博士論文

On the Motion of Inhomogeneous Incompressible Fluids
(非一様非圧縮性流体の運動について)

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Abstract

The purpose of this thesis is to present various results on the well-posedness of strong solutions to some fluid mechanics models such as, for instance, Navier-Stokes equations, magnetohydrodynamics (MHD) equations and Navier-Stokes-Korteweg equations. It is well known that the macroscopic motion of fluids can be described by a system of partial differential equations which are derived from some physical balance laws, such as conservation law of mass, balance law of momentum. Among these fluid mechanics models, the most famous one may be the Navier-Stokes system, which has been studied extensively by many mathematicians and physicists due to its physical importance, mathematical complexity and wide range of applications. In general, the classical fluid mechanics is divided into two types of models corresponding to whether the fluid is homogeneous or not. On the one hand, the classical Navier-Stokes equations in the homogeneous, incompressible case

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P - \mu \triangle u = 0, \\ \mathrm{div} u = 0, \end{cases} \tag{NS}$$

govern the evolution of the velocity field $u(x,t)$ and the pressure function $P(x,t)$ of a homogeneous, incompressible viscous fluid with constant viscosity $\mu > 0$. On the other hand, the evolution of compressible viscous fluids can be written in the following form

$$\begin{cases} \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) - \mu \triangle u - (\mu + \lambda)\nabla \mathrm{div} u = 0, \\ \partial_t \rho + \mathrm{div}(\rho u) = 0, \end{cases} \tag{CNS}$$

where the non-negative function $\rho(x,t)$ stands for the density of the fluid and $P(\rho)(x,t)$ is a given pressure function. μ and λ denote the shear and bulk viscosity coefficients, respectively. In between (NS) and (CNS), we find the inhomogeneous incompressible Navier-Stokes system that governs the evolution of incompressible viscous flows with nonconstant density. That system finds its place in the theory of geophysical flows, where fluids are incompressible but with variable density. Basic examples are mixture of incompressible and non-reactant flows, flows with complex structure like blood flow and models in oceans or rivers, fluids containing a melted substance, etc.. In general, the system can be written in the following form

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P - \mathrm{div}(\mu(\rho)(\nabla u + (\nabla u)^T)) = 0, \\ \mathrm{div} u = 0, \end{cases} \tag{INS}$$

where $\mu(\rho)$ denotes the viscosity, which is a function of density ρ. This thesis mainly investigates the well-posedness of strong solutions to the initial and boundary value

problem or Cauchy problem of the systems (INS) and related inhomogeneous fluid models. Next, we briefly present the results of each chapter.

In Chapter 1, we study an initial and boundary value problem of the inhomogeneous incompressible Navier-Stokes equations (INS) over a bounded smooth domain in $\mathbb{R}^3$. Recently, X. Huang-Y. Wang (2) and J. Zhang (9) proved independently the existence and uniqueness of global strong solutions, provided the initial gradient of velocity ∇u is suitably small in the L^2 norm. After careful observation of their proof, we find this initial and boundary value problem also admits an unique strong solutions under other conditions, in particular, we prove small kinetic energy strong solutions with large L^2 norm of ∇u can exist globally in time, which extends the results of X. Huang-Y. Wang (2) and J. Zhang (9). The point we would like to emphasize is that the original time-weighted energy estimates obtained in X. Huang-Y. Wang (2) and J. Zhang (9) are fail to produce the global strong solution if we only assume that the initial kinetic energy is small sufficiently. To overcome these difficulties, we use a different power of time-weight, combining with some more precise estimates. As a by-product of our analysis, we also prove that the initial and boundary problem of inhomogeneous incompressible Navier-Stokes equations (INS) with vacuum has a global strong solution, provided the lower bound of viscosity coefficient is suitably large, or the upper bound of density is suitably small. The result of this chapter has been published in the second part of paper (3).

In Chapter 2, we discuss the dynamic model of electrically conducting inhomogeneous fluid under the effect of electro-magnetic field, which is described by the inhomogeneous incompressible magnetohydrodynamic (in short, MHD) equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\mu(\rho)(\nabla u + (\nabla u)^T)) + \nabla P - (\nabla \times B) \times B = 0, \\ \partial_t B - \nabla \times (u \times B) + \nabla \times (\lambda(\rho) \nabla \times B) = 0, \\ \mathrm{div} u = 0, \quad \mathrm{div} B = 0, \end{cases} \tag{MHD}$$

where $B(x,t)$ denotes the magnetic field, $\lambda(\rho)$ denotes the resistivity coefficient. As an extension result of the Navier-Stokes model in Chapter 1, we prove the existence and uniqueness of strong solutions to the initial and boundary problem of MHD equations (MHD) in a three-dimensional bounded smooth domain, provided the initial gradients of the velocity and magnetic fields are small sufficiently in some Sobolev space. Compared with the Navier-Stokes model (INS), the necessary time-weighted estimates for the existence of strong solutions will be more difficult to derive since of the apperance of coupling terms of velocity-magnetic fileds and the possible vacuum of density (degeneration in the momentum equation MHD_2). Thanks to the assumption of smallness of initial data, we can overcome these difficulties and successfully extend the local strong solutions to exist globally. The result of this chapter has been published in the first part of paper (3).

The last three chapters are devoted to the inhomogeneous incompressible Navier-Stokes-Korteweg equations

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\mu(\rho)(\nabla u + (\nabla u)^T)) + \nabla P + \mathrm{div}(\kappa(\rho)\nabla\rho \otimes \nabla\rho) = 0, \\ \mathrm{div} u = 0, \end{cases} \tag{NSK}$$

governing the motion of the inhomogeneous capillary fluids, where $\kappa(\rho)$ denotes the capillary coefficient, which is a function of the density ρ. There are many studies on the compressible Navier-Stokes-Korteweg equations, however, to my best knowledge, there are no much results on the study of the inhomogeneous incompressible Navier-Stokes-Korteweg equations. Very recently, T. Wang (8) established the local strong solutions to the initial and boundary value problem if the initial density and velocity satisfy some regularity and compatibility conditions. Next we introduce some new results on the system (NSK) obtained in this thesis.

In Chapter 3, we prove a Serrin type blow-up criterion for the 3D density-dependent Navier-Stokes-Korteweg equations (NSK) with vacuum. It is shown that if the density and velocity field satisfy $\|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)} < \infty$ for some $q > 3$, and any (r,s) satisfying $\frac{2}{s} + \frac{3}{r} \leq 1,\ \ 3 < r \leq \infty$, and L^r_ω denotes the weak L^r space, then the strong solutions to the density-dependent Navier-Stokes-Korteweg equations can exist globally over $[0,T]$. The manuscript (5) containing this result is under review.

Performing the similar calculations in Chater 3, we can obtain a similar blow-up criterion for the strong solutions of the 2D density-dependent Navier-Stokes-Korteweg equations, that is, if the density and velocity field satisfy $\|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)} < \infty$ for some $q > 2$, and any (r,s) satisfying $\frac{2}{s} + \frac{2}{r} \leq 1,\ \ 2 < r \leq \infty$, then the strong solutions to the 2D density-dependent Navier-Stokes-Korteweg equations can exist globally over $[0,T]$. We remark from the basic energy estimate that $\sup_{T>0}(\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2)} + \|\nabla u\|_{L^2(0,T;L^2)})$ is bounded, which implies that $u \in L^4(0,T;L^4)$ if ρ is bounded away from zero. Hence the criterion showed in chapter 3 in fact can be improved to the one only involving the density if the density ρ is bounded away from 0. However, if the density is allowed to vanish, it remains unknown. This is the main problem we shall address in chapter 4. Thanks to a lemma proved by Desjardin (Lemma 4.5, Chapter 4) and a logarithmic Gronwall's inequality, Chapter 4 proves a new blow-up criterion for the strong solutions with vacuum to the density-dependent Navier-Stokes-Korteweg equations over a bounded smooth domain in $\mathbb{R}^2$, which only in terms of the density. The result is contained in the paper (4) which will appear in the journal Acta Appl. Math..

The Chapter 3 and 4 are devoted to discussing the motion of fluids over a bounded domain. An interesting problem is to study the motion of fluids over an unbounded domain, a simple example is the whole space $\mathbb{R}^n, n = 2, 3$. For the three-dimensional Cauchy problem, by the Galerkin method, energy method and the domain expansion technique, it is easy to prove the local existence of unique strong solutions for general initial data, in view of the local result of T. Wang (8) for the initial and boundary value problem. However, it is difficult to apply these ideas to the two-dimensonal case. Since the Sobolev inequality in 2D is critical, it seems difficult to bound the L^p-norm of the velocity u just in terms of $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ and $\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^2)}$. Motivated by a lemma involving the spatial weight due to P. L. Lions (7) (Lemma 5.4, Chapter 5), we can obtain the local well-posedness of strong solutions to inhomogeneous Navier-Stokes-Korteweg equations, provided the initial density with some spatial weight belongs to some Sobolev space. The Chapter 5 proves that the 2D Cauchy problem of the inhomogeneous incompressible Navier-Stokes-Korteweg equations with constant viscosity and resistivity admits a unique local strong solution provided the initial density decay not too slow at infinity. The result of this chapter is contained in the manuscript (6) which is in preparation.

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Contents

Abstract **i**

1 Global strong solution to the 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity **1**
1.1 Introduction and main result 1
1.2 Preliminaries 5
1.2.1 Notations and general inequalities 5
1.2.2 Higher order estimates on the velocity 6
1.3 Global existence of strong solution to Navier-Stokes system 7
1.3.1 A priori estimates of different levels 7
1.3.2 Extension of local strong solution 16

2 Global strong solution to the 3D inhomogeneous MHD equations with density-dependent viscosity and resistivity **20**
2.1 Introduction and main result 20
2.2 Preliminaries 23
2.2.1 Notations and general inequalities 23
2.2.2 Two lemmas on higher order estimates on velocity and magnetic fields 24
2.3 Global existence of strong solution to MHD system 26
2.3.1 A priori estimates of different levels 26
2.3.2 Extension of local strong solution 45

3 A blow-up criterion of the 3D inhomogeneous Navier-Stokes-Korteweg equations **48**
3.1 Introduction and main result 48
3.2 Preliminaries 50
3.2.1 Notations and general inequalities 50
3.2.2 Higher order estimates on the velocity 51
3.3 Proof of the blow-up criterion in 3D 53

4 A blow-up criterion of the 2D inhomogeneous Navier-Stokes-Korteweg equations **62**
4.1 Introduction and main result 62
4.2 Preliminaries 64
4.2.1 Notations and general inequalities 64
4.2.2 Higher order estimates on the velocity 64
4.3 Proof of the blow-up criterion in 2D 65

5 On local strong solutions to the Cauchy problem of 2D inhomogeneous Navier-Stokes-Korteweg equations with vacuum 73

5.1 Introduction and main result 73

5.2 Preliminaries 76

5.3 A priori estimates 77

5.4 Local existence and uniqueness of strong solutions 86

Chapter 1

Global strong solution to the 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity

This chapter is devoted to studying the global well-posedness of an initial and boundary value problem of the inhomogeneous incompressible Navier-Stokes equations over a bounded smooth domain $\Omega \subset \mathbb{R}^3$. The global solvability of strong solution was proved when the L^2 norm of initial gradient of velocity field is suitably small by Huang-Wang [10] and Zhang [20], independently. We find that this initial boundary problem also admits a unique global strong solution under other conditions. In particular, we prove small kinetic energy strong solution exists globally in time, which extends the result of Huang-Wang and Zhang.

Keywords: Navier-Stokes; density-dependent viscosity; strong solution; vacuum

1.1 Introduction and main result

In this chapter, we consider the following inhomogeneous incompressible Navier-Stokes equations over a bounded smooth domain $\Omega \subset \mathbb{R}^3$, which are used to describe the motion of inhomogeneous fluids.

$$(1.1.1) \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) d) + \nabla P = 0, \\ \operatorname{div} u = 0. \end{cases}$$

Here ρ, u and P denote the density, velocity field, and pressure of the fluid respectively. The deformation tensor d is given by

$$d = \frac{1}{2}\Big[\nabla u + (\nabla u)^T\Big],$$

where ∇u is the gradient matrix $(\partial u_i/\partial x_j)$ and $(\nabla u)^T$ is its transpose. $\mu = \mu(\rho)$ stands for the viscosity coefficient of fluid, and is a function of density ρ. In this chapter, it is assumed to satisfy

$$(1.1.2) \qquad \mu \in C^1[0,\infty), \quad \text{and } \ \mu \geq \underline{\mu} > 0 \quad \text{on } [0,\infty)$$

for some positive constant $\underline{\mu}$.

We focus on the system (1.1.1)-(1.1.2) with the initial boundary conditions:

$$u = 0, \quad \text{on } \partial\Omega \times [0, T), \tag{1.1.3}$$

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in } \Omega. \tag{1.1.4}$$

There have been many studies on the Navier-Stokes equations and other fluid models due to their physical importance, mathematical challenge and wide range of applications. It is well known that the global existence of weak solution for homogeneous incompressible Navier-Stokes equations was obtained by Leray [14] in 1934. Since on, how to construct the Leray type weak solution for compressible Navier-Stokes equations had been an open problem. However, because of its hyperbolic-parabolic structure and strong nonlinearity of compressible Navier-Stokes equations, the breakthrough for compressible flow was made by P. L. Lions [16] until 1996. As an intermediate model, inhomogeneous incompressible Navier-Stokes equations with variable density has been also studied extensively by many people. The studies on inhomogeneous Navier-Stokes system can be regarded as an attempt for searching for global weak solution to compressible flow.

Let us recall some known results on inhomogeneous incompressible fluids. In around 1970, the study of inhomogeneous Navier-Stokes equations was initiated by the Russian school. They studied the case that $\mu(\rho)$ is a constant and the initial density is bounded away from 0. In the absence of vacuum, global existence of weak solution was established by Kazhikhov [11] . The uniqueness of local strong solutions was first established by Ladyzhenskaya and Solonnikov [12] for the initial boundary problem in the framework of L^p theory for p larger than space dimension. Furthermore, unique strong solution was proved to be global in two dimension. H. Okamoto [18] improved their result in L^2 framework and proved the global existence of strong solution when the initial data is suitably small in 3D. Very recently, Abidi and Zhang [1] proved that the 3D incompressible innhomogeneous Navier-Stokes system has a unique global strong solution when $\|\nabla u_0\|_{L^2}\|u_0\|_{L^2}$ and $\|\mu(\rho_0) - 1\|_{L^\infty}$ are both suitably small in the whole space and the viscosity is a function of density. However, the vacuum is still not admitted.

When initial vacuum is taken into account, Simon [23] proved the existence of weak solutions of finite energy (See also P. L. Lions [15] for variable viscosity case). Later, Choe, Kim [5] and Cho, Kim [4] proposed some compatibility conditions on initial data to establish local strong solution for constant viscosity and variable viscosity respectively. Global strong solution with vacuum in 2D was proved by Huang and Wang [8] in constant viscosity case.(The proof was given for 2D MHD equations, it just need to take $B \equiv 0$ for Navier-Stokes in their discussion). As a sequent work, for the density dependent viscosity with positive lower bound, they [9] obtained the global strong solution when the initial norm of gradient viscosity $\|\nabla\mu(\rho_0)\|_{L^q}$ is suitably small in 2D. Very recently, they [10] obtained the global strong solution under smallness of $\|\nabla u_0\|_{L^2}$ in 3D.

First we give the local existence result of strong solution to the Navier-Stokes system (1.1.1)-(1.1.4) due to Cho and Kim [4], where the vacuum is allowed.

Theorem 1.1. *Assume that the initial data* (ρ_0, u_0) *satisfies the regularity condition*

$$0 \le \rho_0 \in W^{1,q}, \quad 3 < q < \infty, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \tag{1.1.5}$$

and the compatibility condition

$$-\operatorname{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) + \nabla P_0 = \rho_0^{1/2} g, \tag{1.1.6}$$

for some $(P_0, g) \in H^1 \times L^2$*. Then there exist a small time* T *and a unique strong solution* (ρ, u, P) *to the initial boundary value problem* (1.1.1)-(1.1.4) *such that*

$$\begin{aligned} &\rho \in C([0,T); W^{1,q}), \quad \nabla u, P \in C([0,T); H^1) \cap L^2(0,T; W^{1,s}), \\ &\rho_t \in C([0,T); L^q), \quad \sqrt{\rho} u_t \in L^\infty(0,T; L^2), \quad u_t \in L^2(0,T; H^1_0), \end{aligned} \tag{1.1.7}$$

for any s *with* $1 \le s < q$*. Furthermore, if* T^* *is the maximal existence time of the local strong solution* (ρ, u)*, then either* $T^* = \infty$ *or*

$$\sup_{0 \le t < T^*} (\|\nabla \rho(t)\|_{L^q} + \|\nabla u(t)\|_{L^2}) = \infty. \tag{1.1.8}$$

Recently, the local strong solution established in Theorem 1.1 was extended to a global one when the initial L^2 norm of gradient of velocity is suitably small by Huang and Wang [10]. Their main result reads as follows:

Theorem 1.2. *Assume that the initial data* (ρ_0, u_0) *satisfies* (1.1.5)-(1.1.6)*, and* $0 \le \rho_0 \le \bar{\rho}$*. Then there exists a small positive constant* ϵ_0 *depending on* $\Omega, q, \bar{\rho}, \bar{\mu}, \underline{\mu}$ *and* $\|\nabla \mu(\rho_0)\|_{L^q}$*, such that if*

$$\|\nabla u_0\|_{L^2} \le \epsilon_0,$$

then the initial boundary value problem (1.1.1)-(1.1.4) *admits a unique strong solution* (ρ, u, P)*, with*

$$\begin{aligned} &\rho \in C([0,\infty); W^{1,q}), \quad \nabla u, P \in C([0,\infty); H^1) \cap L^2(0,\infty; W^{1,s}), \\ &\rho_t \in C([0,\infty); L^q), \quad \sqrt{\rho} u_t \in L^\infty_{loc}(0,\infty; L^2), \quad u_t \in L^2_{loc}(0,\infty; H^1_0), \end{aligned} \tag{1.1.9}$$

for any s *with* $1 \le s < q$*.*

For the initial boundary problem (1.1.1)-(1.1.4) of inhomogeneous Navier-Stokes equations, in Theorem 1.2 Huang and Wang gave a sufficient condition to guarantee the global existence of strong solution. We find that the strong solution can also exist globally in time under other conditions. In particular, the smallness of $\|\nabla u_0\|_{L^2}$ in Theorem 1.2 can be replaced by the smallness of initial kinetic energy, where the initial kinetic energy is defined as

$$C_0 := \int_\Omega \frac{1}{2} \rho_0 |u_0|^2 dx. \tag{1.1.10}$$

Now we can state our main result in this chapter as follows:

Theorem 1.3. *For given positive numbers $\bar{\rho}, M$ and N_0, suppose that the initial data (ρ_0, u_0) satisfies* (1.1.5)-(1.1.6)*, and $0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}$, $\|\nabla u_0\|_{L^2} \leq N_0$, and $\|\nabla \mu(\rho_0)\|_{L^q} \leq M$. Then there exists some small positive constant ϵ_0, independent of $\bar{\rho}, \bar{\mu} := \sup_{[0,\bar{\rho}]} \mu(\rho)$, $\underline{\mu}, M$ and N_0, such that if*

(1.1.11)
$$\max\left\{ \frac{\bar{\mu}}{\underline{\mu}^3}(M_2^2 + M^4 M_2^6)\bar{\rho}^3 N_0^2 C_0, \right.$$
$$\left. M_r \bar{\rho}^{\frac{5r-6}{4r}} \underline{\mu}^{-\frac{3(r-2)}{4r}} \left(\left(1 + \frac{\bar{\rho}}{\underline{\mu}}\right)^{\frac{3}{4}} \Theta_1 \right)^{\frac{1}{2}} \cdot \exp\{C\Theta_2\} + \frac{\bar{\mu}^{(2r-3)/r}}{\underline{\mu}^{3(r-1)/r}} (M_r \bar{\rho})^{\frac{5r-6}{r}} N_0^{\frac{4r-6}{r}} C_0 \right\} \leq \epsilon_0,$$

where $3 < r < q$, then the initial boundary value problem (1.1.1)-(1.1.4) *admits a unique global strong solution (ρ, u, P), with*

$$\begin{aligned} &\rho \in C([0,\infty); W^{1,q}), \quad \nabla u, P \in C([0,\infty); H^1) \cap L^2_{loc}(0,\infty; W^{1,s}), \\ &\rho_t \in C([0,\infty); L^q), \quad \sqrt{\rho} u_t \in L^\infty_{loc}(0,\infty; L^2), \quad u_t \in L^2_{loc}(0,\infty; H^1_0), \end{aligned} \tag{1.1.12}$$

for any s with $1 \leq s < q$. Here,

$$\begin{aligned} M_r &= \frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}} \frac{1}{\underline{\mu}^{1/\theta_r + 1}} \cdot (4M)^{\frac{1}{\theta_r}}, \quad \theta_r = \frac{2r}{5r-6} \cdot \frac{q-3}{q}, \\ \Theta_1 &= \frac{M_2^4 \bar{\rho}^6 \bar{\mu}^{\frac{11}{4}}}{\underline{\mu}^6} N_0^{\frac{11}{2}} C_0^{\frac{9}{4}} + \frac{M^2 M_2^8 \bar{\rho}^8 \bar{\mu}^{\frac{15}{4}}}{\underline{\mu}^7} N_0^{\frac{15}{2}} C_0^{\frac{9}{4}} + \bar{\mu}^{\frac{3}{4}} N_0^{\frac{3}{2}} C_0^{\frac{1}{4}}, \\ \Theta_2 &= \frac{\bar{\rho}^3 \bar{\mu}}{\underline{\mu}^5} N_0^2 C_0 + \frac{M_2^2 \bar{\rho}^3 \bar{\mu}}{\underline{\mu}^3} N_0^2 C_0 + \frac{M^2 M_2^4 \bar{\rho}^2 \bar{\mu}}{\underline{\mu}} N_0^2. \end{aligned} \tag{1.1.13}$$

As an immediate result, Theorem 1.3 obtained in this chapter implies that the initial and boundary problem of inhomogeneous incompressible Navier-Stokes equations with vacuum has a global strong solution, provided the initial kinetic energy is suitably small, or the lower bound of viscosity coefficient is suitably large, or the upper bound of density is suitably small. To illustrate our result, we give some remarks on the Theorem 1.3.

Remark 1.4. The positive constant ϵ_0 depends only on Ω, q and various Sobolev's constants. The relation (1.1.11) indeed gives a sufficient condition for the global solvability of strong solution of (1.1.1)-(1.1.4).

Remark 1.5. When $\bar{\mu} \leq C\underline{\mu}$, for some $C \geq 1$, such as $\mu(\rho) = \underline{\mu} + \rho^\alpha, \alpha > 0$, or $\mu(\rho) = \underline{\mu} \exp \rho$, or $\mu(\rho)$ is just equal to a positive constant. We can easily see that the left hand side of (1.1.11) can be as small as desired provided C_0 is sufficiently small, or $\underline{\mu}$ is sufficiently large, or $\bar{\rho}$ is sufficiently small.

Remark 1.6. As a result of Remark 1.5, for any given generally initial data (ρ_0, u_0) containing vacuum, one also get the global strong solution of (1.1.1)-(1.1.4) provided the lower bound of viscosity is sufficiently large. This is analogous to the well-known results due to Leray [14] for the homogeneous incompressible Navier-Stokes equations, this similar conclusion was also obtained by Deng et al. [6] for compressible Navier-Stokes equations recently.

Remark 1.7. For the inhomogeneous incompressible Navier-Stokes equations, since the initial density is bounded above, then the initial kinetic energy is dominated by the initial L^2 norm of gradient of velocity, this can be seen clearly by Poincaré and Hölder inequalities over bounded domains. Then our result extends the result of Huang and Wang [10].

We now comment on the analysis of this chapter. Our study is mainly motivated by a recent work of Huang and Wang [10], where the authors establish the global strong solution to the inhomogeneous incompressible Navier-Stokes equations when the initial L^2 norm of gradient of velocity is suitably small. The key in their proof is to control the norms of $\|\nabla\rho\|_{L^q}$ and $\|\nabla u\|_{L^2}$. First they assume that $\|\nabla\mu(\rho)\|_{L^q} \leq 4M$ and $\|\nabla u\|_{L^2}^2 \leq 4(\bar{\mu}/\underline{\mu})\|\nabla u_0\|_{L^2}^2$ on $[0,T]$, then they proved $\|\nabla\mu(\rho)\|_{L^q} \leq 2M$ and $\|\nabla u\|_{L^2}^2 \leq 2(\bar{\mu}/\underline{\mu})\|\nabla u_0\|_{L^2}^2$ on $[0,T]$, under the assumption $\|\nabla u_0\|_{L^2}$ is suitably small on $[0,T]$. After more precise observation, we also find that under more general conditions, for instance that the upper bound of density $\bar{\rho}$ is suitably small, or the lower bound of viscosity is suitably large, or the initial kinetic energy is suitably small, $\|\nabla\mu(\rho)\|_{L^q} \leq 2M$ and $\|\nabla u\|_{L^2}^2 \leq 2(\bar{\mu}/\underline{\mu})\|\nabla u_0\|_{L^2}^2$ still hold on $[0,T]$. Therefore by a contractive method, $\|\nabla\rho\|_{L^q}$ and $\|\nabla u\|_{L^2}$ are always bounded as long as strong solution exists. It is worthy to remark that one of the main ingredient is a time independent estimate for $\|\nabla u\|_{L^1L^\infty}$ whose derivation need the time weighted estimates of $\|\sqrt{\rho}u_t\|_{L^2}$ and $\|\nabla u_t\|_{L^2}$. However, under the smallness assumption of initial kinetic energy, the estimates of $\|\sqrt{\rho}u_t\|_{L^2}$ and $\|\nabla u_t\|_{L^2}$ with different time weight are also needed in our calculation, so we modify Huang and Wang's Theorem 1.2 slightly, where their estimates are not available.

The rest of this chapter is organized as follows. Section 1.2 consists of some notations and useful technical lemmas. Section 1.3 is devoted to the proof of our main theorem, where in Subsection 1.3.1, we are devoted to deriving some necessary a priori estimates, in Subsection 1.3.2, the contractive technique is used to obtain the global strong solution.

1.2 Preliminaries

1.2.1 Notations and general inequalities

Ω is a smooth bounded domain in $\mathbb{R}^3$. For notations simplicity below, we omit the integration domain Ω. And for $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r = L^r(\Omega), \quad W^{k,r} = \{f \in L^r : \nabla^k f \in L^r\},$$

$$H^k = W^{k,2}, \quad C_{0,\sigma}^\infty = \{f \in (C_0^\infty)^3 : \mathrm{div} f = 0\}.$$

$$H_0^1 = \overline{C_0^\infty}, \quad H_{0,\sigma}^1 = \overline{C_{0,\sigma}^\infty}, \text{ closure in the norm of } H^1.$$

The following Gagliardo-Nirenberg inequality will be also used frequently.

Lemma 1.8 (Gagliardo-Nirenberg inequality). *Let Ω be a domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For $p \in [2,6], q \in (1,\infty)$ and $r \in (3,\infty)$, there exists some generic constants $C > 0$ that may depend on q and r such that for $f \in H^1$ satisfying $f|_{\partial\Omega} = 0$, or $\int_\Omega f dx = 0$ and*

$g \in L^q \cap D^{1,r}$, *we have*

$$\|f\|_{L^p}^p \le C\|f\|_{L^2}^{(6-p)/2}\|\nabla f\|_{L^2}^{(3p-6)/2}, \tag{1.2.1}$$

$$\|g\|_{L^\infty} \le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{1.2.2}$$

See the proof of this lemma in Ladyzhenskaya et al. [13, P. 62](see also Nirenberg [17, P. 125]).

1.2.2 Higher order estimates on the velocity

High-order a priori estimates of velocity field u rely on the following regularity results for density-dependent Stokes equations.

Lemma 1.9. *Assume that* $\rho \in W^{1,q}, 3 < q < \infty$, *and* $0 \le \rho \le \bar{\rho}$. *Let* $(u, P) \in H^1_{0,\sigma} \times L^2$ *be the unique weak solution to the boundary value problem*

$$-\operatorname{div}(2\mu(\rho)d) + \nabla P = F, \quad \operatorname{div} u = 0 \textit{ in } \Omega, \quad \textit{and} \int \frac{P}{\mu(\rho)}dx = 0, \tag{1.2.3}$$

where $d = \frac{1}{2}[\nabla u + (\nabla u)^T]$ *and*

$$\mu \in C^1[0,\infty), \ \ \underline{\mu} \le \mu(\rho) \le \bar{\mu} \textit{ on } [0,\bar{\rho}].$$

Then we have the following regularity results:

(1) If $F \in L^2$, *then* $(u,P) \in H^2 \times H^1$ *and*

$$\|u\|_{H^2} + \|P/\mu(\rho)\|_{H^1} \le C\left(\frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}^{\frac{1}{\theta_2}+2}}\|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_2}}\right)\|F\|_{L^2}, \tag{1.2.4}$$

where θ_2 *satisfies*

$$\frac{1}{2} - \frac{1}{q} = \frac{\theta_2}{3} + \frac{1}{6}, \quad i.e.\ \theta_2 = \frac{q-3}{q}.$$

(2) If $F \in L^r$ *for some* $r \in (2,q)$, *then* $(u,P) \in W^{2,r} \times W^{1,r}$ *and*

$$\|u\|_{W^{2,r}} + \|P/\mu(\rho)\|_{W^{1,r}} \le C\left(\frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}^{\frac{1}{\theta_r}+2}}\|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_r}}\right)\|F\|_{L^r}, \tag{1.2.5}$$

where θ_r *satisfies*

$$\theta_r = \frac{2r}{5r-6} \cdot \frac{q-3}{q}.$$

Here the constant C *in* (1.2.4) *and* (1.2.5) *depends on* Ω, q, r.

The proof of Lemma 1.9 has been given by Huang and Wang in [10]. And refer to Lemma 2.1 in their paper.

1.3 Global existence of strong solution to Navier-Stokes system

The proof of Theorem 1.3 is composed of two parts. The first part contains a priori time-weighted estimates of different levels. Upon these estimates, the second part uses a contradiction induction process to extend the local strong solution.

1.3.1 A priori estimates of different levels

In this subsection, we establish some a priori estimates of different levels. In order to control the $L_t^1 L_x^\infty$ norm of gradient of velocity, some time weighted estimates are also necessary. The initial velocity belongs to H^1, some higher-order estimates independent of time are required. To achieve that, we take some power of time as a weight. The idea is based on the parabolic property of the system.

The key step in the proofs of Theorem 1.3 is to prove the following key a priori estimate of (ρ, u).

Proposition 1.10. *For given positive numbers $\bar{\rho}$, M and N_0, assume that*

$$0 \le \inf \rho_0 \le \sup \rho_0 \le \bar{\rho}, \quad \|\nabla \mu(\rho_0)\|_{L^q} \le M, \quad \|\nabla u_0\|_{L^2} \le N_0. \tag{1.3.1}$$

Suppose (ρ, u, P) is the unique local strong solution to (1.1.1)-(1.1.4) on $\Omega \times [0,T]$, and if it satisfies

$$\sup_{t\in[0,T]} \|\nabla \mu(\rho(t))\|_{L^q} \le 4M, \quad \sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2}^2 \le 4\frac{\bar{\mu}}{\underline{\mu}} N_0^2, \tag{1.3.2}$$

then one has

$$\sup_{t\in[0,T]} \|\nabla \mu(\rho(t))\|_{L^q} \le 2M, \quad \sup_{t\in[0,T]} \|\nabla u(t)\|_{L^2}^2 \le 2\frac{\bar{\mu}}{\underline{\mu}} N_0^2, \tag{1.3.3}$$

provided that (1.1.11)*, together with* (1.1.13)*, holds.*

Proofs of Proposition 1.10 are based on a series of lemmas. First, it is easy to deduce from $(1.1.1)_1$ and $\mathrm{div} u = 0$ that the following lemma holds.

Lemma 1.11. *Suppose (ρ, u, P) is the unique local strong solution to* (1.1.1)-(1.1.4) *on $\Omega \times [0,T]$, with the initial data (ρ_0, u_0), then there it holds that*

$$0 \le \rho(x,t) \le \bar{\rho}, \quad \textit{for every } (x,t) \in \Omega \times [0,T].$$

Applying the standard energy estimate to (1.1.1) gives

Lemma 1.12. *Suppose (ρ, u, P) is the unique local strong solution to* (1.1.1)-(1.1.4) *on $\Omega \times [0,T]$, with the initial data (ρ_0, u_0), then there it holds that*

$$\int \frac{1}{2}\rho |u(t)|^2 dx + 2\int_0^t \int \mu(\rho)|d|^2 dx ds \le C_0, \quad \textit{for every } t \in [0,T], \tag{1.3.4}$$

since $\mu(\rho) \geq \underline{\mu}$, we aslo have

$$\int \frac{1}{2}\rho|u(t)|^2dx + \underline{\mu}\int_0^t\int |\nabla u|^2dxds \leq C_0, \quad \text{for every } t \in [0,T]. \tag{1.3.5}$$

Theorem 1.13. *Suppose (ρ, u, P) is the unique local strong solution to* (1.1.1)-(1.1.4) *on $\Omega \times [0,T]$, with the initial data (ρ_0, u_0), then there exists positive number C_1, depending only on Ω, q such that if*

$$C_1 \cdot \frac{\bar{\mu}}{\underline{\mu}^3}(M_2^2 + M^4M_2^6)\bar{\rho}^3N_0^2C_0 \leq \log 2, \tag{1.3.6}$$

then

$$\frac{1}{\underline{\mu}}\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2dt + \sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^2 \leq 2\frac{\bar{\mu}}{\underline{\mu}}N_0^2. \tag{1.3.7}$$

provided (1.3.2) *holds.*

Before proving Theorem 1.13, we insert the following lemma, which is derived from the auxillary Lemma 1.9.

Lemma 1.14. *Suppose (ρ, u, P) is the unique local strong solution to* (1.1.1)-(1.1.4) *on $\Omega \times [0,T]$, with the initial data (ρ_0, u_0), and it satisfies*

$$\sup_{t\in[0,T]}\|\nabla\mu(\rho(t))\|_{L^q} \leq 4M, \tag{1.3.8}$$

then it holds that

$$\|\nabla u\|_{H^1} \leq CM_2\|\rho u_t\|_{L^2} + CM_2^2\bar{\rho}^2\|\nabla u\|_{L^2}^3. \tag{1.3.9}$$

Proof. We can rewrite the momentum equations as follows,

$$-2\text{div}(\mu(\rho)d) + \nabla P = -\rho u_t - \rho u \cdot \nabla u.$$

It follows from Lemma 1.9 and Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|\nabla u\|_{H^1} &\leq CM_2(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2})\\ &\leq CM_2\|\rho u_t\|_{L^2} + CM_2\bar{\rho}\|u\|_{L^6}\|\nabla u\|_{L^3}\\ &\leq CM_2\|\rho u_t\|_{L^2} + CM_2\bar{\rho}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}.\end{aligned}$$

By Young's inequality,

$$\|\nabla u\|_{H^1} \leq CM_2\|\rho u_t\|_{L^2} + CM_2^2\bar{\rho}^2\|\nabla u\|_{L^2}^3.$$

□

Proof of Theorem 1.13. Using the fact $\text{div}u = 0$ and the mass equation, it is easy to obtain the equation for $\mu(\rho)$,

$$\partial_t[\mu(\rho)] + u\cdot\nabla\mu(\rho) = 0.$$

Multiplying the momentum equations $(1.1.1)_2$ by u_t, and integrating over Ω, we have

$$\begin{aligned}&\int \rho|u_t|^2dx + \frac{d}{dt}\int \mu(\rho)|d|^2dx\\ &\le |\int \rho u\cdot\nabla u\cdot u_t dx| + C\int |\nabla\mu(\rho)||u||\nabla u|^2dx.\end{aligned}$$

Applying Gagliardo-Nirenberg inequality and Lemma 1.14,

$$\begin{aligned}&\left|\int \rho u\cdot\nabla u\cdot u_t dx\right|\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^6}^2\|\nabla u\|_{L^3}^2\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\bar{\rho}\|\nabla u\|_{L^2}^3\|\nabla u\|_{H^1}\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\bar{\rho}\|\nabla u\|_{L^2}^3[M_2\|\rho u_t\|_{L^2} + M_2^2\bar{\rho}^2\|\nabla u\|_{L^2}^3],\end{aligned}$$

and similarly,

$$\begin{aligned}&\int |\nabla\mu(\rho)||u||\nabla u|^2dx\\ &\le C\|\nabla\mu(\rho)\|_{L^3}\|u\|_{L^6}\|\nabla u\|_{L^4}^2\\ &\le C\|\nabla\mu(\rho)\|_{L^3}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{3}{2}}\\ &\le CM\|\nabla u\|_{L^2}^{\frac{3}{2}}[M_2\|\rho u_t\|_{L^2} + M_2^2\bar{\rho}^2\|\nabla u\|_{L^2}^3]^{\frac{3}{2}}.\end{aligned}$$

Hence, by Young's inequality,

$$\begin{aligned}&\int \rho|u_t|^2dx + \frac{d}{dt}\int \mu(\rho)|d|^2dx\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + CM_2^2\bar{\rho}^3\|\nabla u\|_{L^2}^6 + \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2\\ &\quad + C(MM_2^{\frac{3}{2}}\bar{\rho}^{\frac{3}{4}}\|\nabla u\|_{L^2}^{\frac{3}{2}})^4 + CMM_2^3\bar{\rho}^3\|\nabla u\|_{L^2}^6\\ &\le \frac{3}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\left(M_2^2 + M^4M_2^6 + MM_2^3\right)\bar{\rho}^3\|\nabla u\|_{L^2}^6.\end{aligned}$$

Thus we have

$$\begin{aligned}&\int \rho|u_t|^2dx + \frac{d}{dt}\int \mu(\rho)|d|^2dx\\ &\le C\left(M_2^2 + M^4M_2^6 + MM_2^3\right)\bar{\rho}^3\|\nabla u\|_{L^2}^6\\ &\le C\underline{\mu}^{-1}\left(M_2^2 + M^4M_2^6\right)\bar{\rho}^3\|\nabla u\|_{L^2}^4\cdot\int \mu(\rho)|d|^2dx.\end{aligned} \tag{1.3.10}$$

Applying Gronwall's inequality,

$$\begin{aligned}&\frac{1}{\underline{\mu}}\int_0^T\int \rho|u_t|^2dx + \sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2\\ &\le \frac{\bar{\mu}}{\underline{\mu}}\|\nabla u_0\|_{L^2}^2\cdot\exp\left\{C\underline{\mu}^{-1}\left(M_2^2 + M^4M_2^6\right)\bar{\rho}^3\int_0^T\|\nabla u\|_{L^2}^4dt\right\}.\end{aligned}$$

According to the Lemma 1.12 and the assumption (1.3.2),

$$\begin{aligned}\int_0^T \|\nabla u\|_{L^2}^4 dt &\le \sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2 \cdot \int_0^T \|\nabla u\|_{L^2}^2 dt \\ &\le C \cdot \frac{\bar{\mu}}{\underline{\mu}} N_0^2 \cdot \frac{C_0}{\underline{\mu}} \\ &\le C \cdot \frac{\bar{\mu}}{\underline{\mu}^2} N_0^2 C_0.\end{aligned} \tag{1.3.11}$$

Hence, we arrive at

$$\begin{aligned}&\frac{1}{\underline{\mu}} \int_0^T \int \rho |u_t|^2 dx + \sup_{t\in[0,T]} \|\nabla u\|_{L^2} \\ &\le \frac{\bar{\mu}}{\underline{\mu}} \|\nabla u_0\|_{L^2}^2 \cdot \exp\left\{ C_1 \frac{\bar{\mu}}{\underline{\mu}^3} \left(M_2^2 + M^4 M_2^6\right) \bar{\rho}^3 N_0^2 C_0 \right\}.\end{aligned} \tag{1.3.12}$$

Now it is clear that (1.3.7) holds, provided (1.3.6) holds. □

As a byproduct of the estimate in the proof, we have the following time weighted estimate.

Theorem 1.15. *Suppose* (ρ, u, P) *is the unique local strong solution to* (1.1.1)-(1.1.4) *on* $\Omega \times [0, T]$*, with the initial data* (ρ_0, u_0)*, and it satisfies the assumption* (1.3.6) *as in Theorem 1.13. Then*

$$\frac{1}{\underline{\mu}} \int_0^T t \|\sqrt{\rho} u_t\|_{L^2}^2 dt + \sup_{t\in[0,T]} t \|\nabla u(t)\|_{L^2}^2 \le \frac{C \cdot C_0}{\underline{\mu}}, \tag{1.3.13}$$

provided (1.3.2) *holds.*

Proof. Multiplying (1.3.10) by t, as shown in the last proof, one has

$$\begin{aligned}&\frac{1}{\underline{\mu}} \int_0^T t \|\sqrt{\rho} u_t\|_{L^2}^2 dt + \sup_{t\in[0,T]} t \|\nabla u\|_{L^2}^2 \\ &\le \frac{1}{\underline{\mu}} \int_0^T \int \mu(\rho) |d|^2 dx dt \cdot \exp\left\{ C_1 \frac{\bar{\mu}}{\underline{\mu}^3} \left(M_2^2 + M^4 M_2^6\right) \bar{\rho}^3 N_0^2 C_0 \right\}.\end{aligned} \tag{1.3.14}$$

According to Theorem 1.12,

$$\int_0^T \int \mu(\rho) |d|^2 dx dt \le C C_0. \tag{1.3.15}$$

Hence, owing to the assumption (1.3.2),

$$\begin{aligned}&\frac{1}{\underline{\mu}} \int_0^T t \|\sqrt{\rho} u_t\|_{L^2}^2 dt + \sup_{t\in[0,T]} t \|\nabla u\|_{L^2}^2 \\ &\le \frac{C C_0}{\underline{\mu}} \cdot \exp\left\{ C_1 \frac{\bar{\mu}}{\underline{\mu}^3} \left(M_2^2 + M^4 M_2^6\right) \bar{\rho}^3 N_0^2 C_0 \right\} \\ &\le \frac{C C_0}{\underline{\mu}}.\end{aligned}$$

□

In the later analysis, different from the proof of Huang and Wang [10], we need some time weighted estimates with different power to collect the information of initial kinetic energy. The following time weighted estimates can be obtained by interpolation methods as a corollary of Theorem 1.13 and 1.15.

Theorem 1.16. *Suppose* (ρ, u, P) *is the unique local strong solution to* (1.1.1)-(1.1.4) *on* $\Omega \times [0, T]$*, with the initial data* (ρ_0, u_0)*, and it satisfies the assumption* (1.3.6) *as in Theorem 1.13. Then*

$$\frac{1}{\underline{\mu}}\int_0^T t^{\frac{1}{4}}\|\sqrt{\rho}u_t\|_{L^2}^2 dt + \sup_{t\in[0,T]} t^{\frac{1}{4}}\|\nabla u(t)\|_{L^2}^2 \le \frac{C\cdot \bar{\mu}^{\frac{3}{4}} C_0^{\frac{1}{4}} N_0^{\frac{3}{2}}}{\underline{\mu}}, \tag{1.3.16}$$

provided (1.3.2) *holds.*

Theorem 1.17. *Suppose* (ρ, u, P) *is the unique local strong solution to* (1.1.1)-(1.1.4) *on* $\Omega \times [0, T]$*, with the initial data* (ρ_0, u_0)*, and it satisfies the assumption* (1.3.6) *as in Theorem 1.13. Then*

$$\begin{aligned} &\sup_{t\in[0,T]} t^{\frac{5}{4}}\|\sqrt{\rho}u_t\|_{L^2}^2 + \underline{\mu}\int_0^T t^{\frac{5}{4}}\|\nabla u_t\|_{L^2}^2 dt \\ &\le C\Theta_1\cdot \exp\{C\Theta_2\}, \end{aligned} \tag{1.3.17}$$

and

$$\begin{aligned} &\sup_{t\in[0,T]} t^{2}\|\sqrt{\rho}u_t\|_{L^2}^2 + \underline{\mu}\int_0^T t^{2}\|\nabla u_t\|_{L^2}^2 dt \\ &\le C\left(\frac{\bar{\rho}}{\bar{\mu}}\right)^{\frac{3}{4}}\cdot \Theta_1\cdot \exp\{C\Theta_2\}, \end{aligned} \tag{1.3.18}$$

where

$$\begin{aligned} \Theta_1 &= \frac{M_2^4\bar{\rho}^6\bar{\mu}^{\frac{11}{4}}}{\underline{\mu}^6} N_0^{\frac{11}{2}} C_0^{\frac{9}{4}} + \frac{M^2M_2^8\bar{\rho}^8\bar{\mu}^{\frac{15}{4}}}{\underline{\mu}^7} N_0^{\frac{15}{2}} C_0^{\frac{9}{4}} + \bar{\mu}^{\frac{3}{4}} N_0^{\frac{3}{2}} C_0^{\frac{1}{4}}, \\ \Theta_2 &= \frac{\bar{\rho}^3\bar{\mu}}{\underline{\mu}^5} N_0^2 C_0 + \frac{M_2^2\bar{\rho}^3\bar{\mu}}{\underline{\mu}^3} N_0^2 C_0 + \frac{M^2M_2^4\bar{\rho}^2\bar{\mu}}{\underline{\mu}} N_0^2, \end{aligned} \tag{1.3.19}$$

provided (1.3.2) *holds.*

Proof. Performing similar calculation as Huang and Wang [10] (remark here we need the estimates with different power of time weight from [10], therefore we multiply by $t^{\frac{5}{4}}u_t$ in the time-differentiated momentum equations), one has,

$$\begin{aligned} &\frac{d}{dt}\left(t^{\frac{5}{4}}\int \rho|u_t|^2 dx\right) + \underline{\mu} t^{\frac{5}{4}}\|\nabla u_t\|_{L^2}^2 \\ &\le C\frac{\bar{\rho}^3}{\underline{\mu}^3} t^{\frac{5}{4}}\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4 + \frac{CM_2^2\bar{\rho}^3}{\underline{\mu}} t^{\frac{5}{4}}\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4 + \frac{5}{4}t^{\frac{1}{4}}\int \rho|u_t|^2 dx \\ &\quad + \frac{CM_2^4\bar{\rho}^6}{\underline{\mu}} t^{\frac{5}{4}}\|\nabla u\|_{L^2}^{10} + \frac{CM^2M_2^4\bar{\rho}^2}{\underline{\mu}} t^{\frac{5}{4}}\|\sqrt{\rho}u_t\|_{L^2}^4 + \frac{CM^2M_2^8\bar{\rho}^8}{\underline{\mu}} t^{\frac{5}{4}}\|\nabla u\|_{L^2}^{12}. \end{aligned} \tag{1.3.20}$$

Applying Gronwall's inequality,

$$\begin{aligned}&\sup_{t\in[0,T]}\left(t^{\frac54}\int\rho|u_t|^2dx\right)+\underline{\mu}\int_0^T t^{\frac54}\|\nabla u_t\|_{L^2}^2dt\\ &\le\left[\int_0^T\Big(\frac{CM_2^4\bar\rho^6}{\underline{\mu}}t^{\frac54}\|\nabla u\|_{L^2}^{10}+\frac{CM^2M_2^8\bar\rho^8}{\underline{\mu}}t^{\frac54}\|\nabla u\|_{L^2}^{12}+\frac54 t^{\frac14}\|\sqrt{\rho}u_t\|_{L^2}^2\Big)dt\right]\\ &\quad\times\exp\left\{\int_0^T\Big[\Big(\frac{C\bar\rho^3}{\underline{\mu}^3}+\frac{CM_2^2\bar\rho^3}{\underline{\mu}}\Big)\|\nabla u\|_{L^2}^4+\frac{CM^2M_2^4\bar\rho^2}{\underline{\mu}}\|\sqrt{\rho}u_t\|_{L^2}^2\Big]dt\right\}.\end{aligned}$$

Taking some previous estimates into account,

$$\begin{aligned}&\sup_{t\in[0,T]}\left(t^{\frac54}\int\rho|u_t|^2dx\right)+\underline{\mu}\int_0^T t^{\frac54}\|\nabla u_t\|_{L^2}^2dt\\ &\le\left[\int_0^T\Big(\frac{CM_2^4\bar\rho^6}{\underline{\mu}}t^{\frac54}\|\nabla u\|_{L^2}^{10}+\frac{CM^2M_2^8\bar\rho^8}{\underline{\mu}}t^{\frac54}\|\nabla u\|_{L^2}^{12}+\frac54 t^{\frac14}\|\sqrt{\rho}u_t\|_{L^2}^2\Big)dt\right]\\ &\quad\times\exp\left\{C\Big(\frac{\bar\rho^3\bar\mu}{\underline{\mu}^5}N_0^2C_0+\frac{M_2^2\bar\rho^3\bar\mu}{\underline{\mu}^3}N_0^2C_0+\frac{M^2M_2^4\bar\rho^2\bar\mu}{\underline{\mu}}N_0^2\Big)\right\}.\end{aligned}$$

According to Theorem 1.12 and the assumption (1.3.2),

$$\begin{aligned}&\int_0^T t^{\frac54}\|\nabla u\|_{L^2}^{10}dt\\ &\le\sup_{t\in[0,T]}t^{\frac14}\|\nabla u(t)\|_{L^2}^2\sup_{t\in[0,T]}t\|\nabla u(t)\|_{L^2}^2\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^4\int_0^T\|\nabla u\|_{L^2}^2dt\\ &\le C\cdot\frac{\bar\mu^{\frac34}C_0^{\frac14}N_0^{\frac32}}{\underline{\mu}}\cdot\frac{C_0}{\underline{\mu}}\cdot\left(\frac{\bar\mu}{\underline{\mu}}N_0^2\right)^2\cdot\frac{C_0}{\underline{\mu}}\\ &\le C\frac{\bar\mu^{\frac{11}{4}}}{\underline{\mu}^5}N_0^{\frac{11}{2}}C_0^{\frac94}.\end{aligned}$$

Similarly,

$$\int_0^T t^{\frac54}\|\nabla u\|_{L^2}^{12}dt\le C\frac{\bar\mu^{\frac{15}{4}}}{\underline{\mu}^6}N_0^{\frac{15}{2}}C_0^{\frac94}.$$

Hence, in view of Theorem 1.16 for the last term, we have

$$\begin{aligned}&\sup_{t\in[0,T]}\left(t^{\frac54}\int\rho|u_t|^2dx\right)+\underline{\mu}\int_0^T t^{\frac54}\|\nabla u_t\|_{L^2}^2dt\\ &\le C\left(\frac{M_2^4\bar\rho^6\bar\mu^{\frac{11}{4}}}{\underline{\mu}^6}N_0^{\frac{11}{2}}C_0^{\frac94}+\frac{M^2M_2^8\bar\rho^8\bar\mu^{\frac{15}{4}}}{\underline{\mu}^7}N_0^{\frac{15}{2}}C_0^{\frac94}+\bar\mu^{\frac34}N_0^{\frac32}C_0^{\frac14}\right)\\ &\quad\times\exp\left\{C\Big(\frac{\bar\rho^3\bar\mu}{\underline{\mu}^5}N_0^2C_0+\frac{M_2^2\bar\rho^3\bar\mu}{\underline{\mu}^3}N_0^2C_0+\frac{M^2M_2^4\bar\rho^2\bar\mu}{\underline{\mu}}N_0^2\Big)\right\}.\end{aligned}\tag{1.3.21}$$

On the other hand, multiplying the time-differentiated momentum equations by t^2 and performing the same calculation as in Huang and Wang [10], one has

$$\begin{aligned}
&\frac{d}{dt}\left(t^2\int \rho|u_t|^2dx\right)+\underline{\mu}t^2\|\nabla u_t\|_{L^2}^2\\
&\le C\frac{\bar\rho^3}{\underline{\mu}^3}t^2\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4+\frac{CM_2^2\bar\rho^3}{\underline{\mu}}t^2\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4\\
&\quad+\frac{CM_2^4\bar\rho^6}{\underline{\mu}}t^2\|\nabla u\|_{L^2}^{10}+\frac{CM^2M_2^4\bar\rho^2}{\underline{\mu}}t^2\|\sqrt{\rho}u_t\|_{L^2}^4\\
&\quad+\frac{CM^2M_2^8\bar\rho^8}{\underline{\mu}}t^2\|\nabla u\|_{L^2}^{12}+2t\int\rho|u_t|^2dx.
\end{aligned}$$

Applying Gronwall's inequality,

$$\begin{aligned}
&\sup_{t\in[0,T]}t^2\int\rho|u_t|^2dx+\underline{\mu}\int_0^T t^2\|\nabla u_t\|_{L^2}^2dt\\
&\le\left[\int_0^T\Big(\frac{CM_2^4\bar\rho^6}{\underline{\mu}}t^2\|\nabla u\|_{L^2}^{10}+\frac{CM^2M_2^8\bar\rho^8}{\underline{\mu}}t^2\|\nabla u\|_{L^2}^{12}+2t\|\sqrt{\rho}u_t\|_{L^2}^2\Big)dt\right]\\
&\quad\times\exp\left\{C\Big(\frac{\bar\rho^3\bar\mu}{\underline{\mu}^5}N_0^2C_0+\frac{M_2^2\bar\rho^3\bar\mu}{\underline{\mu}^3}N_0^2C_0+\frac{M^2M_2^4\bar\rho^2\bar\mu}{\underline{\mu}}N_0^2\Big)\right\}.
\end{aligned}$$

According to Theorem 1.12 and 1.15,

$$\begin{aligned}
\int_0^T t^2\|\nabla u\|_{L^2}^{10}dt&\le\sup_{t\in[0,T]}t^2\|\nabla u(t)\|_{L^2}^4\cdot\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^4\cdot\int_0^T\|\nabla u\|_{L^2}^2dt\\
&\le C\cdot\left(\frac{C_0}{\underline{\mu}}\right)^2\left(\frac{\bar\mu}{\underline{\mu}}N_0^2\right)^2\frac{C_0}{\underline{\mu}}\\
&\le\frac{\bar\mu^2}{\underline{\mu}^5}N_0^4C_0^3,
\end{aligned}$$

similarly,

$$\begin{aligned}
\int_0^T t^2\|\nabla u\|_{L^2}^{12}dt&\le\sup_{t\in[0,T]}t^2\|\nabla u(t)\|_{L^2}^4\cdot\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2}^6\cdot\int_0^T\|\nabla u\|_{L^2}^2dt\\
&\le C\cdot\left(\frac{C_0}{\underline{\mu}}\right)^2\left(\frac{\bar\mu}{\underline{\mu}}N_0^2\right)^3\frac{C_0}{\underline{\mu}}\\
&\le C\frac{\bar\mu^3}{\underline{\mu}^6}N_0^6C_0^3,
\end{aligned}$$

and

$$\int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2dt\le C\cdot C_0.$$

Hence,

$$
\begin{aligned}
&\sup_{t\in[0,T]} t^2\int \rho|u_t|^2dx + \underline{\mu}\int_0^T t^2\|\nabla u_t\|_{L^2}^2dt \\
&\le C\left(\frac{M_2^4\bar\rho^6\bar\mu^2}{\underline{\mu}^6}N_0^4C_0^3 + \frac{M^2M_2^8\bar\rho^8\bar\mu^3}{\underline{\mu}^7}N_0^6C_0^3 + C_0\right) \\
&\qquad \times \exp\left\{C\Big(\frac{\bar\rho^3\bar\mu}{\underline{\mu}^5}N_0^2C_0 + \frac{M_2^2\bar\rho^3\bar\mu}{\underline{\mu}^3}N_0^2C_0 + \frac{M^2M_2^4\bar\rho^2\bar\mu}{\underline{\mu}}N_0^2\Big)\right\} \\
&\le C\frac{C_0^{\frac34}}{N_0^{\frac32}\bar\mu^{\frac34}}\left(\frac{M_2^4\bar\rho^6\bar\mu^{\frac{11}{4}}}{\underline{\mu}^6}N_0^{\frac{11}{2}}C_0^{\frac94} + \frac{M^2M_2^8\bar\rho^8\bar\mu^{\frac{15}{4}}}{\underline{\mu}^7}N_0^{\frac{15}{2}}C_0^{\frac94} + \bar\mu^{\frac34}N_0^{\frac32}C_0^{\frac14}\right) \qquad (1.3.22)\\
&\qquad \times \exp\left\{C\Big(\frac{\bar\rho^3\bar\mu}{\underline{\mu}^5}N_0^2C_0 + \frac{M_2^2\bar\rho^3\bar\mu}{\underline{\mu}^3}N_0^2C_0 + \frac{M^2M_2^4\bar\rho^2\bar\mu}{\underline{\mu}}N_0^2\Big)\right\} \\
&\le C\left(\frac{\bar\rho}{\bar\mu}\right)^{\frac34}\left(\frac{M_2^4\bar\rho^6\bar\mu^{\frac{11}{4}}}{\underline{\mu}^6}N_0^{\frac{11}{2}}C_0^{\frac94} + \frac{M^2M_2^8\bar\rho^8\bar\mu^{\frac{15}{4}}}{\underline{\mu}^7}N_0^{\frac{15}{2}}C_0^{\frac94} + \bar\mu^{\frac34}N_0^{\frac32}C_0^{\frac14}\right) \\
&\qquad \times \exp\left\{C\Big(\frac{\bar\rho^3\bar\mu}{\underline{\mu}^5}N_0^2C_0 + \frac{M_2^2\bar\rho^3\bar\mu}{\underline{\mu}^3}N_0^2C_0 + \frac{M^2M_2^4\bar\rho^2\bar\mu}{\underline{\mu}}N_0^2\Big)\right\},
\end{aligned}
$$

which completes the proof of the Theorem 1.17. □

Lemma 1.18. *Suppose (ρ, u, P) is the unique local strong solution to* (1.1.1)-(1.1.4) *on $\Omega\times[0,T]$, with the initial data (ρ_0, u_0), and it satisfies the assumptions* (1.3.6) *as in Theorem 1.13. Then for any $r\in(3,\max\{4,q\})$,*

$$
\int_0^T \|\nabla u\|_{L^\infty}dt \le C_*(M,\bar\rho,\underline{\mu},\bar\mu,N_0,C_0), \qquad (1.3.23)
$$

provided (1.3.2) *holds. Here*

$$
\begin{aligned}
C_*(M,\bar\rho,\underline{\mu},\bar\mu,N_0,C_0) := C_2\cdot\Bigg[&M_r\bar\rho^{\frac{5r-6}{4r}}\underline{\mu}^{-\frac{3(r-2)}{4r}}\left(\left(1+\frac{\bar\rho}{\bar\mu}\right)^{\frac34}\Theta_1\right)^{\frac12}\cdot\exp\{C\Theta_2\} \\
&+\frac{\bar\mu^{(2r-3)/r}}{\underline{\mu}^{3(r-1)/r}}(M_r\bar\rho)^{\frac{5r-6}{r}}N_0^{\frac{4r-6}{r}}C_0\Bigg].
\end{aligned}
$$

Proof. By virtue of Lemma 1.9, one has for $r\in(3,\max\{4,q\})$

$$
\begin{aligned}
\|\nabla u\|_{W^{1,r}} \le& CM_r(\|\rho u_t\|_{L^r} + \|\rho u\cdot\nabla u\|_{L^r}) \\
\le& CM_r\Big(\|\rho u_t\|_{L^2}^{\frac{6-r}{2r}}\cdot\|\rho u_t\|_{L^6}^{\frac{3(r-2)}{2r}} + \bar\rho\|u\|_{L^6}\|\nabla u\|_{L^{6r/(6-r)}}\Big) \\
\le& CM_r\Big(\|\rho u_t\|_{L^2}^{\frac{6-r}{2r}}\cdot\|\rho u_t\|_{L^6}^{\frac{3(r-2)}{2r}} + \bar\rho\|\nabla u\|_{L^2}^{\frac{6(r-1)}{5r-6}}\cdot\|\nabla u\|_{W^{1,r}}^{\frac{4r-6}{5r-6}}\Big).
\end{aligned}
$$

Applying Young's inequality and Sobolev inequality,

$$
\|\nabla u\|_{W^{1,r}} \le CM_r\bar\rho^{\frac{5r-6}{4r}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\cdot\|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}} + CM_r^{\frac{5r-6}{r}}\bar\rho^{\frac{5r-6}{r}}\|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}}.
$$

Hence,

$$\begin{aligned}
&\int_0^T \|\nabla u\|_{L^\infty} dt \\
&\leq C \int_0^T \|\nabla u\|_{W^{1,r}} dt \\
&\leq C \int_0^T \left(M_r \bar{\rho}^{\frac{5r-6}{4r}} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2r}} \cdot \|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}} + M_r^{\frac{5r-6}{r}} \bar{\rho}^{\frac{5r-6}{r}} \|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}} \right) dt.
\end{aligned}$$

Denote $\sigma(T) = \min\{1, T\}$, for $T \geq 0$, then according to Theorem 1.17,

$$\begin{aligned}
&\int_0^T \|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}} dt \\
&= \int_0^{\sigma(T)} \|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}} dt + \int_{\sigma(T)}^T \|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}} dt \\
&\leq C(\sup_{t\in[0,\sigma(T)]} t^{\frac{5}{8}} \|\sqrt{\rho} u_t\|_{L^2})^{\frac{6-r}{2r}} \left(\int_0^{\sigma(T)} t^{\frac{5}{4}} \|\nabla u_t\|_{L^2}^2 dt \right)^{\frac{3(r-2)}{4r}} \left(\int_0^{\sigma(T)} t^{-\frac{5r}{2(r+6)}} dt \right)^{\frac{r+6}{4r}} \\
&\quad + C(\sup_{t\in[\sigma(T),T]} t \|\sqrt{\rho} u_t\|_{L^2})^{\frac{6-r}{2r}} \left(\int_{\sigma(T)}^T t^2 \|\nabla u_t\|_{L^2}^2 dt \right)^{\frac{3(r-2)}{4r}} \left(\int_{\sigma(T)}^T t^{-\frac{4r}{r+6}} dt \right)^{\frac{r+6}{4r}} \\
&\leq C \underline{\mu}^{-\frac{3(r-2)}{4r}} \Theta_1^{\frac{1}{2}} \cdot \exp\{C\Theta_2\} + C \underline{\mu}^{-\frac{3(r-2)}{4r}} \left(\frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{3}{8}} \Theta_1^{\frac{1}{2}} \cdot \exp\{C\Theta_2\} \\
&\leq C \underline{\mu}^{-\frac{3(r-2)}{4r}} \left(1 + \frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{3}{8}} \Theta_1^{\frac{1}{2}} \cdot \exp\{C\Theta_2\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^T \|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}} dt &\leq \sup_{t\in[0,T]} \|\nabla u\|_{L^2}^{\frac{(4r-6)}{r}} \int_0^T \|\nabla u\|_{L^2}^2 dt \\
&\leq C \cdot \left(\frac{\bar{\mu}}{\underline{\mu}} \right)^{(2r-3)/r} N_0^{\frac{4r-6}{r}} \cdot \frac{C_0}{\underline{\mu}} \\
&\leq C \cdot \frac{\bar{\mu}^{(2r-3)/r}}{\underline{\mu}^{3(r-1)/r}} N_0^{\frac{4r-6}{r}} C_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^T \|\nabla u\|_{L^\infty} dt \\
&\leq C_2 \left[M_r \bar{\rho}^{\frac{5r-6}{4r}} \underline{\mu}^{-\frac{3(r-2)}{4r}} \left(\left(1 + \frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{3}{4}} \Theta_1 \right)^{\frac{1}{2}} \cdot \exp\{C\Theta_2\} + \frac{\bar{\mu}^{(2r-3)/r}}{\underline{\mu}^{3(r-1)/r}} (M_r \bar{\rho})^{\frac{5r-6}{r}} N_0^{\frac{4r-6}{r}} C_0 \right] \\
&=: C_*(M, \bar{\rho}, \underline{\mu}, \bar{\mu}, N_0, C_0).
\end{aligned}$$

□

Theorem 1.19. *Suppose* (ρ, u, P) *is the unique local strong solution to* (1.1.1)-(1.1.4) *on* $\Omega \times [0, T]$*, with the initial data* (ρ_0, u_0)*, and it satisfies the assumption* (1.3.2)*. There exists a* *positive number ϵ_0, which is independent of $\bar{\rho}, \underline{\mu}, \bar{\mu}, M, N_0$ and C_0, such that if* (1.1.11) *holds then*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho(t))\|_{L^q} \leq 2M, \tag{1.3.24}$$

and

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_{L^q} \leq 2\|\nabla\rho_0\|_{L^q}.$$

Proof. Consider the x_i derivative of the equation for $\mu(\rho)$,

$$(\partial_i\mu(\rho))_t + (\partial_i u\cdot\nabla)\mu(\rho) + u\cdot\nabla\partial_i\mu(\rho) = 0.$$

It implies that for every $t \in [0,T]$,

$$\begin{aligned}\|\nabla\mu(\rho)(t)\|_{L^q} \leq& \|\nabla\mu(\rho_0)\|_{L^q}\cdot\exp\left\{\int_0^t \|\nabla u\|_{L^\infty} ds\right\}\\ \leq& \|\nabla\mu(\rho_0)\|_{L^q}\cdot\exp\left\{C_*(M,\bar{\rho},\underline{\mu},\bar{\mu},N_0)\right\}.\end{aligned}$$

Choose some small positive ϵ_0, satisfying

$$C_1\frac{\bar{\mu}}{\underline{\mu}^3}(M_2^2 + M^4M_2^6)\bar{\rho}^3N_0^2C_0 \leq \epsilon_0,$$

and

$$C_*(M,\bar{\rho},\underline{\mu},\bar{\mu},N_0) \leq \epsilon_0.$$

In view of Lemma 1.18,

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho(t))\|_{L^q} \leq 2M.$$

Similarly,

$$\|\nabla\rho(t)\|_{L^q} \leq \|\nabla\rho_0\|_{L^q}\cdot\exp\left\{\int_0^t \|\nabla u\|_{L^\infty} ds\right\} \leq 2\|\nabla\rho_0\|_{L^q}.$$

Therefore, Theorem 1.19 is proved. □

Combining Theorem 1.13 and Theorem 1.19, one immediately arrives at the desired result of Proposition 1.10.

1.3.2 Extension of local strong solution

With the a priori estimates in Subsection 1.3.1 in hand, we are now in a position to prove the Theorem 1.3.

According to Theorem 1.1, there exists a $T_* > 0$ such that the density-dependent Navier-Stokes equations (1.1.1)-(1.1.4) has a unique local strong solution (ρ, u, P) on $[0, T_*]$. We plan to extend the local solution to a global one.

Since $\|\nabla\mu(\rho_0)\|_{L^q} = M < 4M$, and due to the continuity of $\nabla\mu(\rho)$ in L^q and ∇u in L^2, there exists $T_1 \in (0, T_*)$ such that $\sup_{0\leq t\leq T_1}\|\nabla\mu(\rho(t))\|_{L^q} \leq 4M$, and at the same time, $\sup_{0\leq t\leq T_1}\|\nabla u(t)\|_{L^2} \leq 2(\bar{\mu}/\underline{\mu})^{1/2}\|\nabla u_0\|_{L^2}$. Set

$$T^* = \sup\{T|(\rho,u,P) \text{ is a strong solution to (1.1.1)} - \text{(1.1.4) on } [0,T]\},$$

$$T_1^* = \sup\big\{T|(\rho, u, P) \text{ is a strong solution to (1.1.1)} - \text{(1.1.4) on } [0,T],$$
$$\sup_{0\le t\le T} \|\nabla\mu(\rho(t))\|_{L^q} \le 4M, \sup_{0\le t\le T} \|\nabla u(t)\|_{L^2} \le 2(\bar{\mu}/\underline{\mu})^{1/2}\|\nabla u_0\|_{L^2}\big\}.$$

Then $T_1^* \geq T_1 > 0$. Recalling Proposition 1.10, it is easy to verify

$$T^* = T_1^*, \tag{1.3.25}$$

provided that (1.1.11) holds as assumed.

We claim that $T^* = \infty$. Otherwise, assume that $T^* < \infty$. By virtue of Proposition (1.10), for every $t \in [0, T^*)$, it holds that

$$\|\nabla\rho(t)\|_{L^q} \le 2\|\nabla\rho_0\|_{L^q}, \qquad \|\nabla u(t)\|_{L^2} \le \sqrt{2(\bar{\mu}/\underline{\mu})}N_0, \tag{1.3.26}$$

which contradicts the blowup criterion (1.1.8). Hence we finish the proof of Theorem 1.3.

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Chapter 2

Global strong solution to the 3D inhomogeneous MHD equations with density-dependent viscosity and resistivity

This chapter is devoted to studying an initial boundary value problem for the three dimensional inhomogeneous incompressible magnetohydrodynamic equations with density-dependent viscosity and resistivity coefficients over a bounded smooth domain. Global in time unique strong solution is proved to exist when the L^2 norms of initial vorticity and current density are both suitably small with arbitrary large initial mass density, and the vacuum of initial density is also allowed.

Keywords: magnetohydrodynamics; strong solution; vacuum; time-weighted energy estimates

2.1 Introduction and main result

The magnetohydrodynamic equations (in short, MHD) are usually used to describe the motion of electrically conducting fluids under the effect of electromagnetic field. In particular, for the study of compound of several incompressible immiscible electrically conducting fluids without surface tension, the following density-dependent MHD equations act as a model on some bounded smooth domain $\Omega \subset \mathbb{R}^3$.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) d) + \nabla P - (\nabla \times B) \times B = 0, \\ \partial_t B - \nabla \times (u \times B) + \nabla \times (\lambda(\rho) \nabla \times B) = 0, \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{2.1.1}$$

Here ρ, u, P and B denote the density, velocity field, pressure of the fluid and magnetic field respectively. As in the first chapter,

$$d = \frac{1}{2}\Big[\nabla u + (\nabla u)^T\Big]$$

is the deformation tensor, where ∇u is the gradient matrix $(\partial u_i/\partial x_j)$ and $(\nabla u)^T$ is its transpose. $\mu = \mu(\rho)$ and $\lambda = \lambda(\rho)$ stand for the viscosity and resistivity coefficients of fluid respectively, and are both functions of density ρ. In this chapter, they are assumed to satisfy

$$\mu \in C^1[0,\infty), \quad \text{and } \mu \geq \underline{\mu} > 0 \quad \text{on } [0,\infty) \tag{2.1.2}$$

for some positive constant $\underline{\mu}$, and

$$\lambda \in C^1[0,\infty), \quad \text{and } \lambda \geq \underline{\lambda} > 0 \quad \text{on } [0,\infty) \tag{2.1.3}$$

for some positive constant $\underline{\lambda}$. The positive resistivity coefficient $\lambda(\rho)$ represents the magnetic diffusivity which is inversely proportional to the electrical conductivity coefficient in physics.

We focus on the system (2.1.1)-(2.1.3) with the initial boundary conditions:

$$u = 0, \quad B \cdot n = 0, \quad (\nabla \times B) \times n = 0, \quad \text{on } \partial\Omega \times [0,T), \tag{2.1.4}$$

$$(\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0) \quad \text{in } \Omega, \tag{2.1.5}$$

where n is the unit outward normal vector to $\partial\Omega$.

The inhomogeneous incompressible MHD system (2.1.1) is a combination of the inhomogeneous Navier-Stokes equations of fluid mechanics and the Maxwell equations of electromagnetism. And it studies the dynamics of electrically conducting fluids and the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field. Let us recall some known results on inhomogeneous incompressible MHD system (2.1.1). When the initial density has a positive lower bound, Gerbeau, Le Bris [6] and Desjardins, Le Bris [4] studied the global existence of weak solutions of finite energy in the whole space or in the torus respectively. For the constant viscosity, Chen et al. [3] established the local strong solution. Later, Huang and Wang [8] extended the local solution to global in 2D in presence of vacuum. For the density-dependent viscosity, Wu [13] established the local strong solution by imposing some similar compatibility condition. Global existence of strong solutions with small initial data in some Besov spaces was considered by Abidi and Paicu [1]. Moreover, they allowed variable viscosity and conductivity coefficients, but required an essential assumption that there is no vacuum (more precisely, the initial data is close to a constant state).

First we give a local existence result which concerns the local existence of strong solution to the MHD system (2.1.1)-(2.1.5) due to Wu [13].

Theorem 2.1. *Assume that the initial data (ρ_0, u_0, B_0) satisfies the regularity condition*

$$0 \leq \rho_0 \in W^{1,q}, \quad 3 < q < \infty, \quad u_0 \in H_0^1 \cap H^2, \quad B_0 \in H^2 \tag{2.1.6}$$

with $\mathrm{div} u_0 = \mathrm{div} B_0 = 0$, *and the compatibility condition*

$$-\mathrm{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) - (B_0 \cdot \nabla) B_0 + \nabla P_0 = \rho_0^{1/2} g, \tag{2.1.7}$$

for some $(P_0, g) \in H^1 \times L^2$. *Then there exist a small time* T *and a unique strong solution* (ρ, u, P, B) *to the initial boundary value problem* (2.1.1)-(2.1.5) *such that*

$$\begin{aligned}
&\rho \in C([0,T); W^{1,q}), \quad \nabla u, \nabla B, P \in C([0,T); H^1) \cap L^2(0,T; W^{1,s}),\\
&\rho_t \in C([0,T); L^q), \quad \sqrt{\rho} u_t \in L^\infty(0,T; L^2), \quad u_t \in L^2(0,T; H_0^1),\\
&B_t \in L^\infty(0,T; L^2) \cap L^2(0,T; H^1),
\end{aligned} \tag{2.1.8}$$

for any s *with* $1 \le s < q$.

Our result in this chapter proves the existence of global strong solution for MHD system (2.1.1)-(2.1.5), provided $\|\nabla u_0\|_{L^2}$ and $\|\nabla B_0\|_{L^2}$ are both suitably small.

Theorem 2.2. *Assume that the initial data* (ρ_0, u_0, B_0) *satisfies* (2.1.6)-(2.1.7), *and* $0 \le \rho_0 \le \bar{\rho}$. *Then there exists some small positive constant* ϵ_0 *depending on* $\Omega, q, \bar{\rho}, \bar{\mu} := \sup_{[0,\bar{\rho}]} \mu(\rho), \underline{\mu}$, $\bar{\lambda} := \sup_{[0,\bar{\rho}]} \lambda(\rho), \underline{\lambda}$, $\|\nabla \mu(\rho_0)\|_{L^q}$ *and* $\|\nabla \lambda(\rho_0)\|_{L^q}$, *such that if*

$$\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2} \le \epsilon_0,$$

then the initial boundary value problem (2.1.1)-(2.1.5) *admits a unique global strong solution* (ρ, u, P, B), *with*

$$\begin{aligned}
&\rho \in C([0,\infty), W^{1,q}), \quad \nabla u, \nabla B, P \in C([0,\infty), H^1) \cap L^2_{loc}(0,\infty; W^{1,s}),\\
&\rho_t \in C([0,\infty), L^q), \quad \sqrt{\rho} u_t \in L^\infty_{loc}(0,\infty; L^2), \quad u_t \in L^2_{loc}(0,\infty; H_0^1),\\
&B_t \in L^\infty_{loc}(0,\infty; L^2) \cap L^2_{loc}(0,\infty; H^1),
\end{aligned} \tag{2.1.9}$$

for any s *with* $1 \le s < q$.

Remark 2.3. We use $\omega = \nabla \times u$ and $j = \nabla \times B$ to represent the vorticity and the current density, respectively. And it is easy to show that the L^2 norm of gradient is equal to the L^2 norm of vorticty for a divergence free vector function (See also Lemma 2.7). Then Theorem 2.2 implies that initial boundary value problem for MHD system (2.1.1)-(2.1.5) admits a unique global strong solution provided that the L^2 norms of initial vorticity and current density are both suitably small.

Remark 2.4. Motivated by our work in chapter 1 for Navier-Stokes model, we can also obtain the global in time strong solution for MHD system, provided that the initial energy $E_0 = \int \frac{1}{2}(\rho_0|u_0|^2 + |B_0|^2)dx$ is small sufficiently. In fact, this conclusion has been proved by Yu et. al. [14], which can be seen as an extension of our work in chapter 1.

For the magnetohydrodynamic model, some additional difficulties will arise since the magnetic force $(\nabla \times B) \times B$ and the convection term $\nabla \times (u \times B)$ need to be dealt with. In particular, $\|\sqrt{\rho} u_t\|_{L^2}$ is not equivalent to the term $\|u_t\|_{L^2}$ when the vacuum is allowed, therefore the coupling of u and B is a trouble for us. Thanks to the assumption of smallness of initial data, we can overcome these difficulties. And in our analysis, we can also treat the case that the resistivity coefficient is also a function of density since the density satisfies a transport equation.

The rest of this chapter is organized as follows. In Section 2.2, we first state some elementary facts and useful analytic tools which will be needed in later analysis. Then two lemmas are given for estimating the higher order derivatives of velocity and magnetic fields. In Section 2.3, some a priori estimates are derived for MHD equations and the global existence of strong solution is obtained.

2.2 Preliminaries

2.2.1 Notations and general inequalities

Ω is a smooth bounded domain in $\mathbb{R}^3$. For notations simplicity below, we omit the integration domain Ω. And for $1 \le r \le \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r = L^r(\Omega), \quad W^{k,r} = \{f \in L^r : \nabla^k f \in L^r\},$$

$$H^k = W^{k,2}, \quad C_{0,\sigma}^\infty = \{f \in (C_0^\infty)^3 : \mathrm{div} f = 0\}.$$

$$H_0^1 = \overline{C_0^\infty}, \quad H_{0,\sigma}^1 = \overline{C_{0,\sigma}^\infty}, \ \text{ closure in the norm of } H^1.$$

The following Gagliardo-Nirenberg inequality will be also used frequently.

Lemma 2.5 (Gagliardo-Nirenberg inequality)**.** *Let Ω be a domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For $p \in [2,6], q \in (1,\infty)$ and $r \in (3,\infty)$, there exists some generic constants $C > 0$ that may depend on q and r such that for $f \in H^1$ satisfying $f|_{\partial\Omega} = 0$, or $\int_\Omega f dx = 0$ and $g \in L^q \cap D^{1,r}$, we have*

$$\|f\|_{L^p}^p \le C\|f\|_{L^2}^{(6-p)/2}\|\nabla f\|_{L^2}^{(3p-6)/2}, \tag{2.2.1}$$

$$\|g\|_{L^\infty} \le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2.2}$$

See the proof of this lemma in Ladyzhenskaya et al. [11, P. 62](see also Nirenberg [12, P. 125]).

Remark 2.6. Under the assumption that $\partial\Omega$ is smooth, we have the following fact which is a consequence of boundary condition $B \cdot n|_{\partial\Omega} = 0$ and divergence free property of magnetic field B.

$$\int_\Omega B dx = \int_\Omega \mathrm{div}(x \otimes B) dx = \int_{\partial\Omega} x(B \cdot n) dS = 0.$$

Here $x \otimes B$ is a matrix with i, j component $x_i B_j$. This observation is borrowed from Antontsev et al. [2, P. 27].

In our later analysis, the following elementary fact is also often used.

Lemma 2.7. *Let u, B belong to Sobolev space H_0^1, and $\mathrm{div} u = \mathrm{div} B = 0$, then the following equalities hold.*

$$2\|d\|_{L^2}^2 = \|\nabla u\|_{L^2}^2, \qquad \|\nabla \times B\|_{L^2}^2 = \|\nabla B\|_{L^2}^2.$$

Here $d = \frac{1}{2}[\nabla u + (\nabla u)^T]$.

2.2.2 Two lemmas on higher order estimates on velocity and magnetic fields

This subsection also provides high-order a priori estimates of velocity field u, which has played an important role in the first chapter for the Navier-Stokes model. To make this chapter self-contained, we write the following lemma again.

Lemma 2.8. *Assume that $\rho \in W^{1,q}, 3 < q < \infty$, and $0 \le \rho \le \bar{\rho}$. Let $(u, P) \in H^1_{0,\sigma} \times L^2$ be the unique weak solution to the boundary value problem*

$$-\operatorname{div}(2\mu(\rho)d) + \nabla P = F, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad \text{and} \int \frac{P}{\mu(\rho)} dx = 0, \tag{2.2.3}$$

where $d = \frac{1}{2}[\nabla u + (\nabla u)^T]$ and

$$\mu \in C^1[0, \infty), \ \ \underline{\mu} \le \mu(\rho) \le \bar{\mu} \text{ on } [0, \bar{\rho}].$$

Then we have the following regularity results:

(1) If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and

$$\|u\|_{H^2} + \|P/\mu(\rho)\|_{H^1} \le C \left(\frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}^{\frac{1}{\theta_2}+2}} \|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_2}} \right) \|F\|_{L^2}, \tag{2.2.4}$$

where θ_2 satisfies

$$\frac{1}{2} - \frac{1}{q} = \frac{\theta_2}{3} + \frac{1}{6}, \quad i.e.\ \theta_2 = \frac{q-3}{q}.$$

(2) If $F \in L^r$ for some $r \in (2, q)$, then $(u, P) \in W^{2,r} \times W^{1,r}$ and

$$\|u\|_{W^{2,r}} + \|P/\mu(\rho)\|_{W^{1,r}} \le C \left(\frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}^{\frac{1}{\theta_r}+2}} \|\nabla[\mu(\rho)]\|_{L^q}^{\frac{1}{\theta_r}} \right) \|F\|_{L^r}, \tag{2.2.5}$$

where θ_r satisfies

$$\theta_r = \frac{2r}{5r-6} \cdot \frac{q-3}{q}.$$

Here the constant C in (2.2.4) *and* (2.2.5) *depends on Ω, q, r.*

The proof of Lemma 2.8 has been given by Huang and Wang in [7]. And refer to Lemma 2.1 in their paper.

For the high-order a priori estimates of magnetic field B, we also have the following regularity results which can be derived from the standard elliptic estimates.

Lemma 2.9. *Assume that $\rho \in W^{1,q}, 3 < q < \infty$, and $0 \le \rho \le \bar{\rho}$. Let $B \in H^1_{0,\sigma}$ be the unique weak solution to the boundary value problem*

$$\nabla \times (\lambda(\rho) \nabla \times B) = F, \quad \operatorname{div} B = 0, \ \text{in } \Omega, \quad B \cdot n = 0, \ (\nabla \times B) \times n = 0, \ \text{in } \partial\Omega, \tag{2.2.6}$$

where $\lambda \in C^1[0, \infty), \quad \underline{\lambda} \le \lambda \le \bar{\lambda}, \ \text{on } [0, \bar{\rho}]$.

Then we have the following regularity results:

(1) If $F \in L^2$, then $B \in H^2$ and

$$\|B\|_{H^2} \leq C \left(\frac{1}{\underline{\lambda}} + \frac{1}{\underline{\lambda}^{\frac{1}{\theta_2}+1}} \|\nabla[\lambda(\rho)]\|_{L^q}^{\frac{1}{\theta_2}} \right) \|F\|_{L^2}, \tag{2.2.7}$$

where θ_2 satisfies

$$\frac{1}{2} - \frac{1}{q} = \frac{\theta_2}{3} + \frac{1}{6}, \quad i.e.\ \theta_2 = \frac{q-3}{q}.$$

(2) If $F \in L^r$ for some $r \in (2, q)$, then $B \in W^{2,r}$ and

$$\|B\|_{W^{2,r}} \leq C \left(\frac{1}{\underline{\lambda}} + \frac{1}{\underline{\lambda}^{\frac{1}{\theta_r}+1}} \|\nabla[\lambda(\rho)]\|_{L^q}^{\frac{1}{\theta_r}} \right) \|F\|_{L^r}, \tag{2.2.8}$$

where θ_r satisfies

$$\theta_r = \frac{2r}{5r-6} \cdot \frac{q-3}{q}.$$

Here the constant C in (2.2.7) *and* (2.2.8) *depends on Ω, q, r.*

Proof. For the existence and uniqueness of weak solution, it can be derived from the standard theory of elliptic equation. We give the a priori estimate here. Assume that $F \in L^2$. Multiplying the first equation of (2.2.6) by B and integrate over Ω, then use Poincaré inequality, we have

$$\int \lambda(\rho)|\nabla \times B|^2 dx = \int F \cdot B dx \leq \|F\|_{L^2}\|B\|_{L^2} \leq C\|F\|_{L^2}\|\nabla B\|_{L^2}.$$

Noting that from Lemma 2.7, one has $\|\nabla B\|_{L^2} = \|\nabla \times B\|_{L^2}$, hence

$$\|\nabla B\|_{L^2} \leq C\underline{\lambda}^{-1}\|F\|_{L^2}. \tag{2.2.9}$$

The first equation of (2.2.6) can be re-written as

$$-\triangle B = \frac{F}{\lambda(\rho)} - \frac{(\nabla\lambda(\rho)) \times (\nabla \times B)}{\lambda(\rho)}.$$

By virtue of classical theory of elliptic estimates and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
\|B\|_{H^2} \leq & C\left(\left\| \frac{F}{\lambda(\rho)} \right\|_{L^2} + \left\| \frac{(\nabla\lambda(\rho)) \times (\nabla \times B)}{\lambda(\rho)} \right\|_{L^2} \right) \\
\leq & C(\underline{\lambda}^{-1}\|F\|_{L^2} + \underline{\lambda}^{-1}\|\nabla\lambda(\rho)\|_{L^q}\|\nabla \times B\|_{L^{\frac{2q}{q-2}}}) \\
\leq & C(\underline{\lambda}^{-1}\|F\|_{L^2} + \underline{\lambda}^{-1}\|\nabla\lambda(\rho)\|_{L^q}\|\nabla B\|_{L^2}^{\theta_2}\|\nabla B\|_{H^1}^{1-\theta_2}),
\end{aligned}$$

where θ_2 satisfies

$$\frac{1}{2} - \frac{1}{q} = \frac{1}{2}\theta_2 + \frac{1}{6}(1-\theta_2), \qquad \theta_2 = \frac{q-3}{q}.$$

By Young's inequality,

$$
\begin{aligned}
\|B\|_{H^2} \leq & C(\underline{\lambda}^{-1}\|F\|_{L^2} + \underline{\lambda}^{-\frac{1}{\theta_2}}\|\nabla\lambda(\rho)\|_{L^q}^{\frac{1}{\theta_2}}\|\nabla B\|_{L^2}) \\
\leq & C(\underline{\lambda}^{-1}\|F\|_{L^2} + \underline{\lambda}^{-\frac{1}{\theta_2}-1}\|\nabla\lambda(\rho)\|_{L^q}^{\frac{1}{\theta_2}}\|F\|_{L^2}) \\
\leq & C\left(\frac{1}{\underline{\lambda}} + \frac{1}{\underline{\lambda}^{\frac{1}{\theta_2}+1}}\|\nabla[\lambda(\rho)]\|_{L^q}^{\frac{1}{\theta_2}}\right)\|F\|_{L^2}.
\end{aligned} \tag{2.2.10}
$$

Similarly,

$$
\|B\|_{W^{2,r}} \leq C\left(\frac{1}{\underline{\lambda}} + \frac{1}{\underline{\lambda}^{\frac{1}{\theta_r}+1}}\|\nabla[\lambda(\rho)]\|_{L^q}^{\frac{1}{\theta_r}}\right)\|F\|_{L^r}, \tag{2.2.11}
$$

where

$$
\theta_r = \frac{2r}{5r-6}\cdot\frac{q-3}{q}.
$$

□

2.3 Global existence of strong solution to MHD system

This section is composed of two parts. The first part contains a priori time-weighted estimates of different levels. Upon these estimates, the second part uses a contradiction induction process to extend the local strong solution globally in time. The two parts are presented in Section 2.3.1 and 2.3.2, respectively.

2.3.1 A priori estimates of different levels

In this subsection, we establish some a priori time-weighted estimates. The initial velocity and magneto belong to H^1, but some higher-order estimates independent of time are required. To achieve it, we take some power of time as a weight. The idea is based on the parabolic property of the system. In this subsection, the constant C will denote some positive constant which may depend on Ω, q, but be independent of ρ_0, u_0, B_0, and may change line to line.

First, as the density satisfies the transport equation and making use of the divergence free property of velocity u, one has the following lemma.

Lemma 2.10. *Suppose (ρ, u, P, B) is the unique local strong solution to* (2.1.1)-(2.1.5) *on $\Omega\times[0,T]$, with the initial data (ρ_0, u_0, B_0), then there it holds that*

$$
0 \leq \rho(x,t) \leq \bar{\rho}, \quad \textit{for every } (x,t) \in \Omega\times[0,T].
$$

Next the basic energy inequality of the system (2.1.1)-(2.1.5) reads

Lemma 2.11. *Suppose (ρ, u, P, B) is the unique local strong solution to* (2.1.1)-(2.1.5) *on $\Omega\times[0,T]$, with the initial data (ρ_0, u_0, B_0), then there it holds that*

$$
\begin{aligned}
&\int(\rho|u(t)|^2 + |B(t)|^2)dx + \int_0^t\int(\mu(\rho)|d|^2 + \lambda(\rho)|\nabla\times B|^2)dxds \\
&\leq C\bar{\rho}\|u_0\|_{L^2}^2 + C\|B_0\|_{L^2}^2,
\end{aligned} \tag{2.3.1}
$$

for every $t \in [0, T]$, or in other words,

$$\int (\rho|u(t)|^2 + |B(t)|^2)dx + \int_0^t \int (\underline{\mu}|\nabla u|^2 + \underline{\lambda}|\nabla B|^2)dxds \le C\bar{\rho}\|u_0\|_{L^2}^2 + C\|B_0\|_{L^2}^2, \tag{2.3.2}$$

for every $t \in [0, T]$.

Proof. Multiplying the momentum equation $(2.1.1)_2$ by u and integrating over Ω yield

$$\frac{d}{dt}\int \frac{1}{2}\rho|u(t)|^2 dx + 2\int \mu(\rho)|d|^2 dx = \int (B\cdot\nabla)B\cdot u dx.$$

Multiplying the magneto equation $(2.1.1)_3$ by B and integrating over Ω yield

$$\frac{d}{dt}\int \frac{1}{2}|B(t)|^2 dx + \int \lambda(\rho)|\nabla\times B|^2 dx = \int (B\cdot\nabla)u\cdot B dx.$$

Taking the sum of the above two equalities, and remarking that

$$\int (B\cdot\nabla)B\cdot u dx + \int (B\cdot\nabla)u\cdot B dx = 0,$$

we have

$$\frac{1}{2}\frac{d}{dt}\int (\rho|u(t)|^2 + |B(t)|^2)dx + \int (2\mu(\rho)|d|^2 + \lambda(\rho)|\nabla\times B|^2)dx \le 0.$$

Integrating with respect to time on $[0, t]$ gives

$$\int (\rho|u(t)|^2 + |B(t)|^2)dx + \int_0^t \int (\mu(\rho)|d|^2 + \lambda(\rho)|\nabla\times B|^2)dxds \le C\bar{\rho}\|u_0\|_{L^2}^2 + C\|B_0\|_{L^2}^2,$$

since $\mu(\rho) \ge \underline{\mu}$ and $\lambda(\rho) \ge \underline{\lambda}$, in view of Lemma 2.7, one also has

$$\int (\rho|u(t)|^2 + |B(t)|^2)dx + \int_0^t \int (\underline{\mu}|\nabla u|^2 + \underline{\lambda}|\nabla B|^2)dxds \le C\bar{\rho}\|u_0\|_{L^2}^2 + C\|B_0\|_{L^2}^2.$$

□

Denote

$$M = \|\nabla\mu(\rho_0)\|_{L^q}, \qquad N = \|\nabla\lambda(\rho_0)\|_{L^q},$$

and

$$M_r = \frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\underline{\mu}}\frac{1}{\underline{\mu}^{1/\theta_r+1}}\cdot(4M)^{\frac{1}{\theta_r}}, \quad N_r = \frac{1}{\underline{\lambda}} + \frac{1}{\underline{\lambda}^{1/\theta_r+1}}\cdot(4N)^{\frac{1}{\theta_r}}, \quad r\in[2,q).$$

It is convenient to introduce the sums of M and N, M_r and N_r, which will frequently appear in our calculation.

$$L = M + N, \quad L_r = M_r + N_r, \quad r\in[2,q).$$

Theorem 2.12. *Suppose* (ρ, u, P, B) *is the unique local strong solution to* (2.1.1)-(2.1.5) *on* $\Omega \times [0, T]$*, with the initial data* (ρ_0, u_0, B_0)*, and it satisfies*

(2.3.3) $$\sup_{t\in[0,T]} \|\nabla\mu(\rho(t))\|_{L^q} \le 4M, \quad \sup_{t\in[0,T]} \|\nabla\lambda(\rho(t))\|_{L^q} \le 4N,$$

and

(2.3.4)
$$\sup_{t\in[0,T]} (\|\nabla u(t)\|_{L^2} + \|\nabla B(t)\|_{L^2})^2 \le 16\frac{(\bar\mu + \bar\lambda)(\underline{\mu} + \underline{\lambda})}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})^2 \le 1.$$

There exist positive numbers C_1, C_2*, depending only on* Ω, q *such that if*

(2.3.5) $$C_1 \cdot \left(\frac{\underline{\mu} + \underline{\lambda}}{\underline{\mu}\underline{\lambda}}\right)^2 \left(L_2^2 + L^4 L_2^6\right)(\bar\rho + 1)^4(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})^2 \le \log\frac{4}{3},$$

and

(2.3.6) $$C_2 \cdot \left(\frac{(\bar\mu + \bar\lambda)(\underline{\mu} + \underline{\lambda})}{\underline{\mu}^2\underline{\lambda}^2}\right)^{1/2} (\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2}) \le 1.$$

Then

(2.3.7) $$\begin{aligned} &4\frac{\underline{\mu} + \underline{\lambda}}{\underline{\mu}\underline{\lambda}}\int_0^T (\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^2 dt + \sup_{t\in[0,T]} (\|\nabla u(t)\|_{L^2} + \|\nabla B(t)\|_{L^2})^2 \\ &\le 8\frac{(\bar\mu + \bar\lambda)(\underline{\mu} + \underline{\lambda})}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})^2. \end{aligned}$$

Remark 2.13. In comparison with the corresponding Lemma 3.8 in Huang and Wang [7] for inhomogeneous Navier-Stokes equations, the expression (2.3.5) is different from (3.18) in [7], where the upper and lower bound of viscosity and resistivity coefficients are all involved in (2.3.5). And we find (3.18) in [7] seems not to be correct in their calculation when the viscosity is a function of density. See also the Chapter 1 for the discussion of Navier-Stokes model.

Before proving Theorem 2.12, let us introduce an auxiliary lemma which is a result of the $W^{2,2}$ estimates in the previous Lemma 2.8 and 2.9.

Lemma 2.14. *Suppose* (ρ, u, P, B) *is the unique local strong solution to* (2.1.1)-(2.1.5) *on* $\Omega \times [0, T]$*, with the initial data* (ρ_0, u_0, B_0)*, and it satisfies* (2.3.3)*. Then it holds that*

(2.3.8) $$\begin{aligned} &\|\nabla u\|_{H^1} + \|\nabla B\|_{H^1} \\ &\le CL_2(\sqrt{\bar\rho} + 1)(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2}) + CL_2^2(\bar\rho + 1)^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^3. \end{aligned}$$

Proof. The momentum equations $(2.1.1)_2$ can be written as follows,

(2.3.9) $$-2\text{div}(\mu(\rho)d) + \nabla P = -\rho u_t - (\rho u \cdot \nabla)u + (B \cdot \nabla)B,$$

following from Lemma 2.8 and Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|\nabla u\|_{H^1} \leq & CM_2(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2} + \|B\cdot\nabla B\|_{L^2})\\ \leq & CM_2\|\rho u_t\|_{L^2} + CM_2\bar\rho\|u\|_{L^6}\|\nabla u\|_{L^3} + CM_2\|B\|_{L^6}\|\nabla B\|_{L^3}\\ \leq & CM_2\|\rho u_t\|_{L^2} + CM_2\bar\rho\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}} + CM_2\|\nabla B\|_{L^2}^{\frac{3}{2}}\|\nabla B\|_{H^1}^{\frac{1}{2}}.\end{aligned}$$

The magneto equation $(2.1.1)_3$ can be written as follows,

$$\nabla\times(\lambda(\rho)\nabla\times B) = -B_t - (u\cdot\nabla)B + (B\cdot\nabla)u, \tag{2.3.10}$$

also following from Lemma 2.9 and Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|\nabla B\|_{H^1} \leq & CN_2(\|B_t\|_{L^2} + \|u\cdot\nabla B\|_{L^2} + \|B\cdot\nabla u\|_{L^2})\\ \leq & CN_2(\|B_t\|_{L^2} + \|u\|_{L^6}\|\nabla B\|_{L^3} + \|B\|_{L^6}\|\nabla u\|_{L^3})\\ \leq & CN_2(\|B_t\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla B\|_{L^2}^{\frac{1}{2}}\|\nabla B\|_{H^1}^{\frac{1}{2}} + \|\nabla B\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}),\end{aligned}$$

taking the sum of the above two inequalities and by use of Young's inequality, we get

$$\begin{aligned}\|\nabla u\|_{H^1} + \|\nabla B\|_{H^1} \leq & C(M_2+N_2)(\sqrt{\bar\rho}+1)(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})\\ & + C(M_2+N_2)^2(\bar\rho+1)^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^3.\end{aligned}$$

□

Proof of Theorem 2.12. Multiplying the momentum equation $(2.1.1)_2$ by u_t and integrating over Ω yield

$$\begin{aligned}&\int \rho|u_t|^2dx + \int 2\mu(\rho)d:\nabla u_t dx - \int (B\cdot\nabla B)\cdot u_t dx + \int \rho u\cdot\nabla u\cdot u_t dx\\ &= 0.\end{aligned} \tag{2.3.11}$$

For the third term of left hand side of (2.3.11),

$$\begin{aligned}&\int (B\cdot\nabla B)\cdot u_t dx\\ =&\frac{d}{dt}\int (B\cdot\nabla B)\cdot u dx - \int (B_t\cdot\nabla B)\cdot u dx - \int (B\cdot\nabla B_t)\cdot u dx\\ =&\frac{d}{dt}\int (B\cdot\nabla B)\cdot u dx - \int (B_t\cdot\nabla B)\cdot u dx + \int (B\cdot\nabla u)\cdot B_t dx.\end{aligned} \tag{2.3.12}$$

For the second term of left hand side of (2.3.11), use the fact that

$$\partial_t[\mu(\rho)] + u\cdot\nabla\mu(\rho) = 0,$$

which is a consequence of mass equation and the fact $\mathrm{div} u = 0$. Then

$$\begin{aligned}\int 2\mu(\rho)d:\nabla u_t dx &= \frac{d}{dt}\int 2\mu(\rho)d:\nabla u dx - \int 2\mu(\rho)_t d:\nabla u dx\\ &= \frac{d}{dt}\int 2\mu(\rho)d:\nabla u dx + \int 2u\cdot\nabla\mu(\rho)d:\nabla u dx\\ &= \frac{d}{dt}\int 2\mu(\rho)|d|^2 dx + \int 2u\cdot\nabla\mu(\rho)d:\nabla u dx.\end{aligned}\tag{2.3.13}$$

Multiplying the magneto equation $(2.1.1)_3$ by B_t and integrating over Ω yield

$$\begin{aligned}&\int |B_t|^2 dx + \frac{d}{dt}\int \lambda(\rho)|\nabla\times B|^2 dx\\ &= -\int u\cdot\nabla B\cdot B_t dx + \int B\cdot\nabla u\cdot B_t dx - \int 2u\cdot\nabla\lambda(\rho)|\nabla\times B|^2 dx.\end{aligned}\tag{2.3.14}$$

Here we also used the fact that $\partial_t[\lambda(\rho)] + u\cdot\nabla\lambda(\rho) = 0$.

Thus, adding (2.3.14) to (2.3.11), together with (2.3.12)-(2.3.13) gives

$$\begin{aligned}&\int(\rho|u_t|^2 + |B_t|^2)dx + \frac{d}{dt}K_0\\ &= \int(\rho|u_t|^2 + |B_t|^2)dx + \frac{d}{dt}\int(2\mu(\rho)|d|^2 + \lambda(\rho)|\nabla\times B|^2 - (B\cdot\nabla B)\cdot u)dx\\ &\le \left|\int \rho u\cdot\nabla u\cdot u_t dx\right| + C\int|\nabla\mu(\rho)||u||\nabla u|^2 dx + C\int|\nabla\lambda(\rho)||u||\nabla\times B|^2 dx\\ &\quad + \left|\int(B_t\cdot\nabla B)\cdot u dx\right| + \left|\int(u\cdot\nabla B)\cdot B_t dx\right| + \left|\int B\cdot\nabla u\cdot B_t dx\right|\\ &:= \sum_{i=1}^{6} K_i.\end{aligned}\tag{2.3.15}$$

Here $K_0 = \int(2\mu(\rho)|d|^2 + \lambda(\rho)|\nabla\times B|^2 - (B\cdot\nabla B)\cdot u)dx$.

Applying Gagliardo-Nirenberg inequality,

$$\begin{aligned}K_1 &= \left|\int \rho u\cdot\nabla u\cdot u_t dx\right|\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^6}^2\|\nabla u\|_{L^3}^2\\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\bar{\rho}\|\nabla u\|_{L^2}^3\|\nabla u\|_{H^1},\end{aligned}$$

and

$$\begin{aligned}K_2 + K_3 &= C\int|\nabla\mu(\rho)|\cdot|u|\cdot|\nabla u|^2 dx + C\int|\nabla\lambda(\rho)||u||\nabla\times B|^2 dx\\ &\le C\|\nabla\mu(\rho)\|_{L^3}\|u\|_{L^6}\|\nabla u\|_{L^4}^2 + C\|\nabla\lambda(\rho)\|_{L^3}\|u\|_{L^6}\|\nabla B\|_{L^4}^2\\ &\le C\|\nabla\mu(\rho)\|_{L^3}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{3}{2}} + C\|\nabla\lambda(\rho)\|_{L^3}\|\nabla u\|_{L^2}\|\nabla B\|_{L^2}^{\frac{1}{2}}\|\nabla B\|_{H^1}^{\frac{3}{2}}\\ &\le CM\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{3}{2}} + CN\|\nabla u\|_{L^2}\|\nabla B\|_{L^2}^{\frac{1}{2}}\|\nabla B\|_{H^1}^{\frac{3}{2}}\\ &\le CL(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{\frac{3}{2}}(\|\nabla u\|_{H^1} + \|\nabla B\|_{H^1})^{\frac{3}{2}}.\end{aligned}$$

The remaining terms can be estimated in a similar way,

$$\begin{aligned}K_4 + K_5 &= \left|\int (B_t \cdot \nabla B) \cdot u dx\right| + \left|\int (u \cdot \nabla B) \cdot B_t dx\right| \\ &\le \frac{1}{8}\|B_t\|_{L^2}^2 + C\|u\|_{L^6}^2\|\nabla B\|_{L^3}^2 \\ &\le \frac{1}{8}\|B_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla B\|_{L^2}\|\nabla B\|_{H^1},\end{aligned}$$

and

$$\begin{aligned}K_6 &= \left|\int B \cdot \nabla u \cdot B_t dx\right| \\ &\le \frac{1}{8}\|B_t\|_{L^2}^2 + C\|B\|_{L^6}^2\|\nabla u\|_{L^3}^2 \\ &\le \frac{1}{8}\|B_t\|_{L^2}^2 + C\|\nabla B\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}.\end{aligned}$$

Owing to the assumption (2.3.5), we have

$$\begin{aligned}K_0 &= \int (2\mu(\rho)|d|^2 + \lambda(\rho)|\nabla \times B|^2 - (B \cdot \nabla B) \cdot u) dx \\ &\ge \int (2\mu(\rho)|d|^2 + \lambda(\rho)|\nabla \times B|^2) dx - \|B\|_{L^3}\|\nabla B\|_{L^2}\|u\|_{L^6} \\ &\ge \int (\underline{\mu}|\nabla u|^2 + \underline{\lambda}|\nabla B|^2) dx - C\|\nabla B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla u\|_{L^2} \\ &\ge \underline{\mu}\|\nabla u\|_{L^2}^2 + \underline{\lambda}\|\nabla B\|_{L^2}^2 \\ &\quad - C_2 \cdot \left(\frac{(\bar{\mu}+\bar{\lambda})(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}\right)^{1/2} (\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})\|\nabla B\|_{L^2}\|\nabla u\|_{L^2} \\ &\ge \frac{1}{2}(\underline{\mu}\|\nabla u\|_{L^2}^2 + \underline{\lambda}\|\nabla B\|_{L^2}^2) \\ &\ge \frac{1}{4}\frac{\underline{\mu}\underline{\lambda}}{\underline{\mu}+\underline{\lambda}}(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^2,\end{aligned} \tag{2.3.16}$$

provided that

$$C_2 \cdot \left(\frac{(\bar{\mu}+\bar{\lambda})(\underline{\mu}+\underline{\lambda})}{\underline{\mu}^2\underline{\lambda}^2}\right)^{1/2} (\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2}) \le 1. \tag{2.3.17}$$

With the Lemma 2.14, then

$$
\begin{aligned}
\sum_{i=1}^{6} K_i \leq & \frac{1}{4}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|B_t\|_{L^2}^2) + C\bar{\rho}\|\nabla u\|_{L^2}^3\|\nabla u\|_{H^1} + C\|\nabla u\|_{L^2}^2\|\nabla B\|_{L^2}\|\nabla B\|_{H^1} \\
& + C\|\nabla B\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} + CL(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{\frac{3}{2}}(\|\nabla u\|_{H^1} + \|\nabla B\|_{H^1})^{\frac{3}{2}} \\
\leq & C(\bar{\rho}+1)(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^3 \cdot L_2(\sqrt{\bar{\rho}}+1)(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2}) \\
& + C(\bar{\rho}+1)(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^3 \cdot L_2^2(\bar{\rho}+1)^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^3 \\
& + CL(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{\frac{3}{2}}[L_2(\sqrt{\bar{\rho}}+1)(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})]^{\frac{3}{2}} \\
& + CL(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{\frac{3}{2}}[L_2^2(\bar{\rho}+1)^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^3]^{\frac{3}{2}} \\
& + \frac{1}{4}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|B_t\|_{L^2}^2) \\
\leq & \frac{1}{2}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|B_t\|_{L^2}^2) + C\left(L_2^2 + L^4L_2^6 + LL_2^3\right)(\bar{\rho}+1)^3 \cdot (\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^6.
\end{aligned}
$$

Combining all the obtained estimates, we have

$$
\begin{aligned}
& \int (\rho|u_t|^2 + |B_t|^2)dx + \frac{d}{dt}K_0 \\
& \leq C\left(L_2^2 + L^4L_2^6 + LL_2^3\right)(\bar{\rho}+1)^3(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^6 \\
& \leq C\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\left(L_2^2 + L^4L_2^6\right)(\bar{\rho}+1)^3(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^4 \cdot K_0,
\end{aligned} \tag{2.3.18}
$$

Applying Gronwall's inequality

$$
\begin{aligned}
& 4\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\int_0^T \int (\rho|u_t|^2 + |B_t|^2)dx + \sup_{t\in[0,T]}(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^2 \\
& \leq 4\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}K_0|_{t=0} \cdot \exp\left\{C\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\left(L_2^2 + L^4L_2^6\right)(\bar{\rho}+1)^3\int_0^T(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^4dt\right\} \\
& \leq 6\frac{(\bar{\mu}+\bar{\lambda})(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})^2 \\
& \quad \times \exp\left\{C\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\left(L_2^2 + L^4L_2^6\right)(\bar{\rho}+1)^3\int_0^T(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^4dt\right\}.
\end{aligned}
$$

According to the Lemma 2.11 and the assumption (2.3.3)-(2.3.4),

$$
\begin{aligned}
& \int_0^T(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^4dt \\
& \leq \sup_{t\in[0,T]}(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^2 \cdot \int_0^T(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^2dt \\
& \leq C\frac{(\underline{\mu}+\underline{\lambda})(\bar{\rho}+1)}{\underline{\mu}\underline{\lambda}}(\|u_0\|_{L^2} + \|B_0\|_{L^2})^2 \\
& \leq C\frac{(\underline{\mu}+\underline{\lambda})(\bar{\rho}+1)}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})^2.
\end{aligned} \tag{2.3.19}
$$

Hence, we arrive at

(2.3.20)

$$4\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\int_0^T\int(\rho|u_t|^2+|B_t|^2)dx+\sup_{t\in[0,T]}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2$$
$$\leq 6\cdot\frac{(\bar{\mu}+\bar{\lambda})(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2\cdot\exp\left\{\Pi\cdot(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2\right\}$$

Here, $\Pi=C_1\cdot\left(\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\right)^2\left(L_2^2+L^4L_2^6\right)(\bar{\rho}+1)^4$. Now it is clear that (2.3.7) holds, provided (2.3.5) holds. □

As a byproduct of the estimate in the proof, we have the following time weighted estimate.

Theorem 2.15. *Suppose* (ρ,u,P,B) *is the unique local strong solution to* (2.1.1)-(2.1.5) *on* $\Omega\times[0,T]$, *with the initial data* (ρ_0,u_0,B_0), *and it satisfies the assumptions* (2.3.3)-(2.3.6) *as in Theorem 2.12. Then*

(2.3.21)
$$\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\int_0^T t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2dx+\sup_{t\in[0,T]}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2$$
$$\leq C\frac{(\underline{\mu}+\underline{\lambda})(\bar{\rho}+1)}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2.$$

Proof. Multiplying (2.3.18) by t, as shown in the last proof, one has

(2.3.22)
$$\int t(\rho|u_t|^2+|B_t|^2)dx+\frac{d}{dt}(tK_0)$$
$$\leq K_0+C\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\left(L_2^2+L^4L_2^6\right)(\bar{\rho}+1)^3(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4\cdot tK_0,$$

Applying Gronwall's inequality,

$$4\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\int_0^T t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2dt+\sup_{t\in[0,T]}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2$$
$$\leq 4\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\int_0^T K_0(t)dt\cdot\exp\left\{\Pi\cdot(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2\right\}$$

According to Theorem 2.11 and assumption (2.3.5),

(2.3.23)
$$\int_0^T K_0(t)dt=\int_0^T\int(2\mu(\rho)|d|^2+\lambda(\rho)|\nabla B|^2-(B\cdot\nabla)B\cdot u)dxdt$$
$$\leq C(\bar{\rho}+1)(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2.$$

Hence, owing to the assumption (2.3.4),

$$\begin{aligned}
&\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\int_0^T t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2dx+\sup_{t\in[0,T]}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2\\
&\le C\frac{(\underline{\mu}+\underline{\lambda})(\bar{\rho}+1)}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2\cdot\exp\left\{\Pi\cdot(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2\right\}\\
&\le C\frac{(\underline{\mu}+\underline{\lambda})(\bar{\rho}+1)}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2.
\end{aligned}$$

□

Theorem 2.16. *Suppose (ρ,u,P,B) is the unique local strong solution to* (2.1.1)-(2.1.5) *on $\Omega\times[0,T]$, with the initial data (ρ_0,u_0,B_0), and it satisfies the assumptions* (2.3.3)-(2.3.6), *there exists a positive number C_3 depending only on Ω,q, such that if*

$$C_3\cdot\left(\frac{(\bar{\mu}+\bar{\lambda})(\underline{\mu}+\underline{\lambda})}{\underline{\mu}^2\underline{\lambda}^2}\right)^{1/2}\cdot(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})\le\frac{1}{8}, \tag{2.3.24}$$

then

$$\begin{aligned}
&\sup_{t\in[0,T]}t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2+\int_0^T t(\underline{\mu}\|\nabla u_t\|_{L^2}^2+\underline{\lambda}\|\nabla B_t\|_{L^2}^2)\\
&\le C(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2\cdot\Theta_1\cdot\exp\{C\Theta_2\},
\end{aligned} \tag{2.3.25}$$

and

$$\begin{aligned}
&\sup_{t\in[0,T]}t^2(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2+\int_0^T t^2(\underline{\mu}\|\nabla u_t\|_{L^2}^2+\underline{\lambda}\|\nabla B_t\|_{L^2}^2)\\
&\le C\frac{(\underline{\mu}+\underline{\lambda})(\bar{\rho}+1)}{\underline{\mu}\underline{\lambda}}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2\cdot\Theta_1\cdot\exp\{C\Theta_2\},
\end{aligned} \tag{2.3.26}$$

where

$$\begin{aligned}
\Theta_1&=\frac{L_2^4(\bar{\rho}+1)^8(\underline{\mu}+\underline{\lambda})^2}{\underline{\mu}^3\underline{\lambda}^2}+\frac{L^2L_2^8(\bar{\rho}+1)^{10}(\underline{\mu}+\underline{\lambda})^3}{\underline{\mu}^3\underline{\lambda}^3}+(\bar{\mu}+\bar{\lambda}),\\
\Theta_2&=\frac{(\bar{\rho}+1)^4(\underline{\mu}+\underline{\lambda})^4}{\underline{\mu}^4\underline{\lambda}^4}+\frac{L_2^2(\bar{\rho}+1)^4(\underline{\mu}+\underline{\lambda})}{\underline{\mu}^2\underline{\lambda}}+\frac{L^2L_2^4(\bar{\rho}+1)^2(\bar{\mu}+\bar{\lambda})(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}.
\end{aligned} \tag{2.3.27}$$

Proof. Differentiate the momentum equations $(2.1.1)_2$ with respect to t, one has

$$\begin{aligned}
&\rho u_{tt}+(\rho u)\cdot\nabla u_t-\operatorname{div}(2\mu(\rho)d_t)+\nabla P_t\\
=&-\rho_t u_t-(\rho u)_t\cdot\nabla u+\operatorname{div}(2\mu(\rho)_t d)+(B\cdot\nabla)B_t+(B_t\cdot\nabla)B.
\end{aligned} \tag{2.3.28}$$

Multiplying (2.3.28) by tu_t and integrating over Ω, we get after integration by parts that

$$
\begin{aligned}
&\frac{t}{2}\frac{d}{dt}\int \rho|u_t|^2dx + 2t\int \mu(\rho)|d_t|^2dx \\
&= -t\int \rho_t|u_t|^2dx - t\int(\rho u)_t\cdot\nabla u\cdot u_tdx - t\int 2\mu(\rho)_t\cdot d\cdot\nabla u_tdx \\
&\quad + t\int(B\cdot\nabla)B_t\cdot u_tdx + t\int(B_t\cdot\nabla)B\cdot u_tdx.
\end{aligned}
\tag{2.3.29}
$$

Differentiate the magneto equations $(2.1.1)_3$ with respect to t, one has

$$
\begin{aligned}
&B_{tt} + \nabla\times(\lambda(\rho)\nabla\times B_t) + (u\cdot\nabla)B_t + (u_t\cdot\nabla)B \\
&= -\nabla\times(\lambda(\rho)_t\nabla\times B) + (B\cdot\nabla)u_t + (B_t\cdot\nabla)u.
\end{aligned}
\tag{2.3.30}
$$

Multiplying (2.3.30) by tB_t and integrating over Ω, we get after integration by parts that

$$
\begin{aligned}
&\frac{t}{2}\frac{d}{dt}\int |B_t|^2dx + t\int \lambda(\rho)|\nabla\times B_t|^2dx \\
&= -t\int(u_t\cdot\nabla)B\cdot B_tdx + t\int(B\cdot\nabla)u_t\cdot B_tdx + t\int(B_t\cdot\nabla)u\cdot B_tdx \\
&\quad - t\int \lambda(\rho)_t(\nabla\times B)\cdot(\nabla\times B_t)dx.
\end{aligned}
\tag{2.3.31}
$$

Adding (2.3.31) to (2.3.29), one has

$$
\begin{aligned}
&\frac{t}{2}\frac{d}{dt}\int(\rho|u_t|^2 + |B_t|^2)dx + t\int(2\mu(\rho)|d_t|^2 + \lambda(\rho)|\nabla\times B_t|^2)dx \\
&= -t\int \rho_t|u_t|^2dx - t\int(\rho u)_t\cdot\nabla u\cdot u_tdx - t\int 2\mu(\rho)_t\cdot d\cdot\nabla u_tdx \\
&\quad - t\int \lambda(\rho)_t\nabla\times B\cdot\nabla\times B_tdx - t\int(u_t\cdot\nabla)B\cdot B_tdx \\
&\quad + t\int(B_t\cdot\nabla)B\cdot u_tdx + t\int(B_t\cdot\nabla)u\cdot B_tdx \\
&=: \sum_{i=1}^{7} I_i.
\end{aligned}
\tag{2.3.32}
$$

Let us estimate the terms on the right hand side. First, utilizing the mass equation and Poincaré inequality, one has

$$
\begin{aligned}
I_1 &= -t\int \rho_t|u_t|^2dx \\
&= -2t\int \rho u\cdot\nabla u_t\cdot u_tdx \\
&\le C\bar\rho^{\frac12}t\|\sqrt\rho u_t\|_{L^3}\|\nabla u_t\|_{L^2}\|u\|_{L^6} \\
&\le C\bar\rho^{\frac12}t\|\sqrt\rho u_t\|_{L^2}^{\frac12}\|\sqrt\rho u_t\|_{L^6}^{\frac12}\|\nabla u_t\|_{L^2}\|u\|_{L^6} \\
&\le C\bar\rho^{\frac34}t\|\sqrt\rho u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\|\nabla u\|_{L^2} \\
&\le \frac18\underline{\mu}t\|\nabla u_t\|_{L^2}^2 + C\underline{\mu}^{-3}\bar\rho^3t\|\sqrt\rho u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4.
\end{aligned}
\tag{2.3.33}
$$

Second, utilizing the equation for $\mu(\rho)$,

$$\begin{aligned} I_3 &= -t\int 2\mu(\rho)_t\cdot d\cdot\nabla u_t dx\\ &\le Ct\int |u||\nabla\mu(\rho)||d||\nabla u_t|dx\\ &\le Ct\|\nabla\mu(\rho)\|_{L^3}\|\nabla u_t\|_{L^2}\|d\|_{L^6}\|u\|_{L^\infty}\\ &\le CMt\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1}^2\\ &\le \frac{1}{8}\underline{\mu}t\|\nabla u_t\|_{L^2}^2 + C\frac{M^2}{\underline{\mu}}t\|\nabla u\|_{H^1}^4. \end{aligned} \tag{2.3.34}$$

Similarly, utilizing the equation for $\lambda(\rho)$,

$$\begin{aligned} I_4 &= -t\int \lambda(\rho)_t\cdot(\nabla\times B)\cdot(\nabla\times B_t)dx\\ &\le Ct\int |u||\nabla\lambda(\rho)||\nabla\times B||\nabla\times B_t|dx\\ &\le Ct\|\nabla\lambda(\rho)\|_{L^3}\|\nabla\times B_t\|_{L^2}\|\nabla\times B\|_{L^6}\|u\|_{L^\infty}\\ &\le CNt\|\nabla B_t\|_{L^2}\|\nabla u\|_{H^1}\|\nabla B\|_{H^1}\\ &\le \frac{1}{8}\underline{\lambda}t\|\nabla B_t\|_{L^2}^2 + C\frac{N^2}{\underline{\lambda}}t\|\nabla u\|_{H^1}^2\|\nabla B\|_{H^1}^2. \end{aligned} \tag{2.3.35}$$

It follows from Lemma 2.14 that

(2.3.36)

$$\begin{aligned} I_3+I_4 &= -t\int 2\mu(\rho)_t\cdot d\cdot\nabla u_t dx - t\int\lambda(\rho)_t(\nabla\times B)\cdot(\nabla\times B_t)dx\\ &\le \frac{1}{8}t(\underline{\mu}\|\nabla u_t\|_{L^2}^2+\underline{\lambda}\|\nabla B_t\|_{L^2}^2) + \frac{CL^2(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}L_2^8(\bar\rho+1)^8 t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{12}\\ &\quad + \frac{CL^2(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}L_2^4(\bar\rho+1)^2 t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^4. \end{aligned}$$

Taking account into the mass equation again, we arrive at

$$\begin{aligned} I_2 &= -t\int(\rho u)_t\cdot\nabla u\cdot u_t dx\\ &= -t\int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)dx - t\int\rho u_t\cdot\nabla u\cdot u_t dx\\ &\le t\int\rho|u|\cdot|\nabla u|^2\cdot|u_t|dx + Ct\int\rho|u|^2\cdot|\nabla^2 u|\cdot|u_t|dx\\ &\quad + t\int\rho|u|^2\cdot|\nabla u|\cdot|\nabla u_t|dx + t\int\rho|u_t|^2\cdot|\nabla u|dx\\ &=: I_{21}+I_{22}+I_{23}+I_{24}. \end{aligned} \tag{2.3.37}$$

Hence, it follows from Sobolev embedding inequality, Gagliardo-Nirenberg inequality, and Lemma 2.14 that

(2.3.38)
$$
\begin{aligned}
I_{21} \leq & C\bar{\rho}t\|u_t\|_{L^6}\|u\|_{L^6}\|\nabla u\|_{L^3}^2 \\
\leq & C\bar{\rho}t\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1} \\
\leq & C\bar{\rho}t\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2 L_2(\sqrt{\bar{\rho}}+1)(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2}) \\
& + C\bar{\rho}t\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2 L_2^2(\bar{\rho}+1)^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^3 \\
\leq & \frac{1}{8}\underline{\mu}t\|\nabla u_t\|_{L^2}^2 + \frac{CL_2^2(\bar{\rho}+1)^3}{\underline{\mu}}t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2\|\nabla u\|_{L^2}^4 \\
& + \frac{CL_2^4(\bar{\rho}+1)^6}{\underline{\mu}}t\|\nabla u\|_{L^2}^4(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^6 \\
\leq & \frac{1}{8}\underline{\mu}t\|\nabla u_t\|_{L^2}^2 + \frac{CL_2^2(\bar{\rho}+1)^3}{\underline{\mu}}t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4 \\
& + \frac{CL_2^4(\bar{\rho}+1)^6}{\underline{\mu}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{10}.
\end{aligned}
$$

Similarly,

(2.3.39)
$$
\begin{aligned}
I_{22} \leq & C\bar{\rho}t\|u_t\|_{L^6}\|\nabla^2 u\|_{L^2}\|u\|_{L^6}^2 \\
\leq & C\bar{\rho}t\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1} \\
\leq & \frac{1}{8}\underline{\mu}t\|\nabla u_t\|_{L^2}^2 + \frac{CL_2^2(\bar{\rho}+1)^3}{\underline{\mu}}t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4 \\
& + \frac{CL_2^4(\bar{\rho}+1)^6}{\underline{\mu}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{10},
\end{aligned}
$$

and

(2.3.40)
$$
\begin{aligned}
I_{23} \leq & C\bar{\rho}t\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^6}\|u\|_{L^6}^2 \\
\leq & C\bar{\rho}t\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1} \\
\leq & \frac{1}{8}\underline{\mu}t\|\nabla u_t\|_{L^2}^2 + \frac{CL_2^2(\bar{\rho}+1)^3}{\underline{\mu}}t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4 \\
& + \frac{CL_2^4(\bar{\rho}+1)^6}{\underline{\mu}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{10},
\end{aligned}
$$

and

$$
\begin{aligned}
I_{24} \leq & Ct\|\sqrt{\rho}u_t\|_{L^4}^2\|\nabla u\|_{L^2} \\
\leq & Ct\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\bar{\rho}^{\frac{3}{4}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2} \\
\leq & \frac{1}{8}\underline{\mu}t\|\nabla u_t\|_{L^2}^2 + C\underline{\mu}^{-3}\bar{\rho}^3 t\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4.
\end{aligned}
\tag{2.3.41}
$$

Let us continue to estimate the remaining three terms,
(2.3.42)

$$\begin{aligned}I_5+I_6\leq&\left|t\int(u_t\cdot\nabla)B\cdot B_tdx\right|+\left|t\int(B_t\cdot\nabla)B\cdot u_tdx\right|\\ \leq&Ct\|u_t\|_{L^6}\|\nabla B_t\|_{L^2}\|B\|_{L^3}+Ct\|B_t\|_{L^6}\|\nabla u_t\|_{L^2}\|B\|_{L^3}\\ \leq&C\|\nabla B\|_{L^2}\cdot t\|\nabla u_t\|_{L^2}\|\nabla B_t\|_{L^2}\\ \leq&C_3\cdot\left(\frac{(\bar\mu+\bar\lambda)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}\right)^{1/2}\cdot(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})\cdot t\|\nabla u_t\|_{L^2}\|\nabla B_t\|_{L^2}\\ \leq&\frac{1}{8}t(\underline{\mu}\|\nabla u_t\|_{L^2}^2+\underline{\lambda}\|\nabla B_t\|_{L^2}^2)\end{aligned}$$

provided that

$$C_3\cdot\left(\frac{(\bar\mu+\bar\lambda)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}^2\underline{\lambda}^2}\right)^{1/2}\cdot(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})\leq\frac{1}{8}. \tag{2.3.43}$$

Finally,

$$\begin{aligned}I_7=&t\int(B_t\cdot\nabla)u\cdot B_tdx\\ \leq&Ct\|B_t\|_{L^4}^2\|\nabla u\|_{L^2}\\ \leq&Ct\|B_t\|_{L^2}^{\frac12}\|\nabla B_t\|_{L^2}^{\frac32}\|\nabla u\|_{L^2}\\ \leq&\frac18\underline{\lambda}t\|\nabla B_t\|_{L^2}^2+C\underline{\lambda}^{-3}\|B_t\|_{L^2}^2\|\nabla u\|_{L^2}^4\end{aligned} \tag{2.3.44}$$

Combine all the above estimates (2.3.33)-(2.3.45),

$$\begin{aligned}&\frac{d}{dt}\left(t\int(\rho|u_t|^2+|B_t|^2)dx\right)+\int t(\underline{\mu}|\nabla u_t|^2+\underline{\lambda}|\nabla B_t|^2)dx\\ &\leq C\frac{(\underline{\mu}+\underline{\lambda})^3(\bar\rho+1)^3}{\underline{\mu}^3\underline{\lambda}^3}t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4\\ &+\frac{CL_2^2(\bar\rho+1)^3}{\underline{\mu}}t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4\\ &+\frac{CL_2^4(\bar\rho+1)^6}{\underline{\mu}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{10}+\int(\rho|u_t|^2+|B_t|^2)dx\\ &+\frac{CL^2L_2^4(\bar\rho+1)^2(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^4\\ &+\frac{CL^2L_2^8(\bar\rho+1)^8(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{12}.\end{aligned} \tag{2.3.45}$$

Applying Gronwall's inequality,

$$\begin{aligned}
&\sup_{t\in[0,T]} t\int(\rho|u_t|^2+|B_t|^2)dx+\int_0^T t(\underline{\mu}\|\nabla u_t\|_{L^2}^2+\underline{\lambda}\|\nabla B_t\|_{L^2}^2)dt\\
&\le C\bigg[\int_0^T\Big(\frac{L_2^4(\bar\rho+1)^6}{\underline{\mu}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{10}\\
&\quad+\frac{L^2L_2^8(\bar\rho+1)^8(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{12}+(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2\Big)dt\bigg]\\
&\quad\times\exp\bigg\{\int_0^T\Big[\Big(\frac{C(\underline{\mu}+\underline{\lambda})^3(\bar\rho+1)^3}{\underline{\mu}^3\underline{\lambda}^3}+\frac{CL_2^2(\bar\rho+1)^3}{\underline{\mu}}\Big)(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4\\
&\qquad+\frac{CL^2L_2^4(\bar\rho+1)^2(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2\Big]dt\bigg\}.
\end{aligned}$$

Taking some previous estimates into account,

$$\begin{aligned}
&\sup_{t\in[0,T]} t\int(\rho|u_t|^2+|B_t|^2)dx+\int_0^T t(\underline{\mu}\|\nabla u_t\|_{L^2}^2+\underline{\lambda}\|\nabla B_t\|_{L^2}^2)dt\\
&\le C\bigg[\int_0^T\Big(\frac{L_2^4(\bar\rho+1)^6}{\underline{\mu}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{10}\\
&\quad+\frac{L^2L_2^8(\bar\rho+1)^8(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{12}+(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2\Big)dt\bigg]\\
&\times\exp\bigg\{C\Big(\frac{(\underline{\mu}+\underline{\lambda})^4(\bar\rho+1)^4}{\underline{\mu}^4\underline{\lambda}^4}+\frac{L_2^2(\bar\rho+1)^4(\underline{\mu}+\underline{\lambda})}{\underline{\mu}^2\underline{\lambda}}+\frac{L^2L_2^4(\bar\rho+1)^2(\underline{\mu}+\underline{\lambda})(\bar\mu+\bar\lambda)}{\underline{\mu}\underline{\lambda}}\Big)\bigg\},
\end{aligned}$$

According to Theorem 2.15 and the assumption (2.3.4),

$$\begin{aligned}
&\int_0^T t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{10}dt\\
&\le\sup_{t\in[0,T]} t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2\cdot\sup_{t\in[0,T]}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^6\\
&\quad\times\int_0^T(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2dt\\
&\le\frac{C(\bar\rho+1)^2(\underline{\mu}+\underline{\lambda})^2}{\underline{\mu}^2\underline{\lambda}^2}(\|u_0\|_{L^2}+\|B_0\|_{L^2})^2\\
&\le\frac{C(\bar\rho+1)^2(\underline{\mu}+\underline{\lambda})^2}{\underline{\mu}^2\underline{\lambda}^2}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^T t(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{12}dt\\
&\le\frac{C(\bar\rho+1)^2(\underline{\mu}+\underline{\lambda})^2}{\underline{\mu}^2\underline{\lambda}^2}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2.
\end{aligned}$$

And by virtue of Theorem 2.12,

$$\int_0^T (\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^2 dt \leq 2(\bar{\mu} + \bar{\lambda})(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})^2.$$

Hence,

(2.3.46)

$$\begin{aligned}
&\sup_{t\in[0,T]} t\int (\rho|u_t|^2 + |B_t|^2)dx + \int_0^T t(\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \underline{\lambda}\|\nabla B_t\|_{L^2}^2)dt \\
&\leq C(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})^2 \\
&\times \Big(\frac{L_2^4(\bar{\rho}+1)^8(\underline{\mu}+\underline{\lambda})^2}{\underline{\mu}^3\underline{\lambda}^2} + \frac{L^2L_2^8(\bar{\rho}+1)^{10}(\underline{\mu}+\underline{\lambda})^3}{\underline{\mu}^3\underline{\lambda}^3} + (\bar{\mu}+\bar{\lambda})\Big) \\
&\times \exp\Big\{C\Big(\frac{(\underline{\mu}+\underline{\lambda})^4(\bar{\rho}+1)^4}{\underline{\mu}^4\underline{\lambda}^4} + \frac{L_2^2(\bar{\rho}+1)^4(\underline{\mu}+\underline{\lambda})}{\underline{\mu}^2\underline{\lambda}} + \frac{L^2L_2^4(\bar{\rho}+1)^2(\underline{\mu}+\underline{\lambda})(\bar{\mu}+\bar{\lambda})}{\underline{\mu}\underline{\lambda}}\Big)\Big\}.
\end{aligned}$$

On the other hand, multiplying (2.3.45) by t, one has

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{t^2}{2}\int (\rho|u_t|^2 + |B_t|^2)dx\right) + \int t^2(\underline{\mu}|\nabla u_t|^2 + \underline{\lambda}|\nabla B_t|^2)dx \\
&\leq C\frac{(\underline{\mu}+\underline{\lambda})^3(\bar{\rho}+1)^3}{\underline{\mu}^3\underline{\lambda}^3}t^2(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^4 \\
&+ \frac{CL_2^2(\bar{\rho}+1)^3}{\underline{\mu}}t^2(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^4 \\
&+ \frac{CL_2^4(\bar{\rho}+1)^6}{\underline{\mu}}t^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{10} + t\int (\rho|u_t|^2 + |B_t|^2)dx \\
&+ \frac{CL^2L_2^4(\bar{\rho}+1)^2(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}t^2(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^4 \\
&+ \frac{CL^2L_2^8(\bar{\rho}+1)^8(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}t^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{12}.
\end{aligned}$$

Applying Gronwall's inequality,

$$\begin{aligned}
&\sup_{t\in[0,T]} t^2\int (\rho|u_t|^2 + |B_t|^2)dx + \int_0^T t^2(\underline{\mu}\|\nabla u_t\|_{L^2}^2 + \underline{\lambda}\|\nabla B_t\|_{L^2}^2)dt \\
&\leq C\Big[\int_0^T \Big(\frac{L_2^4(\bar{\rho}+1)^6}{\underline{\mu}}t^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{10} \\
&\quad + \frac{L^2L_2^8(\bar{\rho}+1)^8(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}t^2(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{12} + t(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^2\Big)dt\Big] \\
&\quad \times \exp\Big\{\int_0^T \Big[\Big(\frac{C(\underline{\mu}+\underline{\lambda})^3(\bar{\rho}+1)^3}{\underline{\mu}^3\underline{\lambda}^3} + \frac{CL_2^2(\bar{\rho}+1)^3}{\underline{\mu}}\Big)(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^4 \\
&\quad + \frac{CL^2L_2^4(\bar{\rho}+1)^2(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^2\Big]dt\Big\}.
\end{aligned}$$

According to Theorem 2.14,

$$\begin{aligned}
&\int_0^T t^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{10}dt\\
&\le \sup_{t\in[0,T]} t^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4\cdot \sup_{t\in[0,T]}(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^4\\
&\quad\times\int_0^T(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^2dt\\
&\le \frac{C(\bar\rho+1)^3(\underline{\mu}+\underline{\lambda})^3}{\underline{\mu}^3\underline{\lambda}^3}(\|u_0\|_{L^2}+\|B_0\|_{L^2})^2\\
&\le \frac{C(\bar\rho+1)^3(\underline{\mu}+\underline{\lambda})^3}{\underline{\mu}^3\underline{\lambda}^3}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2.
\end{aligned}$$

Similarly,

$$\int_0^T t^2(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2})^{12}dt\le \frac{C(\bar\rho+1)^3(\underline{\mu}+\underline{\lambda})^3}{\underline{\mu}^3\underline{\lambda}^3}(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2,$$

and

$$\int_0^T t(\|\sqrt{\rho}u_t\|_{L^2}+\|B_t\|_{L^2})^2dt\le C(\bar\rho+1)(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2.$$

Hence,

(2.3.47)

$$\begin{aligned}
&\sup_{t\in[0,T]} t^2\int(\rho|u_t|^2+|B_t|^2)dx+\int_0^T t^2(\underline{\mu}\|\nabla u_t\|_{L^2}^2+\underline{\lambda}\|\nabla B_t\|_{L^2}^2)dt\\
&\le C(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})^2\cdot\Big(\frac{L_2^4(\bar\rho+1)^9(\underline{\mu}+\underline{\lambda})^3}{\underline{\mu}^4\underline{\lambda}^3}+\frac{L^2L_2^8(\bar\rho+1)^{11}(\underline{\mu}+\underline{\lambda})^4}{\underline{\mu}^4\underline{\lambda}^4}+(\bar\rho+1)\Big)\\
&\times\exp\left\{C\left(\frac{(\underline{\mu}+\underline{\lambda})^4(\bar\rho+1)^4}{\underline{\mu}^4\underline{\lambda}^4}+\frac{L_2^2(\bar\rho+1)^4(\underline{\mu}+\underline{\lambda})}{\underline{\mu}^2\underline{\lambda}}+\frac{L^2L_2^4(\bar\rho+1)^2(\underline{\mu}+\underline{\lambda})(\bar\mu+\bar\lambda)}{\underline{\mu}\underline{\lambda}}\right)\right\},
\end{aligned}$$

which completes the proof of the Theorem 2.16. □

Lemma 2.17. *Suppose* (ρ,u,P,B) *is the unique local strong solution to* (2.1.1)-(2.1.5) *on* $\Omega\times[0,T]$, *with the initial data* (ρ_0,u_0,B_0), *and it satisfies the assumptions* (2.3.3)-(2.3.6) *and* (2.3.24), *then for any* $r\in(3,\min\{q,6\})$

(2.3.48)

$$\int_0^T(\|\nabla u\|_{L^\infty}+\|\nabla B\|_{L^\infty})dt\le C(\|\nabla u_0\|_{L^2}+\|\nabla B_0\|_{L^2})\cdot C_4(M,N,\bar\rho,\underline{\mu},\bar\mu,\underline{\lambda},\bar\lambda),$$

where

$$\begin{aligned}
&C_4(M,N,\bar\rho,\underline{\mu},\bar\mu,\underline{\lambda},\bar\lambda)\\
&:=L_r(\bar\rho+1)^{\frac{5r-6}{4r}}\left(\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\right)^{\frac{3(r-2)}{4r}}\left(1+\frac{(\bar\rho+1)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}\right)^{\frac12}\cdot\Theta_1^{\frac12}\cdot\exp\{C\Theta_2\}\\
&\quad+L_r^{\frac{5r-6}{r}}(\bar\rho+1)^{\frac{6(r-1)}{r}}\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}},
\end{aligned}$$

$\Theta_i (i = 1, 2)$ *is given by* (2.3.27).

Proof. By virtue of Lemma 2.8 and 2.9, one has for $r \in (3, \min\{q, 6\})$

$$\begin{aligned}
\|\nabla u\|_{W^{1,r}} \le& CM_r(\|\rho u_t\|_{L^r} + \|\rho u \cdot \nabla u\|_{L^r} + \|B \cdot \nabla B\|_{L^r}) \\
\le& CM_r\Big(\|\rho u_t\|_{L^2}^{\frac{6-r}{2r}} \cdot \|\rho u_t\|_{L^6}^{\frac{3(r-2)}{2r}} + \bar\rho \|u\|_{L^6}\|\nabla u\|_{L^{6r/(6-r)}} \\
&+ \|B\|_{L^6}\|\nabla B\|_{L^{6r/(6-r)}}\Big) \\
\le& CM_r\Big(\|\rho u_t\|_{L^2}^{\frac{6-r}{2r}} \cdot \|\rho u_t\|_{L^6}^{\frac{3(r-2)}{2r}} + \bar\rho \|\nabla u\|_{L^2}^{\frac{6(r-1)}{5r-6}} \cdot \|\nabla u\|_{W^{1,r}}^{\frac{4r-6}{5r-6}} \\
&+ \|\nabla B\|_{L^2}^{\frac{6(r-1)}{5r-6}} \cdot \|\nabla B\|_{W^{1,r}}^{\frac{4r-6}{5r-6}}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla B\|_{W^{1,r}} \le& CN_r(\|B_t\|_{L^r} + \|u \cdot \nabla B\|_{L^r} + \|B \cdot \nabla u\|_{L^r}) \\
\le& CN_r\Big(\|B_t\|_{L^2}^{\frac{6-r}{2r}} \cdot \|B_t\|_{L^6}^{\frac{3(r-2)}{2r}} + \|u\|_{L^6}\|\nabla B\|_{L^{6r/(6-r)}} \\
&+ \|B\|_{L^6}\|\nabla u\|_{L^{6r/(6-r)}}) \\
\le& CN_r\Big(\|B_t\|_{L^2}^{\frac{6-r}{2r}} \cdot \|B_t\|_{L^6}^{\frac{3(r-2)}{2r}} + \|\nabla u\|_{L^2}\|\nabla B\|_{L^2}^{\frac{r}{5r-6}} \cdot \|\nabla B\|_{W^{1,r}}^{\frac{4r-6}{5r-6}} \\
&+ \|\nabla B\|_{L^2}\|\nabla u\|_{L^2}^{\frac{r}{5r-6}} \cdot \|\nabla u\|_{W^{1,r}}^{\frac{4r-6}{5r-6}}\Big).
\end{aligned}$$

Taking the sum of above two inequalities, applying Young's inequality and Sobolev inequality,

$$\begin{aligned}
&\|\nabla u\|_{W^{1,r}} + \|\nabla B\|_{W^{1,r}} \\
\le& CL_r(\bar\rho+1)^{\frac{5r-6}{4r}}(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^{\frac{6-r}{2r}} \cdot (\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})^{\frac{3(r-2)}{2r}} \\
&+ CL_r^{\frac{5r-6}{r}}(\bar\rho+1)^{\frac{5r-6}{r}}(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{\frac{6(r-1)}{r}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^\infty})dt \\
&\le C\int_0^T (\|\nabla u\|_{W^{1,r}} + \|\nabla B\|_{W^{1,r}})dt \\
&\le C\int_0^T \Big[L_r(\bar\rho+1)^{\frac{5r-6}{4r}}(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^{\frac{6-r}{2r}} \cdot (\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})^{\frac{3(r-2)}{2r}} \\
&\quad + L_r^{\frac{5r-6}{r}}(\bar\rho+1)^{\frac{5r-6}{r}}(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{\frac{6(r-1)}{r}}\Big]dt.
\end{aligned}$$

Define $\sigma(T) = \min\{1, T\}$, for $T \geq 0$, then according to Theorem 2.16,

$$\begin{aligned}
&\int_0^T (\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^{\frac{6-r}{2r}} \cdot (\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})^{\frac{3(r-2)}{2r}} dt \\
&= \int_0^{\sigma(T)} (\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^{\frac{6-r}{2r}} \cdot (\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})^{\frac{3(r-2)}{2r}} dt \\
&\quad + \int_{\sigma(T)}^T (\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^{\frac{6-r}{2r}} \cdot (\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})^{\frac{3(r-2)}{2r}} dt \\
&= \int_0^{\sigma(T)} [t^{\frac{1}{2}}(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})]^{\frac{6-r}{2r}} \cdot [t^{\frac{1}{2}}(\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})]^{\frac{3(r-2)}{2r}} t^{-\frac{1}{2}} dt \\
&\quad + \int_{\sigma(T)}^T [t(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})]^{\frac{6-r}{2r}} \cdot [t(\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})]^{\frac{3(r-2)}{2r}} t^{-1} dt \\
&\leq C[\sup_{t\in[0,\sigma(T)]} t(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^2]^{\frac{6-r}{4r}} \\
&\quad \times \left[\int_0^{\sigma(T)} t(\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})^2 dt\right]^{\frac{3(r-2)}{4r}} \left[\int_0^{\sigma(T)} t^{-\frac{2r}{r+6}} dt\right]^{\frac{r+6}{4r}} \\
&\quad + C[\sup_{t\in[\sigma(T),T]} t^2(\|\sqrt{\rho}u_t\|_{L^2} + \|B_t\|_{L^2})^2]^{\frac{6-r}{4r}} \\
&\quad \times \left[\int_{\sigma(T)}^T t^2(\|\nabla u_t\|_{L^2} + \|\nabla B_t\|_{L^2})^2 dt\right]^{\frac{3(r-2)}{4r}} \left[\int_{\sigma(T)}^T t^{-\frac{4r}{r+6}} dt\right]^{\frac{r+6}{4r}} \\
&\leq C\left(\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\right)^{\frac{3(r-2)}{4r}} (\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2}) \left(1 + \frac{(\bar{\rho}+1)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}\right)^{\frac{1}{2}} \Theta_1^{\frac{1}{2}} \exp\{C\Theta_2\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_0^T (\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{\frac{6(r-1)}{r}} dt \\
&\leq \sup_{t\in[0,T]} (\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^{\frac{(4r-6)}{r}} \int_0^T (\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2})^2 dt \\
&\leq \frac{C(\bar{\rho}+1)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}} (\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^\infty}) dt \\
&\leq C(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2}) \Bigg[L_r(\bar{\rho}+1)^{\frac{5r-6}{4r}} \left(\frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}}\right)^{\frac{3(r-2)}{4r}} \left(1 + \frac{(\bar{\rho}+1)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}\right)^{\frac{1}{2}} \\
&\quad \cdot \Theta_1^{\frac{1}{2}} \cdot \exp\{C\Theta_2\} + L_r^{\frac{5r-6}{r}} (\bar{\rho}+1)^{\frac{6(r-1)}{r}} \frac{\underline{\mu}+\underline{\lambda}}{\underline{\mu}\underline{\lambda}} \Bigg] \\
&=: C(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2}) \cdot C_4(M, N, \bar{\rho}, \underline{\mu}, \bar{\mu}, \underline{\lambda}, \bar{\lambda}).
\end{aligned}$$

□

Theorem 2.18. *Suppose* (ρ, u, P, B) *is the unique local strong solution to* (2.1.1)-(2.1.5) *on* $\Omega \times [0,T]$*, with the initial data* (ρ_0, u_0, B_0)*, and it satisfies the assumptions* (2.3.3)-(2.3.6) *and* (2.3.24)*. There exists a positive number* ϵ_0*, depending on* $\Omega, q, M, N, \bar\rho, \underline{\mu}, \bar\mu, \underline{\lambda}$ *and* $\bar\lambda$ *such that if*

$$\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2} \le \epsilon_0,$$

then

$$\sup_{t\in[0,T]} \|\nabla \mu(\rho(t))\|_{L^q} \le 2M, \qquad \sup_{t\in[0,T]} \|\nabla \lambda(\rho(t))\|_{L^q} \le 2N, \tag{2.3.49}$$

and

$$\sup_{t\in[0,T]} \|\nabla \rho\|_{L^q} \le 2\|\nabla \rho_0\|_{L^q}.$$

Proof. Consider the x_i derivative of the equation for $\mu(\rho)$,

$$(\partial_i \mu(\rho))_t + (\partial_i u \cdot \nabla)\mu(\rho) + u \cdot \nabla \partial_i \mu(\rho) = 0.$$

It implies that for every $t \in [0,T]$,

$$\begin{aligned}\|\nabla \mu(\rho)(t)\|_{L^q} \le & \|\nabla \mu(\rho_0)\|_{L^q} \cdot \exp\left\{\int_0^t \|\nabla u\|_{L^\infty} ds\right\} \\ \le & \|\nabla \mu(\rho_0)\|_{L^q} \cdot \exp\left\{C_4(M, N, \bar\rho, \underline{\mu}, \bar\mu, \underline{\lambda}, \bar\lambda)(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})\right\}.\end{aligned}$$

Choose some small positive constant ϵ_0, satisfying

$$16 \frac{(\bar\mu + \bar\lambda)(\underline{\mu} + \underline{\lambda})}{\underline{\mu}\underline{\lambda}} \epsilon_0^2 \le 1, \quad C_1 \cdot \left(\frac{\underline{\mu} + \underline{\lambda}}{\underline{\mu}\underline{\lambda}}\right)^2 (L_2^2 + L^4 L_2^6)(\bar\rho + 1)^4 \cdot \epsilon_0^2 \le \log \frac{4}{3},$$

and

$$C_2 \cdot \left(\frac{(\bar\mu + \bar\lambda)(\underline{\mu} + \underline{\lambda})}{\underline{\mu}^2\underline{\lambda}^2}\right)^{1/2} \epsilon_0 \le 1, \quad C_3 \cdot \left(\frac{(\bar\mu + \bar\lambda)(\underline{\mu} + \underline{\lambda})}{\underline{\mu}^2\underline{\lambda}^2}\right)^{1/2} \epsilon_0 \le \frac{1}{8},$$

and

$$C_4(M, N, \bar\rho, \underline{\mu}, \bar\mu, \underline{\lambda}, \bar\lambda) \cdot \epsilon_0 \le \log 2.$$

In view of Theorem 2.17, if $\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2} \le \epsilon_0$, then $\sup_{t\in[0,T]} \|\nabla \mu(\rho(t))\|_{L^q} \le 2M$.

Similarly,

$$\|\nabla \lambda(\rho(t))\|_{L^q} \le \|\nabla \lambda(\rho_0)\|_{L^q} \cdot \exp\left\{\int_0^t \|\nabla u\|_{L^\infty} ds\right\} \le 2\|\nabla \lambda(\rho_0)\|_{L^q},$$

and

$$\|\nabla \rho(t)\|_{L^q} \le \|\nabla \rho_0\|_{L^q} \cdot \exp\left\{\int_0^t \|\nabla u\|_{L^\infty} ds\right\} \le 2\|\nabla \rho_0\|_{L^q}.$$

Therefore, Theorem 2.18 is proved. □

2.3.2 Extension of local strong solution

With the a priori estimates in Section 2.3.1 in hand, we are now in a position to prove the Theorem 2.2.

According to Theorem 2.1, there exists a $T_* > 0$ such that the density-dependent MHD equations (2.1.1) has a unique local strong solution (ρ, u, B) on $[0, T_*]$. We plan to extend the local solution to a global one.

Since $\|\nabla\mu(\rho_0)\|_{L^q} = M < 4M$, $\|\nabla\lambda(\rho_0)\|_{L^q} = N < 4N$, and due to the continuity of $\nabla\mu(\rho), \nabla\lambda(\rho)$ in L^q and $\nabla u, \nabla B$ in L^2, there exists $T_1 \in (0, T_*)$ such that $\sup_{0\le t\le T_1} \|\nabla\mu(\rho(t))\|_{L^q} \le 4M$, $\sup_{0\le t\le T_1} \|\nabla\lambda(\rho(t))\|_{L^q} \le 4N$ and at the same time, $\sup_{0\le t\le T_1}(\|\nabla u(t)\|_{L^2} + \|\nabla B(t)\|_{L^2}) \le 4\sqrt{\frac{(\bar\mu+\bar\lambda)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}}(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})$. Set

$$T^* = \sup\{T|(\rho, u, B, P) \text{ is a strong solution to (2.1.1)} - \text{(2.1.5) on } [0, T]\},$$

$$\begin{aligned} T_1^* = \sup\Bigg\{ & T|(\rho, u, B, P) \text{ is a strong solution to (2.1.1)} - \text{(2.1.5) on } [0, T], \\ & \sup_{0\le t\le T} \|\nabla\mu(\rho(t))\|_{L^q} \le 4M, \sup_{0\le t\le T_1} \|\nabla\lambda(\rho(t))\|_{L^q} \le 4N, \\ & \sup_{0\le t\le T}(\|\nabla u(t)\|_{L^2} + \|\nabla B(t)\|_{L^2}) \le 4\sqrt{\frac{(\bar\mu+\bar\lambda)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}}(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2})\Bigg\}. \end{aligned}$$

Then $T_1^* \ge T_1 > 0$. Recalling Theorem 2.12 and 2.18, it is easy to verify

$$T^* = T_1^*, \tag{2.3.50}$$

provided $\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2} \le \epsilon_0$ as assumed.

We claim that $T^* = \infty$. Otherwise, assume that $T^* < \infty$. By virtue of Theorem 2.12 and 2.18, for every $t \in [0, T^*)$, it holds that

$$\begin{aligned} & \|\nabla\rho(t)\|_{L^q} \le 2\|\nabla\rho_0\|_{L^q}, \\ & \|\nabla u(t)\|_{L^2} + \|\nabla B(t)\|_{L^2} \le 2\sqrt{2\frac{(\bar\mu+\bar\lambda)(\underline{\mu}+\underline{\lambda})}{\underline{\mu}\underline{\lambda}}}(\|\nabla u_0\|_{L^2} + \|\nabla B_0\|_{L^2}), \end{aligned} \tag{2.3.51}$$

we find that at time T^* the functions $(\rho, u, B)|_{t=T^*} = \lim_{t\to T^*}(\rho, u, B)$ satisfy the conditions imposed on the initial data in the local existence Theorem 2.1. Hence we can take $(\rho, u, B)|_{t=T^*}$ as the initial data at time T^* and by applying Theorem 2.1., we can extend the local solution beyond T^* in time which contradicts the maximality of T^*, thus the strong solution exists globally. This completes the proof of Theorem 2.2.

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Chapter 3

A blow-up criterion of the 3D inhomogeneous Navier-Stokes-Korteweg equations

This chapter is devoted to proving a Serrin type blow-up criterion for the 3D density-dependent Navier-Stokes-Korteweg equations with vacuum. It is shown that if the density and velocity field (ρ, u) satisfy $\|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)} < \infty$ for some $q > 3$, and any (r, s) satisfying $\frac{2}{s} + \frac{3}{r} \leq 1, \ \ 3 < r \leq \infty$, here L^r_ω denotes the weak L^r space, then the strong solutions to the density-dependent Navier-Stokes-Korteweg equations can exist globally over $[0, T]$.

Keywords: 3D Navier-Stokes-Korteweg; Serrin's blow-up criterion; strong solution;

3.1 Introduction and main result

From this chapter on, we will discuss the dynamic model of another type of fluid, that is, capillary fluid. The time evolution of the density $\rho(x,t)$, velocity field $u(x,t) = (u_1, u_2, u_3)(x,t)$ and pressure $P(x,t)$ of a general viscous capillary fluid is governed by the inhomogeneous incompressible Navier-Stokes-Korteweg equations

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)d) + \nabla P + \mathrm{div}(\kappa(\rho)\nabla\rho \otimes \nabla\rho) = 0, \\ \mathrm{div} u = 0, \end{cases} \tag{3.1.1}$$

in $\Omega\times(0,\infty)$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^3$ in this chapter.

$$d = \frac{1}{2}\Big[\nabla u + (\nabla u)^T\Big]$$

is the deformation tensor, where ∇u is the gradient matrix $(\partial u_i/\partial x_j)$ and $(\nabla u)^T$ is its transpose. $\kappa = \kappa(\rho)$ and $\mu = \mu(\rho)$ stand for the capillary and viscosity coefficients of the fluid respectively, and are both functions of the density ρ. In this chapter, they are assumed to satisfy

$$\kappa, \mu \in C^1[0,\infty), \quad \text{and} \ \ \kappa \geq 0, \mu \geq \underline{\mu} > 0 \quad \text{on } [0,\infty) \tag{3.1.2}$$

for some positive constant $\underline{\mu}$.

We focus on the system (3.1.1)-(3.1.2) with the following initial and boundary conditions:

(3.1.3) $$u = 0, \quad \text{on } \partial\Omega \times [0, T),$$

(3.1.4) $$(\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in } \Omega.$$

When $\kappa(\rho) \equiv 0$, the system (3.1.1)-(3.1.4) reduces to the famous inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity. Early in this century, Cho and Kim [2] proved the local existence of unique strong solution with vacuum for all initial data satisfying a compatibility condition. Later, Huang and Wang [10] proved the strong solution exists globally in time when the initial gradient of the velocity is suitably small in some Sobolev space. For the related progress of the Navier-Stokes model, see Chapter 1, references [8]-[10] and therein.

Let us come back to the fluids with capillary effect, that is, $\kappa(\rho)$ depends on the density ρ. As far as I know, the first local existence and uniqueness of strong solutions was obtained by Tan and Wang [14] when the capillary coefficient κ is a nonnegative constant and the viscosity μ is a positive constant. Very recently, Wang [15] extended their result to the case when $\kappa(\rho)$ is a C^1 function of the density.

The purpose of this chapter is to prove a blow-up criterion for the strong solutions to the initial and boundary value problem (3.1.1)-(3.1.4). First we give the definition of strong solutions to the initial and boundary problem (3.1.1)-(3.1.4) as follows.

Definition 3.1 (Strong solution)**.** A pair of functions $(\rho \geq 0, u, P)$ is called a strong solution to the problem (3.1.1)-(3.1.4) in $\Omega \times (0, T)$, if for some $q_0 \in (3, 6]$,

(3.1.5) $$\begin{aligned} &\rho \in C([0,T]; W^{2,q_0}), \quad u \in C([0,T]; H_0^1 \cap H^2), \quad \nabla^2 u \in L^2(0,T; L^{q_0}), \\ &\rho_t \in C([0,T]; W^{1,q_0}), \quad \nabla P \in C([0,T]; L^2) \cap L^2(0,T; L^{q_0}), \quad u_t \in L^2(0,T; H_0^1), \end{aligned}$$

and (ρ, u, P) satisfies (3.1.1) a.e. in $\Omega \times (0, T)$.

In the case when the initial data may vanish in an open subset of Ω, that is, the initial vacuum is allowed, the following local well-posedness of strong solution to (3.1.1)-(3.1.4) was obtained by Wang [15].

Theorem 3.2. *Assume that the initial data* (ρ_0, u_0) *satisfies the regularity condition*

(3.1.6) $$0 \leq \rho_0 \in W^{2,q}, \quad 3 < q \leq 6, \quad u_0 \in H_{0,\sigma}^1 \cap H^2,$$

and the compatibility condition

(3.1.7) $$-\operatorname{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) + \nabla P_0 + \operatorname{div}(\kappa(\rho_0)\nabla\rho_0 \otimes \nabla\rho_0) = \rho_0^{1/2} g,$$

for some $(P_0, g) \in H^1 \times L^2$*. Then there exist a small time* T *and a unique strong solution* (ρ, u, P) *to the initial boundary value problem* (3.1.1)-(3.1.4)*.*

Remark 3.3. In fact, the strong solution obtained in Theorem 3.2 is a little stronger than the original one in Wang [15], where he did not discuss the time continuity of (ρ, u) at initial time. More precisely, the regularity class of solutions he proved is

$$\begin{aligned}&\rho \in L^\infty(0,T;W^{2,q}), \quad u \in L^\infty(0,T;H_0^1 \cap H^2), \quad \nabla^2 u \in L^1(0,T;L^q),\\ &\rho_t \in L^\infty(0,T;W^{1,q}), \quad \nabla P \in L^\infty(0,T;L^2) \cap L^2(0,T;L^q), \quad u_t \in L^2(0,T;H_0^1),\end{aligned} \tag{3.1.8}$$

and (3.1.8) can be improved to (3.1.5) if we slightly modify the discussion in Sec. 3.2 in Cho and Kim [2]. Similar treatment is widely used for other fluid models. Refer to [1],[5] .

Motivated by the work of Kim [11], in which Kim proved a Serrin type blow-up criterion for the 3D inhomogeneous incompressible Navier-Stokes flow, our main purpose is to derive a similar blow-up criterion for the inhomogeneous Navier-Stokes-Korteweg equations with density-dependent viscosity and capillary coefficients. More precisely, our result can be stated as follows.

Theorem 3.4. *Assume that the initial data (ρ_0, u_0) satisfies the regularity condition* (3.1.6) *and the compatibility condition* (3.1.7). *Let (ρ, u, P) be a strong solution of the problem* (3.1.1)-(3.1.4) *satisfying* (3.1.5). *If $0 < T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \to T^*} (\|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)}) = \infty. \tag{3.1.9}$$

for any r and s satisfying

$$\frac{2}{s} + \frac{3}{r} \leq 1, \quad 3 < r \leq \infty, \tag{3.1.10}$$

where L^r_ω denotes the weak L^r space.

The proof of Theorem 3.4 is based on the contradiction argument. In view of the local existence result, to prove Theorem 3.4, it suffices to verify that (ρ, u) satisfy (3.1.6) and (3.1.7) at the time T^* under the assumption that the left hand side of (3.1.9) is finite. Unlike the Navier-Stokes equations treated in Kim [11], the use of weak Lesbegue space makes it more difficult to obtain some estimates because of the appearance of capillary effect. To overcome the difficulties, we make good use of the finiteness of $\|\nabla\rho\|_{W^{1,q}}$ and other interpolation techniques in Lorentz space.

The remainder of this chapter is arranged as follows. In Sec. 3.2, we give some auxiliary lemmas which is useful in our later analysis. The proof of Theorem 3.4 will be done by combining the contradiction argument with the estimates derived in Sec. 3.3.

3.2 Preliminaries

3.2.1 Notations and general inequalities

Ω is a smooth bounded domain in $\mathbb{R}^3$. For notations simplicity below, we omit the integration domain Ω. For $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r = L^r(\Omega), \quad W^{k,r} = \{f \in L^r : \nabla^k f \in L^r\},$$

$$H^k = W^{k,2}, \quad C^\infty_{0,\sigma} = \{f \in (C_0^\infty)^3 : \mathrm{div} f = 0\}.$$

$$H_0^1 = \overline{C_0^\infty}, \quad H_{0,\sigma}^1 = \overline{C_{0,\sigma}^\infty}, \text{ closure in the norm of } H^1.$$

The following Gagliardo-Nirenberg inequality will be also used frequently.

Lemma 3.5 (Gagliardo-Nirenberg inequality). *Let Ω be a domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For $p \in [2,6], q \in (1,\infty)$ and $r \in (3,\infty)$, there exists some generic constants $C > 0$ that may depend on q and r such that for $f \in H^1$ satisfying $f|_{\partial\Omega} = 0$, and $g \in L^q \cap D^{1,r}$, we have*

$$\|f\|_{L^p}^p \le C\|f\|_{L^2}^{(6-p)/2}\|\nabla f\|_{L^2}^{(3p-6)/2}, \tag{3.2.1}$$

$$\|g\|_{L^\infty} \le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{3.2.2}$$

See the proof of this lemma in Ladyzhenskaya et al. [13, P. 62].

Denote the Lorentz space and its norm by $L^{p,q}$ and $\|\cdot\|_{L^{p,q}}$, respectively, where $1 < p < \infty$ and $1 \le q \le \infty$. We recall the weak-L^p space L^p_ω which is defined as follows:

$$L^p_\omega := \{f \in L^1_{loc} : \|f\|_{L^p_\omega} = \sup_{\lambda>0} \lambda|\{|f(x)| > \lambda\}|^{\frac{1}{p}} < \infty\}.$$

And it should be noted that

$$L^p \subsetneqq L^p_\omega, \quad L^\infty_\omega = L^\infty, \quad L^p_\omega = L^{p,\infty}.$$

For the details of Lorentz space, we refer to the first chapter in Grafakos [7]. The following lemma involving the weak Lesbegue spaces has been proved in Kim [11], Xu and Zhang [16], which will play an important role in the subsequent analysis.

Lemma 3.6. *Assume $g \in H^1$, and $f \in L^r_\omega$ with $r \in (3,\infty]$, then $f \cdot g \in L^2$. Furthermore, for any $\epsilon > 0$, we have*

$$\|f \cdot g\|_{L^2}^2 \le \epsilon\|g\|_{H^1}^2 + C(\epsilon)(\|f\|_{L^r_\omega}^s + 1)\|g\|_{L^2}^2, \tag{3.2.3}$$

where C is a positive constant depending only on ϵ, r and the domain Ω.

3.2.2 Higher order estimates on the velocity

High-order a priori estimates of velocity field u rely on the following regularity results for the stationary density-dependent Stokes equations.

Lemma 3.7. *Assume that $\rho \in W^{2,q}, 3 < q < \infty$, and $0 \le \rho \le \bar\rho$. Let $(u,P) \in H^1_{0,\sigma} \times L^2$ be the unique weak solution to the boundary value problem*

$$-\operatorname{div}(2\mu(\rho)d) + \nabla P = F, \ \operatorname{div} u = 0 \ \text{ in } \Omega, \text{ and } \int P dx = 0, \tag{3.2.4}$$

where $d = \frac{1}{2}[\nabla u + (\nabla u)^T]$ and

$$\mu \in C^1[0,\infty), \ \underline{\mu} \le \mu(\rho) \le \bar\mu \text{ on } [0,\bar\rho].$$

Then we have the following regularity results:

(1) If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and

$$\|u\|_{H^2} + \|P\|_{H^1} \leq C(1 + \|\nabla\rho\|_{L^\infty})\|F\|_{L^2}. \tag{3.2.5}$$

(2) If $F \in L^r$ for some $r \in (2, \infty)$, then $(u, P) \in W^{2,r} \times W^{1,r}$ and

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \leq C(1 + \|\nabla\rho\|_{L^\infty})\|F\|_{L^r}. \tag{3.2.6}$$

The proof of Lemma 3.7 has been given by Wang [15]. And refer to Lemma 2.1 in his paper. To make our thesis self-contained, we give the sketch of the proof here. We would like to emphasize that the range of r in the conclusion (2) shold be $(2, \infty)$, but not $(2, q)$ in [15], since we assume that $\nabla\rho \in W^{1,q}$ which is stronger than the original version of Cho and Kim [2].

Proof. It is well-known that from the elliptic estimates, $(u, P) \in H^2 \times H^1$ whenever $F \in L^2$. See Giaquinta and Modica [6]. To prove (3.2.5), we have only to derive the stated regularity estimates. First, we will show that

$$\|\nabla u\|_{L^2} + \|P\|_{L^2} \leq C\|F\|_{L^2}. \tag{3.2.7}$$

We multiply the Stokes type equation (3.2.4) by u and integrate by parts over the domain Ω, using Poincaré's inequality, one has

$$\int 2\mu(\rho)|d|^2 dx = \int F \cdot u dx \leq C\|F\|_{L^2}\|\nabla u\|_{L^2}.$$

Recalling that $\underline{\mu} \leq \mu(\rho) \leq \bar{\mu}$ and $2\|d\|_{L^2}^2 = \|\nabla u\|_{L^2}^2$, we can deduce from the above inequality that $\|\nabla u\|_{L^2} \leq C\|F\|_{L^2}$. On the other hand, if we choose $v \in H_0^1$ such that $P = \mathrm{div} v$ and $\|v\|_{H^1} \leq C\|P\|_{L^2}$, then

$$\begin{aligned}\int P^2 dx = -\int \nabla P \cdot v dx &= \int (2\mu(\rho) d : \nabla v - F \cdot v) dx \\ &\leq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + C\|F\|_{L^2}\|v\|_{L^2} \leq C\|F\|_{L^2}\|v\|_{H^1},\end{aligned}$$

which implies $\|P\|_{L^2} \leq C\|F\|_{L^2}$. To prove the lemma, we rewrite (3.2.4) as

$$\begin{cases} -\triangle u + \nabla\tilde{P} = \mu^{-1}(\rho)(F + 2\nabla\mu(\rho) \cdot d - \tilde{P}\nabla\mu(\rho)), \\ \mathrm{div} u = 0 & \text{in} \quad \Omega \times (0, T), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, T), \end{cases} \tag{3.2.8}$$

where $\tilde{P} = P/\mu(\rho)$, then it follows from the regularity results on the classical Stokes equations that

$$\begin{aligned}\|u\|_{H^2} + \|\tilde{P}\|_{H^1} &\leq C(\|F\|_{L^2} + \||\nabla\rho||\nabla u|\|_{L^2} + \||\nabla\rho||\tilde{P}|\|_{L^2} + \|\tilde{P}\|_{L^2}) \\ &\leq C(\|F\|_{L^2} + \|\nabla\rho\|_{L^\infty}(\|\nabla u\|_{L^2} + \|\tilde{P}\|_{L^2}) + \|\tilde{P}\|_{L^2}) \\ &\leq C\|F\|_{L^2}(1 + \|\nabla\rho\|_{L^\infty}).\end{aligned} \tag{3.2.9}$$

The proof of property (3.2.6) is similar to (3.2.5), we just recall that

(3.2.10) $$\|u\|_{W^{2,r}} + \|\tilde{P}\|_{W^{1,r}} \leq C(\|F\|_{L^r} + \||\nabla\rho||\nabla u|\|_{L^r} + \||\nabla\rho||\tilde{P}|\|_{L^r} + \|\tilde{P}\|_{L^r}).$$

□

3.3 Proof of the blow-up criterion in 3D

Let (ρ, u, P) be a strong solution to the initial and boundary value problem (3.1.1)-(3.1.4) as derived in Theorem 3.2. Then it follows from the standard energy estimate that

Lemma 3.8. *For any $T > 0$, it holds that for any $p \in [1, \infty]$,*

(3.3.1) $$\sup_{0\leq t\leq T}(\|\rho\|_{L^p} + \|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\kappa(\rho)}\nabla\rho\|_{L^2}^2) + \int_0^T \int |\nabla u|^2 dxds \leq C.$$

As mentioned in the Section 3.1, the main theorem will be proved by using a contradiction argument. Denote $0 < T^* < \infty$ the maximal existence time for the strong solution to the initial and boundary value problem (3.1.1)-(3.1.4). Suppose that (3.1.9) were false, that is

(3.3.2) $$M_0 := \limsup_{T\to T^*}(\|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)}) < \infty.$$

Under the condition (3.3.2), one will extend the existence time of the strong solutions to (3.1.1)-(3.1.4) beyond T^*, which contradicts the definition of maximum of T^*.

The first key step is to derive the L^2-norm of the first order spatial derivatives of u under the assumptions of initial data and (3.3.2). Here we define the material derivatives $\dot{u} := u_t + u \cdot \nabla u$.

Lemma 3.9. *Under the condition* (3.3.2)*, it holds that for any $0 < T < T^*$,*

(3.3.3) $$\sup_{0\leq t\leq T}\|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}\dot{u}\|_{L^2}^2 dt \leq C.$$

Proof. Multiplying the momentum equations $(3.1.1)_2$ by u_t, and integrating the resulting equations over Ω, we have

(3.3.4)
$$\begin{aligned}
&\int \rho|\dot{u}|^2dx + \frac{d}{dt}\int \mu(\rho)|d|^2dx \\
=& -\int \rho\dot{u}\cdot(u\cdot\nabla u)dx - \int \mu'(\rho)u\cdot\nabla\rho|d|^2dx + \int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u_t dx \\
=&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \int \kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx \\
&+ 2\int \kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho : \nabla u dx - \int \rho\dot{u}\cdot(u\cdot\nabla u)dx - \int \mu'(\rho)u\cdot\nabla\rho|d|^2dx \\
=&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \sum_{k=1}^4 I_k.
\end{aligned}$$

Now let us estimate these terms one by one. Using the assumption (3.3.2), we get

$$\begin{aligned} I_1 &= \int \kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx \\ &\le \|\kappa'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^6}^3\|u\cdot\nabla u\|_{L^2} \\ &\le \|u\cdot\nabla u\|_{L^2}^2 + C. \end{aligned} \tag{3.3.5}$$

Similarly, dividing I_2 into two parts, one has

$$\begin{aligned} I_2 &= \int \kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho : \nabla u dx \\ &\le \|\kappa(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^2}\|u\cdot\nabla u\|_{L^2} \\ &\quad + \|\kappa(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 \\ &\le C\|u\cdot\nabla u\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2). \end{aligned} \tag{3.3.6}$$

For the term I_3, use Cauchy-Schwarz inequality to get

$$\begin{aligned} I_3 &= \int \rho\dot{u}\cdot(u\cdot\nabla u)dx \\ &\le \epsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(\epsilon)\|u\cdot\nabla u\|_{L^2}^2, \end{aligned} \tag{3.3.7}$$

and finally

$$\begin{aligned} I_3 &= \int \mu'(\rho)u\cdot\nabla\rho|d|^2dx \\ &\le \|\mu'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|u\cdot\nabla u\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}^2 + C\|u\cdot\nabla u\|_{L^2}^2. \end{aligned} \tag{3.3.8}$$

Before we close our estimates, we apply (3.2.6) in Lemma 3.7 to get a higher order estimate of $\|\nabla u\|_{H^1}$. Taking $F = -\rho\dot{u} - \operatorname{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)$, we derive that

$$\begin{aligned} \|\nabla u\|_{H^1} + \|P\|_{H^1} &\le C(1+\|\nabla\rho\|_{L^\infty})\|F\|_{L^2} \\ &\le C(1+\|\nabla\rho\|_{L^\infty})\|\rho\dot{u} + \operatorname{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)\|_{L^2} \\ &\le C_*\|\sqrt{\rho}\dot{u}\|_{L^2} + C\|\nabla\rho\|_{L^6}^3 + C\|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^2} \\ &\le C_*\|\sqrt{\rho}\dot{u}\|_{L^2} + C, \end{aligned} \tag{3.3.9}$$

where C_* is a positive number.

Now we substitute (3.3.5)-(3.3.8) into (3.3.4), then deduce

$$\begin{aligned}
&\int \rho|\dot{u}|^2dx + \frac{d}{dt}\int \mu(\rho)|d|^2dx \\
\leq&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \epsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&+ C(\epsilon)\|u\cdot\nabla u\|_{L^2}^2 \\
\leq&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \epsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&+ \delta\|\nabla u\|_{H^1}^2 + C(\epsilon,\delta)(\|u\|_{L^r_\omega}^s+1)\|\nabla u\|_{L^2}^2 \\
\leq&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + \epsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&+ C_*\delta\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C(\epsilon,\delta)(\|u\|_{L^r_\omega}^s+1)\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.3.10}$$

for any (r,s) satisfying $\frac{2}{s}+\frac{3}{r}\leq 1$ with $r>3$. Here we have used Lemma 3.6 in the second inequality, and (3.3.9) has been used to get the third one. Then choosing ϵ,δ small enough, we get

$$\begin{aligned}
&\int \rho|\dot{u}|^2dx + \frac{d}{dt}\int \mu(\rho)|d|^2dx \\
\leq&\frac{d}{dt}\int \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx + C(1+\|\nabla u\|_{L^2}^2)(\|u\|_{L^r_\omega}^s+1).
\end{aligned} \tag{3.3.11}$$

By the assumption (3.3.2) and Cauchy-Schwarz inequality, it is easily seen that

$$C\int |\kappa(\rho)||\nabla\rho\otimes\nabla\rho : \nabla u|dx \leq \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + C. \tag{3.3.12}$$

Taking this into account, we can conclude from (3.3.11) and the Gronwall inequality that (3.3.3) holds for all $0\leq T<T^*$. Therefore we complete the proof of Lemma 3.9. □

Next we prove the boundedness of $\|\sqrt{\rho}u_t\|_{L^2}$, by using the compatibility condition (3.1.7) on the initial data.

Lemma 3.10. *Under the condition* (3.3.2)*, it holds that for any* $0<T<T^*$,

$$\sup_{0\leq t\leq T}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \leq C. \tag{3.3.13}$$

Proof. Differentiating the momentum equations $(3.1.1)_2$ with respect to t, along with the continuity equation $(3.1.1)_1$, we get

$$\begin{aligned}
&\rho u_{tt} + \rho u\cdot\nabla u_t - \mathrm{div}(2\mu(\rho)d_t) + \nabla P_t \\
=&(u\cdot\nabla\rho)(u_t+u\cdot\nabla u) - \rho u_t\cdot\nabla u - \mathrm{div}(2\mu'(\rho)(u\cdot\nabla\rho)d) \\
&+ \mathrm{div}(\kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho) + 2\mathrm{div}(\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho).
\end{aligned} \tag{3.3.14}$$

Multiplying (3.3.14) by u_t and integrating over Ω, we get after integration by parts that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx + 2\int \mu(\rho)|d_t|^2dx = \int -2\rho u\cdot\nabla u_t\cdot u_t dx \\ &+\int (u\cdot\nabla\rho)(u\cdot\nabla u)\cdot u_t dx - \int \rho u_t\cdot\nabla u\cdot u_t dx \\ &+\int 2\mu'(\rho)(u\cdot\nabla\rho)d:\nabla u_t dx - \int \kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho:\nabla u_t dx \\ &-\int 2\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u_t dx =: \sum_{k=1}^{6} J_k. \end{aligned} \tag{3.3.15}$$

Now let us estimate the terms on the right hand side one by one. First,

$$\begin{aligned} J_1 &= \int -2\rho u\cdot\nabla u_t\cdot u_t dx \\ &\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2} \\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2} \\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}^{\frac{3}{2}} \\ &\le \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4 \\ &\le \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2. \end{aligned} \tag{3.3.16}$$

Similarly,

$$\begin{aligned} J_2 &= \int (u\cdot\nabla\rho)(u\cdot\nabla u)\cdot u_t dx \\ &\le C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^6}^2\|u_t\|_{L^6} \\ &\le C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^2}^3\|\nabla u_t\|_{L^2} \\ &\le \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C, \end{aligned} \tag{3.3.17}$$

$$\begin{aligned} J_3 &= -\int \rho u_t\cdot\nabla u\cdot u_t dx \\ &\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|u_t\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u\|_{L^2} \\ &\le C\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}} \\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}} \\ &\le \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2, \end{aligned} \tag{3.3.18}$$

(3.3.19)
$$\begin{aligned} J_4 &= \int 2\mu'(\rho)(u\cdot\nabla\rho)d:\nabla u_t dx \\ &\le C\|\mu'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\|\nabla u_t\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}}\|\nabla u_t\|_{L^2} \\ &\le \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{H^1}^2, \end{aligned}$$

(3.3.20)
$$\begin{aligned} J_5 &= \int \kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho:\nabla u_t dx \\ &\le C\|\kappa'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^3\|u\|_{L^2}\|\nabla u_t\|_{L^2} \\ &\le \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C, \end{aligned}$$

(3.3.21)
$$\begin{aligned} J_6 &= \int 2\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u_t dx \\ &\le C\|\kappa(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2} \\ &\quad + C\|\kappa(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2} \\ &\le \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C, \end{aligned}$$

here note that we take $q > 3$. It remains to estimate $\|\nabla u\|_{H^1}$ since it appears in the term J_4. Indeed, we can duduce from Lemma 3.7 that

(3.3.22)
$$\begin{aligned} \|\nabla u\|_{H^1} + \|P\|_{H^1} &\le C(1+\|\nabla\rho\|_{L^\infty})\|F\|_{L^2} \\ &\le C(1+\|\nabla\rho\|_{L^\infty})\|\rho u_t + \rho u\cdot\nabla u + \operatorname{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)\|_{L^2} \\ &\le C(\|\sqrt{\rho}u_t\|_{L^2} + \|u\|_{L^6}\|\nabla u\|_{L^3} + \|\nabla\rho\|_{L^6}^3 + \|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^2}) \\ &\le C\|\sqrt{\rho}u_t\|_{L^2} + \frac{1}{2}\|\nabla u\|_{H^1} + C, \end{aligned}$$

which implies

(3.3.23)
$$\|\nabla u\|_{H^1} + \|P\|_{H^1} \le C\|\sqrt{\rho}u_t\|_{L^2} + C.$$

Combining all the estimates (3.3.16)-(3.3.21) and using (3.3.23), we deduce

(3.3.24)
$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2 dx + 2\int \mu(\rho)|d_t|^2 dx \\ \le& \frac{1}{2}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C(1+\|\sqrt{\rho}u_t\|_{L^2}^2). \end{aligned}$$

Remarking that $2\|d_t\|_{L^2}^2 = \|\nabla u_t\|_{L^2}^2$, we obtain (3.3.13) by applying the Gronwall inequality. Then the proof of Lemma 3.10 is completed. □

Lemma 3.11. *Under the condition* (3.3.2)*, it holds that for any* $0 < T < T^*$*,*

(3.3.25)
$$\sup_{0\le t\le T}(\|\rho_t\|_{W^{1,q}} + \|u\|_{H^2} + \|P\|_{H^1}) + \int_0^T (\|u\|_{W^{2,q}}^2 + \|P\|_{W^{1,q}}^2)dt \le C.$$

Proof. By (3.3.23) and Lemma 3.10, it is easy to deduce

$$\sup_{0\le t\le T}(\|u\|_{H^2}+\|P\|_{H^1})\le C, \tag{3.3.26}$$

and, together with $(3.1.1)_1$, we yield

$$\begin{aligned}\|\rho_t\|_{W^{1,q}} &\le C(\|\rho_t\|_{L^q}+\|\nabla\rho_t\|_{L^q})\\ &\le C(\|u\cdot\nabla\rho\|_{L^q}+\|\nabla(u\cdot\nabla\rho)\|_{L^q})\\ &\le C(\|u\|_{L^\infty}\|\nabla\rho\|_{L^q}+\|u\|_{L^\infty}\|\nabla^2\rho\|_{L^q}+\|\nabla u\|_{L^6}\|\nabla\rho\|_{L^{\frac{6q}{6-q}}})\\ &\le C\|u\|_{H^2}\|\nabla\rho\|_{W^{1,q}}\le C.\end{aligned} \tag{3.3.27}$$

Finally, applying (3.2.6) in Lemma 3.7 with $F=-\rho u_t-\rho u\cdot\nabla u-\operatorname{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)$, we get

(3.3.28)

$$\begin{aligned}&\|\nabla u\|_{W^{1,q}}+\|P\|_{W^{1,q}}\\ &\le C(1+\|\nabla\rho\|_{L^\infty})(\|\rho u_t\|_{L^q}+\|\rho u\cdot\nabla u\|_{L^q}+\|\kappa(\rho)|\nabla^2\rho||\nabla\rho|\|_{L^q}+\|\kappa'(\rho)|\nabla\rho|^3\|_{L^q})\\ &\le C(\|\rho u_t\|_{L^q}+\|\rho u\cdot\nabla u\|_{L^q}+1)\\ &\le C(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{3q-6}{2q}}+\|\nabla u\|_{L^2}^{\frac{6(q-1)}{5q-6}}\|\nabla u\|_{W^{1,q}}^{\frac{4q-6}{5q-6}}+1).\end{aligned}$$

Applying Young's inequality and Sobolev embedding inequality,

$$\begin{aligned}\|\nabla u\|_{W^{1,q}}^2+\|P\|_{W^{1,q}}^2 &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{q}}\|\nabla u_t\|_{L^2}^{\frac{3(q-2)}{q}}+C\|\nabla u\|_{L^2}^{\frac{12(q-1)}{q}}+C\\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{q}}\|\nabla u_t\|_{L^2}^{\frac{3(q-2)}{q}}+C.\end{aligned} \tag{3.3.29}$$

Hence

$$\begin{aligned}\int_0^T(\|\nabla u\|_{W^{1,q}}^2+\|P\|_{W^{1,q}}^2)dt &\le C\int_0^T\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{q}}\|\nabla u_t\|_{L^2}^{\frac{3(q-2)}{q}}dt+C\\ &\le C(\sup_{0\le t\le T}\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{6-q}{2q}}\int_0^T\|\nabla u_t\|_{L^2}^2dt+C\\ &\le C,\end{aligned} \tag{3.3.30}$$

here the second inequality holds since $\frac{3(q-2)}{q}\le 2$. Therefore we complete the proof of Lemma 3.11. □

Proof of Theorem 3.4. In fact, in view of (3.3.2) and (3.3.25), it is easy to see that the functions $(\rho,u)(x,t=T^*)=lim_{t\to T^*}(\rho,u)$ have the same regularities imposed on the initial data (3.1.6) at the time $t=T^*$. Furthermore,

$$\begin{aligned}&-\operatorname{div}(2\mu(\rho)d)+\nabla P+\operatorname{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)|_{t=T^*}\\ &=\lim_{t\to T^*}\rho^{\frac12}(\rho^{\frac12}u_t+\rho^{\frac12}u\cdot\nabla u)=:\rho^{\frac12}g|_{t=T^*}\end{aligned}$$

with $g=(\rho^{\frac12}u_t+\rho^{\frac12}u\cdot\nabla u)|_{t=T^*}\in L^2$ due to (3.3.13). Thus the functions $(\rho,u)|_{t=T^*}$ satisfy the compatibility condition (3.1.7) at time T^*. Therefore we can take $(\rho,u)|_{t=T^*}$

as the initial data and apply the local existence theorem (Theorem 3.2) to extend the local strong solution beyond T^*. This contradicts the definition of maximal existence time T^*, and thus, the proof of Theorem 3.4 is completed. □

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Chapter 4

A blow-up criterion of the 2D inhomogeneous Navier-Stokes-Korteweg equations

In this chapter, we prove a blow-up criterion for the strong solutions with vacuum to the density-dependent Navier-Stokes-Korteweg equations over a bounded smooth domain in $\mathbb{R}^2$, which only in terms of the density.

Keywords: 2D Navier-Stokes-Korteweg; blow-up criterion; vacuum

4.1 Introduction and main result

In this chapter, we continue to consider the model of inhomogeneous incompressible Navier-Stokes-Korteweg equations, which are used to describe the motion of a general viscous capillary fluid:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)d) + \nabla P + \operatorname{div}(\kappa(\rho)\nabla\rho \otimes \nabla\rho) = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{4.1.1}$$

in $\Omega \times (0, \infty)$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^2$. Here ρ, u and P denote the density, velocity and pressure of the fluid, respectively.

$$d = \frac{1}{2}\Big[\nabla u + (\nabla u)^T\Big]$$

is the deformation tensor, where ∇u is the gradient matrix $(\partial u_i/\partial x_j)$ and $(\nabla u)^T$ is its transpose. $\kappa = \kappa(\rho)$ and $\mu = \mu(\rho)$ stand for the capillary and viscosity coefficients of the fluid respectively, and are both functions of density ρ. In this chapter, they are assumed to satisfy

$$\kappa, \mu \in C^1[0, \infty), \quad \text{and } \kappa \geq 0, \mu \geq \underline{\mu} > 0 \quad \text{on } [0, \infty) \tag{4.1.2}$$

for some positive constant $\underline{\mu}$.

We focus on the system (4.1.1)-(4.1.2) with the initial and boundary conditions:

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega \times [0, T). \tag{4.1.3}$$

In the last chapter, we proved a Serrin's type blow-up criterion for the strong solutions of the inhomogeneous Navier-Stokes-Korteweg equations in dimensions three. For the two-dimensional case, it follows from the energy inequality the solution satisfies that $\sup_{0<T<T^*}(\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2)} + \|\nabla u\|_{L^2(0,T;L^2)})$ is bounded, which implies that $u \in L^4(0,T;L^4)$ if ρ is bounded away from zero. Hence the criterion showed in chapter 3 in fact can be improved to the one only involving the density if the density ρ is bounded away from 0. However, if the density is allowed to vanish, it remains unknown. This is the main problem we shall address in this chapter. The purpose of this chapter is to prove a blow-up criterion for the strong solutions to the problem (4.1.1)-(4.1.3). First we give the definition of strong solution to the initial and boundary problem (4.1.1)-(4.1.3) as follows (two dimensional version).

Definition 4.1 (Strong solution)**.** A pair of functions $(\rho \geq 0, u, P)$ is called a strong solution to the problem (4.1.1)-(4.1.3) in $\Omega \times (0,T)$, if for some $q_0 \in (2,\infty)$,

$$\begin{aligned}&\rho \in C([0,T];W^{2,q_0}), \quad u \in C([0,T];H_0^1 \cap H^2) \cap L^2(0,T;W^{2,q_0}),\\&\rho_t \in C([0,T];W^{1,q_0}), \quad \nabla P \in C([0,T];L^2) \cap L^2(0,T;L^{q_0}), \quad u_t \in L^2(0,T;H_0^1),\end{aligned} \tag{4.1.4}$$

and (ρ, u, P) satisfies (4.1.1) a.e. in $\Omega \times (0,T)$.

In the case when the initial data may vanish in an open subset of Ω, that is, the initial vacuum is allowed, the following local well-posedness of strong solution to (4.1.1)-(4.1.3) was obtained by Wang [11] in a three dimensional bounded domain. In fact, the local existence of unique strong solution with vacuum to the system (4.1.1) in a two dimensional bounded domain can be established in the same manner as Wang [11] and Cho and Kim [1], also see the remark 2 in Tan and Wang [10].

Theorem 4.2. *Assume that the initial data (ρ_0, u_0) satisfies the regularity condition*

$$0 \leq \rho_0 \in W^{2,q}, \quad 2 < q < \infty, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \tag{4.1.5}$$

and the compatibility condition

$$-\operatorname{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) + \nabla P_0 + \operatorname{div}(\kappa(\rho_0)\nabla\rho_0 \otimes \nabla\rho_0) = \rho_0^{1/2} g, \tag{4.1.6}$$

for some $(P_0, g) \in H^1 \times L^2$. Then there exist a small time T and a unique strong solution (ρ, u, P) to the initial boundary value problem (4.1.1)-(4.1.3).

Motivated by the work of Huang and Wang [5], which proved a new type blow-up criterion for the 2D inhomogeneous incompressible Navier-Stokes flow only involving the density. The main purpose is to derive a similar blow-up criterion for the inhomogeneous Navier-Stokes-Korteweg equations with density-dependent viscosity and capillary coefficients. More precisely, our main result can be stated as follows.

Theorem 4.3. *Assume that the initial data (ρ_0, u_0) satisfies the regularity condition* (4.1.5) *and the compatibility condition* (4.1.6)*, as in Theorem 4.2 . Let (ρ, u, P) be a strong solution of the problem* (4.1.1)-(4.1.3) *satisfying* (4.1.4)*. If $0 < T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T\to T*} \|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} = \infty. \tag{4.1.7}$$

Remark 4.4. It is still unknown that if we can extend the local strong solution to a global one for any arbitrary large initial data when the viscosity and capillary coefficients are constants, since our blow-up criterion involves the gradient of density but not the gradient of viscosity or capillary. We will consider the problem whether we can replace the density with viscosity or capillary in our blow-up criterion in the future work.

The proof of Theorem 4.3 is based on the contradiction argument. In view of the local existence result, to prove Theorem 4.3, it suffices to verify that (ρ, u) satisfy (4.1.5) and (4.1.6) at the time T^* under the assumption of the left hand side of (4.1.7) is finite.

Similar to the arrangement of chapter 3, in Sec. 4.2, we give some auxiliary lemmas which is useful in our later analysis. The proof of Theorem 4.3 will be done by combining the contradiction argument with the estimates derived in Sec. 4.3.

4.2 Preliminaries

4.2.1 Notations and general inequalities

Ω is a smooth bounded domain in $\mathbb{R}^2$. For notations simplicity below, we omit the integration domain Ω. And for $1 \le r \le \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way,

$$L^r = L^r(\Omega), \quad W^{k,r} = \{f \in L^r : \nabla^k f \in L^r\},$$

$$H^k = W^{k,2}, \quad C^\infty_{0,\sigma} = \{f \in (C_0^\infty)^3 : \mathrm{div} f = 0\}.$$

$$H_0^1 = \overline{C_0^\infty}, \quad H^1_{0,\sigma} = \overline{C^\infty_{0,\sigma}}, \text{ closure in the norm of } H^1.$$

The following Ladyzhenskaya inequality in 2D case will be often used.

$$\|u\|_{L^4}^2 \le C\|u\|_{L^2}\|\nabla u\|_{L^2}. \tag{4.2.1}$$

However, to deal with a inhomogeneous problem with vacuum, some interpolation inequality for u with degenerate weight like $\sqrt{\rho}$ is required. We look for a similar estimate for $\sqrt{\rho}u$ as in (4.2.1). This technique can be found in the paper of Desjardin [2].

Lemma 4.5. *Assume that* $0 \le \rho \le \bar{\rho}, u \in H_0^1$*; then*

$$\|\sqrt{\rho}u\|_{L^4}^2 \le C(1 + \|\rho u\|_{L^2})\|\nabla u\|_{L^2}\sqrt{\log(2 + \|\nabla u\|_{L^2}^2)}, \tag{4.2.2}$$

where C *is a positive constant depending only on* $\bar{\rho}$ *and the domain* Ω.

4.2.2 Higher order estimates on the velocity

High-order a priori estimates of velocity field u rely on the following regularity results for the stationary density-dependent Stokes equations.

Lemma 4.6. *Assume that* $\rho \in W^{2,q}, 2 < q < \infty$, *and* $0 \le \rho \le \bar{\rho}$. *Let* $(u, P) \in H^1_{0,\sigma} \times L^2$ *be the unique weak solution to the boundary value problem*

$$-\,\mathrm{div}(2\mu(\rho)d) + \nabla P = F,\ \mathrm{div} u = 0 \textit{ in } \Omega, \textit{ and} \int P dx = 0, \tag{4.2.3}$$

where $d = \frac{1}{2}[\nabla u + (\nabla u)^T]$ *and*

$$\mu \in C^1[0,\infty), \ \ \underline{\mu} \le \mu(\rho) \le \bar{\mu} \text{ on } [0,\bar{\rho}].$$

Then we have the following regularity results:

(1) If $F \in L^2$*, then* $(u,P) \in H^2 \times H^1$ *and*

$$\|u\|_{H^2} + \|P\|_{H^1} \le C(1 + \|\nabla\rho\|_{L^\infty})\|F\|_{L^2}. \tag{4.2.4}$$

(2) If $F \in L^r$ *for some* $r \in (2,\infty)$*, then* $(u,P) \in W^{2,r} \times W^{1,r}$ *and*

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \le C(1 + \|\nabla\rho\|_{L^\infty})\|F\|_{L^r}. \tag{4.2.5}$$

The proof of Lemma 4.6 has been given by Wang [11]. And refer to Lemma 2.1 in his paper.

4.3 Proof of the blow-up criterion in 2D

Let (ρ, u, P) be a strong solution to the initial and boundary value problem (4.1.1)-(4.1.3) as derived in Theorem 4.2. Then it follows from the standard energy estimate that

Lemma 4.7. *For any* $T > 0$*, it holds that for any* $p \in [1,\infty]$*,*

$$\sup_{0\le t\le T}(\|\rho\|_{L^p} + \|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\kappa(\rho)}\nabla\rho\|_{L^2}^2) + \int_0^T \int |\nabla u|^2 dxds \le C. \tag{4.3.1}$$

The proof of Lemma 4.7 is completely the same with that of Lemma 3.8 in chapter 3. Thus we omit the proof here.

As mentioned in the Section 4.1, the main theorem will be proved by using a contradiction argument. Denote $0 < T^* < \infty$ the maximal existence time for the strong solution to the initial and boundary value problem (4.1.1)-(4.1.3). Suppose that (4.1.7) were false, that is

$$M_0 := \limsup_{T\to T^*} \|\nabla\rho\|_{L^\infty(0,T;W^{1,q})} < \infty. \tag{4.3.2}$$

Under the condition (4.3.2), one will extend the existence time of the strong solutions to (4.1.1)-(4.1.3) beyond T^*, which contradicts the definition of maximum of T^*.

The following estimate can be derived quickly from the Lemma 4.6, which is used later.

Lemma 4.8. *Under the assumption* (4.3.2)*, it holds for all* $0 < T < T^*$*,*

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2} + C, \tag{4.3.3}$$

and consequently by Sobolev embedding,

$$\|\nabla u\|_{H^1} \le C\|\rho u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C. \tag{4.3.4}$$

Proof. According to the Lemma 4.6 and the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\|\nabla u\|_{H^1} &\le C(1+\|\nabla\rho\|_{L^\infty})(\|\rho u_t\|_{L^2}+\|\rho u\cdot\nabla u\|_{L^2}+1)\\ &\le C\|\rho u_t\|_{L^2}+C\|\rho u\|_{L^4}\|\nabla u\|_{L^2}^{\frac12}\|\nabla u\|_{H^1}^{\frac12}+C\\ &\le C\|\rho u_t\|_{L^2}+C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}+C+\frac12\|\nabla u\|_{H^1},\end{aligned}$$

which complete the proof of (4.3.3). Applying the Ladyzhenskaya and Poincaré inequalities, we get

$$\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}\le C\|u\|_{L^4}^2\|\nabla u\|_{L^2}\le C\|\nabla u\|_{L^2}^3,$$

therefore we proved (4.3.4), then the proof of Lemma 4.8 is completed. □

The key step is to derive the L^2-norm of the first order spatial derivatives of u under the assumption of initial data and (4.3.2).

Lemma 4.9. *Under the condition* (4.3.2), *it holds that for any* $0<T<T^*$,

$$\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2+\int_0^T\|\sqrt{\rho}u_t\|_{L^2}^2dt\le C. \tag{4.3.5}$$

Proof. Multiplying the momentum equations $(4.1.1)_2$ by u_t, and integrating the resulting equations over Ω, we have

$$\begin{aligned}&\int\rho|u_t|^2dx+\frac{d}{dt}\int\mu(\rho)|d|^2dx\\ =&\int(\rho u\cdot\nabla u)\cdot u_tdx-\int u\cdot\nabla\mu(\rho)|d|^2dx+\int\kappa(\rho)\nabla\rho\otimes\nabla\rho:\nabla u_tdx\\ =&-\frac{d}{dt}\int\kappa(\rho)\nabla\rho\otimes\nabla\rho:\nabla udx+\int\kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho:\nabla udx\\ &+2\int\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla udx-\int(\rho u\cdot\nabla u)\cdot u_tdx-\int u\cdot\nabla\mu(\rho)|d|^2dx\\ =&\frac{d}{dt}\int\kappa(\rho)\nabla\rho\otimes\nabla\rho:\nabla udx+\sum_{k=1}^4I_k.\end{aligned} \tag{4.3.6}$$

Now let us estimate these terms one by one, by use of the Poincaré inequality, we get

$$\begin{aligned}I_1&=\int\kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho:\nabla udx\\ &\le\|\kappa'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^3\|u\|_{L^2}\|\nabla u\|_{L^2}\\ &\le C\|\nabla u\|_{L^2}^2.\end{aligned} \tag{4.3.7}$$

Similarly, dividing I_2 into two parts,

$$\begin{aligned}I_2&=\int\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla udx\\ &\le\|\kappa(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^q}\|u\|_{L^{q^*}}\|\nabla u\|_{L^2}+\|\kappa(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}^2\\ &\le C\|\nabla u\|_{L^2}^2,\end{aligned} \tag{4.3.8}$$

here $\frac{1}{q}+\frac{1}{q^*}=\frac{1}{2}$. and $q^*>2$. For the term I_3, using Cauchy-Schwarz inequality to get

$$\begin{aligned} I_3 &= \int \rho u_t \cdot (u\cdot\nabla u)dx \\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^4}^2\|\nabla u\|_{L^4}^2 \\ &\le \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^4}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} \\ &\le \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u\|_{L^4}^4\|\nabla u\|_{L^2}^2, \end{aligned} \tag{4.3.9}$$

and finally

$$\begin{aligned} I_4 &= \int u\cdot\nabla\mu(\rho)|d|^2dx \\ &\le \|\mu'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|u\|_{L^2}\|\nabla u\|_{L^4}^2 \\ &\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1} \\ &\le C\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1} \\ &\le C\|\nabla u\|_{L^2}^2\|\rho u_t\|_{L^2} + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}^3 + C\|\nabla u\|_{L^2}^2 \\ &\le \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\rho u\|_{L^4}^4\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2. \end{aligned} \tag{4.3.10}$$

Note that Lemma 4.5 tells us that

$$\begin{aligned} \|\sqrt{\rho}u\|_{L^4}^4 &\le C(1+\|\rho u\|_{L^2}^2)\|\nabla u\|_{L^2}^2\cdot\log(2+\|\nabla u\|_{L^2}^2) \\ &\le C\|\nabla u\|_{L^2}^2\cdot\log(2+\|\nabla u\|_{L^2}^2). \end{aligned} \tag{4.3.11}$$

Insert the estimates (4.3.7)-(4.3.10) into (4.3.6) to obtain

$$\begin{aligned} &\frac{1}{2}\int\rho|u_t|^2dx + \frac{d}{dt}\int(\mu(\rho)|d|^2 - \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u)dx \\ &\le C\|\nabla u\|_{L^2}^2(1+\|\nabla u\|_{L^2}^2)(1+\log(2+\|\nabla u\|_{L^2}^2)) \end{aligned} \tag{4.3.12}$$

and we know that

$$\frac{3}{4}\underline{\mu}\|\nabla u\|_{L^2}^2 - C_0 \le \int(\mu(\rho)|d|^2 - \kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u)dx \le \frac{5}{4}\underline{\mu}\|\nabla u\|_{L^2}^2 + C_0, \tag{4.3.13}$$

owing to the following estimate

$$\begin{aligned} \int\kappa(\rho)\nabla\rho\otimes\nabla\rho : \nabla u dx &\le \|\sqrt{\kappa(\rho)}\nabla\rho\|_{L^\infty}\|\sqrt{\kappa(\rho)}\nabla\rho\|_{L^2}\|\nabla u\|_{L^2} \\ &\le \frac{1}{4}\underline{\mu}\|\nabla u\|_{L^2}^2 + C\|\sqrt{\kappa(\rho)}\nabla\rho\|_{L^\infty}^2\|\sqrt{\kappa(\rho)}\nabla\rho\|_{L^2}^2 \\ &\le \frac{1}{4}\underline{\mu}\|\nabla u\|_{L^2}^2 + C_0. \end{aligned}$$

Taking this into account, we can conclude from (4.3.12) and the logarithmic type Gronwall inequality that (4.3.5) holds for all $0\le T<T^*$. Therefore we complete the proof of Lemma 4.9. □

Before we prove the boundedness of $\|\sqrt{\rho}u_t\|_{L^2}$, we insert the following estimate on the L^∞-norm of u.

Lemma 4.10. *Under the condition* (4.3.2)*, it holds that for any* $0 < T < T^*$*,*

$$\sup_{0\le t\le T}(\|u\|_{L^2(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^\infty)}) \le C. \tag{4.3.14}$$

Proof. By the Gagliardo-Nirenberg inequality and Lemma 4.8, we have

$$\begin{aligned}\int_0^T \|u\|_{L^\infty}^4 dt &\le C\int_0^T \|u\|_{L^2}^2\|\nabla u\|_{H^1}^2 dt\\ &\le C\int_0^T (\|\nabla u\|_{L^2}^2\|\rho u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^8 + \|\nabla u\|_{L^2}^2)dt,\end{aligned} \tag{4.3.15}$$

which completes the proof of (4.3.14), owing to Lemma 4.9. □

Now we can give the proof of the boundedness of $\|\sqrt{\rho}u_t\|_{L^2}$, by use of the compatibility condition (4.1.6) on the initial data.

Lemma 4.11. *Under the condition* (4.3.2)*, it holds that for any* $0 < T < T^*$*,*

$$\sup_{0\le t\le T}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \le C. \tag{4.3.16}$$

Proof. Differentiating the momentum equations $(4.1.1)_2$ with respect to t, along with the continuity equation $(4.1.1)_1$, we get

$$\begin{aligned}&\rho u_{tt} + \rho u\cdot\nabla u_t - \mathrm{div}(2\mu(\rho)d_t) + \nabla P_t\\ =&(u\cdot\nabla\rho)(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u - \mathrm{div}(2\mu'(\rho)(u\cdot\nabla\rho)d)\\ &+ \mathrm{div}(\kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho) + 2\mathrm{div}(\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho).\end{aligned} \tag{4.3.17}$$

Multiplying (4.3.17) by u_t and integrating over Ω, we get after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|u_t|^2dx + 2\int\mu(\rho)|d_t|^2dx = \int -2\rho u\cdot\nabla u_t\cdot u_t dx\\ &+\int(u\cdot\nabla\rho)(u\cdot\nabla u)\cdot u_t dx - \int\rho u_t\cdot\nabla u\cdot u_t dx\\ &+\int 2\mu'(\rho)(u\cdot\nabla\rho)d:\nabla u_t dx - \int\kappa'(\rho)(u\cdot\nabla\rho)\nabla\rho\otimes\nabla\rho:\nabla u_t dx\\ &-\int 2\kappa(\rho)\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u_t dx =: \sum_{k=1}^{6} J_k.\end{aligned} \tag{4.3.18}$$

Now let us estimate the terms on the right hand side one by one. First

$$\begin{aligned}J_1 &= \int -2\rho u\cdot\nabla u_t\cdot u_t dx\\ &\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\\ &\le \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2.\end{aligned} \tag{4.3.19}$$

Similarly,

$$
\begin{aligned}
J_2 &= \int (u \cdot \nabla\rho)(u \cdot \nabla u) \cdot u_t dx \\
&\leq C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^\infty}^2\|u_t\|_{L^2} \\
&\leq C\|\nabla\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^\infty}^2\|\nabla u_t\|_{L^2} \\
&\leq \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|u\|_{L^\infty}^4\|\nabla u\|_{L^2}^2,
\end{aligned}
\tag{4.3.20}
$$

$$
\begin{aligned}
J_3 &= -\int \rho u_t \cdot \nabla u \cdot u_t dx \\
&\leq C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|u_t\|_{L^4}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^4} \\
&\leq C\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{H^1} \\
&\leq \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{H^1}^2, \\
&\leq \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^4 + C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^6,
\end{aligned}
\tag{4.3.21}
$$

$$
\begin{aligned}
J_4 &= \int 2\mu'(\rho)(u \cdot \nabla\rho)d : \nabla u_t dx \\
&\leq C\|\mu'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\leq \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2,
\end{aligned}
\tag{4.3.22}
$$

$$
\begin{aligned}
J_5 &= \int \kappa'(\rho)(u \cdot \nabla\rho)\nabla\rho \otimes \nabla\rho : \nabla u_t dx \\
&\leq C\|\kappa'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^3\|u\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\leq \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2,
\end{aligned}
\tag{4.3.23}
$$

$$
\begin{aligned}
J_6 &= \int 2\kappa(\rho)\nabla(u \cdot \nabla\rho) \otimes \nabla\rho : \nabla u_t dx \\
&\leq C\|\kappa(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\quad + C\|\kappa(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^q}\|u\|_{L^{q^*}}\|\nabla u_t\|_{L^2} \\
&\leq \frac{1}{8}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2,
\end{aligned}
\tag{4.3.24}
$$

here note that we take $q > 2$. Substituting all the estimates (4.3.19)-(4.3.24) into (4.3.18), we deduce

$$
\begin{aligned}
&\frac{d}{dt}\int \rho|u_t|^2 dx + \int \mu(\rho)|d_t|^2 dx \\
\leq& C\|u\|_{L^\infty}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^6)\|\sqrt{\rho}u_t\|_{L^2}^2 \\
&+ \|\sqrt{\rho}u_t\|_{L^2}^4 + C(1 + \|u\|_{L^\infty}^4)\|\nabla u\|_{L^2}^2,
\end{aligned}
\tag{4.3.25}
$$

consequently, it follows from Gronwall inequality and Lemma 4.9, 4.10 that

$$\sup_{0\le t\le T}\|\sqrt{\rho}u_t\|_{L^2}^2+\int_0^T\|\nabla u_t\|_{L^2}^2dt\le C.$$

□

Lemma 4.12. *Under the condition* (4.3.2)*, it holds that for any* $0<T<T^*$*,*

$$\sup_{0\le t\le T}(\|\rho_t\|_{W^{1,q}}+\|u\|_{H^2}+\|P\|_{H^1})+\int_0^T(\|u\|_{W^{2,q}}^2+\|P\|_{W^{1,q}}^2)dt\le C. \tag{4.3.26}$$

Proof. By Lemma 4.6 and (4.3.4) , it is easy to deduce

$$\|u\|_{H^2}+\|P\|_{H^1}\le C\|\rho u_t\|_{L^2}+C\|\nabla u\|_{L^2}^3+C\le C, \tag{4.3.27}$$

with the aid of Lemma 4.9 and 4.11.

And, together with $(4.1.1)_1$, yields

$$\begin{aligned}\|\rho_t\|_{W^{1,q}}&\le C(\|\rho_t\|_{L^q}+\|\nabla\rho_t\|_{L^q})\\&\le C(\|u\cdot\nabla\rho\|_{L^q}+\|\nabla(u\cdot\nabla\rho)\|_{L^q})\\&\le C(\|u\|_{L^\infty}\|\nabla\rho\|_{L^q}+\|u\|_{L^\infty}\|\nabla^2\rho\|_{L^q}+\|\nabla u\|_{L^2}\|\nabla\rho\|_{L^{\frac{2q}{q-2}}})\\&\le C\|u\|_{H^2}\|\nabla\rho\|_{W^{1,q}}\le C.\end{aligned} \tag{4.3.28}$$

At last, applying (4.2.5) in Lemma 4.6 with $F=-\rho u_t-\rho u\cdot\nabla u-\operatorname{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)$, we get

(4.3.29)

$$\begin{aligned}&\|\nabla u\|_{W^{1,q}}+\|P\|_{W^{1,q}}\\&\le C(1+\|\nabla\rho\|_{L^\infty})(\|\rho u_t\|_{L^q}+\|\rho u\cdot\nabla u\|_{L^q}+\|\kappa(\rho)|\nabla^2\rho||\nabla\rho|\|_{L^q}+\|\kappa'(\rho)|\nabla\rho|^3\|_{L^q})\\&\le C(\|\rho u_t\|_{L^q}+\|\rho u\cdot\nabla u\|_{L^q}+1)\\&\le C(\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^1}^2+1).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^T(\|\nabla u\|_{W^{1,q}}^2+\|P\|_{W^{1,q}}^2)dt&\le C\int_0^T(\|\nabla u_t\|_{L^2}^2+\|\nabla u\|_{H^1}^4)dt+C\\&\le C\end{aligned} \tag{4.3.30}$$

Therefore we complete the proof of Lemma 4.12. □

Proof of Theorem 4.3. In fact, in view of (4.3.2) and (4.3.27), it is easy to see that the functions $(\rho,u)(x,t=T^*)=lim_{t\to T^*}(\rho,u)$ have the same regularities imposed on the initial data (4.1.5) at the time $t=T^*$. Furthermore,

$$\begin{aligned}&-\operatorname{div}(2\mu(\rho)d)+\nabla P+\operatorname{div}(\kappa(\rho)\nabla\rho\otimes\nabla\rho)|_{t=T^*}\\&=\lim_{t\to T^*}\rho^{\frac12}(\rho^{\frac12}u_t+\rho^{\frac12}u\cdot\nabla u)=:\rho^{\frac12}g|_{t=T^*}\end{aligned}$$

with $g = (\rho^{\frac{1}{2}} u_t + \rho^{\frac{1}{2}} u \cdot \nabla u)|_{t=T^*} \in L^2$ due to (4.3.16). Thus the functions $(\rho, u)|_{t=T^*}$ satisfy the compatibility condition (4.1.6) at time T^*. Therefore we can take $(\rho, u)|_{t=T^*}$ as the initial data and apply the local existence theorem (Theorem 4.2) to extend the local strong solution beyond T^*. This contradicts the definition of maximal existence time T^*, and thus, the proof of Theorem 4.3 is completed. □

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Chapter 5

On local strong solutions to the Cauchy problem of 2D inhomogeneous Navier-Stokes-Korteweg equations with vacuum

This chapter concerns the Cauchy problem of the inhomogeneous incompressible Navier-Stokes-Korteweg equations on the two-dimensional space with vacuum as the far field density. We prove that the 2D Cauchy problem of the inhomogeneous incompressible Navier-Stokes-Korteweg equations admits a unique local strong solution provided the initial density decays not too slow at infinity.

Keywords: Navier-Stokes-Korteweg; local strong solution; weighted density; uniqueness

5.1 Introduction and main result

We continue the topic of fluid mechanics model of the Korteweg type. In the chapter 3 and 4, we proved two blow-up criteria of the initial and boundary value problem of the inhomoegeneous Navier-Stokes-Korteweg equations over some bounded smooth domain in $\mathbb{R}^n, n = 2, 3$. An interesting problem is to consider the motion of fluids of Korteweg type over an unbounded domain, the typical example is the whole space $\mathbb{R}^n, n = 2, 3$. In this chapter, we consider the inhomoegeneous Navier-Stokes-Korteweg equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \triangle u + \nabla P + \kappa \operatorname{div}(\nabla \rho \otimes \nabla \rho) = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{5.1.1}$$

in $\mathbb{R}^2 \times (0, T)$, where $t \geq 0$ is time, $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ is spatial coordinate. The unknown functions $\rho(x,t), u(x,t) = (u_1(x,t), u_2(x,t))$ and $P(x,t)$ represent the density, velocity field and pressure of the fluid, respectively. The constants $\kappa > 0$ and $\mu > 0$ stand for the capillary and viscosity coefficients of the fluid respectively.

Let $\Omega = \mathbb{R}^2$ and we consider the Cauchy problem for (5.1.1) with the far field behavior condition(in the weak sense):

$$(\rho, u) \to (0, 0), \text{ as } |x| \to \infty, \tag{5.1.2}$$

and initial data:

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in } \mathbb{R}^2. \tag{5.1.3}$$

The Navier-Stokes-Korteweg equations are widely studied by many mathematicians since of its physical importance and mathematical complexity, especially a great of efforts have been devoted to the mathematical theory for compressible capillary fluids, see the references [3, 5, 6] and therein. In particular, if there is no capillary effect, that is $\kappa \equiv 0$, the Navier-Stokes-Korteweg system reduces to the well-konown Navier-Stokes equations, which have been studied extensively, see the Chapter 1 for more details on the Navier-Stokes model. When the capillary coefficient $\kappa > 0$, the study of the Navier-Stokes-Korteweg becomes rather difficult than the Navier-Stokes model since of the appearance of capillary effect. For the inhomogeneous incompressible Navier-Stokes-Korteweg equations (5.1.1) over a bounded domain Ω with smooth boundary $\partial\Omega$, under the following compatibility condition on the initial data:

$$-\operatorname{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) + \nabla P_0 + \operatorname{div}(\kappa(\rho_0)\nabla\rho_0 \otimes \rho_0) = \rho_0^{\frac{1}{2}} g, \quad \operatorname{div} u_0 = 0, \quad \text{in } \Omega, \tag{5.1.4}$$

for some $(P_0, g) \leq H^1(\Omega) \times L^2(\Omega)$, Tan-Wang [11] and Wang [12] established the local strong solutions to the initial and boundary value problem of inhomogeneous Navier-Stokes-Korteweg equations when the capillary $\kappa(\rho)$ and viscosity $\mu(\rho)$ are positive constants and variable functions of the density, respectively.

To my best knowledge, there is no any further results of establishing solutions to the Cauchy problem of the Navier-Stokes-Korteweg equations (5.1.1). Using the ideas of Chen-Tan-Wang [1] for 3D Cauchy problem of the inhomogeneous MHD system (2.1.1), the local strong solutions of the 3D Cauchy problem of Navier-Stokes-Korteweg equations (5.1.1) can be established in a similar way. However, some difficulties will bring out when we apply these ideas to the 2D case, since the Sobolev inequality is critical. Recently, Li-Liang [7] established the local strong solutions to the 2D Cauchy problem of the compressible Navier-Stokes equations with vacuum as far field density by deriving some spatial weighted energy estimates. Motivated by their work, Liang [8] proved the local existence of strong solutions to the 2D Cauchy problem of the inhomogeneous incompressible Navier-Stokes equations, that is (5.1.1) with $\kappa \equiv 0$. The purpose of this chapter is to establish local strong solutions to the Cauchy problem (5.1.1)-(5.1.3) as an extension of Liang's work [8] to Navier-Stokes-Korteweg model. First we give the definition of strong solutions to the Cauchy problem (5.1.1)-(5.1.3) as follows.

Definition 5.1 (Strong solution)**.** If all derivatives involved in (5.1.1) are regular distributions, and equations (5.1.1) hold almost everywhere in $\mathbb{R}^2 \times (0, T)$, then (ρ, u, P) is called a strong solution to (5.1.1).

Now we are ready to state the main result of this chapter, and we would like to point out that, in this section, for $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, we denote the standard Lesbegue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{R}^2), \quad W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}.$$

Theorem 5.2. *Let η_0 be a positive constant and*

$$\bar{x} := (e + |x|^2)^{\frac{1}{2}} \log^{1+\eta_0}(e + |x|^2). \tag{5.1.5}$$

For constants $q > 2$ and $a > 1$, assume that the initial data (ρ_0, u_0) satisfies

$$0 \le \bar{x}^a \rho_0 \in L^1 \cap H^2 \cap W^{2,q}, \sqrt{\rho} u_0 \in L^2, \nabla u_0 \in L^2 \ and \ \mathrm{div} u_0 = 0. \tag{5.1.6}$$

Then there exist a small time T_0 and a unique strong solution (ρ, u, P) to the Cauchy problem (5.1.1)-(5.1.3) *on $\mathbb{R}^2 \times (0, T_0]$ satisfying*

$$\begin{cases} 0 \le \rho \in C([0,T_0]; L^1 \cap H^2 \cap W^{2,q}), \\ \bar{x}^a \rho \in L^\infty(0, T_0; L^1 \cap H^2 \cap W^{2,q}), \\ \sqrt{\rho} u, \nabla u, \bar{x}^{-1} u, \sqrt{t}\sqrt{\rho} u_t, \sqrt{t} \nabla P, \sqrt{t} \nabla^2 u \in L^\infty(0, T_0; L^2), \\ \nabla u \in L^2(0, T_0; H^1) \cap L^{\frac{q+1}{q}}(0, T_0; W^{1,q}), \\ \nabla P \in L^2(0, T_0; L^2) \cap L^{\frac{q+1}{q}}(0, T_0; L^q), \\ \sqrt{t} \nabla u \in L^2(0, T_0; W^{1,q}), \\ \sqrt{\rho} u_t, \sqrt{t} \nabla u_t, \sqrt{t} \bar{x}^{-1} u_t \in L^2(\mathbb{R}^2 \times (0, T_0)), \end{cases} \tag{5.1.7}$$

and

$$\inf_{0 \le t \le T_0} \int_{B_N} \rho(x,t) dx \ge \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x) dx, \tag{5.1.8}$$

for some constant $N > 0$ and $B_N = \{x \in \mathbb{R}^2 | |x| < N\}$.

We now make some comments on the key ingredients of the analysis of this chapter. It should be pointed out that, for the whole two-dimensional space, it seems difficult to bound the $L^p(\mathbb{R}^2)$-norm of u just in terms of $\|\sqrt{\rho} u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$. Furthermore, the appearance of capillary term will bring out some new difficulties. In order to overcome these difficulties, we will make goof use of some key ideas due to [7, 8] where they deal with the compressible and inhomogeneous Navier-Stokes equations, repectively. On the other hand, motivated by [7], it is enough to bound the $L^p(\mathbb{R}^2)$-norm of the momentum ρu instead just of u. More precisely, using a Hardy-type inequality which is originally due to Lions [9], together with some careful analysis on the spatial weighted estimate of the density, we can obtain the desired estimates on the $L^p(\mathbb{R}^2)$-norm of the momentum ρu. Next, we then construct approximate solutions to (5.1.1) with density strictly positive, consider an initial and boundary value problem in any bounded ball B_R with radius R. Finally, combining all key points mentioned before with the similar arguments as in [2, 7, 8], we derive the desired bounds on the gradient of velocity and spatial weighted density, which are independent of both the radius of the balls B_R and the lower bound of the initial density.

Remark 5.3. After this work was completed, we found a recent work of Y. Liu, W. Wang and S. N. Zheng [10] closely related to ours. They also prove the local well-posedness of strong solution with vacuum to the Cauchy problem of two-dimensional nonhomogeneous incompressible Navier-Stokes-Korteweg equations. However, as is discussed in detail, see Remark 5.10, they need a stronger assumption on the initial data than ours, that is, except for the same regularity condition (5.1.6), the following compatibility condition on (ρ_0, u_0) is also necessary.

$$-\mu\triangle u_0 + \nabla P_0 + \kappa \mathrm{div}(\nabla\rho_0 \otimes \nabla\rho_0) = \rho_0^{\frac{1}{2}} g, \tag{5.1.9}$$

for some $(P_0, g) \in H^1 \times L^2(\mathbb{R}^2)$.

The rest of the chapter is arranged as follows. In Section 2, we collect some elementary facts and inequalities which will be needed in the later analysis. In Section 3, we will derive some a priori estimates which are used to obtain the local existence and uniqueness of strong solutions. The proof of main result Theorem 5.2 will be given in Section 4.

5.2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be frequently used later. First of all, if the initial density is strictly away from vacuum, the following local existence theorem on bounded balls can be shown by similar arguments as in [2, 11].

Lemma 5.4. *For $R > 0$ and $B_R = \{x \in \mathbb{R}^2 | |x| < R\}$, assume that (ρ_0, u_0) satisfies*

$$\rho_0 \in H^3(B_R), u_0 \in H^2(B_R), \inf_{x\in B_R} \rho_0(x) > 0, \mathrm{div} u_0 = 0. \tag{5.2.1}$$

Then there exists a small time $T_R > 0$ such that the equations (5.1.1) *with the following initial and boundary conditions*

$$\begin{aligned} &(\rho, u)(x, t = 0) = (\rho_0, u_0), && x \in B_R, \\ &u(x, t) = 0, && x \in \partial B_R, t > 0, \end{aligned} \tag{5.2.2}$$

has a unique strong solution (ρ, u, P) on $B_R \times (0, T_R]$ satisfying

$$\rho \in C([0, T_R]; H^3), \quad (\nabla u, P) \in C([0, T_R]; H^2) \cap L^2(0, T_R; H^3), \tag{5.2.3}$$

where we denote $H^k = H^k(B_R)$ for positive integer k.

Next, for $\Omega \subset \mathbb{R}^2$, the following weighted L^m-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega) := \{v \in H^1_{loc}(\Omega) | \nabla v \in L^2(\Omega)\}$ can be found in [9].

Lemma 5.5. *For $m \in [2, \infty)$ and $\theta \in (1 + \frac{m}{2}, \infty)$, there exists a positive constant C independent of Ω such that for either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ with $R \geq 1$ and for any $v \in \tilde{D}^{1,2}(\Omega)$,*

$$\left(\int_\Omega \frac{|v|^m}{e + |x|^2} (\log(e + |x|^2))^{-\theta} dx\right)^{\frac{1}{m}} \leq C\|v\|_{L^2(B_1)} + C\|\nabla v\|_{L^2(\Omega)}. \tag{5.2.4}$$

A useful consequence of Lemma 5.5 is the following crucial weighted bounds for elements of $\tilde{D}^{1,2}(\Omega)$, which has been proved in [7].

Lemma 5.6. *Let Ω be as in Lemma 5.5, and $\bar{x}$ and η_0 be as in* (5.1.5). *Assume that $\rho \in L^1 \cap L^\infty(\Omega)$ is a non-negative function such that*

$$\int_{B_{N_1}} \rho dx \geq M_1, \quad \|\rho\|_{L^1\cap L^\infty(\Omega)} \leq M_2, \tag{5.2.5}$$

for positive constants M_1, M_2 and $N_1 \geq 1$ with $B_{N_1} \subset \Omega$. Then for $\epsilon > 0$ and $\eta > 0$, there is a positive constant C depending only on $\epsilon, \eta, M_1, M_2, N_1$, and η_0 such that every $v \in \tilde{D}^{1,2}(\Omega)$ satisfies

$$\|v\bar{x}^{-\eta}\|_{L^{(2+\epsilon)/\tilde{\eta}}(\Omega)} \leq C\|\rho^{1/2}v\|_{L^2(\Omega)} + C\|\nabla v\|_{L^2(\Omega)} \tag{5.2.6}$$

with $\tilde{\eta} = \min\{1, \eta\}$.

5.3 A priori estimates

Throughout this section, we omit the integration domain B_R with $R > 0$ below for notations simplicity. For $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, the Lesbegue and Sobolev spaces on some ball B_R are defined in a standard way:

$$L^r = L^r(B_R), \quad W^{k,r} = W^{k,r}(B_R), \quad H^k = W^{k,2}.$$

Moreover, for $R > 4N_0 \geq 4$, assume that (ρ_0, u_0) satisifes, in addition to (5.2.1), that

$$\frac{1}{2} \leq \int_{B_{N_0}} \rho_0(x)dx \leq \int_{B_R} \rho_0(x)dx \leq \frac{3}{2}. \tag{5.3.1}$$

Lemma 5.4 thus yield that there exists some $T_R > 0$ such that the initial and boundary value problem (5.1.1) and (5.2.2) has a unique strong solution (ρ, u, P) on $B_R \times [0, T_R]$ satisfying (5.2.3).

Let $\bar{x}, \eta_0, a$ and q be as in Theorem 5.2, the main goal of this section is to derive the following key a priori estimate on $\psi(t)$ defined by

$$\psi(t) := 1 + \|\rho^{1/2}u\|_{L^2} + \|\nabla u\|_{L^2} + \|\bar{x}^a\rho\|_{L^1\cap H^2\cap W^{2,q}}. \tag{5.3.2}$$

Proposition 5.7. *Assume that (ρ_0, u_0) satisfies* (5.2.1) *and* (5.3.1). *Let (ρ, u, P) be the solution to the initial and boundary value problem* (5.1.1) *and* (5.2.2) *on $B_R\times(0, T_R]$ obtained by Lemma 5.4. Then there exist positive constants T_0 and M both depending only on $\mu, \kappa, q, a, \eta_0, N_0$ and E_0 such that*

$$\begin{aligned} &\sup_{0\leq t\leq T_0} \left(\psi(t) + \sqrt{t}\|\sqrt{\rho}u_t\|_{L^2} + \sqrt{t}\|\nabla^2 u\|_{L^2} + \sqrt{t}\|\nabla P\|_{L^2}\right) \\ &+ \int_0^{T_0} \left(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}}\right) dt \\ &+ \int_0^{T_0} \left(t\|\nabla^2 u\|_{L^q}^2 + t\|\nabla P\|_{L^q}^2 + t\|\nabla u_t\|_{L^2}^2\right) dt \leq M, \end{aligned} \tag{5.3.3}$$

where

$$E_0 := \|\sqrt{\rho_0}u_0\|_{L^2} + \|\nabla u_0\|_{L^2} + \|\bar{x}^a\rho_0\|_{L^1\cap H^2\cap W^{2,q}}.$$

The proof of Proposition 5.7 is composed of some lemmas. First, we give the following standard energy estimate for (ρ, u, P) and the estimate on the L^p-norm of the density.

Lemma 5.8. *Under the conditions of Proposition 5.7, let (ρ, u, P) be a solution to the initial and boundary problem* (5.1.1) *and* (5.2.2). *Then for any $t > 0$,*

$$\sup_{0\le s\le t}(\|\rho\|_{L^1\cap L^\infty} + \|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2) + \int_0^t\int |\nabla u|^2 dxds \le C. \tag{5.3.4}$$

Proof. First, it is easy to deduce from $(5.1.1)_1$ and $\mathrm{div}u = 0$ that

$$\sup_{0\le s\le t}\|\rho\|_{L^1\cap L^\infty} \le C. \tag{5.3.5}$$

Then applying the standard energy estimate to (5.1.1) gives

$$\sup_{0\le s\le t}(\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2) + \int_0^t\int |\nabla u|^2 dxds \le C. \tag{5.3.6}$$

This together with (5.3.5) yields (5.3.4) and completes the proof of Lemma 5.8. □

Next, we will derive the key estimate on the $\|\nabla u\|_{L^2(0,t;L^2)}$.

Lemma 5.9. *Under the conditions of Proposition 5.7, let (ρ, u, P) be a solution to the initial and boundary problem* (5.1.1) *and* (5.2.2). *Then there exists a $T_1 = T_1(N_0, E_0) > 0$ such that for all $t \in (0, T_1]$,*

$$\sup_{0\le s\le t}(\|\bar{x}^a\rho\|_{L^1} + \|\nabla u\|_{L^2}^2) + \int_0^t \|\sqrt{\rho}u_s\|_{L^2}^2 ds \le C + C\int_0^t \psi^\alpha(s)ds. \tag{5.3.7}$$

Proof. First, for $N > 1$, define a family of functions $\phi_N \in C_0^\infty(B_N)$ satisfying

$$0 \le \phi_N \le 1, \quad \phi_N(x) = 1, \text{ if } |x| \le N/2, \quad |\nabla^k\phi_N| \le CN^{-k}, \ k \in \mathbb{N}, \tag{5.3.8}$$

it follows from $(5.1.1)_1$ and (5.3.4) that

$$\begin{aligned}\frac{d}{dt}\int \rho\phi_{2N_0}dx &= \int \rho u\cdot\nabla\phi_{2N_0}dx\\ &\ge -CN_0^{-1}\left(\int\rho dx\right)^{1/2}\left(\int\rho|u|^2dx\right)^{1/2}\\ &\ge \tilde{C}(E_0N_0),\end{aligned} \tag{5.3.9}$$

where we used the fact $\int\rho dx = \int\rho_0 dx$ in the last inequality.

Integrating (5.3.9) over the time interval $(0, t)$ and using (5.3.1) gives

$$\inf_{0\le t\le T_1}\int_{B_{2N_0}}\rho dx \ge \inf_{0\le t\le T_1}\int\rho\phi_{2N_0}dx \ge \int\rho_0\phi_{2N_0}dx - \tilde{C}T_1 \ge \frac{1}{4}. \tag{5.3.10}$$

where we take $T_1 := \min\{1, (4\tilde{C})^{-1}\}$. From now on, we will always assume that $t \le T_1$. The combination of (5.3.10), (5.3.4) and (5.2.6) yields that for $\epsilon > 0$ and $\eta > 0$, every

$v \in \tilde{D}^{1,2}(B_R)$ satisfies

$$\|v\bar{x}^{-\eta}\|_{L^{(2+\epsilon)/\tilde{\eta}}} \le C\|\rho^{1/2}v\|_{L^2} + C\|\nabla v\|_{L^2} \tag{5.3.11}$$

with $\tilde{\eta} = \min\{1, \eta\}$.

Next, multiplying $(5.1.1)_1$ by $\bar{x}^a$ and integrating by parts imply that

$$\begin{aligned} \frac{d}{dt}\int \bar{x}^a \rho dx &\le C\int \rho|u|\bar{x}^{a-1}\log^{1+\eta_0}(e+|x|^2)dx \\ &\le C\|\rho\bar{x}^{a-1+\frac{8}{8+a}}\|_{L^{\frac{8+a}{7+a}}}\|u\bar{x}^{-\frac{4}{8+a}}\|_{L^{8+a}} \\ &\le C\|\rho\|_{L^\infty}^{\frac{1}{8+a}}\|\rho\bar{x}^a\|_{L^1}^{\frac{7+a}{8+a}}(\|\rho^{1/2}u\|_{L^2} + \|\nabla u\|_{L^2}) \\ &\le C(1+\|\rho\bar{x}^a\|_{L^1})(1+\|\nabla u\|_{L^2}) \end{aligned} \tag{5.3.12}$$

due to (5.3.4) and (5.3.11). This combined with Gronwall inequality and (5.3.4) lead to

$$\sup_{0\le s\le t}\|\rho\bar{x}^a\|_{L^1} \le C\exp\left\{C\int_0^t (1+\|\nabla u\|_{L^2}^2)ds\right\} \le C. \tag{5.3.13}$$

Now we are prepared to estimate the first order derivatives of the velocity. Multiplying $(5.1.1)_2$ by u_t and integrating by parts, one has

$$\begin{aligned} &\int \rho|u_t|^2dx + \mu\frac{d}{dt}\int|\nabla u|^2dx \\ =& -\int(\rho u\cdot\nabla u)\cdot u_t dx + \kappa\int\nabla\rho\otimes\nabla\rho : \nabla u_t dx \\ =&\kappa\frac{d}{dt}\int\nabla\rho\otimes\nabla\rho:\nabla u dx + 2\int\kappa\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u dx \\ &-\int(\rho u\cdot\nabla u)\cdot u_t dx. \end{aligned} \tag{5.3.14}$$

First, it follows from (5.3.4), (5.3.11), and (5.3.13) that for any $\epsilon > 0$ and $\eta > 0$,

$$\begin{aligned} \|\rho^\eta v\|_{L^{(2+\epsilon)/\tilde{\eta}}} &\le C\|\rho^\eta\bar{x}^{\frac{3\tilde{\eta}a}{4(2+\epsilon)}}\|_{L^{4(2+\epsilon)/3\tilde{\eta}}}\|v\bar{x}^{-\frac{3\tilde{\eta}a}{4(2+\epsilon)}}\|_{L^{4(2+\epsilon)/\tilde{\eta}}} \\ &\le C\left(\int\rho^{\frac{4(2+\epsilon)\eta}{3\tilde{\eta}}-1}\rho\bar{x}^a dx\right)^{\frac{3\tilde{\eta}}{4(2+\epsilon)}}\|v\bar{x}^{-\frac{3\tilde{\eta}a}{4(2+\epsilon)}}\|_{L^{4(2+\epsilon)/\tilde{\eta}}} \\ &\le C\|\rho\|_{L^\infty}^{\frac{4(2+\epsilon)\eta-3\tilde{\eta}}{4(2+\epsilon)}}\|\rho\bar{x}^a\|_{L^1}^{\frac{3\tilde{\eta}}{4(2+\epsilon)}}(\|\rho^{1/2}v\|_{L^2}+\|\nabla v\|_{L^2}) \\ &\le C\|\rho^{\frac{1}{2}}v\|_{L^2} + C\|\nabla v\|_{L^2}, \end{aligned} \tag{5.3.15}$$

where $\tilde{\eta} = \min\{1,\eta\}$ and $v \in \tilde{D}^{1,2}(B_R)$. In particular, this together with (5.3.4) and (5.3.11) derives

$$\|\rho^\eta u\|_{L^{(2+\epsilon)/\tilde{\eta}}} + \|u\bar{x}^{-\eta}\|_{L^{(2+\epsilon)/\tilde{\eta}}} \le C(1+\|\nabla u\|_{L^2}). \tag{5.3.16}$$

Then we estimate the terms in the right hand side of (5.3.14). First, the combination of the Cauchy-Schwarz and Gagliardo-Nirenberg inequalities yields

$$\begin{aligned}\int(\rho u\cdot\nabla u)\cdot u_t dx &\le \frac{1}{4}\int\rho|u_t|^2dx + C\int\rho|u|^2|\nabla u|^2dx\\ &\le \frac{1}{4}\int\rho|u_t|^2dx + C\|\rho^{\frac{1}{2}}u\|_{L^8}^2\|\nabla u\|_{L^{8/3}}^2\\ &\le \frac{1}{4}\int\rho|u_t|^2dx + C\|\rho^{\frac{1}{2}}u\|_{L^8}^2\|\nabla u\|_{L^2}^{3/2}\|\nabla u\|_{H^1}^{1/2}\\ &\le \frac{1}{4}\int\rho|u_t|^2dx + C\psi^\alpha+\epsilon\|\nabla^2u\|_{L^2}^2,\end{aligned} \tag{5.3.17}$$

where (and in what follows) we use $\alpha > 1$ to denote a generic constant, which may different from line to line.

For the second term on the right hand side of (5.3.14), integration by parts together with Gagliardo-Nirenberg inequality deduces that

$$\begin{aligned}&\int\kappa\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u dx\\ \le{}& C\int|\nabla\rho|^2|\nabla u|^2dx + C\int|\nabla^2\rho||\nabla\rho||u||\nabla u|dx\\ \le{}& C\|\nabla\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + C\|\nabla\rho\|_{L^\infty}\|\bar{x}^a\nabla^2\rho\|_{L^q}\|\bar{x}^{-a}u\|_{L^{q^*}}\|\nabla u\|_{L^2}\\ \le{}& C\psi^\alpha.\end{aligned} \tag{5.3.18}$$

Here $\frac{1}{q}+\frac{1}{q^*}=1.$ and $q^*>2$. Inserting (5.3.17) and (5.3.18) into (5.3.14) gives

$$\begin{aligned}&\frac{1}{2}\int\rho|u_t|^2dx + \frac{d}{dt}\int(\mu|\nabla u|^2-\kappa\nabla\rho\otimes\nabla\rho:\nabla u)dx\\ \le{}&\epsilon\|\nabla^2u\|_{L^2}^2+C\psi^\alpha.\end{aligned} \tag{5.3.19}$$

Differentiating the continuity equation $(5.1.1)_1$ with respect to $x_i, i=1,2$, we get

$$(\partial_{x_i}\rho)_t + u\cdot\nabla(\partial_{x_i}\rho) + \partial_{x_i}u\cdot\nabla\rho = 0, \tag{5.3.20}$$

multiplying (5.3.20) by $4|\partial_{x_i}\rho|^2\partial_{x_i}\rho$, integration by parts over the domain B_R yields

$$\begin{aligned}\frac{d}{dt}\|\partial_{x_i}\rho\|_{L^4}^4 &\le C\int|\nabla u||\nabla\rho||\partial_{x_i}\rho|^3dx\\ &\le C\|\nabla u\|_{L^2}\|\nabla\rho\|_{L^2}\|\partial_{x_i}\rho\|_{L^\infty}^3\\ &\le C\psi^\alpha(t).\end{aligned} \tag{5.3.21}$$

Integrating (5.3.21) over the time interval $(0,t)$ lead to

$$\sup_{0\le s\le t}\|\nabla\rho\|_{L^4}^4 \le C + C\int\psi^\alpha ds. \tag{5.3.22}$$

On the other hand, since (ρ, u, P) satisfies the following Stokes system,

$$-\mu\triangle u + \nabla P = -\rho u_t - \rho u\cdot\nabla u - \kappa\mathrm{div}(\nabla\rho\otimes\nabla\rho),$$

applying the standard L^p-estimate, then

$$\begin{aligned}
&\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \\
&\le C\|\rho u_t\|_{L^2}^2 + C\|\rho u\cdot\nabla u\|_{L^2}^2 + C\||\nabla\rho||\nabla^2\rho|\|_{L^2}^2 \\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{H^1} + C\|\nabla\rho\|_{L^\infty}^2\|\nabla^2\rho\|_{L^2}^2 \\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{2}\|\nabla^2 u\|_{L^2}^2 + C\psi^\alpha,
\end{aligned} \tag{5.3.23}$$

which implies that,

$$\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha. \tag{5.3.24}$$

Substituting (5.3.24) into (5.3.19) and choosing ϵ suitably small, one gets

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 + \int_0^t\int \rho|u_t|^2 dxds &\le C + C\|\nabla\rho\|_{L^4}^4 + C\int_0^t \psi^\alpha ds \\
&\le C + C\int_0^t \psi^\alpha ds,
\end{aligned} \tag{5.3.25}$$

where in the second inequality we have used (5.3.22). Thus we complete the proof of Lemma 5.9.

□

Remark 5.10. Here we want to give some comments on the proof of Lemma 5.9. As the same with Y. Liu et. al. [10], this lemma is used to derive the L^∞-estimate on $\|\nabla u\|_{L^2}$. The different part is the treatment of the capillary term $\int \operatorname{div}(\nabla\rho\otimes\nabla\rho)\cdot u_t dx$. In the paper of Y. Liu et. al. [10], they remark from the divergence free property of the velocity that $\int \operatorname{div}(\nabla\rho\otimes\nabla\rho)\cdot u_t dx = \int \triangle\rho\nabla\rho\cdot u_t dx$. Then combining the Hardy-type inequality and Hölder inequality, they complete the estimate in terms of $\psi(t)$ and $\|\nabla u_t\|_{L^2}$. In order to close the estimate, they have to derive the estimate of $\sup\|\sqrt{\rho}u_t\|_{L^2}$ in the next step, therefore the initial value of $\sup\|\sqrt{\rho}u_t\|_{L^2}$ will be involved, to bound this term, the compatibility condition (5.1.9) is necessary. My way is different, we observe that

$$\int \nabla\rho\otimes\nabla\rho : \nabla u_t dx = \frac{d}{dt}\int \nabla\rho\otimes\nabla\rho : \nabla u dx + 2\int \nabla(u\cdot\nabla\rho)\otimes\nabla\rho : \nabla u dx, \tag{5.3.26}$$

and to bound the first term of the right-hand side in (5.3.26), we also derive a new estimate for the density, see (5.3.22).

Lemma 5.11. *Let (ρ, u, P) and T_1 be as in Lemma 5.9. Then there exists a positive constant $\alpha > 1$, such that for all $t \in (0, T_1]$,*
(5.3.27)

$$\sup_{0\le s\le t}(s\|\sqrt{\rho}u_s\|_{L^2}^2) + \int_0^t (\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + s\|\nabla^2 u\|_{L^q}^2 + s\|\nabla u_s\|_{L^2}^2)dt \le C\exp\left(C\int_0^t \psi^\alpha ds\right).$$

Proof. Differentiating the momentum equations $(5.1.1)_2$ with respect to t, using the continuity equation $(5.1.1)_1$, we derive

$$\begin{aligned}
&\rho u_{tt} + \rho u\cdot\nabla u_t - \mu\triangle u_t + \nabla P_t \\
&= -\rho_t(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u - \kappa\triangle\rho\nabla(u\cdot\nabla\rho) - \kappa\triangle(u\cdot\nabla\rho)\nabla\rho.
\end{aligned} \tag{5.3.28}$$

Multiplying (5.3.28) by u_t, we get after integration by parts over B_R that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx + \mu\int |\nabla u_t|^2dx \\
&\le C\int \rho|u|(|\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u|)dx + C\int \rho|u|^2|\nabla u||\nabla u_t|dx \\
&\quad + C\int \rho|u_t|^2|\nabla u|dx + \int |\triangle\rho||\nabla(u\cdot\nabla\rho)||u_t|dx + |\int \triangle(u\cdot\nabla\rho)(\nabla\rho\cdot u_t)dx| \\
&:= \sum_{i=1}^{5} J_i.
\end{aligned} \tag{5.3.29}$$

Now let us estimate the terms on the right hand side of (5.3.29) one by one. First

$$\begin{aligned}
J_1 &\le C\|\rho^{\frac{1}{2}}u\|_{L^6}\|\rho^{\frac{1}{2}}u_t\|_{L^2}^{\frac{1}{2}}\|\rho^{\frac{1}{2}}u_t\|_{L^6}^{\frac{1}{2}}(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^4}^2) \\
&\quad + C\|\rho^{\frac{1}{4}}u\|_{L^{12}}^2\|\rho^{\frac{1}{2}}u_t\|_{L^2}^{\frac{1}{2}}\|\rho^{\frac{1}{2}}u_t\|_{L^6}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2} \\
&\le C(1+\|\nabla u\|_{L^2}^2)\|\rho^{\frac{1}{2}}u_t\|_{L^2}^{\frac{1}{2}}(\|\rho^{\frac{1}{2}}u_t\|_{L^2} + \|\nabla u_t\|_{L^2})^{\frac{1}{2}} \\
&\quad \times (\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} + \|\nabla^2 u\|_{L^2}) \\
&\le \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\psi^\alpha\|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha + C(1+\|\nabla u\|_{L^2}^2)\|\nabla^2 u\|_{L^2}^2.
\end{aligned} \tag{5.3.30}$$

Then, Hölder inequality combined with (5.3.16) leads to

$$\begin{aligned}
J_2 + J_3 &\le C\|\rho^{\frac{1}{2}}u\|_{L^8}^2\|\nabla u\|_{L^4}\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\rho^{\frac{1}{2}}u_t\|_{L^6}^{\frac{3}{2}}\|\rho^{\frac{1}{2}}u_t\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\psi^\alpha\|\sqrt{\rho}u_t\|_{L^2}^2 + C(\psi^\alpha + \|\nabla^2 u\|_{L^2}^2).
\end{aligned} \tag{5.3.31}$$

Next,

$$\begin{aligned}
J_4 &\le C\int |\nabla^2\rho||u||\triangle\rho||u_t|dx + C\int |\nabla\rho||\nabla u||\triangle\rho||u_t|dx \\
&\le C\|\bar{x}^a\nabla^2\rho\|_{L^q}\|\bar{x}^a\triangle\rho\|_{L^q}\|\bar{x}^{-a}u\|_{L^{q^*}}\|\bar{x}^{-a}u_t\|_{L^{q^*}} \\
&\quad + C\|\nabla\rho\|_{L^\infty}\|\bar{x}^a\triangle\rho\|_{L^q}\|\nabla u\|_{L^2}\|\bar{x}^{-a}u_t\|_{L^{q^*}} \\
&\le C\psi^\alpha(1+\|\nabla u\|_{L^2})(\|\rho^{\frac{1}{2}}u_t\|_{L^2} + \|\nabla u_t\|_{L^2}) + C\psi^\alpha(\|\rho^{\frac{1}{2}}u_t\|_{L^2} + \|\nabla u_t\|_{L^2}) \\
&\le \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\psi^\alpha\|\sqrt{\rho}u_t\|_{L^2}^2.
\end{aligned} \tag{5.3.32}$$

Finally,

$$\begin{aligned}
J_5 &= |\int \triangle(u\cdot\nabla\rho)(\nabla\rho\cdot u_t)dx| \\
&\le C\int |\nabla^2\rho|^2|u||u_t|dx + \int |\nabla\rho||\nabla^2\rho||\nabla u||u_t|dx \\
&\quad + \int |\nabla^2\rho||\nabla\rho||u||\nabla u_t|dx + \int |\nabla\rho|^2|\nabla u||\nabla u_t|dx \\
&\le \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\psi^\alpha\|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha.
\end{aligned} \tag{5.3.33}$$

Inserting the estimates (5.3.30)-(5.3.33) into (5.3.29), we get

$$\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx + \mu\int |\nabla u_t|^2dx \le C\psi^\alpha(1+\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2). \tag{5.3.34}$$

Multiplying (5.3.34) by t, using Gronwall inequality, we get

$$\sup_{0\le s\le t}(s\|\sqrt{\rho}u_s\|_{L^2}^2) + \int_0^t (s\|\nabla u_s\|_{L^2}^2)dt \le C\exp\left(C\int_0^t \psi^\alpha ds\right). \tag{5.3.35}$$

Finally, we show that

$$\begin{aligned}&\int_0^t \left(\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} + s\|\nabla^2 u\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2\right)ds\\ &\le C\exp\left\{C\int_0^t \psi^\alpha(s)ds\right\}.\end{aligned} \tag{5.3.36}$$

Applying the Stokes estimate and Gagliardo-Nirenberg inequality, one has

(5.3.37)

$$\begin{aligned}\|\nabla^2 u\|_{L^q} + \|\nabla P\|_{L^q} &\le C(\|\rho u_t\|_{L^q} + \|\rho u\cdot\nabla u\|_{L^q} + \||\nabla\rho||\nabla^2\rho|\|_{L^q})\\ &\le C(\|\rho u_t\|_{L^q} + \|\rho u\|_{L^{2q}}\|\nabla u\|_{L^{2q}} + \|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^q})\\ &\le C\|\rho u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\rho u_t\|_{L^{q^2}}^{\frac{q^2-2q}{q^2-2}} + C\psi^\alpha(1+\|\nabla^2 u\|_{L^2}^{\frac{q-1}{q}})\\ &\le C(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\nabla u_t\|_{L^2}^{\frac{q^2-2q}{q^2-2}} + \|\sqrt{\rho}u_t\|_{L^2}) + C\psi^\alpha(1+\|\nabla^2 u\|_{L^2}^{\frac{q-1}{q}}),\end{aligned}$$

which together with (5.3.7) and (5.3.35) implies that

$$\begin{aligned}&\int_0^t(\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}})ds\\ &\le C\int_0^t s^{-\frac{q+1}{2q}}(s\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{q^2-1}{q(q^2-2)}}(s\|\nabla u_t\|_{L^2}^2)^{\frac{(q-2)(q+1)}{2(q^2-2)}}ds\\ &\quad + \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^{\frac{q+1}{q}}ds + C\int_0^t \psi^\alpha(1+\|\nabla^2 u\|_{L^2}^{\frac{q^2-1}{q^2}})ds\\ &\le C\sup_{0\le s\le t}(s\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{q^2-1}{q(q^2-2)}}\int_0^t s^{-\frac{q+1}{2q}}(s\|\nabla u_t\|_{L^2}^2)^{\frac{(q-2)(q+1)}{2(q^2-2)}}ds\\ &\quad + C\int_0^t(\psi^\alpha + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2)ds\\ &\le C\exp\left\{C\int_0^t \psi^\alpha(s)ds\right\}\left(1+\int_0^t(s^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}} + s\|\nabla u_t\|_{L^2}^2)ds\right)\\ &\le C\exp\left\{C\int_0^t \psi^\alpha(s)ds\right\}.\end{aligned} \tag{5.3.38}$$

and

$$
\begin{aligned}
&\int_0^t (s\|\nabla^2 u\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2) ds \\
&\le C\int_0^t s\|\sqrt{\rho}u_t\|_{L^2}^2 ds + C\int_0^t (s\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{2(q-1)}{q^2-2}} (s\|\nabla u_t\|_{L^2}^2)^{\frac{q^2-2q}{q^2-2}} ds \\
&\quad + C\int_0^t s\psi^\alpha (1 + \|\nabla^2 u\|_{L^2}^{\frac{2(q-1)}{q}}) ds \\
&\le C\int_0^t s\|\sqrt{\rho}u_t\|_{L^2}^2 ds + C\int_0^t s\|\nabla u_t\|_{L^2}^2 ds + C\int_0^t (\psi^\alpha + s\|\nabla^2 u\|_{L^2}^2) ds \\
&\le C\exp\left\{C\int_0^t \psi^\alpha(s) ds\right\}.
\end{aligned} \tag{5.3.39}
$$

Therefore we complete the proof of Lemma 5.11. □

Lemma 5.12. *Let (ρ, u, P) and T_1 be as in Lemma 5.11. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$
\sup_{0\le s\le t} \|\bar{x}^a \rho\|_{L^1\cap H^2\cap W^{2,q}} \le \exp\left\{C\exp\left\{C\int_0^t \psi^\alpha ds\right\}\right\}. \tag{5.3.40}
$$

Proof. First, it follows from Sobolev inequality and that for $\delta \in (0,1)$,

$$
\begin{aligned}
\|u\bar{x}^{-\delta}\|_{L^\infty} &\le C(\delta)\left(\|u\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} + \|\nabla(u\bar{x}^{-\delta})\|_{L^3}\right) \\
&\le C(\delta)\left(\|u\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} + \|\nabla u\|_{L^3} + \|u\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^{\frac{12}{4-3\delta}}}\right) \\
&\le C(\delta)(\psi^\alpha + \|\nabla^2 u\|_{L^2}).
\end{aligned} \tag{5.3.41}
$$

Multiplying the continuity equation $(5.1.1)_1$ by $\bar{x}^a$, after some simple calculation, we get

$$
\partial_t(\bar{x}^a\rho) + u\cdot\nabla(\bar{x}^a\rho) - a\bar{x}^a\rho u\cdot\nabla\log\bar{x} = 0. \tag{5.3.42}
$$

To obtain the estimate of first order spatial derivatives of $\bar{x}^a\rho$, we differentiate (5.3.42) with respect to $x_i, i = 1, 2$:

$$
\begin{aligned}
&\partial_t\partial_{x_i}(\bar{x}^a\rho) + u\cdot\nabla\partial_{x_i}(\bar{x}^a\rho) + \partial_{x_i}u\cdot\nabla(\bar{x}^a\rho) \\
&\qquad - a\partial_{x_i}(\bar{x}^a\rho)u\cdot\nabla\log\bar{x} - a\bar{x}^a\rho\partial_{x_i}(u\cdot\nabla\log\bar{x}) = 0.
\end{aligned} \tag{5.3.43}
$$

Multiplying (5.3.43) by $r|\partial_i(\bar{x}^a\rho)|^{r-2}\partial_i(\bar{x}^a\rho)$ for $r \in [2, q]$, and integrating the resulting equality over B_R, we get

$$
\begin{aligned}
\frac{d}{dt}\|\nabla(\bar{x}^a\rho)\|_{L^r} &\le C(1 + \|\nabla u\|_{L^\infty} + \|u\cdot\nabla\log\bar{x}\|_{L^\infty})\|\nabla(\bar{x}^a\rho)\|_{L^r} \\
&\qquad + C\|\bar{x}^a\rho\|_{L^\infty}\|\nabla(u\nabla\log\bar{x})\|_{L^r}.
\end{aligned} \tag{5.3.44}
$$

To obtain the second order spatial derivatives of $\bar{x}^a\rho$, differentiate the equation (5.3.43) with respect to $x_j, j = 1, 2$, after some calculation, one has

$$\begin{aligned}&\partial_t\partial_{x_i}\partial_{x_j}(\bar{x}^a\rho) + u\cdot\nabla\partial_{x_i}\partial_{x_j}(\bar{x}^a\rho) + \partial_j u\cdot\nabla(\partial_{x_i}(\bar{x}^a\rho)) + \partial_i u\cdot\nabla(\partial_{x_j}(\bar{x}^a\rho))\\&+\partial_{x_i}\partial_{x_j}u\cdot\nabla(\bar{x}^a\rho) - a\partial_{x_i}\partial_{x_j}(\bar{x}^a\rho)u\cdot\nabla\log\bar{x} - a\partial_{x_i}(\bar{x}^a\rho)\partial_{x_j}(u\cdot\nabla\log\bar{x})\\&-a\partial_{x_j}(\bar{x}^a\rho)\partial_{x_i}(u\cdot\nabla\log\bar{x}) - a(\bar{x}^a\rho)\partial_{x_i}\partial_{x_j}(u\cdot\nabla\log\bar{x}) = 0,\end{aligned} \tag{5.3.45}$$

multiplying (5.3.45) by $r|\partial_i\partial_j(\bar{x}^a\rho)|^{r-2}\partial_i\partial_j(\bar{x}^a\rho)$ for $r\in[2,q]$, and integrating the resulting equality over B_R, and using $(5.1.1)_1$, we derive

$$\begin{aligned}\frac{d}{dt}\|\nabla^2(\bar{x}^a\rho)\|_{L^r} \le\ & C(1+\|\nabla u\|_{L^\infty}+\|u\cdot\nabla\log\bar{x}\|_{L^\infty})\|\nabla^2(\bar{x}^a\rho)\|_{L^r}\\&+C\|\bar{x}^a\rho\|_{L^\infty}\|\nabla^2(u\nabla\log\bar{x})\|_{L^r}\\&+C\|\nabla(\bar{x}^a\rho)\|_{L^\infty}(\|\nabla^2 u\|_{L^r}+\|\nabla(u\nabla\log\bar{x})\|_{L^r}),\end{aligned} \tag{5.3.46}$$

combining it with (5.3.44), and summing up for $i, j = 1, 2$, leads to

$$\begin{aligned}\frac{d}{dt}\|\nabla(\bar{x}^a\rho)\|_{W^{1,r}} \le\ & C(1+\|\nabla u\|_{L^\infty}+\|u\cdot\nabla\log\bar{x}\|_{L^\infty})\|\nabla(\bar{x}^a\rho)\|_{W^{1,r}}\\&+C\|\bar{x}^a\rho\|_{L^\infty}(\|\nabla(u\nabla\log\bar{x})\|_{L^r}+\|\nabla^2(u\nabla\log\bar{x})\|_{L^r})\\&+C\|\nabla(\bar{x}^a\rho)\|_{L^\infty}(\|\nabla^2 u\|_{L^r}+\|\nabla(u\nabla\log\bar{x})\|_{L^r})\\ \le\ & C(\psi^\alpha+\|\nabla^2 u\|_{L^2\cap L^q})(1+\|\nabla(\bar{x}^a\rho)\|_{W^{1,r}}+\|\nabla(\bar{x}^a\rho)\|_{W^{1,q}})\end{aligned} \tag{5.3.47}$$

Using (5.3.7), (5.3.36), (5.3.13), (5.3.44), (5.3.47), and Gronwall inequality, one thus get (5.3.40) , therefore we complete the proof of Lemma 5.12. □

Now, we are in a position to give a proof of Proposition 5.7, which is a direct consequence of Lemmas 5.8-5.12.

Proof of Proposition 5.7. It follows from (5.3.4), (5.3.7), and (5.3.40) that

$$\psi(t)\le\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}.$$

Standard arguments yield that for $M := e^{Ce}$ and $T_0 := \min\{T_1, (CM^\alpha)^{-1}\}$,

$$\sup_{0\le t\le T_0}\psi(t)\le M.$$

This combines with (5.3.24) and (5.3.27) gives

$$\sup_{0\le t\le T_0}(t\|\nabla^2 u\|_{L^2}^2+t\|\nabla P\|_{L^2}^2)\le C(M),$$

which together with (5.3.7), (5.3.27), (5.3.40) gives (5.3.3). Therefore the proof of Proposition 5.7 is completed. □

5.4 Local existence and uniqueness of strong solutions

This section is devoting to prove the main result Theorem 5.2 with the aid of the a priori estimates obtained in Section 3.

Let (ρ_0, u_0) be as in Theorem 5.2. Without loss of generality, the initial density ρ_0 is assumed to satisfy

$$\int_{\mathbb{R}^2} \rho_0 dx = 1,$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 dx = \frac{3}{4}. \tag{5.4.1}$$

We construct $\rho_0^R = \hat{\rho}_0^R + R^{-1} e^{-|x|^2}$, where $0 \leq \hat{\rho}_0^R \in C_0^\infty(\mathbb{R}^2)$ satisfies

$$\begin{cases} \displaystyle\int_{B_{N_0}} \hat{\rho}_0^R dx \geq \frac{1}{2}, \\ \bar{x}^a \hat{\rho}_0^R \to \bar{x}^a \rho_0 \qquad \text{in } L^1(\mathbb{R}^2) \cap H^2(\mathbb{R}^2) \cap W^{2,q}(\mathbb{R}^2), \text{as } R \to \infty. \end{cases} \tag{5.4.2}$$

Since $\nabla u_0 \in L^2(\mathbb{R}^2)$, choosing $v_i^R \in C_0^\infty(B_R)(i = 1, 2)$ such that

$$\lim_{R \to \infty} \|v_i^R - \partial_i u_0\|_{L^2(\mathbb{R}^2)} = 0, \quad i = 1, 2. \tag{5.4.3}$$

We consider the unique smooth solution u_0^R of the following elliptic problem:

$$\begin{cases} -\triangle u_0^R + \rho_0^R u_0^R + \nabla P_0^R = \sqrt{\rho_0^R} h^R - \partial_i v_i^R, & \text{in } B_R, \\ \operatorname{div} u_0^R = 0, & \text{in } B_R, \\ u_0^R = 0, & \text{on } \partial B_R, \end{cases} \tag{5.4.4}$$

where $h^R = (\sqrt{\rho_0} u_0) * j_{1/R}$ with j_δ being the standard mollifying kernel with width δ.

Extending u_0^R to $\mathbb{R}^2$ by defining 0 outside B_R and denoting it by $\tilde{u}_0^R$, we claim that

$$\lim_{R \to \infty} \left(\|\nabla(\tilde{u}_0^R - u_0)\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0^R} \tilde{u}_0^R - \sqrt{\rho_0} u_0\|_{L^2(\mathbb{R}^2)} \right) = 0. \tag{5.4.5}$$

In fact, it is easy to find that $\tilde{u}_0^R$ is also a solution of (5.4.4) in $\mathbb{R}^2$. Multiplying (5.4.4) by $\tilde{u}_0^R$ and integrating the resulting equation over $\mathbb{R}^2$ lead to

$$\begin{aligned} &\int_{\mathbb{R}^2} \rho_0^R |\tilde{u}_0^R|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{u}_0^R|^2 dx \\ &\leq \|\sqrt{\rho_0^R} \tilde{u}_0^R\|_{L^2(B_R)} \|h^R\|_{L^2(B_R)} + C \|v_i^R\|_{L^2(B_R)} \|\partial_i \tilde{u}_0^R\|_{L^2(B_R)} \\ &\leq \frac{1}{2} \|\nabla \tilde{u}_0^R\|_{L^2(B_R)}^2 + \frac{1}{2} \int_{\mathbb{R}^2} \rho_0^R |\tilde{u}_0^R|^2 dx + C \|h^R\|_{L^2(B_R)}^2 + C \|v_i^R\|_{L^2(B_R)}^2, \end{aligned} \tag{5.4.6}$$

which implies

$$\int_{\mathbb{R}^2} \rho_0^R |\tilde{u}_0^R|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{u}_0^R|^2 dx \leq C \tag{5.4.7}$$

for some C independent of R. This together with (5.4.2) yields that there exist a subsequence $R_j \to \infty$ and a function $\tilde{u}_0 \in \{\tilde{u}_0 \in H^1_{loc}(\mathbb{R}^2) | \sqrt{\rho_0}\tilde{u}_0 \in L^2(\mathbb{R}^2), \nabla\tilde{u}_0 \in L^2(\mathbb{R}^2)\}$ such that

$$\begin{cases} \sqrt{\rho_0^{R_j}}\,\tilde{u}_0^{R_j} \rightharpoonup \sqrt{\rho_0}\tilde{u}_0 \ \text{ weakly in } \ L^2(\mathbb{R}^2), \\ \nabla\tilde{u}_0^{R_j} \rightharpoonup \nabla\tilde{u}_0 \ \text{ weakly in } \ L^2(\mathbb{R}^2). \end{cases} \tag{5.4.8}$$

Next we will show

$$\tilde{u}_0 = u_0. \tag{5.4.9}$$

Indeed, multiplying (5.3.12) by a test function $\pi \in C_0^\infty(\mathbb{R}^2)$ with $\mathrm{div}\pi = 0$, it holds that

$$\int_{\mathbb{R}^2} \partial_i(\tilde{u}_0^{R_j} - u_0) \cdot \partial_i \pi dx + \int_{\mathbb{R}^2} \sqrt{\rho_0^{R_j}} (\sqrt{\rho_0^{R_j}}\,\tilde{u}_0^{R_j} - h^{R_j}) \cdot \pi dx = 0. \tag{5.4.10}$$

Let $R_j \to \infty$, it follows from (5.4.2), (5.4.3) and (5.4.8) that

$$\int_{\mathbb{R}^2} \partial_i(\tilde{u}_0 - u_0) \cdot \partial_i \pi dx + \int_{\mathbb{R}^2} \rho_0(\tilde{u}_0 - u_0) \cdot \pi dx = 0, \tag{5.4.11}$$

which implies (5.4.9).

Furthermore, multiplying (5.4.4) by $\tilde{u}_0^R$ and integrating the resulting equation over $\mathbb{R}^2$, by the same arguments as (5.4.11), we have

$$\lim_{R_j\to\infty} \int_{\mathbb{R}^2} (\rho_0^{R_j}|\tilde{u}_0^{R_j}|^2 + |\nabla\tilde{u}_0^{R_j}|^2) dx = \int_{\mathbb{R}^2} (\rho_0|u_0|^2 + |\nabla u_0|^2) dx,$$

which combined with (5.4.8) leads to

$$\lim_{R_j\to\infty} \int_{\mathbb{R}^2} |\nabla\tilde{u}_0^{R_j}|^2 dx = \int_{\mathbb{R}^2} |\nabla\tilde{u}_0|^2 dx, \quad \lim_{R_j\to\infty} \int_{\mathbb{R}^2} \rho_0^{R_j}|\tilde{u}_0^{R_j}|^2 dx = \int_{\mathbb{R}^2} \rho_0|\tilde{u}_0|^2 dx.$$

This, along with (5.4.9) and (5.4.8), gives (5.4.5).

Hence, by virtue of Lemma 5.4, the initial and boundary value problem (5.1.1) and (5.2.2) with the initial data (ρ_0^R, u_0^R) has a classical solution (ρ^R, u^R, P^R) on $B_R \times [0, T_R]$. Moreover, Proposition 5.7 shows that there exists a T_0 independent of R such that holds for (ρ^R, u^R, P^R).

For simplicity, in what follows, we denote

$$L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2).$$

Extending (ρ^R, u^R, P^R) by zero on $\mathbb{R}^2/B_R$ and denoting it by

$$(\tilde{\rho}^R = \phi_R\rho^R, \tilde{u}^R, \tilde{P}^R)$$

with ϕ_R satisfying (5.3.8). First, (5.3.3) leads to

$$\sup_{0\le t\le T_0} \left(\|\sqrt{\tilde{\rho}^R}\tilde{u}^R\|_{L^2} + \|\nabla\tilde{u}^R\|_{L^2} \right) \le \sup_{0\le t\le T_0} \left(\|\sqrt{\rho^R}u^R\|_{L^2(B_R)} + \|\nabla u^R\|_{L^2(B_R)} \right) \le C, \tag{5.4.12}$$

and

$$\sup_{0\le t\le T_0}\|\bar{x}^a\tilde{\rho}^R\|_{L^1\cap L^\infty}\le C. \tag{5.4.13}$$

Similarly, it follows from (5.3.3) that for $q>2$,

$$\begin{aligned}&\sup_{0\le t\le T_0} t^{\frac12}\left(\|\sqrt{\tilde{\rho}^R}\tilde{u}_t^R\|_{L^2}+\|\nabla^2\tilde{u}^R\|_{L^2}\right)\\ &+\int_0^{T_0}\left(\|\sqrt{\tilde{\rho}^R}\tilde{u}_t^R\|_{L^2}^2+\|\nabla^2\tilde{u}^R\|_{L^2}^2+\|\nabla^2\tilde{u}^R\|_{L^q}^{\frac{q+1}{q}}\right)dt\\ &+\int\left(t\|\nabla^2\tilde{u}^R\|_{L^q}^2+t\|\nabla\tilde{u}_t^R\|_{L^2}^2\right)dt\le C.\end{aligned} \tag{5.4.14}$$

Next, for $p\in[2,q]$, we obtain from (5.3.3) and (5.3.40) that

$$\begin{aligned}\sup_{0\le t\le T_0}\|\nabla^2(\bar{x}^a\tilde{\rho}^R)\|_{L^p}&\le C\sup_{0\le t\le T_0}(\|\nabla^2(\bar{x}^a\rho^R)\|_{L^p(B_R)}\\ &\quad+R^{-1}\|\nabla(\bar{x}^a\rho^R)\|_{L^p(B_R)}+R^{-2}\|\bar{x}^a\rho^R\|_{L^p(B_R)})\\ &\le C\sup_{0\le t\le T_0}\|\bar{x}^a\rho^R\|_{H^2(B_R)\cap W^{2,p}(B_R)})\le C,\end{aligned} \tag{5.4.15}$$

which together with (5.3.41) and (5.3.3) yields

$$\begin{aligned}\int_0^{T_0}\|\partial_t(\bar{x}\tilde{\rho}^R)\|_{L^p}^2dt&\le C\int_0^{T_0}\|\bar{x}|u^R||\nabla\rho^R|\|_{L^p(B_R)}^2dt\\ &\le C\int_0^{T_0}\|\bar{x}^{1-a}u^R\|_{L^\infty}^2\|\bar{x}^a\nabla\rho^R\|_{L^p(B_R)}^2dt\\ &\le C.\end{aligned} \tag{5.4.16}$$

By virtue of the same arguments as those of (5.3.27) and (5.3.36) , one gets

$$\sup_{0\le t\le T_0} t^{\frac12}\|\nabla\tilde{P}^R\|_{L^2}+\int_0^{T_0}(\|\nabla\tilde{P}^R\|_{L^2}^2+\|\nabla\tilde{P}^R\|_{L^q}^{\frac{q+1}{q}})dt\le C. \tag{5.4.17}$$

With the estimates (5.4.13)-(5.4.17) at hand, we find that the sequence $(\tilde{\rho}^R,\tilde{u}^R,\tilde{P}^R)$ converges, up to the extraction of subsequences, to some limit (ρ,u,P) in the weak sence, that is, as $R\to\infty$, we have

$$\bar{x}\tilde{\rho}^R\to\bar{x}\rho,\text{in } C^1(\overline{B_N}\times[0,T_0]),\text{for any } N>0, \tag{5.4.18}$$

$$\bar{x}^a\tilde{\rho}^R\rightharpoonup\bar{x}^a\rho,\text{weakly * in } L^\infty(0,T_0;H^2\cap W^{2,q}), \tag{5.4.19}$$

$$\sqrt{\tilde{\rho}^R}\tilde{u}^R\rightharpoonup\sqrt{\rho}u,\nabla\tilde{u}^R\rightharpoonup\nabla u,\text{weakly * in } L^\infty(0,T_0;L^2) \tag{5.4.20}$$

$$\nabla^2\tilde{u}^R\rightharpoonup\nabla^2u,\nabla\tilde{P}^R\rightharpoonup\nabla P,\text{weakly in } L^{\frac{q+1}{q}}(0,T_0;L^q)\cap L^2(\mathbb{R}^2\times(0,T_0)), \tag{5.4.21}$$

(5.4.22) $$\sqrt{t}\nabla^2\tilde{u}^R \rightharpoonup \sqrt{t}\nabla^2 u, \text{weak in } L^2(0,T_0;L^q), \text{weak * in } L^\infty(0,T_0;L^2),$$

(5.4.23) $$\sqrt{t}\sqrt{\tilde{\rho}}\tilde{u}_t^R \rightharpoonup \sqrt{t}\sqrt{\rho}u_t, \sqrt{t}\nabla\tilde{P}^R \rightharpoonup \sqrt{t}\nabla P, \text{weak * in } L^\infty(0,T_0;L^2),$$

(5.4.24) $$\sqrt{t}\nabla^2\tilde{u}_t^R \rightharpoonup \sqrt{t}\nabla^2 u_t, \text{weak * in } L^2(\mathbb{R}^2\times(0,T_0)),$$

with

(5.4.25) $$\bar{x}^a\rho \in L^\infty(0,T_0;L^1), \quad \inf_{0\le t\le T_0}\int_{B_{2N_0}}\rho(x,t)dx \ge \frac{1}{4}.$$

Then letteing $R \to \infty$, standard arguments together with (5.4.18)-(5.4.25) show that (ρ,u,P) is a strong solution of on $\mathbb{R}^2\times(0,T_0)$ satisfying (5.1.7) and (5.1.8). Indeed, the existence of a pressure P follows immediately from $(5.1.1)_1$ $(5.1.1)_3$ and by a classical consideration. The proof of the existence part of Theorem 5.2 is finished.

The final work is only to prove the uniqueness of the strong solution satisfying (5.1.7) and (5.1.8). Let (ρ,u,P) and $(\bar{\rho},\bar{u},\bar{P})$ be two strong solutions satisfying (5.1.7) and (5.1.8) with the same initial data, and denote

$$\Theta := \rho-\bar{\rho}, U := u-\bar{u}.$$

First, subtracting the mass equation satisfied by (ρ,u,P) and $(\bar{\rho},\bar{u},\bar{P})$ gives

(5.4.26) $$\Theta_t + \bar{u}\cdot\nabla\Theta + U\cdot\nabla\rho = 0.$$

Multiplying (5.4.26) by $2\Theta\bar{x}^{2r}$ for $r\in(1,\tilde{a})$ with $\tilde{a}=\min\{2,a\}$, and integrating by parts yield

(5.4.27)
$$\begin{aligned}
&\frac{d}{dt}\int|\Theta\bar{x}^r|^2dx\\
&\le C\|\bar{u}\bar{x}^{-\frac{1}{2}}\|_{L^\infty}\|\Theta\bar{x}^r\|_{L^2} + C\|\Theta\bar{x}^r\|_{L^2}\|U\bar{x}^{-(\tilde{a}-r)}\|_{L^{\frac{2q}{(q-2)(\tilde{a}-r)}}}\|\bar{x}^{\tilde{a}}\nabla\rho\|_{L^{\frac{2q}{q-(q-2)(\tilde{a}-r)}}}\\
&\le C(1+\|\nabla\bar{u}\|_{W^{1,q}}\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Theta\bar{x}^r\|_{L^2}(\|\nabla U\|_{L^2}+\|\sqrt{\rho}U\|_{L^2})
\end{aligned}$$

due to Sobolev inequality, (5.1.8), (5.3.16), (5.3.41). This combined with Gronswall inequality shows that for all $0\le t\le T_0$,

(5.4.28) $$\|\Theta\bar{x}^r\|_{L^2} \le C\int_0^t(\|\nabla U\|_{L^2}+\|\sqrt{\rho}U\|_{L^2})ds.$$

Next, taking the gradient in (5.4.26), multiplying the resulting equation by $\nabla\Theta$, and integrating over the $\mathbb{R}^2$, we get

(5.4.29)
$$\frac{1}{2}\frac{d}{dt}\int|\nabla\Theta|^2dx + \int(\nabla\Theta\cdot\nabla\bar{u})\cdot\nabla\Theta dx + \int(\nabla\rho\cdot\nabla U)\cdot\nabla\Theta dx + \int(\nabla^2\rho\cdot U)\cdot\nabla\Theta dx = 0.$$

Observe that

$$-\int \triangle\Theta\nabla\rho\cdot U dx=\int\nabla\Theta\cdot(\nabla\rho\cdot U)dx=\int\nabla\Theta\cdot(\nabla^2\rho\cdot U)dx+\int\nabla\Theta\cdot(\nabla\rho\cdot\nabla U)dx. \tag{5.4.30}$$

Next, subtracting the momentum equation satisfied by (ρ,u,P) and $(\bar\rho,\bar u,\bar P)$ leads to

$$\rho U_t+\rho u\cdot\nabla U-\mu\triangle U=-\rho U\cdot\nabla\bar u-\Theta(\bar u_t+\bar u\cdot\nabla\bar u)-\nabla(P-\bar P)+\kappa\triangle\Theta\nabla\rho+\kappa\triangle\bar\rho\nabla\Theta. \tag{5.4.31}$$

Multiplying by U, integration by parts and combine with (5.4.29) yield

$$\begin{aligned}&\frac{d}{dt}\int(\frac12\rho|U|^2+\frac{\kappa}{2}|\nabla\Theta|^2)dx+\int\frac{\mu}{2}|\nabla U|^2dx\\&=\int-\rho U\cdot\nabla\bar u\cdot U-\Theta(\bar u_t+\bar u\cdot\nabla\bar u)\cdot U-\kappa\triangle\bar\rho\nabla\Theta\cdot U-\kappa(\nabla\Theta\cdot\bar u)\cdot\nabla\Theta dx\\&\le C\|\nabla\bar u\|_{L^\infty}\int(\rho|U|^2+|\nabla\Theta|^2)dx+C\int|\Theta||U|(|\bar u_t|+|\bar u||\nabla\bar u|)dx\\&\quad+C\int|\triangle\bar\rho||\nabla\Theta||U|dx.\end{aligned} \tag{5.4.32}$$

To finish the proof, we estimate the last two terms on the right hand side of (5.4.32). First,

$$\begin{aligned}\int|\Theta||U|(|\bar u_t|+|\bar u||\nabla\bar u|)dx&\le C\|\Theta\bar x^r\|_{L^2}\|U\bar x^{-r/2}\|_{L^4}(\|\bar u_t\bar x^{-r/2}\|_{L^4}+\|\nabla\bar u\|_{L^\infty}\|\bar u\bar x^{-r/2}\|_{L^4})\\&\le C(\epsilon)(\|\sqrt{\bar\rho}\bar u_t\|_{L^2}^2+\|\nabla\bar u_t\|_{L^2}^2+\|\nabla\bar u\|_{L^\infty}^2)\|\Theta\bar x^r\|_{L^2}^2\\&\quad+\epsilon(\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2)\\&\le C(\epsilon)(1+t\|\nabla\bar u_t\|_{L^2}^2+t\|\nabla^2\bar u\|_{L^q}^2)\int_0^t(\|\nabla U\|_{L^2}^2+\|\sqrt{\rho}U\|_{L^2}^2)ds\\&\quad+\epsilon(\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2),\end{aligned} \tag{5.4.33}$$

and

$$\begin{aligned}\int|\triangle\bar\rho||\nabla\Theta||U|dx&\le C\|\bar x^r\triangle\rho\|_{L^q}\|U\bar x^{-r/2}\|_{L^{q^*}}\|\nabla\Theta\|_{L^2}\\&\le C(\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2)\|\bar x^r\triangle\rho\|_{L^q}\|\nabla\Theta\|_{L^2}\\&\le\epsilon(\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2)+C(\epsilon)\|\bar x^r\triangle\rho\|_{L^q}^2\|\nabla\Theta\|_{L^2}^2.\end{aligned} \tag{5.4.34}$$

Denoting

$$G(t):=\|\sqrt{\rho}U\|_{L^2}^2+\int_0^t(\|\nabla U\|_{L^2}^2+\|\sqrt{\rho}U\|_{L^2}^2)ds,$$

then substituting the above into (5.4.32) and choosing ϵ suitably small lead to

$$G'(t)\le C(1+\|\bar x^r\triangle\rho\|_{L^q}^2+\|\nabla\bar u\|_{L^\infty}+t\|\nabla\bar u_t\|_{L^2}^2+t\|\nabla^2u\|_{L^q}^2)G(t),$$

which together with Gronwall inequality and (5.1.7) implies that $G(t)=0$. Hence, $U(x,t)=0$ for almost everywhere $(x,t)\in\mathbb{R}^2\times(0,T)$. Finally, one can deduce from

(5.4.28) that $\Theta = 0$ for almost everywhere $(x, t) \in \mathbb{R}^2 \times (0, T)$. The proof of Theorem 5.2 is completed.

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