

# 博士論文

論文題目

**Bridgeland stability conditions and the moduli  
spaces of coherent sheaves**  
(Bridgeland 安定性条件と連接層のモジュライ空間の研究)

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# Introduction

This thesis consists of two parts. In Part 1, we study geometry of the moduli spaces of coherent sheaves on blow-ups at codimension two loci. In Part 2, we study Bridgeland stability conditions on threefolds.

## 1. Results in Part 1

**1.1. Main result.** In Part 1, we consider the following natural question:

QUESTION 1.1. Let  $X \dashrightarrow Y$  be a birational map between varieties. What is the relation between the moduli space of stable sheaves on  $Y$  and that of  $X$ ?

There are several works answering Question 1.1 in various situations ([10, 11, 37, 39, 40, 41, 51, 52]). In particular, Nakajima and Yoshioka proved the following theorem:

THEOREM 1.2 ([40]). *Let  $(Y, H)$  be a polarized smooth projective surface,  $f: X \rightarrow Y$  the blow-up at a point. Let  $v = (v_0, v_1, v_2) \in H^{2*}(X; \mathbb{Q})$  be the Chern character of a sheaf with  $v_0 > 0$ ,  $\gcd(v_0, f^*H.v_1) = 1$ .*

*Then there exists a diagram of projective schemes*

$$(1.1) \quad \begin{array}{ccccc} \cdots & M^m(v) & & M^{m+1}(v) & \cdots \\ & \searrow & \xrightarrow{\xi_m^-} & \xleftarrow{\xi_m^+} & \searrow \\ & & M^{m,m+1}(v) & & \end{array}$$

such that

- (1) For an integer  $m \in \mathbb{Z}_{\geq 0}$ , the scheme  $M^m(v)$  is the moduli space of  $m$ -stable sheaves with Chern character  $v$ , and  $M^{m,m+1}(v)$  is a scheme whose closed points correspond to  $m$ -stable and  $(m+1)$ -stable sheaves with various Chern characters (see Definition 4.7 for the notion of  $m$ -stability).
- (2) If there exists an element  $w \in H^{2*}(Y; \mathbb{Q})$  with  $v = f^*w$ , then  $M^0(v)$  is isomorphic to the moduli space of stable sheaves on  $Y$ .
- (3) For every sufficiently large  $m$ ,  $M^m(v)$  is isomorphic to the moduli space of stable sheaves on  $X$ .
- (4) For every integer  $m \in \mathbb{Z}_{\geq 0}$ , the fiber over  $\xi_m^\pm$  is the Grassmann variety.

In the above theorem, the notion of *perverse coherent sheaves* plays an important role. In particular, it leads us to define natural stability conditions indexed by  $m \in \mathbb{Z}_{\geq 0}$ , called  *$m$ -stability*. The  $m$ -stability is similar to the Gieseker stability. However, an  $m$ -stable sheaf may have a torsion subsheaf supported on the  $f$ -exceptional curve. Such a generalization of stability enables us to connect the moduli space of stable sheaves on  $X$  and that of  $Y$ .

The notion of perverse coherent sheaves was introduced by Bridgeland (cf. [11]). A perverse coherent sheaf is an element of the heart of a certain bounded t-structure (called *perverse heart* and denoted by  $\text{Per}(X/Y)$ ) in the derived category of coherent sheaves on  $X$ . The heart  $\text{Per}(X/Y)$  encodes the information of the morphism  $f$  and it can be defined more general situation. In particular, if we have the blow-up  $f: X \rightarrow Y$  of a smooth projective variety along its codimension two smooth closed subvariety, we can define the perverse heart  $\text{Per}(X/Y) \subset D^b(X)$ . In this setting, we generalize the result of the paper [40]. The precise statement of our main theorem of Part 1 is the following:

THEOREM 1.3. *Let  $(Y, H)$  be a polarized smooth projective variety of dimension  $n \geq 2$ ,  $f: X \rightarrow Y$  the blow-up along its codimension two smooth closed subvariety. Let  $v = (v_0, v_1, \dots) \in H^*(X; \mathbb{Q})$  be a Chern character with  $v_0 > 0$ ,*

$\gcd(v_0, f^*H^{n-1}.v_1) = 1$ . Then there exists a diagram of projective schemes as in (1.1) such that

- (1) The scheme  $M^m(v)$  is the moduli space of  $m$ -stable sheaves with Chern character  $v$ , and  $M^{m,m+1}(v)$  is a scheme whose closed points corresponds to  $m$ -stable and  $(m+1)$ -stable sheaves with various Chern characters ( $m \in \mathbb{Z}_{\geq 0}$ ).
- (2) If there exists an element  $w \in H^{2*}(Y; \mathbb{Q})$  with  $v = f^*w$ , then  $M^0(v)$  is isomorphic to the moduli space of stable sheaves on  $Y$ .
- (3)' For every sufficiently large  $m$ , the moduli space of stable sheaves on  $X$  is embedded into  $M^m(v)$  as an open and closed subscheme.
- (4)' For every integer  $m \in \mathbb{Z}_{\geq 0}$ , the fiber over  $\xi_m^\pm$  is the certain Quot scheme.

The above theorem is a summary of Corollary 6.2, Proposition 6.4, Proposition 7.5, and Proposition 7.6. See the next subsection for the summary of differences between Theorem 1.2 and Theorem 1.3.

Using Theorem 1.3, we also study birational geometry of Hilbert scheme of two points:

**THEOREM 1.4** (Corollary 8.12, Collollary 8.16). *Let  $f: X \rightarrow Y$  be as in Theorem 2.1, let  $v := (1, 0, \dots, 0, -2) \in H^{2*}(X; \mathbb{Q})$ . Assume that  $H^1(\mathcal{O}_Y) = 0$ . Then we have a diagram of projective varieties*

$$\begin{array}{ccccc}
 & \widetilde{M}^1(v) & & \text{Hilb}^2(X) \subset M^2(v) & \\
 & \swarrow \xi_0 & \searrow \xi_1^- & \swarrow \xi_1^+ & \\
 \text{Hilb}^2(Y) & & M^{1,2}(v) & & 
 \end{array}$$

such that

- (1) When  $\dim Y = 2$  (resp.  $3, \geq 4$ ),  $\text{Hilb}^2(X) \dashrightarrow \widetilde{M}^1(v)$  is a flip (resp. a flop, an anti-flip).
- (2) The morphism  $\xi_0$  is the contraction of a  $K$ -negative extremal ray.

Moreover, we will determine all the fibers over  $\xi_0$ . When  $\dim Y \geq 3$ , we see that some fibers of  $\xi_0$  are *not* the Grassmann varieties (see Lemma 8.14). This is the new phenomenon which does not happen in dimension 2 (see Theorem 1.2 (4)).

**1.2. Difference between Theorem 1.2 and Theorem 1.3.** The idea of the proof of Theorem 1.3 is similar to that of [40]. However, we need to modify the proofs in various points, which are not so straightforward. Let us explain about the differences. Let  $f: X \rightarrow Y$  be as in Theorem 1.3,  $D \subset X$  the  $f$ -exceptional divisor.

**0-stability and 1-stability.** One of the key part of the argument is to describe the difference between 0-stability and 1-stability. Here, a coherent sheaf  $E$  is said to be 0-stable if  $E$  is a perverse coherent sheaf and  $f_*E$  is a slope stable torsion free sheaf on  $Y$ . On the other hand, we say that  $E$  is 1-stable if  $E(-D)$  is 0-stable. To explain the argument, let  $E^- \in \text{Coh}(X)$  be a 0-stable sheaf. In the surface case, Nakajima and Yoshioka proved that the obstruction for the 1-stability is captured by looking at the vector space  $V := \text{Hom}(\mathcal{O}_D(-1), E^-)$ . In fact, we can show that the evaluation morphism  $ev: V \otimes \mathcal{O}_D(-1) \rightarrow E$  is always injective and hence we have a short exact sequence

$$(1.2) \quad 0 \rightarrow V \otimes \mathcal{O}_D(-1) \rightarrow E^- \rightarrow F := \text{Coker}(ev) \rightarrow 0$$

in  $\text{Coh}(X)$ . Furthermore, we can also show that  $F$  is 1-stable. Similarly, we can construct a 0-stable sheaf from a 1-stable sheaf  $E^+$  by looking at the vector space  $\text{Hom}(E^+, \mathcal{O}_D(-1))$ . In this way, we get a (set-theoretical) diagram (1.1) by sending  $E^- \in M^0(v)$  to  $F \in M^{0,1}(v)$ , etc.

However, in higher dimension, not only the sheaf  $\mathcal{O}_D(D)$ , but also various subsheaves of  $E^-$  become the obstruction for 1-stability. Hence we should take the *maximum* subsheaf among them. We take such a subsheaf by using *torsion pairs* on  $\text{Coh}(X)$ . See Definition 5.1 for the definition of torsion pairs. Assume that we have a torsion pair on  $\text{Coh}(X)$ . Then we have the canonical decomposition

$$0 \rightarrow T \rightarrow E^- \rightarrow F \rightarrow 0$$

with respect to the torsion pair. Using such a decomposition, we will see that the difference between 0-stability and 1-stability is captured by certain torsion pairs on  $\text{Coh}(X)$ , which are defined in Definition 5.3 and Definition 5.4. In the surface case, we can easily see that the exact sequence (1.2) is nothing but the decomposition with respect to our torsion pair.

**Scheme structure on  $M^{0,1}(v)$ .** The another key point is how to define the scheme structure on  $M^{0,1}(v)$ . As a set,  $M^{0,1}(v)$  is the disjoint union of the moduli spaces of 0-stable and 1-stable sheaves with various Chern characters. In the surface case, Nakajima and Yoshioka used the moduli space of *perverse coherent systems* to define the scheme structure on  $M^{0,1}(v)$ .

Instead, we use the natural morphisms between moduli spaces. More precisely, we will show that the set-theoretical diagram is naturally identified with the diagram

$$\begin{array}{ccc} M^0(v) & & M^1(v) \\ & \searrow \xi=f_* & \swarrow \xi^+=f_* \\ & M^H(v') & \end{array}$$

where  $M^H(v')$  denotes the moduli space of Gieseker stable sheaves on  $Y$ . Hence we define a scheme structure on  $M^{0,1}(v)$  as the union of the scheme-theoretic images of  $\xi$  and  $\xi^+$ . To be more precise, let  $I, I^+ \subset \mathcal{O}_{M^H(v')}$  be ideal sheaves defining the scheme theoretic images of  $\xi, \xi^+$ , respectively. Then the scheme  $M^{0,1}(v)$  is defined to be a closed subscheme of  $M^H(v')$  whose defining ideal is  $I \cap I^+$ . Note that Nakajima and Yoshioka do the essentially same thing and our approach is inspired by them.

**Moduli space of stable sheaves on  $X$ .** We mention about the difference between Theorem 1.2 (3) and Theorem 1.3 (3)'. According to (3)' in Theorem 1.3, we have an open and closed embedding from the moduli space of stable sheaves on  $X$  to the moduli space of  $m$ -stable sheaves with sufficiently large integer  $m \in \mathbb{Z}_{\geq 0}$ . In the surface case, Nakajima and Yoshioka showed that the above embedding is actually an isomorphism. To show that, they used the following speciality of the surface: Let  $E$  be a torsion free sheaf on a surface. Then the quotient  $E^{DD}/E$  is 0-dimensional, where  $E^{DD}$  is the double dual of  $E$ . In particular, we have  $\chi(E^{DD}/E) \geq 0$ . In the higher dimension case, we do not have such a positivity of the quotient sheaf  $E^{DD}/E$ , which is crucial in the proof given by Nakajima and Yoshioka. In the present paper, we only prove that when  $n = 3$  and  $v = (1, 0, 0, -k)$ , the embedding given in Theorem 2.1 (3)' is actually an isomorphism (see Example 6.6).

## 2. Results in Part 2

**2.1. Motivation and results.** The construction of Bridgeland stability conditions on an algebraic variety  $X$  is an important problem. When  $X$  is a surface, the existence of Bridgeland stability conditions on  $X$  is proved by Bridgeland (cf. [13]) and Arcara-Bertram (cf. [1]). It has found many applications to classical problems

in algebraic geometry, especially in the study of birational geometry of the moduli spaces of Gieseker stable sheaves (see e.g. [2, 4, 5, 9, 16, 17, 18, 19, 30, 31]).

When  $X$  is a threefold, the existence of Bridgeland stability conditions is an open problem in general. In the paper [7], Bayer, Macrì, and Toda reduced the problem to the so-called Bogomolov-Gieseker (BG) type inequality conjecture. The BG type inequality conjecture is known to be true for Abelian threefolds (cf. [6, 32, 33]), Fano threefolds of Picard rank one (cf. [28]), some toric threefolds (cf. [8]), product threefolds of projective spaces and Abelian varieties (cf. [26]), and quintic threefolds (cf. [29]). However, counter-examples of the original BG type inequality conjecture are constructed (see e.g. [36]). The failure of the conjecture is related to the existence of a kind of negative effective divisors on a threefold ([36], see Lemma 9.10). The modification of the conjecture is discussed in the paper [8], and they prove that the modified version of the BG type inequality holds when  $X$  is a Fano threefold of arbitrary Picard rank.

On the other hand, we can still expect that the original BG type inequality conjecture will be true if every effective divisor on  $X$  satisfies a certain positivity condition, e.g. if the pseudo-effective cone agrees with the nef cone. Actually, in this paper, we prove that the original conjecture is true for one class of threefolds satisfying this property, namely those with nef tangent bundles:

**THEOREM 2.1.** *Let  $X$  be a smooth projective threefold with nef tangent bundle. Then the original BG type inequality conjecture holds for  $X$ .*

In particular, the above theorem implies the existence of Bridgeland stability conditions on these threefolds:

**THEOREM 2.2.** *Let  $X$  be as in Theorem 2.1. Then there exist Bridgeland stability conditions on  $X$ .*

See Theorem 9.17, Corollary 9.18 and Theorem 9.19 for the precise statements.

**2.2. Relation to existing works.** First recall that threefolds with nef tangent bundles are classified by F. Campana and T. Peternell.

**THEOREM 2.3 ([15]).** *Let  $X$  be a smooth projective threefold with nef tangent bundle. Then up to taking finite étale coverings,  $X$  is one of the following:*

- (1)  $\mathbb{P}^3$ .
- (2) a three dimensional smooth quadric.
- (3)  $\mathbb{P}^1 \times \mathbb{P}^2$ .
- (4)  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .
- (5)  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ .
- (6)  $\mathbb{P}_A(\mathcal{E})$ , where  $A$  is an Abelian surface and  $\mathcal{E}$  is a rank two vector bundle obtained as an extension of two line bundles in  $\text{Pic}^0(A)$ .
- (7)  $\mathbb{P}_C(\mathcal{E})$ , where  $C$  is an elliptic curve and  $\mathcal{E}$  is a rank three vector bundle obtained as extensions of three line bundles of degree zero.
- (8)  $\mathbb{P}_C(\mathcal{E}_1) \times_C \mathbb{P}_C(\mathcal{E}_2)$ , where  $C$  is an elliptic curve and  $\mathcal{E}_i$  are rank two vector bundles obtained as extensions of degree zero line bundles.
- (9) an Abelian threefold.

Among the above threefolds, the existence of Bridgeland stability conditions is known in the following cases:

- $\mathbb{P}^3$  by [7, 35].
- a three dimensional smooth quadric by [46].
- (3) – (5) in Theorem 2.3 by [8].
- an Abelian threefold by [6, 32, 33].



In this paper, we treat the remaining cases, i.e. (6) – (8) in Theorem 2.3. Note that  $\mathbb{P}^1 \times A$ ,  $\mathbb{P}^2 \times C$ , and  $\mathbb{P}^1 \times \mathbb{P}^1 \times C$  are treated in the author’s previous paper [26], which are the special cases of (6) – (8) in Theorem 2.3.

Furthermore, on  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ , we will construct new Bridgeland stability conditions which were not obtained in [8].

**2.3. Outline of the proof.** As mentioned in the last subsection, we mainly treat the cases (6) – (8) in Theorem 2.3. Recall that, if the bundle is a trivial bundle, then the BG type inequality conjecture is known to be true by the author’s previous paper [26]. In the trivial bundle case, the existence of good endomorphisms is crucial in the proof.

When the bundle is non-trivial, we don’t know the existence of the endomorphisms in general. In such cases, we use the technique developed by Bayer et al ([3]), which we now explain: Consider a smooth family  $\mathcal{X} \rightarrow \mathbb{A}^1$  of threefolds over the affine line  $\mathbb{A}^1$ . Assume that for every points  $t, t' \in \mathbb{A}^1 \setminus \{0\}$ , we have  $\mathcal{X}_t \cong \mathcal{X}_{t'} =: X$ . Then according to [3], the BG type inequality conjecture for  $X$  is reduced to that of  $\mathcal{X}_0$ . In our situation, using this technique, we can reduce to the cases of the projectivizations of split vector bundles (see Proposition 10.3). Then for split cases, we can argue as in [26] using good finite morphisms.

In the final section, we will treat the case when  $X = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ . In [8], they used the fact that  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$  is a Fano variety to construct Bridgeland stability conditions. On the other hand, in this paper, we regard it as a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^2$  and use a full exceptional collection on the derived category.

#### 2.4. Open problems.

- (1) As we will see in Conjecture 9.7, the conjectural BG type inequality depends on a class  $B + i\omega \in \text{NS}(X)_{\mathbb{C}}$  with  $\omega$  ample. For threefolds in Theorem 2.3, except for (5), we can prove the inequality for any choice of a class  $B + i\omega \in \text{NS}(X)_{\mathbb{C}}$  with  $\omega$  ample.

On the other hand, for  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ , we can prove it only when  $B$  and  $\omega$  are proportional so far. We can hope the inequality also holds for any choice of  $B + i\omega \in \text{NS}(\mathbb{P}(\mathcal{T}_{\mathbb{P}^2}))_{\mathbb{C}}$ . At this moment, the author doesn’t know how to solve this problem.

- (2) It is expected that the space of Bridgeland stability conditions has complex dimension equal to the rank of the algebraic cohomology (In fact, it is true in the surface case by the works [1, 13, 51]). As proven in the paper [6], the BG type inequality in Conjecture 9.7 implies the existence of a four dimensional subset in the space of Bridgeland stability conditions.

In [43, Theorem 3.21], the full dimensional family of Bridgeland stability conditions on Abelian threefolds was constructed. Proving the same statement for threefolds treated in this paper is an interesting open problem, which requires the stronger BG type inequality.

### 3. Plan of the paper

Part 1 consists of Section 4 to Section 8. In Section 4, we collect the notions and the properties about perverse coherent sheaves on blow-ups. In Section 5, we describe the diagram (1.1) set-theoretically. In the proofs, we will use certain torsion pairs. In Section 6, we explain the relationship between the  $m$ -stability and the Gieseker stability on both blow-up and blow-down varieties. In Section 7, we realize the diagram (1.1) scheme-theoretically. In Section 8, we study the diagram (1.1) more explicitly in the case of Hilbert scheme of two points.

Part 2 consists of Section 9 to Section 11. In Section 9, we recall about the theory of Bridgeland stability conditions and about threefolds with nef tangent

bundles. In Section 10 we treat varieties in Theorem 2.3 (6) – (8). In Section 11, we will discuss about the stability conditions on  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ .

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NOTATION AND CONVENTION. In this paper we always work over  $\mathbb{C}$ . We use the following notations:

- $\text{ch}^B = (\text{ch}_0^B, \dots, \text{ch}_n^B) := e^{-B} \cdot \text{ch}$ , where  $\text{ch}$  denotes the Chern character and  $B \in \text{NS}(X)_{\mathbb{R}}$ .
- $v^B := \omega \cdot \text{ch}^B := (\omega^n \cdot \text{ch}_0^B, \dots, \omega \cdot \text{ch}_{n-1}^B, \text{ch}_n^B)$ , where  $B, \omega \in \text{NS}(X)_{\mathbb{R}}$ .
- $K(\mathcal{A})$  : the Grothendieck group of an abelian category  $\mathcal{A}$ .
- $\text{hom}(E, F) := \dim \text{Hom}(E, F)$ .
- $\text{ext}^i(E, F) := \dim \text{Ext}^i(E, F)$ .
- $D^b(X) := D^b(\text{Coh}(X))$  : the bounded derived category of coherent sheaves on a smooth projective variety  $X$ .



## Part 1

# Perverse coherent sheaves on blow-ups at codimension two loci

#### 4. Perverse coherent sheaves and their moduli spaces

Throughout Part 1, we use the following notations: Let  $Y$  be a smooth projective variety,  $C \subset Y$  a codimension 2 smooth closed subvariety of  $Y$ . Let  $X := \text{Bl}_C Y$  be the blow-up of  $Y$  along  $C$ ,  $D \subset X$  the exceptional divisor. Hence we have the following diagram:

$$\begin{array}{ccc} D & \xrightarrow{j} & X \\ \pi \downarrow & & \downarrow f \\ C & \xrightarrow{i} & Y. \end{array}$$

**4.1. Perverse coherent sheaves on blow-ups.** In this subsection, we collect the results about the perverse coherent sheaves. First note that the following two conditions hold:

- (1)  $\mathbf{R}f_* \mathcal{O}_X \cong \mathcal{O}_Y$ ,
- (2)  $\dim f^{-1}(y) \leq 1$  for all  $y \in Y$ .

In such a situation, Bridgeland introduced the heart of a bounded t-structure  $\text{Per}(X/Y)$  on  $D^b(X)$  (called *perverse heart*) as follows (cf. [11, 53]):

$$\text{Per}(X/Y) := \left\{ E \in D^b(X) : \begin{array}{l} f_* \mathcal{H}^{-1}(E) = 0, \\ \mathbf{R}^1 f_* \mathcal{H}^0(E) = 0, \text{Hom}(\mathcal{H}^0(E), \mathcal{C}^0) = 0, \\ \mathcal{H}^i(E) = 0, \quad i \neq -1, 0 \end{array} \right\},$$

where  $\mathcal{C} := \{E \in D^b(X) : \mathbf{R}f_* E = 0\}$  and  $\mathcal{C}^0 := \mathcal{C} \cap \text{Coh}(X)$ . In [11], the heart  $\text{Per}(X/Y)$  is denoted as  ${}^{-1}\text{Per}(X/Y)$ . We call an element of  $\text{Per}(X/Y) \cap \text{Coh}(X)$  as a *perverse coherent sheaf*.

We will use the following lemma:

LEMMA 4.1. *We have equalities*

$$\mathcal{C}^0 = \pi^* \text{Coh}(C) \otimes \mathcal{O}(D) = f^* \text{Coh}(C) \otimes \mathcal{O}(D).$$

PROOF. The equality  $\pi^* \text{Coh}(C) \otimes \mathcal{O}(D) = f^* \text{Coh}(C) \otimes \mathcal{O}(D)$  follows from the isomorphism of functors

$$f^* i_* \cong j_* \pi^* : \text{Coh}(C) \rightarrow \text{Coh}(X).$$

To prove the first equality, it is enough to show the equality

$$\mathcal{C} = \pi^* D^b(C) \otimes \mathcal{O}(D).$$

Recall that we have the following semi-orthogonal decomposition (cf. [44])

$$(4.1) \quad D^b(X) = \langle \pi^* D^b(C) \otimes \mathcal{O}(D), \mathbf{L}f^* D^b(Y) \rangle.$$

By the semi-orthogonality and the adjunction, we have

$$0 = \text{Hom}(\mathbf{L}f^* D^b(Y), \pi^* D^b(C) \otimes \mathcal{O}(D)) \cong \text{Hom}(D^b(Y), \mathbf{R}f_*(\pi^* D^b(C) \otimes \mathcal{O}(D))).$$

This proves the inclusion  $\pi^* D^b(C) \otimes \mathcal{O}(D) \subset \mathcal{C}$ . For the converse, take an element  $E \in \mathcal{C}$ . By the semi-orthogonal decomposition (4.1), there exists an exact triangle

$$\mathbf{L}f^* F \rightarrow E \rightarrow \pi^* M \otimes \mathcal{O}(D)$$

for some  $F \in D^b(Y)$ ,  $M \in D^b(C)$ . Applying the functor  $\mathbf{R}f_*$ , we get  $F \cong \mathbf{R}f_* \mathbf{L}f^* F = 0$ , since we have just proven the inclusion  $\pi^* D^b(C) \otimes \mathcal{O}(D) \subset \mathcal{C}$ . Hence we have  $E \cong \pi^* M \otimes \mathcal{O}(D) \in \pi^* D^b(C) \otimes \mathcal{O}(D)$  as required.  $\square$

The following result is due to Van den Bergh:

THEOREM 4.2 ([53, Proposition 3.3.2]). *Let  $\mathcal{E} := \mathcal{O}_X \oplus \mathcal{O}_X(-D)$ . Then we have an equivalence of triangulated categories*

$$\Phi := \mathbf{R}f_* \mathbf{R}\mathcal{H}om(\mathcal{E}, *) : D^b(X) \xrightarrow{\cong} D^b(\mathrm{Coh}(\mathcal{A})),$$

where  $\mathcal{A} := f_* \mathcal{E}nd(\mathcal{E})$ . Furthermore, the functor  $\Phi$  restricts to an equivalence  $\mathrm{Per}(X/Y) \cong \mathrm{Coh}(\mathcal{A})$  of Abelian categories.

From the above theorem, we can easily get the following:

LEMMA 4.3. *Let  $E \in \mathrm{Per}(X/Y)$ . Then we have  $\mathbf{R}^1 f_*(E(D)) = 0$ .*

PROOF. For  $E \in \mathrm{Per}(X/Y)$ , we have  $\Phi(E) = \mathbf{R}f_* E \oplus \mathbf{R}f_*(E(D)) \in \mathrm{Coh}(\mathcal{A})$  by Theorem 4.2. This in particular implies  $\mathbf{R}^1 f_*(E(D)) = 0$ .  $\square$

We give a criterion when a coherent sheaf  $E \in \mathrm{Coh}(X)$  is in  $\mathrm{Per}(X/Y)$ . Before stating the criterion, we recall the following lemma:

LEMMA 4.4 ([40, Lemma 1.2]). *Let  $E$  be a coherent sheaf,  $\phi : f^* f_* E \rightarrow E$  be the adjoint morphism. Then the following statements hold.*

- (1) *We have  $f_*(\mathrm{Image} \phi) \cong f_* E$ , and  $\mathbf{R}^1 f_*(\mathrm{Image} \phi) = 0$ .*
- (2) *We have  $f_*(\mathrm{Coker} \phi) = 0$ , and  $\mathbf{R}^1 f_*(\mathrm{Coker} \phi) \cong \mathbf{R}^1 f_* E$ .*
- (3)  *$E \in \mathrm{Per}(X/Y)$  if and only if  $\mathrm{Coker} \phi = 0$ .*
- (4) *We have  $\mathrm{Ker}(\phi) \in \mathcal{C}^0$ .*

The following criterion will be frequently used in this paper:

LEMMA 4.5 (cf. [40, Proposition 1.9]). *Let  $E$  be a coherent sheaf. Then  $E$  is an object of the category  $\mathrm{Per}(X/Y)$  if and only if for every point  $y$  of  $C$ , we have  $\mathrm{Hom}(E, \mathcal{O}_{L_y}(-1)) = 0$ , where  $L_y := f^{-1}(y) \cong \mathbb{P}^1$  and  $\mathcal{O}_{L_y}(-1) := \mathcal{O}_{\mathbb{P}^1}(-1)$ .*

PROOF. When  $\dim Y = 2$ , the same statement is stated and proven by Nakajima and Yoshioka in [40, Proposition 1.9]. However, their proof does not work in the higher dimension. Hence we give the another proof which works in any dimension.

Assume that  $E \in \mathrm{Per}(X/Y)$ . Then by the definition of  $\mathrm{Per}(X/Y)$ , we have  $\mathrm{Hom}(E, \mathcal{C}^0) = 0$ . In particular, we have  $\mathrm{Hom}(E, \mathcal{O}_{L_y}(-1)) = 0$ .

For the converse, we have to show the following two things:

- (a)  $\mathrm{Hom}(E, \mathcal{C}^0) = 0$ ,
- (b)  $\mathbf{R}^1 f_* E = 0$ .

First we prove (a). We need to show that  $\mathrm{Hom}(E, \pi^* M \otimes \mathcal{O}(D))$  vanishes for all  $M \in \mathrm{Coh}(C)$ . Take an element  $\psi \in \mathrm{Hom}(E, \pi^* M \otimes \mathcal{O}(D))$ . For each point  $y \in C$ , consider the restriction

$$\psi|_{L_y} : E|_{L_y} \rightarrow \pi^*(M|_{\{y\}}) \otimes \mathcal{O}_{L_y}(D) \cong \mathcal{O}_{L_y}(-1)^{\oplus k}$$

( $k \in \mathbb{Z}_{\geq 0}$ ). By our assumption,  $\psi|_{L_y}$  is a zero map for all  $y \in C$ . Hence  $\psi$  itself must be zero. This proves (a).

Next we prove (b). By the formal function theorem, it is enough to show that for every  $y \in C$  and  $n \in \mathbb{N}$ ,  $H^1(L_{y,n}, E_{y,n}) = 0$ . Here,  $L_{y,n} := X \times_Y \mathrm{Spec} \mathcal{O}_{Y,y}/m_y^n$  and  $E_{y,n} := E|_{L_{y,n}}$ .

We argue by induction on  $n$ . First let  $n = 1$ . To obtain a contradiction, suppose that there exists  $y \in C$  such that  $H^1(L_y, E_{y,1}) \neq 0$ . Let us consider the exact sequence

$$0 \rightarrow (E_{y,1})_{\mathrm{tor}} \rightarrow E_{y,1} \rightarrow (E_{y,1})_{\mathrm{fr}} \rightarrow 0,$$

where  $(E_{y,1})_{\mathrm{tor}}$  (resp.  $(E_{y,1})_{\mathrm{fr}}$ ) is the torsion part (resp. torsion free part) of  $E_{y,1}$ . Since  $L_y \cong \mathbb{P}^1$ , there exist integers  $a_i \in \mathbb{Z}$  ( $i = 1, \dots, l$ ) such that  $(E_{y,1})_{\mathrm{fr}} \cong \bigoplus_{i=1}^n \mathcal{O}_{L_y}(a_i)$ . Since  $\bigoplus_{i=1}^n H^1(L_y, \mathcal{O}_{L_y}(a_i)) \cong H^1(L_y, E_{y,1}) \neq 0$ , there exists  $i_0$

such that  $a_{i_0} \leq -2$ . On the other hand, by the surjection  $E \rightarrow E_{y,1} \rightarrow \mathcal{O}_{L_y}(a_{i_0})$ , we have

$$\mathrm{Hom}(\mathcal{O}_{L_y}(a_{i_0}), \mathcal{O}_{L_y}(-1)) \subset \mathrm{Hom}(E, \mathcal{O}_{L_y}(-1)) = 0,$$

which is a contradiction. Hence when  $n = 1$ , we have  $H^1(L_y, E_{y,1}) = 0$  for every  $y \in C$ .

Next assume that for a fixed integer  $n \in \mathbb{N}$ ,  $H^1(L_{y,n}, E_{y,n}) = 0$  holds ( $y \in C$ ). Consider the exact sequence

$$0 \rightarrow m_y^n/m_y^{n+1} \rightarrow \mathcal{O}_{Y,y}/m_y^{n+1} \rightarrow \mathcal{O}_{Y,y}/m_y^n \rightarrow 0.$$

Note that  $m_y^n/m_y^{n+1} \cong (\mathcal{O}_{Y,y}/m_y)^{\oplus k}$  for some  $k \in \mathbb{N}$ . Applying the functor  $f^*(-) \otimes E$  to the above exact sequence, we have the exact sequence

$$E_{y,1}^{\oplus k} \rightarrow E_{y,n+1} \rightarrow E_{y,n} \rightarrow 0.$$

Split this exact sequence into two short exact sequences:

$$\begin{aligned} 0 \rightarrow M \rightarrow E_{y,1}^{\oplus k} \rightarrow K \rightarrow 0, \\ 0 \rightarrow K \rightarrow E_{n+1} \rightarrow E_n \rightarrow 0. \end{aligned}$$

From the first exact sequence, we have the exact sequence

$$0 = H^1(X, E_{y,1})^{\oplus k} \rightarrow H^1(X, K) \rightarrow H^2(X, M) = 0.$$

Note that  $H^1(X, E_{y,1}) = 0$  follows from the argument of  $n = 1$  case, while  $H^2(X, M) = 0$  holds since  $\dim \mathrm{Supp}(M) \leq 1$ . Hence we also have  $H^1(X, K) = 0$ . Then by the second exact sequence and the induction hypothesis, we conclude that  $H^1(E_{y,n+1}) = 0$  as required.  $\square$

**COROLLARY 4.6.** *Let  $E \in \mathrm{Per}(X/Y) \cap \mathrm{Coh}(X)$  be a perverse coherent sheaf. Then  $E(-D)$  is also a perverse coherent sheaf.*

**PROOF.** Take a point  $y \in C$ . Noting the isomorphism  $\mathcal{O}_{L_y}(D) \cong \mathcal{O}_{L_y}(-1)$ , we have

$$\mathrm{Hom}(E(-D), \mathcal{O}_{L_y}(-1)) = \mathrm{Hom}(E, \mathcal{O}_{L_y}(-2)) \subset \mathrm{Hom}(E, \mathcal{O}_{L_y}(-1)) = 0.$$

Hence by Lemma 4.5, we have  $E(-D) \in \mathrm{Per}(X/Y)$ .  $\square$

**4.2. Moduli space of  $m$ -stable sheaves.** In this subsection, we recall the notion of  $m$ -stability and the moduli space of  $m$ -stable sheaves introduced by Nakajima and Yoshioka in their paper [40]. Let  $H$  be an ample divisor on  $Y$ ,  $v = (v_0, v_1, \dots) \in H^{2*}(X; \mathbb{Q})$  such that  $v_0 > 0$  and  $\gcd(v_0, v_1 \cdot f^* H^{n-1}) = 1$ .

**DEFINITION 4.7.** Let  $E \in \mathrm{Coh}(X)$  be a coherent sheaf.

- (1) We say that  $E$  is *0-stable* if  $E \in \mathrm{Per}(X/Y)$  and  $f_* E \in \mathrm{Coh}(Y)$  is  $\mu_H$ -stable.
- (2) Let  $m \in \mathbb{Z}_{>0}$  be a positive integer. We say that  $E$  is  *$m$ -stable* if  $E(-mD)$  is 0-stable.

**THEOREM 4.8** ([40, Theorem 2.9]). *Let  $m \in \mathbb{Z}_{\geq 0}$ . Then there exists the projective coarse moduli scheme  $M^m(v)$  of  $m$ -stable sheaves with Chern character  $v$ .*

**REMARK 4.9.** In [40, Section 2], the notion of  $m$ -stability and the existence of the coarse moduli space are discussed without assuming  $\gcd(v_0, v_1 \cdot f^* H^{n-1}) = 1$ . However, in the following, we use this assumption almost everywhere. In particular, the following fact will be used frequently: for  $E \in \mathrm{Per}(X/Y) \cap \mathrm{Coh}(X)$  with  $\mathrm{ch}(E) = v$ ,  $f_* E$  is  $\mu_H$ -semistable if and only if it is  $\mu_H$ -stable.

### 5. Wall-crossing

In this section, we always fix an ample divisor  $H$  on  $Y$  and the Chern character  $v = (v_0, v_1, \dots) \in H^{2*}(X; \mathbb{Q})$  of a sheaf such that  $v_0 > 0$  and  $\gcd(v_0, v_1, f^*H^{n-1}) = 1$ . We will describe the difference between  $m$ -stability and  $(m+1)$ -stability. To do that, we may assume  $m = 0$  since we have an isomorphism

$$(-) \otimes \mathcal{O}(-mD): M^m(v) \rightarrow M^0(v.e^{-mD}).$$

**5.1. Torsion pairs.** To construct the diagram (1.1), we use *torsion pairs* (cf. [23]).

**DEFINITION 5.1.** Let  $\mathcal{A}$  be an Abelian category,  $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$  full subcategories of  $\mathcal{A}$ . Then the pair  $(\mathcal{T}, \mathcal{F})$  is a *torsion pair* on  $\mathcal{A}$  if the following two conditions hold:

- (1)  $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ .
- (2) For every object  $E \in \mathcal{A}$ , there exists objects  $T \in \mathcal{T}$ ,  $F \in \mathcal{F}$ , and an short exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0.$$

Note that by the property (1), the exact sequence (2) is unique up to isomorphism. First we recall the following easy property of torsion pairs.

**LEMMA 5.2.** *Let  $(\mathcal{T}, \mathcal{F})$  be a torsion pair on an Abelian category  $\mathcal{A}$ . Then  $\mathcal{T}$  is closed under taking quotients.*

**PROOF.** Take an element  $E \in \mathcal{T}$  and a surjective map  $E \twoheadrightarrow Q$  in  $\mathcal{A}$ . By the definition of a torsion pair, there exists an exact sequence

$$0 \rightarrow T \rightarrow Q \rightarrow F \rightarrow 0$$

with  $T \in \mathcal{T}$ ,  $F \in \mathcal{F}$ . Then we get a surjective map  $E \twoheadrightarrow Q \twoheadrightarrow F$ . On the other hand, we have  $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$  by definition. Hence we must have  $F = 0$ , i.e.  $Q = T \in \mathcal{T}$ .  $\square$

In this paper, we use the following two torsion pairs on  $\text{Coh}(X)$ .

**DEFINITION 5.3.** Define the full subcategories of  $\text{Coh}(X)$  as

$$\begin{aligned} \mathcal{T} &:= \{T \in \text{Coh}(X) : \mathbf{R}^1 f_* T = 0, \text{Hom}(T, \mathcal{C}^0) = 0\}, \\ \mathcal{F} &:= \{F \in \text{Coh}(X) : f_* F = 0\}. \end{aligned}$$

**DEFINITION 5.4.** We define the full subcategories  $\mathcal{T}_D, \mathcal{F}_D \subset \text{Coh}(X)$  as follows:

$$\begin{aligned} \mathcal{T}_D &:= \{T \in \text{Coh}(X) : T(-D) \in \text{Per}(X/Y), \text{Supp}(T) \subset D\}, \\ \mathcal{F}_D &:= \{F \in \text{Coh}(X) : \text{Hom}(\mathcal{T}_D, F) = 0\}. \end{aligned}$$

**LEMMA 5.5.** *The pairs  $(\mathcal{T}, \mathcal{F})$  and  $(\mathcal{T}_D, \mathcal{F}_D)$  are torsion pairs on  $\text{Coh}(X)$ .*

**PROOF.** The assertion for the pair  $(\mathcal{T}, \mathcal{F})$  has been proved for example in [40, Lemma 1.3]. For the pair  $(\mathcal{T}_D, \mathcal{F}_D)$ , it is enough to show that the subcategory  $\mathcal{T}_D \subset \text{Coh}(X)$  is closed under taking quotients and extensions by [50, Lemma 2.15], since  $\text{Coh}(X)$  is noetherian. The conditions  $\text{Supp}(T) \subset D$  and  $\text{Hom}(T(-D), \mathcal{C}^0) = 0$  are clearly closed under taking quotients and extensions. Furthermore, since the morphism  $f: X \rightarrow Y$  has relative dimension one, we can also see that the condition  $\mathbf{R}^1 f_* T(-D) = 0$  is closed under taking quotients and extensions.  $\square$

**REMARK 5.6.** Recall that we have  $\text{Per}(X/Y) = \langle \mathcal{F}[1], \mathcal{T} \rangle$ . Similarly, we define

$$\widetilde{\text{Per}}(X/Y) := \langle \mathcal{F}_D[1], \mathcal{T}_D \rangle.$$

By the general theory of torsion pairs and tilting, the category  $\widetilde{\text{Per}}(X/Y)$  is the heart of a bounded t-structure on  $D^b(X)$  (cf. [23]).



LEMMA 5.7. *We have  $\mathcal{C}^0 \subset \mathcal{T}_D$ . Furthermore, for a coherent sheaf  $E \in \text{Coh}(X)$  which is torsion free outside  $D$ , the following statements hold.*

- (1) *The sheaf  $f_*(E(-D))$  is torsion free if and only if  $\text{Hom}(\mathcal{T}_D, E) = 0$ , i.e.  $E \in \mathcal{F}_D$ .*
- (2) *If the sheaf  $f_*(E(-D))$  is torsion free, then  $f_*(E)$  is also torsion free.*

PROOF. For the first assertion, recall from Lemma 4.1 that we have  $\mathcal{C}^0 = f^* \text{Coh}(C) \otimes \mathcal{O}(D)$ . From this, we can see that  $\text{Supp}(T) \subset D$  for any element  $T \in \mathcal{C}^0$ . Moreover, we have  $\mathcal{C}^0 \otimes \mathcal{O}(-D) \cong f^* \text{Coh}(C) \subset \mathcal{T}$ . This proves the inclusion  $\mathcal{C}^0 \subset \mathcal{T}_D$ .

(1) First assume that  $\text{Hom}(\mathcal{T}_D, E) = 0$ . Since we assume  $E$  is torsion free outside  $D$ , it is enough to show that  $\text{Hom}(\text{Coh}(C), f_*(E(-D))) = 0$ . We can compute as

$$\begin{aligned} \text{Hom}(\text{Coh}(C), f_*(E(-D))) &\cong \text{Hom}(f^* \text{Coh}(C), E(-D)) \\ &= \text{Hom}(f^* \text{Coh}(C) \otimes \mathcal{O}(D), E) \\ &= 0. \end{aligned}$$

Note that the last equality holds by the inclusion  $\mathcal{C}^0 \subset \mathcal{T}_D$ .

For the converse, assume that  $f_*(E(-D))$  is torsion free. Let  $T \in \mathcal{T}_D$ . Then

$$\begin{aligned} \text{Hom}(T, E) &= \text{Hom}(T(-D), E(-D)) \\ &\subset \text{Hom}(f^* f_*(T(-D)), E(-D)) \\ &\cong \text{Hom}(f_*(T(-D)), f_*(E(-D))) \\ &= 0. \end{aligned}$$

Note that the adjoint map  $f^* f_*(T(-D)) \rightarrow T(-D)$  is surjective since  $T(-D) \in \text{Per}(X/Y)$  by definition. Hence we have the inclusion  $\text{Hom}(T(-D), E(-D)) \subset \text{Hom}(f^* f_*(T(-D)), E(-D))$ . Note also that the last equality holds since  $f_*(T(-D))$  is torsion.

(2) By (1), it is enough to show that  $\text{Hom}(\mathcal{T}_D, E(D)) = 0$ . Let  $T \in \mathcal{T}_D$ . Then we have  $T(-D) \in \mathcal{T}_D$  by Corollary 4.6. Hence

$$\text{Hom}(T, E(D)) = \text{Hom}(T(-D), E) = 0.$$

Here, the last equality holds again by (1). □

## 5.2. From 0-stability to 1-stability.

PROPOSITION 5.8. *Let  $E^- \in \text{Coh}(X) \cap \text{Per}(X/Y)$  be a perverse coherent sheaf with Chern character  $\text{ch}(E^-) = v$ . Let*

$$0 \rightarrow T \rightarrow E^- \rightarrow F \rightarrow 0$$

*be the unique exact sequence in  $\text{Coh}(X)$  with  $T \in \mathcal{T}_D, F \in \mathcal{F}_D$ . Then the following hold:*

- (1) *If  $f_* E^-$  is torsion free, then  $T \in \mathcal{C}^0$ .*
- (2) *If  $E^-$  is 0-stable, then  $F$  is 0-stable and 1-stable.*

PROOF. First we prove (1). Since  $T(-D) \in \text{Per}(X/Y)$ , we have  $\mathbf{R}^1 f_* T = 0$  by Lemma 4.3. On the other hand, we have the injection  $0 \rightarrow f_* T \rightarrow f_* E^-$  in  $\text{Coh}(Y)$ . Since  $T$  is supported on  $D$  and  $f_* E^-$  is torsion free, we have  $f_* T = 0$ . This proves (1).

Next assume that  $E^-$  is 0-stable. Since we have a surjection  $E^- \rightarrow F \rightarrow 0$  in  $\text{Coh}(X)$ , we have  $F \in \text{Per}(X/Y) \cap \text{Coh}(X) = \mathcal{T}$  by Lemma 5.2. Furthermore, the isomorphism  $f_* E^- \cong f_* F$  implies  $F$  is 0-stable. It remains to show that  $F$  is 1-stable. By Corollary 4.6, we have  $F(-D) \in \text{Per}(X/Y)$ .

CLAIM 5.9.  *$f_*(F(-D))$  is torsion free.*

PROOF. Let  $M \in \text{Coh}(C)$ . We must show that  $\text{Hom}(M, f_*(F(-D))) = 0$ . By Lemma 4.1 and Lemma 5.7, we have  $f^*M \otimes \mathcal{O}(D) \in \mathcal{C}^0 \subset \mathcal{T}_D$ . Hence by adjunction, we get

$$\text{Hom}(M, f_*(F(-D))) \cong \text{Hom}(f^*M \otimes \mathcal{O}(D), F) = 0.$$

□

Now since  $f_*(F(-D)) \rightarrow f_*F$  is an isomorphism outside  $C$  and they are torsion free, it follows that  $0 \rightarrow f_*(F(-D)) \rightarrow f_*F$  is injective in  $\text{Coh}(Y)$ . Furthermore, by the facts that  $f_*F$  is  $\mu$ -stable and  $\mu(f_*(F(-D))) = \mu(f_*F)$ , we conclude that  $f_*(F(-D))$  is  $\mu$ -stable. This means that  $F$  is 1-stable. □

PROPOSITION 5.10. *Let  $F \in M^0(v) \cap M^1(v)$  be a 0-stable and 1-stable sheaf with Chern character  $\text{ch}(F) = v$ . Let  $T \in \mathcal{C}^0$  with a surjective map  $\psi: F \rightarrow T[1] \rightarrow 0$  in  $\text{Per}(X/Y)$ . Let  $E^- := \text{Ker } \psi \in \text{Per}(X/Y)$ . Then  $E^-$  is 0-stable but not 1-stable.*

PROOF. By shifting the exact triangle  $E^- \rightarrow F \rightarrow T[1]$ , we get an exact triangle  $T \rightarrow E^- \rightarrow F$ . Since  $T, F \in \text{Coh}(X)$ , we also have  $E^- \in \text{Coh}(X)$ . Furthermore, since  $E^- \in \text{Per}(X/Y)$  and  $f_*E^- \cong f_*F$ ,  $E^-$  is 0-stable. On the other hand, since  $T \in \mathcal{C}^0$  and  $\text{Hom}(T, E^-) \neq 0$ ,  $E^-$  is not 1-stable by Lemma 5.7. □

### 5.3. From 1-stability to 0-stability.

PROPOSITION 5.11. *Let  $E^+ \in M^1(v)$  be a 1-stable sheaf with Chern character  $\text{ch}(E^+) = v$ . Let*

$$0 \rightarrow F \rightarrow E^+ \rightarrow T \rightarrow 0$$

*be the unique exact sequence in  $\text{Coh}(X)$  with  $F \in \mathcal{T}$ ,  $T \in \mathcal{F}$ . Then we have*

- (1)  $T \in \mathcal{C}^0$ .
- (2)  $F$  is 0-stable and 1-stable.

PROOF. (1) By definition, we have  $f_*T = 0$ . Since  $E^+(-D) \in \text{Per}(X/Y) \cap \text{Coh}(X) = \mathcal{T}$ , it follows that  $T(-D) \in \text{Per}(X/Y)$  by Lemma 5.2. Hence by Lemma 4.3, we get  $\mathbf{R}^1f_*T = 0$ .

(2) By definition,  $F \in \text{Per}(X/Y)$ . By Corollary 4.6, we also have  $F(-D) \in \text{Per}(X/Y)$ . First we will show that  $F$  is 1-stable. We have an exact sequence

$$0 \rightarrow f_*(F(-D)) \rightarrow f_*(E^+(-D)) \rightarrow f_*(T(-D))$$

in  $\text{Coh}(Y)$ . Since  $\text{Supp}(f_*(T(-D))) \subset C$  and  $f_*(E^+(-D))$  is  $\mu$ -stable, it follows that  $f_*(F(-D))$  is also  $\mu$ -stable. We conclude that  $F$  is 1-stable.

It remains to show that  $F$  is 0-stable. As in the argument of the previous proposition, it is enough to show that  $f_*F$  is torsion free. But that follows from Lemma 5.7 (3). □

PROPOSITION 5.12. *Let  $F \in M^0(v) \cap M^1(v)$  be a 0-stable and 1-stable sheaf with Chern character  $\text{ch}(F) = v$ . Let  $T \in \mathcal{C}^0$  with an injective map  $0 \rightarrow T \rightarrow F[1]$  in  $\widetilde{\text{Per}}(X/Y)$ . Let  $E^+[1]$  be its cokernel. Then  $E^+$  is 1-stable but not 0-stable.*

PROOF. First note that  $F \in \mathcal{F}_D$  and hence  $F[1] \in \widetilde{\text{Per}}(X/Y)$ . Furthermore,  $\mathcal{F}_D[1]$  is a torsion part of the torsion pair  $(\mathcal{F}_D[1], \mathcal{T}_D)$  on  $\widetilde{\text{Per}}(X/Y)$ . Hence  $\mathcal{F}_D[1]$  is closed under taking quotients in the abelian category  $\text{Per}(X/Y)$ . In particular,  $E^+ \in \mathcal{F}_D$  and hence  $f_*(E^+(-D))$  is torsion free. Applying  $f_*$  to the exact sequence

$$0 \rightarrow F \rightarrow E^+ \rightarrow T \rightarrow 0$$

in  $\text{Coh}(X)$ , we have an injection  $0 \rightarrow f_*(F(-D)) \rightarrow f_*(E^+(-D))$  which is isomorphism outside  $C$ . Hence the  $\mu$ -stability of  $f_*(F(-D))$  implies that  $f_*(E^+(-D))$

is  $\mu$ -stable. On the other hand, since  $T \in \mathcal{C}^0$  and  $\text{Hom}(E^+, T) \neq 0$ , we have  $E^+ \notin \text{Per}(X/Y)$ . In particular,  $E^+$  is not 0-stable.  $\square$

**5.4. Set-theoretical wall-crossing.** Define  $\mathcal{S} := \{\text{ch}(T) : T \in \mathcal{C}^0\}$ . First we define the following two notions. Note that for a 0-stable and 1-stable sheaf  $F$ , we have  $F \in \text{Per}(X/Y)$  by definition. Furthermore, by Lemma 5.7, we have  $F \in \mathcal{F}_D$  and thus  $F[1] \in \widetilde{\text{Per}}(X/Y)$  (see Remark 5.6 for the definition of  $\widetilde{\text{Per}}(X/Y)$ ).

**DEFINITION 5.13.** Let  $F$  be a 0-stable and 1-stable sheaf and  $\beta \in \mathcal{S}$ . We define  $\text{P-Sub}(F, \beta) := \{0 \rightarrow E \rightarrow F \rightarrow T[1] \rightarrow 0 \text{ exact in } \text{Per}(X/Y) : T \in \mathcal{C}^0, \text{ch}(T) = \beta\}$ ,  $\widetilde{\text{P-Quot}}(F, \beta) := \{0 \rightarrow T \rightarrow F[1] \rightarrow E[1] \rightarrow 0 \text{ exact in } \widetilde{\text{Per}}(X/Y) : T \in \mathcal{C}^0, \text{ch}(T) = \beta\}$ .

Summarizing the results in the previous subsections, we get:

**PROPOSITION 5.14.** *We have a diagram of sets*

$$\begin{array}{ccc} M^0(v) & & M^1(v) \\ & \searrow \xi_0^- & \swarrow \xi_0^+ \\ & M^{0,1}(v) & \end{array}$$

such that

- $M^{0,1}(v) := \coprod_{\beta \in \mathcal{S}} (M^0(v - \beta) \cap M^1(v - \beta))$ ,
- the fibre of  $F \in M^0(v - \beta) \cap M^1(v - \beta)$  over  $\xi_0^-$  is  $\text{P-Sub}(F, \beta)$ ,
- the fibre of  $F \in M^0(v - \beta) \cap M^1(v - \beta)$  over  $\xi_0^+$  is  $\widetilde{\text{P-Quot}}(F, \beta)$ .

**PROOF.** We define the map  $\xi_0^-$  as follows: Take an element  $E^- \in M^0(v)$ , let  $0 \rightarrow T \rightarrow E^- \rightarrow F \rightarrow 0$  be the canonical exact sequence with  $T \in \mathcal{T}_D$ ,  $F \in \mathcal{F}_D$ . Then  $F$  is an element of  $M^{0,1}(v)$  by Proposition 5.8. Hence we define  $\xi_0^-(E^-) := F$ . By the converse construction given in Proposition 5.10, the fiber of  $\xi_0^-$  is given by  $\text{P-Sub}(F, \beta)$ .

The map  $\xi_0^+$  is defined as follows: for an element  $E^+ \in M^1(v)$ , we have an exact sequence  $0 \rightarrow F \rightarrow E^+ \rightarrow T \rightarrow 0$  with  $F \in \mathcal{T}$  and  $T \in \mathcal{F}$ . Now we define as  $\xi_0^+(E^+) := F$ . Then the assertion follows from Proposition 5.11 and Proposition 5.12.  $\square$

In Section 7, we will construct the scheme-theoretic wall-crossing diagram.

## 6. Moduli space of Gieseker stable sheaves

In this section, we will see the relationship between  $m$ -stability and the Gieseler stability on both  $X$  and  $Y$ .

**6.1. Moduli space of Gieseker stable sheaves on  $Y$ .** Take an element  $v = (v_0, v_1, \dots) \in H^{2*}(X, \mathbb{Q})$  with  $\gcd(f^* H^{n-1} \cdot v_1, v_0) = 1$ , and let  $w := f_*(v \cdot \text{td}_X) \cdot \text{td}_Y^{-1}$ . Denote by  $M^H(w)$  the moduli space of Gieseker  $H$ -stable sheaves on  $Y$  with Chern character  $w$ . Then we have a morphism of schemes:

$$\xi : M^0(v) \rightarrow M^H(w), \quad E \mapsto \mathbf{R}f_* E = f_* E.$$

Similarly, we have

$$\xi^+ : M^1(v) \rightarrow M^H(w), \quad E \mapsto \mathbf{R}f_* E = f_* E.$$

Note that  $\xi^+$  is well-defined. Indeed, take an element  $E \in M^1(v)$ . Then by Lemma 5.7,  $f_* E$  is torsion free. Furthermore, since  $f_* E(-D)$  is  $\mu$ -stable and is isomorphic to  $f_* E$  in codimension 1,  $f_* E$  is also  $\mu$ -stable. Hence  $\xi^+$  is actually a morphism from  $M^1(v)$  to  $M^H(w)$ .

LEMMA 6.1. *Let  $F \in \text{Coh}(Y)$  be a torsion free sheaf on  $Y$ . Then we have  $\mathbf{L}f^*F = f^*F$ .*

PROOF. First we claim that  $\mathbf{L}f^*F \in \text{Per}(X/Y)$ . We need to check the following three conditions (cf. [11, Lemma 3.2]):

- (a)  $\mathbf{R}f_*(\mathbf{L}f^*F) \in \text{Coh}(Y)$ ,
- (b)  $\text{Hom}(\mathbf{L}f^*F, \mathcal{C}^{>-1}) = 0$ ,
- (c)  $\text{Hom}(\mathcal{C}^{<-1}, \mathbf{L}f^*F) = 0$ .

(a), (b) are clear. We will show (c). Let  $T \in \mathcal{C}^0$ ,  $i \geq 1$ ,  $f_! := \mathbf{R}f_*(- \otimes \omega_X) \otimes \omega_Y^{-1}$ . Then we have

$$\text{Hom}(T[i], \mathbf{L}f^*F) \cong \text{Hom}(f_!T[i], F).$$

By the description of  $f_!$ , we know that  $f_!T$  is a two term complex concentrated in degree 0 and 1. Hence there exist  $T', T'' \in \text{Coh}(Y)$  and an exact triangle

$$T'[i] \rightarrow f_!T[i] \rightarrow T''[i-1].$$

Applying  $\text{Hom}(-, F)$ , we get an exact sequence

$$\text{Hom}(T''[i-1], F) \rightarrow \text{Hom}(f_!T[i], F) \rightarrow \text{Hom}(T'[i], F).$$

Since  $T', T''$  are torsion sheaves and  $F$  is torsion free sheaf, we conclude that

$$(6.1) \quad \text{Hom}(T[i], \mathbf{L}f^*F) \cong \text{Hom}(f_!T[i], F) = 0.$$

Hence  $\mathbf{L}f^*F \in \text{Per}(X/Y)$ .

Next we claim that  $\mathbf{L}^{-1}f^*F \in \mathcal{C}^0$ . Note that, if so, together with the equation (6.1), we must have  $\mathbf{L}^{-1}f^*F = 0$ , i.e.,  $\mathbf{L}f^*F = f^*F$ . By the spectral sequence

$$\mathbf{R}^p\mathbf{L}^q f^*F \Rightarrow \mathcal{H}^{p+q}(\mathbf{R}f_*\mathbf{L}f^*F) = F,$$

we have

- $f_*(\mathbf{L}^{-1}f_*F) = 0$ ,
- $0 \rightarrow \mathbf{R}^1f_*(\mathbf{L}^{-1}f^*F) \rightarrow F \rightarrow f_*f^*F \rightarrow 0$  is exact in  $\text{Coh}(Y)$ .

Since  $\mathbf{R}^1f_*(\mathbf{L}^{-1}f^*F)$  is a torsion sheaf and we assume  $F$  is torsion free, we have  $\mathbf{R}^1f_*(\mathbf{L}^{-1}f^*F) = 0$ . This proves that  $\mathbf{L}f^*F = f^*F$ .  $\square$

COROLLARY 6.2. *Assume that  $v \in f^*H^*(Y; \mathbb{Q}) \subset H^*(X; \mathbb{Q})$ . Then the morphism  $\xi$  is an isomorphism.*

PROOF. By Lemma 6.1, we can define a morphism  $\eta$  as

$$\eta: M^H(w) \rightarrow M^0(v), \quad F \mapsto \mathbf{L}f^*F = f^*F.$$

Then by the projection formula, we have  $\xi \circ \eta = \text{id}$ . On the other hand, let  $E \in M^0(v)$ . Then we have an exact sequence

$$0 \rightarrow K \rightarrow f^*f_*E \rightarrow E \rightarrow 0$$

in  $\text{Coh}(X)$ , since  $E \in \text{Per}(X/Y)$ . By Lemma 4.4 (4), we have  $K \in \mathcal{C}^0$ . In particular,  $\text{ch}(K) \notin f^*H^*(Y; \mathbb{Q})$ . Since we assume  $v \in f^*H^*(Y; \mathbb{Q})$ , we must have  $K = 0$ , i.e.,  $\eta \circ \xi = \text{id}$ .  $\square$

**6.2. Moduli space of Gieseker stable sheaves on  $X$ .** First we recall the finiteness result of walls for  $\mu$ -stability:

PROPOSITION 6.3 (cf. [48, Lemma 1.1.7]). *Let  $H$  be an ample divisor on  $Y$ , take  $\epsilon_0 \in \mathbb{Q}_{>0}$  so that  $f^*H - \epsilon_0 D$  is ample on  $X$ . Let*

$$\Delta := \{H_\epsilon := f^*H - \epsilon D : 0 \leq \epsilon \leq \epsilon_0\}.$$

*Then there exist only finitely many walls on  $\Delta$  for  $\mu$ -stability with respect to  $v$ .*

PROOF. The argument is essentially same as [48, Lemma 1.1.7]. However, since  $f^*H$  is not ample, we need to be a little bit careful. The proof needs the following two facts:

- (1) The Bogomolov-Gieseker (BG) inequality for  $\mu_{H_\epsilon}$ -stable sheaves.
- (2) Let  $x \in \text{NS}(X)_{\mathbb{R}}$ . If  $x.H_\epsilon^{n-1} = 0$ , then  $-x^2.H_\epsilon^{n-2} \geq 0$ .

Firstly, (1) holds since the BG inequality holds for all nef divisor (cf. [27]). Next consider the statement (2). If  $\epsilon \neq 0$ , then  $H_\epsilon$  is ample and the statement holds by the Hodge index theorem. Assume  $\epsilon = 0$ , let  $x_\epsilon := x - \frac{x.H_\epsilon^{n-1}}{D.H_\epsilon^{n-1}}D$ . Then

$$x_\epsilon.H_\epsilon^{n-1} = x.H_\epsilon^{n-1} - \frac{x.H_\epsilon^{n-1}}{D.H_\epsilon^{n-1}}D.H_\epsilon^{n-1} = 0$$

and  $\lim_{\epsilon \rightarrow +0} x_\epsilon = x$ . Since  $x_\epsilon$  changes continuously with respect to  $\epsilon$ , we get the required result from the result of the ample case.  $\square$

By Proposition 6.3, there exist  $0 < \epsilon(v)$  such that for every  $0 < \epsilon \leq \epsilon(v)$ ,  $M^{f^*H-\epsilon D}(v)$  is constant.

PROPOSITION 6.4. *There exists an integer  $m(v) > 0$  such that for every  $m \geq m(v)$ , we have an open and closed embedding  $M^{f^*H-\epsilon(v)D}(v) \subset M^m(v)$ .*

PROOF. It is enough to show the following: there exists  $m(v) > 0$  such that for every  $m \geq m(v)$  and  $E \in M^{f^*H-\epsilon(v)D}(v)$ , we have  $E \in M^m(v)$ . Indeed, if so, we have an immersion  $\Phi: M^{f^*H-\epsilon(v)D}(v) \hookrightarrow M^m(v)$ . By the openness of  $\mu$ -stability,  $\Phi$  is an open immersion. On the other hand, since both  $M^{f^*H-\epsilon(v)D}(v)$  and  $M^m(v)$  are projective,  $\Phi$  is also a closed immersion.

First of all, since  $M^{f^*H-\epsilon(v)D}(v)$  is bounded, there exists a fixed sheaf  $U \in \text{Coh}(X)$  such that for every  $E \in M^{f^*H-\epsilon(v)D}(v)$ , we have a surjective map  $U \rightarrow E$  (cf. [25, Lemma 1.7.6]). On the other hand, since  $-D$  is  $f$ -ample, there exists an integer  $m(v) > 0$  such that for every  $m \geq m(v)$ , the adjoint map  $f^*f_*(U(-mD)) \rightarrow U(-mD)$  is surjective. Then we have a commutative diagram

$$\begin{array}{ccc} f^*f_*(U(-mD)) & \twoheadrightarrow & U(-mD) \\ \downarrow & & \downarrow \\ f^*f_*(E(-mD)) & \xrightarrow{\alpha} & E(-mD). \end{array}$$

Hence the adjoint map  $\alpha$  is also surjective, i.e.  $E(-mD) \in \text{Per}(X/Y)$ .

It remains to show that for  $m \geq m(v)$ ,  $f_*(E(-mD))$  is  $\mu_H$ -stable. Note that  $f_*(E(-mD))$  is torsion free since so is  $E$ . Take a non-zero proper subsheaf  $F \subset f_*(E(-mD))$ , and let  $\phi: f^*F \rightarrow E(-mD)$  be the corresponding map. Taking its cone  $T := \text{Cone}(\phi)$ , we have

$$\begin{aligned} 0 \rightarrow \mathcal{H}^{-1}(T) \rightarrow f^*F \rightarrow Q \rightarrow 0, \\ 0 \rightarrow Q \rightarrow E(-mD) \rightarrow \mathcal{H}^0(T) \rightarrow 0. \end{aligned}$$

Here,  $\mathcal{H}^i(T)$  denotes the cohomology with respect to the heart  $\text{Coh}(X)$  for each integer  $i \in \mathbb{Z}$ . Note that since  $\phi$  is injective outside  $D$ ,  $\text{Supp}(\mathcal{H}^{-1}(T)) \subset D$ . Hence we can write  $\text{ch}(\mathcal{H}^{-1}(T)) = (0, lD, \dots)$  with  $l \geq 0$ . Now by the  $\mu_{f^*H-\epsilon(v)D}$ -stability of  $E(-mD)$ , we have

$$\mu_{f^*H-\epsilon D}(Q) < \mu_{f^*H-\epsilon D}(E(-mD))$$

for all  $0 < \epsilon \leq \epsilon(v)$ . Taking the limit  $\epsilon \rightarrow +0$ , we have

$$\mu_H(F) = \mu_{f^*H}(f^*F) = \mu_{f^*H}(Q) \leq \mu_{f^*H}(E(-mD)) = \mu_H(f_*(E(-mD))).$$

This shows that  $E \in M^m(v)$ .  $\square$

REMARK 6.5. As mentioned in the introduction, when  $n = \dim Y = 2$ , the embedding given in Proposition 6.4 is actually an isomorphism (see Theorem 1.2). When  $n \geq 3$ , we do not know whether the inclusion is isomorphism or not in general. However, see the following example.

EXAMPLE 6.6. Let  $n = 3$ ,  $v = (1, 0, 0, -k) \in H^{2*}(X; \mathbb{Q})$  with  $k \in \mathbb{Z}_{>0}$ . We claim that for a sufficiently large integer  $m \in \mathbb{Z}$ , we have an isomorphism between  $M^m(v)$  and  $M^{f^*H-\epsilon D}(v)$ . By Proposition 6.4, it is enough to show that every  $m$ -stable sheaf  $E \in M^m(v)$  is torsion free. Let us consider the exact sequence

$$0 \rightarrow E_{\text{tor}} \rightarrow E \rightarrow E_{\text{fr}} \rightarrow 0,$$

where  $E_{\text{tor}}$  (resp.  $E_{\text{fr}}$ ) is the torsion (resp. torsion free) part of  $E$ .

First assume that  $\dim \text{Supp}(E_{\text{tor}}) = 2$ . Take a general member  $A$  in the linear system  $|H|$  and let  $f_A: A' := f^{-1}(A) \rightarrow A$ . We may assume that

- (1)  $(f_*E(-mD))|_A \cong f_{A*}(E(-mD)|_{A'})$  is torsion free.
- (2)  $E_{\text{tor}}|_{A'} \neq 0$ .

This means  $E|_{A'}$  is an  $m$ -stable sheaf on the smooth projective surface  $A'$  which has a non-trivial torsion part. Hence by Theorem 1.2 (3),  $m$  is bounded from above.

We may now assume that  $\dim \text{Supp}(E_{\text{tor}}) = 1$ . Since  $f_*(E(-mD))$  is torsion free, we can see that

$$E_{\text{tor}} \in \langle \mathcal{O}_{L_y}(a_y) : y \in C, \quad a_y + m < 0 \rangle.$$

In particular, we have  $\text{ch}(E_{\text{fr}}) = \left(1, 0, -\sum_{i=1}^l n_i L_{y_i}, -k - \sum_{i=1}^l n_i(a_i + \frac{1}{2})\right)$ , where  $y_i \in C$ ,  $a_i + m < 0$ . Now consider the exact sequence

$$0 \rightarrow E_{\text{fr}} \rightarrow (E_{\text{fr}})^{DD} \cong \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0.$$

Here we use the speciality of our choice of  $v$ , which implies that  $(E_{\text{fr}})^{DD} \cong \mathcal{O}_X$  for any  $E \in M^m(v)$ . Since  $\mathbf{R}f_*\mathcal{O}_Z = f_*\mathcal{O}_Z$  is zero dimensional, the Riemann-Roch theorem yields

$$0 \leq \chi(\mathbf{R}f_*\mathcal{O}_Z) = k + \sum_{i=1}^l n_i(a_i + \frac{1}{2}).$$

Since  $a_i + m < 0$ , we get the inequality  $k \geq m(\sum_{i=1}^l n_i)$  which bounds  $m$  from above.

## 7. Scheme structure on $M^{0,1}(v)$

In this section, we define the scheme structure on  $M^{0,1}(v)$  connecting  $M^0(v)$  and  $M^1(v)$ . Recall that we have constructed a set-theoretic diagram in Proposition 5.14:

$$(7.1) \quad \begin{array}{ccc} M^0(v) & & M^1(v) \\ & \searrow \xi_0^- \quad \swarrow \xi_0^+ & \\ & M^{0,1}(v) & \end{array}$$

On the other hand, we have a scheme-theoretic diagram

$$(7.2) \quad \begin{array}{ccc} M^0(v) & & M^1(v) \\ & \searrow \xi=f_* \quad \swarrow \xi^+=f_* & \\ & M^H(w) & \end{array}$$

In the following, we will show that these two diagrams are essentially same.

PROPOSITION 7.1. (1) *The morphism*

$$(7.3) \quad \xi|_{M^0(v) \cap M^1(v)}: M^0(v) \cap M^1(v) \rightarrow M^H(w)$$

*is an immersion.*

(2) *We can identify  $\xi_0^-(M^0(v))$  with  $\xi(M^0(v))$ .*

(3) *Under the identification (2), we have  $\xi = \xi_0^-$ .*

PROOF. (1) Since  $M^H(w)$  is a projective scheme, it is enough to show that the morphism is injective and it induces injection between tangent spaces.

First we will show that the morphism (7.3) is injective. Let  $E, E' \in M^0(v) \cap M^1(v)$  with  $\phi: f_*E' \cong f_*E$ . Then we have the commutative diagram

$$(7.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & K' & \longrightarrow & f^*f_*E' & \longrightarrow & E' \longrightarrow 0 \\ & & \downarrow & & \downarrow f^*(\phi) & & \downarrow \psi \\ 0 & \longrightarrow & K & \longrightarrow & f^*f_*E & \longrightarrow & E \longrightarrow 0. \end{array}$$

Note that since  $K' \in \mathcal{C}^0$  and  $E$  is 1-stable, we have  $\text{Hom}(K', E) = 0$ . Hence the composition  $K' \rightarrow f^*f_*E' \rightarrow f^*f_*E$  factors through  $K' \rightarrow K \rightarrow f^*f_*E$ . As a result, we get an exact sequence

$$0 \longrightarrow T \longrightarrow E' \xrightarrow{\psi} E \longrightarrow 0.$$

Applying  $\mathbf{R}f_*$  to the diagram (7.4), we have  $\mathbf{R}f_*(\psi) = \phi: f_*E' \cong f_*E$  and hence  $T \in \mathcal{C}^0$ . Since  $E'$  is 1-stable,  $T$  must be 0, i.e.  $\psi: E' \cong E$ .

It remains to show that for every  $E \in M^0(v) \cap M^1(v)$ , there exists an inclusion between tangent spaces

$$T_{[E]}M^0(v) = \text{Ext}^1(E, E) \xrightarrow{f_*} \text{Ext}^1(f_*E, f_*E) = T_{[f_*E]}M^H(w).$$

Note that since  $E \in M^0(v) \cap M^1(v)$ , we have  $\mathbf{R}f_*E = f_*E$  and  $\mathbf{L}f^*\mathbf{R}f_*E \cong f^*f_*E$  (see Lemma 6.1). Applying  $\text{Hom}(-, E)$  to the second row of the diagram (7.4), we get

$$0 = \text{Hom}(K, E) \rightarrow \text{Ext}^1(E, E) \hookrightarrow \text{Ext}^1(f^*f_*E, E) \cong \text{Ext}^1(f_*E, f_*E).$$

Note that  $\text{Hom}(K, E) = 0$  follows from the 1-stability of  $E$ . By [24, Lemma 1.21], the diagram

$$\begin{array}{ccc} \text{Ext}^1(E, E) & & \\ \downarrow & \searrow f_* & \\ \text{Ext}^1(f^*f_*E, E) & \longrightarrow & \text{Ext}^1(f_*E, f_*E) \end{array}$$

is commutative. Hence we conclude that the tangent map  $f_*$  is injective.

(2) For  $G = \xi(E) \in \xi(M^0(v))$ , we associate an element  $\Phi(G)$  of  $\xi_0^-(M^0(v))$  as follows: we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T' & \longrightarrow & f^*f_*E & \longrightarrow & E \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & T & \longrightarrow & f^*f_*E & \longrightarrow & F \longrightarrow 0 \end{array}$$

in  $\text{Coh}(X)$ , where  $T'$  is the kernel of the adjoint map and the bottom sequence is the decomposition with respect to the torsion pair  $(\mathcal{T}_D, \mathcal{F}_D)$ . Since  $f^*f_*E$  is 0-stable, we have  $T \in \mathcal{C}^0$  and  $F$  is 1-stable by Proposition 5.8. Note that since

$\text{Hom}(\mathcal{C}^0, F) = 0$ , we have a surjective map  $E \rightarrow F \rightarrow 0$ . Furthermore, by the construction, its kernel  $\text{Ker}(E \rightarrow F) \in \mathcal{C}^0$ . In other words,

$$(7.5) \quad \Phi(G) := F = \xi_0^-(E) \in \xi_0^-(M^0(v)).$$

Note that the definition of  $\Phi$  is independent of the choice of  $E \in M^0(v)$  with  $f_*E = G$ , i.e. the map  $\Phi$  is well-defined. Indeed,  $F$  only depends on  $f^*f_*E = f^*G$ .

CLAIM 7.2. The map  $\Phi$  is injective.

PROOF. Let  $G, G' \in \xi(M^0(v))$  with  $\Phi(G) = \Phi(G')$ . Then we have  $G = f_*\Phi(G) = f_*\Phi(G') = G'$ .  $\square$

CLAIM 7.3. The map  $\Phi$  is surjective.

PROOF. Let  $F = \xi_0^-(E) \in \xi_0^-(M^0(v))$ . Then the equation (7.5) shows that  $F = \Phi(\xi(E))$ .  $\square$

(3) The assertion follows from the equation (7.5).  $\square$

PROPOSITION 7.4. (1) We can identify  $\xi_0^+(M^1(v))$  with  $\xi^+(M^1(v))$ .  
 (2) Via the identification (1), we have  $\xi^+ = \xi_0^+$ .

PROOF. The proof is similar to that of Proposition 7.1. Hence we just give an outline of the proof. We define the map  $\Psi: \xi^+(M^1(v)) \rightarrow \xi_0^+(M^1(v))$  as follows: Let  $F = \xi^+(E) \in \xi^+(M^1(v))$ . Then we have  $\phi: f_*E \cong F$  by definition. Take an element  $\alpha \in \text{Hom}(E, f^*F(D))$  corresponding to  $\phi$  via the isomorphism  $\text{Hom}(f_*E, F) \cong \text{Hom}(E, f^*F(D))$ . Let  $C := \text{Cone}(\alpha)$ . Then since  $f_*(\alpha) = \phi$ , we have  $C \in \mathcal{C}$ . Furthermore, since  $E$  is 1-stable, we must have  $\mathcal{H}^{-1}(C) = 0$ , i.e.  $C$  is a sheaf. Hence we have the exact sequence

$$(7.6) \quad 0 \longrightarrow E \xrightarrow{\alpha} f^*F(D) \longrightarrow C \longrightarrow 0$$

in  $\text{Coh}(X)$ . Consider the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G' & \longrightarrow & E & \longrightarrow & T' \longrightarrow 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \\ 0 & \longrightarrow & G & \longrightarrow & f^*F(D) & \longrightarrow & T \longrightarrow 0 \end{array}$$

with  $G', G \in \mathcal{T}$  and  $T', T \in \mathcal{F}$ . Using the fact that  $C = \text{Coker}(\alpha) \in \mathcal{C}^0$ , we can show that  $G' \cong G$ . Now we define as  $\Psi(F) := G = G' = \xi_0^+(E) \in \xi_0^+(M^1(v))$ . As in Proposition 7.1, we can prove that  $\Psi$  is bijection. Furthermore, the second assertion follows from the definition of the map  $\Psi$ .  $\square$

As a summary of Proposition 5.14, Proposition 7.1, and Proposition 7.4, we get:

PROPOSITION 7.5. There exists a diagram of projective schemes

$$\begin{array}{ccccc} \cdots & M^m(v) & & M^{m+1}(v) & \cdots \\ & \searrow & \xrightarrow{\xi_m^-} & \nwarrow \xrightarrow{\xi_m^+} & \\ & & M^{m,m+1}(v) & & \end{array}$$



connecting the moduli spaces of  $m$ -stable sheaves, where the morphisms  $\xi_m^\pm$  are defined as  $\xi_m^\pm := f_*(- \otimes \mathcal{O}(-mD))$ . Moreover, we can also describe the morphisms  $\xi_m^+, \xi_m^-$  in terms of the decompositions with respect to the torsion pairs  $(\mathcal{T} \otimes \mathcal{O}(mD), \mathcal{F} \otimes \mathcal{O}(mD))$ ,  $(\mathcal{T}_D \otimes \mathcal{O}(mD), \mathcal{F}_D \otimes \mathcal{O}(mD))$ , respectively.

We end with the following proposition describing the fibers of  $\xi$  and  $\xi^+$ . We can think them as the scheme structures on  $\text{P-Sub}(F, \beta)$  and  $\widetilde{\text{P-Quot}}(F, \beta)$  defined in Definition 5.13.

**PROPOSITION 7.6.** *Let  $F \in M^H(w)$  be a  $H$ -stable sheaf with Chern character  $\text{ch}(F) = v'$ . Then the following statements hold.*

- (1) *We have  $\xi^{-1}(F) = \text{Quot}(f^*F, v)$ .*
- (2) *We have  $(\xi^+)^{-1}(F) = \text{Sub}(f^*F(D), v)$ .*

**PROOF.** (1) Let  $E \in \xi^{-1}(F)$ . Then we have an exact sequence

$$0 \rightarrow T \rightarrow f^*F = f^*f_*E \rightarrow E \rightarrow 0$$

in  $\text{Coh}(X)$ , i.e.  $E \in \text{Quot}(f^*F, v)$ .

For the converse, let  $E \in \text{Quot}(f^*F, v)$ . Then we have

$$0 \rightarrow K \rightarrow f^*F \rightarrow E \rightarrow 0.$$

Note that  $E \in \text{Per}(X/Y)$ . Since  $K$  is torsion and  $F$  is torsion free, we have  $f_*K = 0$  and  $F \hookrightarrow f_*E$  is injective. Moreover, since  $\text{ch}(F) = \text{ch}(f_*E)$ , we have  $F \cong f_*E$ . Hence  $E \in \xi^{-1}(F)$ . Noting that  $\text{Hom}(f^*F, E) \cong \text{Hom}(F, f_*E) = \mathbb{C}$ , we conclude that  $\xi^{-1}(F) = \text{Quot}(f^*F, v)$ .

(2) Let  $E \in (\xi^+)^{-1}(F)$ . Then by the proof of Proposition 7.4, we have the exact sequence as in (7.6):

$$0 \rightarrow E \rightarrow f^*F(D) \rightarrow C \rightarrow 0.$$

In other words,  $E \in \text{Sub}(f^*F(D), v)$ .

For the converse, let  $E \in \text{Sub}(f^*F(D), v)$ . Then we have

$$0 \rightarrow E \rightarrow f^*F(D) \rightarrow C \rightarrow 0.$$

First we claim that  $C \in \mathcal{C}^0$ . Since  $f^*F \in \text{Per}(X/Y)$ , we have  $C(-D) \in \text{Per}(X/Y)$  by Lemma 5.2. Hence by Lemma 4.3, we have  $\mathbf{R}^1f_*C = 0$ . Furthermore, we have  $\text{ch}(C) \in \mathcal{S}$ . Hence  $\text{ch}(f_*C) = \text{ch}(\mathbf{R}f_*C) = 0$  and so we also have  $f_*C = 0$ .

Next we show that  $E(-D) \in \text{Per}(X/Y)$ . To show this, it is enough to show that for  $y \in C$ , we have  $\text{Hom}(E(-D), \mathcal{O}_{L_y}(-1)) = 0$ . Write  $C(-D) = f^*M$ . Applying  $\text{Hom}(-, \mathcal{O}_{L_y}(-1))$  to the exact sequence

$$0 \rightarrow E(-D) \rightarrow f^*F \rightarrow f^*M \rightarrow 0,$$

we have

$$0 = \text{Hom}(f^*F, \mathcal{O}_{L_y}(-1)) \rightarrow \text{Hom}(E(-D), \mathcal{O}_{L_y}(-1)) \rightarrow \text{Hom}(f^*M, \mathcal{O}_{L_y}(-1)[1]).$$

Hence it is enough to show that  $\text{Hom}(f^*M, \mathcal{O}_{L_y}(-1)[1]) = 0$ . From the exact triangle

$$\mathbf{L}^{-1}f^*M[1] \rightarrow \mathbf{L}f^*M \rightarrow f^*M,$$

we have

$$\begin{aligned} 0 &= \text{Hom}(\mathbf{L}^{-1}f^*M[1], \mathcal{O}_{L_y}(-1)) \rightarrow \text{Hom}(f^*M, \mathcal{O}_{L_y}(-1)[1]) \\ &\rightarrow \text{Hom}(\mathbf{L}f^*M, \mathcal{O}_{L_y}(-1)[1]) \\ &\cong \text{Hom}(M, \mathbf{R}f_*\mathcal{O}_{L_y}(-1)[1]) \\ &= 0. \end{aligned}$$

As a conclusion, we have  $E(-D) \in \text{Per}(X/Y)$ . Furthermore, since  $f^*F(D) \in \mathcal{F}_D$ , we also have  $E \in \mathcal{F}_D$ . Hence by Lemma 5.7,  $f_*E(-D)$  is torsion free.

Since  $f_*E \cong F$  is  $\mu$ -stable, we conclude that  $f_*E(-D)$  is also  $\mu$ -stable, i.e.  $E \in (\xi^+)^{-1}(F)$ . As before, noting that  $\text{Hom}(E, f^*F(D)) \cong \text{Hom}(f_*E, F) = \mathbb{C}$ , we have  $(\xi^+)^{-1}(F) = \text{Sub}(f^*F(D), v)$ .  $\square$

### 8. Hilbert scheme of two points

In this section, we study the birational geometry of Hilbert scheme of two points using the flip-like diagram (1.1) constructed in the previous sections. In the followings, we assume that  $H^1(\mathcal{O}_Y) = 0$  and let  $w := (1, 0, \dots, 0, -2) \in H^*(Y; \mathbb{Q})$ ,  $v := f^*w \in H^*(X; \mathbb{Q})$ . Then we have  $M^H(w) = \text{Hilb}^2(Y)$ ,  $M^{f^*H-\epsilon D}(v) = \text{Hilb}^2(X)$ . We will use the following notations:

- For a 0-dimensional closed subscheme  $Z \subset X$  of length 2, we denote its ideal sheaf as  $I_Z \in \text{Hilb}^2(X)$ .
- $\text{Hilb}^2(D/C) \subset \text{Hilb}^2(X)$  denotes the relative Hilbert scheme parametrizing  $I_Z \in \text{Hilb}^2(X)$  such that  $Z$  is scheme-theoretically contained in a fiber of  $\pi: D \rightarrow C$ , i.e. there exists  $y \in C$  such that  $Z \subset L_y$ .
- $\mathcal{I} \in \text{Coh}(\text{Hilb}^2(X) \times X)$  denotes the universal ideal sheaf on  $\text{Hilb}^2(X)$ .

We start with the following lemma:

LEMMA 8.1. *Let  $I_Z \in \text{Hilb}^2(X)$  be an ideal sheaf of a length two closed subscheme  $Z \subset X$ . Then the following holds:*

- (1) *If  $Z \cap D = \emptyset$ , then  $I_Z$  is 0-stable.*
- (2) *If  $Z \cap D \neq \emptyset$  and  $I_Z \notin \text{Hilb}^2(D/C)$ , then  $I_Z$  is 1-stable but not 0-stable.*
- (3) *If  $I_Z \in \text{Hilb}^2(D/C)$ , then  $I_Z$  is 2-stable but not 1-stable.*

PROOF. By the proof of Proposition 6.4, it is enough to find the smallest  $m \in \mathbb{Z}_{\geq 0}$  such that  $I_Z(-mD) \in \text{Per}(X/Y)$ , i.e.

$$\text{Hom}(I_Z, \mathcal{O}_{L_y}(-m-1)) = \text{Hom}(I_Z(-mD), \mathcal{O}_{L_y}(-1)) = 0$$

for all  $y \in C$ .

Restricting the exact sequence

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$

to  $L_y$ , we get

$$0 \rightarrow T \rightarrow I_Z|_{L_y} \rightarrow \mathcal{O}_{L_y} \rightarrow \mathcal{O}_{Z \cap L_y} \rightarrow 0,$$

where  $T \in \text{Coh}(L_y)$  is some torsion sheaf. Hence we get

$$0 \rightarrow T \rightarrow I_Z|_{L_y} \rightarrow \mathcal{O}_{L_y}(-l) \rightarrow 0,$$

where  $l \in \mathbb{Z}$  is the length of  $Z \cap L_y$ . We conclude that

$$\begin{aligned} \text{Hom}(I_Z, \mathcal{O}_{L_y}(-m-1)) &= \text{Hom}(I_Z|_{L_y}, \mathcal{O}_{L_y}(-m-1)) \\ &= \text{Hom}(\mathcal{O}_{L_y}(-l), \mathcal{O}_{L_y}(-m-1)). \end{aligned}$$

Hence we get the result.  $\square$

COROLLARY 8.2. *The exceptional locus of  $\xi_1^+$  is  $\text{Exc}(\xi_1^+) = \text{Hilb}^2(D/C)$ .*

PROOF. The assertion directly follows from Lemma 8.1.  $\square$

By Lemma 8.1, we have the diagram:

$$(8.1) \quad \begin{array}{ccccc} & \widetilde{M}^1(v) & & \text{Hilb}^2(X) \subset M^2(v) & \\ & \swarrow \xi_0 = f_* & \searrow \xi_1^- & \swarrow \xi_1^+ & \\ \text{Hilb}^2(Y) & & M^{1,2}(v), & & \end{array}$$

where  $\widetilde{M}^1(v)$  denotes the normalization of the connected component of  $M^1(v)$  containing  $\text{Hilb}^2(X) \setminus \text{Hilb}^2(D/C)$ . In fact, we can actually show that the connectedness of  $M^1(v)$  by using Proposition 7.6. However, we omit the proof here.

In the following subsections, we will study the properties of these morphisms in details.

**8.1. The diagram  $\xi_1^\pm$  is a flip.** In this subsection, we will show that the diagram  $\xi_1^\pm$  is a flip. The main technical tool used here is so-called elementary transformation. First we observe the following.

LEMMA 8.3. *Let  $Z \subset L_y$  be a length 2 closed subscheme ( $y \in C$ ). Then the following statements hold:*

(1) *We have an exact sequence*

$$(8.2) \quad 0 \rightarrow I_{L_y} \rightarrow I_Z \rightarrow \mathcal{O}_{L_y}(-2) \rightarrow 0.$$

(2) *We have  $I_{L_y} \in \text{Per}(X/Y)$ .*

*In particular, the sequence (8.2) is the decomposition with respect to the torsion pair  $(\mathcal{T} \otimes \mathcal{O}(D), \mathcal{F} \otimes \mathcal{O}(D))$ .*

PROOF. (1) Since  $Z \subset L_y$  is length 2, we have

$$0 \rightarrow \mathcal{O}_{L_y}(-2) \rightarrow \mathcal{O}_{L_y} \rightarrow \mathcal{O}_Z \rightarrow 0.$$

Hence we get the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \mathcal{O}_{L_y}(-2) & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & I_{L_y} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{L_y} \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \longrightarrow & I_Z & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Z \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow \\
 & & T & & & & 0 \\
 & & \downarrow & & & & \\
 & & 0, & & & & 
 \end{array}$$

which implies  $T = \mathcal{O}_{L_y}(-2)$  as required.

(2) Pulling back the exact sequence

$$0 \rightarrow I_y \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_y \rightarrow 0,$$

we get

$$0 \rightarrow \mathcal{O}_{L_y}(-1) \rightarrow f^*I_y \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{L_y} \rightarrow 0.$$

It gives two short exact sequences

$$\begin{aligned}
 0 &\rightarrow \mathcal{O}_{L_y}(-1) \rightarrow f^*I_y \rightarrow I_{L_y} \rightarrow 0, \\
 0 &\rightarrow I_{L_y} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{L_y} \rightarrow 0.
 \end{aligned}$$

By Lemma 6.1, we have  $f^*I_y \in \text{Per}(X/Y)$ . Since there exists a surjection  $f^*I_y \rightarrow I_{L_y}$ , we also have  $I_{L_y} \in \text{Per}(X/Y)$  by Lemma 5.2.

For the last assertion, we first note that  $\mathcal{O}_{L_y}(-2) \otimes \mathcal{O}(-D) \cong \mathcal{O}_{L_y}(-1) \subset \mathcal{F}$ . Furthermore, by (2) and Corollary 4.6, we have  $I_{L_y}(-D) \in \mathcal{T}$ . Hence the sequence (8.2) is the decomposition with respect to the torsion pair  $(\mathcal{T} \otimes \mathcal{O}(D), \mathcal{F} \otimes \mathcal{O}(D))$ .  $\square$

COROLLARY 8.4. (1) *The restriction of  $\xi_1^+$  to  $\text{Hilb}^2(D/C)$  is given by*

$$\xi_1^+ : \text{Hilb}^2(D/C) \rightarrow C, \quad I_Z \mapsto I_{L_y}.$$

(2) *For every closed point  $y$  of  $C$ , the fiber of  $\xi_1^+$  over  $I_{L_y}$  is given by*

$$(\xi_1^+)^{-1}(I_{L_y}) = \text{Hilb}^2(L_y) \cong \mathbb{P}^2.$$

(3) *For every closed point  $y$  of  $C$ , the fiber of  $\xi_1^-$  over  $I_{L_y}$  is given by*

$$(\xi_1^-)^{-1}(I_{L_y}) = \mathbb{P} \text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2))^\vee.$$

PROOF. (1) Recall from Proposition 5.14 that the morphism  $\xi_1^+$  is defined by using the decomposition with respect to the torsion pair  $(\mathcal{T} \otimes \mathcal{O}(D), \mathcal{F} \otimes \mathcal{O}(D))$ . By Lemma 8.3, we have  $\xi_1^+(I_Z) = I_{L_y}$  as required.

(2) By Corollary 8.2 and (1), for  $I_Z \in \text{Hilb}^2(X)$ , we have  $\xi_1^+(I_Z) = I_{L_y}$  if and only if  $Z$  is the closed subscheme of  $L_y$ , i.e.  $I_Z \in \text{Hilb}^2(L_y)$ . This proves the assertion.

(3) For a 1-stable sheaf  $E \in M^1(v)$ , we have  $E \in (\xi_1^-)^{-1}(I_{L_y})$  if and only if there exists an element  $T \in \mathcal{C}^0 \otimes \mathcal{O}(D)$  such that  $E$  fits into a non-trivial exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow I_{L_y} \rightarrow 0$$

by the construction of the morphism  $\xi_1^-$ . Note that we have  $\text{ch}(T) = \text{ch}(E) - \text{ch}(I_{L_y}) = \text{ch}(I_Z) - \text{ch}(I_{L_y}) = \text{ch}(\mathcal{O}_{L_y}(-2))$ , where  $I_Z \in \text{Hilb}^2(X)$ . Hence the only possibility is  $T = \mathcal{O}_{L_y}(-2)$ . We conclude that  $(\xi_1^-)^{-1}(I_{L_y}) = \mathbb{P} \text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2))^\vee$ .  $\square$

The following lemma determines the dimension of  $\mathbb{P} \text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2))^\vee$ :

LEMMA 8.5. *For every  $y \in C$ , we have  $\text{ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2)) = n$ .*

PROOF. Applying  $\text{Hom}(-, \mathcal{O}_{L_y}(-2))$  to the standard exact sequence

$$0 \rightarrow I_{L_y} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{L_y} \rightarrow 0,$$

the claim is reduced to compute  $\text{ext}^i(\mathcal{O}_{L_y}, \mathcal{O}_{L_y}(-2))$  for  $i = 1, 2$ . Using the local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{O}_{L_y}, \mathcal{O}_{L_y}(-2))) \Rightarrow \text{Ext}^{p+q}(\mathcal{O}_{L_y}, \mathcal{O}_{L_y}(-2))$$

and the isomorphism

$$\begin{aligned} \mathcal{E}xt^q(\mathcal{O}_{L_y}, \mathcal{O}_{L_y}(-2)) &\cong \bigwedge^q \mathcal{N}_{L_y/X} \otimes \mathcal{O}_{L_y}(-2) \\ &= \bigwedge^q (\mathcal{O}_{L_y}^{\oplus n-2} \oplus \mathcal{O}_{L_y}(-1)) \otimes \mathcal{O}_{L_y}(-2) \\ &\cong \mathcal{O}_{L_y}(-2)^{\oplus \binom{n-2}{q}} \oplus \mathcal{O}_{L_y}(-3)^{\oplus \binom{n-2}{q-1}} \end{aligned}$$

(cf. [24, Corollary 11.2]), we have

$$\text{ext}^i(\mathcal{O}_{L_y}, \mathcal{O}_{L_y}(-2)) = \begin{cases} 1 & (i = 1) \\ (n-2) + 2 = n & (i = 2). \end{cases}$$

Using them, we get the result.  $\square$

We next compute the normal bundle  $\mathcal{N}_{\text{Hilb}^2(D/C)/\text{Hilb}^2(X)}$ . To do that, let us define an embedding

$$D \rightarrow C \times X, \quad x \mapsto (f(x), x)$$

and denote the ideal sheaf of  $D$  in  $C \times X$  as  $I_{D/C \times X}$ . We also use the following notations for projections:

$$\begin{array}{ccc} & \text{Hilb}^2(X) \times X & \\ p_H \swarrow & & \searrow p_X \\ \text{Hilb}^2(X) & & X \end{array} \quad \begin{array}{ccc} & C \times X & \\ q_C \swarrow & & \searrow q_X \\ C & & X \end{array}$$

DEFINITION 8.6. We define sheaves  $\mathcal{E}_\pm \in \text{Coh}(C)$  and  $\mathcal{E} \in \text{Coh}(\text{Hilb}^2(X))$  as follows:

$$\begin{aligned} \mathcal{E}_+ &:= \mathcal{E}xt_{q_C}^1(\mathcal{O}_D(2D), I_{D/C \times X}), \quad \mathcal{E}_- := \mathcal{E}xt_{q_C}^1(I_{D/C \times X}, \mathcal{O}_D(2D)), \\ \mathcal{E} &:= \mathcal{E}xt_{p_H}^1(\mathcal{I}, \mathcal{I}). \end{aligned}$$

We recall the following version of semicontinuity theorem:

THEOREM 8.7 ([14, Satz 3]). *Let  $g: M \rightarrow N$  be a flat morphism between smooth projective varieties. Let  $E, F \in \text{Coh}(M)$  be flat sheaves over  $N$ . Fix an integer  $i \in \mathbb{Z}_{\geq 0}$ . Assume that for every  $p \in N$ ,  $\text{ext}^i(E_p, F_p)$  is constant, where  $E_p, F_p \in \text{Coh}(M_p)$  is the restriction of  $E, F$  to the fiber  $M_p := g^{-1}(p)$ . Then the sheaf  $\mathcal{E}xt_g^i(E, F)$  is locally free. Furthermore, for  $q = i - 1, i$ ,  $\mathcal{E}xt_g^q(E, F)$  commutes with the base change, i.e., for every  $p \in N$ , we have an isomorphism*

$$\mathcal{E}xt_g^q(E, F)|_{\{p\}} \cong \text{Ext}^q(E_p, F_p).$$

COROLLARY 8.8. (1) *The sheaves  $\mathcal{E}_\pm, \mathcal{E}$  are locally free.*

(2) *We have  $\text{Hilb}^2(D/C) \cong \mathbb{P}(\mathcal{E}_+^\vee)$ .*

(3) *The Hilbert scheme  $\text{Hilb}^2(X)$  is smooth of dimension  $2n$  and its tangent bundle is given as  $\mathcal{T}_{\text{Hilb}^2(X)} \cong \mathcal{E}$ .*

PROOF. First of all, the smoothness of  $\text{Hilb}^2(X)$  is well known: we have  $\text{Hilb}^2(X) \cong \text{Bl}_{\Delta_X} \text{Sym}^2(X)$ . Moreover,  $\text{Sym}^2(X)$  has only  $\frac{1}{2}(1, \dots, 1)$ -type singularity along the diagonal  $\Delta_X \subset \text{Sym}^2(X)$ . Hence  $\text{Hilb}^2(X)$  is smooth.

(1) Since  $D$  is flat over  $C$ ,  $I_{D/C \times D}$  and  $\mathcal{O}_D(2D)$  are flat over  $C$ . Hence by Lemma 8.5 and Theorem 8.7, the assertion follows.

(2) We have the universal extension sheaf  $\mathcal{F} \in \text{Coh}(\mathbb{P}(\mathcal{E}_+^\vee) \times X)$  which fits into the exact sequence

$$(8.3) \quad 0 \rightarrow (\pi_X^+)^* I_{D/C \times X} \otimes p^* \mathcal{O}_{\pi^+}(1) \rightarrow \mathcal{F} \rightarrow (\pi_X^+)^* \mathcal{O}_D(2D) \rightarrow 0$$

(cf. [25, Example 2.1.12]). Here  $\pi^+: \mathbb{P}(\mathcal{E}_+^\vee) \rightarrow C$  is the structure morphism of the projective space bundle,  $\pi_X^+: \mathbb{P}(\mathcal{E}_+^\vee) \times X \rightarrow C \times X$  is the base change morphism, and  $p: \mathbb{P}(\mathcal{E}_+^\vee) \times X \rightarrow \mathbb{P}(\mathcal{E}_+^\vee)$  is the projection. The sheaf  $\mathcal{F}$  parametrizes all the extensions

$$0 \rightarrow I_{L_y} \rightarrow F \rightarrow \mathcal{O}_{L_y}(-2) \rightarrow 0.$$

By Lemma 8.3, we have  $F \cong I_Z \in \text{Hilb}^2(D/C)$ . Hence by the universality of the Hilbert scheme, we get  $\text{Hilb}^2(D/C) \cong \mathbb{P}(\mathcal{E}_+^\vee)$  as required.

(3) Consider the Kodaira-Spencer map  $KS: \mathcal{T}_{\text{Hilb}^2(X)} \rightarrow \mathcal{E}$ . Since  $\mathcal{E}$  commutes with the base change,  $KS$  restricts to an isomorphism  $KS_p: T_p \text{Hilb}^2(X) \rightarrow \text{Ext}^1(I_Z, I_Z)$  for each  $p = [I_Z] \in \text{Hilb}^2(X)$ . Hence  $KS$  is surjective morphism between locally free sheaves of the same rank. We conclude that  $\mathcal{T}_{\text{Hilb}^2(X)} \cong \mathcal{E}$ .  $\square$

Now we can compute the normal bundle:

LEMMA 8.9 (cf. [21, Proposition 3.7]). *We have an isomorphism*

$$\mathcal{N}_{\text{Hilb}^2(D/C)/\text{Hilb}^2(X)} \cong (\pi^+)^* \mathcal{E}_- \otimes \mathcal{O}_{\pi^+}(-1).$$

PROOF. First we construct a morphism  $\mathcal{E}|_{\text{Hilb}^2(D/C)} \rightarrow (\pi^+)^* \mathcal{E}_- \otimes \mathcal{O}_{\pi^+}(-1)$ . Applying  $\mathbf{R}\pi_*^+ \mathbf{R}\mathcal{H}om(\mathcal{I}|_{\text{Hilb}^2(D/C)}, -)$  and  $\mathbf{R}\pi_*^+ \mathbf{R}\mathcal{H}om(-, (\pi_X^+)^* \mathcal{O}_D(2D))$  to the exact sequence (8.3) and taking its cohomology, we get

$$(8.4) \quad \delta_1: \mathcal{E}|_{\text{Hilb}^2(D/C)} \rightarrow \mathcal{E}xt_{\pi^+}^1(\mathcal{I}|_{\text{Hilb}^2(D/C)}, (\pi_X^+)^* \mathcal{O}_D(2D))$$

and

$$(8.5) \quad \delta_2: \mathcal{E}xt_{\pi^+}^1(\mathcal{I}|_{\text{Hilb}^2(D/C)}, (\pi_X^+)^* \mathcal{O}_D(2D)) \rightarrow \mathcal{E}xt_{\pi^+}^1((\pi_X^+)^* I_{D/C \times X} \otimes p^* \mathcal{O}_{\pi^+}(1), (\pi_X^+)^* \mathcal{O}_D(2D)).$$

Note that we used  $\mathcal{F} \cong \mathcal{I}|_{\text{Hilb}^2(D/C)}$  above. Straightforward computation shows that

$$\mathcal{E}xt_{\pi^+}^1((\pi_X^+)^* I_{D/C \times X} \otimes p^* \mathcal{O}_{\pi^+}(1), (\pi_X^+)^* \mathcal{O}_D(2D)) \cong (\pi^+)^* \mathcal{E}_- \otimes \mathcal{O}_{\pi^+}(-1).$$

Hence we get the morphism

$$\delta := \delta_2 \circ \delta_1: \mathcal{E}|_{\text{Hilb}^2(D/C)} \rightarrow (\pi^+)^* \mathcal{E}_- \otimes \mathcal{O}_{\pi^+}(-1).$$

We will prove that  $\delta$  is surjective and  $\ker \delta \cong \mathcal{T}_{\text{Hilb}^2(D/C)}$ . To show that, it is enough to show the following: For every  $p = [I_Z] \in \text{Hilb}^2(D/C)$  with  $Z \subset L_y$ ,

- (1) the restriction  $\delta_p: T_p \text{Hilb}^2(X)|_{\text{Hilb}^2(D/C)} \rightarrow \text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2))$  is surjective,
- (2)  $\ker(\delta_p) = T_p \text{Hilb}^2(D/C)$ .

In (1), we can actually show that both  $\delta_{1,p}$  and  $\delta_{2,p}$  are surjective by using the spectral sequence argument as in Lemma 8.5.

For (2), it is now enough to show that  $T_p \text{Hilb}^2(D/C) \subset \ker(\delta_p)$  since they are vector spaces of the same dimension. The argument is exactly same as [21, Proposition 3.7] and hence we omit the detail.  $\square$

The following result directly follows from Lemma 8.9:

COROLLARY 8.10. (1) *When  $n = 2$ ,  $\xi_1^+$  is a flipping contraction.*  
 (2) *When  $n = 3$ ,  $\xi_1^+$  is a flopping contraction.*  
 (3) *When  $n \geq 4$ ,  $\xi_1^+$  is an anti-flipping contraction.*

Let  $\mu: M := \text{Bl}_{\text{Hilb}^2(D/C)} \text{Hilb}^2(X) \rightarrow \text{Hilb}^2(X)$ ,  $E \subset M$  be the  $\mu$ -exceptional divisor,  $\nu := \mu|_E: E \rightarrow \text{Hilb}^2(D/C)$ . Note that by Lemma 8.9, we know that  $E \cong \mathbb{P}(\mathcal{E}_-^\vee) \times_C \mathbb{P}(\mathcal{E}_+^\vee)$ . By the elementary transformation, we will construct the family  $\mathcal{G}$  of 1-stable sheaves on  $M$ , which gives us the morphism  $M \rightarrow \widetilde{M}^1(v)$ .

Pulling back the exact sequence (8.3), we get

$$0 \rightarrow \nu_X^*(\pi_X^+)^*(I_{D/C \times X} \otimes (\mathcal{O}_{\pi^+}(1))) \rightarrow \mu_X^* \mathcal{I}|_{E \times X} \rightarrow \nu_X^*(\pi^+)^* \mathcal{O}_D(2D) \rightarrow 0$$

in  $\text{Coh}(M \times X)$ . Now define

$$\mathcal{G} := \ker(\mu_X^* \mathcal{I} \rightarrow \mu_X^* \mathcal{I}|_{E \times X} \rightarrow \nu_X^*(\pi^+)^* \mathcal{O}_D(2D)).$$

LEMMA 8.11. (1) *The sheaf  $\mathcal{G}$  is a flat family of 1-stable sheaves and hence defines the morphism  $\sigma: M \rightarrow \widetilde{M}^1(v)$ .*

- (2) *The restriction  $\sigma|_E: E \rightarrow \sigma(E)$  coincides with the projection morphism  $\mathbb{P}(\mathcal{E}_-^\vee) \times_C \mathbb{P}(\mathcal{E}_+^\vee) \rightarrow \mathbb{P}(\mathcal{E}_-^\vee)$ .*

PROOF. (1) We get the following commutative diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
& \mathcal{K} = \mu_X^* \mathcal{I}(-E \times X) & & & & & \\
& \downarrow & & \downarrow & & & \\
0 \longrightarrow & \mathcal{G} & \longrightarrow & \mu_X^* \mathcal{I} & \longrightarrow & \nu_X^* \mathcal{O}_D(2D) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \parallel & \\
0 \longrightarrow & \mathcal{J} & \longrightarrow & \mu_X^* \mathcal{I}|_{E \times X} & \longrightarrow & \nu_X^*(\pi^+)^* \mathcal{O}_D(2D) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0, & & & 
\end{array}$$

where we put

$$\mathcal{J} := \nu_X^*(\pi^+)^*(I_{D/C \times X} \otimes \mathcal{O}_{\pi^+}(1)).$$

Applying  $(-) \otimes^{\mathbf{L}} \mathcal{O}_{E \times X}$  to the left column in the above diagram, we get the following long exact sequence:

$$\begin{aligned}
0 \rightarrow \mathcal{T}or^1(\mathcal{O}_{E \times X}, \mathcal{G}) &\rightarrow \mathcal{T}or^1(\mathcal{O}_{E \times X}, \mathcal{J}) \rightarrow \\
&\mu_X^* \mathcal{I}(-E \times X)|_{E \times X} \rightarrow \mathcal{G}|_{E \times X} \rightarrow \mathcal{J} \rightarrow 0
\end{aligned}$$

Note that we have  $\mathcal{T}or^1(\mathcal{O}_{E \times X}, \mathcal{K}) = 0$  since  $\mathcal{I}$  is flat over  $\text{Hilb}^2(X)$ . Using the isomorphism  $\mathcal{O}_{E \times X} \cong (\mathcal{O}_{M \times X}(-E \times X) \rightarrow \mathcal{O}_{M \times X})$  in  $D^b(M \times X)$ , we can easily show that  $\mathcal{T}or^1(\mathcal{O}_{E \times X}, \mathcal{G}) = 0$  and the above long exact sequence splits into

$$\begin{aligned}
0 \rightarrow \mathcal{J}(-E \times X) &\rightarrow \mu_X^* \mathcal{I}(-E \times X)|_{E \times X} \rightarrow (\nu_X^*(\pi^+)^* \mathcal{O}_D(2D))(-E \times X) \rightarrow 0 \\
0 \rightarrow (\nu_X^*(\pi^+)^* \mathcal{O}_D(2D))(-E \times X) &\rightarrow \mathcal{G}|_{E \times X} \rightarrow \mathcal{J} \rightarrow 0.
\end{aligned}$$

Hence  $\mathcal{G}$  is flat over  $M$ . Furthermore,  $\mathcal{G}|_{E \times X}$  parametrizes extensions

$$0 \rightarrow \mathcal{O}_{L_y}(-2) \rightarrow G \rightarrow I_{L_y} \rightarrow 0,$$

i.e. 1-stable objects  $G$ . This defines a morphism  $\sigma: M \rightarrow M^1(v)$ . We claim that the image of  $\sigma$  is contained in a connected component  $\widetilde{M}^1(v)$ . Indeed, we have  $\sigma(M \setminus E) \subset \widetilde{M}^1(v)$  by our definition of  $\widetilde{M}^1(v)$ . Since  $\sigma(M)$  is connected, it must be contained in  $\widetilde{M}^1(v)$ .

(2) The assertion follows from [20, Proposition A.2].  $\square$

COROLLARY 8.12. *The scheme  $\widetilde{M}^1(v)$  is a smooth projective variety. Moreover,*

- (1) *When  $n = 2$ ,  $\text{Hilb}^2(X) \dashrightarrow \widetilde{M}^1(v)$  is a flip.*
- (2) *When  $n = 3$ ,  $\text{Hilb}^2(X) \dashrightarrow \widetilde{M}^1(v)$  is a flop.*
- (3) *When  $n \geq 4$ ,  $\text{Hilb}^2(X) \dashrightarrow \widetilde{M}^1(v)$  is an anti-flip.*

PROOF. Note that Lemma 8.11 shows that the diagram

$$\begin{array}{ccc}
& M & \\
\sigma \swarrow & & \searrow \mu \\
\widetilde{M}^1(v) & & \text{Hilb}^2(X)
\end{array}$$

is a family version of the standard flip (cf. [24, page 258]). Since we already know the projectivity of  $\widetilde{M}^1(v)$ , it remains to check that  $\widetilde{M}^1(v)$  is a smooth variety.

The assertion then follows from the Fujiki-Nakano criterion (cf. [22] and [24, page 259]).  $\square$

**8.2.  $\xi_0$  is an extremal contraction.** In this subsection, we describe the fiber of  $\xi_0$  and show that it is the contraction of a  $K$ -negative extremal ray. Recall from Corollary 8.4 (3) that we have

$$\widetilde{M}^1(v) = (\text{Hilb}^2(X) \setminus \text{Hilb}^2(D/C)) \coprod \mathbb{P}(\mathcal{E}_-^\vee).$$

We start with the following lemma:

LEMMA 8.13. *Let  $G \in \mathbb{P}(\mathcal{E}_-^\vee) \subset \widetilde{M}^1(v)$ . Then  $G$  is not 0-stable.*

PROOF. We have an exact sequence

$$0 \rightarrow \mathcal{O}_{L_y}(-2) \rightarrow G \rightarrow I_{L_y} \rightarrow 0$$

for some  $y \in C$ . Applying  $\text{Hom}(-, \mathcal{O}_{L_y}(-1))$ , we get

$$\begin{aligned} \text{Hom}(I_{L_y}, \mathcal{O}_{L_y}(-1)) &\rightarrow \text{Hom}(G, \mathcal{O}_{L_y}(-1)) \rightarrow \text{Hom}(\mathcal{O}_{L_y}(-2), \mathcal{O}_{L_y}(-1)) \\ &\rightarrow \text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-1)) \end{aligned}$$

Since  $I_{L_y} \in \text{Per}(X/Y)$ ,  $\text{Hom}(I_{L_y}, \mathcal{O}_{L_y}(-1)) = 0$ . Moreover, as in Lemma 8.5, we can show that  $\text{ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-1)) = 1$ . Hence  $\text{hom}(G, \mathcal{O}_{L_y}(-1)) = 1 \neq 0$ . This shows that  $G \notin \text{Per}(X/Y)$ .  $\square$

By the above lemma and Lemma 8.1, the exceptional locus of  $\xi_0$  is

$$\text{Exc}(\xi_0) = (\{I_Z \in \text{Hilb}^2(X) : Z \cap D \neq \emptyset\} \setminus \text{Hilb}^2(D/C)) \coprod \mathbb{P}(\mathcal{E}_-^\vee).$$

To study the geometry of  $\xi_0$ , we introduce the following filtration  $B_1 \subset B_2 \subset B_3 = B \subset \text{Hilb}^2(Y)$  of  $\text{Hilb}^2(Y)$ :

- $B_1 := \{I_W \in \text{Hilb}^2(C) : \text{Supp}(I_W) = \text{pt}\},$
- $B_2 := \text{Hilb}^2(C),$
- $B_3 := \{I_W \in \text{Hilb}^2(Y) : W \cap C \neq \emptyset\}.$

LEMMA 8.14. (1) We have  $\xi_0(\text{Exc}(\xi_0)) = B$ .  
 (2) For  $I_W \in B \setminus B_2$ , the fibre of  $\xi_0$  is  $\xi_0^{-1}(I_W) = \mathbb{P}^1$ .  
 (3) For  $I_W \in B_2 \setminus B_1$ , the fibre of  $\xi_0$  is  $\xi_0^{-1}(I_W) = \mathbb{P}^1 \times \mathbb{P}^1$ .  
 (4) For  $I_W \in B_1$ , the fibre of  $\xi_0$  is the weighted projective plane  $\xi_0^{-1}(I_W) = \mathbb{P}(1, 1, 2)$ .

Here, we consider the image and the fibers of  $\xi_0$  with its reduced scheme structures.

PROOF. (1) First let  $I_Z \in \text{Exc}(\xi_0) \cap \text{Hilb}^2(X)$ . Then pushing forward the exact sequence

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0,$$

we get

$$0 \rightarrow f_* I_Z \rightarrow \mathcal{O}_Y \rightarrow f_* \mathcal{O}_Z \rightarrow 0.$$

Hence  $f_* I_Z$  is an ideal sheaf of length 2 closed subscheme  $W$  of  $Y$  with  $\text{Supp } W = f(Z)$ . Since  $Z \cap D \neq \emptyset$ ,  $W \cap C \neq \emptyset$ , i.e.  $f_* I_Z \in B$ .

Next take  $G \in \mathbb{P}(\mathcal{E}_-^\vee)$ . We can easily check that the following natural map  $\alpha: \text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2)) \rightarrow \text{Ext}^1(\mathcal{O}_y, \mathcal{O}_y)$  determines the class of  $f_* G \in \text{Hilb}^2(Y)$ : Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_{L_y}(-1) \rightarrow f^* I_y \rightarrow I_{L_y} \rightarrow 0.$$



Applying  $\text{Hom}(-, \mathcal{O}_{L_y}(-2))$ , we get

$$\begin{aligned} \alpha: \text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2)) &\rightarrow \text{Ext}^1(f^*I_y, \mathcal{O}_{L_y}(-2)) \\ &\cong \text{Hom}(I_y, \mathbf{R}f_*\mathcal{O}_{L_y}(-2)[1]) \\ &\cong \text{Hom}(I_y, \mathcal{O}_y) \\ &\cong \text{Ext}^1(\mathcal{O}_y, \mathcal{O}_y). \end{aligned}$$

Hence  $f_*G \in B_1$ . Furthermore, we have  $\text{Ext}^1(\mathcal{O}_{L_y}(-1), \mathcal{O}_{L_y}(-2)) = 0$  as in Lemma 8.5 and hence  $\alpha$  is injective. Since we know  $\text{ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2)) = \text{ext}^1(\mathcal{O}_y, \mathcal{O}_y) = n$ ,  $\alpha$  is bijective. We conclude that

$$\xi_0|_{\mathbb{P}\text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2))}^\vee: \mathbb{P}\text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2))^\vee \cong \mathbb{P}\text{Ext}^1(\mathcal{O}_y, \mathcal{O}_y)^\vee$$

is an isomorphism.

(2) Take an element  $I_W \in B \setminus B_2$ . First assume that  $W = \{a, b\}$ ,  $a \in C, b \notin C$ . Then from the argument of (1), we have

$$\xi_0^{-1}(I_W) = \{I_{p,q} \in \text{Hilb}^2(X) : f(p) = a, f(q) = b\} \cong L_a.$$

Next assume that  $\text{Supp } W = \{y\}$ , but scheme-theoretically  $W \not\subseteq C$ . Let  $x \in D$  with  $f(x) = y$ . Then the following commutative diagram of the tangent maps determines the morphism

$$\xi_0: \mathbb{P}\text{Ext}_X^1(\mathcal{O}_x, \mathcal{O}_x)^\vee \setminus \mathbb{P}\text{Ext}_{L_y}^1(\mathcal{O}_x, \mathcal{O}_x)^\vee \rightarrow \mathbb{P}\text{Ext}_Y^1(\mathcal{O}_y, \mathcal{O}_y)^\vee.$$

(8.6)

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & T_x L_y & & & 0 & \\ & & \downarrow & & & \downarrow & \\ 0 & \longrightarrow & T_x D & \longrightarrow & T_x X & \longrightarrow & N_{D/X}(x) \longrightarrow 0 \\ & & \downarrow \eta_x & & \downarrow \phi_x & & \downarrow \psi_x \\ 0] & \longrightarrow & T_y C & \longrightarrow & T_y Y & \longrightarrow & N_{C/Y}(y) \longrightarrow 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

where maps  $\eta_x, \phi_x, \psi_x$  are the tangent maps. Note that the point  $[\psi_x(N_{D/X}(x))]^\vee \in \mathbb{P}(N_{C/Y}(y)^\vee) = L_y$  is nothing but  $x \in L_y$ . Now take the 1-dimensional subspace  $\mathbb{C} \cdot (\alpha, \beta) \subset T_y Y = T_y C \oplus N_{C/Y}(y)$  which corresponds to  $I_W$ . Since we assume that  $W \not\subseteq C$ , we have  $0 \neq \beta \in N_{C/Y}(y)$ . Let  $x := [(\mathbb{C} \cdot \beta^\vee)] \in L_y$ . Then we have  $\phi_x^{-1}(\mathbb{C} \cdot (\alpha, \beta)) = T_x L_y \oplus N_{D/X}(x)$ . Moreover, recall from (1) that we have the unique element  $G_W \in \mathbb{P}\text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2))^\vee$  such that  $f_*G_W = I_W$ . Hence we conclude that

$$\begin{aligned} \xi_0^{-1}(I_W) &= \left( \mathbb{P}(T_x L_y \oplus N_{D/X}(x))^\vee \setminus \mathbb{P}(T_x L_y)^\vee \right) \coprod \{G_W\} \\ &= \mathbb{A}^1 \coprod \text{pt}. \end{aligned}$$

Since both  $\widetilde{M}^1(v)$  and  $\text{Hilb}^2(Y)$  are smooth,  $H^1(\mathcal{O}_{\xi_0^{-1}(I_W)}) = 0$ . Hence we must have  $\xi_0^{-1}(I_W) \cong \mathbb{P}^1$ .

(3) Take an element  $I_W \in B_2 \setminus B_1$ . Then by definition,  $W = \{a, b\}$  with  $a, b \in C$ ,  $a \neq b$ . Hence we have

$$\xi_0^{-1}(I_W) = \{I_{p,q} : f(p) = a, f(q) = b\} = L_a \times L_b \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

(4) Let  $I_W \in B_1$ ,  $\text{Supp } W = \{y\} \subset C$ . Take the subspace  $\mathbb{C} \cdot \alpha \subset T_y Y$  corresponding to  $I_W$ . Since  $W \subset C$  as scheme, we have  $\mathbb{C} \cdot \alpha \subset T_y C$ . For each  $x \in L_y$ , we have  $\phi_x^{-1}(\mathbb{C} \cdot \alpha) = \mathbb{C} \cdot \alpha \oplus T_x L_y$ . If we change  $x \in L_y$ , the vector space  $T_x L_y$  changes, but the subspace  $\mathbb{C} \cdot \alpha$  does not change. As before, we also have the unique element  $G_W \in \mathbb{P} \text{Ext}^1(I_{L_y}, \mathcal{O}_{L_y}(-2))^\vee$  such that  $f_* G_W = I_W$ . Hence we conclude that

$$\begin{aligned} \xi_0^{-1}(I_W) &= \left( \mathbb{P} \left( (\mathcal{O}_{L_y} \oplus \mathcal{T}_{L_y})^\vee \right) \setminus \mathbb{P} \left( \mathcal{T}_{L_y}^\vee \right) \right) \coprod \{G_W\} \\ &\cong (\mathbb{P}(\mathcal{O}_{L_y} \oplus \mathcal{O}_{L_y}(-2)) \setminus \mathbb{P}(\mathcal{O}_{L_y}(-2))) \coprod \{G_W\}. \end{aligned}$$

By this description, we can see that  $\xi_0^{-1}(I_W)$  is the proper transform of  $S := \mathbb{P}(\mathcal{O}_{L_y} \oplus \mathcal{O}_{L_y}(-2))$  via the birational map  $\widetilde{M}^1(v) \dashrightarrow \text{Hilb}^2(X)$ . To determine the scheme structure of  $\xi_0^{-1}(I_W)$ , recall the diagram

$$\begin{array}{ccc} & M & \\ \swarrow \sigma & & \searrow \mu \\ \widetilde{M}^1(v) & & \text{Hilb}^2(X). \end{array}$$

Since  $S \cap \text{Hilb}^2(D/C) = \mathbb{P}(\mathcal{O}_{L_y}(-2)) =: b$ , the proper transform of  $S$  by  $\mu$  is  $\mu_*^{-1}(S) \cong S$ . Then the morphism  $\sigma|_S : S \cong \mu_*^{-1}(S) \rightarrow \xi_0^{-1}(I_W)$  is nothing but the contraction of a  $(-2)$ -curve  $b \subset S$  and hence we get  $\xi_0^{-1}(I_W) \cong \mathbb{P}(1, 1, 2)$  as required.  $\square$

LEMMA 8.15. *The relative Picard number  $\rho(\widetilde{M}^1(v)/\text{Hilb}^2(Y))$  is one.*

PROOF. By Lemma 8.14, the only codimension 1 irreducible component in  $\text{Exc}(\xi_0)$  is the closure of  $\xi_0^{-1}(B \setminus B_2)$ . Hence it is enough to show that  $B$  is irreducible. To see that, we may assume  $Y = \mathbb{C}^n = \text{Spec } \mathbb{C}[x_1, \dots, x_n]$ ,  $C = (x_1 = x_2 = 0)$ . Then we can write down the local equations of  $B \subset \text{Hilb}^2(Y)$  by using the fact that  $\text{Hilb}^2(Y) = \text{Bl}_{\Delta_Y} \text{Sym}^2(Y)$ . The Jacobian criterion then shows that  $B$  is smooth. In particular, it is irreducible.  $\square$

COROLLARY 8.16. *The birational morphism  $\xi_0$  is the contraction of a  $K$ -negative extremal ray.*

PROOF. By Lemma 8.15, we know that  $\xi_0$  is the contraction of an extremal ray  $R$ . Hence it is enough to compute the intersection number  $K_{\widetilde{M}^1(v)} \cdot f$  for one element  $f \in R$ . Fix  $y \in C$ ,  $q \in X \setminus D$  and put  $f := \{I_{p,q} : p \in L_y\} \in R$ . Since  $f$  does not intersect with the exceptional divisor of the Hilbert-Chow morphism  $\text{Hilb}^2(X) \rightarrow \text{Sym}^2(X)$ , we have

$$K_{\widetilde{M}^1(v)} \cdot f = K_{\text{Hilb}^2(X)} \cdot f = K_{X \times X} \cdot (L_y \times \{q\}) = -1.$$

$\square$

REMARK 8.17. When  $n = 2$ , the locus  $\text{Hilb}^2(C) = \emptyset$ . Hence all the fibers of  $\xi_0$  are  $\mathbb{P}^1$ . In general, Nakajima and Yoshioka shows that every fibre of the zig-zag diagram (1.1) is the Grassmann variety (see Theorem 1.2).

On the other hand, for  $n \geq 3$ , we have shown that  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $\mathbb{P}(1, 1, 2)$  appear as the fibers of  $\xi_0$ . Of course, they are not Grassmann variety. Furthermore,

$\mathbb{P}(1,1,2)$  is even singular. This shows that for  $n \geq 3$ , more complicated fibers appear in the zig-zag diagram.

## Part 2

# Bridgeland stability conditions on threefolds with nef tangent bundles

## 9. Bridgeland stability conditions

**9.1. Definitions.** In this subsection, we recall the notion of Bridgeland stability conditions on a triangulated category. The reference for this subsection is Bridgeland's original paper [12]. First, we define the notion of stability functions:

DEFINITION 9.1. Let  $\mathcal{A}$  be an Abelian category.

- (1) A *stability function* on  $\mathcal{A}$  is a group homomorphism  $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$  satisfying the condition

$$Z(\mathcal{A} \setminus \{0\}) \subset \mathcal{H} \cup \mathbb{R}_{<0},$$

where  $\mathcal{H}$  is the upper half plane.

- (2) Let  $Z$  be a stability function on  $\mathcal{A}$ . An object  $E \in \mathcal{A}$  is called *Z-stable* (resp. *semistable*) if for every non zero proper subobject  $0 \neq F \subset E$ , we have an inequality

$$-\frac{\Re Z(F)}{\Im Z(F)} < (\text{resp. } \leq) -\frac{\Re Z(E)}{\Im Z(E)}.$$

Here, we define  $-\frac{\Re Z(E)}{\Im Z(E)} := +\infty$  if  $\Im Z(E) = 0$ .

- (3) A stability function  $Z$  on  $\mathcal{A}$  satisfies the *Harder-Narasimhan (HN) property* if the following holds: for every object  $E \in \mathcal{A}$ , there exists a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$$

such that  $F_i := E_i/E_{i-1}$  are  $Z$ -semistable and

$$-\frac{\Re Z(F_1)}{\Im Z(F_1)} > \cdots > -\frac{\Re Z(F_m)}{\Im Z(F_m)}.$$

We now define the notion of stability conditions on a triangulated category:

DEFINITION 9.2. Let  $\mathcal{D}$  be a triangulated category. A *stability condition* on  $\mathcal{D}$  is a pair consisting of the heart  $\mathcal{A}$  of a bounded t-structure on  $\mathcal{D}$  and a stability function  $Z$  on  $\mathcal{A}$  satisfying the HN property. A stability function  $Z$  is called a *central charge*.

**9.2. Bogomolov-Gieseker type inequality conjecture.** In this subsection, we recall the conjectural approach for the construction of stability conditions on threefolds. Let  $X$  be a smooth projective threefold. Fix a class  $B + i\omega \in \text{NS}(X)_{\mathbb{C}}$  with  $\omega$  ample. Conjecturally, there exists a stability condition on  $D^b(X)$  with its central charge given as follows (cf. [7, Conjecture 2.1.2]):

$$Z_{\omega, B} := - \int_X e^{-i\omega} \cdot \text{ch}^B.$$

It is easy to see that the pair  $(Z_{\omega, B}, \text{Coh}(X))$  does not define a stability condition when  $X$  is a threefold. Hence we need to introduce new hearts. Our hearts are obtained by the double-tilting construction [7] which we explain below, see the paper [23] for the general theory of torsion pairs and tilting. In the following, we assume that  $B \in \text{NS}(X)_{\mathbb{Q}}$  and  $\omega = mH$  for some ample divisor  $H$  and  $m \in \mathbb{R}_{>0}$  with  $m^2 \in \mathbb{Q}$ . As in the introduction, we use the following notation:

$$v^B = (v_0^B, v_1^B, v_2^B, v_3^B) := (\omega^3 \cdot \text{ch}_0^B, \omega^2 \cdot \text{ch}_1^B, \omega \cdot \text{ch}_2^B, \text{ch}_3^B).$$

**First tilting:** We define the slope function on  $\text{Coh}(X)$  as follows:

$$\mu_{\omega, B} := \frac{v_1^B}{v_0^B}: \text{Coh}(X) \rightarrow (-\infty, +\infty].$$

Then define the full subcategories  $\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B} \subset \text{Coh}(X)$  as follows:

$$\begin{aligned}\mathcal{T}_{\omega,B} &:= \langle T \in \text{Coh}(X) : T \text{ is } \mu_{\omega,B}\text{-semistable with } \mu_{\omega,B}(T) > 0 \rangle, \\ \mathcal{F}_{\omega,B} &:= \langle F \in \text{Coh}(X) : F \text{ is } \mu_{\omega,B}\text{-semistable with } \mu_{\omega,B}(F) \leq 0 \rangle.\end{aligned}$$

Here,  $\mu_{\omega,B}$ -stability for coherent sheaves is defined in a standard manner, and we denote by  $\langle S \rangle$  the extension closure of a set of objects  $S \subset \text{Coh}(X)$ . We can see that the pair  $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$  defines a torsion pair on  $\text{Coh}(X)$  (cf. Definition 5.1). Now we define a new heart, called tilted heart by

$$\text{Coh}^{\omega,B}(X) := \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle.$$

**Second tilting:** As in the first tilting, we introduce a new slope function and tilting of  $\text{Coh}^{\omega,B}(X)$ : A slope function  $\nu_{\omega,B}$  on  $\text{Coh}^{\omega,B}(X)$  is defined to be

$$\nu_{\omega,B} := \frac{v_2^B - \frac{1}{6}v_0^B}{v_1^B} : \text{Coh}^{\omega,B}(X) \rightarrow (-\infty, +\infty],$$

and the notion of  $\nu_{\omega,B}$ -stability for objects in  $\text{Coh}^{\omega,B}(X)$  is defined similarly as  $\mu_{\omega,B}$ -stability for coherent sheaves. We also refer to  $\nu_{\omega,B}$ -stability as *tilt stability*. Note that the existence of Harder-Narasimhan filtration with respect to  $\nu_{\omega,B}$ -stability is shown in the paper [7]. We define full subcategories of  $\text{Coh}^{\omega,B}(X)$  as

$$\begin{aligned}\mathcal{T}'_{\omega,B} &:= \langle T \in \text{Coh}^{\omega,B}(X) : T \text{ is } \nu_{\omega,B}\text{-semistable with } \nu_{\omega,B}(T) > 0 \rangle, \\ \mathcal{F}'_{\omega,B} &:= \langle F \in \text{Coh}^{\omega,B}(X) : F \text{ is } \nu_{\omega,B}\text{-semistable with } \nu_{\omega,B}(F) \leq 0 \rangle.\end{aligned}$$

Now we reach the definition of the double-tilted heart:

$$\mathcal{A}_{\omega,B} := \langle \mathcal{F}'_{\omega,B}[1], \mathcal{T}'_{\omega,B} \rangle.$$

In the paper [7], Bayer, Macrì, and Toda conjectured the following:

CONJECTURE 9.3 ([7]). The pair  $(Z_{\omega,B}, \mathcal{A}_{\omega,B})$  is a stability condition on  $D^b(X)$ .

Let us denote

$$\overline{\Delta}_{\omega,B}(E) := v_1^B(E)^2 - 2v_0^B(E)v_2^B(E)$$

and

$$\overline{\nabla}_{\omega,B}(E) := 2(v_2^B(E))^2 - 3v_1^B(E)v_3^B(E).$$

The following is the so-called Bogomolov-Gieseker (BG) type inequality conjecture ([7, 6, 45]).

CONJECTURE 9.4 ([45, Conjecture 3.8]). For any  $\nu_{\omega,B}$ -stable object  $E$ , we have the inequality

$$\overline{\Delta}_{\omega,B}(E) + 6\overline{\nabla}_{\omega,B}(E) \geq 0.$$

The BG type inequality conjecture implies the existence of a stability condition:

PROPOSITION 9.5 ([45]). Assume that Conjecture 9.4 holds. Then Conjecture 9.3 also holds.

**9.3. Reduction theorems.** In this subsection, we recall two reduction theorems of the BG type inequality conjecture due to [6, 29, 45].

First we recall the following notion.

DEFINITION 9.6. Fix real numbers  $\alpha_0 > 0$  and  $\beta_0$ . Let  $E \in \text{Coh}^{\alpha_0\omega, B+\beta_0\omega}(X)$  be a  $\nu_{\alpha_0\omega, B+\beta_0\omega}$ -semistable object.

(1) We define a real number  $\bar{\beta}(E)$  as

$$\bar{\beta}(E) := \frac{2v_2^B(E)}{v_1^B(E) + \sqrt{\Delta_{\omega,B}(E)}}.$$

(2)  $E$  is  $\bar{\beta}$ -semistable (resp. stable) if there exists an open neighborhood  $V$  of  $(0, \bar{\beta}(E))$  in the  $(\alpha, \beta)$ -plane such that for every  $(\alpha, \beta) \in V$  with  $\alpha > 0$ ,  $E$  is  $\nu_{\alpha\omega, B+\beta\omega}$ -semistable (resp. stable).

The first reduction is of the following form.

CONJECTURE 9.7 ([45, Conjecture 3.17]). Let  $E$  be a  $\bar{\beta}$ -stable object. Then we have

$$\text{ch}_3^{B+\bar{\beta}(E)\omega}(E) \leq 0.$$

THEOREM 9.8 ([45, Theorem 3.20]). Conjectures 9.4 and 9.7 are equivalent.

Using the same technique, the following result was proved in [29].

THEOREM 9.9 ([29, Theorem 3.2]). Let  $H$  be an ample divisor on  $X$ . Assume that there exists a real number  $\alpha_0 > 0$  such that for every real number  $0 < \alpha < \alpha_0$ , Conjecture 9.4 is true for  $(X, \alpha H, B = 0)$ . Then it also holds for  $(X, \alpha H, \beta H)$  with any choice of  $\alpha \geq \frac{1}{2\sqrt{3}}$  and  $\beta \in \mathbb{R}$ .

**9.4. Counter-examples.** Counter-examples to Conjecture 9.3 are constructed in the papers [26, 36, 47]. In particular, we have the following result:

LEMMA 9.10 ([36, Lemma 3.1]). Let  $H$  be an ample divisor. Assume that there exists an effective divisor  $D$  such that

$$(9.1) \quad D^3 > \frac{(H^2 \cdot D)^3}{4(H^3)^2} + \frac{3}{4} \frac{(H \cdot D^2)^2}{H^2 \cdot D}.$$

Then there exists a pair  $(\alpha, \beta)$  of real numbers such that the pair  $(Z_{\alpha H, \beta H}, \mathcal{A}_{\alpha H, \beta H})$  does not define a stability condition.

REMARK 9.11. Let  $D$  be a nef divisor. We claim that  $D$  does not satisfy the inequality (9.1). By the Hodge index theorem for nef divisors, we have the following inequalities:

$$(9.2) \quad (H^2 \cdot D)^3 \geq (H^3)^2 \cdot D^3$$

$$(9.3) \quad (H \cdot D^2)^3 \geq H^3 \cdot (D^3)^2.$$

On the other hand, by replacing  $H$  with its sufficiently large multiple and taking a smooth member, the Hodge index theorem on  $H$  leads the inequality

$$(9.4) \quad (H^2 \cdot D)^2 = (H|_H \cdot D|_H)^2 \geq (H|_H)^2 \cdot (D|_H)^2 = H^3 \cdot H \cdot D^2.$$

The inequality (9.2) is equivalent to the inequality

$$(9.5) \quad D^3 \leq \frac{(H^2 \cdot D)^3}{(H^3)^2}.$$

Furthermore, by the inequalities (9.3), (9.4), and (9.5), we have

$$(9.6) \quad \begin{aligned} \frac{(H \cdot D^2)^2}{H^2 \cdot D} &\geq \frac{H^3 \cdot (D^3)^2}{H^2 \cdot D \cdot H \cdot D^2} \quad (\text{by (9.3)}) \\ &\geq \frac{(H^2 \cdot D)^2}{H \cdot D^2 \cdot H^3} D^3 \quad (\text{by (9.5)}) \\ &\geq D^3 \quad (\text{by (9.4)}). \end{aligned}$$

By combining the inequalities (9.5) and (9.6), we conclude that  $D$  satisfies the opposite inequality to that in (9.1). Hence we can think the inequality (9.1) as

a kind of negativity conditions on an effective divisor. We can still expect that Conjecture 9.3 and Conjecture 9.4 are true if all effective divisors satisfy some positivity conditions.

**9.5. Threefolds with nef tangent bundles.** In this subsection, we recall results on threefolds with nef tangent bundles, which we will need in this paper.

**PROPOSITION 9.12** ([15, Proposition 2.12]). *Let  $X$  be a smooth projective variety with nef tangent bundle. Then every effective divisor on  $X$  is nef.*

The above proposition, together with Remark 9.11, shows that there does not exist an effective divisor on a threefold with nef tangent bundle satisfying the inequality (9.1) in Lemma 9.10. Furthermore, the above proposition also ensures the tilt-stability of line bundles:

**LEMMA 9.13** ([6, Corollary 3.11]). *Let  $X$  be a smooth projective threefold,  $\omega$  an ample  $\mathbb{R}$ -divisor on  $X$ . Assume that for every effective divisor  $D$  on  $X$ , we have  $\omega \cdot D^2 \geq 0$ . Then for every line bundle  $L$  on  $X$  and  $B \in \text{NS}(X)_{\mathbb{R}}$ ,  $L$  or  $L[1]$  is  $\nu_{\omega, B}$ -stable.*

Next we recall the classification theorem of threefolds with nef tangent bundles due to the paper [15].

**THEOREM 9.14** ([15, Theorem 10.1]). *Let  $X$  be a smooth projective threefold with nef tangent bundle. Then there exists an étale covering  $f: \tilde{X} \rightarrow X$  such that  $\tilde{X}$  is one of the following:*

- (1)  $\mathbb{P}^3$ .
- (2) a three dimensional smooth quadric.
- (3)  $\mathbb{P}^1 \times \mathbb{P}^2$ .
- (4)  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .
- (5)  $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ .
- (6)  $\mathbb{P}_A(\mathcal{E})$ , where  $A$  is an Abelian surface and  $\mathcal{E}$  is a rank two vector bundle obtained as an extension of two line bundles in  $\text{Pic}^0(A)$ .
- (7)  $\mathbb{P}_C(\mathcal{E})$ , where  $C$  is an elliptic curve and  $\mathcal{E}$  is a rank three vector bundle obtained as extensions of three line bundles of degree zero.
- (8)  $\mathbb{P}_C(\mathcal{E}_1) \times_C \mathbb{P}_C(\mathcal{E}_2)$ , where  $C$  is an elliptic curve and  $\mathcal{E}_i$  are rank two vector bundles obtained as extensions of degree zero line bundles.
- (9) an Abelian threefold.

For our purpose, we need the following observation:

**LEMMA 9.15.** *In Theorem 9.14, we can choose an étale covering  $f$  to be a Galois covering.*

**PROOF.** Let  $X$  be a smooth projective threefold with nef tangent bundle. In the proof of [15, Theorem 10.1], they actually show the existence of the following diagram of smooth projective varieties:

$$\begin{array}{ccccc} \tilde{X} := \tilde{Y} \times_Y X & \xrightarrow{f} & X & & \\ \downarrow & & \downarrow & & \\ \tilde{Y} & \xrightarrow{\psi} & Y' & \xrightarrow{\phi} & Y, \end{array}$$

where  $Y'$  is an Abelian variety (possibly of dimension zero),  $\psi$  and  $\phi$  are étale coverings. Note that the morphism  $\tilde{X} \rightarrow \tilde{Y}$  is same as (1) – (9) in Theorem 9.14, i.e.,  $\tilde{Y}$  is  $\text{Spec } \mathbb{C}$ ,  $A$ ,  $C$ , or an Abelian threefold in the notation of Theorem 9.14.



Put  $g := \phi \circ \psi$ . Let us take the Galois closure of  $g$ , i.e. an étale covering  $h: \widehat{Y} \rightarrow \widehat{Y}$  such that the morphism  $h \circ g: \widehat{Y} \rightarrow Y$  is an étale Galois covering. Note that since  $\widehat{Y}$  is an Abelian variety, so is  $\widehat{Y}$ . Hence the base change  $\widehat{X} := \widehat{Y} \times_Y X$  is again one of the threefolds in Theorem 9.14 (1) – (9), and is an étale Galois covering of  $X$ . This completes the proof.  $\square$

REMARK 9.16. Among threefolds in Theorem 9.14, Conjecture 9.7 is known to be true in the following cases:

- $\mathbb{P}^3$  by [7, 35].
- a three dimensional smooth quadric by [46].
- $\mathbb{P}^1 \times \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  with any choice of a class  $B + i\omega$  by [8] (In the paper [8], they only treat the case when  $B$  and  $\omega$  are proportional. Even when they are not proportional, the same proof works according to the formulation given in Conjecture 9.7).
- $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$  with  $B$  and  $\omega$  are proportional to the anti-canonical class by [8].
- an Abelian threefold with any choice of a class  $B + i\omega$  by [6, 32, 33].

The following is our main result in Part 2, which completely solve Conjecture 9.7 for threefolds as in Theorem 9.14 (6) – (8):

THEOREM 9.17. *Let  $X$  be a threefold as in Theorem 9.14 (6), (7), or (8). Then for every class  $B + i\omega \in \text{NS}(X)_{\mathbb{C}}$  with  $\omega$  ample, Conjecture 9.7 holds.*

As a corollary, we obtain:

COROLLARY 9.18. *Let  $X$  be a smooth projective threefold with nef tangent bundle. Then there exist Bridgeland stability conditions on  $D^b(X)$ .*

PROOF. By [6, Proposition 6.1], we may replace  $X$  by an étale Galois covering, thus we can assume it is one of the threefolds in Theorem 9.14 (see Lemma 9.15). Then Theorem 9.17, together with the previous works [6, 8, 32, 33, 35, 46], proves the required statement.  $\square$

We will also have the following result for  $X = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ :

THEOREM 9.19. *Let  $X = \mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ ,  $H$  be an ample divisor on  $X$ . Let  $\alpha > \frac{1}{2\sqrt{3}}$  and  $\beta \in \mathbb{R}$  be real numbers. Then Conjecture 9.4 holds for  $(X, \alpha H, \beta H)$ .*

## 10. Proof of Theorem 9.17

In this section, we prove Theorem 9.17. We use the following terminology.

DEFINITION 10.1. Let  $X$  be as in Theorem 9.14 (6) – (8). Then  $X$  is *split* if the vector bundles defining  $X$  are direct sums of line bundles.

**10.1. Reduction to split cases.** In this subsection, we reduce Theorem 9.17 to the split cases. The key method is the following result announced by Bayer et al [3].

PROPOSITION 10.2 ([3]). *Let  $f: \mathcal{X} \rightarrow D$  be a smooth projective family of threefolds over a smooth curve  $D$  and fix a point  $0 \in D$ . Suppose that  $f$  is a trivial family over  $U := D \setminus \{0\}$ , i.e.  $f^{-1}(U) \cong X \times U$  for some threefold  $X$ . Take an  $f$ -ample  $\mathbb{Q}$ -divisor  $\mathcal{H}$  and an arbitrary  $\mathbb{Q}$ -divisor  $\mathcal{B}$  on  $\mathcal{X}$ . Let  $\mathcal{H}_0, \mathcal{B}_0$  (resp.  $H, B$ ) be restriction of  $\mathcal{H}, \mathcal{B}$  to the special fiber  $f^{-1}(0)$  (resp. the general fiber  $X$ ). If Conjecture 9.7 is true for  $(f^{-1}(0), \mathcal{H}_0, \mathcal{B}_0)$ , then it also holds for  $(X, H, B)$ .*

The above result follows from the existence of the relative moduli spaces of tilt-stable objects over the base  $D$ , satisfying the valuative criterion for universal closedness.

PROPOSITION 10.3. *Assume that Theorem 9.17 holds for every split  $X$ . Then it also holds for every non-split  $X$ .*

PROOF. First we consider the case (6) in Theorem 9.14. Let  $A$  be an Abelian surface,  $\mathcal{E}$  be a rank two vector bundle which fits into the non-split short exact sequence

$$0 \rightarrow \mathcal{O}_A \rightarrow \mathcal{E} \rightarrow \mathcal{O}_A \rightarrow 0.$$

Let  $X := \mathbb{P}_A(\mathcal{E})$ . By applying Proposition 10.2, we will show that to prove Theorem 9.17 for  $X$ , it is enough to show it for  $X_0 := \mathbb{P}^1 \times A$ . Let us take an affine line  $\mathbb{A}^1 \subset \text{Ext}^1(\mathcal{O}_A, \mathcal{O}_A)$  passing through the origin and a point  $[\mathcal{E}] \in \text{Ext}^1(\mathcal{O}_A, \mathcal{O}_A)$ . Over  $\mathbb{A}^1$ , we have a smooth family  $f: \mathcal{X} \rightarrow \mathbb{A}^1$  with the following properties (cf. [25, Lemma 4.1.2]):

- (1) Let  $U := \mathbb{A}^1 \setminus \{0\}$ . Then we have  $\mathcal{X}_U := f^{-1}(U) \cong X \times U$ .
- (2) We have  $\mathcal{X}_0 := f^{-1}(0) \cong X_0$ .

Indeed, the family is constructed as a  $\mathbb{P}^1$ -bundle  $\sigma: \mathcal{X} = \mathbb{P}_{A \times \mathbb{A}^1}(\mathcal{U}) \rightarrow A \times \mathbb{A}^1$ , where  $\mathcal{U}$  fits into the exact sequence

$$0 \rightarrow \mathcal{U} \rightarrow q^* \mathcal{E} \rightarrow i_* \mathcal{E} \rightarrow 0.$$

Here,  $q: A \times \mathbb{A}^1 \rightarrow A$  is a projection and  $i: A \times \{0\} \rightarrow A \times \mathbb{A}^1$  is an inclusion. Let  $p := q \circ \sigma: \mathcal{X} \rightarrow A$  be a projection. We also have to prove that, for a given ample divisor  $H$  on  $X$ , there exists an  $f$ -ample divisor  $\mathcal{H}$  on  $\mathcal{X}$  such that its restriction to  $X$  coincides with  $H$ . Write  $H = \mathcal{O}_\pi(a) \otimes \pi^* N$ , where  $\pi: X \rightarrow A$  is a structure morphism. We put  $\mathcal{H} := \mathcal{O}_\sigma(a) \otimes p^* N$ . Then by the ampleness criterion given in Lemma 10.10, we can see that  $\mathcal{H}$  is  $f$ -ample. Hence the result holds by Proposition 10.2.

Next let  $C$  be an elliptic curve and  $L_i$  be degree zero line bundles on  $C$  ( $i = 1, 2, 3$ ). Consider the case (7) in Theorem 9.14, i.e.  $X = \mathbb{P}_C(\mathcal{E})$ , where  $\mathcal{E}$  is a rank three vector bundle obtained as follows:

$$\begin{aligned} 0 \rightarrow L_1 \rightarrow \mathcal{E}' \rightarrow L_2 \rightarrow 0, \\ 0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow L_3 \rightarrow 0. \end{aligned}$$

As above, by considering a family over the affine line  $\mathbb{A}^1 \subset \text{Ext}^1(L_3, \mathcal{E}')$  passing through the origin and a class  $[\mathcal{E}]$ , we may assume that  $\mathcal{E} = \mathcal{E}' \oplus L_3$ . Then by applying the same argument for  $[\mathcal{E}'] \in \text{Ext}^1(L_2, L_1)$ , we can reduce to the split case.

Finally, consider the case (8) in Theorem 9.14. For  $i = 1, 2$ , let  $\pi_i: Y_i := \mathbb{P}_C(\mathcal{E}_i) \rightarrow C$ , where  $\mathcal{E}_i$  are rank two vector bundles fitting into the short exact sequences

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E}_i \rightarrow L_i \rightarrow 0.$$

Let  $X := Y_1 \times_C Y_2$ . Noting that  $X = \mathbb{P}_{Y_1}(\pi_1^* \mathcal{E}_2)$ , we can first reduce to the case when  $\mathcal{E}_2 = \mathcal{O}_C \oplus L_2$ . Then by regarding as  $X = \mathbb{P}_{Y_2}(\pi_2^* \mathcal{E}_1)$ , we can reduce to the case when  $X$  is split.  $\square$

**10.2. Conclusion.** In this subsection, we explain how to prove Theorem 9.17 in the split cases. We use the following notations:

- $A$  is an Abelian surface,  $C$  is an elliptic curve.
- $L \in \text{Pic}^0(A)$  and  $L_1, L_2 \in \text{Pic}^0(C)$ .
- For  $m \in \mathbb{Z}_{>0}$ ,  $L^{\frac{1}{m}}$  is a line bundle such that  $(L^{\frac{1}{m}})^m \cong L$ .  $L_i^{\frac{1}{m}} \in \text{Pic}^0(C)$  are similarly defined.
- For  $i = 1, 2$ ,  $Y_i := \mathbb{P}_C(\mathcal{O}_C \oplus L_i)$ .
- $X$  is  $\mathbb{P}_A(\mathcal{O}_A \oplus L)$ ,  $\mathbb{P}_C(\mathcal{O}_C \oplus L_1 \oplus L_2)$ , or  $Y_1 \times_C Y_2$ .

- For  $m \in \mathbb{Z}_{>0}$ ,  $Y_i^{(m)} := \mathbb{P}_C(\mathcal{O}_C \oplus L_i^m)$ , and  $Y_i^{(\frac{1}{m})} := \mathbb{P}_C(\mathcal{O}_C \oplus L_i^{\frac{1}{m}})$ .  $X^{(m)}$ ,  $X^{(\frac{1}{m})}$  are defined similarly.

We start with the following easy lemma:

LEMMA 10.4. *Let  $X$  be as in Theorem 9.14 (6) – (8) which is split, let  $m \in \mathbb{Z}_{>0}$  be an positive integer. Then by identifying the tautological classes, we have a ring isomorphism*

$$\Phi: H^{2*}(X^{(\frac{1}{m})}, \mathbb{Q}) \rightarrow H^{2*}(X, \mathbb{Q})$$

between the even cohomology rings.

PROOF. We only treat the case when  $X = \mathbb{P}_A(\mathcal{O}_A \oplus L)$ . Let  $h \in H^2(X, \mathbb{Q})$  (resp.  $h^{(\frac{1}{m})} \in H^2(X^{(\frac{1}{m})}, \mathbb{Q})$ ) be a divisor such that we have  $\mathcal{O}_X(h) = \mathcal{O}_\pi(1)$  (resp.  $\mathcal{O}_{X^{(\frac{1}{m})}}(h^{(\frac{1}{m})}) = \mathcal{O}_{\pi^{(\frac{1}{m})}}(1)$ ). Since  $L \in \text{Pic}^0(A)$ , we have ring isomorphisms

$$\Phi: H^{2*}(X^{(\frac{1}{m})}, \mathbb{Q}) \cong H^{2*}(A, \mathbb{Q})[t]/(t^2) \cong H^{2*}(X, \mathbb{Q}).$$

Here, the isomorphism  $H^{2*}(A, \mathbb{Q})[t]/(t^2) \cong H^{2*}(X, \mathbb{Q})$  sends  $t$  to  $[h]$  and the same is true for  $X^{(\frac{1}{m})}$ . Hence  $\Phi([h^{(\frac{1}{m})}]) = [h]$ .  $\square$

Next we construct finite morphisms which play important roles for our purpose.

PROPOSITION 10.5 (cf. [42, Proposition 5]). *Let  $X$  be a threefold as in Theorem 9.17 which is split. Then, for every positive integer  $m \in \mathbb{Z}_{>0}$ , we have the following commutative diagram*

$$(10.1) \quad \begin{array}{ccccc} X^{(\frac{1}{m})} & & & & \\ & \searrow^{g_m} & & \searrow^{F_m} & \\ & & X^{(m)} & \xrightarrow{h_m} & X \\ & \searrow^{\pi^{(\frac{1}{m})}} & \downarrow^{\pi^{(m)}} & & \downarrow^{\pi} \\ & & Z & \xrightarrow{\underline{m}} & Z, \end{array}$$

where  $Z$  is an Abelian surface  $A$  or an elliptic curve  $C$ .

Furthermore, the pull-back via the morphism  $F_m: X^{(\frac{1}{m})} \rightarrow X$  acts on the even cohomology as follows.

$$(10.2) \quad \Phi \circ F_m^*: H^{2*}(X, \mathbb{Q}) \ni (x, y, z, w) \mapsto (x, m^2 y, m^4 z, m^6 w) \in H^{2*}(X, \mathbb{Q}).$$

PROOF. First consider the case (6) in Theorem 9.14:  $X := \mathbb{P}_A(\mathcal{O}_A \oplus L)$ . Consider the multiplication map  $\underline{m}: A \rightarrow A$ . By [38, p. 71 (iii)], we have  $\underline{m}^* L \cong L^m$ . Hence by base change, we have the morphism  $h_m: X^{(m)} \rightarrow X$ . On the other hand, the natural inclusion

$$(10.3) \quad \mathcal{O}_A \oplus L^m \subset \text{Sym}^{m^2}(\mathcal{O}_A \oplus L^{\frac{1}{m}}) = \mathcal{O}_A \oplus L^{\frac{1}{m}} \oplus \dots \oplus (L^{\frac{1}{m}})^{m^2}$$

induces a morphism  $g_m: X^{(\frac{1}{m})} \rightarrow X^{(m)}$ . Now we get a commutative diagram as in (10.1). Locally over  $A$ , the morphism  $g_m$  is nothing but the toric Frobenius morphism  $\underline{m}^2: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . Hence the pull back  $F_m^*$  acts on the cohomology as stated.

Next consider the case (7) in Theorem 9.14, i.e.,  $X = \mathbb{P}_C(\mathcal{O}_C \oplus L_1 \oplus L_2)$ . Replacing (10.3) by the inclusion

$$\mathcal{O}_C \oplus L_1^m \oplus L_2^m \subset \text{Sym}^{m^2}(\mathcal{O}_C \oplus L_1^{\frac{1}{m}} \oplus L_2^{\frac{1}{m}}),$$

we get the diagram as in (10.1).

Finally, consider the case (8) in Theorem 9.14:  $X = Y_1 \times_C Y_2$ . As above, we can construct the morphisms  $Y_i^{(\frac{1}{m})} \rightarrow Y_i^{(m)}$ , which induce the morphism  $g_m: X^{(\frac{1}{m})} \rightarrow X^{(m)}$ . Hence we get the diagram as in (10.1).  $\square$

REMARK 10.6. By using the inclusion

$$\mathcal{O}_A \oplus L^m \subset \text{Sym}^m(\mathcal{O}_A \oplus L)$$

instead of (10.3), we get an endomorphism  $F'_m: X \rightarrow X$  which is the multiplication map  $\underline{m}: A \rightarrow A$  on the base, and the toric Frobenius morphism  $\underline{m}: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  on the fiber. It seems natural to use the endomorphism  $F'_m$  rather than  $F_m$ . The issue is that the endomorphism  $F'_m$  is not polarized. On the other hand, the morphism  $F_m$  behaves like a polarized endomorphism, although it is not an endomorphism (see the formula (10.2)).

According to the description (10.2) of the pull back  $F_m^*$ , we can prove the following two results:

PROPOSITION 10.7 ([6]). *Let  $X$  be as in Theorem 9.14 (6), (7), or (8) which is split, let  $F_m$  be the morphism constructed in Proposition 10.5. Let  $E \in D^b(X)$  be a two term complex concentrated in degree  $-1$  and  $0$ .*

(1) *If there exists an ample divisor  $H$  on  $X^{(\frac{1}{m})}$  such that*

$$\text{hom}(\mathcal{O}(H), F_m^* E) = 0,$$

*then we have*

$$\text{hom}(\mathcal{O}, F_m^* E) = O(m^4).$$

(2) *If there exists an ample divisor  $H$  on  $X^{(\frac{1}{m})}$  such that*

$$\text{ext}^2(\mathcal{O}(-H), F_m^* E) = 0,$$

*then*

$$\text{ext}^2(\mathcal{O}, F_m^* E) = O(m^4).$$

PROOF. Since we know that the pull back  $F_m^*$  acts on the cohomology as in (10.2), the arguments of Section 7 in [6] prove the result.  $\square$

LEMMA 10.8. *Let  $X$  be as in Theorem 9.14 (6) – (8) which is split,  $m, q \in \mathbb{Z}_{>0}$  be positive integers. Take a divisor  $D$  on  $X$  and let  $D^{(\frac{1}{mq})}$  be a divisor on  $X^{(\frac{1}{mq})}$  such that  $D^{(\frac{1}{mq})} = \Phi^{-1}(D)$  in the cohomology ring. Then for every object  $E \in D^b(X)$ , we have the equality*

$$\text{ch}_3 \left( F_{mq}^* E \otimes \mathcal{O}(-m^2 q D^{(\frac{1}{mq})}) \right) = m^6 q^6 \text{ch}_3^{\frac{1}{q} D}(E) \in \mathbb{Q}$$

*as rational numbers.*

PROOF. Note that  $\Phi(D^{(\frac{1}{mq})}) = D$  by definition. Hence by the formula (10.2), we have

$$\begin{aligned} & \text{ch}_3 \left( F_{mq}^* E \otimes \mathcal{O}(-m^2 q D^{(\frac{1}{mq})}) \right) \\ &= \Phi \left( \text{ch}_3 \left( F_{mq}^* E \otimes \mathcal{O}(-m^2 q D^{(\frac{1}{mq})}) \right) \right) \\ &= -\frac{1}{6} m^6 q^3 D^3 \text{ch}_0(E) + \frac{1}{2} (m^4 q^2 D^2) (m^2 q^2 \text{ch}_1(E)) - (m^2 q D) (m^4 q^4 \text{ch}_2(E)) \\ & \quad + m^6 q^6 \text{ch}_3(E) \\ &= m^6 q^6 \text{ch}_3^{\frac{1}{q} D}(E) \end{aligned}$$

as required.  $\square$

Next we prove a variant of the toric Frobenius splitting of line bundles.

PROPOSITION 10.9 (cf. [49]). *Let  $X$  and  $g_m$  be as in Proposition 10.5. Let  $M$  be a line bundle on  $X^{(\frac{1}{m})}$ . Then the vector bundle  $g_{m*}M$  decomposes into a direct sum of line bundles. Furthermore, the direct summands are explicitly described as follows:*

- (1) *When  $X = \mathbb{P}_A(\mathcal{O}_A \oplus L)$  is as in Theorem 9.14 (6) and  $M = \mathcal{O}_{\pi^{(\frac{1}{m})}}(a) \otimes \pi^{(\frac{1}{m})*}N$ , then each direct summand of  $g_{m*}M$  is of the following form:*

$$\mathcal{O}_{\pi^{(m)}}(i) \otimes \pi^{(m)*}(L^j \otimes N),$$

where  $i = \lfloor \frac{a}{m^2} \rfloor - 1, \lfloor \frac{a}{m^2} \rfloor, 0 \leq j \leq m^2$ .

- (2) *When  $X = \mathbb{P}_C(\mathcal{O}_C \oplus L_1 \oplus L_2)$  is as in Theorem 9.14 (7) and  $M = \mathcal{O}_{\pi^{(\frac{1}{m})}}(a) \otimes \pi^{(\frac{1}{m})*}N$ , then each direct summand of  $g_{m*}M$  is of the following form:*

$$\mathcal{O}_{\pi^{(m)}}(i) \otimes \pi^{(m)*}(L_1^{j_1} \otimes L_2^{j_2} \otimes N),$$

where  $i = \lfloor \frac{a}{m^2} \rfloor - 2, \lfloor \frac{a}{m^2} \rfloor - 1, \lfloor \frac{a}{m^2} \rfloor$ , and  $0 \leq j_1, j_2 \leq m^2$ .

- (3) *When  $X = \mathbb{P}_C(\mathcal{O}_C \oplus L_1) \times_C \mathbb{P}_C(\mathcal{O}_C \oplus L_2)$  is as in Theorem 9.14 (8) and  $M = \mathcal{O}_{\pi^{(\frac{1}{m})}}(a, b) \otimes \pi^{(\frac{1}{m})*}N$ , then each direct summand of  $g_{m*}M$  is of the following form:*

$$\mathcal{O}_{\pi^{(m)}}(i, j) \otimes \pi^{(m)*}(L_1^{k_1} \otimes L_2^{k_2} \otimes N),$$

where  $i = \lfloor \frac{a}{m^2} \rfloor - 1, \lfloor \frac{a}{m^2} \rfloor, j = \lfloor \frac{b}{m^2} \rfloor - 1, \lfloor \frac{b}{m^2} \rfloor$ , and  $0 \leq k_1, k_2 \leq m^2$ .

PROOF. (1) Let  $X = \mathbb{P}_A(\mathcal{O}_A \oplus L)$  be as in Theorem 9.14 (6). Since  $g_{m*}M \cong g_{m*}\mathcal{O}_{\pi^{(\frac{1}{m})}}(a) \otimes \pi^{(m)*}N$ , we may assume that  $M = \mathcal{O}_{\pi^{(\frac{1}{m})}}(a)$ . Furthermore, since we have  $g_m^*\mathcal{O}_{\pi^{(m)}}(1) \cong \mathcal{O}_{\pi^{(\frac{1}{m})}}(m^2)$ , we may assume  $0 \leq a < m^2$ . Now let  $\mathcal{F} := g_{m*}M$ , and consider the adjoint map  $\alpha: \pi^{(m)*}\pi_*^{(m)}\mathcal{F} \rightarrow \mathcal{F}$ . On the fiber of  $\pi^{(m)}$ , the map  $\alpha$  is nothing but the natural inclusion

$$\mathcal{O}_{\mathbb{P}^1}^{\oplus a+1} \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus a+1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus (m^2-a-1)}.$$

Indeed by [49], on the fiber of  $\pi^{(m)}$ , we have an isomorphism

$$\mathcal{F}|_{\mathbb{P}^1} \cong \underline{m}^2 \mathcal{O}_{\mathbb{P}^1}(a) \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus a+1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus (m^2-a-1)},$$

where  $\underline{m}^2$  denotes the toric Frobenius morphism on  $\mathbb{P}^1$ . Moreover, the adjoint map  $\alpha$  restricted to the fiber is nothing but the evaluation map

$$\alpha|_{\mathbb{P}^1}: H^0(\mathbb{P}^1, \mathcal{F}|_{\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{P}^1} \hookrightarrow \mathcal{F}|_{\mathbb{P}^1}.$$

Hence globally, the map  $\alpha$  is injective and we get the short exact sequence

$$(10.4) \quad 0 \rightarrow \pi^{(m)*}\pi_*^{(m)}\mathcal{F} \rightarrow \mathcal{F} \rightarrow \pi^{(m)*}\mathcal{G} \otimes \mathcal{O}_{\pi^{(m)}}(-1) \rightarrow 0$$

for some coherent sheaf  $\mathcal{G} \in \text{Coh}(A)$ . First of all, we have

$$\pi_*^{(m)}\mathcal{F} = \pi_*^{(\frac{1}{m})}\mathcal{O}_{\pi^{(\frac{1}{m})}}(a) = \text{Sym}^a(\mathcal{O}_A \oplus L^{\frac{1}{m}}) = \mathcal{O}_A \oplus L^{\frac{1}{m}} \oplus \cdots \oplus L^{\frac{a}{m}}.$$

Next we will show that  $\mathcal{G}$  is a direct sum of line bundles. Applying the functor  $\pi_*^{(m)}(- \otimes \mathcal{O}_{\pi^{(m)}}(1))$  to the exact sequence (10.4), we have

$$0 \rightarrow \text{Sym}^a(\mathcal{O}_A \oplus L^{\frac{1}{m}}) \otimes (\mathcal{O} \oplus L^m) \xrightarrow{\beta} \text{Sym}^{a+m^2}(\mathcal{O}_A \oplus L^{\frac{1}{m}}) \rightarrow \mathcal{G} \rightarrow 0.$$

Note that the vector bundles  $\text{Sym}^a(\mathcal{O}_A \oplus L^{\frac{1}{m}}) \otimes (\mathcal{O} \oplus L^m)$  and  $\text{Sym}^{a+m^2}(\mathcal{O}_A \oplus L^{\frac{1}{m}})$  are the direct sums of line bundles. By the definition of the morphism  $g_m$ , the map  $\beta$  is the natural inclusion as the direct summand. Hence  $\mathcal{G}$  is isomorphic to the vector bundle

$$L^{\frac{a+1}{m}} \oplus L^{\frac{a+2}{m}} \oplus \cdots \oplus L^{\frac{m^2-2}{m}} \oplus L^{\frac{m^2-1}{m}}.$$

It remains to show that the exact sequence (10.4) splits. Let us first compute the Ext-group:

$$\begin{aligned}
& \text{Ext}^1 \left( \pi^{(m)*} \mathcal{G} \otimes \mathcal{O}_{\pi^{(m)}}(-1), \pi^{(m)*} \pi_*^{(m)} \mathcal{F} \right) \\
& \cong H^1 \left( X^{(m)}, \pi^{(m)*} \left( \mathcal{G}^\vee \otimes \pi_*^{(m)} \mathcal{F} \right) \otimes \mathcal{O}_{\pi^{(m)}}(1) \right) \\
& \cong H^1 \left( A, \mathcal{G}^\vee \otimes \pi_*^{(m)} \mathcal{F} \otimes \pi_*^{(m)} \mathcal{O}_{\pi^{(m)}}(1) \right) \\
& \cong \bigoplus_{\eta} H^1(A, L_{\frac{\eta}{m}}).
\end{aligned}$$

Here, the last isomorphism follows from the descriptions of  $\mathcal{G}, \pi_*^{(m)} \mathcal{F}$  given above, together with the equality  $\pi_*^{(m)} \mathcal{O}_{\pi^{(m)}}(1) = \mathcal{O}_A \oplus L^m$ . Furthermore, by these descriptions, we can see that  $\eta \neq 0$ . Hence if  $L$  is not a torsion line bundle, we have the vanishing  $\text{Ext}^1 \left( \pi^{(m)*} \mathcal{G} \otimes \mathcal{O}_{\pi^{(m)}}(-1), \pi^{(m)*} \pi_*^{(m)} \mathcal{F} \right) = 0$  and thus the sequence (10.4) splits. Assume that  $L^{\frac{1}{m}}$  is  $l$ -torsion, i.e.,  $(L^{\frac{1}{m}})^l \cong \mathcal{O}_A$ . Assume also that  $\text{Ext}^1 \left( \pi^{(m)*} \mathcal{G} \otimes \mathcal{O}_{\pi^{(m)}}(-1), \pi^{(m)*} \pi_*^{(m)} \mathcal{F} \right)$  contains  $H^1(A, (L^{\frac{1}{m}})^l) \cong H^1(A, \mathcal{O}_A)$  as a direct summand. Suppose for a contradiction that the sequence (10.4) does not split. Consider the following Cartesian diagram:

$$\begin{array}{ccccc}
\mathbb{P}^1 \times A & \xrightarrow{\underline{m}^2 \times \text{id}_A} & \mathbb{P}^1 \times A & \xrightarrow{\pi_0} & A \\
u \downarrow & & v \downarrow & & \downarrow \underline{l} \\
X^{(m)} & \xrightarrow{g_m} & X^{(\frac{1}{m})} & \xrightarrow{\pi^{(m)}} & A.
\end{array}$$

Since the morphisms  $u$  and  $v$  are flat, we have an isomorphism

$$\begin{aligned}
v^* g_{m*} \mathcal{O}_{\pi^{(m)}}(a) & \cong (\underline{m}^2 \times \text{id}_A)_* u^* \mathcal{O}_{\pi^{(m)}}(a) \\
& \cong (\underline{m}^2 \times \text{id}_A)_* \mathcal{O}_{\pi_0}(a)
\end{aligned}$$

and hence it is a direct sum of line bundles by the usual toric Frobenius splitting on  $\mathbb{P}^1$ . On the other hand, the pull back of the sequence (10.4) via  $v$  cannot split since  $\underline{l}^*: H^1(A, \mathcal{O}_A) \rightarrow H^1(A, \mathcal{O}_A)$  is an isomorphism. Hence we get a contradiction.

(2) Let  $X = \mathbb{P}_C(\mathcal{O}_C \oplus L_1 \oplus L_2)$  be as in Theorem 9.14 (7). As in (1), we may assume  $M = \mathcal{O}_{\pi^{(\frac{1}{m})}}(a)$  and  $0 \leq a < m^2$ . Let  $\mathcal{F} := g_{m*} M$ . By the same argument as in (1), we have the following exact sequences:

$$\begin{aligned}
(10.5) \quad & 0 \rightarrow \pi^{(m)*} \pi_*^{(m)} \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}' \otimes \mathcal{O}_{\pi^{(m)}}(-1) \rightarrow 0, \\
& 0 \rightarrow \pi^{(m)*} \pi_*^{(m)} \mathcal{F}' \rightarrow \mathcal{F}' \rightarrow \pi^{(m)*} \mathcal{G} \otimes \mathcal{O}_{\pi^{(m)}}(-1) \rightarrow 0,
\end{aligned}$$

which correspond to the toric Frobenius splitting

$$\underline{m}_*^2 \mathcal{O}_{\mathbb{P}^2}(a) \cong \mathcal{O}_{\mathbb{P}^2}^{\oplus k_0} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus k_1} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus k_2}$$

on the fiber of  $\pi^{(m)}$ . Furthermore, we can show that the vector bundles  $\mathcal{F}'$  and  $\mathcal{G}$  are direct sums of line bundles, and that exact sequences (10.5) split again by the same argument as (1).

(3) Let  $X = \mathbb{P}_C(\mathcal{O}_C \oplus L_1) \times_C \mathbb{P}_C(\mathcal{O}_C \oplus L_2)$  be as in Theorem 9.14 (8), let  $Y_i := \mathbb{P}_C(\mathcal{O}_C \oplus L_i)$ . Then the problem is reduced to showing the corresponding statement for  $Y^{(\frac{1}{m})} \rightarrow Y^{(m)}$ . The latter follows from the argument as in (1).  $\square$

We also need the following lemma.

LEMMA 10.10. *Let  $X$  be as in Theorem 9.14 (6), (7), or (8) which is not necessarily split. Then by identifying the tautological classes, the Néron-Severi group of  $X$  is canonically isomorphic to that of  $\mathbb{P}^1 \times A$ ,  $\mathbb{P}^2 \times C$ , or  $\mathbb{P}^1 \times \mathbb{P}^1 \times C$ . Furthermore, the following statements hold:*

- (1) *Under the isomorphism, their nef cones are preserved.*
- (2) *Under the isomorphism, their classes of the canonical divisors are preserved.*
- (3) *If  $X$  is split, then the isomorphism is compatible with the formula given in Proposition 10.9.*

PROOF. Let  $\pi: X = \mathbb{P}_A(\mathcal{E}) \rightarrow A$  be as in Theorem 9.14 (6), where  $\mathcal{E}$  is a rank two vector bundle fitting into an exact sequence

$$0 \rightarrow \mathcal{O}_A \rightarrow \mathcal{E} \rightarrow L \rightarrow 0.$$

We only treat this case. We have  $\text{NS}(X) = \mathbb{Z}[h] \oplus \text{NS}(A)$ , where  $h$  is a divisor such that  $\mathcal{O}(h) = \mathcal{O}_\pi(1)$ . Hence by identifying a class  $[h]$ ,  $\text{NS}(X)$  is isomorphic to  $\text{NS}(\mathbb{P}^1 \times A)$ .

(1) We claim that the line bundle  $M = \mathcal{O}_\pi(a) \otimes \pi^*N$  on  $X$  is nef if and only if  $a \geq 0$  and  $N$  is nef. The ‘if’ direction is clear since  $\mathcal{O}_\pi(1)$  is nef. Let us prove the converse. Let  $h \in |\mathcal{O}_\pi(1)|$  be a section of  $\pi$  and  $f \cong \mathbb{P}^1$  be a fiber of  $\pi$ . Then we have  $M|_h \cong N$ ,  $M|_f \cong \mathcal{O}_{\mathbb{P}^1}(a)$  and they are nef, which proves the claim. This description of the nef cone is independent on the choice of  $L \in \text{Pic}^0(A)$  and on the choice of the extension class  $[\mathcal{E}] \in \text{Ext}^1(L, \mathcal{O}_A)$ .

(2) The canonical line bundle on  $X$  is given as  $\mathcal{O}(K_X) = \mathcal{O}(-2h) \otimes \pi^*L$ . Since  $L \in \text{Pic}^0(A)$ , we have  $[K_X] = -2[h] \in \text{NS}(X)$  in the Néron-Severi group which is independent on the choice of  $L \in \text{Pic}^0(A)$ .

(3) The statement is trivial from the proof of Proposition 10.9, again by noting that  $\text{ch}_1(L) = 0$  for  $L \in \text{Pic}^0(A)$ .  $\square$

Now we can prove our main theorem:

PROOF OF THEOREM 9.17. We only give an outline of the proof since the argument is same as [26]. Let  $X$  be as in Theorem 9.17. By Proposition 10.3, we may assume  $X$  is split. Take a  $\bar{\beta}$ -stable object  $E$  and let  $\bar{B} := B + \bar{\beta}(E)\omega$ .

First assume that  $\bar{B}$  is a  $\mathbb{Q}$ -divisor. Take an integer  $q \in \mathbb{Z}_{>0}$  and an integral divisor  $D$  such that  $\bar{B} = \frac{1}{q}D$ . For each integer  $m \in \mathbb{Z}_{>0}$ , let us consider the morphism  $F_{mq}$  constructed in Proposition 10.5. Let  $D^{(\frac{1}{mq})}$  be the divisor on  $X^{(\frac{1}{mq})}$  such that  $D^{(\frac{1}{mq})} = \Phi^{-1}(D)$  in the cohomology. Then the Riemann-Roch theorem and Lemma 10.8 imply the inequality

$$\begin{aligned} m^6 q^6 \text{ch}_3^{\bar{B}}(E) + \mathcal{O}(m^4) &= \chi \left( \mathcal{O}, F_{mq}^* E \otimes \mathcal{O} \left( -m^2 q D^{(\frac{1}{mq})} \right) \right) \\ &\leq \text{hom} \left( \mathcal{O}, F_{mq}^* E \otimes \mathcal{O} \left( -m^2 q D^{(\frac{1}{mq})} \right) \right) \\ &\quad + \text{ext}^2 \left( \mathcal{O}, F_{mq}^* E \otimes \mathcal{O} \left( -m^2 q D^{(\frac{1}{mq})} \right) \right). \end{aligned}$$

We need to prove that the right hand side of the above inequality is of order  $m^4$ . By Proposition 10.7, to prove  $\text{hom} \left( \mathcal{O}, F_{mq}^* E \otimes \mathcal{O} \left( -m^2 q D^{(\frac{1}{mq})} \right) \right) = \mathcal{O}(m^4)$ , it is enough to find an ample divisor  $H$  such that

$$\text{Hom} \left( \mathcal{O}(H), F_{mq}^* E \otimes \mathcal{O} \left( -m^2 q D^{(\frac{1}{mq})} \right) \right) = 0.$$

By using the Serre duality and the projection formula, we have an isomorphism

$$\begin{aligned} & \operatorname{Hom} \left( \mathcal{O}(H), F_{mq}^* E \otimes \mathcal{O} \left( -m^2 q D^{(\frac{1}{mq})} \right) \right) \\ & \cong \operatorname{Hom} \left( \mathcal{O}(-K_{X^{(m)}}) \otimes g_{mq*} \mathcal{O} \left( H + m^2 q D^{(\frac{1}{mq})} + K_{X^{(\frac{1}{m})}} \right), h_{mq}^* E \right). \end{aligned}$$

By Proposition 10.9, the bundle  $\mathcal{O}(-K_{X^{(m)}}) \otimes g_{mq*} \mathcal{O} \left( H + m^2 q D^{(\frac{1}{mq})} + K_{X^{(\frac{1}{m})}} \right)$  splits into a direct sum of line bundles  $M_j$ . Hence it is enough to show the vanishing  $\operatorname{Hom}(M_j, h_{mq}^* E) = 0$  for all  $j$ . Since we know the tilt stability of  $M_j$  (resp.  $h_m^* E$ ) by Lemma 9.13 (resp. [6, Proposition 6.1]), it is enough to show the inequality  $\nu_{0, h_{mq}^* \overline{B}}(M_j) > \nu_{0, h_{mq}^* \overline{B}}(h_{mq}^* E) = 0$ . The required inequality on the tilt-slope will follow if we can show that  $\operatorname{ch}_1^{h_{mq}^* \overline{B}}(M_j)$  is ample. By Lemma 10.10, the problem is now reduced to the case when  $X$  is  $\mathbb{P}^1 \times A$ ,  $\mathbb{P}^2 \times C$ , or  $\mathbb{P}^1 \times \mathbb{P}^1 \times C$ , which is treated in [26, Lemma 4.6]. The estimate of  $\operatorname{ext}^2$  will also be reduced to [26, Lemma 4.7].

When  $B$  is not a  $\mathbb{Q}$ -divisor but an  $\mathbb{R}$ -divisor, we can argue as in [26, Subsection 4.3] by using Dirichlet approximation theorem.  $\square$

### 11. Proof of Theorem 9.19

In this section, we will treat  $\pi: X := \mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2}) \rightarrow \mathbb{P}^2$ . Recall that  $X$  is isomorphic to a  $(1, 1)$ -divisor in  $\mathbb{P}^2 \times \mathbb{P}^2$  and hence has two projections to  $\mathbb{P}^2$ :

$$(11.1) \quad \begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow \sigma \\ \mathbb{P}^2 & & \mathbb{P}^2. \end{array}$$

Let  $h_1, h_2$  be nef divisors on  $X$  such that  $\mathcal{O}(h_1) = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ ,  $\mathcal{O}(h_2) = \sigma^* \mathcal{O}_{\mathbb{P}^2}(1)$ . Then any line bundle on  $X$  can be written as  $\mathcal{O}(a, b) := \mathcal{O}(ah_1) \otimes \mathcal{O}(bh_2)$  with  $a, b \in \mathbb{Z}$ . In this notation, we have  $\mathcal{O}_\pi(1) = \mathcal{O}(1, 1)$ .

Fix an ample divisor  $H = ah_1 + bh_2$  with  $a, b \in \mathbb{Z}_{>0}$ . For a positive real number  $\alpha > 0$ , let  $\omega := \alpha H$ . We will mainly consider the following central charge and heart:

$$Z_{\alpha, 0, s} := -\operatorname{ch}_3 + s\alpha^2 H^2 \cdot \operatorname{ch}_1 + i \left( \alpha H \cdot \operatorname{ch}_2 - \frac{1}{6} \alpha^3 H^3 \operatorname{ch}_0 \right),$$

and  $\mathcal{A}_{\alpha, 0} := \mathcal{A}_{\omega, 0}$ .

First recall the following result due to [6, 7].

**THEOREM 11.1.** *Fix a positive real number  $\alpha > 0$ . Conjecture 9.4 holds for  $(X, \alpha H, B = 0)$  if and only if for every  $s > \frac{1}{18}$ , the pair  $(Z_{\alpha, 0, s}, \mathcal{A}_\alpha)$  is a stability condition on  $D^b(X)$ .*

**PROOF.** By [7, Corollary 5.2.4], the pair  $(Z_{\alpha, 0, s}, \mathcal{A}_\alpha)$  is a stability condition for every  $s > \frac{1}{18}$  if and only if for every  $\nu_{\alpha, 0}$ -stable object  $E \in \operatorname{Coh}^{\alpha H, 0}(X)$  with  $\nu_{\alpha, 0}(E) = 0$ , we have  $\operatorname{ch}_3 \leq \frac{1}{18} \alpha^2 H^2 \cdot \operatorname{ch}_1(E)$ . Then the latter is equivalent to Conjecture 9.4 by [6, Theorem 4.2].  $\square$

**DEFINITION 11.2.** For a fixed ample divisor  $H = ah_1 + bh_2$ , we define a real number  $\alpha_0 > 0$  as

$$\alpha_0 := \min \left\{ \sqrt{\frac{2}{a(a+b)}}, \sqrt{\frac{18}{a^2 + 6ab + b^2}} \right\}.$$

The goal of this subsection is to prove the following:



PROPOSITION 11.3. *Let  $H = ah_1 + bh_2$  be an ample divisor on  $X$  with  $b > a$ . Then for every  $0 < \alpha < \alpha_0$  and  $s > \frac{1}{18}$ , the pair  $(Z_{\alpha,0,s}, \mathcal{A}_{\alpha,0})$  is a stability condition on  $X$ . In particular, Conjecture 9.4 holds for  $(X, \alpha H, B = 0)$ .*

First we prove that the above proposition implies Theorem 9.19.

PROOF OF THEOREM 9.19. Let  $H = ah_1 + bh_2$  be an ample divisor. By the symmetry of the diagram (11.1), we may assume that  $b \geq a$ . Furthermore, if  $a = b$ , then the result is already known due to [8]. Now we can assume that  $b > a$  and then by Theorem 9.9, Proposition 11.3 implies Theorem 9.19.  $\square$

To prove Proposition 11.3, we use the following result due to the paper [7], and follow the arguments in [35, 46].

PROPOSITION 11.4 ([7, Proposition 8.1.1]). *Assume there exists a heart  $\mathcal{C}$  in  $D^b(X)$  with the following properties:*

- (1) *There exist  $\phi_0 \in (0, 1)$  and  $s_0 \in \mathbb{Q}$  such that*

$$Z_{\alpha,0,s_0}(\mathcal{C}) \subset \{r \exp(\pi \phi i) : r \geq 0, \phi_0 \leq \phi \leq \phi_0 + 1\}.$$

- (2)  *$\mathcal{C} \subset \langle \mathcal{A}_{\alpha,0}, \mathcal{A}_{\alpha,0}[1] \rangle$ .*

- (3) *For any  $x \in X$ , we have  $\mathcal{O}_x \in \mathcal{C}$  and, for all non-zero proper subobjects  $C \subset \mathcal{O}_x$  in  $\mathcal{C}$ , we have  $\Im Z_{\alpha,0,s_0}(C) > 0$ .*

*Then for all  $s > s_0$ , the pair  $(Z_{\alpha,0,s}, \mathcal{A}_{\alpha,0})$  is a stability condition on  $D^b(X)$ .*

Our heart  $\mathcal{C}$  is constructed by using an *Ext-exceptional collection* in the sense of [34, Definition 3.10].

DEFINITION 11.5. An exceptional collection  $E_1, \dots, E_n$  on a triangulated category  $\mathcal{D}$  is *Ext-exceptional* if for all  $i \neq j$ , we have  $\text{Ext}^{\leq 0}(E_i, E_j) = 0$ .

LEMMA 11.6 ([34, Lemma 3.14]). *Let  $E_1, \dots, E_n$  be a full Ext-exceptional collection on a triangulated category  $\mathcal{D}$ . Then the extension closure  $\langle E_1, \dots, E_n \rangle_{ex}$  is the heart of a bounded  $t$ -structure on  $\mathcal{D}$ .*

LEMMA 11.7. *A collection*

$$(11.2) \quad \mathcal{O}(-1, -1)[3], \mathcal{O}(0, -1)[2], \mathcal{O}(1, -1)[1], \mathcal{O}(-1, 0)[2], \mathcal{O}[1], \mathcal{O}(1, 0)$$

*is a full Ext-exceptional collection on  $D^b(X)$ .*

PROOF. Using the equality  $\mathcal{O}_\pi(1) = \mathcal{O}(1, 1)$ , the collection (11.2) can be also written as

$$\begin{aligned} & \pi^* \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_\pi(-1)[3], \pi^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_\pi(-1)[2], \pi^* \mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{O}_\pi(-1)[1], \\ & \pi^* \mathcal{O}_{\mathbb{P}^2}(-1)[2], \pi^* \mathcal{O}_{\mathbb{P}^2}[1], \pi^* \mathcal{O}_{\mathbb{P}^2}(1). \end{aligned}$$

Since we have  $D^b(X) = \langle \mathbf{L}\pi^* D^b(\mathbb{P}^2) \otimes \mathcal{O}_\pi(-1), \mathbf{L}\pi^* D^b(\mathbb{P}^2) \rangle$ , we can see that the collection (11.2) is a full exceptional collection. To prove it is Ext-exceptional, we can use the formula

$$\mathbf{R}\Gamma(X, \pi^* \mathcal{O}_{\mathbb{P}^2}(k) \otimes \mathcal{O}_\pi(l)) = \begin{cases} 0 & (l = -1) \\ \mathbf{R}\Gamma(\mathbb{P}^2, \mathcal{O}(k)) & (l = 0) \\ \mathbf{R}\Gamma(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(k)) & (l = 1). \end{cases}$$

$\square$

Now we can define the following heart:

DEFINITION 11.8. We define a heart  $\mathcal{C} \subset D^b(X)$  as

$$\mathcal{C} := \langle \mathcal{O}(-1, -1)[3], \mathcal{O}(0, -1)[2], \mathcal{O}(1, -1)[1], \mathcal{O}(-1, 0)[2], \mathcal{O}[1], \mathcal{O}(1, 0) \rangle_{ex}.$$

The following will be useful in the rest of the arguments:

LEMMA 11.9. *For integers  $k, l \in \mathbb{Z}$ , we have the following equations.*

- (1)  $H^2 \cdot \text{ch}_1(\mathcal{O}(k, l)) = la^2 + 2(k+l)ab + kb^2$ .
- (2)  $H \cdot \text{ch}_2(\mathcal{O}(k, l)) = (2k+l)la + (k+2l)kb$ .
- (3)  $\text{ch}_3(\mathcal{O}(k, l)) = \frac{1}{2}kl(k+l)$ .

PROOF. By using the equations  $h_1^3 = h_2^3 = 0$  and  $h_1^2 \cdot h_2 = h_1 \cdot h_2^2 = 1$ , the straightforward computation yields the result.  $\square$

LEMMA 11.10. *For  $0 < \alpha < \alpha_0$ , we have  $\mathcal{C} \subset \langle \mathcal{A}_{\alpha,0}, \mathcal{A}_{\alpha,0}[1] \rangle_{ex}$ .*

PROOF. By Lemma 11.9, we have  $H \cdot \text{ch}_1(\mathcal{O}(1, 0)) > 0$  and hence  $\mathcal{O}(1, 0) \in \text{Coh}^{\alpha,0}(X)$ . By assumption on  $\alpha$ , we also have

$$H \cdot \text{ch}_2(\mathcal{O}(1, 0)) - \frac{1}{6}\alpha^2 H^3 \text{ch}_0(\mathcal{O}(1, 0)) = b - \frac{1}{2}\alpha^2 ab(a+b) > 0,$$

i.e.,  $\nu_{\alpha,0}(\mathcal{O}(1, 0)) > 0$ . Since  $\mathcal{O}(1, 0)$  is tilt stable by Lemma 9.13, we conclude that  $\mathcal{O}(1, 0) \in \mathcal{A}_{\alpha,0}$ .

Similar computations yield that

$$\mathcal{O}[1], \mathcal{O}(-1, 0)[1], \mathcal{O}(1, -1), \mathcal{O}(0, -1)[1], \mathcal{O}(-1, -1)[1] \in \text{Coh}^{\alpha H,0}(X)$$

and

$$\mathcal{O}[1], \mathcal{O}(-1, 0)[2], \mathcal{O}(1, -1)[1], \mathcal{O}(0, -1)[2], \mathcal{O}(-1, -1)[2] \in \mathcal{A}_{\alpha,0}.$$

$\square$

LEMMA 11.11. *Let  $0 < \alpha < \alpha_0$ , and let  $\phi_0 \in (0, 1)$  be a real number such that  $Z_{\alpha,0,\frac{1}{18}}(\mathcal{O}(1, 0)) = r_0 \exp(\pi\phi_0 i)$  for some positive real number  $r_0 > 0$ . Then we have*

$$Z_{\alpha,0,\frac{1}{18}}(\mathcal{C}) \subset \{r \exp(\pi\phi i) : r \geq 0, \phi_0 \leq \phi \leq \phi_0 + 1\}.$$

PROOF. Recall that our central charge is written as

$$Z_\alpha := Z_{\alpha,0,\frac{1}{18}} = -\text{ch}_3 + \frac{1}{18}\alpha^2 H^2 \cdot \text{ch}_1 + i \left( \alpha H \cdot \text{ch}_2 - \frac{1}{6}\alpha^3 H^3 \text{ch}_0 \right).$$

By Lemma 11.9 and the proof of Lemma 11.10, we can see that  $Z_\alpha(\mathcal{O}(-1, -1)[3])$  is in the third quadrant,  $Z_\alpha(\mathcal{O}(1, 0))$  is in the first quadrant, and  $Z_\alpha(M)$  is in the second quadrant for other generators  $M$  of the heart  $\mathcal{C}$ . Now it is enough to check the inequality

$$-\frac{\Re Z_\alpha(\mathcal{O}(1, 0))}{\Im Z_\alpha(\mathcal{O}(1, 0))} + \frac{\Re Z_\alpha(\mathcal{O}(-1, -1)[3])}{\Im Z_\alpha(\mathcal{O}(-1, -1)[3])} > 0.$$

We can estimate the left hand side of the above required inequality as follows:

$$\begin{aligned} & -\frac{\frac{1}{18}\alpha^2(2a+b)b}{\alpha(b - \frac{1}{2}\alpha^2 ab(a+b))} + \frac{1 - \frac{1}{18}\alpha^2(a^2 + 4ab + b^2)}{\alpha(3a + 3b - \frac{1}{2}\alpha^2 ab(a+b))} \\ & > \frac{-\frac{1}{18}\alpha^2(2a+b)b + 1 - \frac{1}{18}\alpha^2(a^2 + 4ab + b^2)}{\alpha(3a + 3b - \frac{1}{2}\alpha^2 ab(a+b))} \\ & = \frac{1 - \frac{1}{18}\alpha^2(a^2 + 6ab + 2b^2)}{\alpha(3a + 3b - \frac{1}{2}\alpha^2 ab(a+b))} > 0. \end{aligned}$$

Hence the statement holds.  $\square$

LEMMA 11.12. *Let  $0 < \alpha < \alpha_0$  and  $x \in X$ . Then we have  $\mathcal{O}_x \in \mathcal{C}$ . Moreover, for any non-zero proper subobject  $C \subset \mathcal{O}_x$  in the category  $\mathcal{C}$ , we have  $\Im Z_{\alpha,0,\frac{1}{18}}(C) > 0$ .*

PROOF. Consider the subcategories

$$\begin{aligned}\mathcal{C}_1 &:= \pi^* \langle \mathcal{O}_{\mathbb{P}^2}(-1)[2], \mathcal{O}_{\mathbb{P}^2}[1], \mathcal{O}_{\mathbb{P}^2}(1) \rangle_{ex}, \\ \mathcal{C}_2 &:= \pi^* \langle \mathcal{O}_{\mathbb{P}^2}[2], \mathcal{O}_{\mathbb{P}^2}(1)[1], \mathcal{O}_{\mathbb{P}^2}(2) \rangle_{ex} \otimes \mathcal{O}_{\pi}(-1)[1]\end{aligned}$$

of  $\mathcal{C}$ . Both of these subcategories  $\mathcal{C}_i$  are equivalent to the category  $\text{rep}(Q, I)$  of  $Q$ -representations with certain relations  $I$ . Here  $Q$  is the following quiver:

$$0 \begin{array}{c} \rightrightarrows \\ \leftarrow \end{array} 1 \begin{array}{c} \rightrightarrows \\ \leftarrow \end{array} 2$$

Let  $y := \pi(x)$  and denote  $L_y := \pi^{-1}(y) \cong \mathbb{P}^1$ . Then we have the following exact triangle in  $D^b(X)$

$$\mathcal{O}_{L_y} \rightarrow \mathcal{O}_x \rightarrow \mathcal{O}_{L_y}(-1)[1]$$

with  $\mathcal{O}_{L_y} \in \mathcal{C}_1$  and  $\mathcal{O}_{L_y}(-1)[1] \in \mathcal{C}_2$ . This proves that  $\mathcal{O}_x \in \mathcal{C}$ . Note that the  $Q$ -representations corresponding to  $\mathcal{O}_{L_y} \in \mathcal{C}_1$  and  $\mathcal{O}_{L_y}(-1)[1] \in \mathcal{C}_2$  are the same representation, which has dimension vector  $(1, 2, 1)$  and is generated by the vertex 0. We say that  $\mathcal{O}_x$  has dimension vector  $(1, 2, 1, 1, 2, 1)$ .

To prove the second statement, recall that for an object  $M$  in (11.2), we have  $\Im Z_{\alpha, 0, \frac{1}{18}}(M) < 0$  if and only if  $M = \mathcal{O}(-1, -1)[3] = \pi^* \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\pi}(-1)[3]$ . Hence it is enough to consider a subobject  $C \subset \mathcal{O}_x$  with dimension vector  $(1, a, b, c, d, e)$ . We must prove that  $C = \mathcal{O}_x$  for such a subobject  $C$ . There exists an exact sequence

$$0 \rightarrow T_1 \rightarrow C \rightarrow T_2 \rightarrow 0$$

in  $\mathcal{C}$  with some objects  $T_i \in \mathcal{C}_i$ . Using the definition of the Ext-exceptional collection, we can see that  $T_1 \subset \mathcal{O}_{L_y}$  (resp.  $T_2 \subset \mathcal{O}_{L_y}(-1)[1]$ ) is the subobject in  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ ). Since we assume that the dimension vector of  $C$  is  $(1, a, b, c, d, e)$ , and since  $\mathcal{O}_{L_y}(-1)[1]$  is generated by vertex 0 as a quiver representation, we must have  $T_2 = \mathcal{O}_{L_y}(-1)[1]$ . Now we get the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & C & \longrightarrow & T_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_{L_y} & \longrightarrow & \mathcal{O}_x & \longrightarrow & \mathcal{O}_{L_y}(-1)[1] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

for some  $K \in \mathcal{C}_1$ . However, since  $\text{Hom}(\mathcal{O}_x, \mathcal{C}_1) = 0$ , we must have  $K = 0$ , i.e.,  $C = \mathcal{O}_x$  as required.  $\square$

Now we can prove Proposition 11.3.

PROOF OF PROPOSITION 11.3. By Lemma 11.10, Lemma 11.11, and Lemma 11.12, we can apply Proposition 11.4 to get the result.  $\square$

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