博士論文

論文題目　Brane coproducts and their applications
(ブレーン余積とその応用)

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Brane coproducts and their applications

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Abstract

In this thesis, we introduce two kinds of brane coproducts, which are generalizations of the loop coproduct in string topology. In Part I, we introduce the first one, the symmetric brane coproduct, and prove its associativity, commutativity and the Frobenius compatibility with the brane product. As an application, we give an example of nontrivial composition of the brane product and coproduct in Part II. Finally in Part III, we introduce the other brane coproduct, the non-symmetric brane coproduct. Comparing it with the symmetric one, we prove some vanishing of cup products on the mapping spaces from spheres.

0 Introduction

Chas-Sullivan [CS99] constructed the *loop product* on the homology $H_*(LM)$ of the free loop space $LM = \mathrm{Map}(S^1, M)$ of a connected oriented closed manifold M. The loop product is constructed as a mixture of the Pontrjagin product $H_*(\Omega M \times \Omega M) \to H_*(\Omega M)$ defined by the composition of based loops and the intersection product $H_*(M \times M) \to H_{*-m}(M)$. Note that the intersection product is the shriek map defined by the finiteness of the codimension of the diagonal embedding $\Delta\colon M \to M \times M$. Generalizing the loop product, Cohen-Godin [CG04] constructed a topological quantum field theory (TQFT). This gives a family of operations, called *string operations*, which include the *loop coproduct*. String operations were expected to give rich structures on the homology of the free loop space. But Tamanoi [Tam10] proved that the loop coproduct is "almost" trivial, and that the composition of the loop product and coproduct is completely trivial.

Then two kinds of generalizations of string operations were constructed. One is a generalization to Gorenstein spaces due to Félix-Thomas [FT09]. Here, a space M is a Gorenstein space if its singular cochain algebra $C^*(M)$ satisfies some algebraic condition. This notion generalizes Poincaré duality spaces, and the classifying space of a connected Lie group and some Borel constructions are also Gorenstein spaces. Consider the diagram

$$\begin{tikzcd}
LM \arrow[d, "\mathrm{ev}_0\times\mathrm{ev}_{1/2}"'] & LM \times_M LM \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & LM \times LM \\
M \times M & M, \arrow[l, "\Delta"] &
\end{tikzcd} \tag{0.1}$$

where the square is a pullback diagram. Félix-Thomas constructed the loop coproduct as the composition

$$H_*(LM) \xrightarrow{\mathrm{comp}^!} H_{*-m}(LM \times_M LM) \xrightarrow{\mathrm{incl}_*} H_{*-m}(LM \times LM).$$

Here, $\mathrm{comp}^!$ is defined as a "lift" of the intersection product $\Delta^!\colon H_*(M \times M) \to H_{*-m}(M)$. Note that the intersection product is the shriek map defined from the finiteness of the codimension of the diagonal embedding $\Delta\colon M \to M \times M$.

The other generalization is the *brane product* due to Sullivan-Voronov [CHV06, Section 5]. The brane product is a generalization of the loop product to the sphere spaces $S^kM = \mathrm{Map}(S^k, M)$. This product can be defined as a mixture of the Pontrjagin product on $H_*(\Omega^k M)$ and the intersection product on $H_*(M)$, similarly to the loop product. But the other operations, especially the brane coproduct, were not generalized to the sphere spaces.

Part I: Coproducts in brane topology

In Part I, which is based on the author's paper [Wak], we construct the brane product and coproduct

$$\mu\colon H_*(M^S \times M^T) \to H_{*-m}(M^{S\#T})$$
$$\delta\colon H_*(M^{S\#T}) \to H_{*-\bar{m}}(M^S \times M^T)$$

as generalizations of the loop product and coproduct with coefficients in the field $\mathbb{Q}$ of rational numbers (or any filed of characteristic zero). Here, S and T are k-dimensional manifolds, M is a k-connected space with $\bigoplus_n \pi_n(M) \otimes \mathbb{Q}$ of finite dimension, and $\bar{m} = \dim \Omega^{k-1}M$ is the dimension of $\Omega^{k-1}M$ as a Gorenstein space. We should call it the *symmetric* brane coproduct in order to distinguish it from the non-symmetric brane coproduct, which will be introduced in Part III. Note that, when $S = T = S^k$, they have the form

$$\mu\colon H_*(S^kM \times S^kM) \to H_{*-m}(S^kM)$$
$$\delta\colon H_*(S^kM) \to H_{*-\bar{m}}(S^kM \times S^kM).$$

Now we explain the construction of the brane coproduct. As a generalization of the diagram (0.1), we consider the diagram

$$\begin{tikzcd}
M^{S\#T} \arrow[d, "\mathrm{res}"] & M^S \times_M M^T \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & M^S \times M^T \\
S^{k-1}M & M. \arrow[l, "c"]
\end{tikzcd} \tag{0.2}$$

Using this diagram, we define the brane coproduct by the composition

$$H_*(M^{S\#T}) \xrightarrow{\mathrm{comp}^!} H_{*-\bar{m}}(M^S \times_M M^T) \xrightarrow{\mathrm{incl}_*} H_{*-\bar{m}}(M^S \times M^T).$$

Here we need to define the shriek map $\mathrm{comp}^!\colon H_*(M^{S\#T}) \to H_{*-\bar{m}}(M^S \times_M M^T)$ of the map comp. Using the following theorem, we define this shriek map as a "lift" of (the dual of) the shriek map $c_!$ of the embedding $c\colon M \to S^{k-1}M$ as constant maps.

Theorem 0.3 (Corollary 3.2)**.** *Under the above assumptions, we have an isomorphism*

$$\mathrm{Ext}^l_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^{l-\bar{m}}(M).$$

In particular, for $l = \bar{m}$, we have the generator

$$c_! \in \mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong \mathbb{Q}.$$

Note that this theorem generalizes a result of Félix-Thomas [FT09] which was used to define the loop product and coproduct.

Moreover, some properties of the loop product and coproduct are generalized to the brane product and coproduct.

Theorem 0.4 (Theorem 1.5)**.** *The brane product and coproduct give a structure of a Frobenius algebra on the shifted homology* $\mathbb{H}_*(S^kM) = H_{*+m}(S^kM)$*. That is, the brane product and coproduct are (co)associative and (co)commutative, and satisfy the Frobenius compatibility.*

On the other hand, the brane product and coproduct are both nontrivial even in the case M is a manifold, unlike the loop product and coproduct.

Theorem 0.5 (Theorem 1.6)**.** *For* $M = S^{2n+1}$*, the algebra* $\mathbb{H}_*(S^2M)$ *with respect to the brane product is isomorphic to the exterior algebra* $\wedge(y, z)$ *with generators of degrees* $|y| = -2n-1$, $|z| = 2n-1$*. Moreover, the brane coproduct is given by*

$$\begin{aligned}
\delta(1) &= 1 \otimes yz - y \otimes z + z \otimes y + yz \otimes 1 \\
\delta(y) &= y \otimes yz + yz \otimes y \\
\delta(z) &= z \otimes yz + yz \otimes z \\
\delta(yz) &= -yz \otimes yz.
\end{aligned}$$

Part II: Nontrivial example of the composition of the brane product and coproduct on Gorenstein spaces

In Part II, which is based on [Wak19b], we give an example of nontrivial composition of the brane product and coproduct.

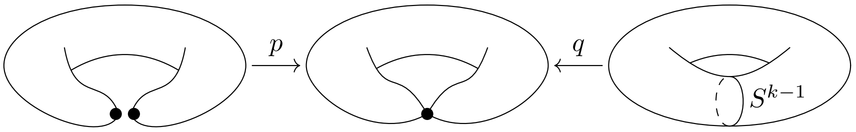
By the same method as Part I, we obtain the brane product and coproduct in the form

$$\begin{aligned}
\mu_U &\colon H_*(M^U) \to H_{*-m}(M^{U_\#}) \\
\delta_U &\colon H_*(M^{U_\#}) \to H_{*-\bar{m}}(M^U).
\end{aligned}$$

Here, U is a k-manifold with two distinct base points, and $U_\#$ is the connected sum of U with itself along the two base points. Note that, if S and T are k-manifolds and we take $U = S \coprod T$ with one base point on S and the other on T, then μ_U and δ_U coincide with μ and δ constructed in Part I.

Then we have an example of nontrivial composition of the brane product and coproduct.

Theorem 0.6 (Theorem 10.3)**.** *Let* k *be a positive even integer. Consider the case* $U = S^k$ *(and hence* $U_\# = S^{k-1} \times S^1$*) and* M *is the Eilenberg-MacLane space* $K(\mathbb{Z}, 2n)$ *with* $n > k/2$*. Then the composition* $\mu_U \circ \delta_U$ *is nontrivial.*



$$\begin{tikzcd}
M^U & M^{U_\vee} \arrow[l, "\text{comp } (= p^*)"'] \arrow[r, "\text{incl } (= q^*)"] & M^{U_\#}
\end{tikzcd}$$

Part III: New construction of the brane coproduct and vanishing of cup products on sphere spaces

In Part III, which is based on [Wak19a], we introduce another kind of the brane coproduct, which we call the *non-symmetric* brane coproduct. By comparing it with the previous one, we obtain the vanishing of some cup products on the cohomology of the sphere spaces.

Let $\mathbb{K}$ be a field (of any characteristic), M a 1-connected Poincaré duality space over $\mathbb{K}$ of dimension m, and T a manifold of dimension k. Now we consider the diagram

$$\begin{tikzcd}
M^T_{f+g} \arrow[d, "\mathrm{res}"] & S^k_f M \times_M M^T_g \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "\mathrm{pr}_1"] & S^k_f M \times M^T_g \\
D^k M & S^k_f M \arrow[l, "\iota"]
\end{tikzcd} \tag{0.7}$$

instead of the diagram (0.2). Here we write $D^kM = \mathrm{Map}(D^k, M)$. The space S^k_fM denotes the path component in S^kM containing the element $f \in S^kM$, and similar for M^T_g and M^T_{f+g}. This diagram leads us to define the non-symmetric brane coproduct as the composition

$$H_*(M^T_{f+g}) \xrightarrow{\mathrm{comp}^!} H_{*-m}(S^k_fM \times_M M^T_g) \xrightarrow{\mathrm{incl}_*} H_{*-m}(S^k_fM \times M^T_g),$$

where $\mathrm{comp}^!$ is a "lift" of (the dual of) the shriek map $\iota_!$ of the embedding ι.

Consider the case where M is k-connected and $S = T = S^k$. Then the dual

$$\delta^\vee_{\mathrm{ns}} \colon H^*(S^kM \times S^kM) \to H^{*+m}(S^kM)$$

of the non-symmetric brane coproduct can be explicitly computed as follows.

Theorem 0.8 (Theorem 19.1)**.** *The non-symmetric brane coproduct is described by*

$$\delta^\vee_{\mathrm{ns}}(u \times v) = \mathrm{ev}_0^*(\omega \cdot c^*(u)) \cdot v,$$

where $u \times v$ *denotes the cross product of* $u \in H^*(S^kM)$ *and* $v \in H^*(S^kM)$*, and* $c\colon M \to S^kM$ *is the embedding as constant maps.*

Now we additionally assume either (a) or (b) of the following.

(a) $k = 1$

(b) k is odd, $\mathrm{ch}\,\mathbb{K} = 0$ and $\bigoplus_n \pi_n(M) \otimes \mathbb{Q}$ is of finite dimension.

Then we can also consider the dual

$$\delta^\vee \colon H^*(S^kM \times S^kM) \to H^{*+\bar{m}}(S^kM)$$

of the symmetric brane coproduct, and compare it with the non-symmetric brane coproduct. The diagram (0.7) is related to the diagram (0.2) by the diagram

$$\begin{tikzcd}
S^kM \arrow[d, "\mathrm{res}"] & S^kM \times_M S^kM \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "\mathrm{pr}_1"] & S^kM \times S^kM \\
D^kM \arrow[d, "\mathrm{res}"] & S^kM \arrow[l, "\iota"] \arrow[d, "\mathrm{ev}"] & \\
S^{k-1}M & M, \arrow[l, "c"] &
\end{tikzcd}$$

where the two squares are pullback diagrams. In this diagram, the upper square coincides with that in (0.7) and the outer square coincides with that in (0.2). We use this diagram to compare the two brane coproducts.

Theorem 0.9 (Proposition 20.2 and Proposition 20.7)**.** *Under the above assumptions, we have* $\delta^\vee = \chi(M)\delta^\vee_{\mathrm{ns}}$.

The non-symmetric brane coproduct $\delta^\vee_{\mathrm{ns}}$ seems to be *non-commutative* by the explicit formula in Theorem 0.8. On the other hand, the symmetric brane coproduct $\delta^\vee$ is *commutative* by Theorem 0.4. In spite of such difference, these coproducts coincide with each other up to the scalar $\chi(M)$. This coincidence gives some nontrivial relations on $H^*(S^kM)$, and hence we obtain the main theorem of this article:

Theorem 0.10 (Theorem 15.2)**.** *Under the above assumptions, for any* $\alpha \in H^{>0}(S^kM)$*, we have*

$$\chi(M)\mathrm{ev}_0^*\omega \cdot \alpha = 0 \in H^{|\alpha|+m}(S^kM).$$

For the case $k = 1$ and M is a manifold, the theorem is proved by [Men13].

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Part I

Coproducts in brane topology

Abstract

We extend the loop product and the loop coproduct to the mapping space from the k-dimensional sphere, or more generally from any k-manifold, to a k-connected space with finite dimensional rational homotopy group, $k \geq 1$. The key to extending the loop coproduct is the fact that the embedding $M \to M^{S^{k-1}}$ is of "finite codimension" in a sense of Gorenstein spaces. Moreover, we prove the associativity, commutativity, and Frobenius compatibility of them.

1 Introduction

Chas and Sullivan [CS99] introduced the loop product on the homology $H_*(LM)$ of the free loop space $LM = \mathrm{Map}(S^1, M)$ of a manifold. Cohen and Godin [CG04] extended this product to other string operations, including the loop coproduct.

Generalizing these constructions, Félix and Thomas [FT09] defined the loop product and coproduct in the case M is a Gorenstein space. A Gorenstein space is a generalization of a manifold in the point of view of Poincaré duality, including the classifying space of a connected Lie group and the Borel construction of a connected oriented closed manifold and a connected Lie group. But these operations tend to be trivial in many cases. Let $\mathbb{K}$ be a field of characteristic zero. For example, Tamanoi showed that the loop coproduct is trivial for a manifold with the Euler characteristic zero in [Tam10, Corollary 3.2], and that the composition of the loop coproduct followed by the loop product is trivial for any manifold in [Tam10, Theorem A]. Similarly, Félix and Thomas [FT09, Theorem 14] proved that the loop product over $\mathbb{K}$ is trivial for the classifying space of a connected Lie group. A space with the nontrivial composition of loop coproduct and product is not found.

On the other hand, Sullivan and Voronov generalized the loop product to the sphere space $S^kM = \mathrm{Map}(S^k, M)$ for $k \geq 1$. This product is called the *brane product*. See [CHV06, Part I, Chapter 5].

In this article, we will generalize the loop coproduct to sphere spaces, to construct nontrivial and interesting operations. We call this coproduct the *brane coproduct*.

Here, we review briefly the construction of the loop product and the brane product. For simplicity, we assume M is a connected oriented closed manifold of dimension m. The loop product is constructed as a mixture of the Pontrjagin product $H_*(\Omega M \times \Omega M) \to H_*(\Omega M)$ defined by the composition of based loops and the intersection product $H_*(M \times M) \to H_{*-m}(M)$. More precisely, we use the following diagram

$$\begin{tikzcd}
LM \times LM \arrow[d, "\mathrm{ev}_1 \times \mathrm{ev}_1"'] & LM \times_M LM \arrow[l, "\mathrm{incl}"'] \arrow[r, "\mathrm{comp}"] \arrow[d] & LM \\
M \times M & M. \arrow[l, "\Delta"']
\end{tikzcd} \tag{1.1}$$

Here, the square is a pullback diagram by the diagonal map Δ and the evaluation map ev_1 at 1, identifying S^1 with the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$, and comp is the map defined by the composition of loops. Since the diagonal map $\Delta\colon M \to M \times M$ is an embedding of finite codimension, we have the shriek map $\Delta^!\colon H_*(M \times M) \to H_{*-m}(M)$, which is called the intersection product. Using the pullback diagram, we can "lift" $\Delta^!$ to $\mathrm{incl}^!\colon H_*(LM \times$

$LM) \to H_{*-m}(LM \times_M LM)$. Then, we define the loop product to be the composition $\mathrm{comp}_* \circ \mathrm{incl}^! \colon H_*(LM \times LM) \to H_{*-m}(LM)$.

The brane product can be defined by a similar way. Let k be a positive integer. We use the diagram

$$\begin{tikzcd}
S^kM \times S^kM \arrow[d] & S^kM \times_M S^kM \arrow[l, "\mathrm{incl}"'] \arrow[r, "\mathrm{comp}"] \arrow[d] & S^kM \\
M \times M & M. \arrow[l, "\Delta"'] &
\end{tikzcd}$$

Since the base map of the pullback diagram is the diagonal map Δ, which is the same as that for the loop product, we can use the same method to define the shriek map $\mathrm{incl}^! \colon H_*(S^kM \times S^kM) \to H_{*-m}(S^kM \times_M S^kM)$. Hence we define the brane product to be the composition $\mathrm{comp}_* \circ \mathrm{incl}^! \colon H_*(S^kM \times S^kM) \to H_{*-m}(S^kM)$.

Next, we review the loop coproduct. Using the diagram

$$\begin{tikzcd}
LM \arrow[d, "\mathrm{ev}_1 \times \mathrm{ev}_{-1}"'] & LM \times_M LM \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & LM \times LM \\
M \times M & M, \arrow[l, "\Delta"'] &
\end{tikzcd} \tag{1.2}$$

we define the loop coproduct to be the composition $\mathrm{incl}_* \circ \mathrm{comp}^! \colon H_*(LM) \to H_{*-m}(LM \times LM)$.

But the brane coproduct cannot be defined in this way. To construct the brane coproduct, we have to use the diagram

$$\begin{tikzcd}
S^kM \arrow[d, "\mathrm{res}"'] & S^kM \times_M S^kM \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & S^kM \times S^kM \\
S^{k-1}M & M. \arrow[l, "c"'] &
\end{tikzcd}$$

Here, $c \colon M \to S^{k-1}M$ is the embedding by constant maps and $\mathrm{res} \colon S^kM \to S^{k-1}M$ is the restriction map to S^{k-1}, which is embedded to S^k as the equator. In a usual sense, the base map c is not an embedding of finite codimension. But using the algebraic method of Félix and Thomas [FT09], we can consider this map as an embedding of codimension $\bar{m} = \dim \Omega^{k-1}M$, which is defined as a finite integer when the iterated loop space $\Omega^{k-1}M$ is a $\mathbb{K}$-Gorenstein space. Hence, under this assumption, we have the shriek map $c^! \colon H_*(S^{k-1}M) \to H_{*-\bar{m}}(M)$ and the lift $\mathrm{comp}^! \colon H_*(S^kM) \to H_{*-\bar{m}}(S^kM \times_M S^kM)$. This enables us to define the brane coproduct to be the composition $\mathrm{incl}_* \circ \mathrm{comp}^! \colon H_*(S^kM) \to H_{*-\bar{m}}(S^kM \times S^kM)$.

Note that, if $\oplus_n \pi_n(M) \otimes \mathbb{K}$ is of finite dimension, then $\Omega^{k-1}M$ is a $\mathbb{K}$-Gorenstein space by a result of Félix, Halperin, and Thomas; see Proposition 2.2 [FHT88, Proposition 3.4]. The converse also holds when $k \geq 2$ by [FHT88, Proposition 1.7].

More generally, using connected sums, we define the product and coproduct for mapping spaces from manifolds. Let S and T be manifolds of dimension k. Let M be a k-connected $\mathbb{K}$-Gorenstein space of finite type. Denote $m = \dim M$. Then we define the (S, T)*-brane product*

$$\mu_{ST} \colon H_*(M^S \times M^T) \to H_{*-m}(M^{S\#T})$$

using the diagram

$$\begin{tikzcd}
M^S \times M^T \arrow[d] & M^S \times_M M^T \arrow[l, "\mathrm{incl}"'] \arrow[r, "\mathrm{comp}"] \arrow[d] & M^{S\#T} \\
M \times M & M. \arrow[l, "\Delta"'] &
\end{tikzcd} \tag{1.3}$$

Assume that M is k-connected and $\Omega^{k-1}M$ is a Gorenstein space (or, equivalently, $\bigoplus_n \pi_n(M)\otimes\mathbb{K}$ is of finite dimension). Denote $\bar{m} = \dim \Omega^{k-1}M$. Then we define the (S,T)-*brane coproduct*

$$\delta_{ST} \colon H_*(M^{S\#T}) \to H_{*-\bar{m}}(M^S \times M^T)$$

using the diagram

$$\begin{tikzcd}
M^{S\#T} \arrow[d] & M^S \times_M M^T \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & M^S \times M^T \\
S^{k-1}M & M. \arrow[l, "c"'] &
\end{tikzcd} \tag{1.4}$$

Note that, if we take $S = T = S^k$, then μ_{ST} and δ_{ST} are the brane product and coproduct, respectively.

Next, we study some fundamental properties of the brane product and coproduct. For the loop product and coproduct on Gorenstein spaces, Naito [Nai13] showed their associativity and the Frobenius compatibility. In this article, we generalize them to the case of the brane product and coproduct. Moreover, we show the commutativity of the brane product and coproduct, which was not known even for the case of the loop product and coproduct on Gorenstein spaces.

Theorem 1.5. *Let M be a k-connected space such that $\Omega^{k-1}M$ is a Gorenstein space. Then the above product and coproduct satisfy following properties.*

(1) *The product is associative and commutative.*

(2) *The coproduct is associative and commutative.*

(3) *The product and coproduct satisfy the Frobenius compatibility.*

In particular, if we take $S = T = S^k$, the shifted homology $\mathbb{H}_*(S^kM) = H_{*+m}(S^kM)$ is a non-unital and non-counital Frobenius algebra, where m is the dimension of M as a Gorenstein space.

Note that M is a Gorenstein space by the assumption $\dim \pi_*(M) \otimes \mathbb{K} < \infty$ (see Proposition 2.2). The associativity of the product holds even if we assume that M is a Gorenstein space instead of assuming $\dim \pi_*(M) \otimes \mathbb{K} < \infty$. But we need the assumption to prove the commutativity of the *product*.

A non-unital and non-counital Frobenius algebra corresponds to a "positive boundary" TQFT, in the sense that TQFT operations are defined only when each component of the cobordism surfaces has a *positive* number of incoming and outgoing boundary components. See a paper of Cohen and Godin [CG04] for details.

See Section 7 for the precise statement and the proof of the associativity, the commutativity and the Frobenius compatibility. It is interesting that the proof of the commutativity of the loop coproduct (ie, $k = 1$) is easier than that of the brane coproduct with $k \geq 2$. In fact, we prove the commutativity of the loop coproduct using the explicit description of the loop coproduct constructed in another article of the author [Wak16]. On the other hand, we prove the commutativity of the brane coproduct with $k \geq 2$ directly from the definition.

Moreover, we compute an example of the brane product and coproduct. Here, we consider the shifted homology $\mathbb{H}_*(S^kM) = H_{*+m}(S^kM)$. We also have the shifts of the brane product and coproduct on $\mathbb{H}_*(S^kM)$ with the sign determined by the Koszul sign convention.

Theorem 1.6. *The shifted homology* $\mathbb{H}_*(S^2S^{2n+1})$, $n \geq 1$, *equipped with the brane product* μ *is isomorphic to the exterior algebra* $\wedge(y, z)$ *with* $|y| = -2n-1$ *and* $|z| = 2n-1$. *The brane coproduct* δ *is described as follows.*

$$\begin{aligned}\delta(1) &= 1 \otimes yz - y \otimes z + z \otimes y + yz \otimes 1\\ \delta(y) &= y \otimes yz + yz \otimes y\\ \delta(z) &= z \otimes yz + yz \otimes z\\ \delta(yz) &= -yz \otimes yz\end{aligned}$$

Note that both of the brane product and coproduct are nontrivial. Moreover, $(\delta \otimes 1) \circ \delta \neq 0$ in contrast with the case of the loop coproduct, in which the similar composition is always trivial [Tam10, Theorem A].

On the other hand, the brane coproduct is trivial in some cases.

Theorem 1.7. *If the minimal Sullivan model* $(\wedge V, d)$ *of* M *is pure and satisfies* $\dim V^{\mathrm{even}} > 0$, *then the brane coproduct on* $H_*(S^2M)$ *is trivial.*

See Definition 6.4 for the definition of a pure Sullivan algebra.

Remark 1.8. If we fix embeddings of disks $D^k \hookrightarrow S$ and $D^k \hookrightarrow T$ instead of assuming S and T being manifolds, we can define the product and coproduct using "connected sums" defined by these embedded disks. Moreover, if we have two disjoint embeddings $i, j\colon D^k \hookrightarrow S$ to the same space S, we can define the "connected sum" along i and j, and hence we can define the product and coproduct using this. We call these (S, i, j)-brane product and coproduct, and give definitions in Section 4.

Section 2 contains brief background material on string topology on Gorenstein spaces. We define the (S, T)-brane product and coproduct in Section 3 and (S, i, j)-brane product and coproduct in Section 4. Here, we defer the proof of Corollary 3.2 to Section 5. In Section 6, we compute examples and prove Theorem 1.6 and Theorem 1.7. Section 7 is devoted to the proof of Theorem 1.5, where we defer the determination of some signs to Section 8 and Section 9.

2 Construction by Félix and Thomas

In this section, we recall the construction of the loop product and coproduct by Félix and Thomas [FT09]. Since the cochain models are good for fibrations, the duals of the loop product and coproduct are defined at first, and then we define the loop product and coproduct as the duals of them. Moreover we focus on the case when the characteristic of the coefficient $\mathbb{K}$ is zero. So we make full use of rational homotopy theory. For the basic definitions and theorems on homological algebra and rational homotopy theory, we refer the reader to [FHT01].

Definition 2.1 ([FHT88])**.** Let $m \in \mathbb{Z}$ be an integer.

(1) An augmented dga (differential graded algebra) (A, d) is called a *($\mathbb{K}$-)Gorenstein algebra of dimension m* if

$$\dim \operatorname{Ext}^l_A(\mathbb{K}, A) = \begin{cases} 1 & (\text{if } l = m)\\ 0 & (\text{otherwise}),\end{cases}$$

where the field $\mathbb{K}$ and the dga (A, d) are (A, d)-modules via the augmentation map and the identity map, respectively.

(2) A path-connected topological space M is called a *($\mathbb{K}$-)Gorenstein space of dimension m* if the singular cochain algebra $C^*(M)$ of M is a Gorenstein algebra of dimension m.

Here, $\mathrm{Ext}_A(M, N)$ is defined using a semifree resolution of (M, d) over (A, d), for a dga (A, d) and (A, d)-modules (M, d) and (N, d). $\mathrm{Tor}_A(M, N)$ is defined similarly. See [FT09, Section 1] for details of semifree resolutions.

An important example of a Gorenstein space is given by the following proposition.

Proposition 2.2 ([FHT88, Proposition 3.4]). *A 1-connected topological space M is a $\mathbb{K}$-Gorenstein space if $\pi_*(M) \otimes \mathbb{K}$ is finite dimensional. Similarly, a Sullivan algebra $(\wedge V, d)$ is a Gorenstein algebra if V is finite dimensional.*

Note that this proposition is stated only for $\mathbb{Q}$-Gorenstein spaces in [FHT88], but the proof can be applied for any $\mathbb{K}$ and Sullivan algebras.

Let M be a 1-connected $\mathbb{K}$-Gorenstein space of dimension m whose cohomology $H^*(M)$ is of finite type. As a preparation to define the loop product and coproduct, Félix and Thomas proved the following theorem.

Theorem 2.3 ([FT09, Theorem 12]). *The diagonal map $\Delta\colon M \to M^2$ makes $C^*(M)$ into a $C^*(M^2)$-module. We have an isomorphism*

$$\mathrm{Ext}^*_{C^*(M^2)}(C^*(M), C^*(M^2)) \cong H^{*-m}(M).$$

By Theorem 2.3, we have $\mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2)) \cong H^0(M) \cong \mathbb{K}$, hence the generator

$$\Delta_! \in \mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2))$$

is well-defined up to the multiplication by a non-zero scalar. We call this element the *shriek map* for Δ.

Using the map $\Delta_!$, we can define the duals of the loop product and coproduct. Then, using the diagram (1.1), we define the dual of the loop product to be the composition

$$\mathrm{incl}_! \circ \mathrm{comp}^*\colon H^*(LM) \xrightarrow{\mathrm{comp}^*} H^*(LM \times_M LM) \xrightarrow{\mathrm{incl}_!} H^{*+m}(LM \times LM).$$

Here, the map $\mathrm{incl}_!$ is defined by the composition

$$\begin{aligned} &H^*(LM \times_M LM) \underset{\cong}{\xleftarrow{\mathrm{EM}}} \mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(LM \times LM)) \\ &\xrightarrow{\mathrm{Tor}_{\mathrm{id}}(\Delta_!, \mathrm{id})} \mathrm{Tor}^{*+m}_{C^*(M^2)}(C^*(M^2), C^*(LM \times LM)) \underset{\cong}{\to} H^{*+m}(LM \times LM), \end{aligned}$$

where the map EM is the Eilenberg-Moore map, which is an isomorphism (see [FHT01, Theorem 7.5] for details). Similarly, using the diagram (1.2), we define the dual of the loop coproduct to be the composition

$$\mathrm{comp}_! \circ \mathrm{incl}^*\colon H^*(LM \times LM) \xrightarrow{\mathrm{incl}^*} H^*(LM \times_M LM) \xrightarrow{\mathrm{comp}_!} H^*(LM).$$

Here, the map $\mathrm{comp}_!$ is defined by the composition

$$\begin{aligned} &H^*(LM \times_M LM) \underset{\cong}{\xleftarrow{\mathrm{EM}}} \mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(LM)) \\ &\xrightarrow{\mathrm{Tor}_{\mathrm{id}}(\Delta_!, \mathrm{id})} \mathrm{Tor}^{*+m}_{C^*(M^2)}(C^*(M^2), C^*(LM)) \underset{\cong}{\to} H^{*+m}(LM). \end{aligned}$$

3 Definition of (S, T)-brane coproduct

Let $\mathbb{K}$ be a field of characteristic zero, S and T manifolds of dimension k, and M a k-connected Gorenstein space of finite type. As in the construction by Félix and Thomas, which we reviewed in Section 2, we construct the duals

$$\mu_{ST}^{\vee}\colon H^*(M^{S\#T}) \to H^{*+\dim M}(M^S \times M^T)$$
$$\delta_{ST}^{\vee}\colon H^*(M^S \times M^T) \to H^{*+\dim \Omega^{k-1}M}(M^{S\#T})$$

of the (S, T)-brane product and the (S, T)-brane coproduct.

The (S, T)-brane product is defined by a similar way to that of Félix and Thomas. Using the diagram (1.3), we define $\mu_{ST}^{\vee}$ to be the composition

$$\mathrm{incl}_! \circ \mathrm{comp}^*\colon H^*(M^{S\#T}) \xrightarrow{\mathrm{comp}^*} H^*(M^S \times_M M^T) \xrightarrow{\mathrm{incl}_!} H^{*+m}(M^S \times M^T).$$

Here, the map $\mathrm{incl}_!$ is defined by the composition

$$\begin{aligned} &H^*(M^S \times_M M^T) \underset{\cong}{\xleftarrow{\mathrm{EM}}} \mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(M^S \times M^T)) \\ &\xrightarrow{\mathrm{Tor}_{\mathrm{id}}(\Delta_!, \mathrm{id})} \mathrm{Tor}^{*+m}_{C^*(M^2)}(C^*(M^2), C^*(M^S \times M^T)) \underset{\cong}{\to} H^{*+m}(M^S \times M^T), \end{aligned}$$

Next, we begin the definition of the (S, T)-brane coproduct. But Theorem 2.3 cannot be applied to this case since the base map of the pullback is $c\colon M \to S^{k-1}M$.

Instead of Theorem 2.3, we use the following theorem to define the (S, T)-brane coproduct. A graded algebra A is *connected* if $A^0 = \mathbb{K}$ and $A^i = 0$ for any $i < 0$. A dga (A, d) is *connected* if A is connected.

Theorem 3.1. *Let $(A \otimes B, d)$ be a dga such that A and B are connected commutative graded algebras, (A, d) is a sub dga of finite type, and $(A \otimes B, d)$ is semifree over (A, d). Let $\eta\colon (A \otimes B, d) \to (A, d)$ be a dga homomorphism. Assume that the following conditions hold.*

- (a) *The restriction of η to A is the identity map of A.*
- (b) *The dga $(B, \bar{d}) = \mathbb{K} \otimes_A (A \otimes B, d)$ is a Gorenstein algebra of dimension $\bar{m}$.*
- (c) *For any $b \in B$, the element $db - \bar{d}b$ lies in $A^{\geq 2} \otimes B$.*

Then we have an isomorphism

$$\mathrm{Ext}^*_{A \otimes B}(A, A \otimes B) \cong H^{*-\bar{m}}(A).$$

This can be proved by a similar method to Theorem 2.3 [FT09, Theorem 12]. The proof is given in Section 9.

Applying to sphere spaces, we have the following corollary.

Corollary 3.2. *Let M be a $(k-1)$-connected (and 1-connected) space of finite type such that $\Omega^{k-1}M$ is a Gorenstein space of dimension $\bar{m}$. Then we have an isomorphism*

$$\mathrm{Ext}^*_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^{*-\bar{m}}(M).$$

To prove the corollary, we need to construct models of sphere spaces satisfying the conditions of Theorem 3.1. This will be done in Section 5.

Note that, since $S^0M = M \times M$, this is a generalization of Theorem 2.3 (in the case that the characteristic of $\mathbb{K}$ is zero).

Assume that M is a k-connected space such that $\Omega^{k-1}M$ is a Gorenstein space.

Then we have $\mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^0(M) \cong \mathbb{K}$, hence the shriek map for $c\colon M \to S^{k-1}M$ is defined to be the generator

$$c_! \in \mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)),$$

which is well-defined up to the multiplication by a non-zero scalar. Using $c_!$ with the diagram (1.4), we define the dual $\delta^\vee_{ST}$ of the (S,T)-brane coproduct to be the composition

$$\mathrm{comp}_! \circ \mathrm{incl}^* \colon H^*(M^S \times M^T) \xrightarrow{\mathrm{incl}^*} H^*(M^S \times_M M^T) \xrightarrow{\mathrm{comp}_!} H^*(M^{S\#T}).$$

Here, the map $\mathrm{comp}_!$ is defined by the composition

$$\begin{aligned} &H^*(M^S \times_M M^T) \underset{\cong}{\xleftarrow{\mathrm{EM}}} \mathrm{Tor}^*_{C^*(S^{k-1}M)}(C^*(M), C^*(M^{S\#T})) \\ &\xrightarrow{\mathrm{Tor}_{\mathrm{id}}(c_!,\mathrm{id})} \mathrm{Tor}^{*+\bar{m}}_{C^*(S^{k-1}M)}(C^*(S^{k-1}M), C^*(M^{S\#T})) \underset{\cong}{\to} H^{*+\bar{m}}(M^{S\#T}). \end{aligned}$$

Note that the Eilenberg-Moore isomorphism can be applied since $S^{k-1}M$ is 1-connected.

4 Definition of (S,i,j)-brane product and coproduct

In this section, we give a definition of (S,i,j)-brane product and coproduct. Let S be a topological space, and i and j embeddings $D^k \to S$. Fix a small k-disk $D \subset D^k$ and denote its interior by D° and its boundary by ∂D. Then we define three spaces $\#(S,i,j)$, $Q(S,i,j)$, and $\bigvee(S,i,j)$ as follows. The space $\#(S,i,j)$ is obtained from $S \setminus (i(D^\circ) \cup j(D^\circ))$ by gluing $i(\partial D)$ and $j(\partial D)$ by an orientation reversing homeomorphism. We obtain $Q(S,i,j)$ by collapsing two disks $i(D)$ and $j(D)$ to two points, respectively. $\bigvee(S,i,j)$ is defined as the quotient space of $Q(S,i,j)$ identifying the two points. Then, since the quotient space D^k/D is homeomorphic to the disk D^k, we identify $Q(S,i,j)$ with S itself. By the above definitions, we have the maps $\#(S,i,j) \to \bigvee(S,i,j)$ and $S = Q(S,i,j) \to \bigvee(S,i,j)$. For a space M, these maps induce the maps $\mathrm{comp}\colon M^{\bigvee(S,i,j)} \to M^{\#(S,i,j)}$ and $\mathrm{incl}\colon M^{\bigvee(S,i,j)} \to M^S$. Moreover, we have diagrams

$$\begin{tikzcd} M^S \arrow[d] & M^{\bigvee(S,i,j)} \arrow[l, "\mathrm{incl}"'] \arrow[r, "\mathrm{comp}"] \arrow[d] & M^{\#(S,i,j)} \\ M \times M & M \arrow[l, "\Delta"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} M^{\#(S,i,j)} \arrow[d] & M^{\bigvee(S,i,j)} \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & M^S \\ S^{k-1}M & M \arrow[l, "c"'] & \end{tikzcd}$$

in which the squares are pullback diagrams. If M is a k-connected space such that $\Omega^{k-1}M$ is a Gorenstein space. we define the (S,i,j)-brane product and coproduct by a similar method to Section 3, using these diagrams instead of the diagrams (1.3) and (1.4). Note that this generalizes (S,T)-brane product and coproduct defined in Section 3.

5 Construction of models and proof of Corollary 3.2

In this section, we give a proof of Corollary 3.2, constructing a Sullivan model of the map $c\colon M \to S^{k-1}M$ satisfying the assumptions of Theorem 3.1.

First, we construct models algebraically. Let $(\wedge V, d)$ be a Sullivan algebra. For an integer $l \in \mathbb{Z}$, let $s^l V$ be a graded module defined by $(s^l V)^n = V^{n+l}$ and $s^l v$ denotes the element in $s^l V$ corresponding to the element $v \in V$.

Define two derivations $s^{(k-1)}$ and $\bar{d}^{(k-1)}$ on the graded algebra $\wedge V \otimes \wedge s^{k-1} V$ by

$$\begin{aligned} &s^{(k-1)}(v) = s^{k-1}v, \quad s^{(k-1)}(s^{k-1}v) = 0, \\ &\bar{d}^{(k-1)}(v) = dv, \text{ and } \quad \bar{d}^{(k-1)}(s^{k-1}v) = (-1)^{k-1}s^{(k-1)}dv. \end{aligned}$$

Then it is easy to see that $\bar{d}^{(k-1)} \circ \bar{d}^{(k-1)} = 0$ and hence $(\wedge V \otimes \wedge s^{k-1}V, \bar{d}^{(k-1)})$ is a dga.

Similarly, define derivations $s^{(k)}$ and $d^{(k)}$ on the graded algebra $\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V$ by

$$\begin{aligned} &s^{(k)}(v) = s^k v, \quad s^{(k)}(s^{k-1}v) = s^{(k)}(s^k v) = 0, \quad d^{(k)}(v) = dv, \\ &d^{(k)}(s^{k-1}v) = \bar{d}^{(k-1)}(s^{k-1}v), \text{ and } \quad d^{(k)}(s^k v) = s^{k-1}v + (-1)^k s^{(k)}dv. \end{aligned}$$

Then it is easy to see that $d^{(k)} \circ d^{(k)} = 0$ and hence $(\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V, d^{(k)})$ is a dga.

Note that the tensor product $(\wedge V, d) \otimes_{\wedge V \otimes \wedge s^{k-1}V} (\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V, d^{(k)})$ is canonically isomorphic to $(\wedge V \otimes \wedge s^k V, \bar{d}^{(k)})$, where $(\wedge V, d)$ is a $(\wedge V \otimes \wedge s^{k-1}V, \bar{d}^{(k-1)})$-module by the dga homomorphism $\phi\colon (\wedge V \otimes \wedge s^{k-1}V, \bar{d}^{(k-1)}) \to (\wedge V, d)$ defined by $\phi(v) = v$ and $\phi(s^{k-1}v) = 0$.

It is clear that, if $V^{\leq k-1} = 0$, the dga $(\wedge V \otimes \wedge s^{k-1}V, \bar{d}^{(k-1)})$ is a Sullivan algebra and, if $V^{\leq k} = 0$, the dga $(\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V, d^{(k)})$ is a relative Sullivan algebra over $(\wedge V \otimes \wedge s^{k-1}V, \bar{d}^{(k-1)})$.

Define a dga homomorphism

$$\tilde{\varepsilon}\colon (\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V, d^{(k)}) \to (\wedge V, d)$$

by $\tilde{\varepsilon}(v) = v$ and $\tilde{\varepsilon}(s^{k-1}v) = \tilde{\varepsilon}(s^k v) = 0$. Then the linear part

$$Q(\tilde{\varepsilon})\colon (V \oplus s^{k-1}V \oplus s^k V, d_0^{(k)}) \to (V, d_0)$$

is a quasi-isomorphism, and hence $\tilde{\varepsilon}$ is a quasi-isomorphism [FHT01, Proposition 14.13].

Define a relative Sullivan algebra $\mathcal{M}_{\mathrm{P}} = (\wedge V^{\otimes 2} \otimes \wedge sV, d)$ over $(\wedge V, d)^{\otimes 2}$ by the formula

$$d(sv) = 1 \otimes v - v \otimes 1 - \sum_{i=1}^{\infty} \frac{(sd)^i}{i!}(v \otimes 1)$$

inductively (see [FHT01, Section 15 (c)] or [Wak16, Appendix A] for details).

For simplicity, we write $\mathcal{M}_{S^k} = (\wedge V \otimes \wedge s^k V, \bar{d}^{(k)})$ for $k \geq 1$ and $\mathcal{M}_{S^0} = (\wedge V, d)^{\otimes 2}$, and $\mathcal{M}_{D^k} = (\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V, d^{(k)})$ for $k \geq 2$ and $\mathcal{M}_{D^1} = \mathcal{M}_{\mathrm{P}}$.

Let $A^*_{\mathrm{PL}}(-)$ be the functor of the algebra of polynomial differential forms. Note that, for a space X, $A^*_{\mathrm{PL}}(X)$ is a *commutative* dga which is naturally quasi-isomorphic to the singular cochain algebra $C^*(X)$ as differential graded algebras. See [FHT01, Section 10] for details.

Using these algebras, we have the following proposition.

Proposition 5.1. *Let $k \geq 2$ be an integer, M a $(k-1)$-connected space of finite type, and $f\colon (\wedge V, d) \to A^*_{\mathrm{PL}}(M)$ its Sullivan model such that $V^{\leq k-1} = 0$ and V is of finite type. Then, for* *any* l *with* $0 \leq l \leq k-1$*, there are quasi-isomorphisms* $f_l\colon \mathcal{M}_{S^l} \xrightarrow{\simeq} A^*_{\mathrm{PL}}(S^lM)$ *and* $g_l\colon \mathcal{M}_{D^l} \to A^*_{\mathrm{PL}}(D^lM)$ *such that the diagrams*

$$\begin{tikzcd}
\mathcal{M}_{S^l} \arrow[r, "\phi"] \arrow[d, "f_l"', "\simeq"] & (\wedge V, d) \arrow[r, hook] \arrow[d, "f"', "\simeq"] & \mathcal{M}_{S^l} \arrow[d, "f_l"', "\simeq"] & \mathcal{M}_{S^{l-1}} \arrow[r, hook] \arrow[d, "f_{l-1}"', "\simeq"] & \mathcal{M}_{D^l} \arrow[d, "g_l"', "\simeq"] \\
A^*_{\mathrm{PL}}(S^lM) \arrow[r, "c^*"] & A^*_{\mathrm{PL}}(M) \arrow[r, "\mathrm{ev}^*"] & A^*_{\mathrm{PL}}(S^lM) & A^*_{\mathrm{PL}}(S^{l-1}M) \arrow[r, "\mathrm{res}^*"] & A^*_{\mathrm{PL}}(D^lM)
\end{tikzcd}$$

commute strictly, where $D^lM = \mathrm{Map}(D^l, M)$*. In particular, the dga homomorphism* $\phi\colon \mathcal{M}_{S^{k-1}} \to (\wedge V, d)$ *is a Sullivan representative of the map* $c\colon M \to S^{k-1}M$ *with strict commutativity* $c^* \circ f_l = f \circ \phi$

Proof. We prove the proposition by induction on l. The case $l = 0$ is well-known, since c is the diagonal map and ϕ is the multiplication map.

Let l be an integer with $1 \leq l \leq k-1$ and assume that we already have f_{l-1} satisfying $c^* \circ f_l = f \circ \phi$. Let $\tilde{c}\colon M \to D^lM$ be the embedding by constant maps, and $\mathrm{res}\colon D^lM \to S^{l-1}M$ the restriction map to the boundary. Since $\mathrm{res} \circ \tilde{c} = c$, the outer square in the following diagram is commutative by the induction hypothesis.

$$\begin{tikzcd}
\mathcal{M}_{S^{l-1}} \arrow[r, "f_{l-1}", "\simeq"'] \arrow[d, hook] & A^*_{\mathrm{PL}}(S^{l-1}M) \arrow[r, "\mathrm{res}^*"] & A^*_{\mathrm{PL}}(D^lM) \arrow[d, two heads, "\simeq"', "\tilde{c}^*"] \\
\mathcal{M}_{D^l} \arrow[r, "\tilde{\varepsilon}", "\simeq"'] \arrow[rru, dashed, "g_l"] & \wedge V \arrow[r, "f", "\simeq"'] & A^*_{\mathrm{PL}}(M)
\end{tikzcd}$$

Here, $\tilde{c}^*$ is a surjective quasi-isomorphism, since the map $\tilde{c}$ is a homotopy equivalence and has a retraction, namely the evaluation map at the base point. Hence, by the lifting property of a relative Sullivan algebra with respect to a surjective quasi-isomorphism, there is a dga homomorphism $g_l\colon \mathcal{M}_{D^l} \to A^*_{\mathrm{PL}}(D^lM)$ which makes *both* of the triangles in the above diagram commute strictly. Note that, when $l = 1$, this diagram is constructed in [Men15, Section 4.5], without the *strict* commutativity of the lower right triangle.

Here the map $c\colon M \to S^lM$ is given by the following pullback diagram.

$$\begin{tikzcd}
M \arrow[rd, dashed, "c"] \arrow[rrd, "\tilde{c}"] \arrow[rdd, "\mathrm{id}"'] & & \\
& S^lM \arrow[r] \arrow[d] & D^lM \arrow[d, "\mathrm{res}"] \\
& M \arrow[r, "c"] & S^{l-1}M
\end{tikzcd}$$

Applying the functor $A^*_{\mathrm{PL}}(-)$ to the diagram and considering its model, we have the diagram

$$\begin{tikzcd}
 & A^*_{\mathrm{PL}}(M) & & & \\
\wedge V \arrow[ur, "f"] & & A^*_{\mathrm{PL}}(S^l M) \arrow[ul, "c^*"'] & & A^*_{\mathrm{PL}}(D^l M) \arrow[ll] \arrow[ulll, "\tilde{c}^*"'] \\
 & \mathcal{M}_{S^l} \arrow[ul, "\phi"] \arrow[ur, dashed, "f_l"] & & \mathcal{M}_{D^l} \arrow[ll] \arrow[ulll, "\tilde{\varepsilon}"'] \arrow[ur, "g_l"] & \\
 & & A^*_{\mathrm{PL}}(M) \arrow[uuul, "\mathrm{id}"] \arrow[uu, "\mathrm{ev}^*"'] & & A^*_{\mathrm{PL}}(S^{l-1}M) \arrow[ll, "c^*"'] \arrow[uu, "\mathrm{res}^*"] \\
 & \wedge V \arrow[uuul, "\mathrm{id}"] \arrow[uu, hook] \arrow[ur, "f"'] & & \mathcal{M}_{S^{l-1}} \arrow[ll, "\phi"] \arrow[uu, hook] \arrow[ur, "f_{l-1}"'] &
\end{tikzcd}$$

where the faces are strictly commutative and the square in the front face is a pushout diagram. By the universality of the pushout, we have the dga homomorphism $f_l\colon \mathcal{M}_{S^l} \to A^*_{\mathrm{PL}}(S^l M)$, which makes the diagram commutative. In particular, it satisfies $f \circ \phi = c^* \circ f_l$. Note that f_l is a quasi-isomorphism by the Eilenberg-Moore theorem [FHT01, Section 15 (c)]. This completes the induction. ☐

Proof of Corollary 3.2. In the case $k = 1$, apply Theorem 3.1 to the multiplication map $(\wedge V, d)^{\otimes 2} \to (\wedge V, d)$. (Note that this case is a result of Félix and Thomas [FT09].)

In the case $k \geq 2$, using Proposition 5.1, apply Theorem 3.1 to the map ϕ. ☐

6 Computation of examples

In this section, we will compute the brane product and coproduct for some examples, which proves Theorem 1.6 and Theorem 1.7.

In [Nai13], the duals of the loop product and coproduct are described in terms of Sullivan models using the torsion functor description of [KMN15]. By a similar method, we can describe the brane product and coproduct as follows.

Theorem 6.1. *Let M be a k-connected $\mathbb{K}$-Gorenstein space of finite type and $(\wedge V, d)$ its Sullivan model such that $V^{\leq k} = 0$ and V is of finite type. Then the dual of the brane product on $H^*(S^k M)$ is induced by the composition*

$$\begin{aligned}
&\mathcal{M}_{S^k} \xrightarrow{\cong} \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \xleftarrow[\simeq]{\tilde{\varepsilon}\otimes\mathrm{id}} \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \xrightarrow{(\phi\otimes\mathrm{id})\otimes_{\phi}(\phi\otimes\mathrm{id})} \mathcal{M}_{S^k} \otimes_{\wedge V} \mathcal{M}_{S^k} \\
&\xrightarrow{\cong} \wedge V \otimes_{\wedge V^{\otimes 2}} {\mathcal{M}_{S^k}}^{\otimes 2} \xleftarrow[\simeq]{\tilde{\varepsilon}\otimes\mathrm{id}} \mathcal{M}_{\mathrm{P}} \otimes_{\wedge V^{\otimes 2}} {\mathcal{M}_{S^k}}^{\otimes 2} \xrightarrow{\delta_!\otimes\mathrm{id}} \wedge V^{\otimes 2} \otimes_{\wedge V^{\otimes 2}} {\mathcal{M}_{S^k}}^{\otimes 2} \xrightarrow{\cong} {\mathcal{M}_{S^k}}^{\otimes 2},
\end{aligned}$$

where $\delta_!$ is a representative of $\Delta_!$. See Section 5 for the definitions of the other maps.

Assume that $\Omega^{k-1}M$ is a Gorenstein space. Then the dual of the brane coproduct is induced *by the composition*

$$\begin{aligned}&{\mathcal{M}_{S^k}}^{\otimes 2} \xrightarrow{\cong} \wedge V^{\otimes 2} \otimes_{{\mathcal{M}_{S^{k-1}}}^{\otimes 2}} {\mathcal{M}_{D^k}}^{\otimes 2} \xrightarrow{\mu \otimes_{\mu'} \eta} \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k})\\ &\xleftarrow[\simeq]{\tilde{\varepsilon} \otimes \mathrm{id}} \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}) \xrightarrow{\gamma_! \otimes \mathrm{id}} \mathcal{M}_{S^{k-1}} \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k})\\ &\xrightarrow{\cong} \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \xrightarrow[\simeq]{\tilde{\varepsilon} \otimes \mathrm{id}} \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \xrightarrow{\cong} \mathcal{M}_{S^k},\end{aligned}$$

where $\gamma_!$ is a representative of $c_!$, the maps μ and μ' are the multiplication maps, and η is the quotient map.

Proof. We omit the detail of the proof for the brane product, since it is the same as that for the usual loop product. Here we give a detailed proof of the construction of the model for the brane coproduct.

Here we use two pullback diagrams

$$\begin{tikzcd} S^kM \times_M S^kM \arrow[r] \arrow[d] & S^kM \arrow[d, "\mathrm{ev}"] & M \times_{S^{k-1}M} S^kM \arrow[r] \arrow[d] & S^kM \arrow[d, "\mathrm{res}"] \\ S^kM \arrow[r, "\mathrm{ev}"] & M & M \arrow[r, "c"] & S^{k-1}M. \end{tikzcd}$$

The spaces $S^kM \times_M S^kM$ and $M \times_{S^{k-1}M} S^kM$ are obviously homeomorphic and hence we identify them outside of this proof, but we distinguish them in this proof in order to specify the pullback diagrams. By a similar method to the proof of Proposition 5.1, we have dga homomorphisms $h_k\colon \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \to A^*_{\mathrm{PL}}(S^kM)$ and $i_k\colon \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}) \to A^*_{\mathrm{PL}}(M \times_{S^{k-1}M} S^kM)$ such that the diagrams

$$\begin{tikzcd} \mathcal{M}_{S^{k-1}} \arrow[r] \arrow[d, "f_{k-1}", "\simeq"'] & \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \arrow[d, "h_k", "\simeq"'] \\ A^*_{\mathrm{PL}}(S^{k-1}M) \arrow[r, "\mathrm{res}^*"] & A^*_{\mathrm{PL}}(S^kM) \end{tikzcd} \tag{6.2}$$

$$\begin{tikzcd} \mathcal{M}_{S^k} \otimes_{\wedge V} \mathcal{M}_{S^k} \arrow[r, "\cong"'] \arrow[d, "f_k \otimes f_k"] & \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}) \arrow[dd, "i_k", "\simeq"'] \\ A^*_{\mathrm{PL}}(S^kM) \otimes_{A^*_{\mathrm{PL}}(M)} A^*_{\mathrm{PL}}(S^kM) \arrow[d] & \\ A^*_{\mathrm{PL}}(S^kM \times_M S^kM) \arrow[r, "\cong"'] & A^*_{\mathrm{PL}}(M \times_{S^{k-1}M} S^kM) \end{tikzcd} \tag{6.3}$$

commute strictly, where the horizontal maps in the second diagram are the canonical isomorphisms.

Using the above maps, we obtain the diagram

$$\begin{tikzcd}
H^*(S^kM \times S^kM) \arrow[d, "\mathrm{incl}^*"] & \mathrm{Tor}_{\mathbb{K}}(A^*_{\mathrm{PL}}(S^kM), A^*_{\mathrm{PL}}(S^kM)) \arrow[l, "\mathrm{EM}", "\cong"'] \arrow[d] & H^*(\mathcal{M}_{S^k} \otimes \mathcal{M}_{S^k}) \arrow[l, "\mathrm{Tor}_{\mathrm{id}}(f_k, f_k)", "\cong"'] \arrow[d] \\
H^*(S^kM \times_M S^kM) \arrow[d, "\cong"] & \mathrm{Tor}_{A^*_{\mathrm{PL}}(M)}(A^*_{\mathrm{PL}}(S^kM), A^*_{\mathrm{PL}}(S^kM)) \arrow[l, "\mathrm{EM}", "\cong"'] & H^*(\mathcal{M}_{S^k} \otimes_{\wedge V} \mathcal{M}_{S^k}) \arrow[l, "\mathrm{Tor}_f(f_k, f_k)", "\cong"'] \arrow[d, "\cong"] \\
H^*(M \times_{S^{k-1}M} S^kM) \arrow[dd, "\mathrm{comp}_!"] & \mathrm{Tor}_{A^*_{\mathrm{PL}}(S^{k-1}M)}(A^*_{\mathrm{PL}}(M), A^*_{\mathrm{PL}}(S^kM)) \arrow[l, "\mathrm{EM}", "\cong"'] \arrow[dd, "\mathrm{Tor}_{\mathrm{id}}(c_!, \mathrm{id})"] & H^*(\wedge V \otimes_{\mathcal{M}_{S^{k-1}}} \overline{\mathcal{M}}_{S^k}) \arrow[l, "\mathrm{Tor}_{f_{k-1}}(f, h_k)", "\cong"'] \\
 & & H^*(\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \overline{\mathcal{M}}_{S^k}) \arrow[u, "\cong"] \arrow[d, "H^*(\gamma_! \otimes \mathrm{id})"] \\
H^*(S^kM) & \mathrm{Tor}_{A^*_{\mathrm{PL}}(S^{k-1}M)}(A^*_{\mathrm{PL}}(S^{k-1}M), A^*_{\mathrm{PL}}(S^kM)) \arrow[l, "\cong"'] & H^*(\mathcal{M}_{S^{k-1}} \otimes_{\mathcal{M}_{S^{k-1}}} \overline{\mathcal{M}}_{S^k}) \arrow[l, "\mathrm{Tor}_{f_{k-1}}(f_{k-1}, h_k)", "\cong"'] \arrow[d, "\cong"] \\
 & & H^*(\mathcal{M}_{S^k}),
\end{tikzcd}$$

where $\overline{\mathcal{M}}_{S^k} = \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}$. The composition of the vertical maps in the left column is the definition of the brane coproduct and the one in the right column is the description in the statement of the theorem. The horizontal maps in the right squares are defined by the strict commutativity of the diagrams in Proposition 5.1 and (6.2). The commutativity of the central square follows from (6.3) and that of the other squares are obvious from the definitions. The commutativity of this diagram proves the theorem. □

As a preparation of computation, recall the definition of a pure Sullivan algebra.

Definition 6.4 (cf [FHT01, Section 32])**.** A Sullivan algebra $(\wedge V, d)$ with $\dim V < \infty$ is called *pure* if $d(V^{\mathrm{even}}) = 0$ and $d(V^{\mathrm{odd}}) \subset \wedge V^{\mathrm{even}}$.

For a pure Sullivan algebra, we have an explicit construction of the shriek map $\delta_!$ and $\gamma_!$. For $\delta_!$, see [Nai13]. For $\gamma_!$, we have the following proposition.

Proposition 6.5. *Let $(\wedge V, d)$ be a pure minimal Sullivan algebra. Take bases $V^{\mathrm{even}} = \mathbb{K}\{x_1, \dots x_p\}$ and $V^{\mathrm{odd}} = \mathbb{K}\{y_1, \dots y_q\}$. Define a $(\wedge V \otimes \wedge sV, d)$-linear map*

$$\gamma_! \colon (\wedge V \otimes \wedge sV \otimes \wedge s^2 V, d) \to (\wedge V \otimes \wedge sV, d)$$

by $\gamma_!(s^2y_1 \cdots s^2y_q) = sx_1 \cdots sx_p$ and $\gamma_!(s^2y_{j_1} \cdots s^2y_{j_l}) = 0$ for $l < q$. Then $\gamma_!$ defines a nontrivial element in $\mathrm{Ext}_{\wedge V \otimes \wedge sV}(\wedge V, \wedge V \otimes \wedge sV)$

Proof. By a straightforward calculation, $\gamma_!$ is a cocycle in $\mathrm{Hom}_{\wedge V \otimes \wedge sV}(\wedge V \otimes \wedge sV \otimes \wedge s^2V, \wedge V \otimes \wedge sV)$. In order to prove the nontriviality, we define an ideal $I = (x_1, \dots, x_p, y_1, \dots, y_q, sy_1, \dots, sy_q) \subset \wedge V \otimes \wedge sV$. By the purity and minimality, we have $d(I) \subset I$. Using this ideal, we have the evaluation map of the form

$$\begin{aligned}
&\mathrm{Ext}_{\wedge V \otimes \wedge sV}(\wedge V, \wedge V \otimes \wedge sV) \otimes \mathrm{Tor}_{\wedge V \otimes \wedge sV}(\wedge V, \wedge V \otimes \wedge sV/I) \\
&\xrightarrow{\mathrm{ev}} \mathrm{Tor}_{\wedge V \otimes \wedge sV}(\wedge V \otimes \wedge sV, \wedge V \otimes \wedge sV/I) \xrightarrow{\cong} \wedge sV^{\mathrm{even}}.
\end{aligned}$$

By this map, the element $[\gamma_!] \otimes [s^2y_1 \cdots s^2y_q \otimes 1]$ is mapped to the element $sx_1 \cdots sx_p$, which is obviously nontrivial. Hence $[\gamma_!]$ is also nontrivial. □

Now, we give proofs of Theorem 1.6 and Theorem 1.7.

Proof of Theorem 1.6. Using the descriptions in Theorem 6.1, we compute the brane product and coproduct for $M = S^{2n+1}$ and $k = 2$. In this case, we can take $(\wedge V, d) = (\wedge x, 0)$ with $|x| = 2n+1$, and have $\mathcal{M}_{S^1} = (\wedge(x, sx), 0)$ and $\mathcal{M}_{D^2} = (\wedge(x, sx, s^2x), d)$ where $dx = dsx = 0$ and $ds^2x = sx$. The computation is straightforward except for the shriek maps $\delta_!$ and $\gamma_!$. The map $\delta_!$ is the linear map $\mathcal{M}_{\mathrm{P}} \to (\wedge x, 0)^{\otimes 2}$ over $(\wedge x, 0)^{\otimes 2}$ determined by $\delta_!(1) = 1 \otimes x - x \otimes 1$ and $\delta_!((sx)^l) = 0$ for $l \geq 1$. By Proposition 6.5, the map $\gamma_!$ is the linear map $\mathcal{M}_{D^k} \to \mathcal{M}_{S^{k-1}}$ over $\mathcal{M}_{S^{k-1}}$ determined by $\gamma_!(s^2x) = 1$ and $\gamma_!(1) = 0$.

Then the dual of the brane product $\mu^\vee$ is a linear map

$$\mu^\vee \colon \wedge(x, s^2x) \to \wedge(x, s^2x) \otimes \wedge(x, s^2x).$$

of degree $m = 2n+1$ over $\wedge(x) \otimes \wedge(x)$, which is characterized by

$$\mu^\vee(1) = 1 \otimes x - x \otimes 1,\ \mu^\vee(s^2x) = (1 \otimes x - x \otimes 1)(s^2x \otimes 1 + 1 \otimes s^2x).$$

Similarly, the dual of the brane coproduct $\delta^\vee$ is a linear map

$$\delta^\vee \colon \wedge(x, s^2x) \otimes \wedge(x, s^2x) \to \wedge(x, s^2x).$$

of degree $\bar{m} = 1 - 2n$ over $\wedge(x) \otimes \wedge(x)$, which is characterized by

$$\delta^\vee(1) = 0,\ \delta^\vee(s^2x \otimes 1) = -1,\ \delta^\vee(1 \otimes s^2x) = 1,\ \delta^\vee(s^2x \otimes s^2x) = -s^2x.$$

Dualizing these results, we get the brane product and coproduct on the homology, which proves Theorem 1.6. ☐

Proof of Theorem 1.7. By Proposition 6.5, we have that $\mathrm{Im}(\gamma_! \otimes \mathrm{id})$ is contained in the ideal $(sx_1, \ldots sx_p)$, which is mapped to zero by the map $\tilde{\varepsilon} \otimes \mathrm{id}$. ☐

7 Proof of the associativity, the commutativity, and the Frobenius compatibility

In this section, we give a precise statement and the proof of Theorem 1.5.

First, we give a precise statement of Theorem 1.5. For simplicity, we omit the statement for (S, i, j)-brane product and coproduct, which is almost the same as that for (S, T)-brane product and coproduct. Let M be a k-connected $\mathbb{K}$-Gorenstein space of finite type such that $\Omega^{k-1}M$ is also a Gorenstein space. Denote $m = \dim M$. Then the precise statement of (1) is that the diagrams

$$\begin{tikzcd}
H^*(M^{S\#T\#U}) \arrow[r, "\mu^\vee_{S\#T,U}"] \arrow[d, "\mu^\vee_{S,T\#U}"] & H^*(M^{S\#T} \times M^U) \arrow[d, "\mu^\vee_{S,T\amalg U}"] \\
H^*(M^S \times M^{T\#U}) \arrow[r, "\mu^\vee_{S\amalg T,U}"] & H^*(M^S \times M^T \times M^U)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^*(M^{T\#S}) \arrow[r, "\mu^\vee_{T,S}"] \arrow[d, "\tau^*_\#"] & H^*(M^T \times M^S) \arrow[d, "\tau^*_\times"] \\
H^*(M^{S\#T}) \arrow[r, "\mu^\vee_{S,T}"] & H^*(M^S \times M^T)
\end{tikzcd}$$

commute by the sign $(-1)^m$. Here, $\tau_\times$ and $\tau_\#$ are defined as the transposition of S and T. Note that the associativity of the product holds even if the assumption that $\Omega^{k-1}M$ is a Gorenstein space is dropped.

Denote $\bar{m} = \dim \Omega^{k-1}M$. Then (2) states that the diagrams

$$\begin{tikzcd}
H^*(M^S \times M^T \times M^U) \arrow[r, "\delta^\vee_{S\amalg T,U}"] \arrow[d, "\delta^\vee_{S,T\amalg U}"] & H^*(M^S \times M^{T\#U}) \arrow[d, "\delta^\vee_{S,T\#U}"] \\
H^*(M^{S\#T} \times M^U) \arrow[r, "\delta^\vee_{S\#T,U}"] & H^*(M^{S\#T\#U})
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^*(M^{T\times S}) \arrow[r, "\delta^\vee_{T,S}"] \arrow[d, "\tau^*_\#"] & H^*(M^T\#M^S) \arrow[d, "\tau^*_\times"] \\
H^*(M^{S\times T}) \arrow[r, "\delta^\vee_{S,T}"] & H^*(M^S\#M^T)
\end{tikzcd}$$

commute by the sign $(-1)^{\bar{m}}$. Similarly, (3) states that the diagram

$$\begin{tikzcd}
H^*(M^S \times M^{T\#U}) \arrow[r, "\delta^\vee_{S,T\#U}"] \arrow[d, "\mu^\vee_{S\#T,U}"] & H^*(M^{S\#T\#U}) \arrow[d, "\mu^\vee_{S\amalg T,U}"] \\
H^*(M^S \times M^T \times M^U) \arrow[r, "\delta^\vee_{S,T\amalg U}"] & H^*(M^{S\#T} \times M^U)
\end{tikzcd} \tag{7.1}$$

commutes by the sign $(-1)^{m\bar{m}}$.

Before proving Theorem 1.5, we give a notation g_α for a shriek map.

Definition 7.2. Consider a pullback diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "p"] & Y \arrow[d, "q"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

of spaces, where q is a fibration. Let α be an element of $\mathrm{Ext}^m_{C^*(B)}(C^*(A), C^*(B))$. Assume that the Eilenberg-Moore map

$$\mathrm{EM}\colon \mathrm{Tor}^*_{C^*(B)}(C^*(A), C^*(Y)) \xrightarrow{\cong} H^*(X)$$

is an isomorphism (eg, B is 1-connected and the cohomology of the fiber is of finite type). Then we define g_α to be the composition

$$g_\alpha\colon H^*(X) \xleftarrow{\cong} \mathrm{Tor}^*_{C^*(B)}(C^*(A), C^*(Y)) \xrightarrow{\mathrm{Tor}(\alpha,\mathrm{id})} \mathrm{Tor}^{*+m}_{C^*(B)}(C^*(B), C^*(Y)) \xrightarrow{\cong} H^{*+m}(Y)$$

Using this notation, we can write the shriek map $\mathrm{incl}_!$ as $\mathrm{incl}_{\Delta_!}$ for the diagram (1.3), and the shriek map $\mathrm{comp}_!$ as $\mathrm{comp}_{c_!}$ for the diagram (1.4).

Now we have the following two propositions as a preparation of the proof of Theorem 1.5.

Proposition 7.3. *Consider a diagram*

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "\varphi"'] \arrow[dd] & & Y \arrow[dd, "q"'] \arrow[dr, "\psi"] & \\
& X' \arrow[rr, "g'"] \arrow[dd] & & Y' \arrow[dd, "q'"] \\
A \arrow[rr] \arrow[dr, "a"'] & & B \arrow[dr, "b"'] & \\
& A' \arrow[rr] & & B',
\end{tikzcd}$$

where q and q' are fibrations and the front and back squares are pullback diagrams. Let $\alpha \in \mathrm{Ext}^m_{C^*(B)}(C^*(A), C^*(B))$ and $\alpha' \in \mathrm{Ext}^m_{C^*(B')}(C^*(A'), C^*(B'))$. Assume that the elements α and α' are mapped to the same element in $\mathrm{Ext}^m_{C^*(B')}(C^*(A'), C^*(B))$ by the morphisms induced by a and b, and that the Eilenberg-Moore maps of two pullback diagrams are isomorphisms. Then the following diagram commutes.

$$\begin{tikzcd}
H^*(X') \arrow[r, "g'_{\alpha'}"] \arrow[d, "\varphi^*"] & H^{*+m}(Y') \arrow[d, "\psi^*"] \\
H^*(X) \arrow[r, "g_\alpha"] & H^{*+m}(Y)
\end{tikzcd}$$

Proposition 7.4. *Consider a diagram*

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}"] \arrow[d, "p"] & Y \arrow[r, "\tilde{g}"] \arrow[d, "q"] & Z \arrow[d, "r"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C,
\end{tikzcd}$$

where the two squares are pullback diagrams. Let α be an element of $\mathrm{Ext}^m_{C^*(B)}(C^*(A), C^*(B))$ and β an element of $\mathrm{Ext}^n_{C^*(C)}(C^*(B), C^*(C))$. Assume that the Eilenberg-Moore maps are isomorphisms for two pullback diagrams. Then we have

$$(\tilde{g} \circ \tilde{f})_{\beta \circ (g_* \alpha)} = \tilde{g}_\beta \circ \tilde{f}_\alpha,$$

where $g_* \colon \mathrm{Ext}^m_{C^*(B)}(C^*(A), C^*(B)) \to \mathrm{Ext}^m_{C^*(C)}(C^*(A), C^*(B))$ is the morphism induced by the map $g \colon B \to C$.

These propositions can be proved by straightforward arguments.

Proof of Theorem 1.5. First, we give a proof for (3). Note that the associativity in (1) and (2) can be proved similarly.

Consider the following diagram.

$$\begin{tikzcd}
H^*(M^S \times M^{T\#U}) \arrow[r, "\mathrm{incl}^*"] \arrow[d, "\mathrm{comp}^*"] & H^*(M^S \times_M M^{T\#U}) \arrow[r, "\mathrm{comp}_{c_!}"] \arrow[d, "\mathrm{comp}^*"] & H^*(M^{S\#T\#U}) \arrow[d, "\mathrm{comp}^*"] \\
H^*(M^S \times M^T \times_M M^U) \arrow[r, "\mathrm{incl}^*"] \arrow[d, "\mathrm{incl}_{\Delta_!}"] & H^*(M^S \times_M M^T \times_M M^U) \arrow[r, "\mathrm{comp}_{(c\times\mathrm{id})_!}"] \arrow[d, "\mathrm{incl}_{(\mathrm{id}\times\Delta)_!}"] & H^*(M^{S\#T} \times_M M^U) \arrow[d, "\mathrm{incl}_{(\mathrm{id}\times\Delta)_!}"] \\
H^*(M^S \times M^T \times M^U) \arrow[r, "\mathrm{incl}^*"] & H^*(M^S \times_M M^T \times M^U) \arrow[r, "\mathrm{comp}_{(c\times\mathrm{id})_!}"] & H^*(M^{S\#T} \times M^U)
\end{tikzcd}$$

Note that the boundary of the whole square is the same as the diagram (7.1). The upper left square is commutative by the functoriality of the cohomology and so are the upper right and lower left squares by Proposition 7.3. Next, we consider the lower right square. Applying Proposition 7.4 to the diagram

$$\begin{tikzcd}
M^S \times_M M^T \times_M M^U \arrow[r, "\mathrm{comp}"] \arrow[d] & M^{S\#T} \times_M M^U \arrow[r, "\mathrm{incl}"] \arrow[d] & M^{S\#T} \times M^U \arrow[d] \\
M \times M \arrow[r, "c\times\mathrm{id}"] & S^{k-1}M \times M \arrow[r, "\mathrm{id}\times\Delta"] & S^{k-1}M \times M^2,
\end{tikzcd}$$

we have

$$\mathrm{incl}_{(\mathrm{id}\times\Delta)_!} \circ \mathrm{comp}_{(c\times\mathrm{id})_!} = (\mathrm{incl} \circ \mathrm{comp})_{(\mathrm{id}\times\Delta)_!\circ((\mathrm{id}\times\Delta)_*(c\times\mathrm{id})_!)}.$$

In order to compute the element

$$(\mathrm{id} \times \Delta)_! \circ ((\mathrm{id} \times \Delta)_*(c \times \mathrm{id})_!) \in \mathrm{Ext}_{C^*(S^{k-1}M\times M^2)}(C^*(M \times M), C^*(S^{k-1}M \times M^2)),$$

we use the models constructed in Section 5. Let $\delta_! \in \mathrm{Hom}^m_{\wedge V^{\otimes 2}}(\mathcal{M}_{\mathrm{P}}, \wedge V^{\otimes 2})$ and $\gamma_! \in \mathrm{Hom}^{\bar{m}}_{\mathcal{M}_{S^{k-1}}}(\mathcal{M}_{D^k}, \mathcal{M}_{S^{k-1}})$ be representatives of the generators:

$$\begin{aligned}
[\delta_!] &= \Delta_! \in \mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2)) \\
[\gamma_!] &= c_! \ \in \mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)).
\end{aligned}$$

Then, using the isomorphism

$$\begin{aligned}
&\mathrm{Ext}^{m+\bar{m}}_{C^*(M^2\times S^{k-1}M)}(C^*(M \times M), C^*(M^2 \times S^{k-1}M)) \\
&\cong H^{m+\bar{m}}(\mathrm{Hom}_{\wedge V^{\otimes 2}\otimes\mathcal{M}_{S^{k-1}}}(\mathcal{M}_{\mathrm{P}} \otimes \mathcal{M}_{D^k}, \wedge V^{\otimes 2} \otimes \mathcal{M}_{S^{k-1}})),
\end{aligned}$$

we have a representation

$$\begin{aligned}
(\mathrm{id} \times \Delta)_! \circ ((\mathrm{id} \times \Delta)_*(c \times \mathrm{id})_!) &= [\mathrm{id} \otimes \delta_!] \circ [\gamma_! \otimes \mathrm{id}] \\
&= [(-1)^{m\bar{m}}\gamma_! \otimes \delta_!]
\end{aligned}$$

as a chain map. Similarly, we compute the other composition to be

$$\mathrm{comp}_{(c\times\mathrm{id})_!} \circ \mathrm{incl}_{(\mathrm{id}\times\Delta)_!} = (\mathrm{comp} \circ \mathrm{incl})_{(c\times\mathrm{id})_!\circ((c\times\mathrm{id})_*)(\mathrm{id}\times\Delta)_!}$$

with

$$(c \times \mathrm{id})_! \circ ((c \times \mathrm{id})_*)(\mathrm{id} \times \Delta)_! = [\gamma_! \otimes \delta_!].$$

This proves the commutativity by the sign $(-1)^{m\bar{m}}$ of the lower right square.

Next, we prove the commutativity of the coproduct in (2). This follows from the commutativity of the diagram

$$\begin{tikzcd}
H^*(M^T \times M^S) \arrow[r, "\mathrm{incl}^*"] \arrow[d, "\tau_\times^*"] & H^*(M^T \times_M M^S) \arrow[r, "\mathrm{comp}_!"] \arrow[d, "\tau_\times^*"] & H^*(M^{T\#S}) \arrow[d, "\tau_\#^*"] \\
H^*(M^S \times M^T) \arrow[r, "\mathrm{incl}^*"] & H^*(M^S \times_M M^T) \arrow[r, "\mathrm{comp}_!"] & H^*(M^{S\#T}).
\end{tikzcd} \tag{7.5}$$

The commutativity of the left square is obvious. If one can apply Proposition 7.3 to the diagram (7.6), we obtain the commutativity of the right square of (7.5).

$$\begin{tikzcd}
M^S \times_M M^T \arrow[rr, "\mathrm{comp}"] \arrow[dr, "\tau_\times"] \arrow[dd] & & M^{S\#T} \arrow[dr, "\tau_\#"] \arrow[dd, "\mathrm{res}"] & \\
& M^T \times_M M^S \arrow[rr, "\mathrm{comp}"] \arrow[dd] & & M^{T\#S} \arrow[dd, "\mathrm{res}"] \\
M \arrow[rr, "c"] \arrow[dr, "\mathrm{id}"] & & S^{k-1}M \arrow[dr, "\tau"] & \\
& M \arrow[rr, "c"] & & S^{k-1}M
\end{tikzcd} \tag{7.6}$$

In order to apply Proposition 7.3, it suffices to prove the equation

$$\mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*)(c_!) = (-1)^{\bar{m}} c_! \tag{7.7}$$

in $\mathrm{Ext}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M))$. Since $\mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong \mathbb{K}$ and $\mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*) \circ \mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*) = \mathrm{id}$, we have (7.7) up to sign. In Section 9, we will determine the sign to be $(-1)^{\bar{m}}$.

Similarly, in order to prove the commutativity of the product in (1), we need to prove the equation

$$\mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*)(\Delta_!) = (-1)^m \Delta_! \tag{7.8}$$

in $\mathrm{Ext}_{C^*(M^2)}(C^*(M), C^*(M^2))$. As above, we have (7.8) up to sign. The sign is determined to be $(-1)^m$ in Section 8.

The same proofs can be applied for (S, i, j)-brane product and coproduct. □

8 Proof of (7.8)

In this section, we will prove (7.8), determining the sign. Here, we need the explicit description of $\Delta_!$ in [Wak16].

Let M be a 1-connected space with $\dim \pi_*(M) \otimes \mathbb{K} < \infty$. By [Wak16, Theorem 1.6], we have a Sullivan model $(\wedge V, d)$ of M which is semi-pure, ie, $d(I_V) \subset I_V$, where I_V is the ideal generated by V^{even}. Let $\varepsilon\colon (\wedge V, d) \to \mathbb{K}$ be the augmentation map and $\mathrm{pr}\colon (\wedge V, d) \to (\wedge V / I_V, d)$ the quotient map. Take bases $V^{\mathrm{even}} = \mathbb{K}\{x_1, \dots x_p\}$ and $V^{\mathrm{odd}} = \mathbb{K}\{y_1, \dots y_q\}$. Recall the relative Sullivan algebra $\mathcal{M}_{\mathrm{P}} = (\wedge V^{\otimes 2} \otimes \wedge sV, d)$ over $(\wedge V, d)^{\otimes 2}$ from Section 6. Note that the relative Sullivan algebra $(\wedge V^{\otimes 2} \otimes \wedge sV, d)$ is a relative Sullivan model of the multiplication map $(\wedge V, d)^{\otimes 2} \to (\wedge V, d)$, Hence, using this as a semifree resolution, we have $\mathrm{Ext}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2}) = H^*(\mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2}))$. By [Wak16, Corollary 5.5], we have a cocycle $f \in \mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2})$ satisfying $f(sx_1 \cdots sx_p) = \prod_{j=1}^{j=q}(1 \otimes y_j - y_j \otimes 1) + u$ for some $u \in (y_1 \otimes y_1, \dots, y_q \otimes y_q)$. Consider the evaluation map

$$\mathrm{ev}\colon \mathrm{Ext}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2}) \otimes \mathrm{Tor}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V/I_V) \to \mathrm{Tor}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2}, \wedge V/I_V) \xrightarrow{\cong} H^*(\wedge V/I_V),$$

where $(\wedge V, d)^{\otimes 2}$, $(\wedge V, d)$, and $(\wedge V/I_V, d)$ are $(\wedge V, d)^{\otimes 2}$-module via id, $\varepsilon \cdot \mathrm{id}$, and $\mathrm{pr} \circ (\varepsilon \cdot \mathrm{id})$, respectively. Here, we use $(\wedge V^{\otimes 2} \otimes \wedge sV, d)$ as a semifree resolution of $(\wedge V, d)$. Then, we have

$$\mathrm{ev}([f] \otimes [sx_1 \cdots sx_p]) = [y_1 \cdots y_q] \neq 0,$$

and hence $[f] \neq 0$ in $\mathrm{Ext}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2})$. Thus, it is enough to calculate $\mathrm{Ext}_t(\mathrm{id}, t)([f])$ to determine the sign in (7.8), where $t\colon (\wedge V, d)^{\otimes 2} \to (\wedge V, d)$ is the dga homomorphism defined by $t(v \otimes 1) = 1 \otimes v$ and $t(1 \otimes v) = v \otimes 1$.

Proof of (7.8). By definition, $\mathrm{Ext}_t(\mathrm{id}, t)$ is induced by the map

$$\mathrm{Hom}_t(\tilde{t}, t)\colon \mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2}) \to \mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2}),$$

where $\tilde{t}$ is the dga automorphism defined by $\tilde{t}|_{\wedge V^{\otimes 2}} = t$ and $\tilde{t}(sv) = -sv$. Since $\tilde{t}(sx_1 \cdots sx_p) = (-1)^p sx_1 \cdots sx_p$ and $t(\prod_{j=1}^{j=q}(1 \otimes y_j - y_j \otimes 1)) = (-1)^q \prod_{j=1}^{j=q}(1 \otimes y_j - y_j \otimes 1)$, we have

$$\mathrm{ev}([\mathrm{Hom}_t(\tilde{t}, t)(f)] \otimes [sx_1 \cdots sx_p]) = \mathrm{ev}([t \circ f \circ \tilde{t}] \otimes [sx_1 \cdots sx_p]) = (-1)^{p+q}[y_1 \cdots y_q].$$

Since the parity of $p + q$ is the same as that of the dimension of $(\wedge V, d)$ as a Gorenstein algebra, the sign in (7.8) is proved to be $(-1)^m$. □

9 Proof of (7.7)

In this section, we give the proof of (7.7), using the spectral sequence constructed in the proof of Theorem 3.1. Although the key idea of the proof of Theorem 3.1 is the same as Theorem 2.3 due to Félix and Thomas, we give the proof here for the convenience of the reader.

Proof of Theorem 3.1. Take a $(A \otimes B, d)$-semifree resolution $\eta\colon (P, d) \xrightarrow{\simeq} (A, d)$. Define $(C, d) = (\mathrm{Hom}_{A \otimes B}(P, A \otimes B), d)$. Then $\mathrm{Ext}_{A \otimes B}(A, A \otimes B) = H^*(C, d)$. We fix a non-negative integer N, and define a complex $(C_N, d) = (\mathrm{Hom}_{A \otimes B}(P, (A/A^{>n}) \otimes B), d)$. We will compute the cohomology of (C_N, d). Define a filtration $\{F^p C_N\}_{p \geq 0}$ on (C_N, d) by $F^p C_N = \mathrm{Hom}_{A \otimes B}(P, (A/A^{>n})^{\geq p} \otimes B)$. Then we obtain a spectral sequence $\{E_r^{p,q}\}_{r \geq 0}$ converging to $H^*(C_N, d)$.

Claim 9.1.

$$E_2^{p,q} = \begin{cases} H^p(A/A^{>N}) & (\textit{if } q = m) \\ 0 & (\textit{if } q \neq m) \end{cases}$$

Proof of Claim 9.1. We may assume $p \leq N$. Then we have an isomorphism of complexes

$$(A^{\geq p}/A^{\geq p+1}, 0) \otimes (\mathrm{Hom}_B(B \otimes_{A \otimes B} P, B), d) \xrightarrow{\cong} (E_0^p, d_0),$$

hence

$$(A^{\geq p}/A^{\geq p+1}) \otimes H^*(\mathrm{Hom}_B(B \otimes_{A \otimes B} P, B), d) \xrightarrow{\cong} E_1^p.$$

Define

$$\bar{\eta}\colon (B,\bar{d})\otimes_{A\otimes B}(P,d)\xrightarrow{1\otimes\eta}(B,\bar{d})\otimes_{A\otimes B}(A,d)\cong\mathbb{K}.$$

Note that the last isomorphism follows from the assumption (a). Then, since η is a quasi-isomorphism, so is $\bar{\eta}$. Hence we have

$$H^q(\mathrm{Hom}_B(B\otimes_{A\otimes B}P,B),d)\cong\mathrm{Ext}^q_B(\mathbb{K},B)\cong\begin{cases}\mathbb{K} & (\text{if } q=m)\\ 0 & (\text{if } q\neq m)\end{cases}$$

by the assumption (b).

Hence we have

$$\begin{aligned}E_1^{p,q}&\cong(A^{\geq p}/A^{\geq p+1})\otimes H^q(\mathrm{Hom}_B(B\otimes_{A\otimes B}P,B),d)\\&\cong A^p\otimes\mathrm{Ext}^q_B(\mathbb{K},B).\end{aligned}$$

Moreover, using the assumption (c) and the above isomorphisms, we can compute the differential d_1 and have an isomorphism of complexes

$$(E_1^{*,q},d_1)\cong(A^*,d)\otimes\mathrm{Ext}^q_B(\mathbb{K},B). \tag{9.2}$$

This proves Claim 9.1. □

Now we return to the proof of Theorem 3.1. We will recover $H^*(C)$ from $H^*(C_N)$ taking a limit. Since $\varprojlim^1_N C_N=0$, we have an exact sequence

$$0\to\varprojlim{}^1_N H^*(C_N)\to H^*(\varprojlim{}_N C_N)\to H^*(\varprojlim{}_N H^*(C_N))\to 0.$$

By Claim 9.1, the sequence $\{H^*(C_N)\}_N$ satisfies the (degree-wise) Mittag-Leffler condition, and hence $\varprojlim^1_N H^*(C_N)=0$. Thus, we have

$$H^l(C)\cong H^l(\varprojlim{}_N C_N)\cong\varprojlim{}_N H^l(C_N)\cong H^{l-m}(A).$$

This proves Theorem 3.1. □

Next, using the above spectral sequence, we determine the sign in (7.7).

Proof of (7.7). If $k=1$, (7.7) is the same as (7.8), which was proved in Section 8. Hence we assume $k\geq 2$. As in Section 7, let M be a k-connected $\mathbb{K}$-Gorenstein space of finite type with $\dim\pi_*(M)\otimes\mathbb{K}<\infty$, and $(\wedge V,d)$ its minimal Sullivan model. Using the Sullivan models constructed in Section 5, we have that the automorphism $\mathrm{Ext}_{\tau^*}(\mathrm{id},\tau^*)$ on $\mathrm{Ext}_{C^*(S^{k-1}M)}(C^*(M),C^*(S^{k-1}M))$ is induced by the automorphism $\mathrm{Hom}_t(\tilde{t},t)$ on $\mathrm{Hom}_{\wedge V\otimes\wedge s^{k-1}V}(\wedge V\otimes\wedge s^{k-1}V\otimes\wedge s^kV,\wedge V\otimes\wedge s^{k-1}V)$, where t and $\tilde{t}$ are the dga automorphisms on $(\wedge V\otimes\wedge s^{k-1}V,d)$ and $(\wedge V\otimes\wedge s^{k-1}V\otimes\wedge s^kV,d)$, respectively, defined by

$$\begin{aligned}&t(v)=v,\ t(s^{k-1}v)=-s^{k-1}v,\\&\tilde{t}(v)=v,\ \tilde{t}(s^{k-1}v)=-s^{k-1}v,\ \text{and }\tilde{t}(s^kv)=-s^kv.\end{aligned}$$

Now, consider the spectral sequence $\{E_r^{p,q}\}$ in the proof of Theorem 3.1 by taking $(A\otimes B,d)=(\wedge V\otimes\wedge s^{k-1}V,d)$ and $(P,d)=(\wedge V\otimes\wedge s^{k-1}V\otimes\wedge s^kV,d)$. Since $k\geq 2$, $\mathrm{Hom}_t(\tilde{t},t)$ induces automorphisms on the complexes C_N and F^pC_N, and hence on the spectral sequence $\{E_r^{p,q}\}$. By the isomorphism (9.2), we have

$$E_2^{p,q}\cong H^p(A)\otimes\mathrm{Ext}^q_{\wedge s^{k-1}V}(\mathbb{K},\wedge s^{k-1}V),$$

and that the automorphism induced on E_2 is the same as $\mathrm{id} \otimes \mathrm{Ext}_{\bar{t}}(\mathrm{id}, \bar{t})$, where $\bar{t}$ is defined by $\bar{t}(s^{k-1}v) = -s^{k-1}v$ for $v \in V$. Since the differential is zero on $\wedge s^{k-1}V$, we have an isomorphism

$$\mathrm{Ext}^*_{\wedge s^{k-1}V}(\mathbb{K}, \wedge s^{k-1}V) \cong \bigotimes_i \mathrm{Ext}^*_{\wedge s^{k-1}v_i}(\mathbb{K}, \wedge s^{k-1}v_i)$$

where $\{v_1, \dots, v_l\}$ is a basis of V. Using this isomorphism, we can identify

$$\mathrm{Ext}_{\bar{t}}(\mathrm{id}, \bar{t}) = \bigotimes_i \mathrm{Ext}_{\bar{t}_i}(\mathrm{id}, \bar{t}_i),$$

where $\bar{t}_i$ is defined by $\bar{t}_i(s^{k-1}v_i) = -s^{k-1}v_i$.

Since $(-1)^{\dim V} = (-1)^{\bar{m}}$, it suffices to show $\mathrm{Ext}_{\bar{t}_i}(\mathrm{id}, \bar{t}_i) = -1$. Taking a resolution, we have

$$\begin{aligned}
&\mathrm{Ext}^*_{\wedge s^{k-1}v_i}(\mathbb{K}, \wedge s^{k-1}v_i) = H^*(\mathrm{Hom}_{\wedge s^{k-1}v_i}(\wedge s^{k-1}v_i \otimes \wedge s^k v_i, \wedge s^{k-1}v_i)) \\
&\mathrm{Ext}_{\bar{t}_i}(\mathrm{id}, \bar{t}_i) = H^*(\mathrm{Hom}_{\bar{t}_i}(\hat{t}_i, \bar{t}_i)),
\end{aligned}$$

where the differential d on $\wedge s^{k-1}v_i \otimes \wedge s^k v_i$ is defined by $d(s^{k-1}v_i) = 0$ and $d(s^k v_i) = s^{k-1}v_i$, and the dga homomorphism $\hat{t}_i$ is defined by $\hat{t}_i(s^{k-1}v_i) = -s^{k-1}v_i$ and $\hat{t}_i(s^k v_i) = -s^k v_i$. Using this resolution, we have the generator $[f]$ of $H^*(\mathrm{Hom}_{\wedge s^{k-1}v_i}(\wedge s^{k-1}v_i \otimes \wedge s^k v_i, \wedge s^{k-1}v_i)) \cong \mathbb{K}$ as follows:

- If $|s^{k-1}v_i|$ is odd, define f by $f(1) = s^{k-1}v_i$ and $f((s^k v_i)^l) = 0$ for $l \geq 1$.
- If $|s^{k-1}v_i|$ is even, define f by $f(1) = 0$ and $f((s^k v_i)) = 1$.

In both cases, we have $\mathrm{Hom}_{\bar{t}_i}(\hat{t}_i, \bar{t}_i)(f) = \bar{t}_i \circ f \circ \hat{t}_i = -f$. This proves $\mathrm{Ext}_{\bar{t}_i}(\mathrm{id}, \bar{t}_i) = -1$ and completes the determination of the sign in (7.7). □

Part II

Nontrivial example of the composition of the brane product and coproduct on Gorenstein spaces

Abstract

We give an example of a space with the nontrivial composition of the brane product and the brane coproduct, which we introduced in a previous article.

10 Introduction

Chas and Sullivan [CS99] introduced the loop product $\mu\colon H_*(LM \times LM) \to H_{*-m}(LM)$ on the homology of the free loop space $LM = \mathrm{Map}(S^1, M)$ of a connected closed oriented manifold M of dimension m. Constructing a 2-dimensional topological quantum field theory without counit, Cohen and Godin [CG04] generalized this product to other string operations, including the loop coproduct $\delta\colon H_*(LM) \to H_{*-m}(LM \times LM)$. But Tamanoi [Tam10] showed that any string operation corresponding to a positive genus surface is trivial. In particular, the composition $\mu \circ \delta$ is trivial. There are many attempts to find nontrivial and interesting operations.

Félix and Thomas [FT09] generalized the loop product and coproduct to the case M is a Gorenstein space. A Gorenstein space is a generalization of a manifold in the point of view of Poincaré duality. For example, connected closed oriented manifolds, classifying spaces of connected Lie groups, and their Borel constructions are Gorenstein spaces. Moreover, any 1-connected space M with $\bigoplus_n \pi_n(M) \otimes \mathbb{Q}$ of finite dimension is a Gorenstein space. In spite of this huge generalization, string operations remain to tend to be trivial. For example, the loop product μ is trivial over a field of characteristic zero for the classifying space of a connected Lie group [FT09, Theorem 14].

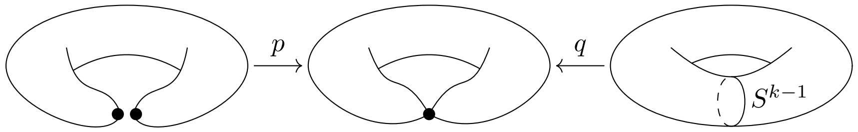
Problem 10.1 ([FT09])**.** *Is there a Gorenstein space such that the composition $\mu \circ \delta$ is nontrivial?*

This is the Gorenstein counterpart of the above result due to Tamanoi. But such an example is not found.

Sullivan and Voronov [CHV06, Part I, Chapter 5] generalized the loop product to the sphere space $S^kM = \mathrm{Map}(S^k, M)$ for $k \geq 1$. This product is called the brane product.

The brane coproduct, a generalization of the loop coproduct to the sphere spaces, is constructed in Part I in the case where the rational homotopy group $\bigoplus_n \pi_n(M) \otimes \mathbb{Q}$ is of finite dimension. In the construction, we assume the "finiteness" of the dimension of the $(k-1)$-fold based loop space $\Omega^{k-1}M$ as a Gorenstein space. Moreover, the product and the coproduct were generalized to the mapping spaces from manifolds, by means of connected sums.

Here we briefly review the brane product and coproduct. See Section 11 for details. Let $\mathbb{K}$ be a field of characteristic zero, S an oriented manifold of dimension k with two disjoint base points, and M a k-connected m-dimensional $\mathbb{K}$-Gorenstein space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Denote the "connected sum" and "wedge sum" of S with itself along the two base points by $S_\#$ and $S_\vee$, respectively. Note that, by the definition of the connected sum, we have the canonical inclusion $S^{k-1} \hookrightarrow S_\#$ and the quotient map $q\colon S_\# \to (S_\#)/S^{k-1} = S_\vee$. Similarly we have $S^0 = \mathrm{pt} \coprod \mathrm{pt} \hookrightarrow S$ and $p\colon S \to S/S^0 = S_\vee$. Hence we have the following diagram



$$\begin{tikzcd} S \arrow[r, "p"] & S_{\vee} & S_{\#} \arrow[l, "q"'] \end{tikzcd}$$

and its dual

$$M^S \xleftarrow{\text{incl}} M^{S_\vee} \xrightarrow{\text{comp}} M^{S_\#}, \tag{10.2}$$

where the maps incl and comp are induced by p and q.

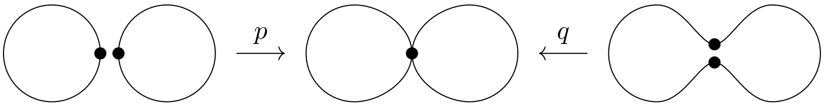
Using this diagram, we can construct two operations, S-brane product μ_S and coproduct δ_S:

$$\mu_S \colon H_*(M^S) \to H_{*-m}(M^{S_\#})$$
$$\delta_S \colon H_*(M^{S_\#}) \to H_{*-\bar{m}}(M^S).$$

Note that, if T and U are oriented k-manifolds and we take $S = T \coprod U$ with one base point on T and the other on U, then μ_S and δ_S have the form

$$\mu_{T \coprod U} \colon H_*(M^T \times M^U) \to H_{*-m}(M^{T\#U})$$
$$\delta_{T \coprod U} \colon H_*(M^{T\#U}) \to H_{*-\bar{m}}(M^T \times M^U).$$

Moreover, if we take $T = U = S^1$, then $\mu_{S^1 \coprod S^1}$ and $\delta_{S^1 \coprod S^1}$ coincide with the usual loop product and coproduct, respectively. Hence the S-brane product and coproduct are generalizations of the loop product and coproduct.



$$\begin{tikzcd} S = S^1 \coprod S^1 \arrow[r, "p"] & S_{\vee} = S^1 \vee S^1 & S_{\#} = S^1 \arrow[l, "q"'] \end{tikzcd}$$

In this article, we give examples that the composition $\mu \circ \delta$ of the brane product and the brane coproduct is nontrivial.

Theorem 10.3. *Let k be a positive even integer. Consider the case $S = S^k$ (and hence $S_\# = S^{k-1} \times S^1$). Let M be the Eilenberg-MacLane space $K(\mathbb{Z}, 2n)$ with $n > k/2$. Then the composition $\mu_{S^k} \circ \delta_{S^k}$ of the S^k-brane product*

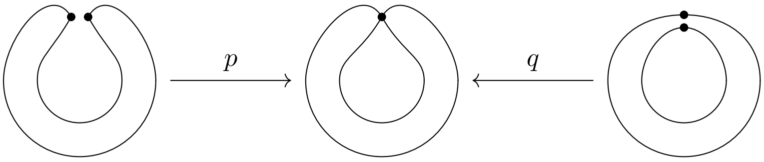
$$\mu_{S^k} \colon H_*(\mathrm{Map}(S^k, M)) \to H_{*+2n-1}(\mathrm{Map}(S^{k-1} \times S^1, M))$$

and the S^k-brane coproduct

$$\delta_{S^k} \colon H_*(\mathrm{Map}(S^{k-1} \times S^1, M)) \to H_{*-2n+k-1}(\mathrm{Map}(S^k, M))$$

is nontrivial.

This gives an answer to Problem 10.1 in the context of brane operations. Here it should be remarked that, the composition $\mu_{S^k} \circ \delta_{S^k}$ corresponds to a cobordism without "genus". In fact, if we take $k = 1$, the composition $\mu_{S^1} \circ \delta_{S^1}$ is equal to the composition $\delta \circ \mu$, not $\mu \circ \delta$, of the loop product μ and coproduct δ.



$$\begin{tikzcd} S = S^1 \arrow[r, "p"] & S_\vee = S^1 \vee S^1 & S_\# = S^1 \coprod S^1 \arrow[l, "q"'] \end{tikzcd}$$

On the other hand, the S-brane coproduct is trivial in some cases.

Theorem 10.4. *Let k be a positive even integer, and M a k-connected (Gorenstein) space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Assume that the minimal Sullivan model of M is pure and has at least one generator of odd degree. Then the S^k-brane coproduct is trivial for M.*

For a connected Lie group G and its closed connected subgroup H, the homogeneous space $M = G/H$ satisfies the assumption if the canonical map $\pi_*(H) \otimes \mathbb{K} \to \pi_*(G) \otimes \mathbb{K}$ is *not* surjective.

By Theorem 10.3 and Theorem 10.4, we have the following corollary.

Corollary 10.5. *Let k be a positive even integer, and M a k-connected (Gorenstein) space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Assume that the minimal Sullivan model of M is pure. Then the composition $\mu_{S^k} \circ \delta_{S^k}$ is nontrivial if and only if M is a finite product $\prod K(\mathbb{Z}, 2n_i)$ of Eilenberg-MacLane spaces of even degrees.*

Section 11 contains brief background material on brane operations. In Section 12, we construct rational models of the S^k-brane product and coproduct, which gives a method of computation. Next we review explicit constructions of the shriek maps in Section 13, which is necessary to accomplish the computation by the above models. Finally, in Section 14, we prove Theorem 10.3 and Theorem 10.4 using the above models.

11 Brane operations for the mapping space from manifolds

In this section, we review the constructions of the S-brane product and coproduct from Part I. Since the cochain models work well for fibrations, we define the duals of the S-brane product and coproduct at first, and then we define the S-brane product and coproduct as the duals of them.

Let $\mathbb{K}$ be a field of characteristic zero. This assumption enables us to make full use of rational homotopy theory. For the basic definitions and theorems on homological algebra and rational homotopy theory, we refer the reader to [FHT01].

Definition 11.1 ([FHT88])**.** Let $m \in \mathbb{Z}$ be an integer.

(1) An augmented dga (differential graded algebra) (A, d) is called a ($\mathbb{K}$-)*Gorenstein algebra* of dimension m if

$$\dim \operatorname{Ext}_A^l(\mathbb{K}, A) = \begin{cases} 1 & (\text{if } l = m) \\ 0 & (\text{otherwise}), \end{cases}$$

where the field $\mathbb{K}$ and the dga (A, d) are (A, d)-modules via the augmentation map and the identity map, respectively.

(2) A path-connected topological space M is called a ($\mathbb{K}$-)*Gorenstein space* of dimension m if the singular cochain algebra $C^*(M)$ of M is a Gorenstein algebra of dimension m.

Here, $\mathrm{Ext}_A(L, N)$ is defined using a semifree resolution of (L, d) over (A, d), for a dga (A, d) and (A, d)-modules (L, d) and (N, d). $\mathrm{Tor}_A(L, N)$ is defined similarly. See [FHT01, Section 1] for details of semifree resolutions.

An important example of a Gorenstein space is given by the following proposition.

Proposition 11.2 ([FHT88, Proposition 3.4]). *A 1-connected topological space M is a $\mathbb{K}$-Gorenstein space if $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ is finite dimensional. Similarly, a Sullivan algebra $(\wedge V, d)$ is a Gorenstein algebra if V is finite dimensional.*

Note that this proposition is proved only for $\mathbb{Q}$-Gorenstein *spaces* in [FHT88], but the proof can be applied for any field $\mathbb{K}$ of characteristic zero and Sullivan *algebras* over $\mathbb{K}$.

We use the following theorem to construct the brane operations.

Theorem 11.3 ([FT09, Theorem 12] for $k = 1$, Corollary 3.2 for $k \geq 2$). *Let M be a $(k-1)$-connected (and 1-connected) space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension, for $k \geq 1$. Then we have an isomorphism*

$$\mathrm{Ext}^*_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^{*-\bar{m}}(M),$$

where $\bar{m}$ is the dimension of $\Omega^{k-1}M$ as a Gorenstein space.

Now we can define the S-brane coproduct as follows. Let S be an oriented manifold with two distinct base points, M a k-connected m-dimensional $\mathbb{K}$-Gorenstein space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Consider the diagram, extending (10.2),

$$\begin{tikzcd} M^{S_\#} \arrow[d, "\mathrm{res}"'] & M^{S_\vee} \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & M^S \\ S^{k-1}M & M. \arrow[l, "c"'] \end{tikzcd} \tag{11.4}$$

Here, the square is a pullback diagram, the map res is the restriction map to S^{k-1}, and c is the embedding as the constant maps. By Theorem 11.3, we have $\mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^0(M) \cong \mathbb{K}$, hence the generator

$$c_! \in \mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M))$$

is well-defined up to the multiplication by a non-zero scalar. Using the map $c_!$ and the diagram (11.4), we can define the shriek map $\mathrm{comp}_!$ as the composition

$$\begin{aligned} &H^*(M^{S_\vee}) \underset{\cong}{\xleftarrow{\mathrm{EM}}} \mathrm{Tor}^*_{C^*(S^{k-1}M)}(C^*(M), C^*(M^{S_\#})) \\ &\xrightarrow{\mathrm{Tor_{id}}(c_!,\mathrm{id})} \mathrm{Tor}^{*+\bar{m}}_{C^*(S^{k-1}M)}(C^*(S^{k-1}M), C^*(M^{S_\#})) \underset{\cong}{\longrightarrow} H^{*+\bar{m}}(M^{S_\#}), \end{aligned}$$

where the map EM is the Eilenberg-Moore map, which is an isomorphism since $S^{k-1}M$ is 1-connected (see [FHT01, Theorem 7.5] for details). By this, we define the dual of the S-brane coproduct as the composition

$$\delta_S^\vee \colon H^*(M^S) \xrightarrow{\mathrm{incl}^*} H^{*+\bar{m}}(M^{S_\vee}) \xrightarrow{\mathrm{comp}_!} H^{*+\bar{m}}(M^{S_\#}).$$

Similarly we can define the S-brane product using the generator

$$\Delta_! \in \operatorname{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2))$$

and the diagram

$$\begin{tikzcd}
M^S \arrow[d] & M^{S_\vee} \arrow[l, "\mathrm{incl}"'] \arrow[r, "\mathrm{comp}"] \arrow[d] & M^{S_\#} \\
M \times M & M. \arrow[l, "\Delta"'] &
\end{tikzcd}$$

Note that, for the brane product and the *loop* coproduct, we can replace the assumption $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ is of finite dimension with the assumption $\pi_*(M) \otimes \mathbb{K}$ is of finite type by using [FT09, Theorem 12] instead of Theorem 11.3.

12 Models of the brane operations

In this section, we consider the case $S = S^k$ and give rational models of the S^k-brane operations, for an integer $k \geq 1$. In Section 14, we will prove Theorem 10.3 and Theorem 10.4 using these models.

Naito [Nai13] constructed a rational model of the duals of the loop product and coproduct in terms of Sullivan models using the torsion functor description of [KMN15]. In Part I, the author constructed a rational model of the duals of the brane product and coproduct as a generalization of it. Here we give a rational model of the S^k-brane operations by a similar method.

12.1 Models of spaces

Let M be a k-connected space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Take a Sullivan model $(\wedge V, d)$ of M with $V^{\leq k} = 0$ and $\dim V < \infty$. For simplicity, we sometimes denote $(\wedge V, d)$ by $\mathcal{M}$. Denote $(S^k)_\# = S^{k-1} \times S^1$ by $T^{(k)}$ and $(S^k)_\vee = (S^{k-1} \times S^1)/S^{k-1}$ by $U^{(k)}$. For an integer $l \in \mathbb{Z}$, let $s^l V$ be a graded module defined by $(s^l V)^n = V^{n+l}$ and $s^l v$ denotes the element in $s^l V$ corresponding to an element $v \in V$. Here we recall models of mapping spaces from the interval, sphere, and disk.

(12.1) Consider s as an derivation on the algebra $\wedge V^{\otimes 2} \otimes \wedge sV$ with $s \circ s = 0$. Define a derivation d on the algebra by

$$d(sv) = 1 \otimes v - v \otimes 1 - \sum_{i=1}^{\infty} \frac{(sd)^i}{i!}(v \otimes 1),$$

inductively. Denote the dga $(\wedge V^{\otimes 2} \otimes \wedge sV, d)$ by $\mathcal{M}(I)$. This is a Sullivan model of the path space M^I $(\simeq M)$. Moreover, define a map $\bar{\varepsilon} \colon \mathcal{M}(I) \to \mathcal{M}$ by $\bar{\varepsilon}(v \otimes 1) = \bar{\varepsilon}(1 \otimes v) = v$ and $\bar{\varepsilon}(sv) = 0$ for $v \in V$. Then it is a relative Sullivan model (resolution) of the product map $\wedge V^{\otimes 2} \wedge sV$. See [FHT01, Section 15 (c)] or [Wak16, Appendix A] for details.

(12.2) Assume $k \geq 2$. Define derivations $s^{(k-1)}$ and d on the graded algebra $\wedge V \otimes \wedge s^{k-1} V$ by

$$\begin{aligned}
&s^{(k-1)}(v) = s^{k-1} v, \quad s^{(k-1)}(s^{k-1} v) = 0, \\
&d(v) = dv, \text{ and } \quad d(s^{k-1} v) = (-1)^{k-1} s^{(k-1)} dv.
\end{aligned}$$

Denote the dga $\wedge V \otimes s^{k-1} V$ by $\mathcal{M}(S^{k-1})$. This is a Sullivan model of the space $M^{S^{k-1}}$. See Section 5 for details.

(12.3) Assume $k \geq 2$. Define derivations $s^{(k)}$ and d on the graded algebra $\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V$ by

$$\begin{aligned} &s^{(k)}(v) = s^k v, \quad s^{(k)}(s^{k-1}v) = s^{(k)}(s^k v) = 0, \quad d(v) = dv, \\ &d(s^{k-1}v) = d(s^{k-1}v), \text{ and } \quad d(s^k v) = s^{k-1}v + (-1)^k s^{(k)} dv. \end{aligned}$$

Denote the dga $\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V$ by $\mathcal{M}(D^k)$. This is a Sullivan model of the space M^{D^k} ($\simeq M$). Moreover, define a map $\tilde{\varepsilon}\colon \mathcal{M}(D^k) \to \mathcal{M}$ by $\tilde{\varepsilon}(v) = v$, $\tilde{\varepsilon}(s^{k-1}v) = \tilde{\varepsilon}[k]v = 0$ for $v \in V$. Then it is a relative Sullivan model (resolution) of the map $\varepsilon\colon \mathcal{M}(S^{k-1}) \to \mathcal{M}$, where $\varepsilon(v) = v$ and $\varepsilon(s^{k-1}v) = 0$. In particular, $\tilde{\varepsilon}$ is a quasi-isomorphism. See Section 5 for details.

Next we construct models of mapping spaces which appear in the definition of brane operations, using the above models.

(12.4) Since $M^{T^{(k)}} = (M^{S^{k-1}})^{S^1}$, we have a Sullivan model $\mathcal{M}(T^{(k)}) = (\wedge V \otimes \wedge s^{k-1}V \otimes \wedge sV \otimes \wedge s s^{k-1}V, d)$ of $M^{T^{(k)}}$ iterating the construction in MyDescription 12.2.

(12.5) Since $U^{(k)}$ is homotopy equivalent to $S^k \vee S^1$, the mapping space $M^{U^{(k)}}$ is homotopy equivalent to $M^{S^k} \times_M M^{S^1}$, and hence we have a Sullivan model $\mathcal{M}(U^{(k)}) = (\wedge V \otimes \wedge s^k V, d) \otimes (\wedge V \otimes \wedge sV, d)$.

12.2 Models of operations

Here we give a model of the S^k-brane product and coproduct in a similar way to [Nai13] and Part I.

First we give a model of the S^k-brane coproduct. Recall that the dual $\delta_{S^k}^{\vee}$ of the S^k-brane coproduct is the composition

$$\delta_{S^k}^{\vee}\colon H^*(M^{S^k}) \xrightarrow{\text{incl}^*} H^{*+\bar{m}}(M^{U^{(k)}}) \xrightarrow{\text{comp}_!} H^{*+\bar{m}}(M^{T^{(k)}}).$$

First the map $\text{incl}^*\colon H^*(M^{S^k}) \to H^{*+\bar{m}}(M^{U^{(k)}})$ is induced by the canonical inclusion $\mathcal{M}(S^k) \to \mathcal{M}(U^{(k)})$, which we also denote by incl^*. Next the map $\text{comp}_!\colon H^{*+\bar{m}}(M^{U^{(k)}}) \to H^{*+\bar{m}}(M^{T^{(k)}})$ is computed as follows. Let

$$\gamma \in \text{Hom}_{\mathcal{M}(S^{k-1})}(\mathcal{M}(D^k), \mathcal{M}(S^{k-1}))$$

be a representative of the nontrivial element (see Theorem 11.3)

$$\begin{aligned} c_! \in\ &\text{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \\ &\cong H^{\bar{m}}(\text{Hom}_{\mathcal{M}(S^{k-1})}(\mathcal{M}(D^k), \mathcal{M}(S^{k-1}))). \end{aligned}$$

Then the map

$$\text{Tor}_{\text{id}}(c_!, \text{id})\colon\ \text{Tor}^*_{C^*(S^{k-1}M)}(C^*(M), C^*(M^{S_\#})) \to \text{Tor}^{*+\bar{m}}_{C^*(S^{k-1}M)}(C^*(S^{k-1}M), C^*(M^{S_\#}))$$

is induced by the cochain map

$$\gamma \otimes \text{id}\colon \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \to \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}),$$

since $\mathcal{M}(D^k)$ is a resolution of $\mathcal{M}$ over $\mathcal{M}(S^{k-1})$. The map $\text{comp}_!$ is computed by this combined with the quasi-isomorphism

$$\tilde{\varepsilon} \otimes \text{id}\colon \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \underset{\simeq}{\to} \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}). \tag{12.6}$$

Hence the dual of the S^k-brane coproduct is induced by the composition

$$\begin{aligned}
&\mathcal{M}(S^k) \xrightarrow{\mathrm{incl}^*} \mathcal{M}(U^{(k)}) \xrightarrow{\cong} \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \\
&\underset{\simeq}{\xleftarrow{\bar{\varepsilon}\otimes\mathrm{id}}} \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \\
&\xrightarrow{\gamma\otimes\mathrm{id}} \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \xrightarrow{\cong} \mathcal{M}(T^{(k)}).
\end{aligned} \tag{12.7}$$

Similarly, the dual of the S^k-brane product is induced by the composition

$$\begin{aligned}
&\mathcal{M}(T^{(k)}) \xrightarrow{\mathrm{comp}^*} \mathcal{M}(U^{(k)}) \xrightarrow{\cong} \mathcal{M} \otimes_{\mathcal{M}^{\otimes 2}} (\mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k)) \\
&\underset{\simeq}{\xleftarrow{\bar{\varepsilon}\otimes\mathrm{id}}} \mathcal{M}(I) \otimes_{\mathcal{M}^{\otimes 2}} (\mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k)) \xrightarrow{\eta\otimes\mathrm{id}} \mathcal{M}^{\otimes 2} \otimes_{\mathcal{M}^{\otimes 2}} (\mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k)) \\
&\xrightarrow{\cong} \mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k) \xrightarrow{\bar{\varepsilon}\otimes\mathrm{id}} \mathcal{M} \otimes_{\mathcal{M}} \mathcal{M}(S^k) \xrightarrow{\cong} \mathcal{M}(S^k)
\end{aligned} \tag{12.8}$$

Here $\eta \in \mathrm{Hom}_{\mathcal{M}^{\otimes 2}}(\mathcal{M}(I), \mathcal{M}^{\otimes 2})$ is a representative of the nontrivial element $\Delta_! \in \mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2))$ and $\mathrm{comp}^* \colon \mathcal{M}(T^{(k)}) \to \mathcal{M}(U^{(k)})$ is the canonical quotient map.

13 Explicit construction of shriek maps

Models of S^k-brane operations are constructed in Section 12 using the representatives of the shriek maps γ and η. They are constructed by Theorem 11.3, which only states the existence of the shriek maps. In this section, we recall methods to construct shriek maps explicitly from [Nai13], [Wak16] and Part I.

Recall the definition of a pure Sullivan algebra. Here we denote $V^{\mathrm{even}} = \bigoplus_n V^{2n}$ and $V^{\mathrm{odd}} = \bigoplus_n V^{2n+1}$.

Definition 13.1 (c.f. [FHT01, Section 32]). A Sullivan algebra $(\wedge V, d)$ with $\dim V < \infty$ is called *pure* if $d(V^{\mathrm{even}}) = 0$ and $d(V^{\mathrm{odd}}) \subset \wedge V^{\mathrm{even}}$.

In the rest of this section, let $(\wedge V, d)$ be a pure minimal Sullivan algebra, $\{x_1, \dots, x_p\}$ a basis of V^{even}, and $\{y_1, \dots, y_q\}$ a basis of V^{odd}.

13.1 Construction of $\Delta_!$

Here we recall the description of $\Delta_!$ in [Wak16], which is a generalization of that of Naito [Nai13]. Note that, although the description holds if the Sullivan model $(\wedge V, d)$ is semi-pure (see [Wak16, Definition 1.5] for the definition), we only refer and use it in the case $(\wedge V, d)$ is pure.

Proposition 13.2 ([Wak16, Theorem 5.6 (2)]). *Take $(\wedge V^{\otimes 2} \otimes \wedge sV, d) = \mathcal{M}(I)$ as in MyDescription 12.1. If a cocycle $\eta \in \mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2})$ satisfies*

$$\eta(sx_1 \cdots sx_p) = (1 \otimes y_1 - y_1 \otimes 1) \cdots (1 \otimes y_q - y_q \otimes 1),$$

then we have

$$\begin{aligned}
[\eta] \neq 0 \in\ & H^*(\mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2})) \\
\cong\ & \mathrm{Ext}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2}).
\end{aligned}$$

This proposition gives a construction of the map $\Delta_!$.

13.2 Construction of $c_!$

Next we recall the description of $c_!$ in Part I. The following proposition gives it completely when k is even.

Proposition 13.3 (Proposition 6.5)**.** *Assume that k is even. Define an element*

$$\gamma \in \mathrm{Hom}_{\mathcal{M}(S^{k-1})}(\mathcal{M}(D^k), \mathcal{M}(S^{k-1}))$$

by $\gamma(s^k y_1 \cdots s^k y_q) = s^{k-1}x_1 \cdots s^{k-1}x_p$ and $\gamma(s^k y_{j_1} \cdots s^k y_{j_l}) = 0$ for $l < q$. Then γ defines a nontrivial element in $\mathrm{Ext}_{\mathcal{M}(S^{k-1})}(\mathcal{M}, \mathcal{M}(S^{k-1}))$.

Note that, although the proposition is proved only when $k = 2$ in Proposition 6.5, the same proof also applies when $k > 2$ as long as k is even.

14 Proof of Theorem 10.3 and Theorem 10.4

In this section, we give a proof of Theorem 10.3 and Theorem 10.4 using the models constructed above.

Proof of Theorem 10.3. We compute the S^k-brane coproduct using (12.7). Since $M = K(\mathbb{Z}, 2n)$, we take the Sullivan model $(\wedge V, d) = (\wedge x, 0)$ where x is the generator of degree $2n$. Note that, in this case, the differentials in $\mathcal{M}(S^k)$ and $\mathcal{M}(T^{(k)})$ are zero, and hence they are identified with the cohomology groups $H^*(M^{S^k})$ and $H^*(M^{T^{(k)}})$.

By Proposition 13.3, we have a representative γ of the shriek map $c_!$ defined by $\gamma(1) = s^{k-1}x$ and $\gamma((s^k x)^l) = 0$ for $l \geq 1$.

Since any Sullivan algebra satisfies the lifting property for a surjective quasi-isomorphism, there is a section φ of $\tilde{\varepsilon} \otimes \mathrm{id}$ in (12.6), which is also a quasi-isomorphism. It is given explicitly by $\varphi(1 \otimes x) = 1 \otimes x$, $\varphi(1 \otimes s^k x) = 1 \otimes ss^{k-1}x$, and $\varphi(1 \otimes sx) = 1 \otimes sx$.

Using these maps, we compute the composition (12.7). Since all maps in the composition are $\wedge V$-linear, it is enough to compute the image for the elements $(s^k x)^n$ for $n \geq 0$. Applying incl^* and the section φ to the element, we have that it is mapped to $1 \otimes (ss^{k-1}x)^n \in \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)})$. Then the map $\gamma \otimes \mathrm{id}$ send it to $s^{k-1}x \otimes (ss^{k-1}x)^n \in \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)})$. Hence the S^k-brane coproduct $\delta^\vee_{S^k}$ is the map determined by $\delta^\vee_{S^k}(\alpha) = s^{k-1}x\iota(\alpha)$, where $\iota\colon \mathcal{M}(S^k) \to \mathcal{M}(T^{(k)})$ is the algebra map defined by $\iota(x) = x$ and $\iota(s^k x) = ss^{k-1}x$.

Similarly we can compute the S^k-brane product. Define a $\wedge V^{\otimes 2}$-linear map $\eta\colon \mathcal{M}(I) \to \wedge V^{\otimes 2}$ by $\eta(1) = 0$ and $\eta(sx) = 1$. By Proposition 13.3, η is a representative of the shriek map $\Delta_!$. We have a section ψ of $\bar{\varepsilon} \otimes \mathrm{id}$ in (12.8), which is defined by $\psi(x \otimes 1) = 1 \otimes (x_1 \otimes 1)$, $\psi(1 \otimes sx) = 1 \otimes (sx \otimes 1) - sx \otimes 1$, and $\psi(1 \otimes s^k x) = 1 \otimes (1 \otimes s^k x)$. Here we denote the element $x \otimes 1 \in \mathcal{M}(I)$ by x_1.

As a result, the S^k-brane product $\mu^\vee_{S^k}$ is the map determined by $\mu^\vee_{S^k}(\beta) = 0$, $\mu^\vee_{S^k}(sx \cdot \beta) = -\rho(\beta)$, and $\mu^\vee_{S^k}(s^{k-1}x \cdot \beta) = \mu^\vee_{S^k}(sx \cdot s^{k-1}x \cdot \beta) = 0$, for $\beta \in \wedge x \otimes \wedge ss^{k-1}x$. Here $\rho\colon \wedge x \otimes \wedge ss^{k-1}x \to \mathcal{M}(S^k)$ is the algebra map defined by $\rho(x) = x$ and $\rho(ss^{k-1}x) = s^k x$.

Composing these two, we have $\delta_{S^k} \circ \mu_{S^k} \neq 0$. In fact, $\delta_{S^k} \circ \mu_{S^k}(sx) = -s^{k-1}x \neq 0 \in \mathcal{M}(T^{(k)}) \cong H^*(M^{T^{(k)}})$. This proves the theorem. □

Next we prove Theorem 10.4.

Proof of Theorem 10.4. Let $(\wedge V, d)$ be the minimal Sullivan model of M, $\{x_1, \dots, x_p\}$ a basis of V^{even}, and $\{y_1, \dots, y_q\}$ a basis of V^{odd}. Consider the part

$$\begin{aligned} \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) &\underset{\simeq}{\xleftarrow{\tilde{\varepsilon}\otimes \mathrm{id}}} \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \\ &\xrightarrow{\gamma\otimes\mathrm{id}} \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \end{aligned}$$

in (12.7). Define a section φ of $\tilde{\varepsilon} \otimes \mathrm{id}$ by $\varphi(1 \otimes v) = 1 \otimes v$, $\varphi(1 \otimes sv) = 1 \otimes sv$, for $v \in V$, $\varphi(1 \otimes ss^{k-1}x_i) = 1 \otimes ss^{k-1}x_i$, and $\varphi(1 \otimes ss^{k-1}y_j) = 1 \otimes ss^{k-1}y_j + (-1)^k s\sigma(dy_j \otimes 1)$. Here, in the last term $s\sigma(dy_j \otimes 1)$, σ is the derivation which sends $v \otimes 1$ to $s^k v \otimes 1$, for $v \in V$, and the other generators to 0. The map s is also the derivation which sends v to sv, $s^{k-1}v$ to $ss^{k-1}v$, and others to 0. Then we have $\operatorname{Im} \varphi \subset \mathcal{N} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)})$, where $\mathcal{N} = \wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k \{x_1, \dots, x_p\} \subset \mathcal{M}(D^k)$. Let γ be the representative of $c_!$ given by Proposition 13.3. Since V has at least one generator of odd degree, γ is zero on $\mathcal{N}$. This implies that the composition $(\gamma \otimes 1) \circ \varphi$ is zero, and hence the brane coproduct δ_{S^k} is zero. □

Part III

New construction of the brane coproduct and vanishing of cup products on sphere spaces

Abstract

Using the loop coproduct, Menichi proved that the cup product with the orientation class vanishes for a closed connected oriented manifold with nontrivial Euler characteristic. We generalize this to the sphere spaces, i.e. the mapping spaces from spheres, using two generalizations of the loop coproduct to sphere spaces. One is constructed in this paper and the other in a previous paper of the author.

15 Introduction

In this article, we give a new construction of the brane coproduct, which we call the *non-symmetric* brane coproduct. Comparing this coproduct with another coproduct constructed in Part I, we prove the vanishing of some cup products on the cohomology of mapping spaces from spheres.

Chas and Sullivan [CS99] introduced the loop product on the homology $H_*(LM)$ of the free loop space $LM = \mathrm{Map}(S^1, M)$ of a manifold M of dimension m. Cohen and Godin [CG04] extended this product to other string operations, including the loop coproduct, whose dual has the form

$$\delta^\vee \colon H^*(LM) \to H^{*+m}(LM \times LM).$$

Although the loop coproduct is almost trivial by [Tam10], Menichi [Men13] used the loop coproduct to obtain the following vanishing result.

Theorem 15.1 ([Men13, Theorem 1])**.** *Let M be a connected, closed oriented manifold of dimension m, $\omega \in H^m(M)$ its orientation class, and $\chi(M)$ its Euler characteristic. Then, for any $\alpha \in H^{>0}(LM)$, we have*

$$\chi(M)\mathrm{ev}_0^*\omega \cdot \alpha = 0 \in H^{|\alpha|+m}(LM),$$

where $\mathrm{ev}_0 \colon LM \to M$ is the evaluation map at the base point $0 \in S^1$.

Moreover, Félix and Thomas [FT09] generalized the loop coproduct to Gorenstein spaces. A Gorenstein space is a generalization of a Poincaré duality space (i.e. a space satisfying Poincaré duality) in an algebraic way. See Definition 17.2 for the definition.

Using the algebraic method due to Félix and Thomas, in Part I, the author constructed a generalization of the loop coproduct, called the *brane coproduct.* Here we explain it along with a little generalization. Let $\mathbb{K}$ be a field, k a positive integer and M a k-connected space with $H^*(M) = H^*(M;\mathbb{K})$ of finite type. Denote by $S^kM = \mathrm{Map}(S^k, M)$ the mapping space from the k-dimensional sphere to M. We fix an arbitrary element $\gamma \in \mathrm{Ext}^l_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M))$, where $C^*(M)$ is the singular cochain algebra on M. Then we can construct (the dual of) the brane coproduct

$$\delta^\vee_\gamma \colon H^*(S^kM \times S^kM) \to H^{*+l}(S^kM).$$

Note that γ will be specified under some assumption on M, and that we can choose l and γ depending on the purpose. See Section 17 for details.

Next we explain the *non-symmetric* brane coproduct, which will be defined in this article. Assume M is a Poincaré duality space (i.e. a space satisfying Poincaré duality) over $\mathbb{K}$ of dimension m. Then we can define the non-symmetric brane coproduct

$$\delta_{\mathrm{ns}}^{\vee}\colon H^*(S^kM \times S^kM) \to H^{*+m}(S^kM).$$

Note that the non-symmetric brane coproduct can be defined for any 1-connected Poincaré duality space, without the assumption of k-connectivity. See Section 18 for details.

The non-symmetric brane coproduct $\delta_{\mathrm{ns}}^{\vee}$ seems to be *non-commutative*, from the explicit formula in Theorem 19.1. On the other hand, the brane coproduct $\delta_{\gamma}^{\vee}$ is *commutative* in the sense of Proposition 20.4. In spite of such difference, these coproducts coincide with each other under some assumptions. This coincidence gives some nontrivial relations on $H^*(S^kM)$, which is the main theorem of this article:

Theorem 15.2. *Let k be a positive integer, M a k-connected Poincaré duality space over $\mathbb{K}$ of dimension m, and $\omega \in H^m(M)$ its orientation class. Assume*

(1) *$k = 1$ or*

(2) *$k \geq 1$ is odd, the characteristic of $\mathbb{K}$ is zero, and $\dim_{\mathbb{K}}(\bigoplus_n \pi_n(M) \otimes \mathbb{K}) < \infty$.*

Then, for any $\alpha \in H^{>0}(S^kM)$, we have

$$\chi(M)\mathrm{ev}_0^*\omega \cdot \alpha = 0 \in H^{|\alpha|+m}(S^kM).$$

Remark 15.3. This theorem generalizes Theorem 15.1 due to Menichi, since we do not assume that M is a manifold and $k = 1$. See Remark 20.10 for the reason why we need the assumption k is odd.

We prove the above theorem using the following general result.

Theorem 15.4. *Let M be a k-connected Poincaré duality space over $\mathbb{K}$ of dimension m, $\omega \in H^m(M)$ its orientation class. We fix an arbitrary element*

$$\gamma \in \mathrm{Ext}^m_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)).$$

Define $\lambda_\gamma \in \mathbb{K}$ by the equation $c^* \circ (H^*(\gamma))(1) = \lambda_\gamma\omega \in H^m(M)$, where $c\colon M \to S^{k-1}M$ is the embedding as constant maps. See Section 16 for the definition of the map $H^*(\gamma)\colon H^*(M) \to H^*(S^{k-1}M)$. Then, for any $\alpha \in H^{>0}(S^kM)$, we have

$$\lambda_\gamma \mathrm{ev}_0^*\omega \cdot \alpha = 0 \in H^{|\alpha|+m}(S^kM).$$

We conjecture that, for any M and k as in Theorem 15.4, there is an element γ satisfying $\lambda_\gamma = \chi(M)$. The assumptions (1) and (2) give sufficient conditions for the existence of such γ.

Throughout this article, $\mathbb{K}$ denotes a field. The characteristic $\mathrm{ch}\,\mathbb{K}$ of the field $\mathbb{K}$ is zero in Subsection 20.3 and Section 21. In other (sub)sections, $\mathrm{ch}\,\mathbb{K}$ can be zero or any prime. For a vector space V over $\mathbb{K}$, we denote the dual of V by $V^{\vee}$. For spaces X and Y, we denote the mapping space from X to Y by Y^X. For $x \in X$, let $\mathrm{ev}_x\colon Y^X \to Y$ be the evaluation map at x. Denote by $[X, Y]$ the homotopy set of maps from X to Y. Base points does not matter since we consider it only when X is 0-connected and Y is 1-connected.

This article is organized as follows. Section 16 contains basic definitions and properties of Ext, which we use in definitions of the brane coproducts. In Section 17, we review the previous construction of the brane coproduct. We define the non-symmetric brane coproduct in Section 18, and, under some assumptions, explicitly compute it in Section 19. In Section 20, we compare two brane coproducts and prove Theorem 15.2 and Theorem 15.4, using explicit construction of shriek maps given in Section 21.

16 Definition and properties of Ext

Let A be a differential graded algebra (dga), and M and N A-modules over a field $\mathbb{K}$ of any characteristic. Then the extension module is defined as $\mathrm{Ext}_A(M,N) = H^*(\mathrm{Hom}_A(P,A))$, where P is a semifree resolution of M over A. See [FHT01, Section 6] for details of semifree resolutions. For an element $\alpha \in \mathrm{Ext}_A(M,N)$, we define $H^*(\alpha)\colon H^*(M) \xleftarrow{\cong} H^*(P) \xrightarrow{H^*(\alpha)} H^*(N)$. This defines a linear map $\mathrm{Ext}_A(M,N) \to \mathrm{Hom}_{H^*(A)}(H^*(M),H^*(N))$; $\alpha \mapsto H^*(\alpha)$.

Consider a pullback diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
A \arrow[r] & B
\end{tikzcd}$$

such that $p\colon E \to B$ is a fibration and B is 1-connected. Let us recall the linear map

$$p^\star\colon\ \mathrm{Ext}^l_{C^*(B)}(C^*(A),C^*(B)) \to \mathrm{Ext}^l_{C^*(E)}(C^*(D),C^*(E))$$

introduced in [FT09, Remark after Theorem 2]. Let P be a semifree resolution of $C^*(A)$ over $C^*(B)$. Then we have a linear map

$$\mathrm{Hom}_{C^*(B)}(P,C^*(B)) \to \mathrm{Hom}_{C^*(E)}(C^*(E)\otimes_{C^*(B)} P, C^*(E)\otimes_{C^*(B)} C^*(B)) \tag{16.1}$$

by sending $\varphi \in \mathrm{Hom}_{C^*(B)}(P,C^*(B))$ to $\mathrm{id}_{C^*(E)} \otimes \varphi$. Here, $C^*(E)\otimes_{C^*(B)} P$ is a semifree $C^*(E)$ module by [FHT01, Lemma 6.2]. Moreover, the Eilenberg-Moore map $C^*(E)\otimes_{C^*(B)} P \to C^*(D)$ is a quasi-isomorphism by the Eilenberg-Moore theorem [Smi67, Theorem 3.2]. Hence $C^*(E)\otimes_{C^*(B)} P$ is a semifree resolution of $C^*(D)$ over $C^*(E)$, and the linear map (16.1) induces the required map $p^\star$.

The above constructions satisfy naturality in the following sense, which can be proved directly from the definitions.

Proposition 16.2. *Consider a diagram*

$$\begin{tikzcd}
& E' \arrow[dl, "\varphi"'] \arrow[dd, "p'"] & & X' \arrow[ll] \arrow[dl, "\psi"] \arrow[dd] \\
E \arrow[dd, "p"'] & & X \arrow[ll] \arrow[dd] & \\
& B' \arrow[dl, "a"] & & A' \arrow[ll] \arrow[dl, "b"] \\
B & & A \arrow[ll] &
\end{tikzcd}$$

and elements $\alpha \in \mathrm{Ext}^m_{C^*(B)}(C^*(A),C^*(B))$ and $\alpha' \in \mathrm{Ext}^m_{C^*(B')}(C^*(A'),C^*(B'))$. Here p and p' are fibrations and the front and back squares are pullback diagrams. Assume that the elements α and α' are mapped to the same element in $\mathrm{Ext}^m_{C^*(B)}(C^*(A),C^*(B'))$ by the morphisms induced by a and b, and that the Eilenberg-Moore maps of two pullback diagrams are isomorphisms. Then the following diagram commutes.

$$\begin{tikzcd}
H^*(X) \arrow[r, "H^*(p^\star\alpha)"] \arrow[d, "\varphi^*"] & H^{*+m}(E) \arrow[d, "\psi^*"] \\
H^*(X') \arrow[r, "H^*(p'^\star\alpha')"] & H^{*+m}(E')
\end{tikzcd}$$

17 Review of the previous construction of the brane coproduct

In this section, we review the previous construction of the brane coprdouct from Part I. Here we explain it in a generalized way, which is necessary for the comparison in Section 20.

First we give a general construction. Let $\mathbb{K}$ be a field of any characteristic, k a positive integer, S and T k-dimensional manifolds, and M a k-connected space. We fix an arbitrary element

$$\gamma \in \operatorname{Ext}^n_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)).$$

To define the brane coproduct, consider the diagram

$$\begin{tikzcd} M^{S\#T} \arrow[d, "\mathrm{res}"] & M^S \times_M M^T \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & M^S \times M^T \\ S^{k-1}M & M, \arrow[l, "c"] & \end{tikzcd} \tag{17.1}$$

where the square is a pullback diagram, the map res is the restriction map to the embedded sphere S^{k-1} which comes from the connected sum $S\#T$, and the map c is an embedding as constant maps.

Then the dual

$$\delta^\vee_\gamma \colon H^*(M^S \times M^T) \to H^{*+n}(M^{S\#T})$$

of the brane coproduct with respect to γ is defined as the composition

$$\mathrm{comp}_! \circ \mathrm{incl}^* \colon H^*(M^S \times M^T) \xrightarrow{\mathrm{incl}^*} H^*(M^S \times_M M^T) \xrightarrow{\mathrm{comp}_!} H^{*+n}(M^{S\#T}).$$

Here the shriek map $\mathrm{comp}_!$ is defined by $\mathrm{comp}_! = H^*(\mathrm{res}^\star(\gamma))$.

Next we specify the element γ under some assumptions, which was considered in Part I. Here we use the notion of a Gorenstein space.

Definition 17.2 ([FHT88]). Let $m \in \mathbb{Z}$ be an integer. A path-connected topological space M is called a *($\mathbb{K}$-)Gorenstein space* of dimension m if

$$\dim \operatorname{Ext}^l_{C^*(M)}(\mathbb{K}, C^*(M)) = \begin{cases} 1, & \text{if } l = m \\ 0, & \text{otherwise.} \end{cases}$$

For example, a Poincaré duality space over $\mathbb{K}$ is a $\mathbb{K}$-Gorenstein space, and its dimension as a Gorenstein space coincides with the one as a Poincaré duality space. Moreover, the following proposition gives an important example of a Gorenstein space.

Proposition 17.3 ([FHT88, Proposition 3.4]). *A 1-connected topological space M is a $\mathbb{K}$-Gorenstein space if $\mathbb{K}$ is a field of characteristic zero and $\pi_*(M) \otimes \mathbb{K}$ is finite dimensional.*

Now we can specify the element γ by the following theorem.

Theorem 17.4 (Corollary 3.2). *Assume $\mathbb{K}$ is a field of characteristic zero. Let M be a $(k-1)$-connected (and 1-connected) space of finite type such that $\Omega^{k-1}M$ is a Gorenstein space of dimension $\bar{m}$. Then we have an isomorphism*

$$\operatorname{Ext}^l_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^{l-\bar{m}}(M)$$

for any $l \in \mathbb{Z}$.

When $l = \bar{m}$, we have the generator

$$c_! \in \mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^0(M) \cong \mathbb{K} \tag{17.5}$$

up to non-zero scalar multiplication. The brane coproduct $\delta^\vee_{c_!}$ for the case $\gamma = c_!$ is the brane coproduct constructed in Part I.

18 New construction of the brane coproduct

In this section, we give a new construction of the brane coproduct, which we call the *non-symmetric* brane coproduct. This is different from the previous one and we will compare them in Section 20.

Let $\mathbb{K}$ be a field of any characteristic, k a positive integer, T a k-dimensional manifold with a base point t_0, and M a 1-connected Poincaré duality space of dimension m. We fix base points $d_0 \in D^k$ and $s_0 \in S^k$ such that d_0 is mapped to s_0 by the quotient map $D^k \twoheadrightarrow S^k$. For an element $g \in M^T$, we denote by M^T_g the component of M^T containing g.

For $f \in S^kM$ and $g \in M^T$, we define a map $f + g \in M^T$ as follows. Fix an embedded k-disk around t_0 in T. Then we have the quotient map $q\colon T \to S^k \vee T$, which is given by pinching the boundary of the embedded disk. Since M is path-connected, there is a map $f' \in S^kM$ such that $f'(s_0) = g(t_0)$ and f' is homotopic to f (without preserving base points). Define $f + g$ to be the composition $T \xrightarrow{q} S^k \vee T \xrightarrow{f' \vee g} M$. Since M is 1-connected, the map $f + g$ is well-defined up to homotopy.

Instead of (17.1), we consider the diagram

$$\begin{tikzcd}
M^T_{f+g} \arrow[d, "\mathrm{res}"] & S^k_fM \times_M M^T_g \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "\mathrm{pr}_1"] & S^k_fM \times M^T_g \\
D^kM & S^k_fM, \arrow[l, "\iota"]
\end{tikzcd} \tag{18.1}$$

where the square is a pullback diagram, the map res is the restriction to the embedded k-disk, and the map ι is the inclusion induced by the quotient map $D^k \to S^k$.

Note that the above diagram is related to the diagram (17.1) in the following way. When M is k-connected, we have the diagram

$$\begin{tikzcd}
M^T \arrow[d, "\mathrm{res}"] & S^kM \times_M M^T \arrow[l, "\mathrm{comp}"'] \arrow[d, "\mathrm{pr}_1"] \\
D^kM \arrow[d, "\mathrm{res}"] & S^kM \arrow[l, "\iota"] \arrow[d, "\mathrm{ev}"] \\
S^{k-1}M & M, \arrow[l, "c"]
\end{tikzcd} \tag{18.2}$$

where the two squares are pullback diagrams (and hence so is the outer square). In this diagram, the upper square coincides with (18.1) and the outer square coincides with (17.1). We use this diagram to compare the two brane coproducts in Section 20.

We define the dual

$$\delta^\vee_{\mathrm{ns}}\colon H^*(S^k_fM \times_M M^T_g) \to H^{*+m}(M^T_{f+g})$$

of the brane coproduct by the composition

$$\mathrm{comp}_! \circ \mathrm{incl}^* \colon H^*(M^T_{f+g}) \xrightarrow{\mathrm{incl}^*} H^*(S^k_f M \times_M M^T_g) \xrightarrow{\mathrm{comp}_!} H^{*+m}(S^k_f M \times M^T_g).$$

Here, $\mathrm{comp}_!$ is the shriek map constructed from the diagram (18.1). In order to define it, we need the corollary of the following proposition.

Proposition 18.3 ([FT09, Lemma 1]). *Let $F \colon X \to N$ be a map between 0-connected spaces. Assume that N is a Poincaré duality space of dimension n. Define a linear map*

$$\Phi \colon \mathrm{Ext}^l_{C^*(N)}(C^*(X), C^*(N)) \to \mathrm{Hom}_{\mathbb{K}}(H^{n-l}(X), H^n(N))$$

by $\Phi(\alpha) = H^*(\alpha)|_{H^{n-l}(X)}$. Then Φ is an isomorphism.

Then we have the following corollary, which is an analogue of Theorem 17.4 for the case of the non-symmetric brane coproduct.

Corollary 18.4. *Consider the same assumption with Proposition 18.3. Additionally assume $l = n$ and $j = 0$. Then we have an isomorphism*

$$\mathrm{Ext}^n_{C^*(N)}(C^*(X), C^*(N)) \xrightarrow{\cong} H^n(N); \quad \alpha \mapsto H^*(\alpha)(1).$$

Applying Corollary 18.4 to the case $F = \iota$ and $n = m$, we have the generator

$$\iota_! \in \mathrm{Ext}^m_{C^*(D^k M)}(C^*(S^k_f M), C^*(D^k M)) \cong H^m(D^k M) \cong \mathbb{K}$$

up to non-zero scalar multiplication. Using this element with the diagram (18.1), we define $\mathrm{comp}_! = H^*(\mathrm{res}^\star(\iota_!))$. This completes the definition of the non-symmetric brane coprdouct.

Next we give more convenient description of $\mathrm{comp}_!$. Consider the commutative diagram

$$\begin{tikzcd}
 & M^T_{f+g} \arrow[dl, "="'] \arrow[dd, "\mathrm{res}"] & & S^k_f M \times_M M^T_g \arrow[ll, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[dl, "\simeq"', "\rho"] \arrow[dd, "\mathrm{pr}_1"] & S^k_f M \times M^T_g \\
M^T_{f+g} \arrow[dd, "\mathrm{ev}_{t_0}"'] & & S^k_f M \times_M M^T_{f+g} \arrow[ll, "\mathrm{pr}_2"'] \arrow[dd, "\mathrm{pr}_1"] & & \\
 & D^k M \arrow[dl, "\simeq"', "\mathrm{ev}_{d_0}"] & & S^k_f M \arrow[ll, "\iota"] \arrow[dl, "="] & \\
M & & S^k_f M, \arrow[ll, "\mathrm{ev}_{s_0}"] & &
\end{tikzcd}$$

where the front and back square are pullback squares. Here $\rho \colon S^k_f M \times_M M^T_g \xrightarrow{\simeq} S^k_f M \times_M M^T_{f+g}$ is defined by $\rho(\varphi, \psi) = (\varphi, \varphi + \psi)$, which is well-defined since we are working on the fiber product over M. By Proposition 16.2, we have $H^*(\mathrm{res}^\star(\iota_!)) \circ \rho^* = H^*({\mathrm{ev}_{t_0}}^\star(\tilde{\iota_!}))$ and hence

$$\mathrm{comp}_! = H^*({\mathrm{ev}_{t_0}}^\star(\tilde{\iota_!})) \circ (\rho^*)^{-1}. \tag{18.5}$$

Here $\tilde{\iota_!} \in \mathrm{Ext}^m_{C^*(M)}(C^*(S^k_f M), C^*(M))$ is the image of $\iota_!$ under the isomorphism induced by ev_{d_0}.

19 Computation of the non-symmetric brane coproduct

In this section, we use the same notation and assumptions as in Section 18. Let $0 \in S^kM$ be the constant map and denote the orientation class of M by $\omega \in H^m(M)$. This section is devoted to the proof of the following formula of the non-symmetric brane coproduct.

Theorem 19.1. *For the case $f = 0 \in [S^k, M]$, the non-symmetric coproduct*

$$\delta_{\mathrm{ns}}^{\vee} \colon H^*(S_0^kM \times M_g^T) \to H^{*+m}(M_g^T)$$

is described by

$$\delta_{\mathrm{ns}}^{\vee}(u \times v) = \mathrm{ev}_{t_0}^*(\omega \cdot c^*(u)) \cdot v,$$

where $u \times v$ denotes the cross product of $u \in H^*(S_0^kM)$ and $v \in H^*(M_g^T)$, and $c \colon M \to S_0^kM$ is the embedding as constant maps.

This is an analogue of [Men13, Theorem 30] in the case of the non-symmetric coproduct. Note that, when $\mathrm{ch}\,\mathbb{K} = 0$, the above formula can be proved easily by using rational models of mapping spaces given in [Ber15].

To prove Theorem 19.1, we need some propositions. First we investigate the map $(\rho^*)^{-1}$ in (18.5). Define $\sigma \colon M_g^T \to S_0^kM \times_M M_g^T$ by $\sigma(\psi) = (c(\psi(t_0)), \psi)$.

Proposition 19.2. *For any $x \in H^*(S_0^kM \times_M M_g^T)$, we have*

$$(\rho^*)^{-1}x - x \in \mathrm{Ker}(\sigma^*).$$

Proof. Let $\bar{\rho}$ be the homotopy inverse of ρ. Then we have $\rho \circ \sigma \simeq \sigma$ and hence $\sigma^*((\rho^*)^{-1}x - x) = \sigma^*(\bar{\rho}^*x - x) = \sigma^*x - \sigma^*x = 0$. ☐

Next we relate $\mathrm{Ker}(\sigma^*)$ with $H^*(\mathrm{ev}_{t_0}{}^{\star}\tilde{\iota}_!)$.

Proposition 19.3. *Consider a pullback diagram*

$$\begin{tikzcd}
E \arrow[d, "p"] & X \arrow[l, "g"'] \arrow[d, "q"] \\
B & A \arrow[l, "f"]
\end{tikzcd}$$

such that the Eilenberg-Moore map is an isomorphism, and take an element $\alpha \in \mathrm{Ext}_{C^*(B)}(C^*(A), C^*(B))$. Let $\sigma \colon E \to X$ and $\tau \colon B \to A$ be sections of g and f, respectively, satisfying $q \circ \sigma = \tau \circ p$. Assume that there is an element $\tilde{\alpha} \in \mathrm{Ext}_{C^*(B)}(C^*(B), C^*(B))$ which is mapped to α by the map induced by τ. Then

$$\mathrm{Ker}(\sigma^*) \subset \mathrm{Ker}(H^*(p^{\star}\alpha)).$$

Proof. Applying Proposition 16.2 to the following diagram, we have $H^*(p^{\star}\alpha) = H^*(p^{\star}\tilde{\alpha}) \circ \sigma^*$, and this proves the proposition.

$$\begin{tikzcd}
 & E \arrow[dl, "="'] \arrow[dd, "p"] & & E \arrow[ll, "="'] \arrow[dl, "\sigma"] \arrow[dd, "p"] \\
E \arrow[dd, "p"'] & & X \arrow[ll, "g"'] \arrow[dd, "q"] & \\
 & B \arrow[dl, "="] & & B \arrow[ll, "="] \arrow[dl, "\tau"] \\
B & & A \arrow[ll, "f"] &
\end{tikzcd}$$

☐

Next, we consider the diagram

$$\begin{tikzcd}
M_g^T \arrow[d, "\mathrm{ev}_{t_0}"'] & S_0^k M \times_M M_g^T \arrow[l, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] \\
M & S_0^k M. \arrow[l, "\mathrm{ev}_{s_0}"]
\end{tikzcd}$$

Note that the maps $\sigma\colon M_g^T \to S_0^k M \times_M M_g^T$ and $c\colon M \to S_0^k M$, are sections of pr_2 and ev_{s_0}, respectively. Recall from (18.5) that we are using $\tilde{\iota}_! \in \mathrm{Ext}^m_{C^*(M)}(C^*(S_0^k M), C^*(M))$ to compute the non-symmetric brane coproduct.

Corollary 19.4. *Under the above notation, we have*

$$\mathrm{Ker}(\sigma^*) \subset \mathrm{Ker}\left(H^*(\mathrm{ev}_{t_0}{}^\star(\tilde{\iota}_!))\right).$$

Proof. By Corollary 18.4, the map c induces an isomorphism

$$\mathrm{Ext}^m_{C^*(M)}(C^*(M), C^*(M)) \xrightarrow{\cong} \mathrm{Ext}^m_{C^*(M)}(C^*(S_0^k M), C^*(M)).$$

Thus we obtain $\tilde{\alpha}$ as in the assumption of Proposition 19.3, and hence it proves the corollary. ☐

By Proposition 19.2 and Corollary 19.4, Theorem 19.1 reduces to the following simple proposition.

Proposition 19.5. *Consider a pullback diagram*

$$\begin{tikzcd}
E \arrow[d, "p"] & X \arrow[l, "g"] \arrow[d, "q"] \\
B & A \arrow[l, "f"]
\end{tikzcd}$$

such that the Eilenberg-Moore map is an isomorphism, and an element $\alpha \in \mathrm{Ext}_{C^*(B)}(C^*(A), C^*(B))$. *Then the composition* $H^*(A \times E) \xrightarrow{\mathrm{incl}^*} H^*(X) \xrightarrow{H^*(p^\star\alpha)} H^*(E)$ *satisfies*

$$H^*(p^\star\alpha) \circ \mathrm{incl}^*(u \times v) = p^*(H^*(\alpha)(u)) \cdot v$$

for any $u \in H^*(A)$ *and* $v \in H^*(E)$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & E \arrow[ld, "{(p,\mathrm{id})}"'] \arrow[dd, "p"] & & X \arrow[ll, "g"] \arrow[ld, "{(q,g)}"'] \arrow[dd, "q"] \\
B \times E \arrow[dd, "\mathrm{pr}_1"'] & & A \times E \arrow[ll, "f \times \mathrm{id}"'] \arrow[dd, "\mathrm{pr}_1"] & \\
 & B \arrow[ld, "="] & & A \arrow[ll, "f"] \arrow[ld, "="] \\
B & & A. \arrow[ll, "f"] &
\end{tikzcd}$$

By Proposition 16.2, we have

$$H^*(p^\star\alpha) \circ \mathrm{incl}^* = (p, \mathrm{id})^* \circ (H^*(\mathrm{pr_1}^\star\alpha)).$$

Since the fibration pr_1 is very simple, we can prove

$$H^*(\mathrm{pr_1}^\star\alpha)(u \times v) = H^*(\alpha)(u) \times v$$

by a direct computation from the definition. □

Now we give a proof of Theorem 19.1 using the above corollary and propositions.

Proof of Theorem 19.1. By (18.5), we have

$$\delta^\vee_{\mathrm{ns}}(u \times v) = H^*(\mathrm{ev}_{t_0}{}^\star(\tilde{\iota}_!)) \circ (\rho^*)^{-1} \circ \mathrm{incl}^*(u \times v).$$

By Proposition 19.2 and Corollary 19.4, we have

$$H^*(\mathrm{ev}_{t_0}{}^\star(\tilde{\iota}_!)) \circ (\rho^*)^{-1} = H^*(\mathrm{ev}_{t_0}{}^\star(\tilde{\iota}_!)).$$

Thus

$$\delta^\vee_{\mathrm{ns}}(u \times v) = H^*(\mathrm{ev}_{t_0}{}^\star(\tilde{\iota}_!)) \circ \mathrm{incl}^*(u \times v),$$

and hence Proposition 19.5 proves the theorem. □

20 Comparison of two brane coproducts

In this section, we compare the two brane coproducts. As an application, we prove Theorem 15.2.

20.1 Proof of Theorem 15.4

In this subsection, we prove Theorem 15.4.

Let $\mathbb{K}$ be a field of any characteristic, k a positive integer, and M a k-connected Poincaré duality space of dimension m. We fix an arbitrary element

$$\gamma \in \mathrm{Ext}^m_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)).$$

Then we have the brane coproduct

$$\delta^\vee_\gamma \colon H^*(S^kM \times S^kM) \to H^{*+m}(S^kM)$$

for the case $S = T = S^k$ by the construction given in Section 17.

Remark 20.1. The degree m of the element γ is different from the degree $\bar{m}$ of $c_!$ in Theorem 17.4. These degrees coincide under the assumption (2) of Theorem 15.2 (see Remark 20.10). This case will be treated in Subsection 20.3 and Section 21.

To compare $\delta_\gamma^\vee$ with $\delta_{\mathrm{ns}}^\vee$, we relate γ with $\iota_!$. As in Theorem 15.4, define $\lambda_\gamma \in \mathbb{K}$ by the equation

$$c^* \circ (H^*(\gamma))(1) = \lambda_\gamma \omega \in H^m(M),$$

where ω is the orientation class of M.

Proposition 20.2. *Under the above notation, we have*

$$\mathrm{res}^\star(\gamma) = \lambda_\gamma \iota_! \in \mathrm{Ext}^m_{C^*(D^kM)}(C^*(S^kM), C^*(D^kM)),$$

where $\mathrm{res}^\star$ *is the lift along the lower pullback square in* (18.2)*. Moreover, this implies*

$$\delta_\gamma^\vee = \lambda_\gamma \delta_{\mathrm{ns}}^\vee \colon H^*(S^kM \times S^kM) \to H^*(S^kM).$$

Proof. Let $\omega \in H^m(M) \cong H^m(D^kM)$ be the orientation class. Recall from Corollary 18.4 that $\iota_!$ is characterized by $H^*(\iota_!)(1) = \omega$. Hence it is enough to prove $H^*(\mathrm{res}^\star(\gamma))(1) = \lambda_\gamma \omega$.

Let $\eta \colon P \xrightarrow{\simeq} C^*(M)$ be a semifree resolution of $C^*(M)$ over $C^*(S^{k-1}M)$, and $u \in P$ a cocycle such that $\eta(u) = 1$. Take a representative $\varphi \in \mathrm{Hom}_{C^*(S^{k-1}M)}(P, C^*(S^{k-1}M))$ of γ. Then we have $[\varphi(u)] = H^*(\gamma)(1) \in H^m(S^{k-1}M)$. By definition, $H^*(\mathrm{res}^\star(\gamma))$ is represented by the chain map $\mathrm{id}_{C^*(D^kM)} \otimes \varphi$ in

$$\mathrm{Hom}_{C^*(D^kM)}(C^*(D^kM) \otimes_{C^*(S^{k-1}M)} P, C^*(D^kM) \otimes_{C^*(S^{k-1}M)} C^*(S^{k-1}M)).$$

Hence we have $H^*(\mathrm{res}^\star(\gamma))(1) = [(\mathrm{id}_{C^*(D^kM)} \otimes \varphi)(1 \otimes u)] = c^*[\varphi(u)] = \lambda_\gamma \omega \in H^m(M)$ under the identification $H^m(M) = H^m(D^kM)$. This proves the proposition. $\square$

Next we consider the commutativity of the coproduct $\delta_\gamma^\vee$. Let $\tau \colon S^{k-1}M \to S^{k-1}M$ be the map induced from the orientation reversing map on S^{k-1}, satisfying $\tau^2 = \mathrm{id}$. Then τ induces the map

$$\begin{aligned} \tau^* = \mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*) \colon\ & \mathrm{Ext}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \\ & \to \mathrm{Ext}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)). \end{aligned}$$

By the definition of λ_γ, we have

$$\lambda_\gamma = \lambda_{\tau^*\gamma}. \tag{20.3}$$

The coproduct is commutative in the following sense.

Proposition 20.4.

$$\delta_\gamma^\vee(\alpha \times \beta) = (-1)^{|\alpha||\beta|} \delta_{\tau^*\gamma}^\vee(\beta \times \alpha)$$

The proposition is proved by the same method with the commutativity of the brane coproduct $\delta_{c_!}^\vee$ Theorem 1.5. Note that we used the equation (7.7) $\tau^* c_! = (-1)^{\bar{m}} c_!$ to prove $\delta_{c_!}^\vee(\alpha \times \beta) = (-1)^{|\alpha||\beta| + \bar{m}} \delta_{c_!}^\vee(\beta \times \alpha)$.

Using the above propositions, we give a proof of Theorem 15.4.

Proof of Theorem 15.4. Since the fibration $\mathrm{ev}_0\colon S^kM \to M$ has a section $c\colon M \to S^kM$, we have a decomposition $H^{>0}(S^kM) \cong H^{>0}(M) \oplus \mathrm{Ker}(c^*)$. When $\alpha \in H^{>0}(M)$, we have $\alpha\omega = 0 \in H^{|\alpha|+m}(M) = 0$. Hence we assume $\alpha \in \mathrm{Ker}(c^*)$. Then, by Theorem 19.1, we have

$$\begin{aligned}\delta_{\mathrm{ns}}^{\vee}(\alpha \times 1) &= \mathrm{ev}_0^*(\omega \cdot c^*(\alpha)) \cdot 1 = 0\\ \delta_{\mathrm{ns}}^{\vee}(1 \times \alpha) &= \mathrm{ev}_0^*(\omega \cdot c^*(1)) \cdot \alpha = \mathrm{ev}_0^*\omega \cdot \alpha.\end{aligned}$$

Moreover, we have

$$\lambda_\gamma \delta_{\mathrm{ns}}^{\vee}(\alpha \times 1) = \delta_{\gamma}^{\vee}(\alpha \times 1) = \pm\delta_{\tau^*\gamma}^{\vee}(1 \times \alpha) = \pm\lambda_\gamma\delta_{\mathrm{ns}}^{\vee}(1 \times \alpha)$$

by (20.3), Proposition 20.4, and Proposition 20.2. These equations prove the theorem. □

20.2 Proof of Theorem 15.2 (1)

In this subsection, we prove Theorem 15.2 under the assumption (1). As a preparation of the proof, we investigate the map Φ in Proposition 18.3.

As in Proposition 18.3, let X be a 0-connected space, N a Poincaré duality space of dimension n, and $F\colon X \to N$ a map. We denote the orientation class of N by $\omega_N \in H^n(N)$ and the fundamental class by $[N] \in H_n(N)$. Then we have $\langle \omega_N, [N]\rangle_N = 1$, where $\langle -, -\rangle_N\colon H^*(N) \otimes H_*(N) \to \mathbb{K}$ denotes the pairing.

Proposition 20.5. *Fix arbitrary elements $x \in H_{n-l}(X)$ and $\nu \in H^j(X)$. Let $\beta_x\colon H^{n-l}(X) \to H^n(N)$ be the linear map defined by $\beta_x(\varphi) = \langle \varphi, x\rangle_X \omega_N$ for $\varphi \in H^{n-l}(X)$. Using the isomorphism Φ in Proposition 18.3, we define*

$$\alpha_x = \Phi^{-1}(\beta_x) \in \mathrm{Ext}^l_{C^*(N)}(C^*(X), C^*(N)).$$

Then the element $H^*(\alpha_x)(\nu) \in H^{l+j}(N)$ is the unique element which satisfies

$$\langle \psi,\ H^*(\alpha_x)(\nu) \cap [N]\rangle_N = (-1)^{l(n-l-j)}\langle F^*\psi \cdot \nu,\ x\rangle_X \tag{20.6}$$

for any $\psi \in H^{n-l-j}(N)$.

Proof. Since the cap product $- \cap [N]$ is an isomorphism by the Poincaré duality, such element is uniquely determined. Since $H^*(\alpha_x)$ is $H^*(N)$-linear, we have $\psi \cdot H^*(\alpha_x)(\nu) = (-1)^{l(n-l-j)}H^*(\alpha_x)(F^*\psi \cdot \nu)$. Using this equation, we can prove (20.6) by a straightforward calculation. □

Now we begin the proof of Theorem 15.2 (1). Let M be a 1-connected Poincaré duality space of dimension m. Here we write $LM = S^1M$ and $\Delta = c\colon M \to M \times M$ as usual. Recall that

$$\Delta_! \in \mathrm{Ext}^m_{C^*(M\times M)}(C^*(M), C^*(M \times M)) \cong \mathbb{K}$$

is the generator, which is defined up to non-zero scalar multiplication.

Proposition 20.7. *The element $H^*(\Delta_!)(1) \in H^m(M \times M)$ is the diagonal class, i.e. the Poincaré dual of the homology class $\Delta_*[M] \in H^m(M \times M)$. In particular, we have*

$$\Delta^* \circ (H^*(\Delta_!))(1) = \chi(M)\omega \in H^*(M).$$

Proof. Since $M \times M$ is also a Poincaré duality space, we can apply Proposition 20.5 for the case $F = \Delta$, $n = 2m$, $l = m$, $j = 0$, $x = [M]$, and $\nu = 1$. Since $\Delta_!$ is defined up to non-zero scalar multiplication, we may assume $\Delta_! = (-1)^m\alpha_{[M]}$. By (20.6), we have

$$\langle \psi,\ H^*(\Delta_!)(1) \cap [M^2]\rangle_{M^2} = \langle \Delta^*\psi \cdot 1,\ [M]\rangle_M = \langle \psi,\ \Delta_*[M]\rangle_{M^2}$$

for any $\psi \in H^m(M^2)$, and hence $H^*(\Delta_!)(1) \cap [M^2] = \Delta_*[M]$.

It is well-known that the diagonal class satisfies the required property (c.f. e.g. [MS74, pp. 127–129, Section 11]). ☐

Now we have the following theorem using the above lemma.

Theorem 20.8 (Theorem 15.2 (1))**.** *Let M be a 1-connected Poincaré duality space over $\mathbb{K}$ and denote its orientation class by $\omega \in H^m(M)$. Then, for any $\alpha \in H^{>0}(LM)$, we have*

$$\chi(M)\mathrm{ev}_0^*\omega \cdot \alpha = 0 \in H^{|\alpha|+m}(LM).$$

Proof. Apply Theorem 15.4 and Proposition 20.7. ☐

Remark 20.9. This theorem generalizes [Men13, Theorem 1] in the sense that our theorem can be applied to Poincaré duality spaces, not only manifolds.

20.3 Proof of Theorem 15.2 (2)

In this section, we prove Theorem 15.2 under the assumption (2).

Let k be a positive *odd* integer and M a k-connected Poincaré duality space over $\mathbb{K}$ of dimension m. Assume $\operatorname{ch}\mathbb{K} = 0$ and $\dim_{\mathbb{K}}(\bigoplus_n \pi_n(M)\otimes\mathbb{K}) < \infty$.

First we explain why we assume k is odd in the assumption (2) in Theorem 15.2.

Remark 20.10. Let $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$ be bases of $\bigoplus_n \pi_{2n}(M)\otimes\mathbb{K}$ and $\bigoplus_n \pi_{2n-1}(M)\otimes\mathbb{K}$, respectively. Then we have the following.

- $\chi(M) \neq 0$ if and only if $p = q$. See Theorem 21.13 for details.
- Define $a_i = |x_i|$ and $b_j = |y_j|$. By [FHT88, Proposition 5.2], we have $m = \dim M = \sum_j b_j + \sum_i (1 - a_i)$. By the same formula, we have

$$\bar{m} = \dim \Omega^{k-1}M = \begin{cases} m - (q-p)(k-1), & \text{if } k \text{ is odd,} \\ -m - (k-2)p + kq, & \text{if } k \text{ is even.} \end{cases}$$

  Thus, except for rare exceptions, $\bar{m}$ coincides with m if and only if k is odd and $p = q$.

Since the statement of Theorem 15.2 is trivial when $\chi(M) = 0$, we are interested only in the case $\chi(M) \neq 0$, i.e. $p = q$. Moreover, since we will compare two brane coproducts, their degrees m and $\bar{m}$ must coincide. Hence we may assume k is odd. This explains why the assumption (2) in Theorem 15.2 is natural one.

Now we give a proposition, which is a key to prove Theorem 15.2 (2).

Proposition 20.11. *Under the assumption* (2) *in Theorem 15.2, there exists an element $\gamma \in \operatorname{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M))$ such that*

$$c^* \circ (H^*(\gamma))(1) = \chi(M)\omega \in H^*(M).$$

We defer the proof of the proposition to Section 21. Applying the proposition and Theorem 15.4, we have (2) of Theorem 15.2.

Theorem 20.12 (Theorem 15.2 (2))**.** *Under the assumption* (2) *in Theorem 15.2, we have*

$$\chi(M)\mathrm{ev}_0^*\omega \cdot \alpha = 0 \in H^{|\alpha|+m}(S^kM)$$

for any $\alpha \in H^{>0}(S^kM)$.

Hence the rest of this article is devoted to the proof of Proposition 20.11.

21 Models of shriek maps

In this section, we give a proof of Proposition 20.11. As a preparation of the proof, we explicitly construct a model of the shriek map $c_!$ when the coefficient is a field $\mathbb{K}$ of characteristic zero. By (17.5), it is enough to construct a nontrivial element in $\mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong \mathbb{K}$. In Subsection 21.1, we construct a candidate of the shriek map, whose nontriviality is proved in Subsection 21.2 under some assumptions.

The construction is a generalization of the ones in [Nai13] and [Wak16], which treat only the case $k = 1$. Note that, in Proposition 6.5, the shriek map is explicitly constructed when k is even and the minimal Sullivan model is pure, which is much simpler than the one in this section.

Throughout this section, we assume $\operatorname{ch}\mathbb{K} = 0$ and make full use of rational homotopy theory. See [FHT01] for basic definitions and theorems.

For a graded vector space V, we define a graded vector space s^kV by $(s^kV)^n = V^{n+k}$. For an element $v \in V$, we denote the corresponding element by $s^kv \in s^kV$. For simplicity, we write $sV = s^1V$.

Let $(\wedge V, d)$ be a Sullivan algebra satisfying $\dim V < \infty$ and $V^1 = 0$. We fix a basis $z_1, \dots, z_r$ of V such that $dz_{t+1} \in \wedge V(t)$, where $V(t) = \mathrm{span}_{\mathbb{K}}\{z_1, \dots, z_t\}$.

21.1 Construction of a chain map

In this subsection, we give an explicit construction of a candidate of the shriek map for $k \geq 1$. The construction is completely analogous to the one in [Wak16].

In this subsection, we assume $V^{\leq k} = 0$ additionally. Write $\mathcal{S}^{k-1} = \mathcal{S}^{k-1}V = \wedge V \otimes \wedge s^{k-1}V$ and $\mathcal{D}^k = \mathcal{D}^kV = \wedge V \otimes \wedge s^{k-1}V \otimes s^kV$. Here we define two Sullivan algebras $(\mathcal{S}^{k-1}, d)$ and $(\mathcal{D}^k, d)$, and two linear maps $\sigma\colon V \to \mathcal{S}^{k-1}$ and $\tau\colon V \to \mathcal{D}^k$. Note that $(\mathcal{S}^{k-1}, d)$ and $(\mathcal{D}^k, d)$ are models of $S^{k-1}M$ and D^kM, respectively.

Let $\tilde{s}^{k-1}\colon \mathcal{S}^{k-1} \to \mathcal{S}^{k-1}$ be the derivation defined by $\tilde{s}^{k-1}(v) = s^{k-1}v$ and $\tilde{s}^{k-1}(s^{k-1}v) = 0$. By an abuse of notation, we write $\tilde{s}^{k-1}$ simply by s^{k-1}. Similarly we define the derivation $s^k\colon \mathcal{D}^k \to \mathcal{D}^k$. Note that these derivations are not equal to the compositions of s^1 (e.g. $s^{k-1} \neq s^1 \circ \cdots s^1$).

First we define the differentials d on $\mathcal{S}^{k-1}$ and $\mathcal{D}^k$ in the case $k = 1$. Then $(\mathcal{S}^0, d)$ is just the tensor product $(\wedge V, d)^{\otimes 2}$. The dga $(\mathcal{D}^1, d)$ is a relative Sullivan algebra over $(\wedge V, d)^{\otimes 2}$, defined by the formula $d(sz_t) = 1 \otimes z_t - z_t \otimes 1 - \sum_{n=1}^{\infty} \frac{(sd)^n}{n!}(z_t \otimes 1)$ inductively on t (see [FHT01, Section 15 (c)] or [Wak16, Appendix A] for details). Then, for $v \in V$, we set $\sigma v = 1 \otimes v - v \otimes 1$ and $\tau v = -\sum_{n=1}^{\infty} \frac{(sd)^n}{n!}(v \otimes 1)$, which satisfy $dsv = \sigma v + \tau v$.

Next we consider the case $k \geq 2$. Define the differential d on $\mathcal{S}^{k-1}$ by the formula $ds^{k-1}v = (-1)^{k-1}s^{k-1}dv$. Set $\sigma v = s^{k-1}v$, $\tau v = (-1)^k s^k dv$. Then we define the relative Sullivan algebra $(\mathcal{D}^k, d)$ over $(\mathcal{S}^{k-1}, d)$ by the formula $ds^kv = \sigma v + \tau v$. See Section 5 for details. By the following proposition, we can use $\mathcal{S}^{k-1}$ and $\mathcal{D}^k$ to construct the shriek map $c_!$.

Proposition 21.1 (Proposition 5.1)**.** *Let M be a k-connected space and $(\wedge V, d)$ be its Sullivan model. Then the above algebras $\mathcal{S}^{k-1}$ and $\mathcal{D}^k$ are Sullivan models of $S^{k-1}M$ and D^kM. In particular, we have*

$$\mathrm{Ext}_{C^*(S^{k-1}M)}\left(C^*(M), C^*(S^{k-1}M)\right) \cong H^*(\mathrm{Hom}_{\mathcal{S}^{k-1}}\left(\mathcal{D}^k, \mathcal{S}^{k-1}\right))$$

Moreover, we define $\mathcal{S}^{k-1}(t) = \wedge V(t) \otimes \wedge s^{k-1}V(t)$ and $\mathcal{D}^k(t) = \wedge V(t) \otimes \wedge s^{k-1}V(t) \otimes s^kV(t)$. Then we have $\sigma\colon V(t) \to \mathcal{S}^{k-1}(t)$ and $\tau\colon V(t) \to \mathcal{D}^k(t-1)$.

Next we give a construction of shriek maps.

Definition 21.2. For $t = 0, \dots, n-1$ define a $\mathbb{K}$-linear map

$$\Phi\colon \operatorname{Hom}_{\mathcal{S}^{k-1}(t-1)}\left(\mathcal{D}^k(t-1), \mathcal{S}^{k-1}(t-1)\right) \to \operatorname{Hom}_{\mathcal{S}^{k-1}(t)}\left(\mathcal{D}^k(t), \mathcal{S}^{k-1}(t)\right)$$

of odd degree as follows.

(1) In the case $|z_t| + k - 1$ is odd, for $f \in \operatorname{Hom}_{\mathcal{S}^{k-1}(t-1)}\left(\mathcal{D}^k(t-1), \mathcal{S}^{k-1}(t-1)\right)$, define

$$\Phi(f) \in \operatorname{Hom}_{\mathcal{S}^{k-1}(t)}\left(\mathcal{D}^k(t), \mathcal{S}^{k-1}(t)\right)$$

by

$$\Phi(f)(\nu) = \sigma z_t \cdot f(\nu) - (-1)^{|f|} f(\tau z_t \cdot \nu), \quad \Phi(f)(\nu \cdot (s z_t)^l) = 0$$

for $\nu \in \wedge sV(t-1)$ and $l \geq 1$.

(2) In the case $|z_t| + k - 1$ is even, for $f \in \operatorname{Hom}_{\mathcal{S}^{k-1}(t-1)}\left(\mathcal{D}^k(t-1), \mathcal{S}^{k-1}(t-1)\right)$, define $\Phi(f)$ by

$$\Phi(f)(\nu \cdot s z_t) = (-1)^{|f|+|\nu|} f(\nu), \quad \Phi(f)(\nu) = 0$$

for $\nu \in \wedge sV(t-1)$.

By a straight-forward calculation, the linear map Φ is a chain map of odd degree. In other words, the map Φ satisfies $d\Phi = -\Phi d$.

Hence we define chain maps

$$\varphi_t \in \operatorname{Hom}_{\mathcal{S}^{k-1}(t)}\left(\mathcal{D}^k(t), \mathcal{S}^{k-1}(t)\right)$$

by $\varphi_0 = \mathrm{id}_{\mathbb{K}}$ and $\varphi_{t+1} = \Phi(\varphi_t)$, inductively.

21.2 The pure case with k odd

Next we investigate the above map in the case $(\wedge V, d)$ is pure and k is odd.

Definition 21.3 ([FHT01, Section 32 (a)])**.** A Sullivan algebra $(\wedge V, d)$ is *pure* if $d(V^{\mathrm{even}}) = 0$ and $d(V^{\mathrm{odd}}) \subset \wedge V^{\mathrm{even}}$.

Here we apply the above construction for the case the basis $z_1, \dots, z_n$ is given by the sequence $x_1, \dots x_p, y_1, \dots, y_q$, where $x_1, \dots x_p$ and $y_1, \dots, y_q$ are (arbitrary) bases of V^{even} and V^{odd}, respectively. That is, $z_i = x_i$ for $1 \leq i \leq p$ and $z_{p+j} = x_j$ for $1 \leq j \leq q$. In this case, we can write $\tau y_j = (-1)^k \sum_i \alpha_{ji} \cdot s^k x_i$ for some elements $\alpha_{ji} \in \mathcal{S}^{k-1}$. Note that $\alpha_{ji} \in \wedge V^{\mathrm{even}}$ when $k \geq 2$, and $\alpha_{ji} \in \wedge V^{\mathrm{even}} \otimes \wedge V^{\mathrm{even}}$ when $k = 1$.

Let $\mu\colon \mathcal{S}^{k-1} \to \wedge V$ be the multiplication map when $k = 1$, and the map defined by $\mu(v) = v$ and $\mu(s^{k-1}v) = 0$ when $k \geq 2$. Then we have

$$\mu(\alpha_{ji}) = \frac{\partial(dy_j)}{\partial x_i} \in \wedge V^{\mathrm{even}}. \tag{21.4}$$

Write $[p] = \{1, 2, \dots, p\}$. For any subset $I = \{i_1, \dots, i_n\} \subset [p]$ with $i_1 < \cdots < i_n$, we define $|I| = i_1 + \cdots i_n$, $l(I) = n$, and $s^k x_I = s^k x_{i_1} \cdots s^k x_{i_n}$. Similarly, for any subset $I = \{j_1, \dots, j_n\} \subset [q]$ with $j_1 < \cdots < j_n$, we define $\sigma y_J = \sigma y_{j_n} \cdots \sigma y_{j_1}$.

For $0 \leq i \leq p$, we can easily compute φ_i by induction on i.

Lemma 21.5. *For any integer i with $0 \leq i \leq p$ and any subset $I \subset [i]$, we have*

$$\varphi_i(s^k x_{[p] \setminus I}) = \begin{cases} 1, & \text{if } I = \emptyset, \\ 0, & \text{if } I \neq \emptyset. \end{cases}$$

Moreover, we have the following formulas for φ_{p+j} for $0 \le j \le q$.

Proposition 21.6. *Let j be an integer with $0 \le j \le q$ and $I \subset [p]$ a subset. Write $n = l(I)$ and $I = \{i_1, \dots, i_n\}$ with $i_1 < \cdots < i_n$. Then the element $\varphi_{p+j}(s^k x_{[p]\setminus I}) \in \mathcal{S}^{k-1}(j)$ satisfies the following.*

1. *If $n = 0$, then we have $\varphi_{p+j}(s^k x_{[p]}) = \sigma y_{[j]}$.*
2. *If $n < j$, then the element $\varphi_{p+j}(s^k(x_{[p]\setminus I}))$ is contained in the ideal $(\sigma y_1, \dots, \sigma y_j) \subset \mathcal{S}^{k-1}(j)$.*
3. *If $n \ge j$, then we have*

$$\varphi_{p+j}(s^k x_{[p]\setminus I}) = \begin{cases} (-1)^{|I|+pj} \det\bigl((\alpha_{t,i_r})_{1\le t,r\le j}\bigr), & \text{if } n = j, \\ 0, & \text{if } n > j. \end{cases}$$

Proof. We prove the formulas by induction on j. The case $j = 0$ is already proved in Lemma 21.5. Assume that $j \ge 1$ and we already have the formulas for φ_{p+j-1}.

By Definition 21.2, we have

$$\begin{aligned} &\varphi_{p+j}(s^k x_{[p]\setminus I}) = \Phi(\varphi_{p+j-1})(s^k x_{[p]\setminus I}) \\ &= \sigma y_j \cdot \varphi_{p+j-1}(s^k x_{[p]\setminus I}) + (-1)^{p+j-1} \varphi_{p+j-1}(\tau y_j \cdot s^k x_{[p]\setminus I}) \\ &= \sigma y_j \cdot \varphi_{p+j-1}(s^k x_{[p]\setminus I}) + (-1)^{p+j-1} \varphi_{p+j-1} \left((-1)^k \sum_{1\le i\le p} \alpha_{ji} \cdot s^k x_i \cdot s^k x_{[p]\setminus I} \right) \\ &= \sigma y_j \cdot \varphi_{p+j-1}(s^k x_{[p]\setminus I}) + (-1)^{p+j} \sum_{1\le r\le j} (-1)^{i_r - r} \alpha_{j,i_r} \cdot \varphi_{p+j-1}\left(s^k x_{[p]\setminus I_r}\right), \end{aligned} \tag{21.7}$$

where $I_r = I \setminus \{i_r\}$.

First we prove (1). Since $|s^k x_i|$ is odd, we have $\tau y_j \cdot s^k x_{[p]} = (-1)^k \sum_i \alpha_{ji} \cdot s^k x_i \cdot s^k x_{[p]} = 0$. Hence we have

$$\begin{aligned} \varphi_{p+j}(s^k x_{[p]}) &= \Phi(\varphi_{p+j-1})(s^k x_{[p]}) \\ &= \sigma y_j \cdot \varphi_{p+j-1}(s^k x_{[p]}) \pm \varphi_{p+j-1}(\tau y_j \cdot s^k x_{[p]}) \\ &= \sigma y_j \cdot \sigma y_{[j-1]} = \sigma y_{[j]}. \end{aligned}$$

Next we prove (2). Assume $n < j$. Then, for any r, we have $\varphi_{p+j-1}\left(s^k x_{[p]\setminus I_r}\right) \in (\sigma y_1, \dots, \sigma y_{j-1})$ by the induction hypothesis, since $l(I_r) = n - 1 < j - 1$. Thus we have $\varphi_{p+j}(s^k(x_{[p]\setminus I})) \in (\sigma y_1, \dots, \sigma y_j)$ by (21.7).

Finally we prove (3). Assume $n \ge j$. Since $l(I) = n > j - 1$, we have $\varphi_{p+j-1}(s^k x_{[p]\setminus I}) = 0$ by the induction hypothesis. Hence (21.7) reduces to the equation

$$\varphi_{p+j}(s^k x_{[p]\setminus I}) = (-1)^{p+j} \sum_{1\le r\le j} (-1)^{i_r - r} \alpha_{j,i_r} \cdot \varphi_{p+j-1}\left(s^k x_{[p]\setminus I_r}\right). \tag{21.8}$$

If $n > j$, since $l(I_r) = n-1 > j-1$, we have $\varphi_{p+j-1}(s^k x_{[p]\setminus I_r}) = 0$ and hence $\varphi_{p+j}(s^k x_{[p]\setminus I}) = 0$ by (21.8). This proves (3) in the case $n > j$.

Next we assume $n = j$. Let $M_{u,r}$ be the minor determinants of the $j \times j$ matrix $A = \left(\alpha_{t,i_s}\right)_{1\leq t,s\leq j}$, i.e. $M_{u,r} = \det\left((\alpha_{t,i_s})_{t\neq u, s\neq r}\right)$. Since $|I_r| = j - 1$, we have $\varphi_{p+j-1}(s^k x_{[p]\setminus I_r}) = (-1)^{|I_r|+p(j-1)} M_{j,r}$ by the induction hypothesis. Hence, by (21.8), we have

$$\begin{aligned}\varphi_{p+j}(s^k x_{[p]\setminus I}) &= (-1)^{p+j} \sum_{1\leq r\leq j} (-1)^{i_r - r} \alpha_{j,i_r} \cdot (-1)^{|I_r|+p(j-1)} M_{j,r} \\ &= (-1)^{|I|+pj} \sum_{1\leq r\leq j} (-1)^{j+r} M_{j,r} \\ &= (-1)^{|I|+pj} \det\bigl((\alpha_{t,i_r})_{1\leq t,r\leq j}\bigr).\end{aligned}$$

This proves (3) in the case $n = j$. □

Proposition 21.9. *If* $\varphi \in \operatorname{Hom}_{\mathcal{S}^{k-1}}\left(\mathcal{D}^k, \mathcal{S}^{k-1}\right)$ *is a chain map satisfying* $\varphi(s^k x_{[p]}) = \sigma y_{[q]}$, *then we have*

$$[\varphi] \neq 0 \in \operatorname{Ext}_{\mathcal{S}^{k-1}}\left(\wedge V, \mathcal{S}^{k-1}\right).$$

Proof. Let $I \subset \mathcal{S}^{k-1}$ be the ideal generated by $x_1 \otimes 1, \ldots, x_p \otimes 1, y_1 \otimes 1, \ldots, y_q \otimes 1, \sigma x_1, \ldots, \sigma x_p$. Note that $d(I) \subset I$ since $(\wedge V, d)$ is pure. Consider the evaluation map

$$\mathrm{ev}\colon \operatorname{Ext}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1}) \otimes \operatorname{Tor}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1}/I) \to \operatorname{Tor}_{\mathcal{S}^{k-1}}(\mathcal{S}^{k-1}, \mathcal{S}^{k-1}/I) \cong \wedge(\sigma y_1, \ldots, \sigma y_q).$$

Using $\mathcal{D}^k$ as a resolution of $(\wedge V, d)$ over $\mathcal{S}^{k-1}$, we have elements $[\varphi] \in \operatorname{Ext}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1})$ and $[s^k x_{[p]} \otimes 1] \in \operatorname{Tor}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1}/I)$. Then we have

$$\mathrm{ev}([\varphi] \otimes [s^k x_{[p]} \otimes 1]) = \sigma y_{[q]} \neq 0 \in \wedge(\sigma y_1, \ldots, \sigma y_q).$$

This proves the proposition. □

Corollary 21.10. *Assume* $p \leq q$, *i.e.* $\dim V^{\mathrm{even}} \leq \dim V^{\mathrm{odd}}$. *Then there is a chain map* $\varphi \in \operatorname{Hom}_{\mathcal{S}^{k-1}}\left(\mathcal{D}^k, \mathcal{S}^{k-1}\right)$ *such that*

(1) $[\varphi] \neq 0 \in \operatorname{Ext}_{\mathcal{S}^{k-1}}\left(\wedge V, \mathcal{S}^{k-1}\right)$.

(2) $$\mu \circ \varphi(1) = \begin{cases} \det\left(\left(\frac{\partial(dy_j)}{\partial x_i}\right)_{1\leq i,j\leq p}\right) \in \wedge V^{\mathrm{even}}, & \textit{if } p = q, \\ 0, & \textit{if } p < q. \end{cases}$$

Proof. Define $\varphi = (-1)^{\frac{1}{2}p(p+3)}\varphi_{2p}$. By Proposition 21.6 (1) and Proposition 21.9, we have $[\varphi] \neq 0 \in \operatorname{Ext}_{\mathcal{S}^{k-1}}\left(\wedge V, \mathcal{S}^{k-1}\right)$. If $p = q$, by (21.4) and Proposition 21.6 (3), we have $\mu \circ \varphi(1) = \det\left(\frac{\partial(dy_j)}{\partial x_i}\right)$. If $p < q$, by Proposition 21.6 (2), we have $\mu \circ \varphi(1) = 0$ since $\sigma y_j \in \operatorname{Ker}\mu$. □

Remark 21.11. We can generalize the nontriviality of the chain map $\varphi = \varphi_{\dim V}$ using the method and notion given in [Wak16]. Let $(\wedge V, d)$ be a *semi-pure* Sullivan algebra, i.e. $\dim V < \infty$ and $d(V^{\mathrm{even}})$ is contained in the ideal $\wedge V \cdot V^{\mathrm{even}}$ generated by V^{even}. Take bases $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$ of V^{even} and V^{odd}, respectively. By induction on $\dim V$, we have $\varphi(s^k x_{[p]}) = \sigma y_{[q]}$ along with $\varphi(\nu) = 0$ for any $\nu \in (s^k y_1, \ldots, s^k y_q) \subset \mathcal{D}^k$. The first equation $[\varphi] \neq 0 \in \operatorname{Ext}_{\mathcal{S}^{k-1}}(\wedge V, \mathcal{S}^{k-1})$, since Proposition 21.9 also holds for a semi-pure Sullivan algebra.

21.3 Proof of Proposition 20.11

In this subsection, we prove Proposition 20.11 using the chain map in Corollary 21.10.

Definition 21.12 ([FHT01, Section 32])**.** A 1-connected space M is *rationally elliptic* if $\dim_{\mathbb{K}} \left(\bigoplus_n H^n(M) \otimes \mathbb{K}\right) < \infty$ and $\dim_{\mathbb{K}} \left(\bigoplus_n \pi_n(M) \otimes \mathbb{K}\right) < \infty$.

First we recall a fundamental theorem on rationally elliptic space.

Theorem 21.13 ([FHT01, Proposition 32.16])**.** *Let M be a rationally elliptic space. Then we have*

- $\chi(M) \geq 0$ *and*
- $\dim_{\mathbb{K}} \left(\bigoplus_n \pi_{2n}(M) \otimes \mathbb{K}\right) \leq \dim_{\mathbb{K}} \left(\bigoplus_n \pi_{2n-1}(M) \otimes \mathbb{K}\right)$.

Moreover, the following conditions are equivalent:

(1) $\chi(M) > 0$.

(2) $\dim_{\mathbb{K}} \left(\bigoplus_n \pi_{2n}(M) \otimes \mathbb{K}\right) < \dim_{\mathbb{K}} \left(\bigoplus_n \pi_{2n-1}(M) \otimes \mathbb{K}\right)$.

(3) *The minimal Sullivan model $(\wedge V, d)$ of M is pure,* $\dim V^{\mathrm{even}} = \dim V^{\mathrm{odd}} = p$*, and $dy_1, \ldots, dy_p$ is a regular sequence in $\wedge V^{\mathrm{even}}$, where $V^{\mathrm{odd}} = \mathrm{span}_{\mathbb{K}}\{y_1, \ldots, y_p\}$.*

Using the theorem with the construction given in Subsection 21.2, we have the following proposition.

Proposition 21.14. *Let M be a rationally elliptic space satisfying the conditions in Theorem 21.13, and $(\wedge V, d)$ its minimal Sullivan model. Write $V^{\mathrm{even}} = \mathrm{span}_{\mathbb{K}}\{x_1, \ldots, x_p\}$ and $V^{\mathrm{odd}} = \mathrm{span}_{\mathbb{K}}\{y_1, \ldots, y_p\}$. Then we have*

$$\left[\det\left(\left(\frac{\partial(dy_j)}{\partial x_i}\right)_{1\leq i,j\leq p}\right)\right] \neq 0 \in H^*(\wedge V) \ \big(\cong H^*(M)\big).$$

Proof. By Corollary 21.10 for $k = 1$, we have a chain map $\varphi \in \mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2})$ such that $[\varphi] \neq 0 \in \mathrm{Ext}^m_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2})$ and $\mu \circ \varphi(1) = \det\left(\frac{\partial(dy_j)}{\partial x_i}\right) \in \wedge V^{\mathrm{even}}$. Since μ is a model of $\Delta\colon M \to M \times M$, we have

$$\Delta^* \circ (H^*([\varphi]))(1) = [\mu \circ \varphi(1)] = \left[\det\left(\frac{\partial(dy_j)}{\partial x_i}\right)\right].$$

Since $\mathrm{Ext}^m_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2}) \cong \mathrm{Ext}^m_{C^*(M\times M)}(C^*(M), C^*(M \times M)) \cong \mathbb{K}$, we have $[\varphi] = \Delta_!$ (up to scalar multiplication). Hence by Proposition 20.7, we have

$$\left[\det\left(\frac{\partial(dy_j)}{\partial x_i}\right)\right] = \chi(M)\omega.$$

Since $\chi(M) \neq 0$ by (3) of Theorem 21.13, this proves the proposition. □

Remark 21.15. The proposition also follows from [Smi82, Proposition 3]. Here we give an alternative proof using an idea coming from string topology.

Now we give a proof of Proposition 20.11, which completes the proof of Theorem 15.2.

Proof of Proposition 20.11. Since the statement is trivial when $\chi(M) = 0$, we may assume $\chi(M) \neq 0$. Then, by Theorem 21.13, the minimal Sullivan model $(\wedge V, d)$ of M satisfies (3). Take $\varphi \in \mathrm{Hom}_{\mathcal{S}^{k-1}}\left(\mathcal{D}^k, \mathcal{S}^{k-1}\right)$ by Corollary 21.10. Then we have $c^* \circ (H^*([\varphi]))(1) \neq 0 \in H^*(\wedge V) \cong H^*(M)$ by Proposition 21.14. Thus $\gamma = [\varphi]$ satisfies the equation (after multiplication of a non-zero scalar, if necessary). □

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