博士論文

論文題目　Fibred Cusp b-Pseudodifferential Operators and Their Applications

(Fibred Cusp b-擬微分作用素とその応用)

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Fibred Cusp b-Pseudodifferential Operators and Their Applications

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Abstract

Let X be a compact C^∞ manifold with corners which has two embedded boundary hypersurfaces $\partial_0 X, \partial_1 X$, and a fibre bundle $\phi : \partial_0 X \to Y$ is given. By using the method of blowing up, we define a pseudodifferential culculus $\Psi^*_{\Phi,b}(X)$ which is suitable to extend the relative index formula of b-calculus to the case of manifold with corners. This calculus contains the Φ-calculus of Mazzeo and Melrose or the (small) b-calculus of Melrose as a special case when $\partial_1 X$ or $\partial_0 X$ is empty. As in the case of b-calculus and cusp calculus, this calculus can be densely embedded into S-calculus of Debord, Lescure and Rochon by using logarithmic blow-up. We discuss the Fredholm condition of those operators and gives an explicit formula for the relative index in terms of the logarithmic residue of the normal operator. As its application, the index theorem of $\mathbb{Z}/k$-manifolds with boundary is proved.

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1 Introduction

As Atiyah and Singer [5] proved, the index problem of an elliptic operator on a closed manifold can be reduced to the topological K-theory. It is because a pseudodifferential operator is Fredholm if and only if its principal symbol is invertible, and the homotopy class of the principal symbol can be determined topologically. For a manifold with boundaries or corners, we need to impose another invertibility to the normal operator to obtain a Fredholm property, which can not be characterized topologically in general. Depending on the normal structure at boundary or corners, various types of pseudodifferential calculi has been studied [8, 14, 15, 16, 17, 18, 21], and its relation to the theory of pseudodifferential calculus on a Lie groupoid [24] also has been studied [4, 9, 23].

What we are particularly interested in here are 0-calculus, b-calculus, scattering calculus and cusp calculus. Let X be a manifold with boundary (without corners) and x be a boundary defining function. The corresponding Lie algebroids are given as follows.

$$\begin{aligned}
\mathcal{V}_0(X) &:= \{V \in \mathcal{V}(X) \mid V|_{\partial X} = 0\} \\
\mathcal{V}_b(X) &:= \{V \in \mathcal{V}(X) \mid V|_{\partial X} \text{ is tangent to } \partial X\} \\
\mathcal{V}_{sc}(X) &:= \{V \in \mathcal{V}(X) \mid V|_{\partial X} = 0 \text{ and } Vx \in x^2 C^\infty(X)\} \\
\mathcal{V}_{cu}(X) &:= \{V \in \mathcal{V}(X) \mid V|_{\partial X} \text{ is tangent to } \partial X \text{ and } Vx \in x^2 C^\infty(X)\}
\end{aligned}$$

When a fibre bundle $\phi : \partial X \to Y$ is given, we can also define edge calculus and fibred cusp calculus. The corresponding Lie algebroid is given by

$$\begin{aligned}
\mathcal{V}_e(X) &:= \{V \in \mathcal{V}(X) \mid V|_{\partial X} \text{ is tangent to the fibre of } \phi\} \\
\mathcal{V}_\Phi(X) &:= \{V \in \mathcal{V}(X) \mid V|_{\partial X} \text{ is tangent to the fibre of } \phi \text{ and } Vx \in x^2 C^\infty(X)\}.
\end{aligned}$$

0-calculus and b-calculus two extreme cases of the edge calculus when $\phi = \mathrm{Id}$ or $\phi = \mathrm{pt}$, while scattering calculus and cusp calculus are two extreme cases of the fibred cusp calculus when $\phi = \mathrm{Id}$ or $\phi = \mathrm{pt}$.

The extension of those calculi to the case of manifold with corners is also considered by many authors. For b-calculus and cusp calculus, there is a straightforward extension [15, 14] due to the interchangeability of the blow-ups. For edge calculus or Φ-calculus, we must impose an additional iterated fibration structure or the higher-depth stratified pseudomanifold structure to extend those calculus [1, 8].

Let $I = 0, b, sc, cu, e$ or Φ. The enveloping algebra $\mathrm{Diff}^*_I(X)$ of $\mathcal{V}_I(X)$ consists of differential operators on X, and the I-calculus $\Psi^*_I(X)$ is an algebra of pseudodifferential operators containing $\mathrm{Diff}^*_I(X)$. For $P \in \Psi^*_I(X)$, P defines a bounded operator between a suitable Sobolev spaces. It is Fredholm if and only if its symbol and normal operator, namely, $\sigma(P)$ and $N(P)$ are invertible (in its closure).

Among those calculi, one of the significant difference is its index problem. Suppose that the symbol σ is given. For b-calculus and cusp calculus, we can always chose a Fredholm realization P, and its index problem is (in a special case) equivalent to the APS index theorem [6]. On the other hand, for 0-calculus and scattering calculus, we can not take a Fredholm realization in general. Because the operator $N(P)$ degenerates to define a vector bundle homomorphism, which is a topological obstruction. Its index problem is reduced to AS index theorem [5]. Thus, in some sense, the index problem of edge and Φ-calculus is in between AS and APS index problem.

The other significant difference is its closedness under the holomorphic functional calculus. Φ-calculus, including scattering and cusp calculus, is closed under the holomorphic functional calculus. On the other hand, edge calculus, including 0 and b-calculus, is *n*ot closed under the the holomorphic functional calculus. In this paper, this difference plays an important role to extend the relative index formula for the manifold with corners.

By using logarithmic blow-up, b-calculus is embedded into a dense subalgebra of cusp calculus: $\Psi_b^*(X) \subset \Psi_{cu}^*(X)$. For an elliptic $P \in \Psi_b^*(X)$, the Fourier (Mellin) transform of the normal operator defines an entire operator-valued holomorphic function, namely, $\widehat{N}(P)(\lambda) \in \Psi^*(\partial X)$ $(\lambda \in \mathbb{C})$. Its relative index formula is given by a logarithmic residue of the normal operator [18]. Namely,

$$\operatorname{ind}(x_1^{\beta_1} P x_1^{-\beta_1}) - \operatorname{ind}(x_1^{\beta_2} P x_1^{-\beta_2}) = \frac{1}{2\pi i} \operatorname{tr} \oint \widehat{N}(P)^{-1}(\lambda) \frac{\partial \widehat{N}(P)}{\partial \lambda}(\lambda) d\lambda, \tag{1}$$

where $\beta_i \notin -\operatorname{ImSpec}(\widehat{N}(P))$ $(i = 1, 2)$, $\beta_2 > \beta_1$ and the path of integral is chosen so that its interior contains all poles of $\widehat{N}(P)^{-1}(\lambda)$ such that $\beta_1 < -\operatorname{Im}(\lambda) < \beta_2$. This formula was used to define an analytic index of the $\mathbb{Z}/k$-manifold in [10].

On the other hand, for a cusp calculus, $\widehat{N}(P)(\lambda)$ is defined only for $\mathbb{R}$ and is only C^∞. As described in [20], to describe the relative index formula for cusp calculus, we need to investigate the regularized trace, which is written in terms of the asymptotic behaviour of the divergent integral. Thus, we can not give a convergent integral formula as in the formula (1).

In this paper, we will define a pseudodifferential calculus which is an appropriate calculus to extend the relative index formula for b-calculus. Let X be a smooth compact manifold with corners which has two embedded boundary hypersurfaces $\partial_0 X, \partial_1 X$, and a fibre bundle $\phi : \partial_0 X \to Y$ is given. Suppose that the fibre Z of ϕ is a closed manifold. Fix a boundary defining function x_0 of $\partial_0 X$ and x_1 of $\partial_1 X$. Such X is called a manifold with boundary. We define a pseudodifferential calculus $\Psi_{\Phi,b}^*(X)$ of fibred cusp b-pseudodifferential operators. This calculus contains the Φ-calculus of Mazzeo-Melrose [17] or the (small) b-calculus of Melrose [18] as a special case when $\partial_1 X$ or $\partial_0 X$ is empty. As b-calculus can be densely embedded into cusp calculus by using logarithmic blow-up, fibred cusp b-calculus also can be densely embedded into a S-calculus of Debord-Lescure-Rochon [8].

Each element $P \in \Psi_{\Phi,b}^0(X)$ defines a bounded operator.

$$P : L^2_{\Phi,b}(X) \to L^2_{\Phi,b}(X)$$

We also define a symbol map σ and two normal maps N_0, N_1 with respect to two boundaries $\partial_0 X, \partial_1 X$, which make the following sequences are exact.

$$0 \to \Psi_{\Phi,b}^{-1}(X) \to \Psi_{\Phi,b}^0(X) \xrightarrow{\sigma} S^0({}^{\Phi,b}T^*X) \to 0$$

$$0 \to x_0 \Psi_{\Phi,b}^0(X) \to \Psi_{\Phi,b}^0(X) \xrightarrow{N_0} \Psi^0_{\mathrm{sus}({}^{\Phi,b}NY)}(\partial_0 X) \to 0$$

$$0 \to x_1 \Psi_{\Phi,b}^0(X) \to \Psi_{\Phi,b}^0(X) \xrightarrow{N_1} \Psi^0_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}) \to 0$$

where ${}^{\Phi,b}T^*X \simeq T^*X$ and ${}^{\Phi,b}NY \simeq \mathbb{R} \oplus^b T^*Y$ are vector bundles, sus is a suspended calculus. $\widetilde{\partial_1 X} \simeq \partial_1 X \times [0, \infty]$ is a compactification of the normal bundle of $\partial_1 X$. As $\widetilde{\partial_1 X}$ is also a manifold with fibred boundary, we can define $\Psi^0_{\Phi,b}(\widetilde{\partial_1 X})$, and "inv" in $\Psi^0_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X})$ means the invariance under the action of $(0, \infty)$.

We say $P \in \Psi_{\Phi,b}^0(X)$ is elliptic when $\sigma(P)$ is invertible, and fully-elliptic when in addition, $N_0(P)$ and $\widehat{N_1}(P)(\lambda)$ $(\lambda \in \mathbb{R})$ are invertible. Where $\widehat{N_1}(P)$ is a Mellin transform of $N_1(P)$, which is a $\Psi^0_\Phi(\partial_1 X)$-valued entire holomorphic function.

As in the case of b-calculus and its variants, the Fredholm condition for this calculus is given as follows.

Theorem 1. *For $P : L^2_{\Phi,b}(X) \to L^2_{\Phi,b}(X)$ is Fredholm if and only if P is fully elliptic.*

We also prove the relative index theorem, which is a generalization of the relative index for b-calculus [18].

Theorem 2. *Let $P \in \Psi^0_{\Phi,b}(X)$ and suppose $\sigma(P)$ and $N_0(P)$ are invertible. Take any $\beta_i \notin -\mathrm{ImSpec}(\widehat{N_1}(P))$ $(i = 1, 2)$, $\beta_2 > \beta_1$. Then,*

$$\mathrm{ind}(x_1^{\beta_1} P x_1^{-\beta_1}) - \mathrm{ind}(x_1^{\beta_2} P x_1^{-\beta_2}) = \frac{1}{2\pi i}\mathrm{tr} \oint \widehat{N_1}(P)^{-1}(\lambda)\frac{\partial \widehat{N_1}(P)}{\partial \lambda}(\lambda)d\lambda, \qquad (2)$$

where the path of integral is chosen so that its interior contains all poles of $\widehat{N_1}(P)^{-1}(\lambda)$ such that $\beta_1 < -\mathrm{Im}(\lambda) < \beta_2$.

Briefly, the proof of the relative index theorem is given as follows. As the calculus of symbols S^0 and the calculus of the suspended operators Ψ^0_{sus} is closed under holomorphic functional calculus, $\sigma(P)^{-1}$ and $N_0(P)^{-1}$ lies in the same calculus. Thus, we can construct a parametrix Q such that $R := \mathrm{Id} - PQ \in x_0^\infty \Psi^{-\infty}_{\Phi,b}(X)$. As R vanishes at ∞ order at $\partial_0 X$, it blows down to define a b-pseudodifferential operator. By using this fact, the relative index theorem can be proved in the same way as the b-calculus case.

As the application of the relative index theorem, we will prove the index theorem for a $\mathbb{Z}/k$-manifold (possibly with boundary), which is a generalization of the index the for a closed $\mathbb{Z}/k$-manifold by Freed and Melrose [10].

The setting is given as follows. Suppose X is a $\mathbb{Z}/k$ manifold, i.e. X is a manifold with corner and $\partial X = \partial_0 X \cup \partial_1 X$, $\angle X = \partial_0 X \cap \partial_1 X$ and the diffeomorhpism $\partial_1 X \simeq kZ$ is given, where Z is a manifold with boundary and kZ is a disjoint union of k copies of Z. For $\phi = \mathrm{Id} : \partial_0 X \to \partial_0 X$, we regard X as a manifold with fibred boundary. And we write $\Psi^0_{sc,b}(X; E, F) = \Psi^0_{\Phi,b}(X; E, F)$ in this case. A vector bundle E over X is called $\mathbb{Z}/k$-vector bundle if $E|_{\partial_1 X} = kE_Z$ for some vector bundle $E_Z \to Z$. Let E, F are $\mathbb{Z}/k$- vector bundle over X.

We define

$$\Psi^0_{sc,b,\mathbb{Z}/k}(X; E, F) := \{P \in \Psi^0_{sc,b}(X; E, F) \mid \; N_1(P) = kQ \text{ for some } Q \in \Psi^0_{sc,b}(\widetilde{Z}; E, F) \;\}. \qquad (3)$$

For $P \in \Psi^0_{sc,b,\mathbb{Z}/k}(X; E, F)$ such that $\sigma(P)$ and $N_0(P)$ are invertible, $\mathrm{ind}(x_1^\beta P x_1^{-\beta})$ mod $k \in \mathbb{Z}/k$ is independent of β because the right hand side of (2) is always a multiple of k.

On the other hand, we can define a map

$$s : \{P \in \Psi^0_{sc,b,\mathbb{Z}/k}(X; E, F) \mid \; \sigma(P) \text{ and } N_0(P) \text{ are invertible}\}/\text{homotopy} \to K(D(\overline{TX}), \partial_0 D(\overline{TX})), \qquad (4)$$

where the overlines mean the identification of k copies, and $\overline{TX} \to \overline{X}$ is a vector bundle. $\partial D(TX) = S(TX) \cup D(TX|_{\partial_0 X}) \cup D(TX|_{\partial_1 X})$, and $\partial_0 D(\overline{TX}) := S(\overline{TX}) \cup D(\overline{TX|_{\partial_0 X}})$. As in the case of Atiyah-Singer [5] or Freed-Melrose [10], there exists a topological index map t-ind : $K(D(\overline{TX}), \partial_0 D(\overline{TX})) \to \mathbb{Z}/k$ [25]. The index theorem is given as follows.

Theorem 3. *Let $P \in \Psi^0_{sc,b,\mathbb{Z}/k}(X; E, F)$ and suppose that $\sigma(P)$ and $N_0(P)$ are invertible, then* $\mathrm{ind}(x_1^\beta P x_1^{-\beta}) \mod k = \text{t-ind}(s(P)) \in \mathbb{Z}/k$, $\beta \notin -\mathrm{ImSpec}(\widehat{N_1}(\lambda))$.

We will demonstrate two different proofs for the index theorem. One way is, as in Atiyah-Singer or Freed-Melrose, to prove that analytic index satisfies several axioms. The other way is to reduce to the case of $\partial_0 X = \phi$ by using excision.

This paper is organized as follows. In section 2, we review some preliminary results. In section 3, we define the fibred cusp b-calculus and investigate its properties. We will describe the Fredholm criterion and prove the relative index theorem. As its application, index theorem for a $\mathbb{Z}/k$-manifold (possibly with boundary) is proved in section 4.

2 Preliminaries

2.1 Blow-up of a manifold with corners

In this section, following [19, 9], we define a spherical blow-up and a normal cone deformation of a manifold with corners.

Let X be a manifold with embedded corners. A subset $Y \subset X$ is called a p-submanifold of X, if the inclusion is locally diffeomorphic to the embedding $\mathbb{R}^n \times 0 \times \mathbb{R}^m_{\geq 0} \times 0 \hookrightarrow \mathbb{R}^{n+k} \times \mathbb{R}^{m+l}_{\geq 0}$. Importantly, a tubular neighbourhood $N_+Y \hookrightarrow X$ can be defined for a p-submanifold, where N_+Y is the inward pointing normal bundle. We define the (spherical) blow-up of X at Y by

$$[X; Y] := SN_+Y \amalg X \setminus Y,$$

where SN_+Y is the sphere bundle of the inward pointing normal bundle. $[X; Y]$ has a unique C^∞ structure such that for each tubular neighbourhood $N_+Y \hookrightarrow X$, the map $SN_+Y \times \mathbb{R}_{\geq 0} \to [X; Y]$ defined by

$$(y, \eta, t) \in SN_+Y \times \mathbb{R}_{\geq 0} \mapsto \begin{cases} (y, \eta) \in SN_+Y & (t = 0) \\ (y, t\eta) \in N_+Y \setminus Y \hookrightarrow X \setminus Y & (t \neq 0) \end{cases}$$

is a deffeomorphism onto the image. In this way, $[X, Y]$ is a manifold with corners which has the *new* boundary hypersurface SN_+Y compared to X. The smooth map $\beta := \pi \amalg \iota : [X, Y] \to X$ is called a blow-down map, where $\pi : SN_+Y \to Y \subset X$ is the projection map and $\iota : X \setminus Y \to X$ is the inclusion map.

Similarly, we define the normal cone deformation of X at Y by

$$DNC(X, Y) := N_+Y \amalg X \times \mathbb{R}^*.$$

$DNC(X, Y)$ has a unique C^∞ structure such that for each tubular neighbourhood $N_+Y \hookrightarrow X$, the map $N_+Y \times \mathbb{R} \to DNC(X, Y)$ defined by

$$(y, \eta, t) \in N_+Y \times \mathbb{R} \mapsto \begin{cases} (y, \eta) \in N_+Y & (t = 0) \\ (y, t\eta, t) \in N_+Y \times \mathbb{R}^* \hookrightarrow X \times \mathbb{R}^* & (t \neq 0) \end{cases}$$

is a deffeomorphism onto the image. $Y \times \mathbb{R}$ is naturally embedded into $DNC(X, Y)$ by

$$(y, t) \in Y \times \mathbb{R} \mapsto \begin{cases} y \in Y \subset N_+Y \subset DNC(X, Y) & (t = 0) \\ (y, t) \in Y \times \mathbb{R}^* \subset DNC(X, Y) & (t \neq 0). \end{cases}$$

There is a map $\beta := \pi \amalg \iota : DNC(X, Y) \to X \times \mathbb{R}$, where $\pi : N_+Y \to Y \subset X \times \{0\} \subset X \times \mathbb{R}$ is the projection map and $\iota : X \times \mathbb{R}^* \to X \times \mathbb{R}$ is the inclusion map.

$s \in \mathbb{R}^*$ acts on $(y,\eta) \in N_+Y$ by $(y, s^{-1}\eta)$ and acts on $(x,t) \in X \times \mathbb{R}^*$ by (x, st). These actions smoothly glue and defines the gauge action of $\mathbb{R}^*$ on $DNC(X,Y)$. The relation of the blow-up and the normal cone deformation is given by

$$[X;Y] = \left(DNC_{\geq 0}(X,Y) \setminus Y \times \mathbb{R}_{\geq 0}\right)/\mathbb{R}_{>0},$$

where $DNC_{\geq 0}(X,Y) := N_+Y \amalg X \times \mathbb{R}_{\geq 0}$ is a codimension 0 submanifold of $DNC(X,Y)$.

2.2 Blow-up of a Lie groupoid

In this section, following [9], we discuss a spherical blow-up and a normal cone deformation of a Lie groupoid.

Let $\mathcal{G}$ and X be smooth manifolds with corners, and suppose that the Lie groupoid structure $\mathcal{G} \rightrightarrows X$ is given. In this paper we assume that each fibre of the domain map $\mathcal{G}_x := \{g \in \mathcal{G} \mid d(g) = x\}$ is a (possibly non-compact) manifold *without* boundaries (or corners).

Let $\mathcal{H} \rightrightarrows Y$ be a Lie subgroupoid of $\mathcal{G} \rightrightarrows X$, and suppose that $\mathcal{H} \subset \mathcal{G}$ and $Y \subset X$ is a p-submanifold. Then by naturality of the DNC construction, $DNC(\mathcal{G},\mathcal{H}) \rightrightarrows DNC(X,Y)$ is a Lie groupiod, where the domain map, range map and multiplication map are given by $DNC(d), DNC(r)$ and $DNC(\mu)$.

This procedure cannot be directly carried out for the blow-up case, because the map $Nd : N\mathcal{H} \to NY$ has a kernel in general, so $SN_+\mathcal{H} \to SN_+Y$ cannot be defined. Thus, we define

$$\widetilde{DNC}(\mathcal{G},\mathcal{H}) := DNC(\mathcal{G},\mathcal{H}) \setminus \left(DNC(d)^{-1}(Y \times \mathbb{R}) \cup DNC(r)^{-1}(Y \times \mathbb{R})\right)$$

$$\widetilde{[\mathcal{G};\mathcal{H}]} := \widetilde{DNC}_{\geq 0}(\mathcal{G},\mathcal{H})/\mathbb{R}_{>0}.$$

Then $\widetilde{DNC}(\mathcal{G},\mathcal{H}) \rightrightarrows DNC(X,Y)$ is an open Lie subgroupoid of $DNC(\mathcal{G},\mathcal{H})$, and $\widetilde{[\mathcal{G};\mathcal{H}]} \rightrightarrows [X;Y]$ is a Lie groupoid.

2.3 The definition of b-calculus and Φ-calculus

In this section, we review the definition of the b-calculus [18] and Φ-calculus [17], and discuss its relation to the notion of the pseudodifferential operators on a Lie groupiod [24]. For simplicity, we assume that the section of an appropriate density bundle is fixed and ignore the density term in the fibre integrals. We also only consider $\mathbb{C}$-valued pseudodifferential operators instead of a general vector bundle setting.

Let X be a compact smooth manifold with boundary and fix a boundary defining function x. Define

$$X_b^2 := [X^2; (\partial X)^2].$$

X_b^2 has three boundary hyper surfaces L, R and F, corresponding to $\partial X \times X$, $X \times \partial X$ and $\partial X \times \partial X$ respectively. X is embedded into X_b^2 by a diagonal map. Then, the (small) b-calculus is defined by

$$\Psi_b^m(X) := \{k \in I^m(X_b^2, X) \mid k \text{ vanishes at infinity order at } L \text{ and } R\},$$

where I^m is the space of m-th order one-step polyhomogenious conormal distribution.

To describe the b-calculus in terms of a Lie groupoid as in [23, 9], let X be embedded in a *closed* manifold Z as a codimension 0 submanifold (e.g. Z can be chosen to be the double of X). Then, $Z^2 \rightrightarrows Z$ and $(\partial X)^2 \rightrightarrows \partial X$ are Lie groupoids, thus $\widetilde{[Z^2; (\partial X)^2]} \rightrightarrows [Z, \partial X]$ is a Lie groupoid as described in section 2.2. Because

∂X is cutting Z into two different components, $\widetilde{[Z, \partial X]} = X \amalg X'$ for some manifold with boundary X'. The restriction of $[Z^2; \widetilde{(\partial X)^2}]$ to X is a Lie groupoid which is diffeomorphic to $X^2_b \setminus L \setminus R$:

$$\mathcal{G}_b := [Z^2; \widetilde{(\partial X)^2}]|_X \simeq X^2_b \setminus L \setminus R \rightrightarrows X.$$

This groupoid is called a puff groupoid in [23]. By the definition of the (compactly supported) pseudodifferential operators on a Lie groupoid [24],

$$\Psi^m_c(\mathcal{G}_b) := \{k \in I^m(X^2_b, X) \mid k \text{ vanishes identically on some neighbourhood of } L \text{ and } R\}$$

and $\Psi^m_c(\mathcal{G}_b) \subset \Psi^m_b(X)$ is a subset, which is dense with respect to the operator norm between the Sobolev spaces.

Suppose further that the fibre bundle $\phi : \partial X \to Y$ is given. Define

$$\Phi := \partial X \underset{Y}{\times} \partial X = \{(x, y) \in \partial X \times \partial X \mid \phi(x) = \phi(y)\}$$

then Φ is naturally embedded into X^2_b and we define

$$X^2_\Phi := [X^2_b; \Phi].$$

X^2_Φ has four boundary hypersurfaces L, R, F and FF where FF is the new face corresponding to Φ. X is embedded diagonally into X^2_Φ. The Φ calculus is defined by

$$\Psi^m_\Phi(X) := \{k \in I^m(X^2_\Phi, X) \mid k \text{ vanishes at infinity order at } L,\ R \text{ and } F\}.$$

$\Phi \rightrightarrows \partial X$ is a subgroupoid of $\mathcal{G}_b \rightrightarrows X$ and $\widetilde{[\mathcal{G}_b; \Phi]} \rightrightarrows [X, \partial X] = X$ is a Lie groupoid which is diffeomorphic to $X^2_\Phi \setminus L \setminus R \setminus F$:

$$\mathcal{G}_\Phi := \widetilde{[\mathcal{G}_b; \Phi]} \simeq X^2_\Phi \setminus L \setminus R \setminus F \rightrightarrows X.$$

Its algebra of pseudodifferential operators are given by

$$\Psi^m_c(\mathcal{G}_\Phi) := \{k \in I^m(X^2_\Phi, X) \mid k \text{ vanishes identically on some neighbourhood of } L, R \text{ and } F\}.$$

Thus $\Psi^m_c(\mathcal{G}_\Phi) \subset \Psi^m_\Phi(X)$ is a dense subset.

2.4 The scattering calculus and Atiyah-Singer index theorem

In this section, we review the Φ-calculus of Mazzeo-Melrose [17], and as a special case, discuss the index problem for the scattering calculus.

Let X be a compact manifold with boundary, and a fibre bundle $\phi : \partial X \to Y$ is given. Fix a boundary defining function x of ∂X. Then we can define a calculus of Φ-pseudodifferential operators (or fibred cusp pseudodifferential operatos) $\Psi^*_\Phi(X)$, which is a filtered $*$-algebra. Each element $P \in \Psi^0_\phi(X)$ defines a bounded operator.

$$P : L^2_\Phi(X) \to L^2_\Phi(X)$$

There exists two homomorphisms, a symbol map σ and a normal map N which make the following sequences exact.

$$0 \to \Psi^{-1}_\Phi(X) \to \Psi^0_\Phi(X) \xrightarrow{\sigma} S^0({}^\Phi T^* X) \to 0$$

$$0 \to x\Psi^{-1}_\Phi(X) \to \Psi^0_\Phi(X) \xrightarrow{N} \Psi^0_{\mathrm{sus}({}^\Phi NY)}(\partial X) \to 0$$

where ${}^{\Phi}T^*X$ is a vector bundle over X which is non-canonically isomorphic to T^*X, ${}^{\Phi}NY$ is a vector bundle over Y which is non-canonically isomorphic to $\mathbb{R} \oplus T^*Y$ and $\Psi^0_{\mathrm{sus}({}^{\Phi}NY)}(\partial X)$ is a space of ${}^{\Phi}NY$-suspended pseudodifferential operators on ∂X of order 0.

We say $P \in \Psi^0_{\Phi}(X)$ is elliptic if $\sigma(P)$ is invertible, and fully elliptic in addition $N(P)$ is invertible. It is shown that $P : L^2_{\Phi}(X) \to L^2_{\Phi}(X)$ is Fredholm if and only if P is fully-elliptic.

In an extreme case when ϕ is an identity map $\phi : \partial X \to \partial X$, the calculus $\Psi^0_{sc}(X) := \Psi^0_{\Phi}(X)$ is called scattering calculus. In this case, as outlined in [21], the index problem of fully elliptic operator is reduced to the Atiyah-Singer index theorem. Let us explain it briefly. We can define a map

$$\{P \in \Psi^0_{sc}(X) \mid P \text{ is fully elliptic}\}/\text{homotopy} \to K(D(TX), \partial D(TX)),$$

where $\partial D(TX) = D(TX|_{\partial X}) \cup S(TX)$ and D or S means a disk or a sphere bundle of the vector bundle. The composition of this map and the topological index map [5] t-ind : $K(D(TX), \partial D(TX)) \to \mathbb{Z}$ gives the index of fully elliptic operator.

2.5 Operator valued logarithmic residual theorem

In this section, following [11, 19], we discuss about the operator valued logarithmic residual theorem.

Let $\Omega \subset \mathbb{C}$ be a bounded domain. For a Hilbert space H let $\mathcal{L}(H)$ be an algebra of bounded linear operators. Let $A : \overline{\Omega} \to \mathcal{L}(H)$ be a continuous map which is holomorphic on Ω Suppose that $A(\lambda)$ is invertible for $\lambda \in \partial\Omega$ and is Fredholm for $\lambda \in \Omega$. Then the map $A^{-1}(\lambda) : \partial\Omega \to \mathcal{L}(H)$ meromorphically extends to Ω. Define

$$\mathrm{Spec}(A) := \{\lambda \in \Omega \mid A(\lambda) \text{ is not invertible }\},$$

then it is a finite set as poles of a meromorphic function are discrete. For each $\lambda_0 \in \mathrm{Spec}(A)$, define a finite dimensional vector space

$$F(A, \lambda_0) := \{f \in \mathcal{M}_{\lambda_0}(H)/\mathcal{O}_{\lambda_0}(H) \mid \ A(\lambda)f(\lambda) \text{ is holomorphic at } \lambda_0 \ \},$$

where $\mathcal{M}_{\lambda_0}(H)$ and $\mathcal{O}_{\lambda_0}(H)$ are the space of H-valued meromorphic and holomorphic germs at λ_0. This set is well-defined because if f is holomophic around λ_0, $A(\lambda)f(\lambda)$ is holomorphic around λ_0.

The operator valued logarithmic residual theorem [11] asserts that

Theorem 4 (Gohberg and Sigal [11])**.**

$$\frac{1}{2\pi i}\mathrm{tr}\oint_{\partial\Omega} A^{-1}(\lambda)\frac{\partial A(\lambda)}{\partial\lambda}d\lambda = \sum_{\lambda_0 \in \mathrm{Spec}(A)} \dim F(A, \lambda_0).$$

Define $\Omega^* := \{\overline{\lambda} \mid \lambda \in \Omega\}$, then the map $A^* : \overline{\Omega^*} \to \mathcal{L}(H)$ defined by $A^*(\lambda) := (A(\overline{\lambda}))^*$ is continuous on $\overline{\Omega^*}$ and holomorphic on Ω^*. Define a sesquilinear form by

$$(f, g) \in F(A, \lambda_0) \times F(A^*, \overline{\lambda_0}) \mapsto B(f, g) := \frac{1}{2\pi i}\oint_C < A(\lambda)f(\lambda), g(\overline{\lambda}) > d\lambda \in \mathbb{C},$$

where C is a small circle around λ_0. If f or g is holomorphic at λ_0 or $\overline{\lambda_0}$, then $< A(\lambda)f(\lambda), g(\lambda) > = < f(\lambda), A^*(\overline{\lambda})g(\overline{\lambda}) >$ is holomorphic at λ_0, thus this sesquilinear form is well-defined. As discussed in [19], this sesquilinear form is non-degenerate.

Fix g and suppose that $B(f,g)=0$ for all f, then for any holomorphic function $u(\lambda)$ around λ_0, $A^{-1}(\lambda)u(\lambda)\in F(A,\lambda_0)$ and the assumption implies that

$$\frac{1}{2\pi i}\oint_C < u(\lambda), g(\overline{\lambda}) > d\lambda = 0.$$

As u is arbitrary, g is holomorphic at $\overline{\lambda_0}$. Similar result can be proved when f and g are interchanged, thus the form is non-degenerate.

2.6 The relative index theorem for b-calculus and the mod k index theorem

In this section, following [19], we briefly review the basic property of b-calculus and the relative index theorem for b-calculus. The application of the relative index theorem for $\mathbb{Z}/k$-manifold [10] is also discussed.

Let X be a compact manifold with boundary, and x be its boundary defining function. Then we can define a small calculus of b-pseudodifferential operators. Each element $P\in\Psi_b^0(X)$ defines a bounded operator.

$$P: L_b^2(X)\to L_b^2(X)$$

There are two important homomorphisms:the symbol map σ and the normal map N (or the indicial map). These maps are $*$-homomorphisms of filtered algebras which make the following sequences exact.

$$0\to\Psi_b^{-1}(X)\to\Psi_b^0(X)\xrightarrow{\sigma}S^0({}^bT^*X)\to 0$$

$$0\to x\Psi_b^{-1}(X)\to\Psi_b^0(X)\xrightarrow{N}\Psi_{b,\mathrm{inv}}^0(\widetilde{\partial X})\to 0$$

where ${}^bT^*X$ is a vector bundle over X which is non-canonically isomorphic to T^*X, and $S^0({}^bT^*X)$ is a space of symbols of order 0 over ${}^bT^*X$. $\widetilde{\partial X}$ is a compactification of the positive normal bundle of $\partial X\hookrightarrow X$ which is non-canonically diffeomorphic to $\partial X\times[0,1]$, and $\Psi_{b,\mathrm{inv}}^m(\widetilde{X})$ is an algebra of b-pseudodifferential operators on $\widetilde{X}$ which are invariant under the action of $(0,\infty)$ on $\widetilde{X}$.

Let $P\in\Psi_b^0(X)$. P is called elliptic if its symbol is invertible. The Mellin transform of the normal operator of P

$$\widehat{N}(P):\mathbb{C}\to\Psi^0(\partial X)$$

is an entire holomorphic function, and if P is elliptic, $\widehat{N}(P)(\lambda)$ is Fredholm for all $\lambda\in\mathbb{C}$.

Theorem 5 (Melrose [19])**.** *Let $P\in\Psi_b^0(X)$, then P defines a bounded linear map $L_b^2(X)\to L_b^2(X)$. This map is Fredholm if and only if P is elliptic and $\widehat{N}(P)(\lambda)$ is invertible for all $\lambda\in\mathbb{R}$*

Let $P\in\Psi_b^0(X)$ and $\beta\in\mathbb{R}$, then $x^\beta P x^{-\beta}\in\Psi_b^0(X)$ and $\widehat{N}(x^\beta P x^{-\beta})(\lambda)=\widehat{N}(P)(\lambda+i\beta)$. Thus, theorem 5 ensures that $x^\beta P x^{-\beta}$ is Fredholm if and only if $\beta\notin-\mathrm{ImSpec}(\widehat{N}(P))$, where Im is the imaginary part map. The relative index theorem asserts that:

Theorem 6 (Melrose [19]). *Let $P \in \Psi_b^0(X)$ be elliptic and $\beta_1, \beta_2 \notin -\mathrm{ImSpec}(\widehat{N}(P)), \beta_2 > \beta_1$. Then,*

$$\mathrm{ind}(x^{\beta_1}Px^{-\beta_1}) - \mathrm{ind}(x^{\beta_2}Px^{-\beta_2}) = \frac{1}{2\pi i}\mathrm{tr}\oint \widehat{N}(P)^{-1}(\lambda)\frac{\partial \widehat{N}(P)}{\partial\lambda}(\lambda)d\lambda, \tag{5}$$

where the path of the integral is chosen so that its interior contains all poles of $\widehat{N}(P)^{-1}(\lambda)$ such that $\beta_1 < -\mathrm{Im}(\lambda) < \beta_2$.

Proof. We only demonstrate the outline of the proof. We can assume that $\beta_2 - \beta_1 < 1$. Let $u \in \ker(x^{\beta_1}Px^{-\beta_1})$, i.e. $u \in x^{\beta_1}L_b^2(X)$ and $Pu = 0$. By hypoellipticity of P, $u \in x^{\beta_1}H_b^\infty(X)$. Let ϕ be a cut-off function, which is supported in the neighbour hood of ∂X and identically 1 around ∂X. Then, the Mellin transform of $\widehat{\phi u}(\lambda)$ is holomorphic on $\mathrm{Im}(\lambda) > -\beta_1$. The condition $Pu = 0$ and $\beta_2 - \beta_1 < 1$ implies that $P(\lambda)\widehat{\phi u}(\lambda)$ is holomorphic up to $\mathrm{Im}(\lambda) > -\beta_2$. In particular, $\widehat{\phi u}(\lambda)$ can be meromorphically extended to $\mathrm{Im}(\lambda) > -\beta_2$. Thus, for each $\lambda_0 \in \mathrm{Spec}(\widehat{N}(P))$, $-\beta_2 < \mathrm{Im}\lambda_0 < -\beta_1$. The map

$$\Gamma : u \in \ker(x^{\beta_1}Px^{-\beta_1}) \mapsto \bigoplus_{\lambda_0}(\text{germ of } \widehat{\phi u} \text{ at } \lambda_0) \in \bigoplus_{\lambda_0} F(\mathrm{Spec}(\widehat{N}(P)), \lambda_0)$$

can be defined. $u \in x^{\beta_2}H_b^\infty(X) \subset x^{\beta_1}H_b^\infty(X)$ if and only if $\widehat{\phi u}(\lambda)$ can be holomophically extended to $\mathrm{Im}(\lambda) > -\beta_2$ if and only if $\Gamma(u) = 0$. In particular,

$$\dim \mathrm{Image}(\Gamma) = \dim\ker(x^{\beta_1}Px^{-\beta_1}) - \dim\ker(x^{\beta_2}Px^{-\beta_2}).$$

Similar discussion for P^* shows that the map

$$\Gamma^* : v \in \ker(x^{-\beta_2}P^*x^{\beta_2}) \mapsto \bigoplus_{\lambda_0}(\text{germ of } \widehat{\phi v} \text{ at } \overline{\lambda_0}) \in \bigoplus_{\lambda_0} F(\mathrm{Spec}(\widehat{N}(P^*)), \overline{\lambda_0})$$

can be defined and

$$\begin{aligned}\dim \mathrm{Image}(\Gamma^*) &= \dim\ker(x^{-\beta_2}P^*x^{\beta_2}) - \dim\ker(x^{-\beta_1}P^*x^{\beta_1})\\ &= \dim\mathrm{coker}\,(x^{\beta_2}Px^{-\beta_2}) - \dim\mathrm{coker}\,(x^{\beta_1}Px^{-\beta_1}).\end{aligned}$$

Then, $\mathrm{Image}(\Gamma)$ and $\mathrm{Image}(\Gamma^*)$ are annihilators of each other with respect to the sesquilinear form b defined in section 2.5 because

$$\begin{aligned}B(\Gamma(u), \Gamma^*(v)) &= \frac{1}{2\pi i}\oint < \widehat{N}(P)(\lambda)\widehat{\phi u}(\lambda), \widehat{\phi v}(\overline{\lambda}) > d\lambda\\ &= \frac{1}{i}\int_{\partial X\times[0,\infty]} < N(P)\phi u, \phi v > - < \phi u, N(P^*)\phi v >\\ &= \frac{1}{i}\int_X < Pu, v > - < u, P^*v > .\end{aligned}$$

In particular,

$$\begin{aligned}\sum_{\lambda_0}\dim F(\mathrm{Spec}(\widehat{N}(P)), \lambda_0) &= \dim\mathrm{Image}(\Gamma) + \dim\mathrm{Image}(\Gamma^*) =\\ &= \mathrm{ind}(x^{\beta_1}Px^{-\beta_1}) - \mathrm{ind}(x^{\beta_2}Px^{-\beta_2})\end{aligned}$$

combined with the operator valued logarithmic residual theorem, the proof is completed. ☐

Recall that a "closed" $\mathbb{Z}/k$-manifold X is a manifold with boundary such that ∂X is a disjoint union of k copies of a closed manifold Z, $\partial X = kZ$. Freed and Melrose [10] introduced a subalgebra of $\Psi_b^0(X)$, consists of $P \in \Psi_b^0(X)$ for which $N(P)$ can be written as a direct sum of k copies of some operator. For such an operator P, $\operatorname{ind}(x^\beta P x^{-\beta}) \mod k \in \mathbb{Z}/k$ is independent of β because right hand side of the formula (5) is always a multiple of k. They proved the index theorem which asserts that this $\mathbb{Z}/k$-valued index can be written in terms of topological K-theory.

2.7 The topological index of a $\mathbb{Z}/k$-manifold

In this section, following [7, 10, 25], we discuss about the topological index of a $\mathbb{Z}/k$-manifold (possibly with boundary).

Let X be a compact smooth manifold with corners with two boundary hypersurfaces $\partial_0 X, \partial_1 X$, i.e. $\angle X = \partial_0 X \cap \partial_1 X$ and $\partial X = \partial_0 X \cup \partial_1 X$. X is called a $\mathbb{Z}/k$-manifold if a diffeomorphism $\partial_1 X \simeq kZ$ is given, where $k > 0$ is an integer, Z is a manifold with boundary and kZ is a disjoint union of k copies of Z. X is called closed as a $\mathbb{Z}/k$-manifold when $\partial_0 X = \phi$. Note that if X is a $\mathbb{Z}/k$-manifold, $\partial_0 X$ also is a $\mathbb{Z}/k$-manifold. Define $\overline{X} = X/\sim$ to be a quotient space obtained by identifying k-copies in $\partial_1 X$.

Lemma 1. *There is an exact sequence*

$$\to K^{*-1}(Z, \partial Z) \to K^*(X, \partial X) \to K^*(\overline{X}, \partial_0 \overline{X}) \to K^*(Z, \partial Z) \to \tag{6}$$

Proof. Consider the exact sequence for the triple $(\overline{X}, \partial\overline{X}, \partial_0\overline{X})$. Then $K^*(\overline{X}, \partial\overline{X}) \simeq K^*(X, \partial X)$ and by excision, $K^*(\partial\overline{X}, \partial_0\overline{X}) \simeq K^*(Z, \partial Z)$ □

A vector bundle $p : E \to X$ is called a $\mathbb{Z}/k$-vector bundle if an isomorphism $E|_{\partial_1 X} \simeq kE_Z$ is given for a vector bundle $E_Z \to Z$ and $p|_{\partial_1 X} = kp$. If $E \to X$ is a $\mathbb{Z}/k$-vector bundle, the quotient $\overline{E} \to \overline{X}$ is a vector bundle (in the usual sense). The disk bundle of E has a $\mathbb{Z}/k$-structure defined by $\partial_0 D(E) := S(E) \cup D(E|_{\partial_0 X})$, $\partial_1 D(E) := D(E|_{\partial_1 X}) \simeq kD(E_Z)$ (strictly speaking, we must smooth the corner $S(E) \cap D(E_{\partial_0 X})$ but it will be omitted in the rest of this paper). TX has a $\mathbb{Z}/k$-vector bundle structure unique up to homotopy. We call X a *Spin*^{c} $\mathbb{Z}/k$-manifold if TX has a *Spin*^{c} structure and is compatible with $\mathbb{Z}/k$ structure.

Definition 1. *Let X, Y be a compact $\mathbb{Z}/k$-manifold and $f : X \to Y$ be a smooth map, then f is called a $\mathbb{Z}/k$ map, if the following conditions are satisfied.*

- $f(\partial_1 X) \subset (\partial_1 Y)$
- $f|_{\partial_1 X} = kf_Z$ *for a smooth map* $f_Z : Z_X \to Z_Y$.
- *The map between normal bundles* $df : \nu(\partial_1 X) \to \nu(\partial_1 Y)$ *is an isomorphism on each fibre.(transversality)*

In obvious way, we can define the category of $\mathbb{Z}/k$-manifolds and $\mathbb{Z}/k$-maps.

Next, we define a "model space" L, which is a terminal object of a homotopy category of $\mathbb{Z}/k$-manifolds. Let $l > 0$ be an integer and take k point in $\operatorname{int} D^l$ then $L := D^l \times I$ is a $\mathbb{Z}/k$-manifold defined by $\partial_1 L :=$ "small neighbourhood of k points" $\simeq kD^l$, $\partial_0 L := \partial L \setminus \operatorname{int} \partial_1 L$. By the exact sequence (6), we can compute that

$$K^*(\overline{L}, \partial_1 \overline{L}) \simeq \begin{cases} \mathbb{Z}/k & (* \equiv l+1 \mod 2) \\ 0 & (* \equiv l \mod 2). \end{cases}$$

Lemma 2. *L is a terminal object of a homotopy category of $\mathbb{Z}/k$-manifolds. Namely, let X be a $\mathbb{Z}/k$-manifold, then there exists a $\mathbb{Z}/k$-map $f : X \to L$ which is unique up to homotopy.*

Proof. To prove existence, let x_1 be a boundary defining function of $\partial_1 X$ and $a_1, \dots a_k \in \partial_1 L$ be k points. The neighbourhood of $\partial_1 X$ is diffeomorphic to $kZ \times [0, 1)$. On n-th copy of Z, define $f(y, t) = (a_n, t) \in L$ for $(y, t) \in Z \times [0, 1)$. By using cut-off function it can be easily extended to whole X. Thus existence is proved.

For uniqueness, Let $f, g : X \to L$ be $\mathbb{Z}/k$-maps, then $tf+(1-t)g$ is also a $\mathbb{Z}/k$-map, because L is convex and the transversality is closed under convex combinations. $\square$

As in the case of closed manifold, we can define Gysin maps for $Spin^c$ $\mathbb{Z}/k$-manifolds. Detailed arguments can be found in [25].

Definition 2. *Let X, Y be $Spin^c$ $\mathbb{Z}/k$-manifolds, $f : X \to Y$ be a $\mathbb{Z}/k$-map. Then there is a map $f_! : K^*(\overline{X}, \partial_0\overline{X}) \to K^{*+\dim Y - \dim X}(\overline{Y}, \partial_0\overline{Y})$ which is characterized by following properties.*

- *If $f : X \to Y$ and $g : Y \to W$ are $\mathbb{Z}/k$-maps, then $(g \circ f)_! = g_! \circ f_!$*
- *If $f : X \to Y$ is an embedding, $f_!$ is a composition of the Thom isomorphism $K^*(\overline{X}, \partial_0\overline{X}) \simeq K^{*+\dim Y - \dim X}(\overline{D(\nu(X))}, \partial_0\overline{D(\nu(X))})$ and $K^{*+\dim Y - \dim X}(\overline{D(\nu(X))}, \partial_0\overline{D(\nu(X))}) \to K^{*+\dim Y - \dim X}(\overline{Y}, \partial_0\overline{Y})$.*
- *If $X = D(E_Y)$ for a $\mathbb{Z}/k$ vector bundle $p : E_Y \to Y$ and $f = p|_X$, then $f_!$ is an (inverse of) Thom isomorphism.*

We can now define a topological index as following. Let X be an even-dimensional $Spin^c$-manifold and $f : X \to L$ be a $\mathbb{Z}/k$-map which is unique up to homotopy, then $f_! : K(\overline{X}, \partial_0\overline{X}) \to K^{\dim L}(\overline{L}, \partial_0\overline{L}) = \mathbb{Z}/k$ and we define

$$\text{t-ind}^c_X := f_! : K(\overline{X}, \partial_0\overline{X}) \to \mathbb{Z}/k.$$

For a general $\mathbb{Z}/k$-manifold X, $D(TX)$ is a $Spin^c$ $\mathbb{Z}/k$-manifold and we can define

$$\text{t-ind}_X := \text{t-ind}^c_{D(TX)} : K(\overline{D(TX)}, \partial_0\overline{D(TX)}) \to \mathbb{Z}/k$$

As in the case of Atiyah-Singer, topological index can be characterized by following axiomatic properties. Suppose that for each $\mathbb{Z}/k$-manifold X, $\mathrm{ind}_X : K(\overline{D(TX)}, \partial_0\overline{D(TX)}) \to \mathbb{Z}/k$ is given.

Axiom 1. *The following diagram commutes.*

$$\begin{tikzcd} K(X, \partial X) \arrow[r] \arrow[d, "\mathrm{ind}^{AS}_X"] & K(\overline{X}, \partial_0\overline{X}) \arrow[d, "\mathrm{ind}_X"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/k \end{tikzcd}$$

where ind^{AS}_X is an Atiyah-Singer index map.

Axiom 2. *Let $\iota : X \to Y$ be a codimension zero embedding and $r : K(\overline{X}, \partial_0\overline{X}) \to K(\overline{Y}, \partial_0\overline{Y})$ be a naturally defined map, then $\mathrm{ind}_X = \mathrm{ind}_Y \circ r$*

Let H be a compact Lie group and $P \to X$ be a principal H bundle compatible with a $\mathbb{Z}/k$-structure. Suppose that a compact H manifold F is given and define $Z := P \underset{H}{\times} F$. Then as in [5, 10, 25], we can define a multiplication

$$K(\overline{D(TX)}, \partial_0 \overline{D(TX)}) \otimes K_H(D(TF), \partial D(TF)) \to K(\overline{D(TZ)}, \partial_0 \overline{D(TZ)}).$$

Also there is a map

$$\mu : R(H) \to K(X)$$

defined by $\mu(V) := P \underset{H}{\times} V$.

Axiom 3. *Let H, P, Z as above, then for $a \in K(\overline{D(TX)}, \partial_0 \overline{D(TX)})$, $b \in K_H(D(TF), \partial D(TF))$,*

$$\mathrm{ind}_Z(ab) = \mathrm{ind}_X(a \cdot \mu(\mathrm{ind}_F^{AS}(b)))$$

Proposition 1. *If ind_X satisfies Axiom 1, 2, 3, then $\mathrm{ind}_X = \text{t-ind}_X$*

In [10],the index theorem for a closed $\mathbb{Z}/k$-manifold was proved by showing that analytic index satisfies three similar axioms. In [25], the index theorem for G-equivariant $\mathbb{Z}/k$-manifold was proved similarly. In this paper, we will prove the index theorem for a $\mathbb{Z}/k$-manifold (possibly with boundary) in two different ways, one is to use Proposition 1, the other is to reduce to the case of a closed $\mathbb{Z}/k$-manifold.

3 The fibred cusp b-calculus

In this section, based on the discussion in [17], we define the fibred cusp b-pseudodifferential operators.

3.1 Settings

Let X be a smooth compact manifold with corners which has two embedded boundary hypersurfaces. Following relations hold where $\partial_0 X$ and $\partial_1 X$ are the boundary hypersurfaces.

$$\partial X = \partial_0 X \cup \partial_1 X \ , \ \angle X = \partial_0 X \cap \partial_1 X$$

Suppose a fibre bundle $\phi : \partial_0 X \to Y$ is given, where Y is a compact manifold with boundary and each fibre Z is a compact manifold without boundary. Suppose further ϕ maps the boundary to the boundary, thus restricts to a fibre bundle $\phi|_{\angle X} : \angle X \to \partial Y$, and the following diagram commutes.

$$\begin{tikzcd} \angle X \arrow[r, hook] \arrow[d, "\phi_{\angle X}"] & \partial_0 X \arrow[d, "\phi"] \\ \partial Y \arrow[r, hook] & Y \end{tikzcd}$$

We say X is a manifold with fibred boundary in this case.

The whole following discussion can be applied in the case when each fibre $\phi^{-1}(y)$ varies on connected components of Y, but for simplicity, we assume they are all diffeomorphic to a single closed manifold Z.

Take any boundary defining functions $x_0, x_1 \in C^\infty(X)$ for $\partial_0 X, \partial_1 X$ respectively. We assume that $x_1|_{\partial_0 X}$ is constant on each fibre of ϕ, which is always possible by taking boundary defining function of $\partial Y \subset Y$ and pulling it back to $\partial_0 X$ and extend it to X.

To describe local properties of X, it is convenient to introduce a "model space" M as follows.

$$M := \{(x_0, x_1, y, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^{k-1} \times \mathbb{R}^l\},\ \partial_0 M := \{x_0 = 0\},\ \partial_1 M := \{x_1 = 0\},$$

$$N := \{(x_1, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{k-1}\},\ \phi : \partial_1 M \to N,$$

where ϕ is the projection. Then M is a manifold with fibred boundary (without a compactness assumption).

For any manifold with fibred boundary X and $p \in X$, their exists an open neighbourhood $p \in U$ and diffeomorphism onto open subset of M which preserves the structure of manifold with fibred boundary. Where "preserving the structure" means it preserves ∂_0, ∂_1 and ϕ, and when $p \in \partial_0 X$ or $p \in \partial_1 X$, it preserves the function x_0 or x_1.

We define fibred cusp b-vector fields on X as follows:

$$\mathcal{V}_{\Phi,b}(X) = \{V \in \mathcal{V}(X) \mid V x_0 \in x_0^2 C^\infty(X), V|_{\partial_0 X} \text{ is tangent to the fibres of } \phi,\\ V|_{\partial_1 X} \text{ is tangent to } \partial_1 X\ \}, \quad (7)$$

where $\mathcal{V}(X)$ is the space of smooth vector fields on X.

When $X = M$, it is straightforward to check $\mathcal{V}_{\Phi,b}(M)$ is freely generated by $x_0^2 \dfrac{\partial}{\partial x_0}, x_0 x_1 \dfrac{\partial}{\partial x_1}, x_0 \dfrac{\partial}{\partial y_i}, \dfrac{\partial}{\partial z_j}$ over $C^\infty(M)$. Thus their exist a smooth vector bundle ${}^{\Phi,b}TX$ over X and the isomorphism $\Gamma(X, {}^{\Phi,b}TX) \simeq \mathcal{V}_{\Phi,b}(X)$.

The map ${}^{\Phi,b}TX \to TX$, induced by $\mathcal{V}^{\Phi,b}(X) \hookrightarrow \mathcal{V}(X)$ defines the Lie algebroid structure on X. Such a structure is called a Lie structure at infinity in [3] and [4].

Although the space $\mathcal{V}_{\Phi,b}(X)$ depends on the choice of boundary defining function x_0 of $\partial_0 X$, the full information of x_0 is not needed to determine $\mathcal{V}_{\Phi,b}(X)$, and we can prove the following lemma by direct calculation.

Lemma 3. *Two choices of boundary defining function of $\partial_0 X$, x_0 and $\tilde{x}_0$ defines a same space $\mathcal{V}_{\Phi,b}(X)$ if and only if $\tilde{x}_0 / x_0 = \alpha \in C^\infty(X)$ satisfies $\alpha|_{\partial_0 X} = \phi^* \gamma$ for some $\gamma \in C^\infty(Y)$.*

On the set

$$B := \{x_0 \mid x_0 \text{ is a boundary defining function of } \partial_0 X\},$$

the group

$$G := \{\alpha \in C^\infty(X) \mid \alpha > 0 \text{ and } \alpha|_{\partial_0 X} \text{ is constant on each fibre of } \phi\ \}$$

acts by multiplication. The above lemma implies that fixing $\mathcal{V}_{\Phi,b}(X)$ is equivalent to fix the G orbit in b.

From now on, we fix the Lie algebroid $\mathcal{V}_{\Phi,b}(X)$, or equivalently, the G orbit in b.

Next we define the vector bundle over Y by using the local coordinate:

$${}^{\Phi,b}NY := \mathrm{span}\{x_0^2 \frac{\partial}{\partial x_0}, x_0 x_1 \frac{\partial}{\partial x_1}, x_0 \frac{\partial}{\partial y_i}\}.$$

By definition,

$$\phi^*({}^{\Phi,b}NY) = \ker({}^{\Phi,b}TX|_{\partial_0 X} \to TX|_{\partial_0 X}).$$

We want to describe this vector bundle without using coordinate. Note that if we fix x_0, ${}^{\Phi,b}NY$ is clearly isomorphic to $\underline{\mathbb{R}} \oplus {}^bTY$. If $\tilde{x}_0 = \alpha x_0$ is another choice of boundary defining function where $\alpha \in G$. The coordinate exchange is given as following.

$$x_0^2 \frac{\partial}{\partial x_0} = \frac{1}{\alpha} \cdot \tilde{x}_0^2 \frac{\partial}{\partial \tilde{x}_0}$$

$$x_0 x_1 \frac{\partial}{\partial x_1} = \frac{1}{\alpha^2} \frac{\partial \alpha}{\partial x_1} \cdot \tilde{x}_0^2 x_1 \frac{\partial}{\partial \tilde{x}_0} + \frac{1}{\alpha} \cdot \tilde{x}_0 x_1 \frac{\partial}{\partial x_1}$$

$$x_0 \frac{\partial}{\partial y_i} = \frac{1}{\alpha^2} \frac{\partial \alpha}{\partial y_i} \cdot \tilde{x}_0^2 \frac{\partial}{\partial \tilde{x}_0} + \frac{1}{\alpha} \cdot \tilde{x}_0 \frac{\partial}{\partial y_i}$$

The group G acts on $\underline{\mathbb{R}} \oplus {}^bTY$ by

$$(\alpha, \tau, \eta) \in G \times (\underline{\mathbb{R}} \oplus {}^bTY) \mapsto (\tau/\gamma + d\gamma \cdot \eta/\gamma^2, \eta/\gamma) \in \underline{\mathbb{R}} \oplus {}^bTY,$$

where $\gamma \in C^\infty(Y)$ is defined by $\phi^*\gamma = \alpha^{-1}|_{\partial_0 X}$ and $d\gamma \cdot \eta$ is the paring of T^*Y and ${}^bTY$. By the above formula of coordinate exchange, the map

$$(\tau, \sigma_1 x_1 \frac{\partial}{\partial x_1}, \eta_i \frac{\partial}{\partial y_i}, x_0) \in (\underline{\mathbb{R}} \oplus {}^bTY) \underset{G}{\times} B \mapsto (\tau x_0^2 \frac{\partial}{\partial x_0}, \sigma_1 x_0 x_1 \frac{\partial}{\partial x_1}, x_0 \frac{\partial}{\partial y_i}) \in {}^{\Phi,b}NY,$$

is a well-defined isomorphism. This gives a coordinate-free definition of ${}^{\Phi,b}NY$.

Let X_b^2 be a smooth manifold obtained by blowing up $\partial_0 X \times \partial_0 X$ and $\partial_1 X \times \partial_1 X$ in $X \times X$.

$$X_b^2 = [X^2; (\partial_0 X)^2, (\partial_1 X)^2] \ , \beta_b : X_b^2 \to X^2$$

The order of two blow-ups does not matter because $\partial_0 X \times \partial_0 X$ and $\partial_1 X \times \partial_1 X$ intersects transversely (see [19]). X_b^2 has 6 boundary hypersurfaces L_0, F_0, R_0, L_1, F_1 and R_1, which corresponds to $\partial_0 X \times X, \partial_0 X \times \partial_0 X, X \times \partial_0 X, \partial_1 X \times X, \partial_1 X \times \partial_1 X$, and $X \times \partial_1 X$ respectively, where L, F or R stands for left, front or right.

Define $\Phi := \partial_0 X \underset{Y}{\times} \partial_0 X = \{(w, w') \in \partial_0 X \times \partial_0 X \mid \phi(w) = \phi(w')\} \subset (\partial_0 X)^2$. Then Φ can be lifted to $\Phi_b \subset (\partial_0 X)_b^2$. The smooth function $x_0'/x_0 : X_b^2 \to [0, \infty]$ is independent of the choice of x_0 when restricted to F_0 and there is a diffeomorphism $(\partial_0 X)_b^2 \simeq \{x_0'/x_0 = 1\} \cap F_0$.

By regarding Φ_b as a submanifold of $\{x_0'/x_0 = 1\} \cap F_0$, we define

$$X_{\Phi,b}^2 := [X_b^2; \Phi_b] \ , \beta_\phi : X_{\Phi,b}^2 \to X_b^2 \ , \beta := \beta_b \circ \beta_\phi : X_{\Phi,b}^2 \to X^2.$$

$X_{\Phi,b}^2$ has 7 boundary hypersurfaces $L_0, F_0, R_0, L_1, F_1, R_1$ and FF_0 where the new hypersurface FF_0 corresponds to Φ_b.

For a model space M, we describe these blowing-ups explicitly using coordinates.

$$M^2 = \{(x_0, x_1, y, z, x_0', x_1', y', z') \mid x_0 \geq 0, x_1 \geq 0, x_0' \geq 0, x_1' \geq 0\}$$

First, the coordinate on $M_b^2 \setminus L_0 \setminus L_1$ is given as following.

$$\{(x_0, x_1, y, z, s_0, s_1, y', z') \mid x_0 \geq 0, x_1 \geq 0, s_0 \geq 0, s_1 \geq 0\}$$

$$s_0 = \frac{x_0'}{x_0}, s_1 = \frac{x_1'}{x_1}$$

In this coordinate, $\Phi_b = \{x_0 = 0, s_0 = 1, s_1 = 1, y = y'\}$. Thus, we can give an explicit coordinate on $M_{\Phi,b}^2 \setminus L_0 \setminus R_0$ around FF_0 as following.

$$\{(x_0, x_1, y, z, u_0, u_1, v, w) \mid x_0 \geq 0, x_1 \geq 0, x_0 \cdot \sqrt{1 + u_0^2 + u_1^2 + v^2} < 1\}$$

$$u_0 = \frac{1 - s_0}{x_0} = \frac{x_0 - x_0'}{x_0^2}, \ u_1 = \frac{1 - s_1}{x_0} = \frac{x_1 - x_1'}{x_0 x_1}, \ v = \frac{y - y'}{x_0}, \ w = z - z'$$

3.2 Lie groupoid structure

In this section, by using the above local cooridnate, we give an explicit description of the structure of the Lie group associated to the fibred cusp b-calculus.

Proposition 2. *$\mathcal{G} := X^2_{\Phi,b} \setminus L_0 \setminus R_0 \setminus L_1 \setminus R_1$ has a structure of a Lie groupoid by extending the Lie groupoid structure on $\operatorname{int} X^2$. The set of units of $\mathcal{G}$ is the lifted diagonal $\Delta_{\Phi,b} \subset X^2_{\Phi,b}$, and the associated Lie algebroid $A(\mathcal{G})$ is ${}^{\Phi,b}TX$.*

Proof. Recall that the Lie algebroid structure of $\operatorname{int} X^2$ is given as follows:

$$d(x,x') = x',\ r(x,x') = x$$

$$\mu((x,x'),(x',x'')) = (x,x'')$$
$$u(x) = (x,x)$$
$$\iota(x,x') = (x',x)$$

where d, r, μ, u and ι are domain, range, multiplication, unit, and inversion map respectively.

By using the coordinate on $X^2_{\Phi,b}$ described above, we can compute:

$$x'_0 = x_0 - x_0^2 u_0,\ x'_1 = x_1 - x_0 x_1 u_1,\ y' = y - x_0 v,\ z' = z - w.$$
$$\frac{x_0 - x''_0}{x_0^2} = u_0 + \frac{x'^2_0}{x_0^2} u'_0,\ \frac{x_1 - x''_1}{x_0 x_1} = u_1 + \frac{x'_0 x'_1}{x_0 x_1} u'_1,\ \frac{y - y''}{x_0} = v + \frac{x'_0}{x_0} v',\ z - z'' = w + w', \tag{8}$$

where
$$u'_0 = \frac{x'_0 - x''_0}{x'^2_0},\ u'_1 = \frac{x'_1 - x''_1}{x'_0 x'_1},\ v' = \frac{y' - y''}{x'_0},\ w' = z' - z''.$$

Because $x'_0/x_0 = 1 - x_0 u_0$ and $x'_1/x_1 = 1 - x_0 u_1$ are smooth on $\mathcal{G}$, d, r, μ, u and ι can be extended to $\mathcal{G}$. These maps satisfy the axiom of the Lie groupoid as they satisfy on the dense subset $\operatorname{int} X^2$.

Clearly, the set of units of $\mathcal{G}$ is the lifted diagonal

$$\Delta_{\Phi,b} = \{(x_0, x_1, y, z, u_0, u_1, v, w) \mid u_0 = u_1 = v = w = 0\}.$$

By definition, $A(\mathcal{G})$ is spanned by the restrictions of $\partial/\partial u_0, \partial/\partial u_1, \partial/\partial v$ and $\partial/\partial w$ to $\Delta_{\Phi,b}$.

$$\frac{\partial}{\partial u_0} = -x_0^2 \frac{\partial}{\partial x'_0},\ \frac{\partial}{\partial u_1} = -x_0 x_1 \frac{\partial}{\partial x'_1},\ \frac{\partial}{\partial v_i} = -x_0 \frac{\partial}{\partial y'},\ \frac{\partial}{\partial w} = -\frac{\partial}{\partial z'} \tag{9}$$

Thus, $A(\mathcal{G}) = {}^{\Phi,b}TX$. □

The Lie groupoid structure of $\mathcal{G}$ can be described simpler if we use the notion of blow-up of a Lie groupoid . Let $\mathcal{G}_b \rightrightarrows X$ be the b-groupoid, then, as described in section2.2, $\mathcal{G}_b$ is an open subset of X^2_b and $\mathcal{G}_b \rightrightarrows X$ is a subgroupoid of $\mathcal{G}_b$. Then $\mathcal{G} = [\widetilde{\mathcal{G}_b; \Phi_b}]$.

3.3 The definition of the fibred cusp b-pseudodifferential operators

Definition 3. *Let E and F are vector bundles over X, and $m \in \mathbb{R}$ be an arbitrary real number. Then, the space of fibred cusp b-pseudodifferential operator of order m from E to F is defined as follows.*

$$\Psi^m_{\Phi,b}(X;E,F) := \{P \in I^m(X^2_{\Phi,b}, \Delta_{\Phi,b}(X); \pi_L^* F \otimes \pi_R^* E' \otimes \pi_R^*({}^{\Phi,b}\Omega)),$$
$$P \equiv 0 \text{ on each boundary hypersurface except for } FF_0 \text{ or } F_1\}$$

Where $\pi_L, \pi_R : X^2_{\Phi,b} \to X$ are left and right projection, E' is a dual of E, ${}^{\Phi,b}\Omega := |\Lambda^{\dim X}|({}^{\Phi,b}T^*X)$, I^m is a space of conormal distribution, and $P \equiv 0$ means a vanishing of infinite order.

In this paper we only consider classical or one-step polyhomogeneous conormal distribution.

The space of uniformly supported pseudodifferential operator $\Psi^m_c(\mathcal{G}; E, F)$ defined in [24] and [4] is, by definition,

$$\Psi^m_c(\mathcal{G}; E, F) = \{P \in \Psi^m_{\Phi,b}(X; E, F) \mid P \text{ vanishes identically on the neighborhood of } L_0 \cup R_0 \cup L_1 \cup R_1\}. \quad (10)$$

By definition, $\Psi^m_c(\mathcal{G}; E, F) \subset \Psi^m_{\Phi,b}(X; E, F)$.

Let $\dot{C}^\infty(X;E) := x_0^\infty x_1^\infty C^\infty(X;E)$ be a space of smooth section of E which vanishes in infinite order on $\partial_0 X$ and $\partial_1 X$. By general theory of conormal distributions [19], for $u \in \dot{C}^\infty(X;E)$ we can see that $Pu := (\pi_L)_* P \pi_R^* u$ defines continuous linear operators $\dot{C}^\infty(X;E) \to \dot{C}^\infty(X;F)$ where $(\pi_L)_*$ is a fibre integral.

To give an explicit description of P, we assume P is supported in the coordinate patch $\{(x_0, x_1, y, z, u_0, u_1, v, w)\}$.

By the condition $P \equiv 0$ on $L_0 \cup R_0 \cup L_1 \cup R_1$, P decreases rapidly as $u_0^2 + u_1^2 + |v|^2 \to \infty$. Thus we can Fourier transform with respect to u_0, u_1, v, w , and P can be written as following.

$$P(x_0, x_1, y, z, u_0, u_1, v, w) = \int e^{i\sigma_0 u_0} e^{i\sigma_1 u_1} e^{i\eta \cdot v} e^{i\zeta w} p(x_0, x_1, y, z, \sigma_0, \sigma_1, \eta, \zeta) \, d\sigma_0 d\sigma_1 d\eta d\zeta |du_0 du_1 dv dw| \quad (11)$$

Note that $\pi_R^*({}^{\Phi,b}\Omega)$ is generated by $x_0'^{-k-2} x_1'^{-1} |dx_0' dx_1' dy' dz'|$. Its restriction to the fibre of π_L is $\frac{x_0^{k+2} x_1}{x_0'^{k+2} x_1'} du_0 du_1 dv dw$, and the coefficient $\frac{x_0^{k+2} x_1}{x_0'^{k+2} x_1'}$ is a non-zero smooth function, thus we can absorb this coefficient in the symbol term.

For a function $u(x_0, x_1, y, z)$, the action of P is given by

$$Pu(x_0, x_1, y, z) = \int e^{i\sigma_0 u_0} e^{i\sigma_1 u_1} e^{i\eta \cdot v} e^{i\zeta w} p(x_0, x_1, y, z, \sigma_0, \sigma_1, \eta, \zeta)$$
$$u(x_0 - x_0^2 u_0, x_1 - x_0 x_1 u_1, y - x_0 v, z - w) d\sigma_0 d\sigma_1 d\eta d\zeta |du_0 du_1 dv dw|.$$

For any complex numbers α and β, $x_0^\alpha x_1^\beta P x_0^{-\alpha} x_1^{-\beta} \in \Psi^m_{\Phi,b}(X; E, F)$, because $x_0'/x_0 = 1 - x_0 u_0$, $x_1'/x_1 = 1 - x_0 u_1$ and these derivatives are smooth up to FF_0 and F_1 and at most polynomial order up to other boundary hypersurfaces. Thus P also defines an operator $x_0^\alpha x_1^\beta C^\infty(X;E) \to x_0^\alpha x_1^\beta C^\infty(X;F)$. In particular, we can obtain three operators,

$$P|_{\partial_0 X} : C^\infty(\partial_0 X; E) \to C^\infty(\partial_0 X; F),$$

$$P|_{\partial_1 X} : C^\infty(\partial_1 X; E) \to C^\infty(\partial_1 X; F),$$

and

$$P|_{\angle X} : C^\infty(\angle X; E) \to C^\infty(\angle X; F),$$

such that $(Pu)|_{\partial_0 X} = P|_{\partial_0 X} u|_{\partial_0 X}$, $(Pu)|_{\partial_1 X} = P|_{\partial_1 X} u|_{\partial_1 X}$ and $(Pu)|_{\angle X} = P|_{\angle X} u|_{\angle X}$ for $u \in C^\infty(X; E)$.

Lemma 4. *$P|_{\partial_0 X} \in \Psi^m_{\mathrm{fibre}}(\partial_0 X; E, F)$, $P|_{\partial_1 X} \in \Psi^m_\Phi(\partial_1 X; E, F)$, $P|_{\angle X} \in \Psi^m_{\mathrm{fibre}}(\angle X; E, F)$ where Ψ^m_{fibre} is the space of a family of m-th pseudodifferential operators on each fibre of ϕ which depends smoothly on the base points, and Ψ^m_Φ is the space of m-th fibred cusp pseudodifferential operator.*

Proof. By using partition of unity, we can assume P is supported in the coordinate patch. Let p is a symbol of P as in (11), the symbols of $P|_{\partial_0 X}$, $P|_{\partial_1 X}$ and $P|_{\partial_0 X}$ are $p(0, x_1, y, z, 0, 0, 0, \zeta)$, $p(x_0, 0, y, z, \sigma_0, 0, \eta, \zeta)$ and $p(0, 0, y, z, 0, 0, 0, \zeta)$ respectively. □

As in [17], we can construct a blow-up $X^3_{\Phi,b}$ of X^3

$$X^3_{\Phi,b} := [X^3_b; \Phi_T; \Phi_{FT}; \Phi_{ST}; \Phi_{CT}; \Phi_F; \Phi_S; \Phi_C].$$

By using this manifold, we can prove the following proposition exactly parallel as in [17] or [8].

Proposition 3. *Let E, F, G are vector bundles over X , $m, m' \in \mathbb{R}$, $P \in \Psi^m_{\Phi,b}(X; E, F)$ and $Q \in \Psi^{m'}_{\Phi,b}(X, F, G)$, then $Q \circ P \in \Psi^{m+m'}_{\Phi,b}(X; E, G)$.*

3.4 Symbols and normal operators

In this section, we define the symbol σ and normal operators N_0 and N_1. Essentially, $N_0(P)$ and $N_1(P)$ are restriction of the Schwartz kernel of P to FF_0 and F_1. A similar notion is called a normal operator in [17], an indicial operator in [18], and just a symbol in [8].

As described in [19],[17],[24], we can obtain a symbol homomorphism

$$\sigma : \Psi^m_{\Phi,b}(X; E, F) \to S^m({}^{\Phi,b}T^*X; \mathrm{Hom}\,(E, F)).$$

Where $S^m({}^{\Phi,b}T^*X; \mathrm{Hom}\,(E, F))$ is a space of bundle homomorphisms ${}^{\Phi,b}T^*X \setminus 0 \to \mathrm{Hom}\,(E, F)$ which are homogeneous of degree m. The sequence

$$0 \to \Psi^{m-1}_{\Phi,b}(X; E, F) \to \Psi^m_{\Phi,b}(X; E, F) \xrightarrow{\sigma} S^m({}^{\Phi,b}T^*X; \mathrm{Hom}\,(E, F)) \to 0$$

is exact.

Next, we consider a normal operator at $\partial_0 X$. Let $p \in Y$, $(\tau, \tilde\eta) \in (\underline{\mathbb{R}} \oplus T^*Y)_y$. Fix any real valued $f \in C^\infty(Y)$ such that $f(p) = \tau$, $df(p) = \tilde\eta$, and real valued $\tilde f \in C^\infty(X)$ such that $\phi^* f = \tilde f|_{\partial_0 X}$. Define

$$\begin{aligned}(\tau, \tilde\eta) &\in (\underline{\mathbb{R}} \oplus T^*Y)_y \\ &\mapsto \tilde N_0(\tau, \tilde\eta) := [\exp(-i\tilde f/x_0) P \exp(i\tilde f/x_0)]|_{\phi^{-1}(p)} \in \Psi^m(\phi^{-1}(p); E, F). \quad (12)\end{aligned}$$

Then, for $\tau = f(x_1, y)$,$\tilde{\sigma} = \frac{\partial}{\partial x_1} f(x_1, y)$, $\xi_i = \frac{\partial}{\partial y_i} f(x_1, y)$, the symbol of $\tilde{N}(\tau, \sigma, \xi)$ is given by $p(0, x_1, y, z, -\tau, x_1\tilde{\sigma}, \xi, \eta)$, in particular $\tilde{N}_0$ is well-defined and does not depend on the choice of f or $\tilde{f}$ and depends smoothly in $(\tau, \tilde{\eta})$.

Note that if $(\tilde{\sigma}, \xi_i) = (\tilde{\sigma}dx_1, \sum \xi_i dy_i)$ is a coordinate for T^*Y, then $(x_1\tilde{\sigma}(dx_1/x_1), \sum \xi_i dy_i)$ is a coordinate for ${}^bT^*Y$. So if $\beta_b : TY \to {}^bTY$ is a blow-down map, the above symbol expression implies that there is a unique map $\widehat{N_0} : (\underline{\mathbb{R}} \oplus {}^bT^*Y)_y \to \Psi^m(\phi^{-1}(p); E, F)$ such that $\widehat{N_0} \circ \beta_b = \tilde{N}_0$, i.e. $\widehat{N_0}(\tau, \sigma, \xi) = \tilde{N}(\tau, \sigma/x_1, \xi)$, and its symbol is given by $p(0, x_1, y, z, -\tau, \sigma, \xi, \eta)$.

By the fourier transform, it turns out that $\widehat{N_0}$ defines a suspended pseudodifferential operator (see [17]). $N_0 = N_0(P) \in \Psi^m_{\mathrm{sus}({}^{\Phi,b}NY)}(\partial X)$, ${}^{\Phi}NY \simeq \underline{\mathbb{R}} \oplus {}^bT^*Y$. We say $N_0(P)$ is a normal operator of P on $\partial_0 X$.

We obtain the exact sequence

$$0 \to x_0\Psi^m_{\Phi,b}(X) \to \Psi^m_{\Phi,b}(X) \xrightarrow{N_0} \Psi^m_{\mathrm{sus}({}^{\Phi,b}NY)}(\partial_0 X) \to 0.$$

We consider a normal operator of P at $\partial_1 X$. For $P \in \Psi^m_{\Phi,b}(X; E, F)$, $\lambda \in \mathbb{C}$ define

$$\widehat{N_1}(P)(\lambda) = [x_1^{-i\lambda} P x_1^{i\lambda}]|_{\partial_1 X} \in \Psi^m_{\Phi}(\partial_1 X; E, F).$$

Obviously, $\widehat{N_1} : \mathbb{C} \to \Psi^m_{\Phi}(\partial_1 X; E, F)$ is an entire holomorphic function.

Let $\widetilde{\partial_1 X} \simeq \partial_1 X \times [0, \infty]$ is the compactification of the positive normal bundle of $\partial_1 X \subset X$, then $\widetilde{\partial_1 X}$ obviously has a structure of a manifold with fibred boundary. Define

$$\Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}) := \{B \in \Psi^m_{\Phi,b}(\widetilde{\partial_1 X})| \text{ b is equivariant with respect to the } (0,\infty) \text{ action on } \widetilde{\partial_1 X}\ \}. \quad (13)$$

Note that the first front faces $F_1 \subset X^2_{\Phi,b}$ and $F_1 \subset \widetilde{\partial_1 X}^2_{\Phi,b}$ are canonically diffeomorphic, thus, by Mellin transformation, it turns out that $\widehat{N_1}(P)$ defines $N_1(P) \in \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X})$ and the following sequence is exact (see [18] for the detail).

$$0 \to x_1\Psi^m_{\Phi,b}(X) \to \Psi^m_{\Phi,b}(X) \xrightarrow{N_1} \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}) \to 0$$

In a local coordinate, define $t := \log(1 - x_0 X_1)/x_0$, then t is smooth up to $x_0 = 0$, and $X_1 = (1 - e^{tx_0})/x_0$ is also smooth up to $x_0 = 0$. Then $\frac{x_1'^{i\lambda}}{x_1^{i\lambda}} = e^{ix_0\lambda t}$

Thus by changing coordinate form X_1 to t , $(x_0, x_1, X_0, t, y, Y, z, Z)$ also gives a coordinate. Define the symbol $\tilde{p}$ of P with respect to this coordinate by following.

$$P(x_0, x_1, X_0, t, y, Y, z, Z) = \int e^{i\sigma_0 X_0} e^{i\sigma_1 t} e^{i\eta Y} e^{i\zeta Z} \tilde{p}(x_0, x_1, y, z, \sigma_0, \sigma_1, \eta, \zeta) \, d\sigma_0 d\sigma_1 d\eta d\zeta |dX_0 dt dY dZ|$$

Then the symbol of $\widehat{N_1}(P)(\lambda)$ is $\tilde{p}(x_0, 0, y, z, \sigma_0, -x_0\lambda, \eta, \zeta)$.

The normal operators N_0, N_1 can be thought as the restriction to $\partial_1 X, \partial_0 X$, and the symbol σ can be thought as the restriction to the boundary $S({}^{\Phi,b}T^*X)$ of ${}^{\Phi,b}T^*X$ at infinity. We want to consider further restriction to the intersection of these two.

As in the case of σ, we can define symbol maps

$$\sigma_0 : \Psi^m_{\mathrm{sus}({}^{\Phi,b}NY)}(\partial_0 X; E, F) \to S^m({}^{\Phi,b}T^*X|_{\partial_0 Z}; \mathrm{Hom}\,(E, F))$$

and
$$\sigma_1 : \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}; E, F) \to S^m({}^{\Phi,b}T^*X|_{\partial_1 X}; \mathrm{Hom}\,(E,F)).$$

To consider the restriction to $\angle X$, as $\widetilde{\partial_1 X}$ is also a manifold with fibred boundary, we can define a normal operator on $\partial_0(\widetilde{\partial_1 X}) = \widetilde{\angle X}$, $N_{0,1} : \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}; E, F) \to \Psi^m_{\mathrm{sus}({}^{\Phi,b}N\widetilde{\partial Y})}(\widetilde{\angle X}; E, F)$.

Where $\widetilde{\partial Y} \simeq \partial Y \times [0,\infty]$,$\widetilde{\angle X} \simeq \angle X \times [0,\infty]$ Note that for $Q \in \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}; E, F)$, $N_{0,1}(Q)$ is also equivariant to the action of $(0,\infty)$, and $(0,\infty)$ acts on $\widetilde{\angle X}$ or $\widetilde{\partial Y}$ by multiplication. So the restriction of $N_{0,1}$ to the any fibre of ϕ gives the same value in $\Psi^m_{\mathrm{sus}({}^{\Phi,b}N\partial Y)}(\angle X; E, F)$, where ${}^{\Phi,b}N\partial Y := {}^{\Phi,b}NY|_{\partial Y} \simeq \underline{\mathbb{R}} \oplus \underline{\mathbb{R}} \oplus T^*\partial Y$.

Thus we can define
$$N_{1,0} : \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}; E, F) \to \Psi^m_{\mathrm{sus}({}^{\Phi,b}N\partial Y)}(\angle X; E, F).$$

On the other hand we can define
$$N_{0,1} : \Psi^m_{\mathrm{sus}({}^{\Phi,b}NY)}(\partial_0 X; E, F) \to \Psi^m_{\mathrm{sus}({}^{\Phi,b}N\partial Y)}(\angle X; E, F)$$
by restriction.

We can also define the symbol map.
$$\sigma_{0,1} : \Psi^m_{\mathrm{sus}({}^{\Phi,b}N\partial Y)}(\angle X; E, F) \to S^m({}^{\Phi,b}TX|_{\angle X}; \mathrm{Hom}\,(E,F))$$

Define following maps by restrictions of symbols.
$$S^m({}^{\Phi,b}T^*X; \mathrm{Hom}\,(E,F)) \to S^m({}^{\Phi,b}T^*X|_{\partial_0 X}; \mathrm{Hom}\,(E,F))$$
$$S^m({}^{\Phi,b}T^*X; \mathrm{Hom}\,(E,F)) \to S^m({}^{\Phi,b}T^*X|_{\partial_1 X}; \mathrm{Hom}\,(E,F))$$
$$S^m({}^{\Phi,b}T^*X|_{\partial_1 X}; \mathrm{Hom}\,(E,F)) \to S^m({}^{\Phi,b}T^*X|_{\angle X}; \mathrm{Hom}\,(E,F))$$
$$S^m({}^{\Phi,b}T^*X|_{\partial_0 X}; \mathrm{Hom}\,(E,F)) \to S^m({}^{\Phi,b}T^*X|_{\angle X}; \mathrm{Hom}\,(E,F))$$

In summary, we defined 12 maps $\sigma, \sigma_0, \sigma_1, \sigma_{0,1}, N_0, N_1, N_{0,1}N_{1,0}$ and four restriction maps. It is obvious from the definition that any composition of any composable pair which is defined on same spaces coincides, e.g. $N_{0,1}N_0 = N_{1,0}N_1$ or $\sigma|_{\partial_1 X} = \sigma_0 N_0$. Exact sequences exist for all of these 12 maps as shown in the case σ, N_0 and N_1, but we will omit here.

Finally we can consider the joint symbol J^m which is defined as follows.

$$\begin{aligned} J^m(X; E, F) := \{(s, n_0, n_1) \in S^m({}^{\Phi,b}T^*X; \mathrm{Hom}\,(E,F)) \oplus \Psi^m_{\mathrm{sus}({}^{\Phi,b}NY)}(\partial_0 X; E, F) \\ \oplus \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}; E, F) \mid s|_{\partial_1 X} = \sigma_0(n_0), s|_{\partial_0 X} = \sigma_1(n_1), N_{0,1}(n_0) = N_{1,0}(n_1)\} \end{aligned}$$

Then the following sequence is exact by the diagram chasing.

$$0 \to x_0 x_1 \Psi^{m-1}_{\Phi,b}(X; E, F) \to \Psi^m_{\Phi,b}(X; E, F) \xrightarrow{\sigma \oplus N_0 \oplus N_1} J^m(X; E, F) \to 0 \qquad (14)$$

3.5 The Fredholm criterion

Let X be a manifold with fibred boundary. In this section fix a Riemannian metric g on ${}^{\Phi,b}TX$, then g can be considered as a singular metric on TX and $\mathrm{int}\,X$ is a Riemannian manifold with respect to that metric. We also assume that every complex vector bundle E on X has a hermitian metric h. Then, we can define the L^2 space.

$$L^2_{\Phi,b}(X;E) := \{u \mid u \text{ is a measurable section of } E \text{ and } \int ||u||^2 dg < \infty\}$$

For $u, v \in L^2_{\Phi,b}(X;E)$, we can define the inner product by $(u,v) := \int h(u,v)dg.$ and $L^2_{\Phi,b}(X;E)$ is a Hilbert space with respect to this inner product.

To prove P is bounded on L^2, we need some technical preparations. Consider the normal operator on $\partial_1 X$, $N_1 : \Psi^m_{\Phi,b}(X;E,F) \to \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X};E,F)$, then as illustrated in [22], we can define a section of N_1 as following. Fix a diffeomorphism $\widetilde{\partial_1 X} \simeq \partial_1 X \times [0,\infty]$ and a collar neighbourhood $\partial_1 X \times [0,\infty] \hookrightarrow X$ and take a smooth function ψ on X which is supported in a small neighbourhood of $\partial_1 X$ and identically equal to 1 around the neighbourhood of $\partial_1 X$. Then the multiplication by ψ can be regarded as a operator $M_\psi : C^\infty(\widetilde{\partial_1 X}) \to C^\infty(X)$. Define $S : \Psi^m_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X};E,F) \to \Psi^m_{\Phi,b}(X;E,F)$ by $S(B) = M_\psi B M_\psi^*$. By definition S is smooth with respect to the Fréćht space topology.

As in the case of b-calculus, for $B \in \Psi^0_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X};E,F)$ and $u \in \dot{C}^\infty(\widetilde{\partial_1 X};E)$, the action of b is characterized as follows.

$$\widehat{Bu}(\lambda) = \widehat{B}(\lambda)\widehat{u}(\lambda)$$

Where $\widehat{u}$ and $\widehat{Bu}$ are Mellin transform of u or $\widehat{Bu}$, and $\widehat{B}(\lambda) = \widehat{N_1}(B)(\lambda)$.

By the coordinate representation given in section 3.4, for $\lambda \in \mathbb{R}$, $\widehat{B}(\lambda)$ is bounded with respect to the Fŕchet space topology on $\Psi^0_\Phi(\partial_1 X;E,F)$. By [17], the embedding $\Psi^0_\Phi(\partial_1 X;E,F) \to \mathcal{L}(L^2_\Phi(\partial_1 X;E), L^2_\Phi(\partial_1 X;F))$ is bounded, thus $||\widehat{B}(\lambda)||$ is bounded. Thus b is also bounded because Mellin transform is an isomorphism on a L^2 space.

Proposition 4. *For $P \in \Psi^0_{\Phi,b}(X;E,F)$, P is bonded as an operator $L^2_{\Phi,b}(E) \to L^2_{\Phi,b}(F)$, and the inclusion $\Psi^0_{\Phi,b}(X;E,F) \to \mathcal{L}(L^2_{\Phi,b}(E), L^2_{\Phi,b}(F))$ is a bounded map.*

Proof. First, we show the proposition holds for $P \in x_0^N x_1^N \Psi^{-N}_{\Phi,b}(X;E,F)$ when $N > 0$ is sufficiently large. In this case the Schwartz kernel of P blows down and can be written as a continuous kernel on X^2. Thus the boundedness is obvious.

Secondly, we show the proposition holds for $P \in x_0^\epsilon x_1^\epsilon \Psi^{-\epsilon}_{\Phi,b}(X;E,F)$ for any $\epsilon > 0$. Because $||P||^2 = ||P^*P||$ and $P^*P \in x_0^{2\epsilon} x_1^{2\epsilon} \Psi^{-2\epsilon}$, using this discussion recursively, the boundedness follows by first step.

Lastly, we consider the general case $P \in \Psi^0_{\Phi,b}(X;E,F)$. As discussed above, $N_1(P)$ is bounded with respect to the operator norms, so $S(N_1(P)) = M_\psi N_1(P) M_\psi^*$ is also bounded with respect to operator norms, because M_ψ is obviously bounded with respect to operator norms. By replacing P by $P - S(N_1(P))$, we can assume that $N_1(P) = 0$ i.e. $P \in x_1 \Psi^0_{\Phi,b}(X;E,F)$

Take sufficiently large $C > 0$ so that $C - N_0(P^*P)$ and $C - \sigma(P^*P)$ are positive. Then, we can find a formally self adjoint operator $A \in \Psi^0_{\Phi,b}(X;E,E)$ such that $N_0(A) = \sqrt{C - N_1(P^*P)}$, $N_1(A) = \sqrt{C}$ and $\sigma(A) = \sqrt{C - \sigma(P^*P)}$. Note that $\sqrt{C - N_1(P^*P)}$ can be defined because the calculus of the suspended pseudodifferential operator is closed under holomorphic functional calculus. Set $B := C - P^*P - A^2$, then $N_0(B) = N_1(B) = \sigma(B) = 0$ so $B \in x_0 x_1 \Psi^{-1}_{\Phi,b}(X;E,E)$. Thus b is L^2

bounded by second step. $||Pu||^2 = (P^*Pu, u) = -(Bu, u) + C||u||^2 - ||Au||^2 \leq (||B|| + ||C||) \cdot ||u||^2$

□

As $\Psi^0_c(\mathcal{G}; E, F) \subset \Psi^0_{\Phi,b}(X; E, F)$ is obviously dense with respect to the operator norm, following propositions can be reduced to the general theory of pseudodifferential operator of a groupoid [24, 13].

Proposition 5. *Any one of the 12 maps in the last part of the section 3.4, including σ, N_0 and N_1, is bounded with respect to the operator norm.*

Theorem 7. *$P \in \Psi^0_{\Phi,b}(X; E, F)$ is Fredholm if and only if $\sigma(P)$ and $N_0(P)$ are invertible and $\widehat{N_1}(P)(t)$ is invertible for all $t \in \mathbb{R}$.*

Proof. As described in [13], by diagram chasing, the L^2 completion of the exact sequence (14) is also exact,

$$0 \to \mathcal{K}(X; E, F) \to \overline{\Psi}^0_{\Phi,b}(X; E, F) \xrightarrow{j} \overline{J}^0(X; E, F) \to 0$$

where $\mathcal{K}$ is the space of compact operators.

Thus P is Fredholm iff $j(P)$ is invertible iff $\sigma(P), N_0(P), N_1(P)$ are invertible in its L^2 closure.

Because $S^0({}^{\Phi,b}T^*X; E, F)$ and $\Psi^0_{\mathrm{sus}({}^{\Phi,b}NY)}(\partial_0 X; E, F)$ are closed under holomorphic functional calculus, $\sigma(P)$ and $N_0(P)$ are invertible if and only if they are invertible in its completion.

For $N_1(P) \in \Psi^0_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}; E, F)$, there is an injective $*$-homomorphism defined by the Mellin transform.

$$B \in \Psi^0_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}; E, F) \to \widehat{B}|_{\mathbb{R}} \in C_b(\mathbb{R}, \Psi^0_\Phi(\partial_1 X; E, F))$$

where $C_b(\mathbb{R}, \Psi^0_\Phi(\partial_1 X; E, F))$ is a space of bounded continuous function from $\mathbb{R}$ to $\Psi^m_\Phi(\partial_1 X; E, F)$. Obviously, the completion of $C_b(\mathbb{R}, \Psi^0_\Phi(\partial_1 X; E, F))$ is $C_b(\mathbb{R}, \overline{\Psi}^0_\Phi(\partial_1 X; E, F))$ and the above map extends to an injective *-homomorphism.

$$\overline{\Psi}^0_{\Phi,b,\mathrm{inv}}(\widetilde{\partial_1 X}; E, F) \to C_b(\mathbb{R}, \overline{\Psi}^0_\Phi(\partial_1 X; E, F)).$$

Thus $N_1(P)$ is invertible in its completion if and only if its image in $C_b(\mathbb{R}, \overline{\Psi}^0_\Phi(\partial_1 X; E, F))$ is invertible, and the claim follows. □

3.6 The relative index theorem

Lemma 5. *Let $P \in \Psi^0_{\Phi,b}(X; E, F)$ and suppose that $\sigma(P)$ and $N_0(P)$ are invertible. Then there is a parametrix $Q \in \Psi^0_{\Phi,b}(X; F, E)$ such that $PQ - \mathrm{Id} \in x_0^\infty \Psi^{-\infty}_{\Phi,b}(X; F, F)$, $QP - \mathrm{Id} \in x_0^\infty \Psi^{-\infty}_{\Phi,b}(X; E, E)$.*

Proof. We construct the right parametrix Q inductively. Take $Q_0 \in \Psi^0_{\Phi,b}(X; F, E)$ so that $\sigma_0(Q_0) = \sigma_0(P)^{-1}$, $N_0(Q_0) = \sigma(P)^{-1}$. Then $PQ_0 - \mathrm{Id} \in x_0 \Psi^{-1}_{\Phi,b}(X; F, F)$.

Set $R_0 := (PQ_0 - \mathrm{Id})/x_0 \in \Psi^{-1}_{\Phi,b}(X; E, E)$. Take $Q_1 \in \Psi^{-1}_{\Phi,b}(X; F, E)$ such that $\sigma_{-1}(Q_1) = -\sigma_0(P)^{-1}\sigma_{-1}(R_0)$, $N_0(Q_1) = -N_0(P)^{-1}N_0(R_0)$. By definition of Q_1, $P(Q_0 + x_0 Q_1) - \mathrm{Id} = x_0(R_0 + PQ_1) \in x_0^2 \Psi^{-2}_{\Phi,b}(X; F, F)$.

Suppose we constructed $Q_1, \ldots Q_n$ such that $Q_m \in \Psi^{-m}_{\Phi,b}(X; F, E)$ and $P(\sum_0^n x_0^m Q_m) - \mathrm{Id} \in x_0^{n+1} \Psi^{-n-1}_{\Phi,b}(X; F, F)$. Set $R_n := (P(\sum_0^n x_0^m Q_m) - \mathrm{Id})/x_0^{n+1}$. Take $Q_{n+1} \in$

$\Psi^{-n-1}_{\Phi,b}(X;F,E)$ such that $\sigma_{-n-1}(Q_1) = -\sigma_0(P)^{-1}\sigma_{-n-1}(R_n)$, $N_0(Q_1) = -N_0(P)^{-1}N_0(R_n)$. Then as above, $P(\sum_0^{n+1} x_0^m Q_m) - \mathrm{Id} \in x_0^{n+2}\Psi^{-n-1}_{\Phi,b}(X;F,F)$.

Finally define $Q := \sum_0^\infty Q_m$ by an asymptotic sum. Then $Q \in \Psi^0_{\Phi,b}(X;F,E)$ and $PQ - \mathrm{Id} \in \bigcap_m x_0^m \Psi^{-m}_{\Phi,b}(X;E,E) = x_0^\infty \Psi^{-\infty}(X;F,F)$. As we can construct left parametrix similarly, Q is actually right and left parametrix.

□

For the above parametrix Q, define $S = \mathrm{Id} - PQ \in x_0^\infty \Psi^{-\infty}_{\Phi,b}(X;F,F)$. Note that $x_0^\infty \Psi^{-\infty}_{\Phi,b}(X;F,F) = x_0^\infty \Psi^{-\infty}_b(X;F,F)$, because the Schwartz kernel of any element of $x_0^\infty \Psi^{-\infty}_{\Phi,b}(X;F,F)$ vanishes on FF_0 in infinite order and blows down to the kernel on X_b^2.

We can see $\widehat{N_1}(S)(\lambda) = \mathrm{Id} - \widehat{N_1}(P)(\lambda)\widehat{N_1}(Q)(\lambda)$ rapidly decreases as $|\mathrm{Re}\lambda| \to \infty$ by the theory of b-calculus. Thus as in [18], we can prove the following lemma.

Lemma 6. *Let $P \in \Psi^0_{\Phi,b}(X;E,F)$ and suppose that $\sigma(P)$ and $N_0(P)$ are invertible. $\widehat{N_1}(P)(\lambda)^{-1}$ is a meromorphic map from $\mathbb{C}$ to $\Psi^0_{\Phi,b}(X;F,E)$. Furthermore, for any $N > 0$, there exists $C > 0$ such that $\widehat{N_1}(P)^{-1}(\lambda)$ exists and bounded on $\{\lambda \in \mathbb{C} \mid |\mathrm{Re}\lambda| > C$ and $|\mathrm{Im}\lambda| < N\ \}$.*

In particular,the number of poles in the strip $\{\lambda \in \mathbb{C} \mid |\mathrm{Im}\lambda| < N\}$ is finite.

Let $P \in \Psi^0_{\Phi,b}(X;E,F)$, and suppose that $\sigma(P)$ and $N_0(P)$ are invertible. Obviously, $\sigma(x_1^\alpha P x_1^{-\alpha}) = \sigma(P)$. Because x_1 is constant on each fibre of ϕ , $N_0(P)$ commutes with x_1^α and $N_0(x_1^\alpha P x_1^{-\alpha}) = N_0(P)$.

For $\beta \in \mathbb{R}$, $\widehat{N_1}(x_1^\beta P x_1^{-\beta})(\lambda) = \widehat{N_1}(\lambda + i\beta)$. By theorem 7 $x_1^\beta P x_1^{-\beta}$ is Fredholm if and only if $\beta \notin -\mathrm{ImSpec}(\widehat{N_1}(P))$. Where $\mathrm{Spec}(\widehat{N_1}(P)) := \{\lambda \in \mathbb{C} \mid\ \widehat{N_1}(P)(\lambda)$ is not invertible$\}$ which is discrete by lemma 6.

Thus, exactly as in [18], we can prove the relative index theorem.

Theorem 8. *Let $P \in \Psi^0_{\Phi,b}(X;E,F)$ and suppose that $\sigma(P)$ and $N_0(P)$ are invertible, $\beta_i \notin -\mathrm{ImSpec}_b(P)$ $(i = 1,2)$ $\beta_2 > \beta_1$ .Then,*

$$\mathrm{ind}(x_1^{\beta_1} P x_1^{-\beta_1}) - \mathrm{ind}(x_1^{\beta_2} P x_1^{-\beta_2}) = \frac{1}{2\pi i}\mathrm{tr}\oint \widehat{N_1}(P)^{-1}(\lambda)\frac{\partial \widehat{N_1}(P)}{\partial\lambda}(\lambda)d\lambda,$$

where ind *is the index of Fredholm operator,* tr *is the trace, and the integral path is chosen so that its interior contains all poles of $\widehat{N}(P)^{-1}(\lambda)$ such that $\beta_1 < -\mathrm{Im}(\lambda) < \beta_2$.*

For $P \in \Psi^0_{\Phi,b}(X;E,F)$ and $I := [\delta,\gamma] \subset \mathbb{R}$, $\delta < \gamma$ be a closed interval. Define a norm by $||P||_I := \sup_{\alpha\in I}||x_1^\alpha P x_1^{-\alpha}||$, and $\overline{\Psi^0_{\Phi,b}}^I(X;E,F)$ be a completion with respect to that norm. Then $\widehat{N_1}$ extends.

$$\widehat{N_1} : \overline{\Psi^0_{\Phi,b}}^I(X;E,F) \to \mathrm{Hol}_b(\mathbb{R} \times iI, \overline{\Psi}^0_\Phi(\partial_1 X;E,F)),$$

where $\mathbb{R} \times iI = \{\lambda \in \mathbb{C} \mid \delta \le \mathrm{Im}(\lambda) \le \gamma\}$ and Hol_b is a space of bounded continuous function which is holomorphic in the interior.

To see $\widehat{N_1}$ extends to the completion, let $P \in \overline{\Psi^0_{\Phi,b}}^I(X;E,F)$ and $P_n \in \Psi^0_{\Phi,b}(X;E,F)$ such that $||P - P_n||_I \to 0$, then $\widehat{N_1}(P_n)|_{\mathbb{R}\times iI}$ is a Cauchy sequence by the definition of the norm, and uniformly converges to some $\widehat{N_1}(P)$. Because uniform limit of holomorphic function is holomorphic, $\widehat{N_1}(P)$ is holomorphic in the interior.

Note that σ and N_0 also extends because $\sigma(x_1^{\alpha}Px_1^{-\alpha}) = \sigma(P)$ and $N_0(x_1^{\alpha}Px_1^{-\alpha}) = N_0(P)$.

And theorem 8 can be extended to the completion $P \in \overline{\Psi^0_{\Phi,b}}^{I}(X;E,F)$, because both hand side of the equality are continuous with respect to $||\cdot||_I$ and is an integer.

4 Application to $\mathbb{Z}/$k-manifolds

In this section we fix an isomorphism $TX = {}^{\Phi,b}T^*X$ for simplicity. Suppose X is a $\mathbb{Z}/k$ manifold, i.e. X is a manifold with corner and $\partial X = \partial_0 X \cup \partial_1 X$, $\angle X = \partial_0 X \cap \partial_1 X$ and the diffeomorhpism $\partial_1 X \simeq kZ$ is given, where Z is a manifold with boundary and kZ is a disjoint union of k copies of Z. For $\phi = \mathrm{Id} : \partial_0 X \to \partial_0 X$, we regard X as a manifold with fibred boundary. And we write $\Psi^0_{sc,b}(X;E,F) = \Psi^0_{\Phi,b}(X;E,F)$ in this case.

A vector bundle E over X is called $\mathbb{Z}/k$-vector bundle if $E|_{\partial_1 X} = kE_Z$ for some vector bundle $E_Z \to Z$. Fix a $\mathbb{Z}/k$-vector bundle structure on TX. Define $\overline{X}$ be a quotient of X obtained by identifying k copies of Z in X. Then $TX \to X$ descends to a vector bundle $\overline{TX} \to \overline{X}$.

Let E, F are $\mathbb{Z}/k$- vector bundle over X, and $P \in \Psi^0_{\Phi,b}(X;E,F)$. Define

$$\begin{aligned}\Psi^0_{sc,b,\mathbb{Z}/k}(X;E,F) := \{P \in \Psi^0_{sc,b}(X;E,F) \mid \ N_1(P) = kQ \\ \text{for some } Q \in \Psi^0_{sc,b}(\widetilde{Z};E,F)\ \}\end{aligned} \tag{15}$$

Where $kQ = Q \oplus Q \cdots \oplus Q \in \Psi^0_{sc,b}(\widetilde{\partial_1 X};E,F)$ is defined by using isomorphism $\partial_1 X \simeq kZ$, and $\widetilde{Z} \simeq Z \times [0,\infty]$.

$\widehat{N_0}(P)$ is a bundle homomorphism over $\partial_0 X$, $\widehat{N_0}(P) : \mathbb{R} \oplus T(\partial_1 X) \to \bigcup_{y\in\partial_0 X} \Psi^0(\phi^{-1}(y);E,F)$. Note that $\phi^{-1}(y)$ is one point set in this case. So $\Psi^0(\phi^{-1}(y);E,F) \simeq \mathrm{Hom}\,(E,F)$. Under this identification, by the compatibility of N_0 and σ, $\sigma(P)(\xi) = \lim_{t\to\infty} \widehat{N_0}(t\xi)$ for $\xi \in S(TX|_{\partial_0 X})$.

By above observations, there is a map

$$\begin{aligned}s : P \in \{P \in \Psi^0_{sc,b,\mathbb{Z}/k}(X;E,F) \mid \ \sigma(P) \text{ and } N_0(P) \text{ are invertible }\} \\ \mapsto [E, \sigma(P) \cup N_0(P), F] \in K(D(\overline{TX}), S(\overline{TX}) \cup D(\overline{TX})|_{\overline{\partial_0 X}}).\end{aligned} \tag{16}$$

Where we regard $\sigma(P) \cup N_0(P)$ as a bundle isomorphism between E and F over $S(\overline{TX}) \cup D(\overline{TX})|_{\overline{\partial_0 X}}$.

Let $P \in \Psi^0_{sc,b,\mathbb{Z}/k}(X;E,F)$ and suppose that $\sigma(P)$ and $N_0(P)$ are invertible. Then the right hand side of the equality in theorem 8 is always a multiple of k, so $\mathrm{ind}(x_1^{\beta}Px_1^{-\beta}) \mod k \in \mathbb{Z}/k$ does not depends on $\beta \notin -\mathrm{ImSpec}(\widehat{N_1}(P)(\lambda))$. More strongly, following lemma holds.

Lemma 7. *Let $P, Q \in \Psi^0_{sc,b,\mathbb{Z}/k}(X;E,F)$ and suppose that $\sigma(P), \sigma(Q)$ and $N_0(P), N_1(P)$ are invertible. If $(\sigma(P), N_0(P))$ and $(\sigma(Q), N_0(Q))$ are homotopic in the space of invertible joint symbols, then* $\mathrm{ind}(x_1^{\beta}Px_1^{-\beta}) \equiv \mathrm{ind}(x_1^{\beta}Qx_1^{-\beta}) \mod k$ *for* $\beta \notin -\mathrm{ImSpec}(\widehat{N_1}(P)(\lambda)) \cup -\mathrm{ImSpec}(\widehat{N_1}(Q)(\lambda))$

Proof. Let (s_t, n_t), $0 \le t \le 1$ be a homotopy such that $(s_0, n_0) = (\sigma(P), N_0(P))$, $(s_1, n_1) = (\sigma(Q), N_0(Q))$. Take any lift R_t of (s_t, n_t) , i.e. $(\sigma(R_t), N_0(R_t)) = (s_t, n_t)$. Combining the homotopies $(1-t)P + tR_0$ and $tQ + (1-t)R_1$, we can get a homotopy S_t such that $S_0 = P$, $S_1 = Q$ and $(\sigma(S_t), N_0(S_t))$ is invertible for all $0 \le t \le 1$.

Subdivide the interval sufficiently small $[t_0, t_1], \dots, [t_{m-1}, t_m]$,$0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1$ so that we can choose $\beta_0, \dots, \beta_{m-1}$ such that $x^{\beta_i} S_t x^{-\beta_i}$ is Fredholm on $[t_i, t_{i+1}]$.

Then $\mathrm{ind}(x^{\beta_i} S_t x^{-\beta_i})$ is constant on $[t_i, t_{i+1}]$ and $\mathrm{ind}(x^{\beta_i} S_{t_i} x^{-\beta_i}) \equiv \mathrm{ind}(x^{\beta_{i-1}} S_{t_i} x^{-\beta_{i-1}})$ mod k. Thus $\mathrm{ind}(x^{\beta_0} P x^{-\beta_0}) \equiv \mathrm{ind}(x^{\beta_{m-1}} Q x^{-\beta_{m-1}}) \mod k$ and the claim is proved. □

As in [10] or [25], we can define a topological index map.

$$\text{t-ind} : K(D(\overline{TX}), S(\overline{TX}) \cup D(\overline{TX})|_{\overline{\partial_0 X}}) \to \mathbb{Z}/k$$

And the index theorem can be proved.

Theorem 9. *Let $P \in \Psi^0_{sc,b,\mathbb{Z}/k}(X; E, F)$ and suppose that $\sigma(P)$ and $N_0(P)$ are invertible, then* $\mathrm{ind}(x_1^\beta P x_1^{-\beta}) \mod k = \text{t-ind}(s(P)) \in \mathbb{Z}/k$, $\beta \notin -\mathrm{ImSpec}(\widehat{N_1}(\lambda))$.

Proof. We will demonstrate two different ways to prove the theorem.

The first method is to reduce to the case when $\partial_0 X$ is empty as in [21]. Embed X into Y ,where Y is $\mathbb{Z}/k$- manifold such that $\partial_0 Y = \phi$, e.g. we can take Y as a double of X. Choose G so that $F \oplus G \simeq \underline{\mathbb{C}}^n$, by replacing P by $P \oplus \mathrm{Id}_G$, we can assume $F = \underline{\mathbb{C}}^n$ is a trivial bundle. Because $D(TX|_{\partial_0 X})$ is homotopy equivalent to $\partial_0 X$, replacing P by a homotpic element, we can assume that $\widehat{N_1}(P) : TX|_{\partial_0 X} \to \mathrm{Hom}\,(E, F)$ is constant on each fibre and given by some bundle isomorphism $\theta : E \simeq F = \mathbb{C}^n$.

Using θ, we can extend E onto Y in the obvious way. Take a cut-off function ϕ such that $\phi \equiv 1$ near $\partial_0 X$. If we choose ϕ with sufficiently small support, $Q := \theta\phi + P(1 - \phi)$ is homotopic to P via linear homotopy. Q can be extended to a b-pseudodifferential operator $\tilde{Q}$ on Y And by construction, under the excision map $K(D(\overline{TX}), S(\overline{TX}) \cup D(\overline{TX})|_{\overline{\partial_0 X}}) \to K(D(\overline{TY}), S(\overline{TY}))$, $\sigma(Q)$ is mapped to $\sigma(\tilde{Q})$ thus, $\text{t-ind}(s(Q)) = \text{t-ind}(s(\tilde{Q}))$. Obviously, $\mathrm{ind}(x_1^\beta Q x_1^{-\beta}) = \mathrm{ind}(x_1^\beta \tilde{Q} x_1^{-\beta})$ for all $\beta \notin -\mathrm{ImSpec}(\widehat{Q}(\lambda))$.

Because $\partial_0 Y$ is empty, by [10], $\text{t-ind}(s(\tilde{Q})) = \mathrm{ind}(x_1^\beta \tilde{Q} x_1^{-\beta}) \mod k$ and the theorem is proved.

For the second method, we only give an outline. We define the analytic index a-ind : $K(D(\overline{TX}), S(\overline{TX}) \cup D(\overline{TX})|_{\overline{\partial_0 X}}) \to \mathbb{Z}/k$ by $\text{a-ind}(s(P)) = \mathrm{ind}(x^\beta P x^{-\beta})$ mod k , $\beta \notin -\mathrm{ImSpec}(\widehat{P}(\lambda))$. Then it is well-defined. And we can prove that a-ind satisfies the axioms as in [5], [10] or [25].

For the part of the axiom about multiplication, we need to be careful. Let W be a closed manifold, and $P \in \Psi^m_{\Phi,b}(X; E, F)$, $m > 0$. Then in general, as in [5], $P \boxtimes \mathrm{Id}_W \notin \Psi^m_{\Phi,b}(X \times W; E, F)$ but it is contained in the completion $\overline{\Psi^m_{\Phi,b}}^I(X \times Z; E, F)$ defined in section 3.5 .

□

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