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博士論文

Bosonization with background U(1) gauge field and its realization in one-dimensional quantum many-body systems

(バックグラウンド U(1)ゲージ場存在下でのボソン化法とその一次元量子多体系での実現)

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Abstract

Analysis of the many-body system using field theory is one of the most fundamental aspects of contemporary physics and mathematical physics. Bosonization is an exemplar, as it enables one to analyze fermionic theory, typically one spatial dimension, as bosonic field theory whose low energy behavior can be easier to obtain.

In this thesis, we demonstrate the nontriviality of Dirac fermion coupled with $U(1)$ gauge field and its bosonization. We propose a new bosonization scheme which resolves all the difficulties of earlier works, such as global anomaly and mass condensation paradox. Moreover a spin chain with a twisted boundary condition, such as the XXZ spin chain with a twisted boundary condition, is successfully analyzed by our bosonization scheme.

As a related problem, we have numerically calculated the expectation value of the polarization operator, or polarization amplitude, of 1d spin chains which are described by Tomonaga-Luttinger liquid. This polarization amplitude is the expectation value of the twist operator which generates the large gauge transformation of the system. We found nontrivial power-law scalings of the polarization amplitude. Our bosonization scheme naturally leads to the correct bosonized expression of the twist operator, which gives scaling behaviors consistent with our numerical results. These findings reveal new connections between gauge field theory and lattice models.

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0.2 Introduction

Bosonization of the fermionic system coupled with gauge field is one of the most important problems in condensed matter physics and high energy physics during these decades. There exist a lot of established works related to this problem. Gauged Wess-Zumino-Witten (WZW) model is one of the most famous models which is proposed in those works [1]. We mainly discuss conformal field theory which corresponds to critical or gapless many-body systems, but there exist a lot of related established works on gapped topological systems.

Historically, the correspondence between bosonic field theory and fermionic theory was established in the research of massive-Thiring model and sine-Gordon models [2, 3]. In this correspondence, we can identify all of the field theoretic quantities, like energy spectrum and operator contents, of both of the theories. There exist lattice models which realize this correspondence, for example $S = \frac{1}{2}$ XXZ spin chain [4]. In general, bosonization is practically useful to analyze fermionic systems, because the bosonization enables one to calculate correlation function or critical exponent of the interacting system which are hard to study in fermionic representation. In mathematical terms, the bosonization called Wakimoto free field representation is the significant object of quantum representation theory which is closely related to the representation of quantum spin chains and statistical mechanical models [5, 6].

As a generalization of this correspondence, the nonabelian bosonization was established by Witten [7]. Moreover gauged WZW model was proposed and analyzed as a bosonic field coupled to gauge fields. Goodard-Kent-Olive (GKO) coset WZW model is obtained by integrating out the gauge fields of this gauged WZW models [8]. GKO coset WZW model gives a construction of a wide class of conformal field theories and it gives an efficient clue and understanding of the lattice models at criticality.

Conversely, it is also important to analyze the lattice models themselves. The analysis of the lattice models can have nontrivial implications on the effective field theories. From this perspective, the relation between quantum field theory and Lieb-Shultz-Mattis (LSM) theorem is studied recently [9, 10, 11]. The central obeject of LSM type argument is the expectation value of the following twist operator for closed system,

$$U = \exp\left(\frac{2\pi i}{L}\sum_j j n_j\right) \tag{0.2.1}$$

where n_j is fermion (or boson) number operator at site j and L is the system size. For the later use, we note here the same operator for spin systems,

$$U = \exp\left(\frac{2\pi i}{L}\sum_j j S_j^z\right) \tag{0.2.2}$$

where S_j^z is the z component of the $SU(2)$ generator at site j.

For condensed matter physicists, how to characterize quantum phases is one of the most important problems. The LSM theorem and its generalizations have been explored, because they can impose strong constraints on gapped quantum phases [12]. Related to quantum field theory, it was proposed that the one-site translation symmetry of the lattice model corresponds to global internal symmetry of the field theory in the thermodynamic limit [10, 11, 13]. In this correspondence, we can understand the phase factor of twist operator of LSM type argument as quantum anomaly of this global internal symmetry. We explain here the details of these connections between the LSM argument and the field theory.

The key assumption of the LSM type argument is the insensitivity of the gapped system under flux insertion or twisted boundary condition. More precisely, we assume there exists no gap closing under flux insertion. This condition may be usual for the gapped system, but it is not trivial to prove. Recently, Watanabe has probed such insensitivity of energy spectrum under the existence of the gap without flux [14]. He demonstrated the effect of twisted boundary to the energy gap is exponentially small. Moreover, he has also proved the insensitivity of correlation function under flux insertion.

With respect to the renormalization group, it might be natural to guess such insensitivity under flux insertion at a fixed point. However, the analysis of the critical spin chain, such as XXZ model, shows the insensitivity is far from trivial. There exist papers which show the behavior of spin chain under flux insertion or twisted boundary condition by numerical calculation and Bethe ansatz [15, 16, 17]. They have shown the sensitivity of the theory under flux insertion in the degree of the conformal anomaly and the spectrum. The degree of these change is linear to twist angle Φ. Historically, the study of twisted fermion and twisted spin chain were done independently. The former was done in high energy community [18], and the later was done in condensed matter and integrable model community. A phenomenological field theoretic understanding of these results was proposed by Kitazawa [19]. However, there has ever existed no research which establishes his work with respect to gauge theory and bosonization. With respect to the perturbation theory, the twisted boundary condition induces a marginal perturbation with conformal dimension 2 which keeps the system at criticality.

Here we explain the property of the twist operator which is the central object in the LSM type argument. This operator, denoted as U, induces a momentum shift. If we interpret it with respect to gauge theory, it can be identified as large gauge transformation[12]. The characteristic of this operator ensures the equivalence of two pictures, flux insertion and twisted boundary condition. By assuming one-site translation symmetry for the system, one can show this operator obtains the phase factor under one site translation. Recently, as we have noted, some condensed matter physicists have shown that this phase factor can be related to the global anomaly of (fermionic) quantum field theory [11, 13]. The correspondence is based on the representation of twist operator as monopole insertion of fermionic field theory. However, the bosonized expression of this operator is still unclear in spite of its importance [20]. Therefore the unified understanding with respect to bosonized gauge theory is desirable, but it was still lacking. Moreover, there exist some confusions about the bosonized representation of fermion coupled with gauge field.

Turning back to the gauge theory and CFT, it might be natural to expect the equivalence of the model between fermion coupled with gauge field and gauged WZW model. However, the equivalence of such a "conventionally gauged" WZW model with an action $S_{\mathrm{WZW}}[A]$ and the adjoint-represented fermion in the corresponding gauge field with $S_{\mathrm{Dirac}}[A]$ was questioned by Smilga (and Nekrasov) [21, 22]. It stems from an apparent contradiction in the behaviors of the fermionic bilinear condensation between these two models. More specifically speaking, the Dirac mass term does not gain expectation value in the presence of more than two instantons, while the bosonized term corresponding to the Dirac mass is always condensed. Thus the conventional gauged WZW models cannot be bosonization of complex fermions in the presence of general gauge field configurations, e.g. gauge field with nonzero instantons [21, 22]. Moreover, it was shown that the $U(1)$ boson obtains mass term by integrating out background gauge field. This result itself is quite different from G/G coset WZW model description which results in topological field theory by integrating out the background gauge field [23, 24]. Therefore, it is necessary to modify the bosonic gauged WZW models so that they can reproduce a consistent path integral with their fermionic counterparts. Furthermore, as we will see in this paper, the quantum anomaly, e.g. global chiral anomaly of Dirac fermions coupled to background gauge field cannot be reproduced in the gauged WZW model. Unfortunately, related to LSM theorem, almost no condensed matter physicist has ever paid attention to such an inconsistency of bosonization. However, the functional bosonization, which results in the same form of the bosonization by Smilga in some cases, is considered and applied to some variety of fermionic systems coupled with gauge fields in higher dimensions [25].

The field theoretic model we consider is Dirac fermion coupled with $U(1)$ background gauge field described by the following form of the action,

$$S_{\mathrm{Dirac}}[A] = \int dtdx\, i\psi^\dagger\gamma^0\gamma^\mu(\partial_\mu - ieA_\mu)\psi. \tag{0.2.3}$$

This form of the action is in almost all of the textbook of quantum field theory, but few works consider the effect of the gauge field.

In this thesis, we concentrate on the two aspects of these problems about bosonization of fermion coupled with gauge field. One is the consistency of the field theory itself. The second is the realization of field theory on the lattice model, especially the description of large gauge transformation, and twisted boundary condition of XXZ chain.

The rest of this thesis is organized as follows.

First, in chapter 1, we review and introduce the most basic bosonization of Dirac fermion and its application to 1d spin chain with twisted boundary condition.

In chapter 2, we show the problem of existing bosonization, related to the global anomaly, the definition of current and the operator contents of the theory. To resolve these problems, we propose the following form of action as the bosonization of Dirac fermion coupled with $U(1)$ gauge field.

$$\begin{aligned} S_{\text{boson}}[A] &= \int_{T^2} \left[\frac{1}{8\pi} (\partial\varphi)^2 - i\frac{e}{2\pi} \varphi dA \right] \\ &\quad - i\frac{\epsilon^{\mu\nu}}{2\pi} \left(\int_{\text{cycle}_\mu} d\varphi \right) \left(\int_{\text{cycle}_\nu} eA \right). \end{aligned} \tag{0.2.4}$$

The same form of the bosonized action was derived earier in [26] by using complex algeraic geometry. This thesis verifies their results by modern understanding of the correspondence between field theory and lattice models. For example, we show our formalism is consistent with the Bethe ansatz results of spin chain with twisted boundary condition. This chapter is the main part of this thesis. It suggests the new and general framework of bosonization which is consistent with lattice model and the anomaly of fermionic theory.

Next, in chapter 3, we discuss other realization of fermion coupled with gauge field. More precisely, we show the nontriviality of the expectation value of LSM twist operator. This expectation value is known as polarization amplitude (or polarization) in the modern theory of polarization [27, 28]. The consistency of field theory and the numerical results of the lattice models is checked with respect to polarization.

We devote the first section of chapter 2 and 3, section 2.1 and 3.1, to summarize the problems of existing works.

Finally, we mention some future problems and the conclusion in chapter 4.

Chapter 1

Bosonization and twisted boundary condition of spin chain

1.1 Bosonization without gauge field

In this section, we introduce the most basic example of bosonization of Dirac fermion without gauge field. The equality of two representations is shown in the level of the torus partition function.

To make the paper self-contained, we give the Minkowskian action of free boson and Dirac fermion:

$$\begin{aligned} S_0^{(\mathrm{b})} &= -\int dtdx \frac{1}{8\pi}\left[\partial_\mu\varphi(t,x)\partial^\mu\varphi(t,x)\right] \\ S_0^{(\mathrm{f})} &= \int dtdx\, i\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi, \end{aligned} \tag{1.1.1}$$

so that $Z = \int \mathscr{D}(\psi,\bar{\psi})\exp(iS)$, and the Minkowskian signature takes the form as $\eta = \mathrm{diag}(-1,+1)$ with $\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}$ and γ's being real, e.g. $\gamma^0 = \sigma_1$ and $\gamma^1 = i\sigma_2$ where $\vec{\sigma}$ denotes the Pauli matrices. Then the chirality can be defined as $\gamma^3 \equiv \gamma^0\gamma^1 = -\sigma_3$.

Next, we will normalize several constants and fix the conventions.

When $A_\mu = 0$, i.e. the external charge $U(1)$ electromagnetic field is vanishing, the bosonization takes the form as:

$$\psi(z) = \frac{1}{\sqrt{L}} :\exp[-i\phi(z)]:, \tag{1.1.2}$$

$$\bar{\psi}(\bar{z}) = \frac{1}{\sqrt{L}} :\exp[i\bar{\phi}(\bar{z})]:, \tag{1.1.3}$$

$$\varphi \equiv \phi(z) + \bar{\phi}(\bar{z}), \tag{1.1.4}$$

$$\varphi \sim \varphi + 2\pi, \tag{1.1.5}$$

in which "::" denotes the normal ordering and $z \equiv x^1 + ix^0$ in Euclidean signature, namely $(x^0, x^1) \equiv (it, x)$, and the system scale L is included so that the $\psi(z)$ and $\bar{\psi}(\bar{z})$

have a scaling independent correlation function, where bars denote complex conjugation and

$$\begin{aligned} S_0^{(\mathrm{b})} &= \int \frac{1}{8\pi}(\partial\varphi)^2; &(1.1.6)\\ S_0^{(\mathrm{f})} &= \int i\psi^\dagger\gamma^0\left(\gamma^0 i\partial_0+\gamma^1\partial_1\right)\psi, &(1.1.7)\end{aligned}$$

where we write γ^μ in its Minkowskian form while space-time coordinates in the Euclidean signature, which is the reason that the form of $S_0^{(\mathrm{f})}$ is asymmetric, and $Z = \int\exp(-S)$. We can also write down the correspondence of $U(1)$ electromagnetic current operators:

$$eJ^\nu = \frac{e}{2\pi}\epsilon^{\mu\nu}\partial_\mu\varphi. \tag{1.1.8}$$

Bosonization of Dirac fermion is completed by showing the two theories are equivalent in the algebraic sense on the torus T^2 parametrized by τ, namely possessing the same spectrum. To do so, we must sum up all the winding numbers on the bosonic side:

$$\begin{aligned} &Z_0^{(\mathrm{b})} = \sum_{n,n'\in\mathbb{Z}}\int \mathscr{D}\varphi\exp(-S_0^{(\mathrm{b})}[\varphi]|_{n,n'}),\\ &\varphi(z,\bar z) = \varphi(z+1,\bar z+1)-2\pi n;\\ &\varphi(z,\bar z) = \varphi(z+\tau,\bar z+\bar\tau)-2\pi n'. \end{aligned}\tag{1.1.9}$$

It has been proven that the spectrum is equivalent with the fermionic one as long as we sum up the spin structures of Dirac fermion:

$$\begin{aligned} &Z_0^{(\mathrm{f})} = \sum_{s_1,s_2\in\{-1,+1\}}\int \mathscr{D}(\psi,\bar\psi)\exp(-S_0^{(\mathrm{f})}[\psi,\bar\psi]|_{s_1,s_2});\\ &\psi(z+1) = -s_1\psi(z);\ \ \bar\psi(\bar z+1) = -s_1\bar\psi(\bar z);\\ &\psi(z+\tau) = -s_2\psi(z);\ \ \bar\psi(\bar z+\bar\tau) = -s_2\bar\psi(\bar z). \end{aligned}\tag{1.1.10}$$

Then

$$Z_0^{(\mathrm{b})}(\tau) = Z_0^{(\mathrm{f})}(\tau) = \sum_{i=1}^{4}\frac{1}{2}\left(\left|\frac{\theta_i(\tau)}{\eta(\tau)}\right|^2\right), \tag{1.1.11}$$

where the Dedekind function is defined as $\eta(\tau)\equiv q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ and $\theta_i(\tau)$'s are the Theta functions [29] with $q\equiv\exp(i2\pi\tau)$ and $\bar q = \exp(-i2\pi\bar\tau)$. It should be noted in advance that, when there is a nonzero background gauge field, the summation weight in Eq. (1.1.11) is not equal for every spin structure to be shown later.

1.2 Field theory for XXZ spin chain with twisted boundary condition

The effect of the twisted boundary condition of spin $\frac{1}{2}$ XXZ chain has been considered in the context of integrability. Numerical calculation of this model was achieved by the seminal work by [15]. The excitation spectrum is considered in [30]. The field theoretic analysis and their relation to quantum group were considered by [31, 32].

The more combinatorial approach on this effect was considered in the context of polynomial and integrable field theory. Combinatorial equivalence of this boson-fermion correspondence is called Rogers-Ramanujan identity [33, 34]. This identity relates the character of minimal model to q deformed fermionic sum.

However, except for the work by Kitazawa [17, 19], such a boson-fermion correspondence has not been considered in the presence of a background gauge field. The reason for the lack of studies may be related to the interpretations of background gauge fields in the communities. The interpretation of such gauge transformation for condensed matter physicists is different from that of high energy physicists in the sense that the former does not integrate out the background gauge field. Kitazawa has numerically and phenomenologically shown that the effect of twisted boundary condition, which is induced by background gauge field, can be described by the effect of the background charge of free boson. At this stage, it is difficult to understand the equivalence of fermion with flux and boson with background charge. Hence the more systematic derivation of his results is desirable.

We review here the discussion by Kitazawa, and point out some subtlety in his argument. First of all, the Hamiltonian of XXZ Heisenberg model with length L is,

$$H_{XXZ} = J \sum_{j=1}^{L} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + (\cos\gamma)\, S_j^z S_{j+1}^z \right). \tag{1.2.1}$$

γ is the anisotropy paramter of the system. The low energy effective action of this model is given by free boson with Lagrangian,

$$S = \frac{1}{8\pi K} \int dx dt \left((\partial_t \varphi)^2 + (\partial_x \varphi)^2 \right). \tag{1.2.2}$$

$$K = \frac{\pi}{2(\pi - \gamma)}. \tag{1.2.3}$$

This Lagrangian gives a class of $c = 1$ conformal field theory which is parametrized by the Luttinger parameter K. In condensed matter physics, the system described by this Lagrangian is called Tomonaga-Luttinger liquid (TLL). We mention here several special points in the TLL class, corresponding to specific values of the Luttinger parameter. $K = 1$ corresponds to free fermionic point. At this point, the system can be described by free fermion by applying the Jordan-Wigner transformation. For discussions later, it should be noted that the Lagrangian does not contain any perturbation even at the level of quantum field theory. Apart from this point, the system contains the perturbation effect of Umklapp term $\cos 2n\phi, n =$ integer. $K = \frac{1}{2}$ point is $SU(2)$ point. The model is expected to be described by $SU(2)_1$ WZW model but it suffers from logarithmic correction. Haldane-Shastry model is a lattice model described by $SU(2)_1$ WZW without logarithmic correction. We treat the generalization of this model for general Luttinger parameter in chapter 3.

Here, we introduce the deformed Hamiltonian with flux insertion as,

$$H_{XXZ,\frac{\Phi}{L}} = J\sum_{j=1}^{L}\left(e^{-i\frac{\Phi}{L}}S_j^+S_{j+1}^- + e^{i\frac{\Phi}{L}}S_j^+S_{j+1}^- + \cos\gamma S_j^zS_{j+1}^z\right). \tag{1.2.4}$$

The field theoretic description of this model is the central concern of this thesis.

Then the transformation of magnetic and electric charge n, m under flux insertion $\frac{\Phi}{L}$ were assumed:

$$n \to n, \tag{1.2.5}$$

$$m \to m + \frac{\Phi}{2\pi} \tag{1.2.6}$$

The corresponding magnetic and electric operator is $V_{n,m} = e^{in\theta + im\varphi}$. θ is the dual field of free boson with compactification $\theta = \theta + 2\pi$. For the later use, we show here the Hamiltonian of this model,

$$H = \frac{K}{2\pi}\int dx\,(\partial_x\theta)^2 + \frac{1}{8\pi K}\int dx\,(\partial_x\phi)^2\,. \tag{1.2.7}$$

with the commutation relation,

$$[\phi(x), \partial_x\theta(y)] = 2\pi i\delta(x-y). \tag{1.2.8}$$

We have chosen the units so that the spin-wave velocity, which depends on the details of the model, becomes unity.

Let us notify here the expectation value of the most basic operators,

$$\langle\varphi(x)\varphi(y)\rangle = K\log|x-y|. \tag{1.2.9}$$

$$\langle\theta(x)\theta(y)\rangle = \frac{1}{4K}\log|x-y|. \tag{1.2.10}$$

Hence the vertex operator $V_{n,m}$ has the conformal dimension $h_{n,m} = Km^2 + \frac{1}{4K}n^2$

In spin chain, the flux insertion assigns the boundary condition of the XXZ spin chain to the following form by the unitary transformation of the Hamiltonian, $U^{-\frac{\Phi}{2\pi}}H_{XXZ,\frac{\Phi}{L}}U^{\frac{\Phi}{2\pi}}$,

$$S_{L+1}^{\pm} = e^{\pm i\Phi}S_1^{\pm}, \tag{1.2.11}$$

with $0 < \Phi < \pi$ for simplicity in this section and U as (0.2.2).

Next we assume the folowing form of Lagrangian which is consistent with the previous transformation law of the vertex operator under infinite cylinder geometry,

$$S = \frac{1}{8\pi K}\int dxdt \left\{ (\partial_t \varphi)^2 + (\partial_x \varphi)^2 \right\} + i\frac{\Phi}{2\pi}\left(\varphi\left(t=\infty\right) - \varphi\left(-t=\infty\right)\right) \tag{1.2.12}$$

Actually, this form of Lagrangian is quite similar to that of Dotsenko-Fateev models [5] with the action,

$$S = \frac{1}{8\pi}\int dxdt \left\{ (\partial_t \varphi)^2 + (\partial_x \varphi)^2 \right\} + 2i\alpha_0 \varphi(\infty). \tag{1.2.13}$$

Hence the central charge of this bosonic model with flux insertion can be obtained as the following form,

$$c = 1 - 12\left(\frac{\Phi}{2\pi}\right)^2 K. \tag{1.2.14}$$

At this stage, we will introduce the state corresponding to the vertex operator. The action gives the transformation law of state from usual free boson $|n, m\rangle_0$ to twisted one as,

$$|n, m\rangle_\Phi \rightarrow |n, m + \frac{\Phi}{2\pi}\rangle_0. \tag{1.2.15}$$

Energy excitation from vaccum without twist is,

$$E_{n,m}(\Phi) - E_{0,0}(0) = Km^2 + \frac{n^2}{4K} \tag{1.2.16}$$

The similarity between (1.2.12) and (1.2.13) suggests some correspondence between twisted XXZ chain and Q state Potts model, because the latter can describe some part of Q state Potts models. Moreover, the partition function of Q state Potts model on torus was calculated by twisted free boson [35]. More recently, it was proposed the analytical continuation to $Q > 4$ is also valid [36].

For the later discussion, we discuss the transformation of $\theta \rightarrow \theta + \frac{2\Phi x\pi}{L}$. This transformation changes the action to free bosonic theory without charge insertion. However, the winding condition and momentum are changed by this transformation. This transformation is called large gauge transformation. In the fermionic theory, this large gauge transformation for the lattice model is known to be described by (0.2.1),

$$U = \exp\left(\frac{2\pi i}{L}\sum_j j n_j\right). \tag{1.2.17}$$

with n_j as fermion number operator at site j.

Finally, in that paper [19] and related papers [15, 16], the correspondence between numerical results and the analytical calculation for central charge and critical exponents are checked with consistency. However, the derivation of the bosonic theory without numerical result is still lacking.

Chapter 2

Bosonization of fermion coupled with gauge field

2.1 Problems in the bosonized representation of gauged fermion

As we have discussed in the previous introduction part or chapter 0, there exist several problems for the bosonization of Dirac fermion coupled with gauge field. In this thesis, we propose the new bosonization which is consistent with the following characteristics of fermionic theory.

1. Gauge invariance
2. Anomaly
3. Definition of current
4. Mass condensation of fermion

Here we summarize the discussion of this section for the comprehensive understanding of the readers.

For the candidate of bosonized action, the following two forms of action was considered by preceding works [21, 22].

$$S'[A] = S_0^{(b)} + i\int \frac{e}{2\pi} A_\nu \epsilon^{\mu\nu}\partial_\mu\varphi. \tag{2.1.1}$$

$$S''[A] \;=\; S'[A] - i\int \frac{e}{2\pi}\epsilon^{\mu\nu}\partial_\mu[\varphi A_\nu] = S_0^{(\mathrm{b})} - i\int \frac{e}{2\pi}\varphi F_{01}. \tag{2.1.2}$$

We note here the difficulties of these two actions. The action $S'[A]$ suffers from gauge noninvariance, mismatch of the anomaly to gauged fermion, and mass condensation paradox. The action $S''[A]$ was proposed by Smilga to resolve these problems, but it does not give the correct definition of $U(1)$ current.

To resolve all of these problems, we introduce a new bosonized action. Finally, in the next section, the equivalence between our bosonized theory and gauged fermion is shown by the equivalence of these partition function.

2.1.1 Bosonization with background gauge field

For bosonization on torus without background gauge field, it is sufficient to consider bosonization on a Riemann sphere and the calculations can be related to the case on torus by a standard treatment [29]. However, it turns out there exists no straightforward approach to bosonization with background gauge field.

In this part, we will discuss the bosonization of a single complex Dirac fermion with external background electromagnetic field in the Minkowskian signature:

$$S_{\text{Dirac}}[A] \;=\; \int dtdx\, i\psi^\dagger\gamma^0\gamma^\mu(\partial_\mu - ieA_\mu)\psi, \tag{2.1.3}$$

which, in the Euclidean space-time, becomes

$$S_{\text{Dirac}}[A] = S_0^{(\text{f})} + ie\!\int\!\left[(\psi^\dagger\gamma^0\gamma^0\psi)A_0 + (-i\psi^\dagger\gamma^0\gamma^1\psi)A_1\right],$$

where we have fixed the notation that only γ matrices are Minkowskian while all vector fields A_μ and space-time are Wick-rotated as Euclidean. Since the additional term is $i\int eJ^\mu A_\mu$ and observing Eq. (1.1.8), one reasonable candidate of the bosonized action is

$$S'[A] = S_0^{(b)} + i\int \frac{e}{2\pi}A_\nu\epsilon^{\mu\nu}\partial_\mu\varphi. \tag{2.1.4}$$

Actually this form of action can be thought of as bosonic coupling to background gauge field. More detailed calculation can be seen in [37]. However, $S'[A]$ has the following two problems that 1) $S'[A]$ is gauge-dependent [22] and 2) it has no chiral anomaly factor. In other words, we cannot think of this action as "bosonization".

2.1.2 Gauge non-invariance and mismatching of anomaly and partition function of the preceding action $S'[A]$

To see the gauge dependence of $S'[A]$, let us introduce a uniform electromagnetic tensor field:

$$F_{01} \equiv \epsilon^{\mu\nu}\partial_\mu A_\nu = \frac{2\pi}{|\text{Im}\tau|}, \tag{2.1.5}$$

where $|\text{Im}\tau|$ is the area of the spactime torus. To get a local expression of A_μ, we introduce a Dirac-string singularity, e.g. at x^* by some gauge choice. Then

$$\begin{aligned} & i\int \frac{e}{2\pi}A_\nu\epsilon^{\mu\nu}\partial_\mu\varphi \\ = \;& i\int \frac{e}{2\pi}\epsilon^{\mu\nu}\left[\partial_\mu(\varphi A_\nu) - (\partial_\mu A_\nu)\varphi\right] \\ = \;& ie[\varphi(x^*) - \varphi_{\text{ave}}], \end{aligned} \tag{2.1.6}$$

where $\varphi_{\text{ave}} \equiv \int \varphi/|\text{Im}\tau|$ is the average value of φ upon the torus. We could see that the naive imposing the duality mapping of current in Eq. (1.1.8) does not give a gauge invariant theory on the bosonic side.

Furthermore, the action $S'[A]$ suffers from the chiral anomaly mismatch of the fermionic action. The chiral transformation for the bosonic field is $\varphi \to \varphi + \text{const.}$. Obviously, the bosonic theory and the partition function defined by $S'[A]$ is invariant under such a transformation. Therefore, its chiral anomaly does not match that of a single complex fermion. One direct result from such an anomaly mismatching is the discrepancy between the partition function obtained from integrating out $\mathscr{D}\varphi$ with $S'[A]$ and the fermionic partition function in the appearance of nonzero flux: $\int eF_{01}/(2\pi) \neq 0$. By Atiyah-Singer index theorem, there must exist at least one zero mode of the Dirac operator. Then, formally,

$$Z_{\text{Dirac}}[A] \propto \prod_{k\in K} \lambda_k = 0, \tag{2.1.7}$$

where $\{\lambda_k\}_{k\in K}$ is the spectrum of Dirac operator.

However, $\int \mathscr{D}\varphi \exp(-S'[A]) \neq 0$ generically. As a typical example, let us choose the following gauge-field configuration $\{\tilde{A}_\mu\}$:

$$\begin{cases} e\tilde{A}_0^{\text{I}} = 0,\, e\tilde{A}_1^{\text{I}} = \frac{2\pi}{\epsilon}(x_0 - \tilde{x}_0), & \text{if } \vec{x} \in U_{\text{I}}; \\ e\tilde{A}_0^{\text{II}} = e\tilde{A}_1^{\text{II}} = 0, & \text{if } \vec{x} \in U_{\text{II}}. \end{cases} \tag{2.1.8}$$

Here $U_{\text{I}} \equiv [\tilde{x}_0, \tilde{x}_0 + \epsilon] \times [0, L_1)$ and $U_{\text{II}} = \bar{U}_{\text{I}}$ is its complement. It can be calculated that $\int \tilde{F}_{01}/(2\pi) = 1$ thereby $Z^{(\text{f})}[\tilde{A}] = 0$.

When we take $\epsilon \to 0^+$, it is straightforward to check that $Z'[\tilde{A}] \equiv \int \mathscr{D}\varphi \exp(-S'[\tilde{A}]) = \int \mathscr{D}\varphi \exp(-S_0^{(b)}) = Z_0^{(b)}$, where $Z_0^{(b)}$ is the partition function of free boson without background gauge field in Eq. (1.1.11) which is nonzero. Therefore, $Z'[\tilde{A}] \neq Z^{(\text{f})}[\tilde{A}]$.

Thus we come to the second candidate of bosonization by a total derivative addition as a counterterm:

$$S''[A] \;=\; S'[A] - i\int \frac{e}{2\pi}\epsilon^{\mu\nu}\partial_\mu[\varphi A_\nu] = S_0^{(\text{b})} - i\int \frac{e}{2\pi}\varphi F_{01},$$

which is explicitly gauge-invariant since the curvature tensor F_{01} is gauge-independent. This form of action was first introduced in [38] and extensively considered in [21]. It is valid if we think about the trivial topological sector or zero winding number of φ. Moreover, it is consistent with functional bosonization. We can obtain a similar form of action for the representation of polarization [39]. Actually it is consistent with the anomaly argument of LSM theorem [11, 13].

2.1.3 Mismatch of $U(1)$ electromagnetic current of the preceding action $S''[A]$

However, since φ is not single-valued, the current J^μ might not be properly coupled with A_μ in $S''[A]$. Indeed, let us take the functional derivative:

$$\begin{aligned}\delta_{ieA_\rho} S''[A] &= \frac{1}{2\pi}\epsilon^{\mu\rho}\partial_\mu\varphi - \delta_{A_\rho}\int \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\mu[\varphi A_\nu] \\ &= J^\rho + n'\delta(x_0+0^+)\epsilon^{0\rho} + n\delta(x_1+0^+)\epsilon^{1\rho},\end{aligned}$$

where, without loss of generality, just for simplicity, we have assumed the (Euclidean) rectangular spacetime (before quotiented to the torus) $[0,L_0]\times[0,L_1]$. The form of δ-functions depends how we distribute the unity between two equivalent boundary point and, by no means, will affect the following results. To see why the term "$-\delta_{A_\rho}\int\epsilon^{\mu\nu}\partial_\mu[\varphi A_\nu]/2\pi$" only gives the additional boundary current "$n'\delta(x_0-L_0^-)\epsilon^{0\rho}+n\delta(x_1-L_1^-)\epsilon^{1\rho}$", we perform the integration in the following form:

$$\begin{aligned}\int_M \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\mu[\varphi A_\nu] &= \int_M \frac{1}{2\pi} d(\varphi A) \\ &= \int_{\partial M}\frac{1}{2\pi}\varphi A + \sum_i \int_{\partial U_i}\frac{1}{2\pi}\varphi A_{(i)}.\end{aligned} \tag{2.1.9}$$

where we take M as a rectangular from which the torus is made by conventional pasting procedure. We can see that the rest bulk part "$\sum_i\int_{\partial U_i}\frac{1}{2\pi}\varphi A_{(i)}$" is gauge dependent, in which U_i's, where $A_{(i)}$ is locally well-defined depending on gauge choices, cover M. Then,

$$\sum_i \int_{\partial U_i}\frac{1}{2\pi}\varphi A_{(i)} = \sum_{i,j}{}' \int_{\partial U_i\cap\partial U_j}\frac{1}{2\pi}\varphi\, t_{ij}^{-1}dt_{ij}, \tag{2.1.10}$$

where $\sum'_{i,j}$ denote no double-counting with proper orientations of $\partial U_i\cap\partial U_j$'s, and t_{ij} is the transition function defined by $A_i = A_j + t_{ij}^{-1}dt_{ij}$. On the other hand, $\delta_{A_\rho}(t_{ij}^{-1}dt_{ij}) = \delta_{A_\rho}(A_i - A_j) = 0$, which implies such a *gauge-dependent* bulk contribution induced by Eq. (2.1.10) vanishes: $\delta_{A_\rho}\left[\sum_i\int_{\partial U_i}\varphi A_{(i)}/(2\pi)\right] = 0$. The first term $\int_{\partial M}\varphi A/2$ in the last line of Eq. (2.1.9) gives the "boundary" current:

$$\begin{aligned}-\delta_{A_\rho}\int_{\partial M}\frac{1}{2\pi}\varphi A &= \delta_{A_\rho}\left[\int_{z=0}^{z=1} n'A - \int_{z=0}^{z=\tau} nA\right] \\ &= n'\delta(x_0+0^+)\epsilon^{0\rho} + n\delta(x_1+0^+)\epsilon^{1\rho}.\end{aligned}$$

2.2 New bosonization scheme

2.2.1 Cancellation of boundary current

To cancel the additional "boundary" coupling which induces the boundary current $(-n'\delta(x_0+0^+)\epsilon^{0\rho} - n\delta(x_1+0^+)\epsilon^{1\rho})$, we *tentatively* take into consideration the following modified action S_{boson} so that $(1/i)\delta_{A_\rho}S_{\text{boson}} = J^\rho$:

$$\begin{aligned} S_{\text{boson}}|_{n,n'} &= S''[A] + i\int eA_\rho\left[\delta_{A_\rho}\int_{\partial M}\frac{1}{2\pi}\varphi A\right] \\ &= \int\left[\frac{1}{8\pi}(\partial\varphi)^2 - i\frac{e}{2\pi}\varphi\epsilon^{\mu\nu}\partial_\mu A_\nu\right] + i2\pi(-n'\alpha + n\beta), \end{aligned} \tag{2.2.1}$$

where

$$\begin{aligned} \alpha &\equiv \frac{e}{2\pi}\int_{\text{cycle}_1} dx^1 A_1(x^0 = 0, x^1); \\ \beta &\equiv \frac{e}{2\pi}\int_{\text{cycle}_0} dx^0 A_0(x^0, x^1 = 0), \end{aligned} \tag{2.2.2}$$

where "$\text{cycle}_{0,1}$" are two generating cycles of the underlying torus along real axis and τ direction, respectively. Alternatively in a compact way,

$$\begin{aligned} S_{\text{boson}}[A] &= S_0^{(\text{b})} - i\frac{e}{2\pi}\int_{T^2}\varphi dA \\ &\quad - i\frac{\epsilon^{\mu\nu}}{2\pi}\left(\int_{\text{cycle}_\mu} d\varphi\right)\left(\int_{\text{cycle}_\nu} eA\right). \end{aligned} \tag{2.2.3}$$

This is the new action of the bosonized theory we propose and verify in this thesis [26].

Actually, this form of action is consistent with the previous discussion by Kitazawa. If we take uniform flux under $\beta = 0$, and $\tau \to \infty$, then this action is,

$$S_{boson}[A] = S_0^b - i\frac{\alpha}{2\pi}\left(\int_{\text{cycle}_0} d\varphi\right) \tag{2.2.4}$$

2.2.2 Gauge invariance, matchings of anomaly and equation of motion

As one of the several necessary checks, $S_{\text{boson}}[A]$ obviously still has the correct chiral anomaly factor $\exp\left(i\nu\int eF_{01}\right)$ as $S''[A]$ by the chiral transformation $\varphi \to \varphi + 2\pi\nu$. In addition, it is also gauge-invariant since the coefficient of $2\pi\alpha$ and $2\pi\beta$ is integer despite of the fact that α and β are only gauge invariant modulo $\mathbb{Z}$ or only ($\alpha \mod \mathbb{Z}$) and ($\beta \mod \mathbb{Z}$) are gauge invariant. Furthermore, $S_{\text{boson}}[A]$ gives a correct equation of motion: $\delta_\varphi S_{\text{boson}}[A] = -\partial^2\varphi/(4\pi) - ieF_{01}/(2\pi) = 0$ beause n' and n are integer-valued which implies they are insensitive and invariant for any infinitesimal variation: $\delta_\varphi n' = \delta_\varphi n = 0$. Thus, the equation of motion $\partial^2\varphi/(4\pi) = -ieF_{01}/(2\pi)$ is exactly the equation of motion of axial current on the fermionic side and the appearance of "i" on the right-hand side is due to the Wick rotation of A_μ.

2.2.3 Resolution of Dirac mass condensation paradox

It is an appropriate point to resolve the Dirac mass condensation paradox mentioned in chapter 0. We first restate or generalize that paradox below.

Assume we have N of $U(1)$ instantons in the spacetime T^2 and, for simplicity, they are localized at spacetime points $\{x_k\}_{k=1,\cdots,N}$ or

$$eF_{01}(x) = \sum_{k=1}^{N} 2\pi\delta^2(x - x_k). \tag{2.2.5}$$

Let us evaluate the path-integral (P-T) expectation value of a series of Dirac mass bilinear term:

$$\left\langle \prod_{j=1}^{M} \bar{\Psi}(y_j)\Psi(y_j) \right\rangle_{\text{P-T}} \equiv \left\langle \prod_{j=1}^{M} \bar{\Psi}(y_j)\Psi(y_j) \right\rangle_{\text{P-T: } S_{\text{Dirac}}} \tag{2.2.6}$$

in which $\bar{\Psi}$ denotes the Dirac adjoint of Ψ and it should be distinguished from the complex conjugation notation used before for $\bar{\psi}(\bar{z})$. To evaluate the above expectation value, we expand Ψ and $\bar{\Psi}$ into their eigen-function of Dirac operator $D \equiv \gamma^\mu(\partial_\mu - ieA_\mu)$: $D\Psi_n = \lambda_n\Psi_n$ with

$$\Psi = \sum_n a_n\Psi_n,\ \bar{\Psi} = \sum_n \bar{a}_n\bar{\Psi}_n,\ \int \bar{\Psi}_m\Psi_n = \delta_{m,n}, \tag{2.2.7}$$

where $\{\bar{a}_n\}$ and $\{a_n\}$ are independent Grassmanian numbers. Then

$$\left\langle \prod_{j=1}^{M} \bar{\Psi}(y_j)\Psi(y_j) \right\rangle_{\text{P-T}} \begin{cases} \neq 0, & \text{if } M \geq N \& M = N \text{ mod } 2; \\ = 0, & \text{otherwise}, \end{cases} \tag{2.2.8}$$

where we have made use of the Atiyah-Singer index theorem which implies that the number of zero mode of Dirac operator is the instanton number N, and the "mod 2" results from the fact that, for any λ_n with Ψ_n in the spectrum of Dirac operator D, we have

$$D\gamma^3\Psi_n = -\lambda_n\gamma^3\Psi_n, \tag{2.2.9}$$

in which $\{D, \gamma^3\} = 0$ is made of use and γ^3 is canonically well-defined on any spin manifold.

Then the paradox follows: if we assume the bosonization of the fermion model is $S'[A]$ defined in Eq. (2.1.1), applying the operator correspondence $\bar{\Psi}\Psi \propto \cos\varphi$, we obtain $\left\langle \prod_{j=1}^{M} \bar{\Psi}(y_j)\Psi(y_j) \right\rangle_{\text{P-T}} \propto \left\langle \prod_{j=1}^{M} \cos\varphi(y_j) \right\rangle_{\text{P-T: } S'} \neq 0$ generically for any M value independing on N, which fails to match the fermionic statement in Eq. (2.2.8).

Thus the conventional gauged bosonic model $S'[A]$ or its generalization gauged WZW model is problematic and inconsistent with its presumed fermionic partner.

We will solve the inconsistency or paradox above by our proposed bosonization $S_{\text{boson}}[A]$ defined in Eq. (2.2.3). Due to the localized gauge-field configuration in Eq. (2.2.5), we have $S_{\text{boson}}[A] = S_0^{(\text{b})} - i\sum_{k=1}^{N}\varphi(x_k)$, and thus

$$\begin{aligned}&\left\langle \prod_{j=1}^{M}\bar{\Psi}(y_j)\Psi(y_j)\right\rangle_{\text{P-T}} \propto \left\langle \prod_{j=1}^{M}\cos\varphi(y_j)\right\rangle_{\text{P-T: }S_{\text{boson}}} \\ = &\left\langle \prod_{j,k}\frac{\exp[i\varphi(y_j)]+\exp[-i\varphi(y_j)]}{2}\exp[i\varphi(x_k)]\right\rangle_{\text{P-T: }S_0^{(\text{b})}} \\ &\begin{cases}\neq 0, & \text{if } M \geq N \& M = N \mod 2;\\ = 0, & \text{otherwise,}\end{cases}\end{aligned} \tag{2.2.10}$$

which is exactly the fermionic result in Eq. (2.2.8), and we have applied the neutrality condition for φ's path integral upon action $S_0^{(\text{b})}$. We can see that the constraint given by Atiyah index theorem on the fermionic side precisely corresponds to that by neutrality condition. Therefore, with our new bosonization $S_{\text{boson}}[A]$, the paradox brought by wrong $S'[A]$ has been resolved successfully. The similar argument may be also straightforward to be applied for higher symmetries with nontrivial fundamental homotopy group, e.g. $SU(N)/\mathbb{Z}_N$.

2.3 Verification of new bosonization

2.3.1 Spectrum with a flat background gauge field

We will take a simple case so that the duality could be seen readily. The background gauge field will be taken flat so that $F_{01} = 0$. For later convenience, let us take a more general Luttinger parameter as $1/8\pi \to 1/8\pi K$ or $S_0^{(\text{b})} \to S_{\text{T-L}}^0 \equiv S_0^{(\text{b})}/K$, though the current interest is $K = 1$. Such a generalized model corresponds to a type of interacting fermions and we will later show that the rest terms of $S_{\text{boson}}[A]$ indeed do not gain renormalization by K due to topological reasons.

We can calculate the partition function associated with $S_{\text{boson}}[K;\alpha,\beta]$ as

$$\begin{aligned}&Z_{\text{boson}}[K, A_{\text{flat}}] \\ = &\sum_{m,n}\exp(-i2\pi n\beta)q^{\frac{K}{2}\left(m+\frac{n}{2K}+\alpha\right)^2}\bar{q}^{\frac{K}{2}\left(m-\frac{n}{2K}+\alpha\right)^2}/|\eta(\tau)|^2.\end{aligned} \tag{2.3.1}$$

We can identify the charge of $\mathbb{Z}_2$ symmetry generated by $(-1)^{Q_{\mathbb{Z}_2}}: \Psi \to -\Psi$ the fermion number parity transformation or $\varphi \to \varphi + \pi$ bosonically, of a certain operator labelled by (n, m) as

$$Q_{\mathbb{Z}_2} = n, \tag{2.3.2}$$

due to $\beta = 1/2$ giving $\exp(-i2\pi n\beta) = (-1)^n$ which is its fermion number parity.

Then we arrive at the following result at the free fermion point $K = 1$:

$$\begin{aligned} Z_{\text{boson}}[A_{\text{flat}}] &= \sum_{f_{0,1}\in\{0,1/2\}} \frac{(-1)^{\delta_{f_0+f_1,1}}}{2|\eta(\tau)|^2} \left\{ \left| \vartheta\begin{bmatrix} \alpha + f_1 \\ -(\beta + f_2) \end{bmatrix}(\tau) \right|^2 \right\} \\ &= \frac{1}{2}\left\{ Z^{+,+}_{\text{Dirac}} + Z^{+,-}_{\text{Dirac}} + Z^{-,+}_{\text{Dirac}} - Z^{-,-}_{\text{Dirac}} \right\}[\alpha, \beta], \end{aligned} \tag{2.3.3}$$

where $Z^{s_1,s_2}_{\text{Dirac}}$ labels the the Dirac partition function [18] with the spin structure (s_1, s_2) defined in Eq. (1.1.10) and $\vartheta\begin{bmatrix} \alpha \\ -\beta \end{bmatrix}(\tau) \equiv \sum_{n\in\mathbb{Z}} \exp\left[i\pi(n+\alpha)^2\tau - i2\pi\beta\right]$ is the generalized Theta function. To obtain the fermionization or the inverse of the bosonization which completes the bosonization procedure, we can make use of the $\mathbb{Z}_2$ transformation defined above, whose charge is obtained in Eq. (2.3.2). We can apply this Z_2 transformation onto the Hilbert space as an operator, equivalent to inserting $(-1)^{Q_{\mathbb{Z}_2}} = \exp\left(-i\int_{\text{cycle}_1} d\varphi/2\right)$ into the bosonic path integral. Similarly, we can also apply this $\mathbb{Z}_2$ transformation to twist the bosonic wave function spatially by a defect line operator, equivalently inserting $I_{\mathbb{Z}_2} \equiv \exp\left(i\int_{\text{cycle}_0} d\varphi/2\right)$ into the bosonic path integral.

Then we can label the corresponding $\mathbb{Z}_2$ sectors by $\mathbb{Z}^{w_1,w_2}_{\text{boson}}$ where $w_1 \& w_2 \in \{\pm\}$ with "+" no $\mathbb{Z}_2$ twisting whereas "−" a $\mathbb{Z}_2$ twisting, denotes whether the $\mathbb{Z}_2$ generator is operated spatially and temporally, respectively. Specially, $Z_{\text{boson}}[A_{\text{flat}}]$ in Eq. (2.3.3) is $Z^{+,+}_{\text{boson}}$. Therefore,

$$\begin{pmatrix} Z^{+,+}_{\text{boson}} \\ Z^{+,-}_{\text{boson}} \\ Z^{-,+}_{\text{boson}} \\ Z^{-,-}_{\text{boson}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} Z^{+,+}_{\text{Dirac}} \\ Z^{+,-}_{\text{Dirac}} \\ Z^{-,+}_{\text{Dirac}} \\ Z^{-,-}_{\text{Dirac}} \end{pmatrix} \tag{2.3.4}$$

for the flat background gauge field and we can define a matrix $W_{(w_1,w_2),(s_1,s_2)}$ so that

$$Z^{w_1,w_2}_{\text{boson}}[\alpha, \beta] = \sum_{(s'_1,s'_2)} W_{(w_1,w_2),(s'_1,s'_2)} Z^{s'_1,s'_2}_{\text{Dirac}}[\alpha, \beta], \tag{2.3.5}$$

and similarly, the fermionization takes the form as

$$Z^{s_1,s_2}_{\text{Dirac}}[\alpha, \beta] = \sum_{(w'_1,w'_2)} W^{-1}_{(s_1,s_2),(w'_1,w'_2)} Z^{w'_1,w'_2}_{\text{boson}}[\alpha, \beta], \tag{2.3.6}$$

where numerically $W = W^{-1}$. The bosonization and fermionization above exactly reproduce the same results with $\alpha = \beta = 0$ except for the last column of W matrix which does not matter with a vanishing background gauge field [40, 41, 42, 43].

2.3.2 Bosonization: Duality of partition function

An observation of the flat-connection case in Eq. (2.3.3) implies the bosonic partition function cannot be dualized to some fermionic one unless all the possible spin structures are summed up by a weight determined by matrix $W_{(w_1,w_2),(s_1,s_2)}$ and its inverse. Hence, assuming these weights only depend on spin structures, we could propose that, for general fluctuating $\{A_\mu\}$ gauge field configurations,

$$\begin{aligned} &Z^{s_1,s_2}_{\text{Dirac}}[A] = \int \mathscr{D}(\psi,\bar\psi)\exp(-S_{\text{Dirac}}[\psi,\bar\psi,A]|_{s_1,s_2}); \\ &\psi(z+1) = -s_1\psi(z);\ \bar\psi(\bar z+1) = -s_1\bar\psi(\bar z); \\ &\psi(z+\tau) = -s_2\psi(z);\ \bar\psi(\bar z+\bar\tau) = -s_2\bar\psi(\bar z) \end{aligned} \tag{2.3.7}$$

and its dual

$$\begin{aligned} &Z^{w_1,w_2}_{\text{boson}}[A] = \sum_{n,n'\in\mathbb{Z}}\int_{w_1,w_2}\mathscr{D}\varphi\exp(-S_{\text{boson}}[\varphi,A]|_{n,n'}), \\ &\varphi(z,\bar z) = \varphi(z+1,\bar z+1) - 2\pi n; \\ &\varphi(z,\bar z) = \varphi(z+\tau,\bar z+\bar\tau) - 2\pi n', \end{aligned} \tag{2.3.8}$$

are related by matrix W defined by Eq. (2.3.4):

$$\begin{aligned} Z^{w_1,w_2}_{\text{boson}}[A] &= \sum_{(s_1',s_2')} W_{(w_1,w_2),(s_1',s_2')} Z^{s_1',s_2'}_{\text{Dirac}}[A]; \\ Z^{s_1,s_2}_{\text{Dirac}}[A] &= \sum_{(w_1',w_2')} W^{-1}_{(s_1,s_2),(w_1',w_2')} Z^{w_1',w_2'}_{\text{boson}}[A], \end{aligned} \tag{2.3.9}$$

which can be generalized to the interacting fermion with a general K not necessarily 1.

Let us furthermore gauge the bosonic $\mathbb{Z}_2$ symmetry or equivalently impose the identification $\theta \sim \theta + \pi$. Then

$$Z^{\mathbb{Z}_2\text{-gauge}}_{\text{boson}}[A] = \frac{1}{2}\sum_{w_{1,2}} Z^{w_1,w_2}_{\text{boson}}[A] = \frac{1}{2}\sum_{s_{1,2}} Z^{s_1,s_2}_{\text{Dirac}}[A], \tag{2.3.10}$$

which is exactly the Dirac fermion gauged by $\mathbb{Z}_2$ fermion number parity.

2.3.3 Large gauge transformation for fermion

For the later discussion, we comment on the large gauge transformation for fermionic field theory. First, let us assume the uniform flux with the following form,

$$A_1 = \frac{2\pi\alpha}{eL}, A_0 = 0 \tag{2.3.11}$$

with $\beta = 0$.

Hence it contains an additional term which represents the coupling of gauge field and fermion. Here we apply the following large gauge transformation,

$$\psi \to \exp\left(\frac{2\pi i\alpha x}{L}\right)\psi \tag{2.3.12}$$

Under this transformation, the action changes into the original action without gauge field, with extended boundary condition $\psi(L) = \exp(2\pi i\alpha)\,\psi(0)$.

There exist two points which are significant for the later discussion. The first is the large gauge transformation for fermionic system relates fermion coupled with gauge field and fermion with twisted boundary condition. The second is this transformation shift the momentum by $\frac{2\pi\alpha}{L}$.

2.4 Twisted XXZ chain

2.4.1 Correspondences between quantum XXZ chain and Potts model

In this part, we will see how the partition function of the low-energy twisted XXZ chain can give properties of the thermal operator in Q-state Potts model for $Q \leq 4$. First we see that partition function of twisted XXZ chain is exactly Eq. (2.3.1) with $\alpha = \Phi/2\pi$ and $\beta = 0$:

$$Z_{\mathrm{TLL}}(K,\Phi) \;=\; \frac{1}{|\eta(\tau)|^2}\sum_{m,n\in\mathbb{Z}} q^{\frac{K}{2}\left(m+\frac{n}{2K}+\frac{\Phi}{2\pi}\right)^2}\bar{q}^{\frac{K}{2}\left(m-\frac{n}{2K}+\frac{\Phi}{2\pi}\right)^2}. \tag{2.4.1}$$

Correspondence between twisted free boson and Potts model has been considered by [35]. In other words, our formalism is spin chain version of this work. If we think about these facts, we should be careful about Z_n projection which is considered in boundary states of symmetry protected trivial (SPT) phases, because this projection can change the conformal anomaly or the central charge of the underlying CFT [44].

The quantum Q state Potts model takes the form as

$$H^{\mathrm{P}}_{\text{Q-Potts}} = -\sum_{i=1}^{L}\sum_{k}^{Q-1}\Omega_i^k - \sum_{i=1}^{L}\sum_{k=1}^{Q-1} R_i^k R_{i+1}^{Q-k}, \tag{2.4.2}$$

with PBC: $R_{L+1} = R_1$ denoted by "P" in the superscript in "$H^{\mathrm{P}}_{\text{Potts}}$" and the $Z(q)$ algebra ($\omega \equiv \exp(i2\pi/Q)$) is satisfied by R's and Q's:

$$\Omega_i R_i = \omega^{-1} R_i \Omega_i;\;\; \Omega_i R_i^\dagger = \omega R_i^\dagger \Omega_i;\;\; \Omega_i^Q = R_i^Q = 1. \tag{2.4.3}$$

$H^{\mathrm{P}}_{\text{Q-Potts}}$ can be diagonalized into blocks labelled by $H^{\mathrm{P},q}_{\text{Q-Potts}}$ with $\prod_{i=1}^{L}\Omega_i = \omega^q$:

$$H^{\mathrm{P}}_{\text{Q-Potts}} = \mathrm{diag}\left[H^{\mathrm{P},0}_{\text{Potts}}, H^{\mathrm{P},1}_{\text{Q-Potts}}, \cdots, H^{\mathrm{P},Q-1}_{\text{Q-Potts}}\right]. \tag{2.4.4}$$

Thermal operator: ε

It has been proven that, on the operator level, the following correspondence between the ground-state sector of Potts model and twisted XXZ chain up to an irrelevant constant shift:

$$H^{\text{P},0}_{\text{Q-Potts}} = H_{\text{XXZ}}(\gamma, \Phi = 2\gamma) \tag{2.4.5}$$

by an appropriate normalization of coupling constant J_{XXZ} and setting $\gamma = \arccos(\sqrt{Q}/2)$ [15].

By finite-size scaling of correlation length, the thermal operator "ε" of Q-state Potts model lies exactly at the first excited state of the sub-Hamiltonian $H^{\text{P},0}_{\text{Q-Potts}}$. By the correspondence in Eq. (2.4.5) above, we see that the conformal properties of ε can be extracted out by the partition in Eq. (2.4.1):

$$\begin{aligned} & Z_{\text{TL}}\left(\frac{\pi}{2(\pi-\gamma)}, 2\gamma\right) \\ = \; & \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{\pi}{4(\pi-\gamma)}\left[m+\frac{n(\pi-\gamma)}{\pi}+\frac{\gamma}{\pi}\right]^2} \bar{q}^{\frac{\pi}{4(\pi-\gamma)}\left[m-\frac{n(\pi-\gamma)}{\pi}+\frac{\gamma}{\pi}\right]^2}, \end{aligned} \tag{2.4.6}$$

from which we can read off the conformal anomaly defined as the lowest conformal weight of the critical Q-state Potts model after setting $\gamma = \arccos(\sqrt{Q}/2)$:

$$c = 1 - \frac{6\arccos(\sqrt{Q}/2)^2}{\pi(\pi - \arccos(\sqrt{Q}/2))}, \tag{2.4.7}$$

and the conformal weight of ε by setting its first excited energy eigenstate labelled by $(k, n) = (1, 0)$:

$$\Delta_\varepsilon \;=\; \bar{\Delta}_\varepsilon = \frac{\pi + 2\arccos(\sqrt{Q}/2)}{4(\pi - \arccos(\sqrt{Q}/2))}, \tag{2.4.8}$$

where Δ_ε and $\bar{\Delta}_\varepsilon$ are, respectively, holomorphic and anti-holomorphic conformal dimensions of ϵ. These properties exactly match those of conformal field theories of low-energy Q-state Potts model.

Order operator σ

For the other operators such as order parameter and para-fermion operator, their location are only empirically identified in the spectrum of twisted XXZ spin chains with other twisted boundary conditions. More specifically, the order parameter σ of Potts model can be found in the spectrum of XXZ chain with twisted angle as $\Phi = \pi$ with the partition function as

$$\begin{aligned} & Z_{\text{TL}}\left(\frac{\pi}{2(\pi-\gamma)}, \pi\right) \\ = \; & \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{\pi}{4(\pi-\gamma)}\left[m+\frac{n(\pi-\gamma)}{\pi}+\frac{1}{2}\right]^2} \bar{q}^{\frac{\pi}{4(\pi-\gamma)}\left[m-\frac{n(\pi-\gamma)}{\pi}+\frac{1}{2}\right]^2}. \end{aligned}$$

The conformal dimensions of σ is empirically determined by the lowest energy eigenstate of $n = 0$ sector of XXZ Hamiltonian, namely $(k, n) = (0, 0)$:

$$\Delta_\sigma + \frac{\gamma^2}{4\pi(\pi-\gamma)} = \frac{\pi}{16(\pi-\gamma)} \quad (2.4.9)$$
$$\bar{\Delta}_\sigma + \frac{\gamma^2}{4\pi(\pi-\gamma)} = \frac{\pi}{16(\pi-\gamma)}, \quad (2.4.10)$$

which are solved as $\Delta_\sigma = \bar{\Delta}_\sigma = (\pi^2 - 4\gamma^2)/[16\pi(\pi-\gamma)]$.

2.4.2 Parafermion operators

Numerically, the parafermion operator with its spin as $\tilde{Q}/Q$ with $\tilde{Q} = 1, 2, \cdots, Q-1$ can find its location in the lowest energy eigenstate of $n = 1$ sector of the spectrum of XXZ chain with twisted angle as $\Phi = 2\pi\tilde{Q}/Q$. To obtain its conformal properties, we write down the corresponding partition function of XXZ chain:

$$Z_{\mathrm{TL}}\left(\frac{\pi}{2(\pi-\gamma)}, \frac{2\pi\tilde{Q}}{Q}\right) = \frac{1}{|\eta(\tau)|^2}\sum_{m,n\in\mathbb{Z}} q^{\frac{\pi}{4(\pi-\gamma)}\left[m+\frac{n(\pi-\gamma)}{\pi}+\frac{\tilde{Q}}{Q}\right]^2} \bar{q}^{\frac{\pi}{4(\pi-\gamma)}\left[m-\frac{n(\pi-\gamma)}{\pi}+\frac{\tilde{Q}}{Q}\right]^2},$$

which implies, after $(m, n) = (0, 1)$ is extracted out,

$$\Delta_{\mathrm{pf}} + \frac{\gamma^2}{4\pi(\pi-\gamma)} = \frac{\pi-\gamma}{4\pi} + \frac{\tilde{Q}}{2Q} + \frac{\pi\tilde{Q}^2}{4Q^2(\pi-\gamma)}; \quad (2.4.11)$$
$$\bar{\Delta}_{\mathrm{pf}} + \frac{\gamma^2}{4\pi(\pi-\gamma)} = \frac{\pi-\gamma}{4\pi} - \frac{\tilde{Q}}{2Q} + \frac{\pi\tilde{Q}^2}{4Q^2(\pi-\gamma)}, \quad (2.4.12)$$

which are solved as

$$\Delta_{\mathrm{pf}} = \frac{\pi-\gamma}{4\pi} + \frac{\pi^2\tilde{Q}^2 - \gamma^2 Q^2}{4\pi Q^2(\pi-\gamma)} + \frac{\tilde{Q}}{2Q}, \quad (2.4.13)$$
$$\bar{\Delta}_{\mathrm{pf}} = \frac{\pi-\gamma}{4\pi} + \frac{\pi^2\tilde{Q}^2 - \gamma^2 Q^2}{4\pi Q^2(\pi-\gamma)} - \frac{\tilde{Q}}{2Q}, \quad (2.4.14)$$

which exactly imply the spin as $\Delta_{\mathrm{pf}} - \bar{\Delta}_{\mathrm{pf}} = \tilde{Q}/Q$, namely the spin of the parafermion operator.

2.5 An analysis of twisted higher spin XXZ model

Our formulation of bosonization strongly indicates a profound structure of the fermionic system with flux insertion. For example, our formulation explains the result for higher spin XXZ model by Sogo [45]. It indicates the change of central charge of the spin chain by twist is universal for general systems. Here we only show spin 1 Hamiltonian of this model but the spin $S > 1$ Hamiltonian can also be constructed with respect to integrability. The spin 1 Hamiltonian is,

$$H^1_{XXZ}(\gamma) = \sum_i^L \left(\sigma_i - (\sigma_i)^2 - 2\left(\cos\gamma - 1\right)\left(\sigma_i^{\pm}\sigma_i^z + \sigma_i^z\sigma_i^{\pm}\right) - 2\sin^2\gamma\left(\sigma_i^z - (\sigma_i^z)^2 + 2\left(S_i^z\right)^2 - 2\right)\right), \tag{2.5.1}$$

with

$$\sigma_i = S_i^+S_{i+1}^- + S_i^-S_{i+1}^+ + S_i^zS_{i+1}^z, \sigma_i^{\pm} = S_i^+S_{i+1}^- + S_i^-S_{i+1}^+, \sigma_i^z = S_i^zS_{i+1}^z. \tag{2.5.2}$$

First, we assume the following decomposition of $SU(2)_k$ WZW model as parafermion and $U(1)$ WZW model, with energy mometum tensor,

$$T_{SU(2)_k} = T_{Z_k} + T_{U(1)}. \tag{2.5.3}$$

Here we assume this $U(1)$ part is described by fermionic field theory. As we have discussed in the previous section, it is possible to deform the theory by flux insertion for this sector. Therefore it may be natural to guess spin chain described by $SU(2)_k$ WZW model should change the conformal anomaly by flux insertion. Actually, higher spin XXZ model is an example of this phenomena[16].

Hence it might be reasonable to consider the deformed model described by,

$$T = T_{Z_k} + T_{DF} \tag{2.5.4}$$

T_{DF} is energy momentum tensor for free boson with background charge or Dotsenko-Fateev model. Actually, this form of action coincides with that of $\frac{SU(2)_k\times SU(2)_l}{SU(2)_{k+l}}$ coset WZW models [46].

Hence it may be natural to guess some correspondence between $\frac{SU(2)_k\times SU(2)_l}{SU(2)_{k+l}}$ coset WZW model and higher spin XXZ model because we have shown a relation between them for spin $\frac{1}{2}$-XXZ model as the simplest case of this correspondence. A correspondence between these two models with respect to the conformal anomaly and part of the excitation spectrum is shown by numerical calculation [30].

It was proposed and shown the conformal anomaly of the higher spin S-XXZ spin chain with twisted angle Φ coincides with

$$c = \frac{3S}{S+1}\left(1 - \frac{(S+1)\Phi^2}{\pi(\pi - 2S\gamma)}\right) = c_{Z_{2S}} + 1 - \frac{3S\Phi^2}{S\pi(\pi - 2S\gamma)} = c_{Z_{2S}} + 1 - 12\left(\frac{\Phi}{2\pi}\right)^2\frac{\pi S}{(\pi - 2S\gamma)} \tag{2.5.5}$$

Hence we can interprete this theory as the composition of Z_{2S} parafermion model and twisted free boson with Luttinger parameter $K = \frac{\pi S}{(\pi - 2S\gamma)}$. For $S = 1$ case, this interpretation perfectly explains the numerical and Bethe ansatz results [47]. The excitation indexed by n and m from trivial boundary condition is obtained by the composition of the spectrum of Ising model and coulomb gas model. More generally, the specturm is,

$$E_{m,n} = \Delta_{Z_{2S}} + \overline{\Delta_{Z_{2S}}} + K\left(\frac{m}{2S} + \frac{\Phi}{2\pi}\right)^2 + \frac{n^2}{4K}, \tag{2.5.6}$$

$$. \tag{2.5.7}$$

where $\Delta_{Z_{2S}}$, $\overline{\Delta_{Z_{2S}}}$ are conformal dimensions of Z_{2S} parafermion.

A similar description is also valid for usual spin-S XXZ Heisenberg model with S:half integer [48]. In this model, the central charge is one and the conformal dimensions are,

$$E_{m,n} = K\left(m + \frac{\Phi}{2\pi}\right)^2 + \frac{n^2}{4K}, \tag{2.5.8}$$

$$. \tag{2.5.9}$$

with Luttinger parameter $K = \frac{\pi S}{\pi - \gamma}$.

2.6 Summary of chapter 2

In this chapter, we have shown the new bosonization which is consistent with the characteristics of the original fermionic theory. It was also shown that the existing field theoretic description of XXZ chain can be derived by our formalism. As a preparation for the next chapter, we mention the relationship between twisted boundary condition and large gauge transformation here again. The expectation value of large gauge transformation U is described by,

$$\langle U \rangle = Z_{Dirac}[A]. \tag{2.6.1}$$

$$A^1 = \frac{-2\pi x}{L}\delta(t), A^0 = 0, \tag{2.6.2}$$

if we assume the large gauge transformation is consistent with both of fermionic field theory and latice models (1.2.17).

This theory has a global anomaly,

$$e^{-2\pi i\nu}, \tag{2.6.3}$$

with $\nu = \frac{1}{2}$ in this chapter. We have introduced the parameter ν which corresponds to the transformation $\varphi \to \varphi + 2\pi\nu$. This transformation is identified as the bosonic representation of one site translation of the lattice model. As we will show, this phase factor exactly coincides with the phase of twist operator under one site translation of the lattice model.

Moreover, this U should induce momentum shift of the theory. It enables us to analyze flux inserted system as a system of twisted boundary condition. That is the case for both fermionic and bosonized theory. In the next chapter, we will show the bosonized expression of this operator which is derived from bosonization of $Z_{Dirac}[A]$.

Chapter 3

Nontriviality of the bosonized representation of large gauge transformation

3.1 Resta polarization amplitude

In this section, we introduce the modern theory of polarization and its original background motivation in condensed matter.

One of the most important problems in condensed matter physics is to identify the electrical conduction properties of each material. As pointed out by Kohn [49], localization of electrons and the presence of a dielectric polarization density are two related essential features common to all insulating ground states of materials. As a consequence, the electric polarization could be utilized for the classification of conductor and insulator. Based on several earlier studies [50, 51, 52, 53], Resta [27] proposed a compact definition of electric polarization, which can be naturally applied to interacting systems [28] as well as to non-interacting electrons in one dimension. In Resta's framework, the polarization for a ground state of a one-dimensional periodic lattice system with length L is defined as $\mathrm{Im} z$, where

$$z := \langle\psi_0|U|\psi_0\rangle, \tag{3.1.1}$$

which we call polarization amplitude. Here $|\psi_0\rangle$ is a ground state, and

$$U := \exp\left(\frac{2\pi i}{L}\sum_{j=1}^{L} jn_j\right), \tag{3.1.2}$$

where n_j is the fermion particle number operator at site j. The argument of the exponential in Eq. (3.1.2) is proportional to the center of mass of the particles, which is related to the polarization. The exponential form makes U invariant under $j \to j+L$ and naturally compatible with the periodic boundary condition. U is nothing but the

Lieb-Schultz-Mattis twist operator, or the large gauge transformation operator [9, 12, 54]. Although it is interesting to consider extensions to higher dimensions, in this thesis we focus on one-dimensional systems.

It was argued [27, 28] that the amplitude z serves as a good indicator of electron localization in both non-interacting and interacting systems. Intuitively, the polarization would be well-defined in an insulating phase when each electron is localized around nucleus, because one can define a local dipolar vector at each site, and many-body polarization is just defined by summing it over the whole system. On the other hand, in a conducting phase electrons are moving itinerantly and polarization would be ill-defined. Then it is natural to expect that the polarization amplitude z can be an "order parameter" that distinguishes an insulating phase from conducting one. Resta conjectured that if the system is a conductor $z = 0$ and an insulator $z \neq 0$ in the thermodynamic limit $L \to \infty$.

It is easy to see this in free fermion systems. Since U induces momentum shift by $2\pi/L$ for each particle, if one operates U on a ground state of a gapless system, one particle is shifted from a Fermi point to another Fermi point, creating a particle-hole excitation. This excited state is clearly orthogonal to the initial Fermi sea ground state, thus $z = 0$. On the other hand, if the system is a band insulator, $U|\psi_0\rangle$ remains the ground state up to phase, and thus $|z| \to 1$ in the thermodynamic limit [27, 55].

However, in the presence of a lattice translation symmetry, one can immediately see that the simple criterion based on z fails when the ground state is fractionally-filled. The lattice translation operator T satisfies [12],

$$TUT^{-1} = e^{-2\pi i\nu}U, \tag{3.1.3}$$

where ν is a filling factor, i.e., the number of fermions per a unit cell. We have used the relations,

$$Tn_jT^{-1} = n_{j+1}, \tag{3.1.4}$$
$$\exp(2\pi i n_j) = 1. \tag{3.1.5}$$

This transformation law (3.1.3) can be interpreted as quantum anomaly of fermionic quantum field theory coupled with $U(1)$ gauge field [11] as we have introduced in section 2.6. It follows that

$$\begin{aligned}\langle\psi_0|U|\psi_0\rangle &= \langle\psi_0|T^{-1}TUT^{-1}T|\psi_0\rangle \\ &= e^{-2\pi i\nu}\langle\psi_0|U|\psi_0\rangle,\end{aligned} \tag{3.1.6}$$

and thus $z = 0$ when ν is not integer. In fact, this observation is fundamental in the proof of the celebrated Lieb-Schultz-Mattis (LSM) theorem [9] and some of its generalizations [12].

In a naive interpretation of z, $z = 0$ would imply that the system is always conductor when it is fractionally-filled, but it is of course not true. Indeed, the system can become a Mott insulator for any rational filling, if accompanied by a spontaneous breaking of the translation symmetry as required by the LSM theorem. Based on this observation, Aligia and Ortiz [56, 57] proposed using U^q instead of U when $\nu = p/q$ (p and q are coprime integers), i.e., they argued that the definition of polarization should be replaced by

$$z^{(q)} := \langle\psi_0|U^q|\psi_0\rangle, \tag{3.1.7}$$

so that the simple criterion $z^{(q)} \neq 0$ could be used to characterize insulators at any rational filling. They have indeed confirmed that its consistency with Kohn's criterion for insulators based on the Drude weight [49].

The behavior of $z^{(q)}$ and its generalization has been studied [54, 58, 59, 60] in various insulating states, including the VBS state, the Néel ordered state, the gapped phase of bond-alternating Heisenberg chain, and the Mott insulating phase of the extended Hubbard model. Analytical and numerical results confirmed that $z^{(q)} \neq 0$ in the thermodynamic limit. However, a comprehensive study of $z^{(q)}$ in gapless conducting phases of interacting particles has been lacking. The expected vanishing of $z^{(q)}$ in a conducting phase is already nontrivial for interacting systems. In a generic interacting system, $z^{(q)}$ does not vanish exactly in a finite-size system. Nevertheless, we expect that $z^{(q)}$ vanishes in the thermodynamic limit. If this is the case, we can ask how precisely $z^{(q)}$ vanishes as the system size increases, namely its scaling property. We may hope that the scaling of $z^{(q)}$ characterizes various gapless conducting phases.

Toward this goal, in this chapter, we study the polarization amplitude $z^{(q)}$ and its scaling in the $S = 1/2$ XXZ chain

$$H = J\sum_{j=1}^{L}\left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z\right), \tag{3.1.8}$$

for $J > 0$, with the periodic boundary condition $S_{L+1}^\alpha \equiv S_1^\alpha$ ($\alpha = x, y, z$). We will also study a few generalizations of the XXZ chain.

3.1.1 Free fermion

For free fermionic point, the vanishing of polarization amplitude can be shown exactly. The technique we use is only the Fourier transformation and anti-commutation relation.

The $S = 1/2$ XY chain, which corresponds to the special case of $\Delta = 0$ of the XXZ chain (3.1.8), can be mapped to the free fermion model

$$H = -\frac{J}{2}\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + \text{h.c.}), \tag{3.1.9}$$

by the Jordan-Wigner transformation followed by a gauge transformation

$$
\begin{aligned}
S_j^+ &= (-1)^j \exp\left(i\pi\sum_{k=1}^{j-1} c_k^\dagger c_k\right) c_j^\dagger, \\
S_j^- &= c_j \exp\left(-i\pi\sum_{k=1}^{j-1} c_k^\dagger c_k\right) (-1)^j, \\
S_j^z &= c_j^\dagger c_j - \frac{1}{2}.
\end{aligned} \tag{3.1.10}
$$

We introduce here the Hamiltonian with flux inserion as,

$$
H_\phi = -\frac{J}{2}\sum_{j=1}^{L}(e^{-i\phi}c_j^\dagger c_{j+1} + \text{h.c.}), \tag{3.1.11}
$$

For convenience in later discussions, we take $L = 4N$ (N: integer). The ground state of the Hamiltonian (3.1.9) is clearly the Fermi sea state

$$
|\psi_0\rangle = \prod_{-k_F<q<k_F} c_q^\dagger|0\rangle, \tag{3.1.12}
$$

where k_F is a Fermi momentum: $k_F = \pi/2$, and the momentum q takes values

$$
q = \frac{(2n+1)\pi}{L}, \tag{3.1.13}
$$

with $n = -N, -N+1, \ldots, N-1$. To see that $z^{(q)} \equiv 0$, we remark that U induces the momentum shift of each fermion by $2\pi/L$:

$$
Uc_q^\dagger U^{-1} = c_{q+2\pi/L}^\dagger, \tag{3.1.14}
$$

where

$$
c_q^\dagger = \frac{1}{\sqrt{L}}\sum_{j=1}^{L} e^{iqj}c_j^\dagger. \tag{3.1.15}
$$

Then, we can see that

$$
U^q|\psi_0\rangle \propto \prod_{n=1}^{q} c_{k_F+\frac{(2n-1)\pi}{L}}^\dagger c_{-k_F+\frac{(2n-1)\pi}{L}}|\psi_0\rangle, \tag{3.1.16}
$$

which is clearly orthogonal to the initial state $|\psi_0\rangle$. Therefore

$$
z^{(q)} = \langle\psi_0|U^q|\psi_0\rangle = 0 \tag{3.1.17}
$$

for q different than a multiple of L.

The key feature of the polarization operator for the later discussion is that it can be interpreted as the momentum shift operator as we have discussed field theoretic analog in the previous chapter.

Related to flux insertion problem in the previous chapter, the polarization operator induces the following form of transformation law of Hamiltonian,

$$UHU^{-1} = H_{\frac{2\pi}{L}}. \tag{3.1.18}$$

The right-hand side is the Hamiltonian with uniform flux insertion. This equation relates the fermion with flux insertion can be transformed to original fermionic model. As a generalization of this relation, the equivalence between twisted fermion and fermion with flux under polarization operator is expressed as,

$$U^q H(\Phi = 2\pi q)U^{-q} = H_{\frac{2q\pi}{L}}. \tag{3.1.19}$$

3.1.2 Controversial representations of the polarization operator

There exist several controversial representations of polarization amplitude. In this section, we review these representations for the theoretical understanding of polarization amplitude. First, we introduce the standard representation of bosonization for spin chain.

The z-component of spin operator, which corresponds to the particle number operator, is represented as

$$S_j^z = \frac{1}{2\pi}\partial_x \varphi + (-1)^j \cos(\varphi), \tag{3.1.20}$$

in the TLL theory.

It is widely known that this bosonization can explain the behavior of (multi-point) correlation functions [4]. Hence it might be natural to represent the polarization amplitude by this bosonization. This naive investigation leads to the following form of polarization amplitude,

$$\begin{aligned} z^{(q)} &= \left\langle \exp\left(\frac{2\pi q i}{L}\sum_{j=1}^{L} j \cdot S_j^z\right)\right\rangle \\ &\overset{?}{=} \left\langle \exp\left(\frac{2\pi q i}{L}\sum_{j=1}^{L} j \cdot \left(\frac{1}{2\pi}\partial_x\varphi(j) + (-1)^j \cos\varphi(j)\right)\right)\right\rangle. \end{aligned} \tag{3.1.21}$$

This may be computed in a finite-size system by techniques in conformal field theory (CFT). By partial integration, it is approximated as,

$$z^{(q)} \sim \exp\left(iq\varphi(0) - \frac{iq}{L}\int dx\varphi(x)\right) \tag{3.1.22}$$

Then we use the conformal transformation law and calculation of multipoint correlation function. Finally, we find

$$z^{(q)} \overset{?}{\propto} \left(\frac{1}{L}\right)^{q^2 K}. \tag{3.1.23}$$

However, it is not true as we will show in the next sections. In the following part of this section, we review several proposals for the polarization operator.

Earlier, Nakamura and Voit proposed the following expression [54],

$$z^{(q)} = \langle \cos\left(q\varphi(L)\right)\rangle. \tag{3.1.24}$$

However, it does not explain the characteristic of the polarization operator as a momentum shift operator.

Next, Aligia and Batista proposed an alternative description of bosonization of polarization operator based on the fact that the polarization operator induces momentum shift $\partial_x\theta \to \partial_x\theta + \frac{2\pi}{L}$ [39]. To introduce the proposal, we see again the commutation relation of bosonic field and its dual field which we have introduced in the previous chapter 1,

$$[\varphi(x), \partial_x\theta(y)] = 2\pi i\delta(x-y). \tag{3.1.25}$$

$\partial_x\theta$ is the local momentum of the system. In this language, it can be observed that the field φ behaves as local momentum shift operator. Hence by integration, we can obtain the momentum shift operator,

$$\left[-i\int_0^L dx\varphi(x), \partial_x\theta(y)\right] = 2\pi. \tag{3.1.26}$$

It suggests the form of polarization as the momentum shift operator,

$$z^{(q)} = \left\langle \exp\left(-qi\frac{1}{L}\int_0^L dx\varphi(x)\right)\right\rangle. \tag{3.1.27}$$

In this representation, we can conclude it should go to zero in the thermodynamic limit only by Ward identity or charge neutrality condition of correlation function, especially for free fermion point. It is consistent with the discussion of the previous section and the mass condensation in chapter 2.2.3. Moreover, this form is consistent with lattice translation symmetry. It is widely believed that lattice translation goes to internal transformation $\varphi \to \varphi + \pi$ in the thermodynamic limit. It is easy to see the behavior of $z^{(q)}$ under this transformation and it coincides with the lattice version of this operator.

We will see the more detailed meaning and verification of this expression in section 3.4.

3.2 XXZ model with a weak interaction

This section, we show peturbative calculation of polarization amplitude of XXZ model near free fermion point. Similar calculation for J_1-J_2 model is shown in the Appendix.

The XXZ chain (3.1.8) is generally mapped to the model of interacting fermions

$$H = -\frac{J}{2}\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + \text{h.c.}) + J\Delta\sum_{j=1}^{L}\left(c_j^\dagger c_j - \frac{1}{2}\right)\left(c_{j+1}^\dagger c_{j+1} - \frac{1}{2}\right), \tag{3.2.1}$$

by the transformation (3.1.10). When Δ is small, we can take the interaction as a perturbation, and the ground state $\rangle\psi$ of (3.1.8) is expressed as

$$|\psi\rangle = |\psi_0\rangle + \sum_n |\psi_n\rangle\frac{1}{E_0 - E_n}\langle\psi_n|V|\psi_0\rangle \tag{3.2.2}$$

up to the 1st order perturbation, where $|\psi_0\rangle$ (resp. $\{|\psi_n\rangle\}$) is a ground state (resp. excited states) for $\Delta = 0$, and V is the interaction in z-direction:

$$V = J\Delta\sum_{j=1}^{L} c_j^\dagger c_j c_{j+1}^\dagger c_{j+1}. \tag{3.2.3}$$

Then, the polarization becomes

$$z^{(2)} = \sum_n \langle\psi_0|V|\psi_n\rangle\frac{1}{E_0 - E_n}\langle\psi_n|U^2|\psi_0\rangle + \text{c.c.} \tag{3.2.4}$$

in the leading order of Δ, where we used $\langle\psi_0|U^2|\psi_0\rangle = 0$. $\langle\psi_n|U^2|\psi_0\rangle$ takes nonzero value iff

$$|\psi_n\rangle = U^2|\psi_0\rangle = c_{k_F+\frac{\pi}{L}}^\dagger c_{k_F+\frac{3\pi}{L}}^\dagger c_{-k_F+\frac{3\pi}{L}} c_{-k_F+\frac{\pi}{L}}|\psi_0\rangle, \tag{3.2.5}$$

For this $|\psi_n\rangle$ the energy becomes $E_0 - E_n = -2J(\sin\pi/L + \sin 3\pi/L)$, and hence

$$z^{(2)} = -\frac{1}{2J(\sin\frac{\pi}{L} + \sin\frac{3\pi}{L})}\langle\psi_0|V c_{k_F+\frac{\pi}{L}}^\dagger c_{k_F+\frac{3\pi}{L}}^\dagger c_{-k_F+\frac{3\pi}{L}} c_{-k_F+\frac{\pi}{L}}|\psi_0\rangle + \text{c.c.} \tag{3.2.6}$$

$$= -\frac{\Delta}{\sin\frac{\pi}{L} + \sin\frac{3\pi}{L}}\frac{1}{L}\left(-2 + 2\cos\frac{2\pi}{L}\right) \approx \frac{\pi\Delta}{L^2}, \tag{3.2.7}$$

thus we obtain the scaling law $z^{(2)} \propto 1/L^2$ near $K = 1$. This indeed demonstrates that $z^{(q)}$ can be non-vanishing in a finite-size system and shows a nontrivial power-law of the system size L, once the interaction among fermions is introduced.

Similarly, we can obtain $z^{(2s)}$ in the leading, s-th order of Δ as

$$z^{(2s)} \sim \langle\psi_0|(VR)^s U^{2s}|\psi_0\rangle + \text{c.c.}, \tag{3.2.8}$$

where we introduced an operator R as

$$R = \sum_n |\psi_n\rangle \frac{1}{E_0 - E_n} \langle\psi_n|. \tag{3.2.9}$$

Evaluating Eq. (3.2.8) similarly to Eq. (3.2.4), we find

$$z^{(2s)} \approx 2\left(\frac{\Delta}{4}\right)^s \sum_{\sigma,\tau\in S_{2s}} \epsilon_\sigma\epsilon_\tau \prod_{j=1}^{s} \frac{(k_{\sigma(2j-1)} - k_{\sigma(2j)})(k_{\tau(2j-1)} - k_{\tau(2j)})}{L\left(\sum_{l=1}^{2j}(k_{\sigma(l)} + k_{\tau(l)})\right)}, \tag{3.2.10}$$

where $k_j = (2j-1)\pi/L$, and S_{2s} is the symmetric group of degree $2s$. Each summand in (3.2.10) is proportional to $1/L^{2s}$. Hence we obtain, for $K \sim 1$ and an even integer q, the exponent as,

$$\beta(q) = q, \tag{3.2.11}$$

$$z^{(q)} \sim \left(\frac{1}{L}\right)^{\beta(q)}. \tag{3.2.12}$$

However, it should be noted that, in the present analysis, we cannot rule out the possibility that the RHS of (3.2.10) happens to vanish. We will later confirm that the results of the perturbation theory obtained here are consistent with the numerical results on the XXZ chain.

3.3 Mismatch between naive bosonization analysis and numerical results

As we have discussed in the previous sections, the field theoretic representation of polarization contains some difficulties. Hence it is natural to investigate polarization by numerical calculation of some systems for the consistency of descriptions. For that purpose, we have investigated a class of critical systems, 1d spin chain described by Tomonaga-Luttinger liquid, XXZ spin chain, J_1-J_2 model and Gutzwiller-Jastrow wave function [1]. Each theory can be described by the Tomonaga-Luttinger liquid by choosing the appropriate parameter, but the perturbations from conformal point are different for each case. We have used the exact diagonalization. q dependence of the exponents shows the most significant difference from the analytical calculation.

We have shown $z^{(q)}$ exactly vanishes for the free fermions. In the field theory, the effects of the interaction may appear as (irrelevant) perturbation to the free boson field theory. The XXZ chain has the U(1) symmetry generated by total magnetization

[1]The main contribution of numerical calculation in this section was done by the coauthor Yuya Nakagawa.

$\sum_j S_j^z$. This symmetry, which we always keep in the present paper, forbids the perturbations of the form $\cos(m\theta)$ where θ is the dual field of ϕ. Moreover, the lattice translation symmetry is represented in TLL by $\phi \to \phi + \pi$, which forbids $\cos((2n-1)\phi)$. Thus the effective action including the allowed perturbations reads

$$S[\phi] = S_0 + \sum_{n=1}^{\infty} g_{2n} \int dx d\tau \ \cos(2n\phi) + \dots . \tag{3.3.1}$$

The vertex operators $\cos(2n\phi)$ represent the Umklapp processes of various orders, with the scaling dimensions $4n^2K$. In the XXZ chain with $-1 < \Delta \leq 1$, $K \geq 1/2$ and thus the Umklapp operator is irrelevant. As long as permitted by symmetries, we generically expect any perturbation to be non-vanishing: $g_{2n} \neq 0$ for any $n = 1, 2, \dots$. For $J_1 - J_2$ model, the first Umklapp term vanishes i.e. $g_2 = 0$.

3.3.1 XXZ chain

First, let us present the results of numerical exact diagonalization of the standard XXZ chain (3.1.8) up to the system size $L = 26$. In the top left and middle left panels of Fig. 3.1, we present $z^{(2)}$ in the ground state of (3.1.8). The power-law decay of $z^{(2)}$ with L is clearly visible for $-0.5 \leq \Delta \leq 1$. However, for $\Delta < -0.5$, the power-law scaling is less clear. Especially, the data of $\Delta = -0.55$ do not show any power-law decay within the system size we can reach ($L = 26$). This seemingly strange change of the behavior across $\Delta = -0.5$ can also be seen in the left bottom panel of Fig. 3.1, where the power-law exponents β estimated from the fitting of the data of $z^{(2)}$ are plotted. Around $K = 1.5$, or $\Delta = -0.5$, the exponent β exhibits non-systematic behavior (we note that the data corresponding to $\Delta = -0.55$ is not plotted in the figure.). We see that the overall behavior of β is explained by $\beta = 4K - 2$, especially for $K < 1$. In the panels of the right column of Fig. 3.1, we present the data of $z^{(4)}$. The behaviors are qualitatively the same as those of $z^{(2)}$ and the exponent of the power-law β might be described by $\beta = 8K - 4$, which is twice of the value of the $q = 2$. Thus we can conjecture

$$\beta(q) = q(2K - 1), \tag{3.3.2}$$

for an even integer q, in the XXZ chain with $K \lesssim 1.5$.

3.3.2 J_1-J_2 XXZ chain tuned at the Gaussian point

In order to study the effect of the leading Umklapp term, next we study the J_1-J_2 model.

$$H = J_1 \sum_{j=1}^{L} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right) + J_2 \sum_{j=1}^{L} \left(S_j^x S_{j+2}^x + S_j^y S_{j+2}^y + \Delta S_j^z S_{j+2}^z \right) \tag{3.3.3}$$

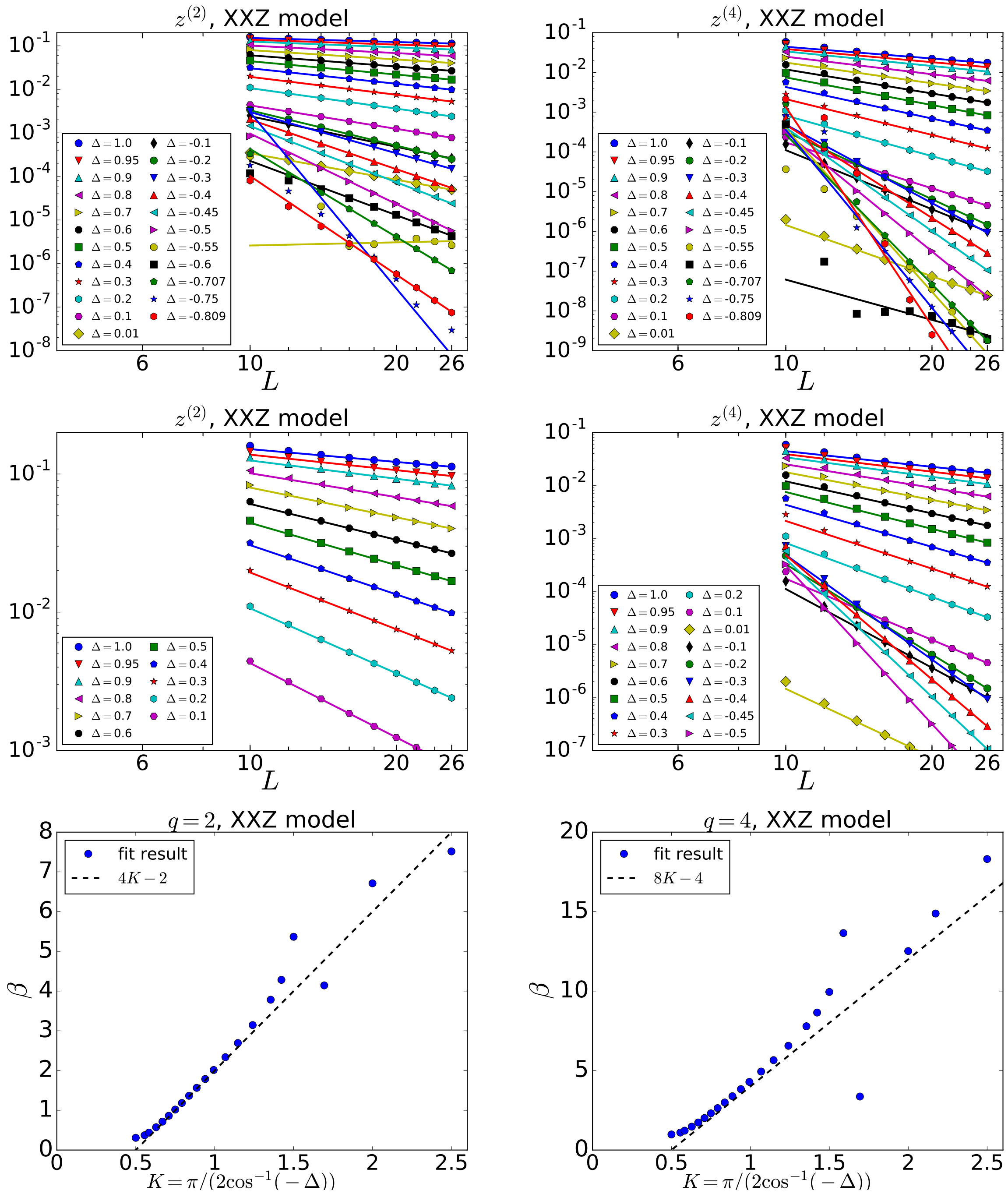

Figure 3.1: (top) Numerical results of $z^{(2)}$ (left) and $z^{(4)}$ (right) for the ground state of the XXZ chain (3.1.8) with the system size L up to $L = 26$. The dots are the numerical data and the lines are numerical fits by a simple power-law $f(L) = a/L^{\beta}$ where a and β are fitting parameters. (middle) The closeups of the top panels. (bottom) The power-law exponent β obtained from the fitting.

First we need to identify the Gaussian point $J_{2,G}(\Delta)$ where the leading Umklapp term vanishes ($g_2 = 0$). In Section A it was done analytically in the lowest order of the perturbation theory in the interaction Δ. For generic values of Δ, no explicit formula for $J_{2,G}(\Delta)$ is available. Therefore, we have to determine $J_{2,G}(\Delta)$ numerically. This was done with the level spectroscopy method [61, 62]. Setting J_2 to $J_{2,G}(\Delta)$ thus obtained, we numerically obtain the amplitude $z^{(q)}$, as we did for the standard XXZ chain. Furthermore, we also determine the Luttinger parameter K by evaluating the energy-level spacing of the system. The top left and middle left panels of Fig. 3.2 shows the results of $z^{(2)}$ obtained by exact diagonalization. $z^{(2)}$ exhibits a clear power-law decay for all values Δ even for $\Delta < -0.5$ in contrast to the XXZ chain (3.1.8) in the previous subsection. In the inset of the top left panel, the value of $J_{2,G}(\Delta)$ is also shown. As for the power-law exponent β, we numerically find that $\beta = 4K$ explains the data well for $K \lesssim 1.5$ (the bottom left panel of Fig. 3.2). We also show the numerical results for $z^{(4)}$ in the panels in the right column of Fig. 3.2, which imply $\beta = 8K$ for $K \lesssim 1.5$. Thus we conjecture

$$\beta(q) = 2qK, \tag{3.3.4}$$

for an even integer q, in the J_1-J_2 chain at the Gaussian point with $K \lesssim 1.5$. Again this is consistent with the weak-coupling result (A.0.11) near the XY point, for $K \sim 1$. Remarkably, a steep change or possible discontinuity of β is observed at $K \sim 1.5$, as in the case of the XXZ chain. Again we do not have a theoretical understanding for this phenomenon at $K \sim 1.5$.

3.3.3 Gutzwiller-Jastrow wave function

Finally, we study the polarization amplitude $z^{(q)}$ in the Gutzwiller-Jastrow wave function (3.3.6).

The lattice realization of the "fixed point" theory without the Umklapp terms is known as the Haldane-Shastry (HS) model with $1/r^2$-interaction [63, 64, 65, 66]. The Hamiltonian for a finite chain of length L reads

$$H = \frac{J\pi^2}{L^2} \sum_{n<m} \frac{\boldsymbol{S}_n \cdot \boldsymbol{S}_m}{\sin^2(\pi(n-m)/L)}. \tag{3.3.5}$$

By identifying the down-spin state as an empty site (vacuum) and the up-spin state as a particle (magnon), the ground state of this model is exactly given by the Gutzwiller-Jastrow wavefunction as a function of the locations $x_i = 1, 2, \ldots, L$ of the magnons ($i = 1, 2, \ldots, M$ is a label of magnons), as

$$\tilde{\Psi}_G(x_1, \ldots, x_M) = \prod_i z_i^{-\frac{L(L-1)}{4K}} \prod_{i<j} (z_i - z_j)^{\frac{1}{K}}, \tag{3.3.6}$$

where

$$z_j = e^{2\pi i x_j/L} = e^{i\theta_j}, \tag{3.3.7}$$

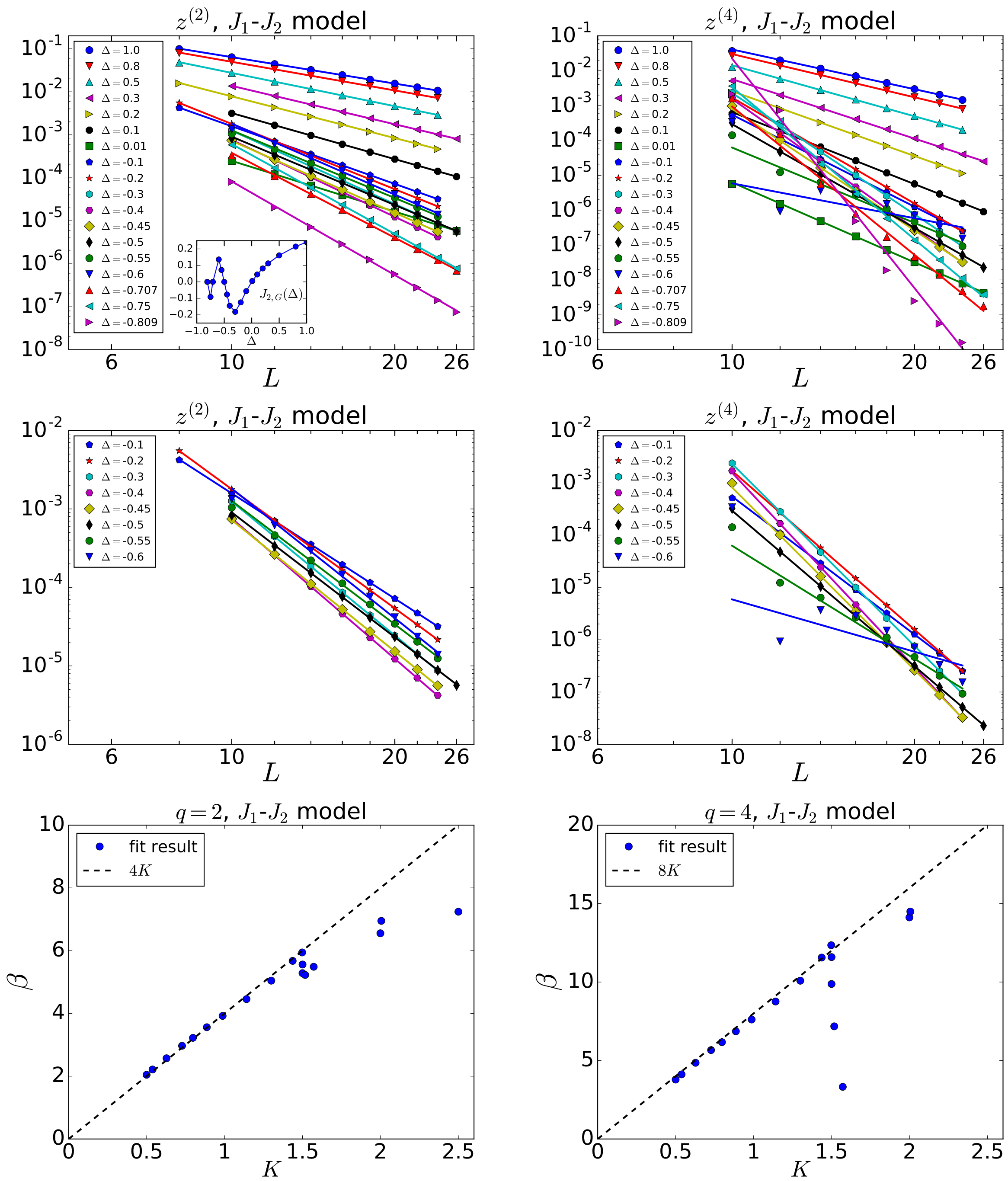

Figure 3.2: (top) Numerical results of $z^{(2)}$ (left) and $z^{(4)}$ (right) for the ground state of the J_1-J_2 XXZ model at the Gaussian fixed point with the system size L. The dots are the numerical data and the lines are numerical fits by a simple power-law $f(L) = a/L^{\beta}$ where a and β are fitting parameters. (middle) The closeups of the top panels. (bottom) The power-law exponent β obtained from the fitting.

and

$$\theta_j = \frac{2\pi x_j}{L}. \tag{3.3.8}$$

The Gutzwiller-Jastrow wavefunction (3.3.6) for a general value of K realizes the TLL with the Luttinger parameter K. It has been found that the wavefunction (3.3.6) have a large ($\sim 99.5\%$) overlap for $L = 20$, with the ground state of XXZ chain [67] and corresponds to the same Luttinger parameter K. This type of wave function also appears in various important systems, such as the Laughlin state of the fractional quantum Hall effect (FQHE) [68], and the Calogero-Sutherland state of hard-core bosons. At the special value $K = 1/2$, the TLL acquires the enhanced SU(2) symmetry, and Eq. (3.3.6) is the exact ground state of the SU(2) symmetric Haldane-Shastry model (3.3.5).

Here we study the same wave function but at different values of K. For generic values of K, we have not found exact results on $z^{(q)}$ and thus we need to evaluate $z^{(q)}$ numerically.

The results of $z^{(2)}$ are shown in the top left and middle left panels of Fig. 3.3, where one can see a clear power-law behavior of $z^{(2)}$ with L. We also present the K-dependence of the exponent of the power-law β in the bottom left panel of Fig. 3.3. For $K \lesssim 1.5$, it seems that $\beta = 4K - 1$ explains the data well. However, for $K \gtrsim 1.5$, the slope of the β-K curve becomes small and $\beta \propto 3.5K$ seems to fit the data. Generally, the finite size effect is strong for large positive K (ferromagnetic-like critical regime) as one can see in the XXZ chain and J_1-J_2 XXZ model described in the previous subsections, but in this case the difference between $K < 1.5$ and $K > 1.5$ is not due to the finite size effect because the power-law behavior is evident even for $K > 1.5$ within the accessible system size $L = 26$ in Fig. 3.3. The results of $z^{(4)}$ are qualitatively the same as those of $z^{(2)}$, so the exponent seems $\beta = 8K - 2$ for $K \lesssim 1.5$ and $\beta \propto 7K$ for $K \gtrsim 1.5$ (see the right column of Fig. 3.3). Thus, for $K \lesssim 1.5$ we conjecture that

$$\beta(q) = q\left(2K - \frac{1}{2}\right), \tag{3.3.9}$$

for an even integer q in the Gutzwiller-Jastrow wave function.

3.4 Analytical derivation

In the recent paper by Furuya and Nakamura [69], they have calculated polarization amplitude by using the argument by Kitazawa [19] and perturbation. Their results perfectly coincide with our numerical ones. However their derivation and discussion contain some ambiguities with respect to momentum shift and global anomaly because they assumed the form of the polarization operator as (3.1.22). In other words, this expression is suffering from the lattice version of mass condensation paradox. Hence, in this section, we demonstrate a more complete derivation of their results from bosonization of Dirac fermion coupled with gauge field.

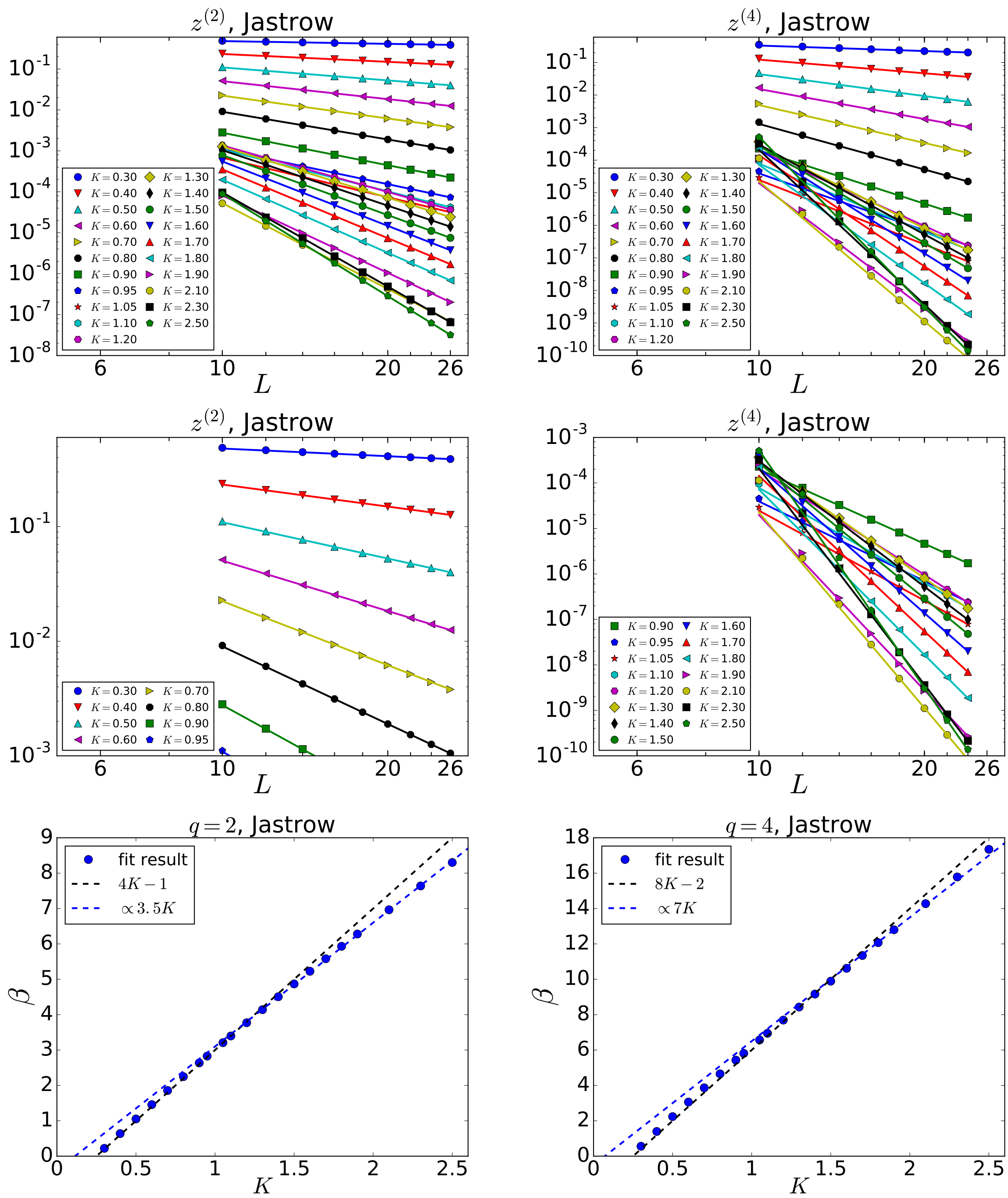

Figure 3.3: (top) Numerical results of $z^{(2)}$ (left) and $z^{(4)}$ (right) for the Gutzwiller-Jastrow wave function state (3.3.6) with the system size L. The dots are the numerical data and the lines are numerical fits by a simple power-law $f(L) = a/L^{\beta}$ where a and β are fitting parameters. (middle) The closeups of the top panels. (bottom) The power-law exponent β obtained from the fitting.

First, let us start from the fermionic expression of polarization operator.

$$\langle U^q \rangle = Z_{Dirac}[A]. \tag{3.4.1}$$

$$A^1 = \frac{-2\pi q x}{L}\delta(t), A^0 = 0. \tag{3.4.2}$$

This choice of gauge field is almost the same as that we have introduced in the section 2.6. This expression satisfies all the necessary condition as the expectation value of twist operator by definition, such as global anomaly matching and exact vanishing at free fermion point.

Then we bosonize the expression. By applying the argument in chapter 2, (2.2.3), we obtain,

$$z^{(q)} = \langle U^q \rangle = \langle e^{-iq\frac{1}{L}\int_0^L dx\phi} \rangle. \tag{3.4.3}$$

This bosonized expression is also consistent with chiral transformation $\phi \to \phi + 2\pi\nu$ which realizes lattice one site translation in quantum field theory. This expression was first introduced by Aligia and Batista as we have explained in the previous section [39].

It is also consistent with the following relation of lattice model,

$$U c_j U^{-1} = c_j \exp\left(-\frac{2\pi i j}{L}\right), \tag{3.4.4}$$

or,

$$U S_j^{\pm} U^{-1} = S_j^{-} \exp\left(\pm\frac{2\pi i j}{L}\right). \tag{3.4.5}$$

This condition is the property of momentum shift of polarization operator as $\theta \to \theta + \frac{2\pi x}{L}$ with the bosonized expression $S_j^{\pm} = \exp\left(\pm i\theta\left(j\right)\right)$. Moreover, this correspondence is true for arbitrary (even rational) power of U and results in the equivalence of flux insertion and twisted boundary condition with an arbitrary twist angle. Hence, it is consistent with our discussion for twisted boundary condition in chapter 1. This relation is significant because it gives a fundamental characteristic for polarization operator when it acts on the local operators.

Next, by using operator-state correspondence and taking the thermodynamic limit, the following expression is obtained,

$$z^{(q)} = \langle e^{-iq\frac{1}{L}\int_0^L dx\varphi(t,x)} \rangle \underset{t\to-\infty}{\sim} \langle 0|{-q} \rangle. \tag{3.4.6}$$

We have taken the charge insertion by polarization operator to $-\infty$ and we have used radial quantisation of CFT,

$$|-q\rangle = e^{-iq\varphi}|0\rangle, \tag{3.4.7}$$

with $|0\rangle$ as CFT vacuum. For convenience, we have omitted the index resulted from the vertex operator $e^{in\theta}$. Hence we can show the vanishing of the polarization amplitude only by the charge neutrality condition when the system does not contain any perturbation like $\cos n\varphi$.

The alternative derivation of this form can be obtained by considering transformation law of operator induced by U [19, 69]. In our formalism the Hamiltonian of the system is changed as,

$$UHU^{-1} = \int dx \frac{K}{2\pi}\left(\partial_x\theta + \frac{2\pi q}{L}\right)^2 + \frac{1}{8\pi K}\left(\partial_x\phi\right)^2. \qquad (3.4.8)$$

Hence we can obtain the partition function, by mapping the geometry to infinite cylinder, as,

$$Z = \int d\varphi \exp\left(-S_{\text{boson}} - iq\left(\varphi(t=\infty) - \varphi(t=-\infty)\right)\right) \qquad (3.4.9)$$

It gives the transformation law of states induced by this operator as we have discussed in the previous sections, especially section 1.2. The point is that we have expressed the transformed Hamiltonian by the original coordinate (θ, ϕ).

Above is the correct and more complete derivation of polarization amplitude using bosonization. In the following, we calculate this expectation value by the perturbation theory. The similar calculation (almost the same for XXZ spin chain case) can be seen in the paper by Furuya and Nakamura [69].

Finally, we calculate this expectation value by evaluating the change of the states $|0\rangle, |-q\rangle$ under perturbation from conformal fixed point or Umklapp term (3.3.1). Assuming the following decomposition of the Hamiltonian, $H = H_{CFT} + H'$. Then the state $|0\rangle, |-q\rangle$ change by H' as a perturbation or Umklapp term. By evaluating this perturbation, the scaling behavior of the polarization amplitude can be obtained successfully.

Let us show more detail of their discussion of the perturbation theory. The states $|0\rangle, |-q\rangle$ are changed by this perturbation,

$$|0\rangle = |0\rangle - \sum_n \frac{\langle n|H'|0\rangle}{E_n - E_0}|n\rangle + ..., \qquad (3.4.10)$$

$$|-q\rangle = |-q\rangle - \sum_n \frac{\langle n|H'|-q\rangle}{E_n - E_{-q}}|n\rangle + ... \qquad (3.4.11)$$

Hence applying the form of perturbation, the polarization amplitude should be obtained by the matrix element $\langle n|H'|m\rangle$. For the simplest example, we treat here the situation with $H' = \int_0^L dx \cos 2\varphi$ which corresponds to XXZ model. The matrix element can be calculated as [70],

$$\langle n|H'|0\rangle = \frac{L}{4\pi}\left(\frac{2\pi}{L}\right)^{4K}\left(\delta_{n,1} + \delta_{n,-1}\right) \qquad (3.4.12)$$

Higher order correction can be obtained similarly. Consequently, it is possible to obtain the polarization amplitude with exponent $q(2K-1)$, or $\langle U^q\rangle \sim \frac{1}{L^{q(2K-1)}}$ with q: even. It coincides with the numerical result of the previous section. By taking $H' = \int dx \left(\partial\varphi\right)^2 \cos 2\varphi$ for J_1-J_2 model and $H' = \int dx \partial\varphi \cos 2\varphi, \int\int dx dy \cos\varphi(x) \cos\varphi(y)$ for

Gutzwiller-Jastrow wavefunction, the analytical results are also consistent with those of the previous section.

As we have explained, the most essential point of the argument is the bosonic expression for the polarization. We can conclude the polarization as a realization of the fermionic field theory coupled with gauge field.

There exist other phenomena which support the validity of our bosonized expression of the polarization operator.

First, the exact vanishing of the operator $\langle U \rangle$ from perturbation theory can be observed with the symmetry $\varphi \to \varphi + \pi$. This is due to the charge neutrality condition and closely related to mass condensation paradox in chapter 2. Hence if we believe this expression, the vanishing of $\langle U \rangle$ should be observed even if the system is described by massive field theory. This is consistent with the original motivation of the polarization operator.

Similarly, we can also show the exact vanishing of $\{\langle U^k \rangle\}_{k=1}^{q-1}$ under the symmetry $\varphi \to \varphi + 2\pi\nu$ for Hamiltonian with $\nu = S - m = \frac{p'}{q'}$ [10, 11]. This symmetry corresponds to one-site translation symmetry of the general lattice models. If we remember this translation symmetry should restrict the perturbation terms to $H' \sim \cos nq'\varphi$ form, the change of the phase diagram with respect to q' can be easily obtained. For example, $K > 2$ is the massless phase for $q' = 1$ and $K > \frac{1}{2}$ for $q' = 2$. In these parameter regions, we can show the vanishing of the polarization amplitude $U^{q'}$ in the thermodynamic limit by the scaling relation $\langle U^{nq'} \rangle \sim L^{n(2-Kq'^2)}$. Hence we can observe the expected TLL description of the general spin XXZ Heisenberg model is changed by S =integer or not [48] under the identification $S = \frac{p'}{q'}$ and this change coincides the scaling behavior of the polarization amplitude. Especially, the connectivity to $SU(2)_1$ point is changed by this symmetry [71]. In other words, the gaplessness of the spin chain was protected by the symmetry. Therefore, we can understand the Haldane conjecture as a consequence of symmetry protection. This is also consistent with non-vanishing of $\langle U^{q'} \rangle$ from perturbation theory and the charge neutrality condition. In the massive phase, the relevant operator should give the mass and lock $\varphi = \text{const}$. Hence the nonvanishing of the expectation value $\langle U^{q'} \rangle$ is implied by perturbation theory even in the thermodynamic limit.

In short, we have explained LSM theorem and Haldane conjecture [72] by symmetry and anomaly from the viewpoint of conformal field theory and its perturbation. Witout any interaction, the charge neutrality or Atiyah index theorem suggest the vanishing of the expectation value of elementary particle excitation $\langle U^q \rangle = \langle e^{-iq\varphi} \rangle$. Then we induce the interaction which preserves the symmetry $\varphi \to \varphi + 2\pi\frac{p'}{q'}$. Even then, perturbation theory and the charge neutrality ensure the vanishing of the expectation value $\langle U^k \rangle$ except for $k = q'n$. This is the consequence of anomaly matching. If we consider the lowest order perturbation for $\cos q'\varphi$, it gives $\langle U^{nq'} \rangle \sim L^{n(2-Kq'^2)}$. Hence the vanishing of this operator in the thermodynamic limit occurs iff $\cos q'\varphi$ is irrelevant. This explains the LSM theorem when we consider the system is described by conformal fixed point and its perturbation (please see Figure 3.4 and check $SU(2)_1$ point or $K = \frac{1}{2}$) .

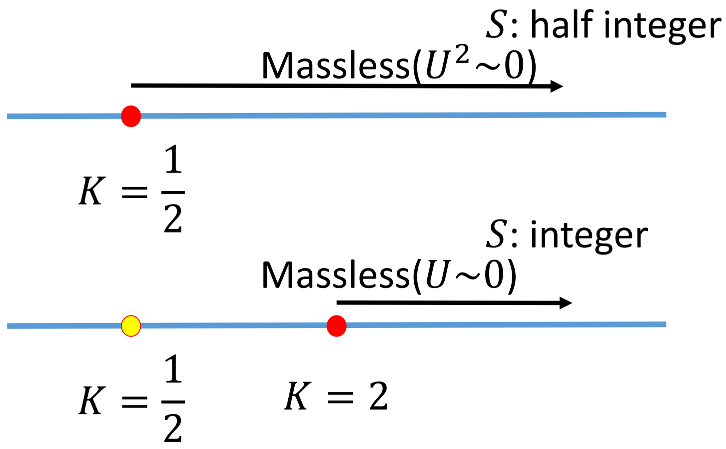

Figure 3.4: Phase diagram of TLL changed by symmetry

3.5 Summary of chapter 3

In this chapter, we have shown the numerical results of the polarization amplitude and its bosonized expression. Consequently, we have to conclude that the polarization amplitude for critical systems shows the nonuniversal behavior of the system. We mean "nonuniversal behavior" as the sensitivity of the scaling determined by the irrelevant perturbations. Moreover, we have explained the effect by using the bosonization of fermion coupled with gauge field we have proposed in the previous chapter. Our bosonized expression of polarization operator which is first proposed by Aligia and Batista is superior to the previous proposals because it reproduces correct global anomaly and momentum shift. Finally, we have analytically calculated polarization amplitude. The results is consistent with numerical results and give the reason why the naive bosonization fails to explain the behavior. Moreover, it is possible to obtain a clearer insight of LSM theorem and symmetry protection by our bosonized expression of polarization operator.

Chapter 4

Conclusion and future problem

In this thesis, we have established nontrivial bosonization of Dirac fermion coupled with $U(1)$ gauge field. Moreover, we have shown it is closely related to LSM theorem and spin chain with twisted boundary conditions. Actually, we can understand the flux insertion for fermion as charge insertion of boson. This identification clarifies the reason why twisted boundary condition induces the change of conformal anomaly [15, 19]. As an example, we have shown the mismatch of numerical result and naive bosonization. This result combined with our bosonization indicates the nontriviality of the description of fermion coupled with gauge field. Applications of our method to lattice models with background gauge field (including experimenal realization of such models by cold atom) may be interesting [73].

Our results suggest fruitful structures of fermionic systems coupled with gauge fields, under the identification of fermionic flux insertion as bosonic background charge insertion. It is because bosonic field theory with background charge is expected to describe a wide class of CFTs. Historically, the first example of such a bosonic representation of CFT was considered by Dotsenko and Fateev [5, 6], and its generalization is known as Wakimoto free field representation [6]. Hence we propose some correspondence between affine Toda field theory and $SU(N)$ spin chain with twisted boundary condition as a generalization of our bosonization [74]. It may also be interesting to consider our bosonization procedure to coupled wire construction of topological phase. Actually, there exist some controversial explanations for fermion coupled with gauge field and its bosonized representation under coupled wire construction [75, 76]. Our formulation suggests this problem may be the result of the nontriviality of bosonization. It may also be interesting to consider the relation with our formulation and fermionic SPT [77].

As a future problem, it may be interesting to establish the relation between (massive) RG flow and our bosonization [78]. Our bosonization shows the sensitivity of fixed point under the flux insertion. However, it should result in some form of the insensitivity of the RG flow if we assume the insensitivity of gapped systems [14]. Although no one has established the influence of the flux insertion on the RG flow, as far as we know, the field theoretic argument of LSM theorem is believed to be true [11]. Moreover, we have shown the important aspects for field theoretic understanding of the polarization amplitude are operator state correspondence and symmetry of the Hamiltonian. These two aspects, combined with polarization amplitude, may shed new light on the analysis of more general lattice models, such as $SU(N)$ spin chain and higher dimensional systems [79] which are expected to described by CFT and its perturbation. Hence the understanding of change of RG flow under a twist and perturbations is important for further understanding of quantum field theory and LSM theorem.

Appendix A

J_1-J_2 model

In this appendix we show the detailed calculation of the polarization amplitude for J_1-J_2 model near free fermion point. The spin-1/2 J_1-J_2 model

$$H = J_1 \sum_{j=1}^{L} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right) + J_2 \sum_{j=1}^{L} \left(S_j^x S_{j+2}^x + S_j^y S_{j+2}^y + \Delta S_j^z S_{j+2}^z \right) \tag{A.0.1}$$

under the periodic boundary condition, can be transformed to a fermion system

$$\begin{aligned} H = &-\frac{J_1}{2} \sum_{j=1}^{L} (c_j^\dagger c_{j+1} + \text{h.c.}) + J_1 \Delta \sum_{j=1}^{L} \left(c_j^\dagger c_j - \frac{1}{2} \right) \left(c_{j+1}^\dagger c_{j+1} - \frac{1}{2} \right) + \\ &\frac{J_2}{2} \sum_{j=1}^{L} (c_j^\dagger (1 - 2c_{j+1}^\dagger c_{j+1}) c_{j+2} + \text{h.c.}) + J_2 \Delta \sum_{j=1}^{L} \left(c_j^\dagger c_j - \frac{1}{2} \right) \left(c_{j+2}^\dagger c_{j+2} - \frac{1}{2} \right), \end{aligned} \tag{A.0.2}$$

by the transformation (3.1.10). This model has several phases [61, 80] such as dimer phase or critical phase depending on Δ and J_2.

When we fix Δ, the coefficient g_2 of the leading Umklapp term is non-zero for general values of J_2 in the critical phase. However, there is a special value J_2, which we denote $J_{2,G}(\Delta)$, where $g_2 = 0$ holds. We call this point as the Gaussian point. Vanishing of the leading Umklapp term $g_2 = 0$ in the field theory can be manifested, for example, in the absence of the logarithmic correction at the critical-dimer phase transition of the Heisenberg ($\Delta = 1$) J_1-J_2 chain [81]. As explained in main text, $z^{(q)} \equiv 0$ for the free fermion system even when the system is finite ($L < \infty$). From this viewpoint, the non-zero value of $z^{(q)}$ for finite system size L may be attributed to the (irrelevant) Umklapp terms. In order to investigate the effect of irrelevant terms, we also perform perturbation analysis for J_1-J_2 model fine-tuned at the Gaussian point.

In the case of $J_1 \gg J_2 \sim \Delta$, we can take the last three terms in (A.0.2) as perturbations, and in the lowest order of Δ, $z^{(2)}$ becomes

$$z^{(2)} = \sum_n \langle\psi_0|V'|\psi_n\rangle \frac{1}{E_0 - E_n} \langle\psi_n|U^2|\psi_0\rangle + \text{c.c.}, \tag{A.0.3}$$

where

$$V' = J_1\Delta \sum_{j=1}^{L} c_j^\dagger c_j c_{j+1}^\dagger c_{j+1} - J_2 \sum_{j=1}^{L} (c_j^\dagger c_{j+1}^\dagger c_{j+1} c_{j+2} + \text{h.c.}), \tag{A.0.4}$$

which is a three-site two-body interaction term of $O(\Delta)$ that appears in (A.0.2). The first term in (A.0.4) corresponds to the J_1 z-direction interaction in the spin model, and we have already considered this type of contribution to $z^{(2)}$ in the previous subsection. The contribution of the second term in (A.0.4) is

$$\frac{J_2}{2J_1(\sin\frac{\pi}{L} + \sin\frac{3\pi}{L})} \langle\psi_0| \left(\sum_{j=1}^{L} (c_j^\dagger c_{j+1}^\dagger c_{j+1} c_{j+2} + \text{h.c.}) \right) c^\dagger_{k_F+\frac{\pi}{L}} c^\dagger_{k_F+\frac{3\pi}{L}} c_{-k_F+\frac{3\pi}{L}} c_{-k_F+\frac{\pi}{L}} \rangle\psi_0 + \text{c.c.} \tag{A.0.5}$$

$$= \frac{2J_2}{J_1(\sin\frac{\pi}{L} + \sin\frac{3\pi}{L})} \frac{1}{L} \left(\cos\frac{6\pi}{L} - 2\cos\frac{4\pi}{L} + \cos\frac{2\pi}{L} \right) \approx -\frac{2\pi J_2}{J_1 L^2}. \tag{A.0.6}$$

Therefore

$$z^{(2)} = \frac{\pi}{L^2} \left(\Delta - \frac{2J_2}{J_1} \right) + O(1/L^4) \tag{A.0.7}$$

near $K = 1$, and one can see that the scaling law becomes $z^{(2)} \propto 1/L^4$ when $J_2/J_1 = \Delta/2$, which corresponds to the Gaussian point of J_1-J_2 model in the limit of small $|\Delta|$.

For $z^{(2s)}$, s-th order perturbation contributes to the leading term. Performing the similar calculation to the previous subsection, one can see that

$$\begin{aligned} z^{(2s)} = \sum_{\sigma,\tau\in S_{2s}} \epsilon_\sigma\epsilon_\tau \prod_{j=1}^{q} \Bigg(& \frac{\Delta(e^{-ik_{\sigma(2j-1)}} - e^{-ik_{\sigma(2j)}})(e^{ik_{\tau(2j-1)}} - e^{ik_{\tau(2j)}})}{4L\left(\sum_{l=1}^{2j}(\sin k_{\sigma(l)} + \sin k_{\tau(l)})\right)} \\ & + \frac{\frac{J_2}{J_1}\left((e^{ik_{\sigma(2j-1)}} - e^{ik_{\sigma(2j)}})(e^{ik_{\tau(2j-1)}} - e^{ik_{\tau(2j)}}) + \text{c.c.}\right)}{4L\left(\sum_{l=1}^{2j}(\sin k_{\sigma(l)} + \sin k_{\tau(l)})\right)} \Bigg) + \text{c.c.}, \end{aligned} \tag{A.0.8}$$

in the lowest order of Δ. When $\Delta \neq 2J_2/J_1$, the scaling is identical to the case of XXZ model: $z^{(2s)} \propto 1/L^{2s}$ and

$$z^{(2s)} = 2\left(\frac{\Delta - \frac{2J_2}{J_1}}{4}\right)^s \sum_{\sigma,\tau\in S_{2s}} \epsilon_\sigma\epsilon_\tau \prod_{j=1}^{s} \frac{(k_{\sigma(2j-1)} - k_{\sigma(2j)})(k_{\tau(2j-1)} - k_{\tau(2j)})}{L\left(\sum_{l=1}^{2j}(k_{\sigma(l)} + k_{\tau(l)})\right)} \tag{A.0.9}$$

in the order of $1/L^{2s}$. On the other hand, at the Gaussian point $\Delta = 2J_2/J_1$, the scaling behavior drastically changes to $z^{(2s)} \propto 1/L^{4s}$, and

$$z^{(2s)} = 2\left(\frac{\Delta}{8}\right)^s \sum_{\sigma,\tau\in S_{2s}} \epsilon_\sigma \epsilon_\tau \prod_{j=1}^{s} \frac{(k^2_{\sigma(2j-1)} - k^2_{\sigma(2j)})(k^2_{\tau(2j-1)} - k^2_{\tau(2j)})}{L\left(\sum_{l=1}^{2j}(k_{\sigma(l)} + k_{\tau(l)})\right)} \tag{A.0.10}$$

in the order of $1/L^{4s}$. Thus we find

$$\beta(q) = 2q, \tag{A.0.11}$$

for an even integer q, in the J_1-J_2 chain at the Gaussian point near the XY limit ($K \sim 1$). Thus we find that the exponent β changes drastically from Eq. (3.2.12) to Eq. (A.0.11) by the fine-tuning of J_2 at the Gaussian point. This is consistent with our expectation that the Umklapp process has an important effect on the amplitude $z^{(q)}$. In fact, fine-tuning away the leading Umklapp term g_2 suppresses $z^{(q)}$ (by making the exponent β larger), as it is naturally expected.

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