaces of Kodaira dimension 1.

These three cases are studied in Sections 3.4-3.7 below.

Remark 3.3.4. Fujimoto and Nakayama gave a complete classification of surfaces which admit non-invertible endomorphisms (cf. [16, Theorem 1.1], [14, Proposition 3.3], [45, Theorem 3], and [15, Appendix to Section 4]).

3.4 Some properties of $\mathbb{P}^1$-bundles over curves

In this section, we recall and prove some properties of $\mathbb{P}^1$-bundles (see [18, Chapter V.2], [20], [21] for details). In this section, let X be a $\mathbb{P}^1$-bundle over a curve C. Let $\pi\colon X \longrightarrow C$ be the projection.

Proposition 3.4.1. *We can represent X as $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank 2 on C such that $H^0(\mathcal{E}) \neq 0$ but $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for all invertible sheaves $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$. The integer $e := -\deg \mathcal{E}$ does not depend on the choice of such $\mathcal{E}$. Furthermore, there is a section $\sigma\colon C \longrightarrow X$ with image C_0 such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$.*

Proof. See [18, Proposition 2.8]. □

Lemma 3.4.2. *The Picard group and the Néron–Severi group of X have the following structure:*

$$\operatorname{Pic}(X) \cong \mathbb{Z} \oplus \pi^* \operatorname{Pic}(C),$$
$$\operatorname{NS}(X) \cong \mathbb{Z} \oplus \pi^* \operatorname{NS}(C) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Furthermore, the image C_0 of the section $\sigma\colon C \longrightarrow X$ in Proposition 3.4.1 generates the first direct factor of $\operatorname{Pic}(X)$ *and* $\operatorname{NS}(X)$*.*

Proof. See [18, V, Proposition 2.3]. □

Lemma 3.4.3. *Let $F \in \operatorname{NS}(X)$ be a fiber $\pi^{-1}(p) = \pi^* p$ over a point $p \in C(\overline{k})$, and e the integer defined in Proposition 3.4.1. Then the intersection numbers of generators of* $\operatorname{NS}(X)$ *are as follows.*

$$F \cdot F = 0,$$
$$F \cdot C_0 = 1,$$
$$C_0 \cdot C_0 = -e.$$

Proof. It is easy to see that the equalities $F \cdot F = 0$ and $F \cdot C_0 = 1$ hold. For the last equality, see [18, V, Proposition 2.9]. □

We say that f *preserves fibers* if there is an endomorphism f_C on C such that $\pi \circ f = f_C \circ \pi$. In our situation, since there is a section $\sigma\colon C \longrightarrow X$, f preserves fibers if and only if, for any point $p \in C$, there is a point $q \in C$ such that $f(\pi^{-1}(p)) \subset \pi^{-1}(q)$.

The following lemma appears in [1, p. 18] in more general form. But we need it only in the case of $\mathbb{P}^1$-bundles on a curve, and the proof in general case is similar to our case. So we deal only with the case of $\mathbb{P}^1$-bundle on a curve.

Lemma 3.4.4. *For any surjective endomorphism f on X, the iterate f^2 preserves fibers.*

Proof. By the projection formula, the fibers of $\pi\colon X \longrightarrow C$ can be characterized as connected curves having intersection number zero with any fiber $F_p = \pi^* p$, $p \in C$. Hence, to check that the iterate f^2 sends fibers to fibers, it suffices to show that $(f^2)^*(\pi^* \operatorname{NS}(C)_{\mathbb{R}}) = \pi^* \operatorname{NS}(C)_{\mathbb{R}}$. Now $\dim \operatorname{NS}(X)_{\mathbb{R}} = 2$ and the set of the numerical classes in X with self-intersection zero forms two lines, one of which is $\pi^* \operatorname{NS}(C)_{\mathbb{R}}$, and f^* fixes or interchanges them. So $(f^2)^*$ fixes $\pi^* \operatorname{NS}(C)_{\mathbb{R}}$. □

The following might be well-known, but we give a proof for the reader's convenience.

Lemma 3.4.5. *A surjective endomorphism f preserves fibers if and only if there exists a non-zero integer a such that $f^*F \equiv aF$. Here, F is the numerical class of a fiber.*

Proof. Assume $f^*F \equiv aF$. For any point $p \in C$, we set $F_p := \pi^{-1}(p) = \pi^* p$. If f does not preserve fibers, there is a point $p \in C$ such that $f(F_p) \cdot F > 0$. Now we can calculate the intersection number as follows:

$$\begin{aligned} 0 = F \cdot aF = F \cdot (f^*F) &= F_p \cdot (f^*F) \\ &= (f_*F_p) \cdot F = \deg(f|_{F_p}) \cdot (f(F_p) \cdot F) > 0. \end{aligned}$$

This is a contradiction. Hence f preserves fibers.

Next, assume that f preserves fibers. Write $f^*F = aF + bC_0$. Then we can also calculate the intersection number as follows:

$$\begin{aligned} b = F \cdot (aF + bC_0) &= F \cdot f^*F = (f_*F) \cdot F \\ &= \deg(f|_F) \cdot (F \cdot F) = 0. \end{aligned}$$

Further, by the injectivity of f^*, we have $a \neq 0$. The proof is complete. □

Lemma 3.4.6. *If $\mathcal{E}$ splits, i.e., if there is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$, the invariant e of $X = \mathbb{P}(\mathcal{E})$ is non-negative.*

Proof. See [18, V, Example 2.11.3]. □

Lemma 3.4.7. *Assume that $e \geq 0$. Then for a divisor $D = aF + bC_0 \in \mathrm{NS}(X)$, the following properties are equivalent.*

- *D is ample.*
- *$a > be$ and $b > 0$.*

In other words, the nef cone of X is generated by F and $eF + C_0$.

Proof. See [18, V, Proposition 2.20]. □

We can prove a result stronger than Lemma 3.4.4 as follows.

Lemma 3.4.8. *Assume that $e > 0$. Then any surjective endomorphism $f \colon X \longrightarrow X$ preserves fibers.*

Proof. By Lemma 3.4.5, it is enough to prove $f^*F \equiv aF$ for some integer $a > 0$. We can write $f^*F \equiv aF + bC_0$ for some integers $a, b \geq 0$.

Since we have

$$aF + bC_0 = (a - be)F + b(eF + C_0)$$

and f preserves the nef cone and the ample cone, either of the equalities $a - be = 0$ or $b = 0$ holds.

We have

$$\begin{aligned} 0 &= \deg(f)(F \cdot F) = (f_* f^* F) \cdot F \\ &= (f^*F) \cdot (f^*F) = (aF + bC_0) \cdot (aF + bC_0) \\ &= 2ab - b^2 e = b(2a - be). \end{aligned}$$

So either of the equalities $b = 0$ or $2a - be = 0$ holds.

If we have $b \neq 0$, we have $a - be = 0$ and $2a - be = 0$. So we get $a = 0$. But since $e \neq 0$, we obtain $b = 0$. This is a contradiction. Consequently, we get $b = 0$ and $f^*F \equiv aF$. □

Lemma 3.4.9. *Fix a fiber $F = F_p$ for a point $p \in C(\overline{k})$. Let f be a surjective endomorphism on X preserving fibers, f_C the endomorphism on C satisfying $\pi \circ f = f_C \circ \pi$, $f_F := f|_F \colon F \longrightarrow f(F)$ the restriction of f to the fiber F. Set $f^*F \equiv aF$ and $f^*C_0 \equiv cF + dC_0$. Then we have $a = \deg(f_C)$, $d = \deg(f_F)$, $\deg(f) = ad$, and $\delta_f = \max\{a, d\}$.*

Proof. Our assertions follow from the following equalities of divisor classes in $\mathrm{NS}(X)$ and of intersection numbers:

$$\begin{aligned} aF &= f^*F = f^*\pi^*p \\ &= \pi^* f_C^* p = \pi^*(\deg(f_C)p) \\ &= \deg(f_C)\pi^*p = \deg(f_C)F, \\ \deg(f)F &= f_* f^* F = f_* f^* \pi^* p \\ &= f_* \pi^* f_C^* p = f_* \pi^*(\deg(f_C)p) \\ &= \deg(f_C) f_* F = \deg(f_C)\deg(f_F) f(F) \\ &= \deg(f_C)\deg(f_F)F \\ \deg(f) &= \deg(f) C_0 \cdot F = (f_* f^* C_0) \cdot F \\ &= (f^*C_0) \cdot (f^*F) = (cF + dC_0) \cdot aF = ad. \end{aligned}$$

The last assertion $\delta_f = \max\{a, d\}$ follows from the functoriality of f^* and the equality $\delta_f = \lim_{n\to\infty} \rho((f^n)^*)^{1/n} = \rho(f^*)$ (cf. Remark 1.1.2 (3)). □

Lemma 3.4.10. *We use the notation in Lemma 3.4.9. Assume that* $e \geq 0$. *Then both* F *and* C_0 *are eigenvectors of* $f^*\colon \mathrm{NS}(X)_{\mathbb{R}} \longrightarrow \mathrm{NS}(X)_{\mathbb{R}}$. *Further, if* e *is positive, then we have* $\deg(f_C) = \deg(f_F)$.

Proof. Set $f^*F = aF$ and $f^*C_0 = cF + dC_0$ in $\mathrm{NS}(X)$. Then we have

$$\begin{aligned} -ead = -e \deg f &= (f_* f^* C_0) \cdot C_0 \\ &= (f^*C_0)^2 = (cF + dC_0)^2 = 2cd - ed^2. \end{aligned}$$

Hence, we get $c = e(d-a)/2$. We have the following equalities in $\mathrm{NS}(X)$:

$$f^*(eF + C_0) = aeF + (cF + dC_0) = (ae + c)F + dC_0.$$

By the fact that f^*D is ample if and only if D is ample, it follows that $eF + C_0$ is an eigenvector of f^*. Thus, we have

$$de = ae + c = ae + e(d-a)/2 = e(d+a)/2.$$

Therefore, the equality $e(d-a) = 0$ holds. So $c = e(d-a)/2 = 0$ holds.

Further, we assume that $e > 0$. Then it follows that $d - a = 0$. So we have $\deg(f_C) = a = d = \deg(f_F)$. □

The following lemma is used in Subsection 3.5.2.

Lemma 3.4.11. *Let* $\mathcal{L}$ *be a non-trivial invertible sheaf of degree* 0 *on a curve* C *with* $g(C) \geq 1$, $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$, *and* $X = \mathbb{P}(\mathcal{E})$. *Let* C_0, C_1 *be sections corresponding to the projections* $\mathcal{E} \longrightarrow \mathcal{L}$ *and* $\mathcal{E} \longrightarrow \mathcal{O}_C$. *If* $\sigma\colon C \longrightarrow X$ *is a section such that* $(\sigma(C))^2 = 0$, *then* $\sigma(C)$ *is equal to* C_0 *or* C_1.

Proof. Note that $e = 0$ in this case and thus $(C_0^2) = 0$. Moreover, $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$ and $\mathcal{O}_X(C_1) \cong \mathcal{O}_X(1) \otimes \pi^*\mathcal{L}^{-1}$. Set $\sigma(C) \equiv aC_0 + bF$. Then $a = (\sigma(C) \cdot F) = 1$ and $2ab = (\sigma(C)^2) = 0$. Thus $\sigma(C) \equiv C_0$. Therefore, $\mathcal{O}_X(\sigma(C)) \cong \mathcal{O}_X(C_0) \otimes \pi^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ of degree 0 on C. Then

$$\begin{aligned} 0 \neq H^0(X, \mathcal{O}_X(\sigma(C))) &= H^0(C, \pi_*\mathcal{O}_X(C_0) \otimes \mathcal{N}) \\ &= H^0(C, (\mathcal{L} \oplus \mathcal{O}_C) \otimes \mathcal{N}) \end{aligned}$$

and this implies $\mathcal{N} \cong \mathcal{O}_C$ or $\mathcal{N} \cong \mathcal{L}^{-1}$. Hence $\mathcal{O}_X(\sigma(C))$ is isomorphic to $\mathcal{O}_X(C_0)$ or $\mathcal{O}_X(C_0) \otimes \pi^*\mathcal{L}^{-1} = \mathcal{O}_X(C_1)$. Since $\mathcal{L}$ is non-trivial, we have $H^0(\mathcal{O}_X(C_0)) = H^0(\mathcal{O}_X(C_1)) = \overline{k}$ and we get $\sigma(C) = C_0$ or C_1. □

3.5 $\mathbb{P}^1$-bundles over curves

In this section, we prove Conjecture 3.1.1 for non-invertible endomorphisms on $\mathbb{P}^1$-bundles over curves. We divide the proof according to the genus of the base curve.

3.5.1 $\mathbb{P}^1$-bundles over $\mathbb{P}^1$

Theorem 3.5.1. *Let $\pi\colon X \longrightarrow \mathbb{P}^1$ be a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and $f\colon X \to X$ be a non-invertible endomorphism. Then Conjecture 3.1.1 holds for f.*

Proof. Take a locally free sheaf $\mathcal{E}$ of rank 2 on $\mathbb{P}^1$ such that $X \cong \mathbb{P}(\mathcal{E})$ and $\deg \mathcal{E} = -e$ (cf. Proposition 3.4.1). Then $\mathcal{E}$ splits (see [18, V. Corollary 2.14]). When X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. the case of $e = 0$, the assertion holds by [47, Theorem 1.3]. When X is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. the case of $e > 0$, the endomorphism f preserves fibers and induces an endomorphism $f_{\mathbb{P}^1}$ on the base curve $\mathbb{P}^1$. By Lemma 3.4.10, we have $\delta_f = \delta_{f_{\mathbb{P}^1}}$. Fix a point $p \in \mathbb{P}^1$ and set $F = \pi^* p$. Let $P \in X(\overline{k})$ be a point whose forward f-orbit is Zariski dense in X. Then the forward $f_{\mathbb{P}^1}$-orbit of $\pi(P)$ is also Zariski dense in $\mathbb{P}^1$. Now the assertion follows from the following computation.

$$\begin{aligned}\alpha_f(P) &\geq \lim_{n\to\infty} h_F(f^n(P))^{1/n} = \lim_{n\to\infty} h_{\pi^*p}(f^n(P))^{1/n} \\ &= \lim_{n\to\infty} h_p(\pi \circ f^n(P))^{1/n} = \lim_{n\to\infty} h_p(f_{\mathbb{P}^1}^n \circ \pi(P))^{1/n} = \delta_{f_{\mathbb{P}^1}} = \delta_f.\end{aligned}$$

□

3.5.2 $\mathbb{P}^1$-bundles over genus one curves

In this subsection, we prove Conjecture 3.1.1 for any endomorphisms on a $\mathbb{P}^1$-bundle on a curve C of genus one.

The following result is due to Amerik. Note that Amerik in fact proved it for $\mathbb{P}^1$-bundles over varieties of arbitrary dimension (cf. [1]).

Lemma 3.5.2. *Let $X = \mathbb{P}(\mathcal{E})$ be a $\mathbb{P}^1$-bundle over a curve C. If X has a fiber-preserving surjective endomorphism whose restriction to a general fiber has degree greater than 1, then $\mathcal{E}$ splits into a direct sum of two line bundles after a finite base change. Furthermore, if $\mathcal{E}$ is semistable, then $\mathcal{E}$ splits into a direct sum of two line bundles after an étale base change.*

Proof. See [1, Theorem 2 and Proposition 2.4]. □

Lemma 3.5.3. *Let E be a curve of genus one with an endomorphism $f\colon E \longrightarrow E$. If $g\colon E' \longrightarrow E$ is a finite étale covering of E, there exists a finite étale covering $h\colon E'' \longrightarrow E'$ and an endomorphism $f'\colon E'' \longrightarrow E''$ such that $f \circ g \circ h = g \circ h \circ f'$. Furthermore, we can take h as satisfying $E'' = E$.*

Proof. At first, since E' is an étale covering of genus one curve E, E' is also a genus one curve. By fixing a rational point $p \in E'(\overline{k})$ and $g(p) \in E(\overline{k})$, these curves E and E' are regarded as elliptic curves, and g can be regarded as an isogeny between elliptic curves. Let $h := \hat{g}\colon E \longrightarrow E'$ be the dual isogeny of g. The morphism f is decomposed as $f = \tau_c \circ \psi$ for a homomorphism ψ and a translation map τ_c by $c \in E(\overline{k})$. Fix a rational point $c' \in E(\overline{k})$ such that $[\deg(g)](c') = c$ and consider the translation map $\tau_{c'}$, where $[\deg(g)]$ is the multiplication by $\deg(g)$. We set $f' = \tau_{c'} \circ \psi$. Then we have the following equalities.

$$\begin{aligned} f \circ g \circ h &= \tau_c \circ \psi \circ g \circ \hat{g} \\ &= \tau_c \circ \psi \circ [\deg(g)] = \tau_c \circ [\deg(g)] \circ \psi \\ &= [\deg(g)] \circ \tau_{c'} \circ \psi = g \circ h \circ f'. \end{aligned}$$

This is what we want. □

Proposition 3.5.4. *Let $\mathcal{E}$ be a locally free sheaf of rank 2 on a genus one curve C and $X = \mathbb{P}(\mathcal{E})$. Suppose Conjecture 3.1.1 holds for any non-invertible endomorphism on X with $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle of degree zero on C. Then Conjecture 3.1.1 holds for any non-invertible endomorphism on $X = \mathbb{P}(\mathcal{E})$ for any $\mathcal{E}$.*

Proof. By Lemma 3.4.4 and Lemma 3.2.1, we may assume that f preserves fibers. We can prove Conjecture 3.1.1 in the case of $\deg(f|_F) = 1$ in the same way as in the case of $g(C) = 0$ since $\deg(f|_F) = 1 \leq \deg(f_C)$. Since we are considering the case of $g(C) = 1$, if $\mathcal{E}$ is indecomposable, then $\mathcal{E}$ is semistable (see [43, 10.2 (c), 10.49] or [18, V. Exercise 2.8 (c)]). By Lemma 3.5.2, if $\deg(f|_F) > 1$ and $\mathcal{E}$ is indecomposable, there is a finite étale covering $g\colon E \longrightarrow C$ satisfying that $E \times_C X \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ for an invertible sheaf $\mathcal{L}$ over E. Furthermore, by Lemma 3.5.3, we can take E equal to C and there is an endomorphism $f'_C\colon C \longrightarrow C$ satisfying $f_C \circ g = g \circ f'_C$. Then by the universality of cartesian product $X \times_{C,g} C$, we have an induced endomorphism $f'\colon X \times_{C,g} C \longrightarrow X \times_{C,g} C$. By Lemma 3.2.2, it is enough to prove Conjecture 3.1.1 for the endomorphism f'. Thus, we may assume that $\mathcal{E}$ is decomposable, i.e., $X \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$. Then the invariant e is non-negative by Lemma 3.4.6. When e is positive, by the same method as the proof of Theorem 3.1.3 in the case of $g(C) = 0$, the proof is complete. When $e = 0$, we have $\deg \mathcal{L} = 0$ and the assertion holds by the assumption. □

In the rest of this subsection, we keep the following notation. Let C be a genus one curve and $\mathcal{L}$ an invertible sheaf on C with degree 0. Let $X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{O}_C \oplus \mathcal{L}))$ and $\pi\colon X \longrightarrow C$ the projection. When $\mathcal{L}$ is trivial, we have $X \cong C \times \mathbb{P}^1$, and by [47, Theorem1.3], Conjecture 3.1.1 is true for X. Thus we may assume $\mathcal{L}$ is non-trivial. In this case, we have two sections of $\pi\colon X \longrightarrow C$ corresponding to the projections $\mathcal{O}_C \oplus \mathcal{L} \longrightarrow \mathcal{L}$ and $\mathcal{O}_C \oplus \mathcal{L} \longrightarrow \mathcal{O}_C$. Let C_0 and C_1 denote the images of these sections. Then we have $\mathcal{O}_X(C_0) = \mathcal{O}_X(1)$ and $\mathcal{O}_X(C_1) = \mathcal{O}_X(1) \otimes \pi^*\mathcal{L}^{-1}$. Since $\mathcal{L}$ is non-trivial, we have $C_0 \neq C_1$. But since $\deg \mathcal{L} = 0$, C_0 and C_1 are numerically equivalent. Thus $(C_0 \cdot C_1) = (C_0^2) = 0$ and therefore $C_0 \cap C_1 = \emptyset$.

Let f be a non-invertible endomorphism on X such that there is a surjective endomorphism $f_C\colon C \longrightarrow C$ with $\pi \circ f = f_C \circ \pi$.

Lemma 3.5.5. *When $\mathcal{L}$ is a torsion element of* $\mathrm{Pic}\, C$*, Conjecture 3.1.1 holds for f.*

Proof. We fix an algebraic group structure on C. Since $\mathcal{L}$ is torsion, there exists a positive integer $n > 0$ such that $[n]^*\mathcal{L} \cong \mathcal{O}_C$. Then the base change of $\pi\colon X \longrightarrow C$ by $[n]\colon C \longrightarrow C$ is the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1 \times C \longrightarrow C$. Applying Lemma 3.5.3 to $g = [n]$, we get a finite morphism $h\colon C \longrightarrow C$ such that the base change of $\pi\colon X \longrightarrow C$ by $h\colon C \longrightarrow C$ is $\mathbb{P}^1 \times C \longrightarrow C$ and there exists a finite morphism $f'_C\colon C \longrightarrow C$ with $f_C \circ h = h \circ f'_C$. Then f induces a non-invertible endomorphism $f'\colon \mathbb{P}^1 \times C \longrightarrow \mathbb{P}^1 \times C$. By [47, Theorem 1.3], Conjecture 3.1.1 holds for f'. By Lemma 3.2.2, Conjecture 3.1.1 holds also for f. □

Now, let F be the numerical class of a fiber of π. By Lemma 3.4.10, we have

$$f^*F \equiv aF,$$
$$f^*C_0 \equiv bC_0$$

for some integers $a, b \geq 1$. Note that $a = \deg f_C$, $b = \deg f|_F$ and $ab = \deg f$ (cf. Lemma 3.4.9).

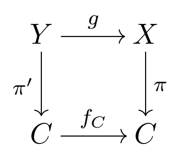
Lemma 3.5.6.

(1) *One of the equalities* $f(C_0) = C_0$, $f(C_0) = C_1$ *and* $f(C_0) \cap C_0 = f(C_0) \cap C_1 = \emptyset$ *holds. The same is true for* $f(C_1)$.

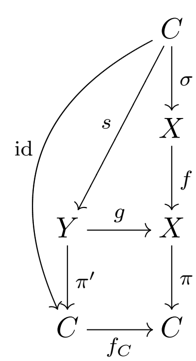
(2) *If* $f(C_0) \cap C_i = \emptyset$ *for* $i = 0, 1$, *then the base change of* $\pi\colon X \longrightarrow C$ *by* $f_C\colon C \longrightarrow C$ *is isomorphic to* $\mathbb{P}^1 \times C$. *In particular,* $f_C^* \mathcal{L} \cong \mathcal{O}_C$ *and* $\mathcal{L}$ *is a torsion element of* $\operatorname{Pic} C$. *The same conclusion holds under the assumption that* $f(C_1) \cap C_i = \emptyset$ *for* $i = 0, 1$.

Proof. (1) Since $f^*C_i \equiv bC_i$, $C_0 \equiv C_1$ and $(C_0^2) = 0$, we have $(f_*C_i \cdot C_j) = 0$ for every i and j. Thus the assertion follows.

(2) Assume $f(C_0) \cap C_i = \emptyset$ for $i = 0, 1$. Consider the following Cartesian diagram.



Then Y is a $\mathbb{P}^1$-bundle over C associated with the vector bundle $\mathcal{O}_C \oplus f_C^*\mathcal{L}$. The pull-backs $C_i = g^{-1}(C_i), i = 0, 1$ are sections of π'. By the projection formula, we have $(C_i'^2) = 0$. Let $\sigma\colon C \longrightarrow X$ be the section with $\sigma(C) = C_0$. Since $\pi \circ f \circ \sigma = f_C$, we get a section $s\colon C \longrightarrow Y$ of π'.



Note that $g(s(C)) = f(C_0) \neq C_0, C_1$. Thus $s(C), C_0', C_1'$ are distinct sections of π'. Moreover, by the projection formula, we have $(s(C) \cdot C_0') = 0$. Thus we have three sections which are numerically equivalent to each other. Then Lemma 3.4.11 implies $f_C^*\mathcal{L} \cong \mathcal{O}_C$ and $Y \cong \mathbb{P}^1 \times C$. Since $f_C^*\colon \operatorname{Pic}^0 C \longrightarrow \operatorname{Pic}^0 C$ is an isogeny, the kernel of f_C^* is finite and thus $\mathcal{L}$ is a torsion element of $\operatorname{Pic} C$. □

Lemma 3.5.7.

(1) *Suppose that*

- *$\mathcal{L}$ is non-torsion in* $\operatorname{Pic} C$,
- $f(C_0) = C_0$ *or* C_1, *and*
- $f(C_1) = C_0$ *or* C_1.

Then $f(C_0) = C_0$ *and* $f(C_1) = C_1$, *or* $f(C_0) = C_1$ *and* $f(C_1) = C_0$.

(2) *If the equalities* $f(C_0) = C_0$ *and* $f(C_1) = C_1$ *hold, then* $f^*C_i \sim_{\mathbb{Q}} bC_i$ *for* $i = 0$ *and* 1.

Proof. (1) Assume that $f(C_0) = C_0$ and $f(C_1) = C_0$. Then $f_*C_0 = aC_0$ and $f_*C_1 = aC_0$ as cycles. Since $f_C^*\colon \operatorname{Pic}^0 C \longrightarrow \operatorname{Pic}^0 C$ is surjective, there exists a degree zero divisor M on C such that $f_C^*\mathcal{O}_C(M) \cong \mathcal{L}$. Then $C_1 \sim C_0 - \pi^* f_C^* M$. Hence

$$aC_0 = f_*C_1 \sim (f_*C_0 - f_*\pi^* f_C^* M) = (aC_0 - f_*\pi^* f_C^* M)$$

and

$$0 \sim f_*\pi^* f_C^* M \sim f_* f^* \pi^* M \sim (\deg f)\pi^* M.$$

Thus π^*M is torsion and so is M. This implies that $\mathcal{L}$ is torsion, which contradicts the assumption.

The same argument shows that the case when $f(C_0) = C_1$ and $f(C_1) = C_1$ does not occur.

(2) In this case, we have $f_*C_0 \sim aC_0$. We can write $f^*C_0 \sim bC_0 + \pi^*D$ for some degree zero divisor D on C. Thus

$$(\deg f)C_0 \sim f_*f^*C_0 \sim abC_0 + f_*\pi^*D = (\deg f)C_0 + f_*\pi^*D$$

and $f_*\pi^*D \sim 0$. Since $f_C^*\colon \operatorname{Pic}^0 C \longrightarrow \operatorname{Pic}^0 C$ is surjective, there exists a degree zero divisor D' on C such that $f_C^* D' \sim D$. Then

$$0 \sim f_*\pi^*D \sim f_*\pi^* f_C^* D' \sim f_* f^* \pi^* D' \sim (\deg f)\pi^* D'.$$

Hence $\pi^*D' \sim_{\mathbb{Q}} 0$ and $D' \sim_{\mathbb{Q}} 0$. Therefore $D \sim_{\mathbb{Q}} 0$ and $f^*C_0 \sim_{\mathbb{Q}} bC_0$.

Similarly, we have $f^*C_1 \sim_{\mathbb{Q}} bC_1$. □

Lemma 3.5.8. *Suppose* $a < b$. *If* $f^*C_i \sim_{\mathbb{Q}} bC_i$ *for* $i = 0, 1$, *the line bundle* $\mathcal{L}$ *is a torsion element of* $\operatorname{Pic} C$.

Proof. Let L be a divisor on C such that $\mathcal{O}_C(L) \cong \mathcal{L}$. Note that $C_1 \sim C_0 - \pi^*L$. Thus

$$f^*\pi^*L \sim f^*(C_0 - C_1) \sim_{\mathbb{Q}} bC_0 - bC_1 \sim b\pi^*L$$

and $f_C^*L \sim_{\mathbb{Q}} bL$ hold.

Thus, from the following lemma, $\mathcal{L}$ is a torsion element. □

Lemma 3.5.9. *Let a, b be integers such that $1 \leq a < b$. Let C be a curve of genus one defined over an algebraically closed field k. Let $f_C\colon C \longrightarrow C$ be an endomorphism of $\deg f_C = a$. If L is a divisor on C of degree 0 satisfying*

$$f_C^* L \sim_{\mathbb{Q}} bL,$$

the divisor L is a torsion element of $\mathrm{Pic}^0(C)$

Proof. By the definition of $\mathbb{Q}$-linear equivalence, we have $f_C^* rL \sim brL$ for some positive integer r. Since the curve C is of genus one, the group $\mathrm{Pic}^0(C)$ is an elliptic curve. Assume the (group) endomorphism

$$f_C^* - [b]\colon\ \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^0(C)$$

is the 0 map. Then we have the equalities $a = \deg f_C = \deg f_C^* = \deg[b] = b^2$. But this contradicts to the inequality $1 \leq a < b$. Hence the map $f_C^* - [b]$ is an isogeny, and $\mathrm{Ker}(f_C^* - [b]) \subset \mathrm{Pic}^0(C)$ is a finite group scheme. In particular, the order of $rL \in \mathrm{Ker}(f_C^* - [b])(k)$ is finite. Thus, L is a torsion element. □

Remark 3.5.10. We can actually prove the following. Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f\colon X \longrightarrow X$ be a surjective morphism over $\overline{\mathbb{Q}}$ with first dynamical degree δ. If an $\mathbb{R}$-divisor D on X satisfies

$$f^* D \sim_{\mathbb{R}} \lambda D$$

for some $\lambda > \delta$, then one has $D \sim_{\mathbb{R}} 0$.

Sketch of the proof. Consider the canonical height

$$\hat{h}_D(P) = \lim_{n\to\infty} h_D(f^n(P))/\lambda^n$$

where h_D is a height associated with D (cf. [7]). If $\hat{h}_D(P) \neq 0$ for some P, then we can prove $\overline{\alpha}_f(P) \geq \lambda$. This contradicts the fact $\delta \geq \overline{\alpha}_f(P)$ and the assumption $\lambda > \delta$. Thus one has $\hat{h}_D = 0$ and therefore $h_D = \hat{h}_D + O(1) = O(1)$. By a theorem of Serre, we get $D \sim_{\mathbb{R}} 0$ (see [49, 2.9. Theorem]). □

Proposition 3.5.11. *Let $\mathcal{L}$ be an invertible sheaf of degree zero on a genus one curve C and $X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$. For any non-invertible endomorphism $f\colon X \longrightarrow X$, Conjecture 3.1.1 holds.*

Proof. By Lemma 3.5.5 and Proposition 3.5.9 we may assume $a \geq b$. In this case, $\delta_f = a$ and Conjecture 3.1.1 can be proved as in the proof of Proposition 3.5.1. □

Proof of Theorem 3.1.3 for $\mathbb{P}^1$-bundles over genus one curves. As we argued at the first of Section 3.3, we may assume that the endomorphism $f\colon X \longrightarrow X$ is not an automorphism. Then the assertion follows from Proposition 3.5.4 and Proposition 3.5.11. □

Remark 3.5.12. In the above setting, the line bundle $\mathcal{L}$ is actually an eigenvector for f_C^* up to linear equivalence. More precisely, for a $\mathbb{P}^1$-bundle $\pi\colon X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$ over a curve C with $\deg \mathcal{L} = 0$ and an endomorphism $f\colon X \longrightarrow X$ that induces an endomorphism $f_C\colon C \longrightarrow C$, there exists an integer t such that $f_C^*\mathcal{L} \cong \mathcal{L}^t$. Indeed, let C_0 and C_1 be the sections defined above. Since $(f^*(C_0) \cdot C_0) = 0$, we can write $\mathcal{O}_X(f^{-1}(C_0)) \cong \mathcal{O}_X(mC_0)\otimes\pi^*\mathcal{N}$ for some integer m and degree zero line bundle $\mathcal{N}$ on C. Since

$$\begin{aligned}0 \neq H^0(\mathcal{O}_X(f^{-1}(C_0))) &= H^0(\mathcal{O}_X(mC_0)\otimes\pi^*\mathcal{N})\\ &= H^0(\mathrm{Sym}^m(\mathcal{O}_C \oplus \mathcal{L})\otimes\mathcal{N}) = \bigoplus_{i=0}^{m} H^0(\mathcal{L}^i\otimes\mathcal{N}),\end{aligned}$$

we have $\mathcal{N} \cong \mathcal{L}^r$ for some $-m \leq r \leq 0$. Thus $f^*\mathcal{O}_X(C_0) \cong \mathcal{O}_X(mC_0)\otimes\pi^*\mathcal{L}^r$. The key is the calculation of global sections using projection formula. Since $\mathcal{O}_X(C_1) \cong \mathcal{O}_X(C_0)\otimes\pi^*\mathcal{L}^{-1}$, we have $\pi_*\mathcal{O}_X(mC_1) \cong \pi_*\mathcal{O}_X(mC_0)\otimes\mathcal{L}^{-m}$. Moreover, since C_0 and C_1 are numerically equivalent, we can similarly get $f^*\mathcal{O}_X(C_1) \cong \mathcal{O}_X(mC_0)\otimes\pi^*\mathcal{L}^s$ for some integer s. Thus, $f^*\pi^*\mathcal{L} \cong \pi^*\mathcal{L}^{r-s}$. Therefore, $\pi^*f_C^*\mathcal{L} \cong \pi^*\mathcal{L}^{r-s}$. Since $\pi^*\colon \operatorname{Pic} C \longrightarrow \operatorname{Pic} X$ is injective, we get $f_C^*\mathcal{L} \cong \mathcal{L}^{r-s}$.

3.5.3 $\mathbb{P}^1$-bundles over curves of genus ≥ 2

By the following proposition, Conjecture 3.1.1 trivially holds in this case.

Proposition 3.5.13. *Let C be a curve with $g(C) \geq 2$ and $\pi\colon X \longrightarrow C$ be a $\mathbb{P}^1$-bundle over C. Let $f\colon X \longrightarrow X$ be a surjective endomorphism. Then there exists an integer $t > 0$ such that f^t is a morphism over C, that is, f^t satisfies $\pi \circ f^t = \pi$. In particular, f admits no Zariski dense orbit.*

Proof. By Lemma 3.4.4, we may assume that f induces a surjective endomorphism $f_C\colon C \longrightarrow C$ with $\pi \circ f = f_C \circ \pi$. Since C is of general type, f_C is an automorphism of finite order and the assertion follows. □

Remark 3.5.14. One can also show that any surjective endomorphism over a curve of genus at least two admits no dense orbit by using the Mordell conjecture (Faltings's theorem).

3.6 Hyperelliptic surfaces

Theorem 3.6.1. *Let X be a hyperelliptic surface and $f\colon X \longrightarrow X$ a non-invertible endomorphism on X. Then Conjecture 3.1.1 holds for f.*

Proof. Let $\pi\colon X \longrightarrow E$ be the Albanese map of X. By the universality of π, there is a morphism $g\colon E \longrightarrow E$ satisfying $\pi \circ f = g \circ \pi$. It is well-known that E is a genus one curve, π is a surjective morphism with connected fibers, and there is an étale cover $\phi\colon E' \longrightarrow E$ such that $X' = X \times_E E' \cong F \times E'$, where F is a genus one curve (cf. [2, Chapter 10]). In particular, X' is an abelian surface. By Lemma 3.5.3, taking a further étale base change, we may assume that there is an endomorphism $h\colon E' \longrightarrow E'$ such that $\phi \circ h = g \circ \phi$. Let $\pi'\colon X' \longrightarrow E'$ and $\psi\colon X' \longrightarrow X$ be the induced morphisms. Then, by the universality of fiber products, there is a morphism $f'\colon X' \longrightarrow X'$ satisfying $\pi' \circ f' = \pi' \circ h$ and $\psi \circ f' = f \circ \psi$. Applying Lemma 3.2.2, it is enough to prove Conjecture 3.1.1 for the endomorphism f'. Since X' is an abelian variety, this holds by [28, Corollary 31] and [51, Theorem 2]. □

3.7 Surfaces with $\kappa(X) = 1$

Let $f\colon X \longrightarrow X$ be a non-invertible endomorphism on a surface X with $\kappa(X) = 1$. In this section we shall prove that f does not admit any Zariski dense forward f-orbit. Although this result is a special case of [46, Theorem A] (see Remark 1.3.5), we will give a simpler proof of it.

By Lemma 3.3.2, X is minimal and f is étale. Since $\deg(f) \geq 2$, we have $\chi(X, \mathcal{O}_X) = 0$.

Let $\phi = \phi_{|mK_X|}\colon X \longrightarrow \mathbb{P}^N = \mathbb{P}H^0(X, mK_X)$ be the Iitaka fibration of X and set $C_0 = \phi(X)$. Since f is étale, it induces an automorphism $g\colon \mathbb{P}^N \longrightarrow \mathbb{P}^N$ such that $\phi \circ f = g \circ \phi$ (cf. [16, Lemma 3.1]). The restriction of g to C_0 gives an automorphism $f_{C_0}\colon C_0 \longrightarrow C_0$ such that $\phi \circ f = f_{C_0} \circ \phi$. Take the normalization $\nu\colon C \longrightarrow C_0$ of C_0. Then ϕ factors as $X \xrightarrow{\pi} C \xrightarrow{\nu} C_0$ and π is an elliptic fibration. Moreover, f_{C_0} lifts to an automorphism $f_C\colon C \longrightarrow C$ such that $\pi \circ f = f_C \circ \pi$.

So we obtain an elliptic fibration $\pi\colon X \longrightarrow C$ and an automorphism f_C on C such that $\pi \circ f = f_C \circ \pi$ In this situation, the following holds.

Theorem 3.7.1. *Let X be a surface with $\kappa(X) = 1$, $\pi\colon X \longrightarrow C$ an elliptic fibration, $f\colon X \longrightarrow X$ a non-invertible endomorphism, and $f_C\colon C \longrightarrow C$ an automorphism such that $\pi \circ f = f_C \circ \pi$. Then $f_C^t = \mathrm{id}_C$ for a positive integer t.*

Proof. Let $\{P_1, \ldots, P_r\}$ be the points over which the fibers of π are multiple fibers (possibly $r = 0$, i.e. π does not have any multiple fibers). We denote by m_i denotes the multiplicity of the fiber $\pi^* P_i$ for every i. Then we have the canonical bundle formula:

$$K_X = \pi^*(K_C + L) + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \pi^* P_i,$$

where L is a divisor on C such that $\deg(L) = \chi(X, \mathcal{O}_X)$. Then $\deg(L) = 0$ because f is étale and $\deg(f) \geq 2$ (cf. Lemma 3.3.2). Since $\kappa(X) = 1$, the divisor $K_C + L + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} P_i$ must have positive degree. So we have

$$2(g(C) - 1) + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} > 0. \qquad (*)$$

For any i, set $Q_i = f_C^{-1}(P_i)$. Then $\pi^* Q_i = \pi^* f_C^* P_i = f^* \pi^* P_i$ is a multiple fiber. So $(f_C)|_{\{P_1, \ldots, P_r\}}$ is a permutation of $\{P_1, \ldots, P_r\}$ since f_C is an automorphism.

We divide the proof into three cases according to the genus $g(C)$ of C:

(1) $g(C) \geq 2$; then the automorphism group of C is finite. So $f_C^t = \mathrm{id}_C$ for a positive integer t.

(2) $g(C) = 1$; by $(*)$, it follows that $r \geq 1$. For a suitable t, all P_i are fixed points of f_C^t. We put the algebraic group structure on C such that P_1 is the identity element. Then f_C^t is a group automorphism on C. So $f_C^{ts} = \mathrm{id}_C$ for a suitable s since the group of group automorphisms on C is finite.

(3) $g(C) = 0$; again by $(*)$, it follows that $r \geq 3$. For a suitable t, all P_i are fixed points of f_C^t. Then f_C^t fixes at least three points, which implies that f_C^t is in fact the identity map. $\square$

Immediately we obtain the following corollary.

Corollary 3.7.2. *Let $f \colon X \longrightarrow X$ be a non-invertible endomorphism on a surface X with $\kappa(X) = 1$. Then there does not exist any Zariski dense f-orbit.*

Therefore Conjecture 3.1.1 trivially holds for non-invertible endomorphisms on surfaces of Kodaira dimension 1.

3.8 Existence of a rational point P satisfying $\alpha_f(P) = \delta_f$

In this section, we prove Theorem 3.1.7 and Theorem 3.1.8. Theorem 3.1.7 follows from the following lemma. A subset $\Sigma \subset V(\overline{k})$ is called a *set of bounded height* if for some (or, equivalently, any) ample divisor A on V, the height function h_A associated with A is a bounded function on Σ.

Lemma 3.8.1. *Let X be a smooth projective variety and $f\colon X \longrightarrow X$ a surjective endomorphism with $\delta_f > 1$. Let $D \not\equiv 0$ be a nef $\mathbb{R}$-divisor such that $f^*D \equiv \delta_f D$. Let $V \subset X$ be a closed subvariety of positive dimension such that $(D^{\dim V} \cdot V) > 0$. Then there exists a non-empty open subset $U \subset V$ and a set $\Sigma \subset U(\overline{k})$ of bounded height such that for every $P \in U(\overline{k}) \setminus \Sigma$ we have $\alpha_f(P) = \delta_f$.*

Remark 3.8.2. By Perron-Frobenius-type result of [4, Theorem], there is a nef $\mathbb{R}$-divisor $D \not\equiv 0$ satisfying the condition $f^*D \equiv \delta_f D$ since f^* preserves the nef cone.

Proof. Fix a height function h_D associated with D. For every $P \in X(\overline{k})$, the following limit exists (cf. [29, Theorem 5]).

$$\hat{h}(P) = \lim_{n\to\infty} \frac{h_D(f^n(P))}{\delta_f^n}$$

The function $\hat{h}$ has the following properties (cf. [29, Theorem 5]).

(i) $\hat{h} = h_D + O(\sqrt{h_H})$ where H is any ample divisor on X and $h_H \geq 1$ is a height function associated with H.

(ii) If $\hat{h}(P) > 0$, then $\alpha_f(P) = \delta_f$.

Since $(D^{\dim V} \cdot V) > 0$, we have $(D|_V{}^{\dim V}) > 0$ and $D|_V$ is big. Thus we can write $D|_V \sim_{\mathbb{R}} A + E$ with an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-divisor E on V. Therefore we have

$$\hat{h}|_{V(\overline{k})} = h_A + h_E + O(\sqrt{h_A})$$

where h_A, h_E are height functions associated with A, E and h_A is taken to be $h_A \geq 1$. In particular, there exists a positive real number $B > 0$ such that $h_A + h_E - \hat{h}|_{V(\overline{k})} \leq B\sqrt{h_A}$. Then we have the following inclusions.

$$\begin{aligned}\{P \in V(\overline{k}) \mid \hat{h}(P) \leq 0\} &\subset \{P \in V(\overline{k}) \mid h_A(P) + h_E(P) \leq B\sqrt{h_A(P)}\} \\ &\subset \operatorname{Supp} E \cup \{P \in V(\overline{k}) \mid h_A(P) \leq B\sqrt{h_A(P)}\} \\ &= \operatorname{Supp} E \cup \{P \in V(\overline{k}) \mid h_A(P) \leq B^2\}.\end{aligned}$$

Hence we can take $U = V \setminus \operatorname{Supp} E$ and $\Sigma = \{P \in U(\overline{k}) \mid \hat{h}(P) \leq 0\}$. □

Corollary 3.8.3. *Let X be a smooth projective variety of dimension N and $f\colon X \longrightarrow X$ a surjective endomorphism. Let C be a irreducible curve which is a complete intersection of ample effective divisors $H_1, \ldots, H_{N-1}$ on X. Then for infinitely many points P on C, we have $\alpha_f(P) = \delta_f$.*

Proof. We may assume $\delta_f > 1$. Let D be as in Lemma 3.8.1. Then $(D \cdot C) = (D \cdot H_1 \cdots H_{N-1}) > 0$ (cf. [29, Lemma 20]). Since $C(\overline{k})$ is not a set of bounded height, the assertion follows from Lemma 3.8.1. □

To prove Theorem 3.1.8, we need the following theorem which is a corollary of the dynamical Mordell–Lang conjecture for étale finite morphisms.

Theorem 3.8.4 (Bell–Ghioca–Tucker [3, Corollary 1.4])**.** *Let $f\colon X \longrightarrow X$ be an étale finite morphism of smooth projective variety X. Let $P \in X(\overline{k})$. If the orbit $\mathcal{O}_f(P)$ is Zariski dense in X, then any proper closed subvariety of X intersects $\mathcal{O}_f(P)$ in at most finitely many points.*

Proof of Theorem 3.1.8. We may assume $\dim X \geq 2$. Since we are working over $\overline{k}$, we can write the set of all proper subvarieties of X as

$$\{V_i \subsetneq X \mid i = 0, 1, 2, \ldots\}.$$

By Corollary 3.8.3, we can take a point $P_0 \in X \setminus V_0$ such that $\alpha_f(P) = \delta_f$. Assume we can construct $P_0, \ldots, P_n$ satisfying the following conditions.

(1) $\alpha_f(P_i) = \delta_f$ for $i = 0, \ldots, n$.

(2) $\mathcal{O}_f(P_i) \cap \mathcal{O}_f(P_j) = \emptyset$ for $i \neq j$.

(3) $P_i \notin V_i$ for $i = 0, \ldots, n$.

Now, take a complete intersection curve $C \subset X$ satisfying the following conditions.

- For $i = 0, \ldots, n$, $C \not\subset \mathcal{O}_f(P_i)$ if $\overline{\mathcal{O}_f(P_i)} \neq X$.
- For $i = 0, \ldots, n$, $C \not\subset \mathcal{O}_{f^{-1}}(P_i)$ if $\overline{\mathcal{O}_{f^{-1}}(P_i)} \neq X$.
- $C \not\subset V_{n+1}$.

By Theorem 3.8.4, if $\mathcal{O}_{f^{\pm}}(P_i)$ is Zariski dense in X, then $\mathcal{O}_{f^{\pm}}(P_i) \cap C$ is a finite set. By Corollary 3.8.3, there exists a point

$$P_{n+1} \in C \setminus \left(\bigcup_{0 \leq i \leq n} \mathcal{O}_f(P_i) \cup \bigcup_{0 \leq i \leq n} \mathcal{O}_{f^{-1}}(P_i) \cup V_{n+1} \right)$$

such that $\alpha_f(P_{n+1}) = \delta_f$. Then $P_0, \ldots, P_{n+1}$ satisfy the same conditions. Therefore we get a subset $S = \{P_i \mid i = 0, 1, 2, \ldots\}$ of X which satisfies the desired conditions. □

Chapter 4

Function filed analogues

(Joint work with Kaoru Sano and Takahiro Shibata.)

4.1 Summary

In this chapter, we consider the function filed analogue of arithmetic degree and Kawaguchi-Silverman conjecture. Let $\overline{k(t)}$ be the algebraic closure of one dimensional function field of characteristic zero. Let X be a smooth projective variety over $\overline{k(t)}$. Fix a height $h_X \geq 1$ on X associated to an ample divisor (cf. Definition 4.2.3). Given a dominant rational self-map $f : X \dashrightarrow X$ on X and a point $P \in X(\overline{k(t)})$, we study how the height $h_X(f^m(P))$ varies as m grows. The (upper/lower) arithmetic degree of f at P are defined in the same way as follows:

$$\overline{\alpha}_f(P) = \limsup_{m\to\infty} h_X(f^m(P))^{1/m},$$

$$\underline{\alpha}_f(P) = \liminf_{m\to\infty} h_X(f^m(P))^{1/m}$$

If $\overline{\alpha}_f(P) = \underline{\alpha}_f(P)$, then we set $\alpha_f(P) = \overline{\alpha}_f(P)$, call it the arithmetic degree of f at P. For details, see Definition 4.3.1 (ii).

The proof of inequality $\overline{\alpha}_f(P) \leq \delta_f$ in (Chapter 2, [39]) works over any field where height functions can be defined. In this chapter, we give another proof of it. This proof works only over function fields of characteristic zero, but it is simple and short.

Theorem A (= Theorem 4.4.1)**.** *Let X be a smooth projective variety over $\overline{k(t)}$ and f a dominant rational self-map on X. Then we have*

$$\overline{\alpha}_f(P) \leq \delta_f$$

for any $P \in X_f$. Here X_f is the set of rational points for which the f-orbits are well-defined (cf. Notation and Conventions below).

Given the above inequality, it is natural to ask when a rational point has maximal arithmetic degree, that is to say, the arithmetic degree of the rational point attains the dynamical degree. Actually, Kawaguchi and Silverman conjecture does not hold in general over $\overline{k(t)}$ (cf. Example 4.3.4). Nevertheless, we obtain a sufficient condition for a rational point to have maximal arithmetic degree as a geometric condition of the corresponding section of a model of X over a curve.

Theorem B (= Theorem 4.5.1)**.** *Let X be a smooth projective variety over $\overline{k(t)}$ and f a dominant rational self-map on X. Let $(X_C \overset{\pi}{\to} C, f_C)$ be a model of (X, f) over a curve C (cf. Definition 4.4.3). Take a rational point $P \in X_f$ corresponding to a section $\sigma : C \to X_C$ of π (cf. Proposition 4.2.4 (i)). Assume that*

- *$\sigma(C) \cap I_{f_C^m} = \varnothing$ for every $m \geq 1$ and*
- *$(E \cdot \sigma(C)) > 0$ for any $E \in \overline{\mathrm{Eff}}(X_C) \setminus \{0\}$.*

Then $\alpha_f(P)$ exists and $\alpha_f(P) = \delta_f$.

For a self-map on a projective space, we will give some other sufficient conditions (Theorem 4.6.2 and Theorem 4.6.4).

We can also consider how many points obtain maximal arithmetic degree. More precisely, we can ask whether there is a Zariski dense set of points with maximal arithmetic degree and pairwise disjoint orbits (Problem 4.7.1). It is only known for some particular cases over number fields (cf. [27, Theorem 3] and [40, Theorem 1.7]). Here we prove the following result, which gives a positive answer of the question when the base field is a function field over an uncountable algebraically closed field of characteristic zero.

Theorem C (= Theorem 4.7.2)**.** *Assume that k is an uncountable algebraically closed field of characteristic zero. Let X be a smooth projective variety over $\overline{k(t)}$ and f a dominant rational self-map on X. Then there exists a subset $S \subset X_f$ such that*

- *$\alpha_f(P)$ exists and $\alpha_f(P) = \delta_f$ for every $P \in S$,*
- *$O_f(P) \cap O_f(Q) = \varnothing$ if $P, Q \in S$ and $P \neq Q$, and*
- *S is a Zariski dense subset of X.*

Moreover, we can take such points over a fixed function field for a self-map on a smooth projective rational variety (Theorem 4.7.5).

To prove our main results, we use the the following geometric interpretation. A variety X over a function field $K = K(C)$ of a curve C can be seen as the generic fiber of a fibration $\pi : \mathcal{X} \to C$, and then a K-rational point P of X corresponds to a section σ of π. In this situation, the height $h_X(P)$ of P is equal to $\deg(\sigma^*\mathcal{H})$, where $\mathcal{H}$ is a π-ample Cartier divisor on $\mathcal{X}$. So arithmetic degrees are described as the limits of the degrees of certain divisors.

Notation and Conventions.

- Throughout this chapter, k denotes an algebraically closed field of characteristic zero, and $\overline{k(t)}$ denotes the algebraic closure of the rational function field of one variable over k.
- In this chapter, a *dynamical system* means the pair (X, f) of a smooth projective variety X and a dominant rational self-map f on X.
- For any $\mathbb{R}$-valued function $h(x)$, we set $h^+(x) = \max\{h(x), 1\}$.
- A *curve* means a smooth projective variety of dimension one unless otherwise stated.

4.2 Height functions for varieties over function fields

In this section, we recall the (Weil) height functions on projective varieties over $\overline{k(t)}$, as well as a description of height using the degree of a divisor on a curve. The content of this section can be found for example in [32, Chapter 3, §3] and [19, B.10].

First, we define the height functions on projective spaces over function fields.

Definition 4.2.1. Let C be a curve over k. Take $P \in \mathbb{P}^n(K(C))$ with homogeneous coordinates $P = (f_0 : f_1 : \cdots : f_n)$, where $f_i \in K(C)$. We define the *height function on* $\mathbb{P}^n(K(C))$ *relative to* $K(C)$ as

$$h_{K(C)}(P) = \sum_{p \in C} -\min\{v_p(f_0), \ldots, v_p(f_n)\},$$

where $v_p(f)$ is the multiplicity of f at $p \in C(k)$.

We define the (absolute) height function on $\mathbb{P}^n(\overline{k(t)})$ as follows:

Definition 4.2.2. Take $P \in \mathbb{P}^n(\overline{k(t)})$. We define the *height function on* $\mathbb{P}^n(\overline{k(t)})$ as

$$h(P) = \frac{1}{[K : k(t)]} h_K(P),$$

where K is a finite extension field of $k(t)$ such that $P \in \mathbb{P}^n(K)$. Note that $h(P)$ is independent of the choice of K.

We also define the height function associated to a Cartier divisor up to the difference of a bounded function.

Definition 4.2.3. Let X be a projective variety over $\overline{k(t)}$.

(i) Let A be a base point free Cartier divisor on X. We define a *height function on X associated to A* as

$$h_A = h \circ \phi_A,$$

where ϕ_A is a morphism associated to $|A|$. h_A is well-defined up to a bounded function.

(ii) Let D be a Cartier divisor on X. We define a *height function on X associated to D* as

$$h_D = h \circ \phi_A - h \circ \phi_B,$$

where A, B are base point free Cartier divisors such that $D \sim A - B$ and ϕ_A, ϕ_B are morphisms associated to $|A|$ and $|B|$ respectively. Note that we can always take such A and B, and h_D is well-defined up to a bounded function.

In what follows, we see that a height on a function field can be described as the degree of certain divisor on a curve. The following proposition follows from an elementary scheme-theoretic argument.

Proposition 4.2.4. *Let C be a curve over k and set $K = K(C)$.*

(i) Let $\pi : X \to C$ be a surjective morphism from a projective variety X to C and X_η the generic fiber of π. Then $X_\eta(K)$ corresponds one-to-one to the set of sections of π.

(ii) Let Y_k be a projective variety over k and set $Y_K = Y_k \times_k K$. Then $Y_K(K)$ corresponds one-to-one to the set of k-morphisms from C to Y_k.

The following is a description of the height on a projective space in terms of the degree of a divisor.

Proposition 4.2.5 (cf. [19, Lemma B.10.1]). *Let C be a curve and set $K = K(C)$. Take $P \in \mathbb{P}^n(K)$ and let $g : C \to \mathbb{P}^n_k$ be the corresponding morphism. Let $h : \mathbb{P}^n(\overline{k(t)}) \to \mathbb{R}$ be the natural height function on the projective space. Then we have*

$$h(P) = \frac{1}{[K : k(t)]} \deg(g^* \mathcal{O}(1)).$$

By Proposition 4.2.5, we have $h(P) \geq 0$ for any $P \in \mathbb{P}^n(K)$. Furthermore, for a rational point $P \in \mathbb{P}^n(K)$ corresponding to a morphism $g : C \to \mathbb{P}^n_k$, $P \in \mathbb{P}^n(k)$ if and only if g is a constant map. So we obtain the following.

Proposition 4.2.6.

(i) $h(P) \geq 0$ *for any* $P \in \mathbb{P}^n(\overline{k(t)})$.

(ii) For $P \in \mathbb{P}^n(\overline{k(t)})$, $h(P) = 0$ *if and only if* $P \in \mathbb{P}^n(k)$.

We give a description of height by the degree of a divisor for a projective variety over $\overline{k(t)}$.

Definition 4.2.7. Let X be a projective variety over $\overline{k(t)}$ and H an ample Cartier divisor on X. We define a function $\tilde{h}_H : X(\overline{k(t)}) \to \mathbb{R}_{\geq 0}$ as follows.

(i) A *model* $(X_C \stackrel{\pi}{\to} C, H_C)$ *of* (X, H) *over a curve* C is a projective variety X_C over k with a surjection $\pi : X_C \to C$ whose geometric generic fiber is X, and a π-ample Cartier divisor H_C on X_C such that $(X \to X_C)^* H_C \sim H$.

(ii) Fix a model $(X_C \stackrel{\pi}{\to} C, H_C)$ of (X, H) over a curve C. For any $P \in X(\overline{k(t)})$, take a curve C_1 with $K(C_1) \supset K(C)$ and the section σ_1 of $\pi_{C_1} : X_C \times_C C_1 \to C_1$ corresponding to P, and set $H_{C_1} = (X_C \times_C C_1 \to X_C)^* H_C$ and

$$\tilde{h}_H(P) = \frac{1}{[K(C_1) : k(t)]} \deg(\sigma_1^* H_{C_1}).$$

Proposition 4.2.8. *Notation is as in Definition 4.2.7. Then* $\tilde{h}_H$ *is a well-defined height function associated to* H.

Proof. Take any point $P \in X(\overline{k(t)})$. Take curves C_i with $K(C_i) \supset K(C)$ and the sections σ_i of $\pi_{C_i} : X_{C_i} = X_C \times_C C_i \to C_i$ for $i = 1, 2$. To see the well-definedness of $\tilde{h}_H$, we may assume that $K(C_2) \supset K(C_1)$.

$$\begin{array}{ccccc}
C_2 & \xrightarrow{\phi_2} & C_1 & & \\
\downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_1} & & \\
X_{C_2} & \xrightarrow{\psi_2} & X_{C_1} & \xrightarrow{\psi_1} & X_C \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\
C_2 & \xrightarrow{\phi_2} & C_1 & \xrightarrow{\phi_1} & C
\end{array}$$

Set $H_{C_1} = \psi_1^* H_C$ and $H_{C_2} = \psi_2^* H_{C_1}$. Then

$$\begin{aligned}
\frac{\deg(\sigma_2^* H_{C_2})}{[K(C_2) : k(t)]} &= \frac{\deg(\phi_2^* \sigma_1^* H_{C_1})}{[K(C_2) : K(C_1)][K(C_1) : k(t)]} \\
&= \frac{\deg(\sigma_1^* H_{C_1})}{[K(C_1) : k(t)]}.
\end{aligned}$$

So it follows that $\tilde{h}_H(P)$ is well-defined.

Take a sufficiently large integer N such that there is a morphism $\iota_C : X_C \to \mathbb{P}_k^n \times C$ over C with $\iota_C^* \mathcal{O}_{\mathbb{P}_C^n}(1) \sim_\pi NH_C$. Take a Cartier divisor D_C on C such that $\iota_C^* \mathcal{O}_{\mathbb{P}_C^n}(1) \sim NH_C + \pi^* D_C$. Let $\iota : X \to \mathbb{P}^n_{\overline{k(t)}}$ be the base change of ι_C by $\operatorname{Spec} \overline{k(t)} \to C$. Then the function $\frac{1}{N} h \circ \iota$ is a height function associated to H. $\iota(P) \in \mathbb{P}^n(\overline{k(t)})$ corresponds to the morphism $\mathrm{pr}_{\mathbb{P}_k^n} \circ \iota_{C_1} \circ \sigma_1 : C_1 \to \mathbb{P}_k^n$, where ι_{C_1} be the base change of ι_C by $C_1 \to C$. We compute

$$\begin{aligned}
\frac{1}{N} h(\iota(P)) &= \frac{\deg((\mathrm{pr}_{\mathbb{P}_k^n} \circ \iota_{C_1} \circ \sigma_1)^* \mathcal{O}_{\mathbb{P}_k^n}(1))}{N[K(C_1) : k(t)]} \quad \text{(by Proposition 4.2.5)} \\
&= \frac{\deg(\sigma_1^*(NH_{C_1} + \psi_1^* \pi^* D_C))}{N[K(C_1) : k(t)]} \\
&= \frac{\deg(\sigma_1^*(NH_{C_1})) + [K(C_1) : K(C)] \deg(D_C)}{N[K(C_1) : k(t)]} \\
&= \tilde{h}_H(P) + \frac{\deg(D_C)}{N[K(C) : k(t)]}.
\end{aligned}$$

So $\tilde{h}_H$ is a height function associated to H. □

4.3 Arithmetic degrees for dynamical systems over function fields

Arithmetic degree is defined in the same way as when the ground field is $\overline{\mathbb{Q}}$.

Definition 4.3.1. Let (X, f) be a dynamical system over $\overline{k(t)}$. Take an ample Cartier divisor H on X and a rational point $P \in X_f$. The *arithmetic degree of f at P* is defined as

$$\alpha_f(P) = \lim_{m\to\infty} h_H^+(f^m(P))^{1/m},$$

where h_H is a height function associated to H. Note that we do not know whether the limit converges. Similarly, $\overline{\alpha}_f(P), \underline{\alpha}_f(P)$ are defined as

$$\overline{\alpha}_f(P) = \limsup_{m\to\infty} h_H^+(f^m(P))^{1/m},$$

$$\underline{\alpha}_f(P) = \liminf_{m\to\infty} h_H^+(f^m(P))^{1/m}.$$

Remark 4.3.2. In the notation of Definition 4.3.1, $\alpha_f(P)$, $\overline{\alpha}_f(P)$ and $\underline{\alpha}_f(P)$ are independent of the choices of H and h_H (cf. [29, Proposition 14]).

As in the number field case, we consider:

Problem 4.3.3. Let (X, f) be a dynamical system over $\overline{k(t)}$. Take a point $P \in X_f$. When the equality $\alpha_f(P) = \delta_f$ holds?

The following examples show that Conjecture 1.3.2 (4) is not true over function fields.

Example 4.3.4. (i) Let $f : \mathbb{P}^1_{\overline{k(t)}} \to \mathbb{P}^1_{\overline{k(t)}}$ be a surjective endomorphism with $\delta_f > 1$. Take a k-valued non-preperiodic point $P \in \mathbb{P}^1(k)$. Then $O_f(P)$ is Zariski dense in $\mathbb{P}^1_{\overline{k(t)}}$, but $\alpha_f(P) = 1 < \delta_f$.

(ii) Define $f : \mathbb{A}^2_{\overline{k(t)}} \to \mathbb{A}^2_{\overline{k(t)}}$ as $f(x, y) = (x^2, y^3)$. Then f naturally extends to the morphism $f : \mathbb{P}^2_{\overline{k(t)}} \to \mathbb{P}^2_{\overline{k(t)}}$ and $\delta_f = 3$. Take a point $P = (t, 2) \in \mathbb{A}^2(k(t))$. Then $f^m(P) = (t^{2^m}, 2^{3^m})$ and

$$\alpha_f(P) = \lim_{m\to\infty} \max\{\deg(t^{2^m}), \deg(2^{3^m})\}^{1/m} = \lim_{m\to\infty} (2^m)^{1/m} = 2.$$

We show that $O_f(P) = \{(t^{2^m}, 2^{3^m})\}_{m=0}^{\infty}$ is dense in $\mathbb{P}^2_{\overline{k(t)}}$. It is enough to show that $O_f(P)$ is dense in $\mathbb{A}^2_{\overline{k(t)}}$. Suppose $O_f(P)$ is contained in the zero locus of a polynomial $\phi(t,x,y) \in k(t)[x,y]$. Multiplying ϕ with a polynomial in $k[t]$, we may assume that $\phi \in k[t,x,y]$. Set $\phi(t,x,y) = \phi_r(t,y)x^r + \phi_{r-1}x^{r-1} + \cdots + \phi_0(t,y)$, $\phi_r(t,y) \neq 0$. By assumption, $\phi(t, t^{2^m}, 2^{3^m}) = 0$ as a polynomial in $k[t]$. Since

$$\deg(t^{2^m r}) > \deg(\phi(t, t^{2^m}, 2^{3^m}) - \phi_r(t, 2^{3^m}) t^{2^m r}) = \deg(-\phi_r(t, 2^{3^m}) t^{2^m r})$$

for sufficiently large m, it follows that $\phi_r(t, 2^{3^m}) t^{2^m r} = 0$ as a polynomial in $k[t]$ for sufficiently large m. Therefore $\phi_r(t,y) = 0$ as a polynomial in $k[t,y]$, which is a contradiction. So $O_f(P)$ is Zariski dense in $\mathbb{P}^2_{\overline{k(t)}}$.

4.4 A fundamental inequality

There is a fundamental inequality between arithmetic degrees and dynamical degrees:

Theorem 4.4.1 ([29, Theorem 4] and [39]). *K denotes an algebraically closed field where heights are well-defined. Let (X,f) be a dynamical system over K. Then*

$$\overline{\alpha}_f(P) \leq \delta_f$$

holds for any $P \in X_f$.

We will give another proof of the inequality over $\overline{k(t)}$.

Theorem 4.4.2. *Let (X,f) be a dynamical system over $\overline{k(t)}$. Then the inequality*

$$\overline{\alpha}_f(P) \leq \delta_f$$

holds for any $P \in X_f$.

To prove Theorem 4.4.2, we prepare some lemmas. To begin with, we define a model of a dynamical system over $\overline{k(t)}$.

Definition 4.4.3. Let (X,f) be a dynamical system over $\overline{k(t)}$. A *model of (X,f) over a curve C* is a pair $(X_C \xrightarrow{\pi} C, f)$ of a surjective morphism $\pi : X_C \to C$ from a smooth projective k-variety X_C to C and a dominant rational self-map $f_C : X_C \dashrightarrow X_C$ over C such that $X_C \times_C \overline{k(t)} = X$ and the base change of f_C along $\operatorname{Spec} \overline{k(t)} \to C$ is equal to f.

Lemma 4.4.4. *Let (X,f) be a dynamical system over $\overline{k(t)}$. Then there exists a model of (X,f) over a curve C.*

Such a model is obtained by resolution of singularities.

Lemma 4.4.5 (cf. [29, Proposition 19])**.** *Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be dominant rational maps of smooth projective varieties. Take a Cartier divisor H on Z and a curve C on X.*

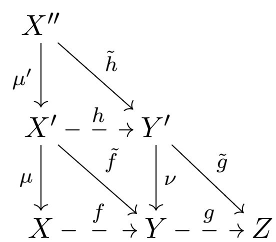
(i) If $C \not\subset I_f$, $f(C) \not\subset I_g$ and H is nef, then

$$((g \circ f)^*H \cdot C) \leq (f^*g^*H \cdot C).$$

(ii) If $C \cap I_f = \varnothing$ and $f(C) \cap I_g = \varnothing$, then

$$((g \circ f)^*H \cdot C) = (f^*g^*H \cdot C).$$

Proof. Since a nef divisor is the limit of a sequence of ample divisors, we may assume that H is ample.



In the above diagram, $\tilde{f} : X' \to Y$ (resp. $\tilde{g} : Y' \to Z$) is an elimination of indeterminacy of f (resp. g) by blowing up smooth centers in I_f (resp. I_g), $h = \nu^{-1} \circ f \circ \mu$, and $\tilde{h} : X'' \to Y'$ is an elimination of indeterminacy of h by blowing up smooth centers in I_h. Then

$$\begin{aligned} &(g \circ f)^*H \cdot C \leq f^*g^*H \cdot C \quad \cdots (1) \\ \iff\ &(\mu \circ \mu')_*(\tilde{g} \circ \tilde{h})^*H \cdot C \leq \mu_*\tilde{f}^*\nu_*\tilde{g}^*H \cdot C \\ \iff\ &\mu_*\mu'_*\tilde{h}^*\tilde{g}^*H \cdot C \leq \mu_*\tilde{f}^*\nu_*\tilde{g}^*H \cdot C. \end{aligned}$$

Here $\mu_*\tilde{f}^*\nu_*\tilde{g}^*H = \mu_*\mu'_*\mu'^*\tilde{f}^*\nu_*\tilde{g}^*H = \mu_*\mu'_*\tilde{h}^*\nu^*\nu_*\tilde{g}^*H$. Set

$$E = \nu^*\nu_*\tilde{g}^*H - \tilde{g}^*H,$$

then (1) is equivalent to the inequality

$$\mu_*\mu'_*\tilde{h}^*E \cdot C \geq 0.$$

By negativity lemma (cf. [31, Lemma 3.39]), E is an effective and ν-exceptional divisor. Take a curve C' on X' such that $\mu(C') = C$ and a curve C'' on X'' such that $\mu'(C'') = C'$.

$\nu(\tilde{h}(C'')) = \tilde{f}(\mu'(C'')) = \tilde{f}(C') = f(C) \not\subset I_g$, so $\tilde{h}(C'') \not\subset \mathrm{Exc}(\nu)$. In particular, $\tilde{h}(C'') \not\subset \operatorname{Supp} E$ and then $C'' \not\subset \operatorname{Supp} \tilde{h}^* E$. Hence

$$C = \mu(\mu'(C'')) \not\subset \mu(\mu'(\operatorname{Supp} \tilde{h}^* E)) = \operatorname{Supp} \mu_* \mu'_* \tilde{h}^* E.$$

This implies (1).

(ii) is obvious since $f|_C : C \to f(C)$ and $g|_{f(C)} : f(C) \to Y$ are morphisms. $\square$

The following lemma is a variant of Lemma 4.4.5.

Lemma 4.4.6. *Let $f : X \dashrightarrow Y$ be a dominant rational map of smooth projective varieties and $g : C \to X$ a morphism from a curve C. Take a Cartier divisor H on Y.*

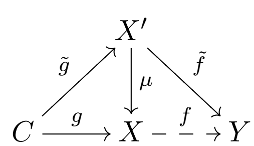
(i) If $g(C) \not\subset I_f$ and H is nef, then

$$\deg((f \circ g)^* H) \leq \deg(g^* f^* H).$$

(ii) If $g(C) \cap I_f = \varnothing$, then

$$\deg((f \circ g)^* H) = \deg(g^* f^* H).$$

Proof. (i) Since a nef divisor is the limit of a sequence of ample divisors, we may assume that H is ample.



In the above diagram, $\tilde{f} : X' \to Y$ is an elimination of indeterminacy of f by blowing up smooth centers in I_f, and we can define the composition $\tilde{g} = \mu \circ g$ by the assumption that $g(C) \not\subset I_f$. Moreover it is a morphism.

We compute

$$\begin{aligned} \deg(g^* f^* H) - \deg((f \circ g)^* H) &= \deg(\tilde{g}^* \mu^* \mu_* \tilde{f}^* H - \tilde{g}^* \tilde{f}^* H) \\ &= \deg(\tilde{g}^* E), \end{aligned}$$

where we set $E = \mu^* \mu_* \tilde{f}^* H - \tilde{f}^* H$. By negativity lemma(cf. [31, Lemma 3.39]), E is an effective and μ-exceptional divisor on X'. Moreover $\tilde{g}(C) \not\subset \operatorname{Supp} E$ since $\mu(\tilde{g}(C)) = g(C) \not\subset I_f$ and $\mu(E) \subset I_f$. So $\deg(\tilde{g}^* E) \geq 0$.

(ii) is obvious since both $g : C \to g(C)$ and $f|_{g(C)} : g(C) \to Y$ are morphisms. $\square$

Lemma 4.4.7. *Let X be a smooth projective variety with an ample Cartier divisor H. $\overline{\mathrm{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$ denotes the pseudo-effective cone of X. Take a 1-cycle $Z \in N_1(X)_{\mathbb{R}}$. Then there is a constant $M > 0$ such that*

$$(E \cdot Z) \leq M(E \cdot H^{\dim X - 1})$$

holds for any $E \in \overline{\mathrm{Eff}}(X)$.

Proof. Note that $(E \cdot H^{\dim X-1}) > 0$ for any $E \in \overline{\mathrm{Eff}}(X) \setminus \{0\}$ (see [29, Lemma 20]). We define a function $f : \overline{\mathrm{Eff}}(X) \setminus \{0\} \to \mathbb{R}$ as

$$f(E) = \frac{(E \cdot Z)}{(E \cdot H^{\dim X - 1})}.$$

Take a norm $||\cdot||$ on $N^1(X)_{\mathbb{R}}$ and set $S = \{E \in \overline{\mathrm{Eff}}(X) | \, ||E|| = 1\}$. Then we can take an upper bound $M > 0$ of $f|_S$ since S is compact. But f satisfies $f(cE) = f(E)$ for $E \in N^1(X)_{\mathbb{R}}$ and $c > 0$, so M is in fact an upper bound of f. This implies the claim. □

Lemma 4.4.8. *Let (X, f) be a dynamical system over $\overline{k(t)}$ with a model $(X_C \stackrel{\pi}{\to} C, f_C)$ over a curve C or k. Then $\delta_f = \delta_{f_C}$.*

Proof. We define the k-th dynamical degree and the k-th relative dynamical degree:

$$\lambda_k(f_C) = \lim_{m \to \infty} ((f_C^m)^* H_C^k \cdot H_C^{n+1-k})^{1/m},$$
$$\lambda_k(f_C|\pi) = \lim_{m \to \infty} ((f_C^m)^* H_C^k \cdot H_C^{n-k} \cdot F)^{1/m}.$$

Note that $\lambda_1(f_C) = \delta_{f_C}$.

Set $n = \dim X$. Take an ample divisor H_C on X_C and a general fiber F of π. Fix an integer $m > 0$. Take an elimination of indeterminacy of f_C^m:

$$\begin{array}{ccc} & \Gamma_C & \\ {}^{p_C}\swarrow & & \searrow^{g_C} \\ X_C & \underset{f_C^m}{\dashrightarrow} & X_C \end{array}$$

Pulling it back along $\overline{k(t)} \to C$, we get the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & X_C \\ {\scriptstyle p}\uparrow & & \uparrow{\scriptstyle p_C} \\ \Gamma & \xrightarrow{\eta_\Gamma} & \Gamma_C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_C} \\ X & \underset{\eta}{\longrightarrow} & X_C \end{array}$$

Set $H = \eta^* H_C$. We can show that $g^*\eta^* = \eta_\Gamma^* g_C^*$ and $p_*\eta_\Gamma^* = \eta^* p_{C*}$. So we have

$$(f^m)^* H = p_* g^* \eta^* H_C = p_* \eta_\Gamma^* g_C^* H_C = \eta^* p_{C*} g_C^* H_C = \eta^* (f_C^m)^* H_C.$$

So $((f^m)^* H \cdot H^{n-1}) = (\eta^*(f_C^m)^* H_C \cdot (\eta^* H_C)^{n-1})$. Hence $((f^m)^* H \cdot H^{n-1})$ is equal to the coefficient of the monomial $t_1 \cdots t_n$ for the numerical polynomial

$$\chi(X, t_1 \eta^*(f_C^m)^* H_C + t_2 \eta^* H_C + \cdots + t_n \eta^* H_C).$$

For any Cartier divisor D on X_C and a general fiber F of π, the equality $\chi(X, \eta^* D) = \chi(F, D|_F)$ holds. So

$$\begin{aligned}&\chi(X, t_1 \eta^*(f_C^m)^* H_C + t_2 \eta^* H_C + \cdots + t_n \eta^* H_C)\\ &= \chi(F, t_1 (f_C^m)^* H_C|_F + t_2 H_C|_F + \cdots + t_n H_C|_F).\end{aligned}$$

Hence we have $(\eta^*(f_C^m)^* H_C \cdot (\eta^* H_C)^{n-1}) = ((f_C^m)^* H_C|_F \cdot (H_C|_F)^{n-1}) = ((f_C^m)^* H_C \cdot H_C^{n-1} \cdot F)$, and so $\lambda_1(f) = \lambda_1(f_C|\pi)$.

On the other hand, by [52, Theorem 1.4],

$$\begin{aligned}\lambda_1(f_C) &= \max\{\lambda_1(f_C|\pi)\lambda_0(\mathrm{id}_C), \lambda_0(f_C|\pi)\lambda_1(\mathrm{id}_C)\}\\ &= \max\{\lambda_1(f_C|\pi), 1\}\\ &= \lambda_1(f_C|\pi).\end{aligned}$$

Note that $\lambda_q(\mathrm{id}_C) = 1$ for all q and $\lambda_0(f_C|\pi) = 1$ by definition. So

$$\delta_f = \lambda_1(f) = \lambda_1(f_C|\pi) = \lambda_1(f_C) = \delta_{f_C}.$$

□

Proof of Theorem 4.4.2. Take $P \in X_f$. Put $n = \dim X$. By Lemma 4.4.4, we can take a model $(X_C \xrightarrow{\pi} C, f_C)$ over a curve C. We may assume that P corresponds to a section $\sigma : C \to X_C$ of π.

Take an ample Cartier divisor H_C on X_C and set $H = (X \to X_C)^* H_C$. By Lemma 4.4.8,

$$\delta_f = \delta_{f_C} = \lim_{m\to\infty} ((f_C^m)^* H_C \cdot H_C^n)^{1/m}.$$

On the other hand, by Proposition 4.2.8,

$$\begin{aligned}\overline{\alpha}_f(P) &= \limsup_{m\to\infty} \tilde{h}_H^+(f^m(P))^{1/m}\\ &= \limsup_{m\to\infty} \deg^+((f_C^m \circ \sigma)^* H_C)^{1/m}.\end{aligned}$$

Note that $\mathrm{Im}(\sigma) \not\subset I_{f_C^m}$ since $P \notin I_{f^m}$. By Lemma 4.4.6 (i),

$$\deg((f_C^m \circ \sigma)^* H_C) \leq \deg(\sigma^*(f_C^m)^* H_C) = ((f_C^m)^* H_C \cdot \sigma_* C).$$

It is obvious that $(f_C^m)^* H_C \in \overline{\mathrm{Eff}}(X_C)$ for every m. So, by Lemma 4.4.7, there is a constant $M > 0$ such that the inequality

$$((f_C^m)^* H_C \cdot \sigma_* C) \leq M((f_C^m)^* H_C \cdot H_C^n)$$

holds for every m. Therefore we have

$$\begin{aligned}
\overline{\alpha}_f(P) &\leq \limsup_{m\to\infty}((f_C^m)^* H_C \cdot \sigma_* C)^{1/m} \\
&\leq \limsup_{m\to\infty}(M((f_C^m)^* H_C \cdot H_C^n))^{1/m} \\
&= \limsup_{m\to\infty}((f_C^m)^* H_C \cdot H_C^n)^{1/m} \\
&= \delta_{f_C} \\
&= \delta_f.
\end{aligned}$$

□

4.5 A sufficient condition

Let (X, f) be a dynamical system over $\overline{k(t)}$. In this section, we give a sufficient condition for a rational point $P \in X_f$ to have maximal arithmetic degree.

Theorem 4.5.1. *Let (X, f) be a dynamical system over $\overline{k(t)}$ and $(X_C \xrightarrow{\pi} C, f_C)$ a model of (X, f) over a curve C. Take a rational point $P \in X_f$ corresponding to a section $\sigma : C \to X_C$ of π. Assume that*

- *$\sigma(C) \cap I_{f_C^m} = \varnothing$ for every $m \geq 1$ and*
- *$(E \cdot \sigma(C)) > 0$ for any $E \in \overline{\mathrm{Eff}}(X_C) \setminus \{0\}$.*

Then $\alpha_f(P)$ exists and $\alpha_f(P) = \delta_f$.

We prepare the following lemma.

Lemma 4.5.2. *Let X be a smooth projective variety and $Z \subset X$ a 1-cycle such that $(E \cdot Z) > 0$ for any $E \in \overline{\mathrm{Eff}}(X) \setminus \{0\}$. We define a non-negative function $||\cdot||_Z : N^1(X)_{\mathbb{R}} \to \mathbb{R}$ as*

$$||v||_Z = \inf\{(v_1 \cdot Z) + (v_2 \cdot Z) \mid v = v_1 - v_2,\ v_1, v_2 \text{ are effective classes}\}.$$

(i) $||v||_Z = (v \cdot Z)$ *for any effective class* $v \in N^1(X)_{\mathbb{R}}$.

(ii) $||\cdot||_Z$ *is a norm on* $N^1(X)_{\mathbb{R}}$.

Proof. (i) For effective classes v_1, v_2 such that $v = v_1 - v_2$, we have $(v \cdot Z) = (v_1 \cdot Z) - (v_2 \cdot Z) \leq (v_1 \cdot Z) + (v_2 \cdot Z)$. So $||v||_Z = (v \cdot Z)$.

(ii) It is easy to see that

- $||cv||_Z = |c| \cdot ||v||_Z$ for any $c \in \mathbb{R}$ and $v \in N^1(X)_{\mathbb{R}}$ and
- $||v + w||_Z \leq ||v||_Z + ||w||_Z$ for any $v, w \in N^1(X)_{\mathbb{R}}$.

Take $v \in N^1(X)_{\mathbb{R}}$ and assume that $||v||_Z = 0$. Then we have

$$\{v_n^+\}_n, \{v_n^-\}_n \subset N^1(X)_{\mathbb{R}}$$

such that $v = v_n^+ - v_n^-$ for every n and

$$\lim_{n\to\infty}((v_n^+ \cdot Z) + (v_n^- \cdot Z)) = 0.$$

So $\lim_{n\to\infty}(v_n^\pm \cdot Z) = 0$. Since $(w \cdot Z) > 0$ for any $w \in \overline{\mathrm{Eff}}(X) \setminus \{0\}$, it follows that $\lim_{n\to\infty} v_n^\pm = 0$. Therefore $v = \lim_{n\to\infty}(v_n^+ - v_n^-) = 0$. So $||\cdot||_Z$ satisfies the conditions of norm. $\square$

Proof of Theorem 4.5.1. Set $n = \dim X$. We have

$$\begin{aligned}\delta_f &= \delta_{f_C} \quad \text{(by Lemma 4.4.8)}\\ &= \lim_{m\to\infty}((f_C^m)^* H_C \cdot H_C^n)^{1/m}\\ &= \lim_{m\to\infty}||(f_C^m)^* H_C||_{H_C^n}^{1/m}. \quad \text{(by Lemma 4.5.2 (i))}\end{aligned}$$

Note that $||\cdot||_{H_C^n}$ is a norm since $(E \cdot H_C^n) > 0$ for every $E \in \overline{\mathrm{Eff}}(X_C) \setminus \{0\}$ (cf. [29, Lemma 20]). We obtain

$$\begin{aligned}\delta_f &= \lim_{m\to\infty}||(f_C^m)^* H_C||_{\sigma(C)}^{1/m} \quad (\text{since } ||\cdot||_{H_C^n} \text{ is equivalent to } ||\cdot||_{\sigma(C)})\\ &= \lim_{m\to\infty}((f_C^m)^* H_C \cdot \sigma(C))^{1/m} \quad \text{(by Lemma 4.5.2 (i))}\\ &= \lim_{m\to\infty}\deg^+(\sigma^*(f_C^m)^* H_C)^{1/m}\\ &= \lim_{m\to\infty}\deg^+((f_C^m \circ \sigma)^* H_C)^{1/m} \quad \text{(by Lemma 4.4.6 (ii))}\\ &= \lim_{m\to\infty}\tilde{h}_H^+(f^m(P))^{1/m} \quad \text{(by Proposition 4.2.8)}\\ &= \alpha_f(P).\end{aligned}$$

$\square$

4.6 Arithmetic degrees for projective spaces

In this section, we study arithmetic degrees for dynamical systems on projective spaces.

At first, we give some sufficient conditions for a rational point to have maximal arithmetic degree.

Lemma 4.6.1. *Let C be a curve. Set $X = \mathbb{P}^n_k \times C$. Take an pseudo-effective Cartier divisor E on X and a general fiber F of $\pi = \mathrm{pr}_C$. Then $\mathcal{O}_X(E) \equiv \mathcal{O}_X(d) \otimes \mathcal{O}_X(eF)$ for some $d, e \in \mathbb{Z}_{\geq 0}$.*

Proof. It is sufficient to prove the claim for effective divisors, so we may assume that E is effective. Since $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(1)$ and $\pi^* \mathrm{Pic}(C)$, there are an integer d and a divisor D_C on C such that $\mathcal{O}_X(E) \cong \mathcal{O}_X(d) \otimes \pi^*\mathcal{O}_C(D_C)$. Set $e = \deg D_C$. Then $\mathcal{O}_X(E) \equiv \mathcal{O}_X(d) \otimes \mathcal{O}_X(eF)$.

Since $E|_F$ is effective and $\mathcal{O}_F(E|_F) \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(d)$, $d \geq 0$. By projection formula,

$$\pi_*(\mathcal{O}_X(d) \otimes \pi^*\mathcal{O}_C(D_C)) \cong \pi_*(\mathcal{O}_X(d)) \otimes \mathcal{O}_C(D_C) \cong S^d(\mathcal{O}_C^{\oplus n+1}) \otimes \mathcal{O}_C(D_C).$$

Since $H^0(C, S^d(\mathcal{O}_C^{\oplus n+1}) \otimes \mathcal{O}_C(D_C)) = H^0(X, \mathcal{O}_X(d) \otimes \pi^*\mathcal{O}_C(D_C)) \neq 0$, D_C is effective. So $e = \deg D_C \geq 0$. □

Theorem 4.6.2. *Let $(X = \mathbb{P}^n_{\overline{k(t)}}, f)$ be a dynamical system over $\overline{k(t)}$ and $(X_C = \mathbb{P}^n_k \times C \overset{\mathrm{pr}_C}{\to} C, f_C)$ a model of (X, f) over a curve C. Take a morphism $g : C \to \mathbb{P}^n_k$ corresponding to a rational point $P_g \in X_f$ and set $\sigma_g = (g, \mathrm{id}_C) : C \to X_C$.*

(i) Assume that g is non-constant and $\mathrm{Im}(\sigma_g) \cap I_{f_C^m} = \varnothing$ for every $m \geq 1$. Then $\alpha_f(P_g)$ exists and $\alpha_f(P_g) = \delta_f$.

(ii) Assume the following conditions.

($*$) For every $m \geq 0$, $\mathrm{pr}_{\mathbb{P}^n_k} \circ f_C^m \circ \sigma_g$ is non-constant.

($**$) There is a sequence of positive integers $m_1 < m_2 < \dots$ such that $\mathrm{Im}(f_C^{m_k} \circ \sigma_g) \cap I_{f_C^{m_{k+1}-m_k}} = \varnothing$ for every $k \geq 1$ and $\lim_{k\to\infty}(m_k/m_{k+1}) = 0$.

($***$) The limit $\alpha_f(P_g)$ exists.

Then the equality $\alpha_f(P_g) = \delta_f$ holds.

Proof. (i) Let F be a general fiber of pr_C. Then

$$\overline{\mathrm{Eff}}(X_C) = \mathbb{R}_{\geq 0}\mathcal{O}_X(F) + \mathbb{R}_{\geq 0}\mathcal{O}_X(1)$$

by Lemma 4.6.1. It is obvious that σ_g satisfies the assuption of Theorem 4.5.1. So (i) follows from Theorem 4.5.1.

(ii) For $m \geq 1$, set $(f_C^m)^*\mathcal{O}_{X_C}(1) \equiv \mathcal{O}_{X_C}(d_m) \otimes \mathcal{O}_{X_C}(e_m F)$. Then $d_m, e_m \geq 0$ by Lemma 4.6.1. It is clear that $(f^m)^*\mathcal{O}_X(1) \equiv \mathcal{O}_X(d_m)$, and so

$$\delta_f = \lim_{m\to\infty}((f^m)^*\mathcal{O}_X(1) \cdot \mathcal{O}_X(1)^{n-1})^{1/m} = \lim_{m\to\infty} d_m^{1/m}.$$

Set $b_m = \deg((f_C^m \circ \sigma_g)^*\mathcal{O}_{X_C}(1))$. By $(*)$, $b_m \geq 1$ for every m. So we have

$$\alpha_f(P_g) = \lim_{m\to\infty} \max\{b_m, 1\}^{1/m} = \lim_{m\to\infty} b_m^{1/m}.$$

For $k \geq 1$, set $l_k = m_{k+1} - m_k$. We compute

$$\begin{aligned}
b_{m_{k+1}} &= \deg((f_C^{m_{k+1}} \circ \sigma_g)^*\mathcal{O}_{X_C}(1)) \\
&= \deg(f_C^{l_k} \circ f_C^{m_k} \circ \sigma_g)^*\mathcal{O}_{X_C}(1)) \\
&= \deg(f_C^{m_k} \circ \sigma_g)^*(f_C^{l_k})^*\mathcal{O}_{X_C}(1)) \quad \text{(by $(**)$ and Lemma 4.4.6 (ii))} \\
&= \deg((f_C^{m_k} \circ \sigma_g)^*\mathcal{O}_{X_C}(d_{l_k}) \otimes \mathcal{O}_{X_C}(e_{l_k}F)) \\
&\geq \deg((f_C^{m_k} \circ \sigma_g)^*\mathcal{O}_{X_C}(d_{l_k})) \\
&= d_{l_k} b_{m_k} \\
&\geq d_{l_k}.
\end{aligned}$$

Note that $\lim_{k\to\infty} l_k = \infty$ by the assumption that $\lim_{k\to\infty}(m_k/m_{k+1}) = 0$. Hence

$$\begin{aligned}
\alpha_f(P_g) &= \lim_{m\to\infty} (b_m)^{\frac{1}{m}} \quad \text{(by Proposition 4.2.8)} \\
&= \lim_{k\to\infty} (b_{m_{k+1}})^{\frac{1}{m_{k+1}}} \\
&\geq \lim_{k\to\infty} (d_{l_k})^{\frac{1}{l_k}\cdot(1-\frac{m_k}{m_{k+1}})} \\
&= \delta_f.
\end{aligned}$$

Combining with Theorem 4.4.1, it follows that $\alpha_f(P_g) = \delta_f$. □

Next, we show that a sufficiently general morphism $g : C \to \mathbb{P}_k^n$ of a given sufficiently large degree corresponds to a rational point of maximal arithmetic degree.

Definition 4.6.3. Let C be a curve of genus $g(C)$ over k and d, n positive integers. $\mathrm{Mor}_d(C, \mathbb{P}^n_k)$ denotes the set of morphisms $g : C \to \mathbb{P}^n_k$ such that $\deg(g^*\mathcal{O}(1)) = d$.

$\mathrm{Mor}_d(C, \mathbb{P}^n_k)$ has a structure of k-variety with the evaluation $e : \mathrm{Mor}_d(C, \mathbb{P}^n_k) \times C \to \mathbb{P}^n_k$ which maps (g, p) to $g(p)$. Moreover, if $\mathrm{Mor}_d(C, \mathbb{P}^n_k)$ is non-empty, we have

$$\dim \mathrm{Mor}_d(C, \mathbb{P}^n_k) \geq (n+1)d + n(1 - g(C))$$

(cf. [31, 1.1]).

Theorem 4.6.4. *Let $(X = \mathbb{P}^n_{\overline{k(t)}}, f)$ be a dynamical system over $\overline{k(t)}$ and $(X_C = \mathbb{P}_k \times C \stackrel{\mathrm{pr}_C}{\to} C, f_C)$ a model of (X, f) over a curve C of genus $g(C)$. Take a positive integer d satisfying $d > \frac{n(g(C)-1)}{n+1}$. $P_g \in X(\overline{k(t)})$ denotes the rational point corresponding to $g \in \mathrm{Mor}(C, \mathbb{P}^n_k)$. Then $\alpha_f(P_g)$ exists and $\alpha_f(P_g) = \delta_f$ for a sufficiently general $g \in \mathrm{Mor}_d(C, \mathbb{P}^n_k)$.*

Proof. Let $M \subset \mathrm{Mor}_d(C, \mathbb{P}^n_k)$ be an irreducible component of maximal dimension. Then $\dim M > 0$ by assumption. Set $\Phi = (e, \mathrm{id}_C) : M \times C \to X_C$, where e is the evaluation. For any $g \in M$ and $\rho \in \mathrm{Aut}(\mathbb{P}^n_k)$, we have $\deg(g^*\rho^*\mathcal{O}_{\mathbb{P}^n}(1)) = \deg(g^*\mathcal{O}_{\mathbb{P}^n}(1)) = d$, so $\mathrm{Aut}(\mathbb{P}^n_k)$ acts on M. Fix $g_0 \in M$. For any $(x, p) \in X_C$, we can take $\rho \in \mathrm{Aut}(\mathbb{P}^n_k)$ such that $\rho(g_0(p)) = x$. Then $\Phi(\rho \circ g_0, p) = ((\rho \circ g_0)(p), p) = (x, p)$. So it follows that Φ is surjective.

For every $m \geq 1$, we compute

$$\begin{aligned}\dim \Phi^{-1}(I_{f_C^m}) &\leq (\dim(M \times C) - \dim X_C) + \dim I_{f_C^m} \\ &\leq (\dim M + 1 - \dim X_C) + \dim X_C - 2 \\ &= \dim M - 1.\end{aligned}$$

Hence $\mathrm{pr}_M(\Phi^{-1}(I_{f_C^m})) \subset M$ is a proper subset of M for every $m \geq 1$.

For $g \in M$, $\sigma_g = (g, \mathrm{id}_C) : C \to X_C$ denotes the corresponding section of pr_C. For $g \in M$, we have

$$\begin{aligned}\sigma_g(C) \cap I_{f_C^m} = \varnothing &\iff \Phi(\{g\} \times C) \cap I_{f_C^m} = \varnothing \\ &\iff \{g\} \times C \cap \Phi^{-1}(I_{f_C^m}) = \varnothing \\ &\iff g \notin \mathrm{pr}_M(\Phi^{-1}(I_{f_C^m})).\end{aligned}$$

Set $(f_C^m)^*\mathcal{O}_{X_C}(1) \equiv \mathcal{O}_{X_C}(d_m) \otimes \mathcal{O}_{X_C}(e_m F)$, where F is a general fiber of pr_C. Then $d_m, e_m \geq 0$ by Lemma 4.6.1. Take $g \in M \setminus \bigcup_{m \geq 1} \mathrm{pr}_M(\Phi^{-1}(I_{f_C^m}))$.

We compute

$$\begin{aligned}
\underline{\alpha}_f(P_g) &= \liminf_{m\to\infty} \deg((f_C^m \circ \sigma_g)^* \mathcal{O}_{X_C}(1))_+^{1/m} \quad \text{(by Proposition 4.2.8)} \\
&= \liminf_{m\to\infty} \deg(\sigma_g^* (f_C^m)^* \mathcal{O}_{X_C}(1))_+^{1/m} \quad \text{(by Lemma 4.4.6 (ii))} \\
&= \liminf_{m\to\infty} (\mathcal{O}_{X_C}(d_m) \otimes \mathcal{O}_{X_C}(e_m F) \cdot \sigma_{g*} C)_+^{1/m} \\
&\geq \liminf_{m\to\infty} (\mathcal{O}_{X_C}(d_m) \cdot \sigma_{g*} C)_+^{1/m} \\
&= \liminf_{m\to\infty} (d_m(\mathcal{O}_{X_C}(1) \cdot \sigma_{g*} C))_+^{1/m} \\
&= \liminf_{m\to\infty} d_m^{1/m} \\
&= \delta_f.
\end{aligned}$$

Note that $(\mathcal{O}_{X_C}(1) \cdot \sigma_{g*} C) = (\mathcal{O}_{\mathbb{P}_k^n}(1) \cdot g_* C) > 0$ since $d = \deg(g) > 0$. Combining with Theorem 4.4.2, it follows that $\alpha_f(P_g)$ exists and $\alpha_f(P_g) = \delta_f$. □

4.7 Construction of orbits

In this section, we consider a problem on the existence of the rational points of maximal arithmetic degree.

Problem 4.7.1. Let (X, f) be a dynamical system over $\overline{\mathbb{Q}}$. Is there a subset $S \subset X_f$ such that

- $\alpha_f(P)$ exists and $\alpha_f(P) = \delta_f$ for every $P \in S$,
- $O_f(P) \cap O_f(Q) = \varnothing$ if $P, Q \in S$ and $P \neq Q$, and
- S is a Zariski dense subset of X.

or not?

The above problem is studied in some papers over $\overline{\mathbb{Q}}$ (cf. [27, Theorem 3] and [40, Theorem 1.7]). We give an affirmative answer for any dynamical system over $\overline{k(t)}$, where k is an uncountable algebraically closed field of characteristic 0.

Theorem 4.7.2. *Assume that k is an uncountable algebraically closed field of characteristic zero. Let (X, f) be a dynamical system over $\overline{k(t)}$. Then there exists a subset $S \subset X_f$ such that*

- *$\alpha_f(P)$ exists and $\alpha_f(P) = \delta_f$ for every $P \in S$,*
- *$O_f(P) \cap O_f(Q) = \varnothing$ if $P, Q \in S$ and $P \neq Q$, and*
- *S is a Zariski dense subset of X.*

Lemma 4.7.3. *Let k be an uncountable algebraically closed field and X an algebraic scheme of positive dimension over k. Let $Z_1, Z_2, \ldots \subset X$ be proper closed subsets of X. Then $\bigcup_i Z_i \neq X$ and there exists a countable set M of k-valued points of $X \setminus \bigcup_i Z_i$ such that M is Zariski dense in X.*

Proof. Replacing X by an affine open subset, we may assume that X is affine. By Noether's normalization lemma, there is a finite cover $\phi : X \to \mathbb{A}_k^n$. Replacing X and $Z_1, Z_2, \ldots$ by $\mathbb{A}_k^n$ and $\phi(Z_1), \phi(Z_2), \ldots$, we may assume that $X = \mathbb{A}_k^n$.

We prove the claim by induction on n. Assume that $n = 1$. Then $Z_i(k)$ is a finite set for every i and $\mathbb{A}_k^1(k) = k$ is uncountable, so $\bigcup_i Z_i \neq \mathbb{A}_k^1$. We take an infinite subset M of $\mathbb{A}_k^1(k) \setminus \bigcup_i Z_i(k)$. Then M is a Zariski dense subset of $\mathbb{A}_k^1$.

Assume that the claim holds for $\mathbb{A}_k^1, \mathbb{A}_k^2, \ldots, \mathbb{A}_k^{n-1}$. Define $p : \mathbb{A}_k^n \to \mathbb{A}_k^{n-1}$ and $q : \mathbb{A}_k^n \to \mathbb{A}_k^1$ as $p(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$ and $q(x_1, \ldots, x_n) = x_n$. Let $\{Z'_j\}_j$ (resp. $\{Z''_k\}_k$) be the subset of $\{Z_i\}_i$ such that $p(Z'_j) \neq \mathbb{A}_k^{n-1}$ (resp. $p(Z''_k) = \mathbb{A}_k^{n-1}$). Let $W_k \subset \mathbb{A}_k^{n-1}$ be the set of points $w \in \mathbb{A}_k^{n-1}$ such that the fiber $(p|_{Z''_k})^{-1}(w) = p^{-1}(w) \cap Z''_k$ of $p|_{Z''_k}$ over w has positive dimension. Then W_k is a proper closed subset of $\mathbb{A}_k^{n-1}$. By induction hypothesis, $\bigcup_j \phi(Z'_j) \cup \bigcup_k W_k \neq \mathbb{A}_k^{n-1}$ and we can take a countable subset $M' \subset \mathbb{A}_k^{n-1}(k) \setminus (\bigcup_j \phi(Z'_j)(k) \cup \bigcup_k W_k(k))$ such that $M' = \{a_m\}_{m=1}^{\infty}$ is Zariski dense in $\mathbb{A}_k^{n-1}$. For every m and k, $p^{-1}(a_m) \cap Z''_k \neq \mathbb{A}_k^1$ since $a_m \notin W_k$. So $\bigcup_{m,k}(p^{-1}(a_m) \cap Z''_k) \not\subset \mathbb{A}_k^1$ and we can take a countable subset $M'' \subset \mathbb{A}_k^1(k) \setminus \bigcup_{m,k}(p^{-1}(a_m)(k) \cap Z''_k(k))$ such that M'' is Zariski dense in $\mathbb{A}_k^1$, by induction hypothesis. Set $M = M' \times M'' \subset \mathbb{A}_k^{n-1} \times \mathbb{A}_k^1$. Then it is clear that M satisfies the claim. □

Proof of Theorem 4.7.2. Take a model $(X_{C_0} \overset{\pi_{C_0}}{\to} C_0, f_{C_0})$ of (X, f) over a curve C_0. For any curve C with a finite morphism $C \to C_0$, $(X_C \overset{\pi_C}{\to} C, f_C)$ denotes the pull-back of $(X_{C_0} \overset{\pi_{C_0}}{\to} C_0, f_{C_0})$ by $C \to C_0$ and $\psi_C : X_C \to X_{C_0}$ denote the projection. For a section $\sigma : C \to X$ and a finite morphism $C' \to C$ of curves, $(\sigma)_{C'} : C' \to X \times_C C'$ denotes the pull-back of σ by $C' \to C$.

By Lemma 4.7.3, we can take a countable subset $M = \{a_i\}_{i=1}^{\infty} \subset X_{C_0}$ such that

- M is Zariski dense in X_{C_0} and
- $a_i \notin I_{f^m_{C_0}}$ for every $m \geq 1$ and $i \geq 1$.

We will construct rational points $P_1, P_2, \ldots \in X$ inductively. Let $C_k \to C_{k-1} \to \cdots \to C_1 \to C_0$ be a sequence of finite morphisms of curves and $P_i \in X$ a rational point corresponding to a section $\sigma_i : C_i \to X_{C_i}$ of π_{C_i} for each $1 \leq i \leq k$. Assume that $P_1, \ldots, P_k \in X$ satisfy the following condition $(*)_k$:

- $P_i \in X_f$ for $1 \leq i \leq k$,
- $\alpha_f(P_i) = \delta_f$ for $1 \leq i \leq k$,
- $O_f(P_i) \cap O_f(P_j) = \varnothing$ if $1 \leq i, j \leq k$ and $i \neq j$, and
- $a_i \in \mathrm{Im}(\psi_{C_i} \circ \sigma_i)$ for $1 \leq i \leq k$.

Set $n = \dim X$. Note that X_{C_k} is smooth outside a finite union of fibers of π_{C_k}.

Let $p_k : X_k \to X_{C_k}$ be a resolution of $(X_{C_k})_{\mathrm{red}}$ whose exceptional locus is contained in a finite union of fibers of π_{C_k}. By blowing up a point in $(p_k \circ \psi_{C_k})^{-1}(a_{k+1})$, we may assume that $(p_k \circ \psi_{C_k})^{-1}(a_{k+1})$ has codimension 1. We take a very ample divisor H on X_k and suitable members $H_1, \ldots, H_n \in |H|$, and set $C_{k+1} = H_1 \cap \cdots \cap H_n$. Let $\iota : C_{k+1} \to X_k$ denote the inclusion. We can choose $H_1, \ldots, H_n$ satisfying

(I) C_{k+1} is a smooth and irreducible curve satisfying $\mathrm{Im}(\pi_{C_k} \circ p_k \circ \iota) = C_k$,

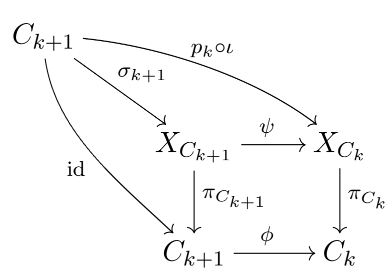
(II) $C_{k+1} \not\subset p_k^{-1}(f^m_{C_k})^{-1}(I_{f_{C_k}})$ for all $m \geq 0$,

(III) $C_{k+1} \cap I_{f^m_k} = \varnothing$ for all $m \geq 1$,

(IV) $C_{k+1} \not\subset (f^{m'}_k)^{-1}(\mathrm{Im}(f^m_k \circ p_k^{-1} \circ (\sigma_i)_{C_k}))$ for all m, m' and $1 \leq i \leq k$, and

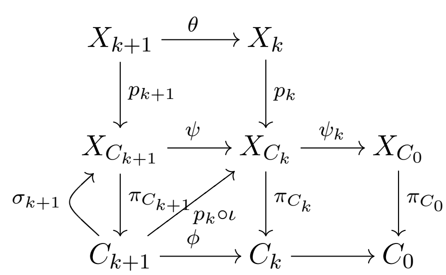
(V) $a_{k+1} \in \mathrm{Im}(\psi_{C_k} \circ p_k \circ \iota)$.

Set $\phi = \pi_{C_k} \circ p_k \circ \iota : C_{k+1} \to C_k$. Then we obtain the following diagram:



Here, $X_{C_{k+1}} = X_{C_k} \times_{C_k} C_{k+1}$ and σ_{k+1} is the unique morphism which makes the above diagram commutative. Let $P_{k+1} \in X$ be the rational point of X corresponding to σ_{k+1}. By (II), $\mathrm{Im}(\sigma_{k+1}) \not\subset (f^m_{C_{k+1}})^{-1}(I_{f_{C_{k+1}}})$ for every $m \geq 0$. Hence, $\mathrm{Im}(f^m_{C_{k+1}} \circ \sigma_{k+1}) \not\subset I_{f_{C_{k+1}}}$ and so $f^m(P_{k+1}) \notin I_f$ for every $m \geq 0$. Therefore, $P_{k+1} \in X_f$.

Let $p_{k+1} : X_{k+1} \to X_{C_{k+1}}$ be a resolution of $(X_{C_{k+1}})_{\mathrm{red}}$ whose exceptional locus is over a finite union of fibers of $\pi_{C_{k+1}}$ and $\theta = p_k^{-1} \circ \psi \circ p_{k+1}$ becomes a morphism. Then we obtain the following diagram:



Set $f_k = p_k^{-1} \circ f_{C_k} \circ p_k$, $f_{k+1} = p_{k+1}^{-1} \circ f_{C_{k+1}} \circ p_{k+1}$ and $\sigma'_{k+1} = p_{k+1}^{-1} \circ \sigma_{k+1}$. Fix a positive integer m. Then it follows that $p_k \circ \theta \circ f^m_{k+1} \circ \sigma'_{k+1} = p_k \circ f^m_k \circ \iota$. Since p_k is birational, we have $\theta \circ f^m_{k+1} \circ \sigma'_{k+1} = f^m_k \circ \iota$. Take an ample divisor A on X_{k+1} such that $A - \theta^* H$ is ample. We compute

$$\begin{aligned}\deg(f^m_{k+1} \circ \sigma'_{k+1})^* A &\geq \deg(f^m_{k+1} \circ \sigma'_{k+1})^* \theta^* H \\ &= \deg(\theta \circ f^m_{k+1} \circ \sigma'_{k+1})^* H \\ &= \deg(f^m_k \circ \iota)^* H.\end{aligned}$$

By (III) and Lemma 4.4.6 (ii), we have

$$\deg(f^m_k \circ \iota)^* H = \deg \iota^* (f^m_k)^* H = ((f^m_k)^* H \cdot H^{n-1}).$$

Now, $(X_{k+1} \xrightarrow{\pi_{C_{k+1}} \circ p_{k+1}} C_{k+1}, f_{k+1})$ is a model of (X, f) and σ'_{k+1} is a section of $\pi_{C_{k+1}} \circ p_{k+1}$ corresponding to P_{k+1}. Therefore

$$\begin{aligned}\underline{\alpha}_f(P_{k+1}) &= \liminf_{m \to \infty} \deg((f^m_{k+1} \circ \sigma'_{k+1})^* A)^{1/m} \quad \text{(by Proposition 4.2.8)} \\ &\geq \liminf_{m \to \infty} ((f^m_k)^* H \cdot H^{n-1})^{1/m} \\ &= \delta_{f_k} = \delta_f.\end{aligned}$$

So $\alpha_f(P_{k+1})$ exists and $\alpha_f(P_{k+1}) = \delta_f$.

Fix $i \in \{0, \ldots, k\}$ and $m, m' \geq 0$. By (IV), $\mathrm{Im}(f^{m'}_k \circ \iota) \neq \mathrm{Im}(f^m_k \circ p_k^{-1} \circ (\sigma_i)_{C_k}) = \mathrm{Im}(f^m_k \circ p_k^{-1} \circ (\sigma_i)_{C_k} \circ \phi)$. Since p_k is birational and the images of both $f_k^{m'} \circ \iota$ and $f_k^m \circ p_k^{-1} \circ (\sigma_i)_{C_k} \circ \phi$ intersects the isomorphic locus of p_k, we have

$$\mathrm{Im}(p_k \circ f_k^{m'} \circ \iota) \neq \mathrm{Im}(p_k \circ f_k^m \circ p_k^{-1} \circ (\sigma_i)_{C_k} \circ \phi).$$

On the other hand,

$$p_k \circ f_k^{m'} \circ \iota = \psi \circ f_{C_{k+1}}^{m'} \circ \sigma_{k+1}$$

and

$$p_k \circ f_k^m \circ p_k^{-1} \circ (\sigma_i)_{C_k} \circ \phi = \psi \circ f_{C_{k+1}}^m \circ (\sigma_i)_{C_{k+1}}.$$

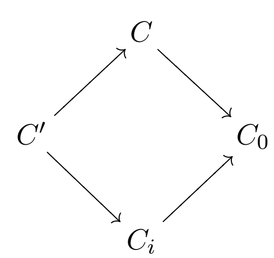
So $\mathrm{Im}(\psi \circ f_{C_{k+1}}^{m'} \circ \sigma_{k+1}) \neq \mathrm{Im}(\psi \circ f_{C_{k+1}}^m \circ (\sigma_i)_{C_{k+1}})$, and so $f_{C_{k+1}}^{m'} \circ \sigma_{k+1} \neq f_{C_{k+1}}^m \circ (\sigma_i)_{C_{k+1}}$. This means that $f^{m'}(P_{k+1}) \neq f^m(P_i)$. Hence $O_f(P_{k+1}) \cap O_f(P_i) = \varnothing$.

Set $\psi_{k+1} = \psi_k \circ \psi$. Then $\psi_{k+1} \circ \sigma_{k+1} = \psi_k \circ p_k \circ \iota$. By (V), $a_i \in \mathrm{Im}(\psi_{k+1} \circ \sigma_{k+1})$. As a consequence, $P_1, \ldots, P_{k+1}$ satisfies $(*)_{k+1}$.

Continuing this process, we obtain a subset $S = \{P_1, P_2, \ldots\} \subset X_f$, a sequence $\cdots \to C_1 \to C_0$ of finite morphisms of curves, and sections $\sigma_i : C_i \to X_{C_i}$ corresponding to P_i for each $i \geq 1$ such that

- $\alpha_f(P_i) = \delta_f$ for every i,
- $O_f(P_i) \cap O_f(P_j) = \varnothing$ if $i \neq j$, and
- $a_i \in \mathrm{Im}(\psi_{C_i} \circ \sigma_i)$ for every i.

So it is enough to show that S is a Zariski dense subset of X. Let $Z \subset X$ be a proper closed subset of X. We take a finite cover $C \to C_0$ such that Z lifts to a proper closed subset $Z_C \subset X_C = X_{C_0} \times_{C_0} C$. Since $\psi_C(Z_C)$ is a proper closed subset of X_{C_0}, $a_i \notin \psi_C(Z_C)$ for some i. Take a curve C' with finite morphisms $C' \to C$ and $C' \to C_i$ which makes the following diagram commutative:



Set $Z_{C'} = Z_C \times_C C' \subset X_{C'}$. Since $a_i \in \mathrm{Im}(\psi_{C'} \circ (\sigma_i)_{C'})$ and $a_i \notin \psi_C(Z_C) = \psi_{C'}(Z_{C'})$, $\mathrm{Im}((\sigma_i)_{C'}) \not\subset Z_{C'}$. So $P_i \notin Z$. Therefore S is a Zariski dense subset of X. $\square$

Theorem 4.7.2 includes the following result.

Corollary 4.7.4. *Assume that k is an uncountable algebraically closed field of characteristic zero. Let (X, f) be a dynamical system over $\overline{k(t)}$. Then there exists a rational point $P \in X_f$ such that $\alpha_f(P) = \delta_f$.*

For a dynamical system on a rational variety, we can take a subset S as in the statement of Theorem 4.7.2 over a fixed function field.

Theorem 4.7.5. *Assume that k is an uncountable algebraically closed field of characteristic zero. Let (X, f) be a dynamical system over $\overline{k(t)}$ such that X is rational. Then there exists a subset $S \subset X_f$ such that*

- *There exists a function field K of a curve over k and a model X_K of X over K such that all points in S are defined over K,*
- *$\alpha_f(P) = \delta_f$ for every $P \in S$,*
- *S is a Zariski dense subset of X, and*
- *$O_f(P) \cap O_f(Q) = \varnothing$ if $P, Q \in S$ and $P \neq Q$.*

We need a result in [40].

Theorem 4.7.6 ([40, Theorem 3.4 (i)])**.** *Let $f\colon X \dashrightarrow X$ and $g\colon Y \dashrightarrow Y$ be dominant rational self-maps on smooth projective varieties and $\phi\colon Y \dashrightarrow X$ a birational map such that $\phi \circ g = f \circ \phi$. Let $V \subset Y$ be an open subset such that $\phi|_V : V \to \phi(V)$ is an isomorphism. Then $\overline{\alpha}_g(Q) = \overline{\alpha}_f(\phi(Q))$ and $\underline{\alpha}_g(Q) = \underline{\alpha}_f(\phi(Q))$ for any $Q \in Y_g \cap \phi^{-1}(X_f)$ satisfying $O_g(Q) \subset V$.*

Proof of Theorem 4.7.5. Let $\phi : Y = \mathbb{P}^n_{\overline{k(t)}} \dashrightarrow X$ be a birational map. Set $g = \phi^{-1} \circ f \circ \phi$. Take open subsets $U \subset X$ and $V \subset Y$ such that $\phi|_V : V \to U$ is isomorphic. We can take a curve C, a model $(X_C \stackrel{\mathrm{pr}_C}{\to} C, f_C)$ (resp. $(Y_C = \mathbb{P}^n_k \times C \stackrel{\mathrm{pr}_C}{\to} C, g_C)$) of (X, f) (resp. (Y, g)), a lift U_C (resp. V_C) of U (resp. V), and a birational map $\phi_C : Y_C \dashrightarrow X_C$ such that $\phi_C|_{V_C} : V_C \to U_C$ is isomorphic.

Set $Z_C = Y_C \setminus V_C$. By Lemma 4.7.3, we can take a countable subset $M = \{a_i = (b_i, c_i)\}_{i=1}^{\infty} \subset V_C$ such that

- M is Zariski dense in Y_C,
- $a_i \notin (g_C^m)^{-1}(I_{g_C} \cup Z_C) \cup \phi_C^{-1}(f_C^m)^{-1}(I_{f_C})$ for every $m \geq 0$, and
- $b_i \notin \mathrm{pr}_{\mathbb{P}^n_k}(I_{g_C^m})$ for every $m \geq 1$ and $i \geq 1$.

Note that $\mathrm{pr}_{\mathbb{P}^n_k}(I_{g^m_C})$ is a proper closed subset of $\mathbb{P}^n_k$ for every m because $\dim I_{g^m_C} \leq n-1$. For $m \geq 1$, let $J_m \subset \mathrm{pr}_{\mathbb{P}^n_k}(I_{g^m})$ be the closed subset over which the fibers of $\mathrm{pr}_{\mathbb{P}^n_k}|_{I_{g^m}}$ have positive dimensions. Then $\mathrm{codim}_{\mathbb{P}^n_k} J_m \geq 2$. We take a point $q \in C$ such that $q \neq c_i$ for all i and $f_C|_{F_q} : F_q \dashrightarrow F_q$ is well-defined and dominant, where F_q denotes the fiber of $\mathrm{pr}_C : Y_C \to C$ over q.

Assume that we have sections $\tau_1, \ldots, \tau_k$ of pr_C corresponding to $Q_1, \ldots, Q_k \in Y$ which satisfy the condition $(*)_k$:

- $Q_i \in Y_g \cap \phi^{-1}(X_f)$ such that $O_g(Q_i) \subset V$ for $1 \leq i \leq k$,
- $\alpha_g(Q_i) = \delta_g$ for $1 \leq i \leq k$,
- $O_g(Q_i) \cap O_g(Q_j) = \varnothing$ if $1 \leq i, j \leq k$ and $i \neq j$, and
- $a_i \in \mathrm{Im}(\tau_i)$ for $1 \leq i \leq k$.

We take general hyperplanes $H_1, \ldots, H_{n-1}$ of $\mathbb{P}^n_k$. Then we have a line $L = H_1 \cap \cdots \cap H_{n-1} \subset \mathbb{P}^n_k$. We can choose $H_1, \ldots, H_{n-1}$ as satisfying

(I) $L \not\subset \mathrm{pr}_{\mathbb{P}^n_k}(I_{g^m_C})$ and $L \cap J_m = \varnothing$ for every $m \geq 1$,

(II) $L \times \{q\} \not\subset (g^{m'}_C|_{F_q})^{-1}(\mathrm{Im}(g^m_C \circ \tau_i) \cap F_q)$ for every $m, m' \geq 0$ and $1 \leq i \leq k$, and

(III) $b_{k+1} \in L$.

Note that $\mathrm{Im}(g^m_C \circ \tau_i) \cap F_q$ is a point since $g^m_C \circ \tau_i$ is a section of pr_C. By (I), $I_{g^m_C} \cap (L \times C) \subset Y$ is a finite set. Set $\bigcup_{m \geq 1}(I_{g^m_C} \cap (L \times C)) = \{(x_j, y_j)\}_j$. We can construct a finite cover $\phi : C \to L$ satisfying

(1) $\phi(c_{k+1}) = b_{k+1}$,

(2) $\phi(y_j) \neq x_j$ for every j, and

(3) $(\phi(q), q) \notin (g^{m'}|_{F_q})^{-1}(\mathrm{Im}(g^m \circ \tau_i) \cap F_q)$ for every m, m' and $1 \leq i \leq k$,

by composing a fixed finite morphism $C \to L$ with a suitable automorphism on L.

Set $\tau_{k+1} : C \overset{(\phi, \mathrm{id}_C)}{\to} L \times C \hookrightarrow X_C$ and let $Q_{k+1} \in \mathbb{P}^n(K)$ be the corresponding rational point. Then $a_{k+1} \in \mathrm{Im}(\tau_{k+1})$ by (1). Since $a_{k+1} \notin (g^m_C)^{-1}(I_{g_C} \cup Z_C) \cup \phi_C^{-1}(f^m_C)^{-1}(I_{f_C})$, $\mathrm{Im}(g^m_C \circ \tau_{k+1}) \not\subset I_{g_C} \cup Z_C$ and $\mathrm{Im}(f^m_C \circ \phi_C \circ \tau_{k+1}) \not\in I_{f_C}$ for every $m \geq 0$. Hence $Q_{k+1} \in Y_g \cap \phi^{-1}(X_f)$ such that $O_g(Q_{k+1}) \subset V$. Furthermore, $\mathrm{Im}(\tau_{k+1}) \cap I_{f^m} = \varnothing$ for every m by (2).

Therefore $\alpha_g(Q_{k+1})$ exists and $\alpha_g(Q_{k+1}) = \delta_g$ by Theorem 4.6.2 (i). Moreover, $(g_C^{m'} \circ \tau_{k+1})(q) \neq (g^m \circ \tau_i)(q)$ for every m, m' and $1 \leq i \leq k$ by (3). In particular, it follows that $g_C^{m'} \circ \tau_{k+1} \neq g_C^m \circ \tau_i$ for every m, m' and $1 \leq i \leq k$. This means that $O_g(Q_{k+1}) \cap O_g(Q_i) = \varnothing$ for $1 \leq i \leq k$. As a consequence, $\{\tau_1, \ldots, \tau_k, \tau_{k+1}\}$ satisfies $(*)_{k+1}$.

Continuing this process, we obtain morphisms $\tau_1, \tau_2, \ldots$ and a subset $T = \{Q_1, Q_2, \ldots\} \subset V \cap Y_g$. Set $P_i = \phi(Q_i)$ and $S = \{P_1, P_2, \ldots\}$. Then P_i corresponds to a section $\sigma_i = \phi_C \circ \tau_i$. Since $Q_i \in Y_g \cap \phi^{-1}(X_f)$ and $O_g(Q_i) \subset V$, $\alpha_g(Q_i) = \alpha_f(P_i)$ by Theorem 4.7.6. So we have $\alpha_f(P_i) = \delta_f$. For i, j with $i \neq j$, $O_g(Q_i) \cap O_g(Q_j) = \varnothing$ implies that $O_f(P_i) \cap O_f(P_j) = \varnothing$. Since $p_i \in \mathrm{Im}(\tau_i)$ for every i, $\bigcup_i \mathrm{Im}(\tau_i)$ is Zariski dense in Y_C and so $\bigcup_i \mathrm{Im}(\sigma_i)$ is Zariski dense in X_C. So S is Zariski dense in X. Therefore S satisfies the claim. $\square$

Chapter 5

Self-morphisms of semi-abelian varieties

(Joint work with Kaoru Sano.)

5.1 Summary

Let X be a smooth projective variety and $f\colon X \dashrightarrow X$ a rational self-map, both defined over $\overline{\mathbb{Q}}$. Fix a Weil height function h_X associated with an ample divisor on X. Let $A(f)$ be the set of the arithmetic degrees of f, i.e.

$$A(f) = \{\alpha_f(x) \mid x \in X \text{ with } f^n(x) \notin I_f \text{ for every } n \geq 0\}$$

where I_f is the indeterminacy locus of f. Determining the set $A(f)$ for a given f is an interesting problem. In [40, Theorem 1.6], we proved that for any surjective morphism f, there exists a point $x \in X$ such that $\alpha_f(x) = \delta_f$. When X is a toric variety and f is a self-rational map on X that is induced by a group homomorphism of the algebraic torus, the set $A(f)$ is completely determined [37, 50].

When X is quasi-projective, the arithmetic degrees and dynamical degrees can be defined by taking a smooth compactification of X. In this chapter, we prove Conjecture 1.3.2 (KSC) for self-morphisms of semi-abelian varieties and determine the set $A(f)$.

Theorem 5.1.1. *Let X be a semi-abelian variety and $f\colon X \longrightarrow X$ a self-morphism (not necessarily surjective), both defined over $\overline{\mathbb{Q}}$.*

(1) *For every $x \in X(\overline{\mathbb{Q}})$, the arithmetic degree $\alpha_f(x)$ exists. Moreover, we can write $f = T_a \circ g$ where T_a is the translation by a point $a \in X(\overline{\mathbb{Q}})$ and g is a group homomorphism (c.f. [6, Lemma 5.4.8]). Then $A(f) = A(g)$.*

(2) *Suppose f is surjective. Then for any point $x \in X(\overline{\mathbb{Q}})$ with Zariski dense f-orbit, we have $\alpha_f(x) = \delta_f$.*

(3) *Suppose f is a group homomorphism. Let $F(t)$ be the monic minimal polynomial of f as an element of* $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ *and*

$$F(t) = t^{e_0} F_1(t)^{e_1} \cdots F_r(t)^{e_r}$$

the irreducible decomposition in $\mathbb{Q}[t]$ where $e_0 \geq 0$ and $e_i > 0$ for $i = 1, \ldots, r$. Let $\rho(F_i)$ be the maximum among the absolute values of the roots of F_i. Then we have

$$A(f) \subset \{1, \rho(F_1), \rho(F_1)^2, \ldots, \rho(F_r), \rho(F_r)^2\}.$$

More precisely, set

$$X_i = f^{e_0} F_1(f)^{e_1} \cdots F_{i-1}(f)^{e_{i-1}} F_{i+1}(f)^{e_{i+1}} \cdots F_r(f)^{e_r}(X).$$

Define

$$A_i = \begin{cases} \{\rho(F_i)\} & \text{if } X_i \text{ is an algebraic torus,} \\ \{\rho(F_i)^2\} & \text{if } X_i \text{ is an abelian variety,} \\ \{\rho(F_i), \rho(F_i)^2\} & \text{otherwise.} \end{cases}$$

Then we have

$$A(f) = \{1\} \cup A_1 \cup \cdots \cup A_r.$$

Remark 5.1.2. Actually, in the situation of Theorem 5.1.1 (3), f is conjugate by an isogeny to a group homomorphism of the form

$$f_0 \times \cdots \times f_r \colon X_0 \times \cdots \times X_r \longrightarrow X_0 \times \cdots \times X_r$$

where $A(f_0) = \{1\}$ and $A(f_i) = \{1\} \cup A_i$ for $i = 1, \ldots, r$.

We can characterize the set of points whose arithmetic degrees are equal to 1 as follows (cf. [48] for related results).

Theorem 5.1.3. *Let X be a semi-abelian variety and $f \colon X \longrightarrow X$ a surjective morphism both defined over $\overline{\mathbb{Q}}$. Write $f = T_a \circ g$ where T_a is the translation by $a \in X(\overline{\mathbb{Q}})$ and g is a surjective group endomorphism of X. Suppose that the minimal polynomial of g has no irreducible factor that is a cyclotomic polynomial. Then there exists a point $b \in X(\overline{\mathbb{Q}})$ such that, for any $x \in X(\overline{\mathbb{Q}})$, the following are equivalent:*

(1) $\alpha_f(x) = 1$;

(2) $\# \, O_f(x) < \infty$;

(3) $x \in b + X(\overline{\mathbb{Q}})_{\mathrm{tors}}$.

Here $X(\overline{\mathbb{Q}})_{\mathrm{tors}}$ *is the set of torsion points.*

Remark 5.1.4. It is easy to see that when f is a surjective group homomorphism, we can take $b = 0$.

Remark 5.1.5. If the minimal polynomial of g has irreducible factor that is a cyclotomic polynomial, then one of f_i in Remark 5.1.2 (applied to $f = g$) has dynamical degree 1.

To prove the above theorems, we calculate the dynamical degrees of self-morphisms of semi-abelian varieties.

Theorem 5.1.6. *Let* X *be a semi-abelian variety over an algebraically closed field of characteristic zero.*

(1) *Let* $f\colon X \longrightarrow X$ *be a surjective group homomorphism. Let*

$$0 \longrightarrow T \longrightarrow X \xrightarrow{\pi} A \longrightarrow 0$$

be an exact sequence where T *is a torus and* A *is an abelian variety. Then* f *induces surjective group homomorphisms*

$$\begin{aligned} f_T := f|_T \colon T \longrightarrow T \\ g\colon A \longrightarrow A \end{aligned}$$

with $g \circ \pi = \pi \circ f$. *Then we have*

$$\delta_f = \max\{\delta_g, \delta_{f_T}\}.$$

Moreover, let P_T *and* P_A *be the monic minimal polynomials of* f_T *and* g *as elements of* $\mathrm{End}(T)_{\mathbb{Q}}$ *and* $\mathrm{End}(A)_{\mathbb{Q}}$ *respectively. Then,* $\delta_{f_T} = \rho(P_T)$ *and* $\delta_g = \rho(P_A)^2$.

(2) *Let* $f\colon X \longrightarrow X$ *be a surjective group homomorphism and* $a \in X$ *a closed point. Then* $\delta_{T_a \circ f} = \delta_f$.

Remark 5.1.7. The description of δ_{f_T} in Theorem 5.1.6(1) is well-known (see for example [50]).

The outline of this chapter is as follows. In §5.2, we fix notation, and summarize basic properties of arithmetic degrees for later use. In §5.3, we prove a lemma that says every group homomorphism of a semi-abelian variety "splits into rather simple ones". In §5.4, we prove our main theorems for isogenies of abelian varieties. We use these to prove the main theorems. In §5.5.1, we calculate the first dynamical degrees of self-morphisms of semi-abelian varieties and prove Theorem 5.1.6. In §5.5.2, we prove Theorem 5.1.1 and 5.1.3.

5.2 Notation and Preliminaries

5.2.1 Notation

In this chapter, the ground field is either $\overline{\mathbb{Q}}$ or an arbitrary algebraically closed field of characteristic zero. A variety is a separated irreducible reduced scheme of finite type over an algebraically closed field k. Let X be a variety over k and $f\colon X \dashrightarrow X$ a rational map. We use the following notation:

$\mathrm{CH}^1(X)$	The group of codimension one cycles on X modulo rational equivalence is denoted by $\mathrm{CH}^1(X)$.
$\rho(T)$	For an endomorphism $T\colon V \longrightarrow V$ of a finite dimensional real vector space V, the maximum among the absolute values of the eigenvalues of T is called the spectral radius of T and denoted by $\rho(T)$.
$\rho(F)$	For a polynomial $F \in \mathbb{C}[t]$, the maximum among the absolute values of the roots of F is denoted by $\rho(F)$ and called the spectral radius of F.
T_a	Let X be a commutative algebraic group and $a \in X(k)$ a point. The translation by a is denoted by T_a.

5.2.2 Arithmetic degrees

In this subsection, the ground field is $\overline{\mathbb{Q}}$. We give the definition of arithmetic degrees when the variety is not necessarily projective.

Definition 5.2.1. Let $f\colon X \dashrightarrow X$ be a rational self-map of a smooth quasi-projective variety.

(1) A point $x \in X_f(\overline{\mathbb{Q}})$ is called f-preperiodic if the orbit $O_f(x) = \{f^n(x) \mid n \geq 0\}$ is a finite set.

(2) Fix a smooth projective variety $\overline{X}$ and an open embedding $X \subset \overline{X}$. Let H be an ample divisor on $\overline{X}$ and take a Weil height function h_H associated with H. The arithmetic degree $\alpha_f(x)$ of f at $x \in X_f(\overline{\mathbb{Q}})$ is defined by

$$\alpha_f(x) = \lim_{n\to\infty} \max\{1, h_H(f^n(x))\}^{1/n}$$

if the limit exists. Since the convergence of this limit is not proved in general, we introduce the following:

$$\overline{\alpha}_f(x) = \limsup_{n\to\infty} \max\{1, h_H(f^n(x))\}^{1/n},$$
$$\underline{\alpha}_f(x) = \liminf_{n\to\infty} \max\{1, h_H(f^n(x))\}^{1/n}.$$

We call $\overline{\alpha}_f(x)$ the upper arithmetic degree and $\underline{\alpha}_f(x)$ the lower arithmetic degree. The definitions of the (upper, lower) arithmetic degrees do not depend on the choice of $\overline{X}, H$ and h_H ([29, Proposition 12] [40, Theorem 3.4]).

(3) Suppose that $\alpha_f(x)$ exists for every $x \in X_f(\overline{\mathbb{Q}})$. Then we write $A(f) = \{\alpha_f(x) \mid x \in X_f(\overline{\mathbb{Q}})\}$.

Remark 5.2.2. By definition, $1 \le \underline{\alpha}_f(x) \le \overline{\alpha}_f(x)$. When x is f-preperiodic, $\alpha_f(x) = 1$.

Lemma 5.2.3. *Let X, Y be smooth quasi-projective varieties and $f\colon X \dashrightarrow X$, $g\colon Y \dashrightarrow Y$ rational maps. Let $x \in X_f(\overline{\mathbb{Q}})$ and $y \in Y_g(\overline{\mathbb{Q}})$. If $\alpha_f(x)$ and $\alpha_g(y)$ exist, then $\alpha_{f\times g}(x, y)$ also exists and*

$$\alpha_{f\times g}(x, y) = \max\{\alpha_f(x), \alpha_g(y)\}.$$

Proof. It is enough to prove when X, Y are projective. Take ample divisors H_X, H_Y on X, Y respectively. Fix associated height functions h_{H_X}, h_{H_Y} so that $h_{H_X} \ge 1$ and $h_{H_Y} \ge 1$. Then $h := h_{H_X} \circ \mathrm{pr}_1 + h_{H_Y} \circ \mathrm{pr}_2$ is an ample height function on $X \times Y$. Then

$$\lim_{n\to\infty} h((f \times g)^n(x, y))^{1/n} = \lim_{n\to\infty} (h_{H_X}(f^n(x)) + h_{H_Y}(g^n(y)))^{1/n}$$
$$= \max\{\lim_{n\to\infty} h_{H_X}(f^n(x))^{1/n}, \lim_{n\to\infty} h_{H_Y}(g^n(y))^{1/n}\} = \max\{\alpha_f(x), \alpha_g(y)\}.$$

□

Lemma 5.2.4. *Consider the following commutative diagram*

$$\begin{array}{ccc} Y & \overset{g}{\dashrightarrow} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \underset{f}{\dashrightarrow} & X \end{array}$$

where X, Y are smooth quasi-projective varieties, f, g rational maps and π a surjective morphism. Let $y \in Y_g(\overline{\mathbb{Q}})$ be a point such that $\pi(y) \in X_f(\overline{\mathbb{Q}})$. Then

$$\overline{\alpha}_g(y) \geq \overline{\alpha}_f(x)$$
$$\underline{\alpha}_g(y) \geq \underline{\alpha}_f(x).$$

Proof. We may assume X, Y are projective. Take an ample divisor H_X on X and fix an associated height function h_{H_X} with $h_{H_X} \geq 1$. Take an ample divisor H_Y on Y so that $H_Y - \pi^* H_X$ is ample. Then we can take a height function h_{H_Y} associated with H_Y so that $h_{H_Y} \geq h_{H_X} \circ \pi$. Then

$$\begin{aligned} \overline{\alpha}_f(x) &= \limsup_{n\to\infty} h_{H_X}(f^n(x))^{1/n} = \limsup_{n\to\infty} h_{H_X}(\pi(g^n(y)))^{1/n} \\ &\leq \limsup_{n\to\infty} h_{H_Y}(g^n(y))^{1/n} = \overline{\alpha}_g(y). \end{aligned}$$

The second inequality can be proved similarly. □

Lemma 5.2.5. *Consider the following commutative diagram*

$$\begin{array}{ccc} Y & \overset{g}{\dashrightarrow} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \underset{f}{\dashrightarrow} & X \end{array}$$

where X, Y are smooth projective varieties and f, g rational maps. Suppose there exists a non-empty open subset $U \subset X$ such that $\pi \colon V := \pi^{-1}(U) \longrightarrow U$ is finite.

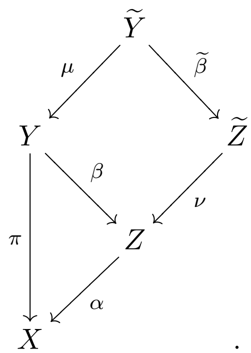
Let $y \in Y(\overline{\mathbb{Q}})$ such that $y \in Y_g(\overline{\mathbb{Q}})$, $x := \pi(y) \in X_f(\overline{\mathbb{Q}})$, $O_g(y) \subset V$ and $O_f(x) \subset U$. If $\alpha_g(y)$ exists, then $\alpha_f(x)$ also exists and $\alpha_g(y) = \alpha_f(x)$.

Proof. Take an ample divisor H on X and let h_H be a height function associated with H. We choose h_H so that $h_H \geq 1$. Then we have

$$h_H(f^n(x)) = h_H(\pi(g^n(y))) = h_{\pi^* H}(g^n(y)). \tag{5.2.1}$$

Here we choose h_{π^*H} so that $h_{\pi^*H} = h_H \circ \pi$.

Let $Y \xrightarrow{\beta} Z \xrightarrow{\alpha} X$ be the Stein factorization of π. Then $\beta \colon V \longrightarrow \alpha^{-1}(U)$ is an isomorphism. Let $\nu \colon \widetilde{Z} \longrightarrow Z$ be a resolution of singularities that is a composite of blow-ups with center in the singular locus of Z. In particular, the blow-up centers do not intersect with $\alpha^{-1}(U)$. Let $\mu \colon \widetilde{Y} \longrightarrow Y$ be a resolution of indeterminacy of $Y \dashrightarrow \widetilde{Z}$ that is a composite of blow-ups along smooth centers outside V. The situation is summarized in the following diagram:



Then, since ν is a sequence of blow-ups and α^*H is ample (because α is finite), there exists an effective ν-exceptional $\mathbb{Q}$-divisor E_ν on $\widetilde{Z}$ such that

$$\nu^*\alpha^*H - E_\nu$$

is ample (cf. [18, II Proposition 7.10 (b)]). Also, since $\widetilde{Z}$ is smooth, there exists an effective $\mathbb{Q}$-divisor $E_{\widetilde{\beta}}$ on $\widetilde{Y}$ that is $\widetilde{\beta}$-exceptional such that

$$\widetilde{\beta}^*(\nu^*\alpha^*H - E_\nu) - E_{\widetilde{\beta}}$$

is ample. (To see this, use [31, Lemma 2.62] or write $\widetilde{\beta}$ as a blow up along an ideal and apply [18, II Proposition 7.10 (b)].) Set $A = \widetilde{\beta}^*(\nu^*\alpha^*H - E_\nu) - E_{\widetilde{\beta}}$ and $E = \widetilde{\beta}^*E_\nu + E_{\widetilde{\beta}}$. Then A is ample, E is effective, $\operatorname{Supp} E \cap \mu^{-1}(V) = \emptyset$ and $\mu^*\pi^*H = A + E$.

Let $\widetilde{y} = \mu^{-1}(y)$. Let $\widetilde{g} = \mu^{-1} \circ g \circ \mu \colon \widetilde{Y} \dashrightarrow \widetilde{Y}$ be the rational map induced by g. (Since $g(y) \in V$ and μ is isomorphic over V, $\mu^{-1} \circ g \circ \mu$ is well-defined.) Then by [40, Theorem 3.4(i)], $\alpha_{\widetilde{g}}(\widetilde{y})$ exists and equals $\alpha_g(y)$ since $\alpha_g(y)$ exists. (Note that [40, Theorem 3.4(i)] works for possibly non-dominant rational maps f, g.) Choose height function h_{A+E} so that $h_{A+E} = h_{\pi^*H} \circ \mu$. Then

$$h_{\pi^*H}(g^n(y)) = h_{A+E}(\widetilde{g}^n(\widetilde{y})). \tag{5.2.2}$$

Since the $\widetilde{g}$-orbit of $\widetilde{y}$ does not intersect with $\operatorname{Supp} E$, we have

$$h_{A+E}(\widetilde{g}^n(\widetilde{y})) \geq h_A(\widetilde{g}^n(\widetilde{y})) + O(1) \tag{5.2.3}$$

where h_A is a height associated with A. Here, we use the fact that any height function associated with an effective divisor is bounded below outside the support of the divisor (cf. [19, Theorem B.3.2.(e)]). If $C > 0$ is a positive number with $CA - (A + E)$ is ample, we have

$$h_{A+E}(\widetilde{g}^n(\widetilde{y})) \leq Ch_A(\widetilde{g}^n(\widetilde{y})) + O(1). \tag{5.2.4}$$

By (5.2.1), (5.2.2), (5.2.3) and (5.2.4), we have

$$h_A(\widetilde{g}^n(\widetilde{y})) + O(1) \leq h_H(f^n(x)) \leq Ch_A(\widetilde{g}^n(\widetilde{y})) + O(1).$$

Since $\alpha_{\widetilde{g}}(\widetilde{y}) = \lim_{n\to\infty} \max\{1, h_A(\widetilde{g}^n(\widetilde{y}))\}^{1/n}$ exists and is equal to $\alpha_g(y)$, we get $\lim_{n\to\infty} h_H(f^n(x))^{1/n} = \alpha_g(y)$. □

5.2.3 Kawaguchi-Silverman conjecture

We restate Kawaguchi-Silverman conjecture (Conjecture 1.3.2) when the variety is not necessarily projective.

Conjecture 5.2.6. *Let X be a smooth quasi-projective variety and $f\colon X \dashrightarrow X$ a dominant rational map, both defined over $\overline{\mathbb{Q}}$. Let $x \in X_f(\overline{\mathbb{Q}})$.*

(1) *The limit defining $\alpha_f(x)$ exists.*

(2) *The arithmetic degree $\alpha_f(x)$ is an algebraic integer.*

(3) *If the orbit $O_f(x) = \{f^n(x) \mid n = 0, 1, 2, \dots\}$ is Zariski dense in X, then $\alpha_f(x) = \delta_f$.*

The following results are used later.

Theorem 5.2.7. *[28, Theorem 4], [50, Theorem 4, Corollary 32], [51, Theorem 2]*

(1) *For any self-morphisms of abelian varieties, Conjecture 5.2.6 is true.*

(2) *Let X be an algebraic torus and $f\colon X \longrightarrow X$ be a group homomorphism. Then Conjecture 5.2.6 is true for f. Moreover, let $F(t)$ be the minimal monic polynomial of f as an element of $\operatorname{End}(X)_{\mathbb{Q}}$ and $F(t) = t^{e_0}F_1(t)^{e_1}\cdots F_r(t)^{e_r}$ the irreducible decomposition. Then $A(f) = \{1, \rho(F_1), \dots, \rho(F_r)\}$.*

5.3 Splitting lemma

In this section, the ground field is an algebraically closed field of characteristic zero. Let X be a a semi-abelian variety, i.e. a commutative algebraic group that is an extension of an abelian variety by an algebraic torus. Note that X is divisible i.e. the morphism $X \longrightarrow X; x \mapsto nx$ is surjective for every $n > 0$.

Lemma 5.3.1. *Let $f\colon X \longrightarrow X$ be a group homomorphism. Let $F(t) \in \mathbb{Z}[t]$ be a polynomial such that $F(f) = 0$ in $\mathrm{End}(X)$. Suppose $F(t) = F_1(t)F_2(t)$ where $F_1, F_1 \in \mathbb{Z}[t]$ are coprime in $\mathbb{Q}[t]$. Set $X_1 = F_2(f)(X)$ and $X_2 = F_1(f)(X)$. Then $X = X_1 + X_2$ and $X_1 \cap X_2$ is finite. In other words, the morphism $X_1 \times X_2 \longrightarrow X; (x_1, x_2) \mapsto x_1 + x_2$ is an isogeny.*

Proof. The proof of [51, Lemma 3.1] works for semi-abelian varieties. □

In the situation of Lemma 5.3.1, write $f_i = f|_{X_i}$. Then $F_i(f_i) = 0$ and we have the following commutative diagram:

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & X_1 \times X_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & X. \end{array}$$

Here π is the isogeny defined by $\pi(x_1, x_2) = x_1 + x_2$.

Since X is divisible, we have $\mathrm{End}(X) \subset \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $f \in \mathrm{End}(X)$ and $F(t) \in \mathbb{Z}[t]$ be the monic minimal polynomial of f as an element of $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. (The monic minimal polynomial has integer coefficients because those of endomorphisms of a torus and an abelian variety have integer coefficients.) Let

$$F(t) = F_0(t)^{e_0} F_1(t)^{e_1} \cdots F_r(t)^{e_r}$$

be the decomposition into irreducible factors where $F_0(t) = t$, $e_0 \geq 0$, $e_i > 0, i = 1, \ldots, r$ and $F_i(t)$ are distinct monic irreducible polynomials. Note that r is possibly zero. Set

$$X_i = F_0(f)^{e_0} \cdots F_{i-1}(f)^{e_{i-1}} F_{i+1}(f)^{e_{i+1}} \cdots F_r(f)^{e_r}(X)$$

and $f_i = f|_{X_i}$. Here, X_i are also (semi-)abelian varieties since they are images of a (semi-)abelian variety. Then we get the commutative diagram

$$\begin{array}{ccc} X_0 \times \cdots \times X_r & \xrightarrow{f_0 \times \cdots \times f_r} & X_0 \times \cdots \times X_r \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow[f]{} & X \end{array}$$

where $\pi(x_0, \ldots, x_r) = x_0 + \cdots + x_r$. Note that the monic minimal polynomial of f_i as an element of $\mathrm{End}(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ is $F_i(t)^{e_i}$. Note that f is surjective if and only if $e_0 = 0$ and if this is the case, we have $\delta_f = \delta_{f_0 \times \cdots \times f_r} = \max\{\delta_{f_1}, \ldots, \delta_{f_r}\}$ (cf. Remark 1.1.2 (5)).

5.4 Arithmetic and dynamical degrees of isogenies of abelian varieties

Theorem 5.4.1 (Theorem 5.1.1(3) for abelian varieties)**.** *Let X be an abelian variety and $f \colon X \longrightarrow X$ be a group homomorphism, both defined over $\overline{\mathbb{Q}}$. Let $F(t)$ be the monic minimal polynomial of f as an element of* $\mathrm{End}(X)_{\mathbb{Q}}$ *and*

$$F(t) = t^{e_0} F_1(t)^{e_1} \cdots F_r(t)^{e_r}$$

the irreducible decomposition in $\mathbb{Q}[t]$ where $e_0 \geq 0$ and $e_i > 0$ for $i = 1, \ldots, r$. Then we have

$$A(f) = \{1, \rho(F_1)^2, \ldots, \rho(F_r)^2\}$$

Theorem 5.4.2 (Theorem 5.1.3 for isogenies of abelian varieties)**.** *Let X be an abelian variety and $f \colon X \longrightarrow X$ a surjective group homomorphism, both defined over $\overline{\mathbb{Q}}$. Suppose that the minimal polynomial of f has no irreducible factor that is a cyclotomic polynomial. Then for any $x \in X(\overline{\mathbb{Q}})$,*

$$\alpha_f(x) = 1 \iff \# O_f(x) < \infty \iff x \in X(\overline{\mathbb{Q}})_{\mathrm{tors}}$$

where $X(\overline{\mathbb{Q}})_{\mathrm{tors}}$ is the set of torsion points.

Lemma 5.4.3. *Let X be an abelian variety of dimension g over an algebraically closed field of characteristic zero and $f \colon X \longrightarrow X$ a surjective group homomorphism. Let $P(t)$ be the monic minimal polynomial of f as an element of* $\mathrm{End}(X)_{\mathbb{Q}}$*, which has integer coefficient, and ρ the maximum among the absolute values of the roots of $P(t)$. Then we have $\delta_f = \rho^2$.*

Remark 5.4.4. The minimal polynomial of f as an element of $\mathrm{End}(X)_{\mathbb{Q}}$ is equal to the minimal polynomial of $T_l(f)$ for every prime number l. If the ground field is $\mathbb{C}$, these are also equal to the minimal polynomial of the analytic representation of f.

Proof. By the Lefschetz principle, we may assume that the ground field is $\mathbb{C}$. Let $X = \mathbb{C}^g/\Lambda$, where Λ is a lattice in $\mathbb{C}^g$. Let $f_r \colon \Lambda \longrightarrow \Lambda$ be the rational representation and $f_a \colon \mathbb{C}^g \longrightarrow \mathbb{C}^g$ the analytic representation of f.

We have a natural isomorphism $H^r(X;\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\bigwedge^r \Lambda, \mathbb{Z})$ (cf. [44, §1 (4)]). If we identify $H^r(X;\mathbb{Z})$ with $\mathrm{Hom}_{\mathbb{Z}}(\bigwedge^r \Lambda, \mathbb{Z})$ by this isomorphism, then $f^* \colon H^r(X;\mathbb{Z}) \longrightarrow H^r(X;\mathbb{Z})$ is $(\bigwedge^r f_r)^*$. Therefore, the eigenvalues of f^* are products of r eigenvalues of f_r. Since $f_a|_\Lambda = f_r$, the characteristic polynomial of f_r as an $\mathbb{R}$-linear map is $Q(t)\overline{Q(t)}$ where $Q(t)$ is the characteristic polynomial of f_a as a $\mathbb{C}$-linear map. (Take a basis $e_1, \ldots, e_g$ of $\mathbb{C}^g$ so that f_a is represented by an upper triangular matrix. Then compute the characteristic polynomial of f_a, f_r using bases $\{e_1, \ldots, e_g\}, \{e_1, ie_1, \ldots, e_g, ie_g\}$ respectively.) Note that the set of roots of $P(t)$ and $Q(t)$ are the same. Therefore, the spectral radius of $f^* \colon H^2(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow H^2(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is equal to the square of spectral radius of f_r. Note that the spectral radius of $f^* \curvearrowright H^2(X;\mathbb{Z})$ is equal to the spectral radius of $f^* \curvearrowright H^{1,1}(X)$ (cf. the inequality above Proposition 4.4 in [13]), this proves the theorem. $\square$

Now, let X be an abelian variety and $f \colon X \longrightarrow X$ a group homomorphism, both defined over $\overline{\mathbb{Q}}$. Let $F(t)$ be the monic minimal polynomial of f and

$$F(t) = t^{e_0} F_1(t)^{e_1} \cdots F_r(t)^{e_r}$$

the decomposition into irreducible factors in $\mathbb{Q}[t]$. Here F_i are distinct monic irreducible polynomial in $\mathbb{Z}[t]$ with $F_i(0) \neq 0$. Write $F_0(t) = t$. Set

$$X_i = F_0(f)^{e_0} \cdots F_{i-1}(f)^{e_{i-1}} F_{i+1}(f)^{e_{i+1}} \cdots F_r(f)^{e_r}(X).$$

Then by §3, we have the following commutative diagram:

$$\begin{array}{ccc} X_0 \times \cdots \times X_r & \xrightarrow{f_0 \times \cdots \times f_r} & X_0 \times \cdots \times X_r \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow[f]{} & X. \end{array}$$

Here, the vertical arrows are isogenies. Note that the minimal polynomial of f_i is $F_i(t)^{e_i}$.

Lemma 5.4.5. *Let $f\colon X \longrightarrow X$ be a surjective group homomorphism over $\overline{\mathbb{Q}}$ such that the minimal polynomial of f is the form of $F(t)^e$ where F is an irreducible monic polynomial in $\mathbb{Z}[t]$. For any $x \in X(\overline{\mathbb{Q}})$, if $\alpha_f(x) < \delta_f$, then x is a torsion point. In particular, x is an f-preperiodic point and $\alpha_f(x) = 1$.*

Remark 5.4.6. Note that $\alpha_f(x) < \delta_f$ happens only if $\delta_f > 1$. In the above situation, $\delta_f = 1$ if and only if $F(t)$ is a cyclotomic polynomial. This follows from Lemma 5.4.3 and the fact that if the absolute value of every root of an irreducible monic polynomial with integer coefficients is less than or equal to one, then the polynomial is cyclotomic.

Proof. We prove the claim by induction on $\dim X$. If $\dim X = 0$, there is nothing to prove. Suppose $\dim X = d > 0$ and the claim holds for dimension $< d$. Take a nef $\mathbb{R}$-divisor D such that $f^*D \equiv \delta_f D$. Let q be the quadratic part of the canonical height of D, i.e. $q(x) = \lim_{n\to\infty} h_D(nx)/n^2$. By [28, Theorem 29, Lemma 31], there exists an f-invariant subabelian variety $Y \subsetneq X$ such that

$$\{x \in X(\overline{\mathbb{Q}}) \mid q(x) = 0\} = Y(\overline{\mathbb{Q}}) + X(\overline{\mathbb{Q}})_{\mathrm{tors}}.$$

Assume $\alpha_f(x) < \delta_f$. Then $x = y + z$ for some $y \in Y(\overline{\mathbb{Q}})$ and some torsion point z. It is enough to show that y is a torsion point. If Y is a point, we are done. Suppose $\dim Y > 0$. Since Y is f-invariant, the minimal polynomial of $f|_Y$ divides $F(t)^e$ and is not equal to 1. Thus $\delta_{f|_Y} = \delta_f > \alpha_f(x) = \alpha_f(y) = \alpha_{f|_Y}(y)$. Here, we use the fact that $\alpha_f(x) = \alpha_f(y+z) = \alpha_f(y)$. This follows from the definition of arithmetic degree and the fact that the Neron-Tate height associated with a symmetric ample divisor is invariant under the translation by a torsion point. By the induction hypothesis, y is a torsion point. □

Proof of Theorem 5.4.1. We use the notation of §3. Set $f_i = f|_{X_i}$. By [51, Lemma 6], $A(f) = A(f_0\times\cdots\times f_r)$. Since $\alpha_{f_i}(0) = 1$ and $\alpha_{f_0\times\cdots\times f_r}(x_0,\ldots,x_r) = \max\{\alpha_{f_0}(x_0),\ldots,\alpha_{f_r}(x_r)\}$ (see Lemma 5.2.3), we have $A(f_0 \times \cdots \times f_r) = A(f_0) \cup \cdots \cup A(f_r)$. Note that $A(f_0) = \{1\}$ since $f_0^{e_0} = 0$. By Lemma 5.4.5 and the fact that there always exists a point whose arithmetic degree equals the dynamical degree (cf. [28, Corollary 32] or [40, Theorem 1.6]), we have $A(f_i) = \{1, \delta_{f_i}\}$ for $i = 1,\ldots,r$. Thus $A(f) = \{1, \delta_{f_1},\ldots,\delta_{f_r}\}$. By Lemma 5.4.3, δ_{f_i} is equal to $\rho(F_i)^2$. □

Proof of Theorem 5.4.2. By §3, we may assume the minimal polynomial of f is the form of $F(t)^e$ where F is an irreducible polynomial that is not cyclotomic. Then $\rho(F)$ is greater than one. Thus $\delta_f > 1$. By Lemma 5.4.5, if $\alpha_f(x) = 1$ then x is a torsion point. □

5.5 Arithmetic and dynamical degrees of self-morphisms of semi-abelian varieties

5.5.1 Dynamical degrees

In this subsection, the ground field is an algebraically closed field of characteristic zero.

Proposition 5.5.1. *Let X be a semi-abelian variety. Let $f\colon X \longrightarrow X$ be a surjective group homomorphism. Let*

$$0 \longrightarrow T \longrightarrow X \xrightarrow{\pi} A \longrightarrow 0$$

be an exact sequence where T is a torus and A is an abelian variety. Then f induces surjective group homomorphisms

$$f_T := f|_T\colon T \longrightarrow T$$
$$g\colon A \longrightarrow A$$

with $g \circ \pi = \pi \circ f$. Then we have

$$\delta_f = \max\{\delta_g, \delta_{f_T}\}$$

Remark 5.5.2. This follows from the product formula of dynamical degrees ([10, Theorem 1.1]) and [10, Remark 3.4]. To apply [10, Remark 3.4], just take the standard compactification of X as in [54, §2 (2.3)]. The proofs of [10, Theorem 1.1] and [10, Remark 3.4] are based on analytic methods, so we give an algebraic proof of Proposition 5.5.1 below.

Lemma 5.5.3. *Let $X \overset{f}{\dashrightarrow} Y \overset{g}{\dashrightarrow} Z$ be rational maps of smooth projective varieties. Suppose $f(X \setminus I_f) \not\subset I_g$ where I_f, I_g are the indeterminacy loci of f, g. Then for any free divisor H on Z, we have*

$$(g \circ f)^* H \leq f^*(g^* H).$$

Here, for divisor classes A and B, $A \leq B$ means $B - A$ is represented by an effective divisor.

Proof. Take resolutions π_X, π_Y as follows:

$$\begin{array}{ccccc} \widetilde{X} & & & & \\ & \searrow^{\widetilde{f}} & & & \\ \downarrow{\scriptstyle \pi_X} & & \widetilde{Y} & & \\ & & \downarrow{\scriptstyle \pi_Y} & \searrow^{\widetilde{g}} & \\ X & \underset{f}{\dashrightarrow} & Y & \underset{g}{\dashrightarrow} & Z \end{array}$$

where $\widetilde{X}, \widetilde{Y}$ are smooth projective varieties and $\pi_Y \colon \pi_Y^{-1}(Y \setminus I_g) \simeq Y \setminus I_g$. Then

$$\begin{aligned} (g \circ f)^* H &= \pi_{X*} \widetilde{f}^* \widetilde{g}^* H \\ f^*(g^* H) &= \pi_{X*} \widetilde{f}^* \pi_Y^* \pi_{Y*} \widetilde{g}^* H. \end{aligned}$$

Since $\widetilde{g}^* H$ is free, the divisor $\pi_Y^* \pi_{Y*} \widetilde{g}^* H - \widetilde{g}^* H$ is represented by an effective divisor with support contained in the exceptional locus $\mathrm{Exc}(\pi_Y)$ of π_Y. Since $\widetilde{f}(\widetilde{X}) \not\subset \mathrm{Exc}(\pi_Y)$, we have $\pi_{X*} \widetilde{f}^* (\pi_Y^* \pi_{Y*} \widetilde{g}^* H - \widetilde{g}^* H) \geq 0$. $\square$

Proof of Proposition 5.5.1. We will write the multiplication of the groups X, A, T by addition. Take a non-empty open subset $U \subset A$ and a section $s \colon U \longrightarrow \pi^{-1}(U)$ of π. (There exists such a section because of the structure theorem of semi-abelian varieties [54, Lemma 2.2].) Then

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad \sim \quad} & U \times T \\ \cup & & \cup \\ P & \longmapsto & (\pi(P), P - s(\pi(P))). \end{array}$$

By this isomorphism, f is conjugate to the rational map

$$\begin{array}{ccc} A \times T & \overset{\widetilde{f}}{\dashrightarrow} & A \times T \\ \cup & & \cup \\ (x, y) & \longmapsto & (g(x), f_T(y) + h(x)) \end{array}$$

where $h(x) = f(s(x)) - s(g(x))$. Note that h is defined on $V := U \cap g^{-1}(U)$ and $h(V) \subset T$. Fix a compactification $T \subset \overline{T} = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. The rational map $A \times \overline{T} \dashrightarrow A \times \overline{T}$ defined by $\widetilde{f}$ is also denoted by $\widetilde{f}$.

Claim 5.5.4. *Let $m \colon \overline{T} \times \overline{T} \dashrightarrow \overline{T}$ be the rational map defined by the multiplication morphism $T \times T \longrightarrow T$. Then, for any divisor D on $\overline{T}$, $m^* D \sim \mathrm{pr}_1^* D + \mathrm{pr}_2^* D$.*

Proof. We can write $m^* D \sim \operatorname{pr}_1^* D_1 + \operatorname{pr}_2^* D_2$ where D_1, D_2 are divisors on $\overline{T}$. Let $1 \in T$ be the neutral element. Let $i \colon \overline{T} \longrightarrow \overline{T} \times \overline{T}$ be the map defined by $i(t) = (t, 1)$. Then, $i(\overline{T}) \cap I_m = \emptyset$. Therefore, we have $D_1 \sim i^* m^* D = (m \circ i)^* D = D$. In the same way, we can show that $D_2 \sim D$. ■

Since $\overline{T}$ is a product of $\mathbb{P}^1$, we have $\mathrm{CH}^1(A \times \overline{T}) = \operatorname{pr}_1^* \mathrm{CH}^1(A) \oplus \operatorname{pr}_2^* \mathrm{CH}^1(\overline{T}) \simeq \mathrm{CH}^1(A) \oplus \mathrm{CH}^1(\overline{T})$.

Claim 5.5.5. *We have*

$$\widetilde{f}^* = \begin{pmatrix} g^* & h^* \\ 0 & f_T^* \end{pmatrix} \colon \mathrm{CH}^1(A) \oplus \mathrm{CH}^1(\overline{T}) \longrightarrow \mathrm{CH}^1(A) \oplus \mathrm{CH}^1(\overline{T}).$$

Proof. It is enough to prove the following two statements:

(1) $\widetilde{f}^* \operatorname{pr}_1^* H_A = \operatorname{pr}_1^* g^* H_A$ for every ample divisor H_A on A.

(2) $\widetilde{f}^* \operatorname{pr}_1^* H_{\overline{T}} = \operatorname{pr}_1^* h^* H_{\overline{T}} + \operatorname{pr}_2^* f_T^* H_{\overline{T}}$ in $\mathrm{CH}^1(A \times \overline{T})$ for every very ample divisor $H_{\overline{T}}$ on $\overline{T}$.

(1) Since pr_1 is a morphism, we have

$$\widetilde{f}^* \operatorname{pr}_1^* H_A = (\operatorname{pr}_1 \circ \widetilde{f})^* H_A = (g \circ \operatorname{pr}_1)^* H_A = \operatorname{pr}_1^* g^* H_A.$$

(2) First, in $\mathrm{CH}^1(A \times \overline{T})$, we have

$$\begin{aligned} \widetilde{f}^* \operatorname{pr}_2^* H_{\overline{T}} &= (\operatorname{pr}_2 \circ \widetilde{f})^* H_{\overline{T}} \\ &= (m \circ (h \times f_T))^* H_{\overline{T}} \\ &\leq (h \times f_T)^* m^* H_{\overline{T}} && \text{by Lemma 5.5.3} \\ &= (h \times f_T)^* (\operatorname{pr}_1^* H_{\overline{T}} + \operatorname{pr}_2^* H_{\overline{T}}) && \text{by Claim 5.5.4} \\ &= \operatorname{pr}_1^* h^* H_{\overline{T}} + \operatorname{pr}_2^* f_T^* H_{\overline{T}}. \end{aligned}$$

Now take an effective divisor E on $A \times \overline{T}$ that represents the class $(h \times f_T)^* m^* H_{\overline{T}} - (m \circ (h \times f_T))^* H_{\overline{T}}$. For a general closed point $a \in A$, $\operatorname{Supp} E$ does not contain $\{a\} \times \overline{T}$. Let $i_a \colon \overline{T} = \{a\} \times \overline{T} \subset A \times \overline{T}$ be the inclusion. Since $i_a^* E$ is effective, we have

$$i_a^* (\widetilde{f}^* \operatorname{pr}_2^* H_{\overline{T}}) \leq i_a^* (\operatorname{pr}_1^* h^* H_{\overline{T}} + \operatorname{pr}_2^* f_T^* H_{\overline{T}}) = f_T^* H_{\overline{T}}.$$

Similarly, if $b \in \overline{T}$ is a general closed point and $j_b \colon A = A \times \{b\} \subset A \times \overline{T}$ is the inclusion, we have

$$j_b^* (\widetilde{f}^* \operatorname{pr}_2^* H_{\overline{T}}) \leq j_b^* (\operatorname{pr}_1^* h^* H_{\overline{T}} + \operatorname{pr}_2^* f_T^* H_{\overline{T}}) = h^* H_{\overline{T}}.$$

Therefore, if we write $\widetilde{f}^* \operatorname{pr}_2^* H_{\overline{T}} = \operatorname{pr}_1^* D_1 + \operatorname{pr}_2^* D_2$ where D_1 and D_2 are divisor classes on A and $\overline{T}$ respectively, we have proved

$$D_1 \leq h^* H_{\overline{T}},\ D_2 \leq f_T^* H_{\overline{T}}.$$

On the other hand, for a general $a \in V \subset A$, $\{a\} \times \overline{T}$ is not contained in the indeterminacy locus of $\widetilde{f}$, and $\operatorname{pr}_2 \circ \widetilde{f} \circ i_a = T_{h(a)} \circ f_T$. Here, the translation $T_{h(a)}$ defines an automorphism on $\overline{T}$ and induces identity on $\mathrm{CH}^1(\overline{T})$. Thus

$$\begin{aligned}
f_T^* H_{\overline{T}} &= f_T^* T_{h(a)}^* H_{\overline{T}} = (T_{h(a)} \circ f_T)^* H_{\overline{T}} \\
&= (\operatorname{pr}_2 \circ \widetilde{f} \circ i_a)^* H_{\overline{T}} = (\widetilde{f} \circ i_a)^* \operatorname{pr}_2^* H_{\overline{T}} \\
&\leq i_a^* \widetilde{f}^* \operatorname{pr}_2^* H_{\overline{T}} && \text{by Lemma 5.5.3} \\
&= i_a^* (\operatorname{pr}_1^* D_1 + \operatorname{pr}_2^* D_2) = D_2.
\end{aligned}$$

Hence we get $D_2 \leq f_T^* H_{\overline{T}} \leq D_2$ and therefore $D_2 = f_T^* H_{\overline{T}}$.

Similarly, since $A \times \{1\}$ is not contained in the indeterminacy locus of $\widetilde{f}$, we have

$$\begin{aligned}
h^* H_{\overline{T}} &= (\operatorname{pr}_2 \circ \widetilde{f} \circ j_1)^* H_{\overline{T}} = (\widetilde{f} \circ j_1)^* \operatorname{pr}_2^* H_{\overline{T}} \\
&\leq j_1^* \widetilde{f}^* \operatorname{pr}_2^* H_{\overline{T}} && \text{by Lemma 5.5.3} \\
&= j_1^* (\operatorname{pr}_1^* D_1 + \operatorname{pr}_2^* D_2) = D_1.
\end{aligned}$$

Thus $D_1 = h^* H_{\overline{T}}$.

■

Note that $\widetilde{(f^n)} = \widetilde{f}^n$. Set $h_n = f^n \circ s - s \circ g^n$. Then we have

$$(\widetilde{f}^n)^* = \begin{pmatrix} (g^n)^* & h_n^* \\ 0 & (f_T^n)^* \end{pmatrix}.$$

Note also that $N^1(A \times \overline{T}) = (\mathrm{CH}^1(A)/\equiv) \oplus \mathrm{CH}^1(\overline{T})$ and the action of $(\widetilde{f}^n)^*$ on $N^1(A \times \overline{T})$ is in the same form. By [29, Theorem 15], $\delta_{\widetilde{f}} = \lim_{n\to\infty} \rho((\widetilde{f}^n)^*|_{N^1(A \times \overline{T})})^{1/n}$. Thus

$$\delta_f = \delta_{\widetilde{f}} = \lim_{n\to\infty} \max\{\rho((g^n)^*|_{N^1(A)}), \rho((f_T^n)^*|_{N^1(\overline{T})})\}^{1/n} = \max\{\delta_g, \delta_{f_T}\}.$$

□

Lemma 5.5.6. *Let $f\colon X \longrightarrow X$ be a surjective group homomorphism of a semi-abelian variety X and $a \in X$ a closed point. Then $\delta_{T_a \circ f} = \delta_f$.*

Proof. Let $\overline{X}$ be the standard compactification of X as in [54, §2 (2.3)]. Then T_a extends to an automorphism of $\overline{X}$, which we also denote by T_a, and the pull-back $T_a^*\colon N^1(\overline{X}) \longrightarrow N^1(\overline{X})$ is the identity. (We can deduce these facts from the description of the group law in terms of the compactification, cf. [54, the proof of Proposition 2.6].) Thus, as an endomorphisms of $N^1(\overline{X})$, we have

$$((T_a \circ f)^n)^* = (T_b \circ f^n)^* = (f^n)^* \circ T_b^* = (f^n)^*$$

where $b = a + f(a) + \cdots + f^{n-1}(a)$. Therefore,

$$\delta_{T_a \circ f} = \lim_{n\to\infty} \|((T_a \circ f)^n)^*\|^{1/n} = \lim_{n\to\infty} \|(f^n)^*\|^{1/n} = \delta_f$$

where $\|\cdot\|$ is a norm on $\mathrm{End}_{\mathbb{R}}(N^1(\overline{X})_{\mathbb{R}})$. □

Proof of Theorem 5.1.6. (2) is Lemma 5.5.6. (1) follows from Proposition 5.5.1, Lemma 5.4.3 and Remark 5.1.7. □

5.5.2 Kawaguchi-Silverman conjecture and arithmetic degrees

In this subsection, the ground field is $\overline{\mathbb{Q}}$.

Lemma 5.5.7. *Let $f\colon X \longrightarrow X$ be a surjective group homomorphism of a semi-abelian variety. Fix an exact sequence*

$$0 \longrightarrow T \longrightarrow X \xrightarrow{\pi} A \longrightarrow 0.$$

The morphisms induced by f is denoted by

$$f_T\colon T \longrightarrow T$$
$$g\colon A \longrightarrow A.$$

Suppose the minimal polynomial of f as an element of $\mathrm{End}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ is the form of $F(t)^e$ where $F(t)$ is a monic irreducible polynomial that is not cyclotomic and $e > 0$. (Note that the minimal polynomial is automatically monic with integer coefficient because it is the case for f_T and g.) Then, for $x \in X(\overline{\mathbb{Q}})$, either

1. *$O_g(\pi(x))$ is infinite and $\alpha_f(x) = \delta_f$ or,*
2. *$\pi(x)$ is a torsion point and $\alpha_f(x) = 1$ or δ_{f_T}.*

Moreover,

$$A(f) = \{1, \delta_{f_T}, \delta_g\} = \begin{cases} \{1, \rho(F)\} & \textit{if } X = T, \\ \{1, \rho(F)^2\} & \textit{if } X = A, \\ \{1, \rho(F), \rho(F)^2\} & \textit{otherwise.} \end{cases}$$

Lemma 5.5.8. *Let X be a semi-abelian variety and $f\colon X \longrightarrow X$ be a surjective group homomorphism. Let $x \in X(\overline{\mathbb{Q}})$ be a point and $n > 0$ a positive integer. If $\alpha_f(nx)$ exists, then $\alpha_f(x)$ also exists and $\alpha_f(nx) = \alpha_f(x)$.*

Proof. Let X'' and X' be smooth projectivization of X such that the multiplication morphism $[n]\colon X \longrightarrow X$ becomes a morphism $\pi\colon X'' \longrightarrow X'$. Let $f''\colon X'' \longrightarrow X''$ and $f'\colon X' \longrightarrow X'$ be the dominant rational maps induced by f. Since f is a group homomorphism, we have $f' \circ \pi = \pi \circ f''$. Moreover, we have $\pi^{-1}(X) = X$ since $[n]\colon X \longrightarrow X$ is finite. By Lemma 5.2.5, we get the assertion. □

Proof of Lemma 5.5.7. First of all, we have

$$\delta_f = \max\{\delta_g, \delta_{f_T}\} = \max\{\rho(F)^2, \rho(F)\} = \rho(F)^2 = \delta_g$$

(see Theorem 5.2.7(2), Proposition 5.5.1, Lemma 5.4.3). By Lemma 5.4.5, we have

$$\alpha_g(\pi(x)) = \begin{cases} 1 & \text{if } \pi(x) \text{ is torsion} \\ \delta_g = \delta_f & \text{otherwise.} \end{cases}$$

Note that, by Lemma 5.2.4, we have $\alpha_g(\pi(x)) \leq \underline{\alpha}_f(x) \leq \delta_f$. Thus, if $\alpha_g(\pi(x)) = \delta_f$, $\alpha_f(x)$ exists and is equal to δ_f.

Now, suppose $\pi(x)$ is a torsion point. Take a positive integer n such that $n\pi(x) = 0$. Then $nx \in T$ and therefore $\alpha_f(nx) = \alpha_{f_T}(nx)$ exists and is equal to 1 or $\rho(F) = \delta_{f_T}$ (Theorem 5.2.7(2)). By Lemma 5.5.8, $\alpha_f(x)$ exists and is equal to 1 or δ_{f_T}.

The claim $A(f) = \{1, \delta_{f_T}, \delta_f\}$ follows from the facts that $A(f_T) = \{1, \delta_{f_T}\}$ (Theorem 5.2.7(2)), $A(g) = \{1, \delta_g\}$ (Lemma 5.4.5) and $\alpha_f(x) \geq \alpha_g(\pi(x))$ (Lemma 5.2.4). □

Lemma 5.5.9. *Let $f\colon X \longrightarrow X$ be a group homomorphism of a semi-abelian variety. Let $F(t)$ be the minimal monic polynomial of f. Assume $F(1) \neq 1$. Let $a \in X(\overline{\mathbb{Q}})$ be any point. Then there exists a point $b \in X(\overline{\mathbb{Q}})$ such that $h := T_b \circ (T_a \circ f) \circ T_{-b}$ is a group homomorphism. For every such b, the minimal polynomial of h is also $F(t)$.*

Proof. Since $F(1) \neq 1$, $f - \mathrm{id}$ is surjective. For any $b \in X(\overline{\mathbb{Q}})$ with $f(b) - b = a$, the morphism $T_b \circ (T_a \circ f) \circ T_{-b}$ is a group homomorphism.

Now we prove the second part. By symmetry, it is enough to prove $F(h) = 0$. We have

$$h^n = T_b \circ (T_a \circ f)^n \circ T_{-b} = T_b \circ T_{a+f(a)+\cdots+f^{n-1}(a)} \circ f^n \circ T_{-b}.$$

Note that since h is a group homomorphism, we have $h(0) = 0$, in other words, $a = (f - \mathrm{id})(b)$. Thus

$$h^n = T_b \circ T_{f^n(b)-b} \circ f^n \circ T_{-b} = T_{f^n(b)} \circ f^n \circ T_{-b}.$$

Therefore, for any $x \in X(\overline{\mathbb{Q}})$

$$F(h)(x) = F(f)(b) + F(f)(x - b) = 0.$$

□

Proof of Theorem 5.1.1. Let X be a semi-abelian variety and first assume $f\colon X \longrightarrow X$ is a group homomorphism. We use the notation of §5.3. Apply Lemma 5.2.5 for a suitable smooth compactification of

$$\begin{array}{ccc} X_0 \times \cdots \times X_r & \xrightarrow{f_0 \times \cdots \times f_r} & X_0 \times \cdots \times X_r \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & X. \end{array}$$

By Lemma 5.5.7, $\alpha_{f_i}(x)$ exists for every i and every point $x \in X_i(\overline{\mathbb{Q}})$. Therefore, by Lemma 5.2.3 and Lemma 5.2.5, $A(f) = A(f_0 \times \cdots \times f_r) = A(f_0) \cup \cdots \cup A(f_r)$. Since f_0 is nilpotent, $A(f_0) = \{1\}$. If F_i is a cyclotomic polynomial, then $\delta_{f_i} = 1$ and $A(f_i) = \{1\}$. Therefore by Lemma 5.5.7, we have

$$\begin{aligned} A(f) &= A(f_1) \cup \cdots \cup A(f_r) \\ &= \{1\} \cup A_1 \cup \cdots \cup A_r. \end{aligned}$$

Now, consider any self-morphism of X. Any self-morphism is the form of $T_a \circ f$ where T_a is the translation by $a \in X(\overline{\mathbb{Q}})$ and f is a group homomorphism (c.f. [6, Lemma 5.4.8]). There exist points $a_i \in X_i(\overline{\mathbb{Q}})$ such that

$\pi(a_0, \ldots, a_r) = a_0 + \cdots + a_r = a$. Then we have the following commutative diagram:

$$\begin{CD}
X_0 \times \cdots \times X_r @>{f_0 \times \cdots \times f_r}>> X_0 \times \cdots \times X_r @>{T_{a_0} \times \cdots \times T_{a_r}}>> X_0 \times \cdots \times X_r \\
@V{\pi}VV @VV{\pi}V @VV{\pi}V \\
X @>>{f}> X @>>{T_a}> X.
\end{CD}$$

As above, we have $A(T_a \circ f) = A\big((T_{a_0} \circ f_0) \times \cdots \times (T_{a_r} \circ f_r)\big)$. Since f_0 is nilpotent, every orbit of $T_{a_0} \circ f_0$ is finite and therefore $A(T_{a_0} \circ f_0) = \{1\} = A(f_0)$. If $F_i(t)$ is a cyclotomic polynomial, by Lemma 5.5.6 we have $\delta_{T_{a_i} \circ f_i} = \delta_{f_i} = 1$ and therefore $A(T_{a_i} \circ f_i) = \{1\} = A(f_i)$. If $F_i(t), i \geq 1$ is not a cyclotomic polynomial, by Lemma 5.5.9, $T_{a_i} \circ f_i$ is conjugate by a translation to a group homomorphism h_i with minimal polynomial $F_i^{e_i}$. In particular, $A(T_{a_i} \circ f_i) = A(h_i) = A(f_i)$. Therefore

$$\begin{aligned}
A\big((T_{a_0} \circ f_0) \times \cdots \times (T_{a_r} \circ f_r)\big) &= A(T_{a_0} \circ f_0) \cup \cdots \cup A(T_{a_r} \circ f_r) \\
&= A(f_0) \cup \cdots \cup A(f_r) = A(f).
\end{aligned}$$

If the $T_a \circ f$-orbit of a point $x \in X(\overline{\mathbb{Q}})$ is Zariski dense, then by Lemma 5.2.3 and Lemma 5.5.7, we have

$$\alpha_f(x) = \max\{\delta_{h_i} = \delta_{f_i} \mid F_i \text{ is not a cyclotomic polynomial}\} = \delta_f.$$

□

Proof of Theorem 5.1.3. Since $F(1) \neq 1$, by Lemma 5.5.9, there exists a point $b \in X(\overline{\mathbb{Q}})$ such that $T_{-b} \circ f \circ T_b$ is a group homomorphism. Thus it is enough to prove the equivalence of (1), (2) and (3) for every group homomorphism f and $b = 0$. (3) $\Rightarrow$ (2). This follows from the fact that the set of n-torsion points of X is finite for each $n > 0$ and that the image of an n-torsion point by a group homomorphism is also an n-torsion point. (2) $\Rightarrow$ (1) is trivial. To prove (1) $\Rightarrow$ (3), let $x \in X(\overline{\mathbb{Q}})$ be a point with $\alpha_f(x) = 1$. By §3, we may assume that the minimal polynomial of f is the form of $F(t)^e$ where F is an irreducible monic polynomial that is not cyclotomic. We use the notation of Lemma 5.5.7. By Theorem 5.4.2 and the inequality $\alpha_f(x) \geq \alpha_g(p(x))$, $p(x)$ is a torsion point. Take $n > 0$ so that $np(x) = 0$. Then $nx \in T$. By Lemma 5.5.8, $\alpha_{f_T}(nx) = \alpha_f(nx) = \alpha_f(x) = 1$. Since the minimal polynomial of f_T divides $F(t)^e$, we can use [50, Proposition 21(d)] and have $nx \in T(\overline{\mathbb{Q}})_{\text{tors}}$. Hence $x \in X(\overline{\mathbb{Q}})_{\text{tors}}$. □

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