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Towards the spontaneous compactification of extra dimensions with generalized gravity

（拡張重力理論による余剰次元の自発的コンパクト化への道程）

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Abstract

The Lovelock theory and the generalized Galileon are the theories of generalized gravity which enable us to describe a variety of healthy models including Einstein's general relativity. The Horndeski theory, which is equivalent to the generalized Galileon in four dimensions, allows anisotropic inflationary solutions to be attractors without anisotropic matter, as opposed to Wald's cosmic no-hair theorem in the general relativity with a positive cosmological constant. The stability of perturbations of the Bianchi-type I universe is investigated and it is shown that propagation through an anisotropic background leads to the birefringence of gravitational waves. Since the perturbative behavior conflicts with observations, our Universe must be in the vicinity of the isotropic attractor in four dimensions with the aforementioned birefringence too small to be observable. This, however, motivates us two new directions of study, namely, the possibility of birefringence in an environment with strong gravity, and application of the theory to higher-dimensional spacetime. To observe the birefringence, we study the perturbations of static and spherically symmetric spacetime and obtain angular stability conditions, which enables us to theoretically test solutions of a black hole. The analysis of the anisotropic attractor in four dimensions suggests that some spatial dimensions expand much slower than the other dimensions if the hierarchical conditions among parameters are assumed. We investigate anisotropic attractors in higher dimensions in the presence of energy contents with isotropic and anisotropic pressure with the Lovelock theory and particular models of the generalized Galileon. The hierarchical conditions realize arbitrarily slow growth of extra dimensions and the universe which evolve as if it obeys the general relativity.

Chapter 1

Introduction

The homogeneity and isotropy on large scales are the most significant features of our universe, which have been observationally ascertained by the redshift surveys [1–3]. It is the evidence for the heliocentric theory that our Earth does not lie in the center of the universe, or more generally, that there is no preferred point nor preferred direction. This is what is called the cosmological principle and is a fundamental viewpoint to study cosmology. We need some mechanism in the early universe in order for the cosmological principle to hold even ten billion years after the big bang, which is called the horizon problem. One of the most convincing mechanisms is cosmic inflation [4–8], which makes the early universe undergo accelerating expansion with a scalar field called inflaton. The observation of the cosmic microwave background (CMB) has revealed that the spectrum of primordial curvature perturbations is slightly red-tilted [9]. If the primordial gravitational waves are detected by the forthcoming observation of the CMB polarization, such as LiteBIRD [10], then it must be the smoking gun that the inflation occurred. These clues, however, do not tell us which model of inflation is correct because the main consequence of the inflation faintly depends on specific models. The research on how to distinguish inflation models is continued even after forty years have passed since the first article on the primordial gravitational waves [11] appeared.

Homogeneous configuration of the inflaton provides isotropic pressure and it makes us expect that the universe would be isotropized. It is not trivial that the inflation always isotropize the universe even in highly warped spacetime. Wald has demonstrated that a positive cosmological constant makes the spatially homogeneous anisotropic universe always approach the de Sitter universe if it starts with expanding volume, except for the case with positive spatial curvature [12]. This guarantees the inflation to isotropize the universe and it is called the cosmological no-hair theorem since the universe loses its information which is likened to a hair. It is rather difficult to produce anisotropy of the universe in inflation. The present CMB observations are consistent with zero anisotropy but still admit about 1%-level quadrupolar modulation of the power spectrum of the curvature perturbations [13, 14]. A naive way to produce such anisotropy is to put in the universe anisotropic matter such as a vector field [15]. With a nonconformal coupling with inflaton, the vector field obtains a nonvanishing expectation value of energy, and it contributes to producing the Bianchi-type-I anisotropy of the universe. Such anisotropic expansion produces statistical anisotropies of fluctuations produced on exiting the Hubble horizon.

Here we propose another way to realize anisotropic inflation using one of the generalized gravity theories, the Horndeski theory [16]. The Horndeski theory is the most general gravity theory with a scalar field which yields second-order field equations. For this feature, the theory can be free from ghost instability stated by the Ostrogradsky theorem in higher-derivative theories [17]. Most models of inflation with a single scalar field, inflaton, are contained in the Horndeki theory [18]. Moreover, it has been shown that there are plenty of possibilities of coupling between the inflaton and the gravitational field. In the general relativity, which is Wald's assumption, we can say that there is only an isotropic attractor and all of the initial states terminates at the isotropic attractor. On the other hand, a part of the Horndeski theory allows the existence of anisotropic attractors as well as the isotropic one and enables anisotropic inflation to occur [19]. It depends on the initial anisotropy of the universe whether it terminates at the isotropic attractor or the anisotropic attractors. This implies that the universe has to start with the sufficiently isotropic state so that it can terminate at the isotropic attractor if the four-dimensional spacetime is given. It stimulates us to think of the hypothesis that our universe initiated with anisotropic and higher-dimensional spacetime, and an anisotropic attractor realized our isotropic and four-dimensional universe.

The Horndeski theory has been tested both in the theoretical ground and in the observational ground [20]. The starting point of constraining theory space is to check the conditions for no ghost and gradient stability in the flat Friedmann universe [18]. Those conditions have been also studied in a static and spherically symmetric spacetime such as black holes with general solutions [21,22] although they have skipped derivation of the gradient stability condition along angular directions because of complicated calculations. Contrary to the no-hair theorems for black holes [23–25], the hairy solutions of a black hole have been discovered [26, 27] and some solutions are realized spontaneously [28, 29]. Their quasi-normal modes have also been investigated (*e.g.* [30, 31]). The future observation of the ringdown after binary black hole mergers would yield constraints on such hairy solutions and consequently, the gravity models could be ruled out. The simultaneous observations of gravitational waves [32] and its electromagnetic counterpart [33] have given constraints on the theory though propagation speed of gravitational waves [34–37], with the formula derived in [18].

Higher dimensions may play an important role in the unification of the fundamental forces. One of the most famous unified theories has been given by Kaluza and Klein in the 1920's [38, 39]. They aimed to unify the gravitational and electromagnetic forces by postulating the fifth dimension of spacetime invisible to us. This model, however, conflicts with observation because such a massless particle has never been detected which is called radion or modulus originating from the metric component of the fifth dimension. Sixty years later, the superstring theory has been advocated and it provides a quantum gravity theory, which requires ten-dimensional spacetime for consistency [40, 41]. It is a strong candidate for a unified theory with extra dimensions compactified [42]. As in the theory of Kaluza and Klein, moduli ought to obtain their mass to be consistent with observation. A standard solution is injecting fluxes along the extra dimensions and the flux stabilizes the moduli fields, which is called flux compactification (see *e.g.* [43]).

It is allowed to deal with the extra dimensions in a more phenomenological manner.

One of the efficient ways to give an effective theory to describe the dynamics of spacetime is to consider conditions for avoiding ghosts. In this thesis, we introduce two generalized theories of gravity which are free from a ghost. One is the Lovelock gravity [44]. It is the theory only with the metric which provides the most generalized tensor having similar properties with the Einstein tensor, and the action contains not only a linear term of the second derivative of the metric but also its higher-order terms. The other is the generalized Galileon, which is a general scalar-tensor theory and also introduces higher-order terms of the metric connection in the action through the second derivatives of the scalar field [45]. The generalized Galileon in four-dimensional spacetime is equivalent to the Horndeski theory, which has been proven in [18]. The Lovelock theory and the generalized Galileon can apply to any number of spatial dimensions and they enable us to analyze the dynamics of higher-dimensional spacetime. The anisotropic dynamics similar to [19] is expected in those two theories because they introduce nonlinear terms of the metric connection in the actions.

There have been several studies on the dynamics in higher dimensions in generalized gravity, aiming to compactify it into large and three-dimensional space [46–57]. The systematic search on the evolution of the universe in the Lovelock theory has been given in [48]. It has been shown that there is a parameter region in which the maximally symmetric spacetime is not allowed to be the solution and that higher dimensions would be compactified by collapsing into a less symmetric spacetime in [46, 52, 55]. Reference [49] has shown that the effect of higher-order Lovelock terms and the spatial curvature can balance and static extra dimensions are realized. As an example of a recent study in a specific case, a numerical example has been given in [57] of anisotropic dynamics of eight-dimensional spacetime in the Einstein-Gauss-Bonnet gravity, which consists of up to second-order terms in the Lovelock theory. The spacetimes compactified similarly to the previous way can exhibit Friedmann-like dynamics [56]. The exactly exponential solution has been investigated in [53] and [54] in the Einstein-Gauss-Bonnet theory and the Lovelock theory, respectively. Anisotropic evolution of higher-dimensional space has been studied in [50, 51] in the Lovelock theory of which parameters originate from higher-order corrections in the superstring theory.

We try to tackle the question of how higher-dimensional spacetime can evolve into a lower-dimensional universe with compactified extra dimensions, considering in higher dimensions the anisotropic attractors discussed in [19]. It has been shown in the four-dimensional case [19] that anisotropic attractor can exhibit such anisotropic expansion that one dimension can expand or contract much more slowly than the other dimensions. If we regard slowly expanding directions as "extra dimensions", we have a $(1 + 1)$-dimensional universe with two-dimensional extra dimensions. It suggests that if we begin with higher-dimensional spacetime *a priori*, we have the anisotropic attractors which enable the spacetime to be compactified into the four-dimensional universe. Here we do not persist in the conventional paradigm that the extra dimensions must be static. We control the expansion or contraction rate of extra dimensions to be slow enough so as not to contradict non-observation of time variation of fundamental physical constants [58]. We aim to freeze the extra dimensions at an arbitrary level compared to the dimensions of the universe.

The thesis is organized as follows. In the next chapter, we introduce the action of two kinds of generalized gravity, the Lovelock theory [44] and the generalized Galileon [45]. Those gravity theories are available for any number of dimensions. With the generalized Galileon, we study the evolution of anisotropic background in four dimensions in Chapter 3 and in higher dimensions in Chapter 6. We develop perturbation theory in four dimensions on different unperturbed spacetimes, the Bianchi-type I model in Chapter 4 and the static and spherically symmetric spacetime in Chapter 5.

As Wald has proven in the general relativity, all Bianchi-type universe can be isotropized in the presence of a positive cosmological constant, except for the Bianchi-type IX. In Chapter 3, we investigate anisotropic solutions with the generalized Galileon in four dimensions, the Horndeski theory [16], and we find the universe approaches anisotropic attractor, which enables the universe to stay anisotropic, even in the presence of a positive cosmological constant. Chapter 4 shows the perturbative behavior on Bianchi-type I background and the dispersion relations of gravitational waves and scalar waves. The analysis around the anisotropic attractor reveals the singular behavior of gravitational waves with even parity and it can provide a way to homogenize the space by enlarging sound horizon in the early universe. As another fact found out in Chapter 4, it is worthy of special mention that the Horndeski theory admits mixing of the dispersion relation between gravitational and scalar waves and it indicates that birefringence of gravitational waves can occur while propagating over anisotropically expanding region. It is also important to study perturbative behavior in a strong-field regime such as the vicinity of black holes. The first half of Chapter 5 is dedicated to reviewing the perturbation theory of the static and spherically symmetric spacetime developed in [21, 22]. In the other half, we improve the way to express lengthy combinations of the coefficients in the action and successfully calculate the dispersion relation including angular directions. Nonlinear terms of expansion rates in the action provide different dynamics of spacetime, which is suggested in Chapter 3, and it motivates us to investigate anisotropic solutions in higher dimensions. In Chapter 6, we use a subclass of the generalized Galileon to study such phenomena in the presence of both isotropic and anisotropic energy contents. We find that isotropic energy contents, such as the cosmological constant, homogeneous scalar field, and nonrelativistic matter, allow the system to approach an anisotropic attractor. With some hierarchy conditions between constant parameters, extra dimensions can freeze and the universe evolves as if it follows the Einstein gravity. In an inflationary era, the higher-dimensional space expands only along three spatial directions and we observe the large three-dimensional space emerges while approaching the anisotropic attractor. We also show that although anisotropic energy contents do not admit the anisotropic attractors to be an actual attractor of the system, the acceleration of the extra dimensions is much smaller than that of the three-dimensional space of the universe, which means that the expansion rate remains small. As a special case, radiation or relativistic matter satisfies the equation-of-state condition for damping down the expansion rate of the extra dimensions. We, therefore, see that the universe recovers Friedmann-like dynamics in the whole cosmic history. Chapter 7 is devoted to summarize this thesis and discuss the physical implications.

Chapter 2

Generalized gravity

Here we introduce two generalized theories of gravity, the Lovelock theory and the generalized Galileon. Common feature of them are that they do not suffer from the ghost which has negative kinetic energy. To avoid Osrtogradsky ghost [17], which always appears when the Lagrangian is not degenerate, higher derivative terms are eliminated from the field equations of both theories. In the first section, we introduce the Lovelock theory, which contains only tensor-type degrees of freedom. In the next section, we review the generalized Galileon with its history from the Galileon. We also show that the action of the Horndeski theory is given by that of the generalized Galileon in four dimensions. In the next section, we write the Horndeski action with the ADM variables to replace the scalar field with the geometrical quantities.

2.1 Gravity only with metric: Lovelock theory

The Lovelock theory of gravity is the most generalized theory which is written only with metric $g_{\mu\nu}$ and its first two derivatives. Its field equation is given by generalization of the Einstein equation.

In order to generalize the Einstein equation $G_{\mu\nu} = T_{\mu\nu}$, Lovelock explored in arbitrary dimensions the most generalized tensor $A_{\mu\nu}$ which has the same appropriate properties as the Einstein tensor $G_{\mu\nu}$ has [44]. He assumed that the generalized Einstein tensor $A_{\mu\nu}$ should have the three properties below:

(a) $A_{\mu\nu}$ is symmetric, i.e.,

$$A_{\mu\nu} = A_{\nu\mu}.$$

(b) $A_{\mu\nu}$ is a function of the metric tensor $g_{\mu\nu}$ and its first and second derivatives, i.e.,

$$A_{\mu\nu} = A_{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,\rho}, g_{\mu\nu,\rho\sigma}).$$

(c) $A_{\mu\nu}$ is divergence-free, i.e.,

$$\nabla^\mu A_{\mu\nu} = 0.$$

The explicit form of all the possible $A_{\mu\nu}$ can be written as below [44]:

$$A^{\mu\nu} = a_0 g^{\mu\nu} + \sum_{p=1}^{m-1} a_p g^{\mu\mu_1\cdots\mu_{2p}\nu\nu_1\cdots\nu_{2p}} R_{\mu_1\mu_2\nu_1\nu_2}\cdots R_{\mu_{2p-1}\mu_{2p}\nu_{2p-1}\nu_{2p}}, \tag{2.1}$$

where a_0 and a_p are arbitrary constants, m is the positive integer which is defined by

$$m \equiv \lceil D/2 \rceil = \frac{1}{2}D \quad \text{if } D \text{ is even}, \tag{2.2}$$

$$= \frac{1}{2}(D+1) \quad \text{if } D \text{ is odd}, \tag{2.3}$$

where D is the number of the dimension of the spacetime, $g^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N}$ is the superscripted generalized Kronecker delta defined by

$$g^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N} = \det\begin{pmatrix} g^{\mu_1\nu_1} & \cdots & g^{\mu_1\nu_N} \\ \vdots & \ddots & \vdots \\ g^{\mu_N\nu_1} & \cdots & g^{\mu_N\nu_N} \end{pmatrix}, \tag{2.4}$$

and $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor. [1] The superscripted generalized Kronecker delta is also denoted in different representation with signature $\epsilon(\sigma)$ of the permutation group S_N

$$g^{\mu_1\cdots\mu_N\nu_1\cdots\nu_N} = \sum_{\sigma\in S_N} \epsilon(\sigma) g^{\mu_{\sigma(1)}\nu_1} g^{\mu_{\sigma(2)}\nu_2}\dots g^{\mu_{\sigma(N)}\nu_N} \tag{2.5}$$

$$= -\frac{1}{(D-m)!}\varepsilon^{\mu_1\mu_2\cdots\mu_m\sigma_1\sigma_2\dots\sigma_{D-m}}\varepsilon^{\nu_1\nu_2\dots\nu_m}{}_{\sigma_1\sigma_2\dots\sigma_{D-m}}, \tag{2.6}$$

$$\varepsilon^{\mu_1\mu_2\dots\mu_D} = -\frac{1}{\sqrt{-g}}\delta_1^{[\mu_1}\delta_2^{\mu_2}\dots\delta_D^{\mu_D]}. \tag{2.7}$$

For $n=4$ as in the universe, the tensor $A_{\mu\nu}$ is reduced to

$$A^{\mu\nu} = a_0 g^{\mu\nu} + a_1 g^{\mu\mu_1\mu_2\nu\nu_1\nu_2} R_{\mu_1\mu_2\nu_1\nu_2} \tag{2.8}$$

$$= a_0 g^{\mu\nu} - 4a_1 G^{\mu\nu}. \tag{2.9}$$

This recovers the Einstein equation with a cosmological constant.

The action which gives the generalized Einstein equation $A_{\mu\nu}=0$ is given by:

$$S = \int d^n x\sqrt{-g}\left(2a_0 + \sum_{p=1}^{m-1} 2a_p g^{\mu_1\cdots\mu_{2p}\nu_1\cdots\nu_{2p}} R_{\mu_1\mu_2\nu_1\nu_2}\cdots R_{\mu_{2p-1}\mu_{2p}\nu_{2p-1}\nu_{2p}}\right), \tag{2.10}$$

which is the whole action of the Lovelock theory.

[1] Here we define the Riemann curvature tensor with any covariant vector field X^μ by $X^\nu_{;\rho\sigma} - X^\nu_{;\sigma\rho} = R_\mu{}^\nu{}_{\rho\sigma} X^\mu$.

2.2 Gravity with a scalar field: Generalized Galileon

The generalized Galileon gives general classes of action of a single scalar field which is coupled with metric. The original Galileon [59] is the scalar field in flat spacetime $g_{\mu\nu} = \eta_{\mu\nu}$ which enjoys the Galilean symmetry

$$\phi \to \phi + b_\mu x^\mu + c. \tag{2.11}$$

All the models we consider here contains at most a single scalar field ϕ and its (covariant) derivatives are denoted by

$$\phi_\mu = \nabla_\mu \phi, \quad \phi_{\mu\nu} = \nabla_\nu \nabla_\mu \phi, \quad \phi_{\mu\nu\rho} = \nabla_\rho \nabla_\nu \nabla_\mu \phi. \tag{2.12}$$

The Lagrangian of the Galileon is

$$\mathcal{L}_1 = \phi, \tag{2.13}$$

$$\mathcal{L}_2 = -\frac{1}{2}\phi_\mu \phi^\mu, \tag{2.14}$$

$$\mathcal{L}_3 = -\frac{1}{2}\phi_\lambda{}^\lambda \phi_\mu \phi^\mu, \tag{2.15}$$

$$\mathcal{L}_4 = -\frac{1}{4}\{(\phi_\mu{}^\mu)^2 \phi_\nu \phi^\nu - 2\phi_\lambda{}^\lambda \phi_\mu \phi^\mu_\nu \phi^\nu - (\phi_\lambda{}^\mu \phi_\mu{}^\lambda)\phi_\nu \phi^\nu + 2\phi_\mu \phi^\mu_\nu \phi^\nu_\rho \phi^\rho\}, \tag{2.16}$$

$$\begin{aligned}\mathcal{L}_5 = -\frac{1}{5}\{&(\phi_\mu{}^\mu)^3 \phi_\nu \phi^\nu - 3(\phi_\lambda{}^\lambda)^2 \phi_\mu \phi^\mu_\nu \phi^\nu - 3(\phi_\rho{}^\rho)(\phi_\lambda{}^\mu \phi_\mu{}^\lambda)\phi_\nu \phi^\nu \\ &+ 6(\phi_\lambda{}^\lambda)\phi_\mu \phi^\mu_\nu \phi^\nu_\rho \phi^\rho + 2(\phi_\lambda{}^\rho \phi_\rho{}^\sigma \phi_\sigma{}^\lambda)\phi_\nu \phi^\nu + 3(\phi_\lambda{}^\rho \phi_\rho{}^\lambda)\phi_\mu \phi^\mu_\nu \phi^\nu - 6\phi_\mu \phi^\mu_\nu \phi^\nu_\rho \phi^\rho_\sigma \phi^\sigma\}.\end{aligned} \tag{2.17}$$

Each Lagrangian term does introduce neither higher-order derivatives term in its Euler-Lagrange equation nor ghost instabilities. Its covariantization [60], introducing gravity in the theory, introduces non-minimal coupling between the scalar field and metric to eliminate higher-order derivatives term in the field equation of metric.

The generalized Galileon [45] gives the most general theory in flat spacetime satisfying the three conditions below:

(i) its Lagrangian contains derivatives of order 2 or less of the scalar field ϕ ;

(ii) its Lagrangian is polynomial in the second derivatives of ϕ ;

(iii) the corresponding field equations are of order 2 or lower in derivatives .

The whole Lagrangian is given by an arbitrary linear combination of the Lagrangians $\mathcal{L}_{n,0}\{f\}$ of the form

$$\mathcal{L}_{n,0}\{f\} = f(\phi, X) \times \left(X g^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \phi_{\mu_1 \nu_1} \cdots \phi_{\mu_n \nu_n}\right), \tag{2.18}$$

where $X \equiv -\partial_\mu \phi \partial^\mu \phi/2$ is a canonical kinetic term [2] and $g^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n}$ is the superscripted generalized Kronecker delta defined in (2.4). As is seen when we take $f =$ const., the

[2] The definition of X in [45] is different from that in this thesis and in [18], the reference of the Horndeski theory. Consequently the value in (2.20) is modified.

Lagrangians (2.18) include (2.17) as a subset. Note, however, that the Lagrangians (2.18) no longer enjoy the Galilean symmetry (2.12) in general. Naive covariantization ($\eta_{\mu\nu} \to g_{\mu\nu}$, $\partial_\mu\partial_\nu\phi \to \nabla_\mu\nabla_\nu\phi$) leads higher-order derivative term like $R_{\mu\nu\rho\sigma,\lambda}$. In order to eliminate such terms, we need to add the compensation terms to get the Lagrangian which is free from higher-derivative terms. (Such a theory is called *healthy.*) [45] gives the healthy covariant action of the form:

$$S = \int d^{D+1}x\sqrt{-g}\sum_{n=0}^{D}\sum_{p=0}^{\lfloor \frac{n}{2} \rfloor}\mathcal{C}_{n,p}\mathcal{L}_{n,p}\{f_n\}, \tag{2.19}$$

$$\mathcal{C}_{n,p} = \frac{4^{-p}n!}{(n-2p)!p!}, \tag{2.20}$$

$$\mathcal{L}_{n,p}\{f\} = g^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}\mathcal{P}_{(p)}\{f\}\left[\prod_{i=1}^{p}R_{\mu_{2i-1}\mu_{2i}\nu_{2i-1}\nu_{2i}}\right]\left[\prod_{i=2p+1}^{n}\phi_{\mu_i\nu_i}\right], \tag{2.21}$$

$$\mathcal{P}_{(p)}\{f\} \equiv \int_{X_0}^{X} dX_1 \int_{X_0}^{X_1} dX_2 \cdots \int_{X_0}^{X_{p-1}} dX_p \left[X_p f(\phi, X_p)\right], \tag{2.22}$$

Interestingly, the covariantization of the general scalar-field theory leads the Lovelock theory. It is easily seen that we recover (2.10) when we set

$$\mathcal{P}_{(p=n/2)}\{f\} = \frac{2^n(n/2)!}{n!}\cdot 2a_p \qquad \text{if } n \text{ is even}, \tag{2.23}$$

$$= 0 \qquad \text{if } n \text{ is odd}. \tag{2.24}$$

2.2.1 In four-dimensional spacetime: Horndesky theory

We live in four dimensional spacetime and we are most interested in four dimension to apply the theories to cosmology. It has been proved in [18] that the generalized Galileon (2.22) in four dimension is equivalent to the Horndeski theory [16]. The Horndeski theory describes the most general couplings between a scalar field ϕ and the metric $g_{\mu\nu}$ which yield second-order field equations.

The theory is characarized by four arbitrary functions, G_2, G_3, G_4 and G_5, of ϕ and its canonical kinetic function $X \equiv -\partial_\mu\phi\partial^\mu\phi/2$ as

$$S = \int \mathrm{d}^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i, \tag{2.25}$$

$$\mathcal{L}_2 = G_2(\phi, X), \tag{2.26}$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \tag{2.27}$$

$$\mathcal{L}_4 = G_4(\phi, X)\mathcal{R} + G_{4X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \tag{2.28}$$

$$\mathcal{L}_5 = G_5(\phi, X)\mathcal{G}_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3], \tag{2.29}$$

where $\mathcal{R}$ is the four-dimensional Ricci scalar, $\mathcal{G}_{\mu\nu}$ is the four-dimensional Einstein tensor, $(\nabla_\mu\nabla_\nu\phi)^2 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi$, $(\nabla_\mu\nabla_\nu\phi)^3 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi$, and $G_{iX} \equiv \partial G_i/\partial X$.

The Horndeski action is given by (2.19) when we set the functions as

$$\mathcal{P}_{(p=0)}\{f_0\} = G_2(\phi, X), \tag{2.30}$$
$$\mathcal{P}_{(p=0)}\{f_1\} = -G_3(\phi, X), \tag{2.31}$$
$$\mathcal{P}_{(p=1)}\{f_2\} = G_4(\phi, X), \tag{2.32}$$
$$\mathcal{P}_{(p=1)}\{f_3\} = -\frac{1}{6}G_5(\phi, X). \tag{2.33}$$

The Horndeski theory contains the general relativity ($G_4 = (16\pi G)^{-1}$ and $G_2 = G_3 = G_5 = 0$) and several well-known models of modified gravity.

2.3 ADM form of Horndeski theory and beyond

The action described by the Arnowitt-Deser-Misner (ADM) variables [61] is more useful to study anisotropic cosmological solutions than the covariant form (2.25). The metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{2.34}$$

where $q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ is the induced metric. (Note that $q_{ij} = q_{\mu\nu}(\partial_i)^\mu(\partial_j)^\nu$ and that $(\partial_i)^\mu$ is a component of the vector ∂_i in the direction of μ.) We take the unitary gauge, $\phi = \phi(t)$, and then X is given by $X = \dot{\phi}^2/2N^2$ with N being the lapse function. If ϕ is a monotonic function of t, this is a very convenient gauge and we can use (t, N) instead of (ϕ, X) to express the action. Next we define the normal vector field $n_\mu \equiv -N\nabla_\mu t$ to a family of the time-constant hypersurfaces Σ_t.

Then, the theory is described only in terms of t and geometrical quantities as

$$S = \int \mathrm{d}t\mathrm{d}^3x N\sqrt{g}\sum_{i=2}^{5}\mathcal{L}_i, \tag{2.35}$$
$$\mathcal{L}_2 = A_2(t, N), \tag{2.36}$$
$$\mathcal{L}_3 = A_3(t, N)K, \tag{2.37}$$
$$\mathcal{L}_4 = A_4(t, N)\left(K^2 - K^i_jK^j_i\right) + B_4(t, N)R, \tag{2.38}$$
$$\mathcal{L}_5 = A_5(t, N)\left(K^3 - 3KK^i_jK^j_i + 2K^i_jK^j_kK^k_i\right) + B_5(t, N)\left(R_{ij} - \frac{1}{2}g_{ij}R\right)K^{ij}. \tag{2.39}$$

uo to the total derivative [62]. K^i_j and R_{ij} are the extrinsic and intrinsic curvature of constant t (constant ϕ) hypersurfaces. The functions A_i, B_i and G_i are related with each other as follows:

$$A_2(t,N) = G_2(\phi,X) - \sqrt{X}\int \frac{G_{3\phi}(\phi,X)}{\sqrt{X}}dX, \tag{2.40}$$

$$A_3(t,N) = \int \sqrt{2X}G_{3X}(\phi,X)dX - 2\sqrt{2X}G_{4\phi}(\phi,X), \tag{2.41}$$

$$A_4(t,N) = -G_4(\phi,X) + 2XG_{4X}(\phi,X) - XG_{5\phi}(\phi,X), \tag{2.42}$$

$$A_5(t,N) = \frac{1}{6}(2X)^{3/2}G_{5X}(\phi,X), \tag{2.43}$$

$$B_4(t,N) = G_4(\phi,X) - \frac{\sqrt{X}}{2}\int \frac{G_{5\phi}(\phi,X)}{\sqrt{X}}dX, \tag{2.44}$$

$$B_5(t,N) = -\int \sqrt{2X}G_{5X}(\phi,X)dX, \tag{2.45}$$

where we identify $X = \dot{\phi}^2(t)/2N^2$. As seen below, among those terms the most crucial ones in this paper are the terms cubic in the extrinsic curvature. In the covariant language they come from $\mathcal{L}_5$ which depends cubically on the second derivatives of the scalar field.

In the Horndeski theory, (A_4, A_5) and (B_4, B_5) are not independent, as is clear from Eqs. (2.42)–(2.45) and also from the fact that we originally have four free functions in the action. However, this point turns out to be not essential in the following discussion. The most important ingredient here is the cubic (or higher) order terms in the extrinsic curvature. This allows us to start from the ADM Lagrangians (2.36)–(2.39) and consider all A_i's and B_i's to be independent free functions, which amounts to employing the so-called "beyond Horndeski" theory [62].

To discuss nontrivial evolution of the background, we use the ADM form (2.36)–(2.39), because the cubic curvature term (2.39) is essential. On the other hand, we would use the covariant form (2.25) in the analysis of perturbations in order to provide the way to calculate perturbations in anisotropic (inhomogeneous) background. The two forms are equivalent and it is easy to work in the other form by using (2.42)–(2.45).

2.4 Summary

We have introduced the several theories of generalized gravity which are used in the following chapters. We use the beyond-Horndeski theory (2.35) in Chapter 3 to calculate the evolution of anisotropic background. In Chapter 4 and Chapter 5, we use the Horndeski action (2.25) to make the dispersion relation be calculable. In Chapter 6, we focus on the Lovelock action (2.10) and subclass of the generalized Galileon action (2.19).

Chapter 3

Isotropic and anisotropic attractors in four dimensions

Here we calculate the evolution of the anisotropic background, especially Bianchi type-I model. The Horndeski and beyond-Horndeski theories contain nonlinear terms of expansion rates in conserved momenta of background anisotropies, and it gives other nontrivial roots than the trivial root which exhibits isotropic expansion. In the first section we review Wald's cosmological no-hair theorem [12]. In the next section, we see the theorem no longer holds in the Horndeski theory in the next section with a few demonstration of numerical calculation. In the next section, we consider more general anisotropy with matter contents and conclude that if the spatial curvature is negligible and if matter content is isotropic then the anisotropic attractor works and it exhibits axial symmetry.

3.1 Cosmological no-hair theorem in general relativity

Wald has proven the cosmological no-hair theorem with the Hamiltonian constraint and an evolution equation which is called the Raychaudhuri equation. The equations reads in the spatially homogeneous universe (referring to (3.35) and (3.37) with $A_2 = -\Lambda/16\pi G$ and $A_4 = -B_4 = -1/16\pi G$)

$$-\Lambda + \frac{2}{3}K^2 - \Sigma^i_j\Sigma^j_i + R = 16\pi G\rho, \tag{3.1}$$

$$-4\dot{K} + 3\Lambda - 2K^2 - 3\Sigma^i_j\Sigma^j_i - R = 16\pi G p, \tag{3.2}$$

where K and Σ denote the trace and trace-free part of the extrinsic curvature, which are defined in (3.36), respectively. From summation of (3.1) and (3.2), we obtain

$$\dot{K} = \frac{\Lambda}{2} - \frac{1}{3}K^2 - \Sigma^i_j\Sigma^j_i - 4\pi G(\rho + p). \tag{3.3}$$

We assume the dominant and strong energy conditions and they imply

$$\rho \geq 0, \quad \rho + p \geq 0. \tag{3.4}$$

Except for Bianchi-type IX model, All the spatially homogeneous spacetime has flat or negative spatial curvature

$$R \leq 0. \tag{3.5}$$

The energy conditions (3.4) reduce the evolution equation (3.3) to

$$\dot{K} \leq \frac{\Lambda}{2} - \frac{1}{3}K^2 - \Sigma^i_j \Sigma^j_i \leq \frac{\Lambda}{2} - \frac{1}{3}K^2 \tag{3.6}$$

With (3.4) and (3.5), the Hamiltonian constraint (3.1) reads

$$\frac{2}{3}K^2 - \Lambda \geq \Sigma^i_j \Sigma^j_i \geq 0. \tag{3.7}$$

From those two equations, we obtain

$$\dot{K} \leq \frac{\Lambda}{2} - \frac{1}{3}K^2 \leq 0. \tag{3.8}$$

The second inequality in (3.8) shows that the trace part of the extrinsic curvature, which corresponds to total-volume expansion rate, has an lower limit once it initiate with a positive value

$$K \geq \sqrt{\frac{3\Lambda}{2}}. \tag{3.9}$$

On the other hand, the first inequality in (3.8) implies that time evolution of K has an upper limit

$$K \leq \sqrt{\frac{3\Lambda}{2}} \left(\tanh \frac{t}{\alpha} \right)^{-1}, \tag{3.10}$$

where α is a typical time scale of evolution defined by $\alpha \equiv \sqrt{6/\Lambda}$. The upper and lower bounds indicate that for $t \gg \alpha$ the trace part of the extrinsic curvature K approaches $\sqrt{3\Lambda/2}$ rapidly. From (3.7), we can tell that the trace-free part of the extrinsic curvature decreases exponentially, which corresponds to anisotropic expansion rate. This is a simpler version of the proof given by Wald. It is difficult to extend the proof into the Horndeski theory in general. We would pay attention to evolution equation of the trace-free part (3.38) in Sec. 3.3, and we see an essential difference between the general relativity, in which the Wald's cosmological no-hair theorem holds, and the Horndeski theory.

3.2 Vacuum Bianchi type-I model

For the simplest case, let us consider the Bianchi type-I model, which is the spatially flat and homogeneous model. Once we diagonalize the spatial metric and its time derivative, off-diagonal components are not generated in this model, so that we can express the metric in the Kasner-type form as

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left[e^{2(\beta_+(t)+\sqrt{3}\beta_-(t))}dx^2 + e^{2(\beta_+(t)-\sqrt{3}\beta_-(t))}dy^2 + e^{-4\beta_+(t)}dz^2\right], \tag{3.11}$$

where $a(t)$ is a scale factor and $\beta_\pm(t)$ show the differences between the expansion rates in different directions. Substituting the metric (3.11) in the ADM form of the action (2.35), we obtain

$$S = \int dt d^3x\ \mathcal{L} \tag{3.12}$$

$$= \int dt d^3x\ Na^3\left[A_2 + 3HA_3 + 6A_4(H^2 - \sigma_+^2 - \sigma_-^2) + 6A_5(H^3 - 3H(\sigma_+^2 + \sigma_-^2) + 2(3\sigma_+\sigma_-^2 - \sigma_+^3))\right], \tag{3.13}$$

where we have defined the Hubble parameter H and the shear $\sigma_\pm$ as

$$H \equiv \frac{1}{N}\frac{d\ln a}{dt}, \qquad \sigma_\pm \equiv \frac{1}{N}\frac{d\beta_\pm}{dt}. \tag{3.14}$$

Since the Bianchi type-I model is spatially flat and consequently Eq. (3.13) depends on $\beta_\pm$ only through their time derivatives, the momenta conjugate to $\beta_\pm$ are conserved.

$$\frac{\delta S}{\delta\beta_\pm} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\beta}_\pm} = \frac{d}{dt}P_{\beta_\pm} = 0. \tag{3.15}$$

The conserved momenta are given by

$$P_{\beta_+} = a^3\left[(A_4 + 3HA_5)\sigma_+ + 3A_5(\sigma_+^2 - \sigma_-^2)\right], \tag{3.16}$$

$$P_{\beta_-} = a^3\left[(A_4 + 3HA_5)\sigma_- - 6A_5\sigma_+\sigma_-\right]. \tag{3.17}$$

up to constant factors, which are irrelevant in the discussion. It is manifest that as the scale factor $a(t)$ increases, the expressions inside the square brackets of Eqs. (3.16) and (3.17) decay toward zero as $[\cdots] = P_{\beta_\pm}a^{-3} \to 0$, and thus σ_+ and σ_- evolve to one of the fixed points. In the present case, there are four fixed points. One is the isotropic solution $\sigma_\pm = 0$, whereas the other three are anisotropic attractors.

Let us look at the trajectories on the (σ_+, σ_-) plane of the phase space. Given initial condition, the constants $P_{\beta_\pm}$ are fixed. Then, $\sigma_\pm$ can be expressed in terms of A_4, A_5, a, H, and $P_{\beta_\pm}$ by solving the algebraic equations (3.16) and (3.17), although the equation is of the fourth degree and thus the explicit form of the solutions is complicated. In order to show the dynamics of the anisotropies in a single figure, we use the normalized shear $\mathcal{A}_\pm$ defined as

$$\mathcal{A}_\pm \equiv \frac{3A_5}{A_4 + 3HA_5}\sigma_\pm, \tag{3.18}$$

instead of σ_+ and σ_-. Here we assumed that $A_4 + 3HA_5 \neq 0$ and $A_5 \neq 0$. It is also convenient to introduce the new time coordinate $\tau \equiv a^3(A_4 + 3HA_5)^2/3A_5$. In an expanding universe, $|\tau|$ is an increasing function of t provided that A_4, A_5, and H depend on t only weakly, which is a natural assumption during inflation. With τ and $\mathcal{A}_\pm$, we can rewrite Eqs. (3.16) and (3.17) simply as

$$P_{\beta_+} = \tau\left[\mathcal{A}_+ + \mathcal{A}_+^2 - \mathcal{A}_-^2\right], \tag{3.19}$$
$$P_{\beta_-} = \tau\left[\mathcal{A}_- - 2\mathcal{A}_+\mathcal{A}_-\right]. \tag{3.20}$$

We show trajectories $(\mathcal{A}_+(\tau), \mathcal{A}_-(\tau))$ for different values of $P_{\beta_\pm}$ in Figure 3.1. As stated above, there are four fixed points in the $(\mathcal{A}_+, \mathcal{A}_-)$ plane: one isotropic solution, $(0, 0)$, and three anisotropic solutions, $(-1, 0)$ and $(1/2, \pm\sqrt{3}/2)$. All of them are attractors (as long as $|\tau|$ is an increasing function of t).

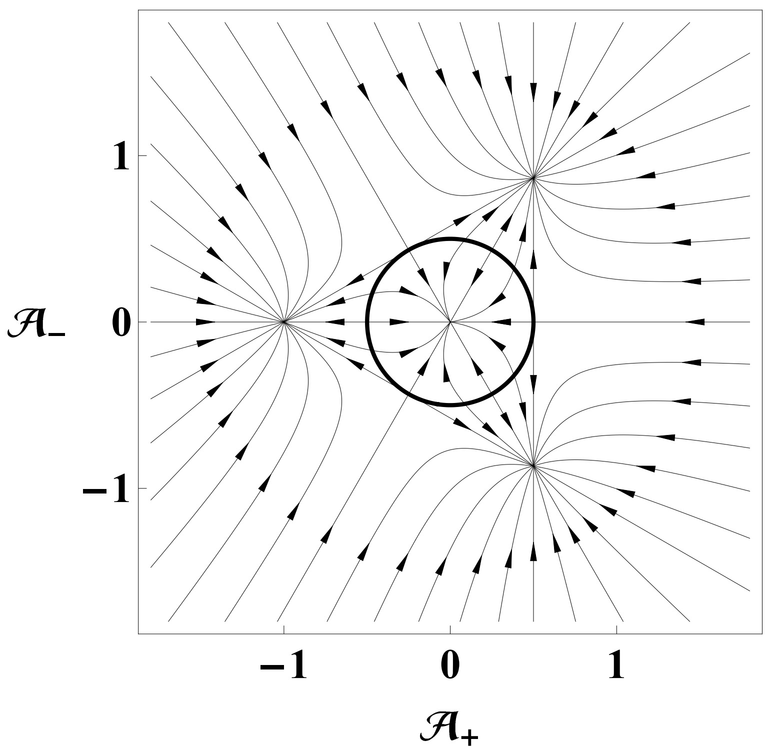

Figure 3.1: Trajectories of the evolution of the normalized shear $(\mathcal{A}_+, \mathcal{A}_-)$. If the initial conditions lie inside the circle given by $\mathcal{A}_+^2 + \mathcal{A}_-^2 = 1/4$, the universe evolves toward the center, $(\mathcal{A}_+, \mathcal{A}_-) = (0, 0)$, as τ increases. If the universe starts from outside of the circle, it goes to the closest one of the anisotropic fixed points on the vertices, $(\mathcal{A}_+, \mathcal{A}_-) = (-1, 0), (1/2, \pm\sqrt{3}/2)$, of the triangle as τ increases.

The initial anisotropies determine which attractor the universe approaches. To see this explicitly, we differentiate Eqs. (3.19) and (3.20) and get

$$\tau \frac{d\mathcal{A}_+}{d\tau} = -\frac{(2\mathcal{A}_+ - 1)(\mathcal{A}_+^2 + \mathcal{A}_-^2 + \mathcal{A}_+)}{4\mathcal{A}_+^2 + 4\mathcal{A}_-^2 - 1}, \tag{3.21}$$

$$\tau \frac{d\mathcal{A}_-}{d\tau} = -\frac{\mathcal{A}_-(2\mathcal{A}_+^2 + 2\mathcal{A}_-^2 - 2\mathcal{A}_+ - 1)}{4\mathcal{A}_+^2 + 4\mathcal{A}_-^2 - 1}. \tag{3.22}$$

Equivalently, one may introduce the polar coordinates $(r(\tau), \theta(\tau))$ defined by $\mathcal{A}_+ = r\cos\theta$ and $\mathcal{A}_- = r\sin\theta$ and write

$$\tau \frac{dr}{d\tau} = -\frac{r[2r^2 + r\cos(3\theta) - 1]}{4r^2 - 1}, \tag{3.23}$$

$$\tau \frac{d\theta}{d\tau} = \frac{r\sin(3\theta)}{4r^2 - 1}. \tag{3.24}$$

The denominators vanish on a circle given by $r^2 = \mathcal{A}_+^2 + \mathcal{A}_-^2 = (1/2)^2$ (the black circle in Fig. 3.1).[1] The fate of the universe depends on whether the initial anisotropies are inside this circle or not: the universe is attracted toward the isotropic solution at the origin if the initial anisotropies lie inside the circle, while it goes away from the circle to the closest one of the anisotropic attractors if outside initially. That is to say, if the universe is sufficiently anisotropic initially, then it converges to the anisotropic attractor.

The exceptional case is the trajectories with $\theta = 0, 2\pi/3, 4\pi/3$. Those constant values of θ solve Eq. (3.24), while Eq. (3.23) leads to $r(\tau) = (\sqrt{C/|\tau| + 1} - 1)/2$, where C is an integration constant. Therefore, for all initial conditions on $\theta = 0, 2\pi/3, 4\pi/3$ the isotropic universe is the attractor.

The structure of Fig. 3.1 will be more transparent in terms of the polar coordinates. Equations (3.23) and (3.24) clearly show that there are discrete rotation symmetry $\theta \to \theta + 2\pi/3$ and reflection symmetry across $\theta = 0, 2\pi/3$, and $4\pi/3$ axes. Because of these symmetries only a sixth part of Fig. 3.1 is physically independent.

Each of the anisotropic attractors corresponds to an axially symmetric space, whose symmetry axis is the x, y or z axis. This axial symmetry is closely related to the degeneracy of the eigenvalues of Σ_i^j discussed in the previous section. The discrete rotation symmetry in the $(\mathcal{A}_+, \mathcal{A}_-)$ plane is the manifestation of the fact that one can always take, say, the z axis as the symmetry axis without loss of generality by a rotation of the spatial coordinates.

So far we have focused only on the shear evolution equations. This is sufficient for the purpose of seeing that the anisotropic fixed points do exist and for initial anisotropies larger than a certain threshold they are indeed the attractors. To determine the precise dynamics of the universe including the evolution of H and ϕ, one needs to solve the full set of the field equations (the trace and trace-free parts of the evolution equations as well as the constraint equation) consistently. In the next subsection we will show a numerical example obtained by solving all the equations consistently.

[1] The shear evolution equations become singular on this circle. However, if we consider the full phase space by taking into account the trace part of the evolution equation, we see that this singularity is only apparent.

It has been pointed out by Wald that in general relativity, all vacuum Bianchi universes with a positive cosmological constant except type IX evolve toward the isotropic attractor, which was proven by using the Hamiltonian constraint and the trace of the Einstein equations [12]. In our case, since the Horndeski action dramatically changes both of them, it must be checked one by one whether a specific model under consideration evolves toward the isotropic or anisotropic attractor. We note that the magnitude of the shear on the anisotropic attractors diverges when we take the general relativity limit $A_5 \to 0$ keeping A_4 constant. In other words, the anisotropic attractors go to infinity in the (σ_+, σ_-) space. In this limit, for all initial conditions the isotropic universe is an attractor (as they are all inside the circle in Fig. 3.1), and thus the standard result of Wald in the general relativity is recovered.

Noting that the background anisotropies of the Bianchi type-I universe can be regarded as gravitational waves with infinitely long wavelengths, we point out that the emergence of anisotropic attractors is closely related to the three-point coupling of gravitational waves in the Horndeski theory. From Eq. (15) of [63], one sees that there are two types of the three-point couplings of the form $hh\partial^2 h$ and $\dot{h}\dot{h}\dot{h}$, giving rise to local and equilateral non-Gaussianity, respectively. The former appears even in general relativity as well as in a generic scalar-tensor theory, while the latter, which obviously comes from K_{ij}^3, emerges only in the class with $A_5 \neq 0$ (i.e., $G_{5X} \neq 0$). The former has spatial derivatives and therefore vanishes in the long-wavelength limit, whereas the latter has only time derivatives and hence does not vanish even in the homogeneous limit.

3.2.1 Examples

Let us present some examples which yield self-anisotropizing Bianchi type-I solutions. The first one is simply given by

$$G_2 = -V_0, \quad G_3 = 0, \quad G_4 = \frac{M^2}{2} + g_4 X, \quad G_5 = g_5 X, \tag{3.25}$$

where V_0, M, g_4, and g_5 are constants. The corresponding ADM form in the unitary gauge is given by

$$\begin{aligned} &A_2 = -V_0, \quad A_3 = 0, \quad A_4 = -\frac{M^2}{2} + \frac{g_4}{2}\left(\frac{\dot{\phi}}{N}\right)^2, \quad A_5 = \frac{g_5}{6}\left(\frac{\dot{\phi}}{N}\right)^3, \\ &B_4 = \frac{M^2}{2} + \frac{g_4}{2}\left(\frac{\dot{\phi}}{N}\right)^2, \quad B_5 = -\frac{g_5}{3}\left(\frac{\dot{\phi}}{N}\right)^3. \end{aligned} \tag{3.26}$$

Figure 3.2 shows the evolution of the Hubble parameter, (the velocity of) the scalar field, and the shear obtained by solving the dynamical and constraint equations numerically with a certain initial condition away from the attractors at $a = 1$. The parameters in this toy example are given by $V_0 = 0.1$, $M = 1$, $g_4 = -0.2$, and $g_5 = 1$. It can be seen that the universe quickly converges to the anisotropic inflationary attractor.

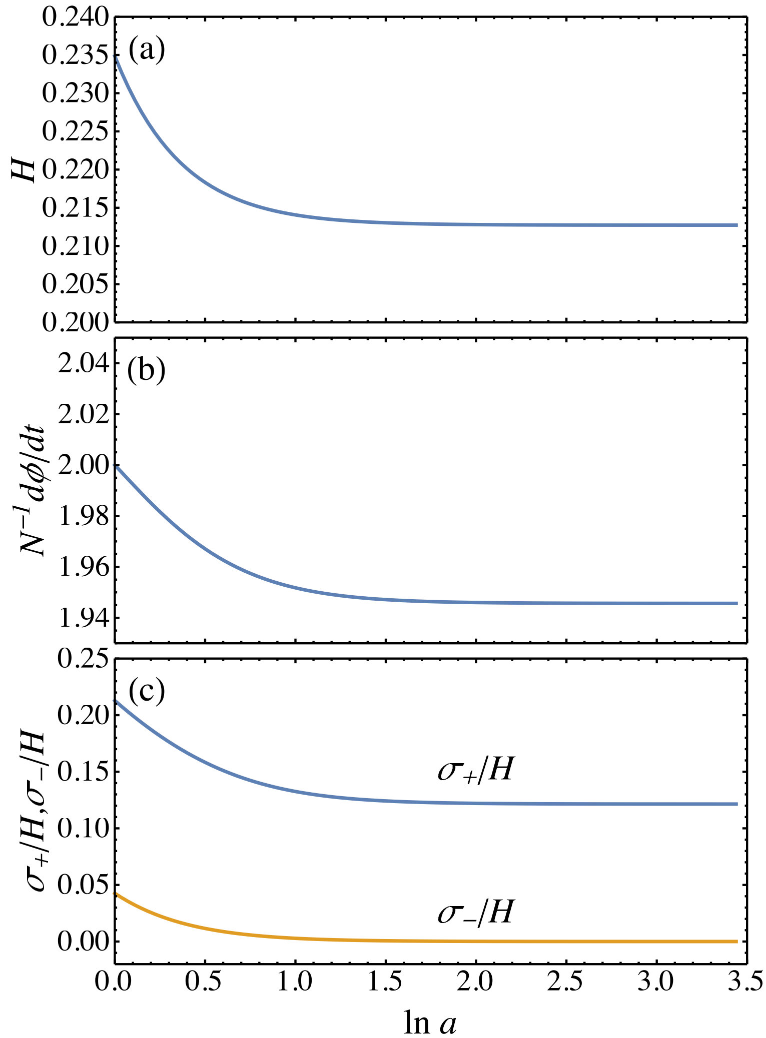

Figure 3.2: Numerical example of a self-anisotropizing Bianchi type-I universe: (a) H; (b) $\dot{\phi}/N$; (c) $\sigma_\pm/H$ as functions of $\ln a$.

Another example with A_5 (or, equivalently, G_{5X}) is the Gauss-Bonnet term coupled to a scalar field, and the total Lagrangian is of the form

$$\mathcal{L} = f(\phi)\mathcal{R} + P(\phi, X) + \xi(\phi)\left(\mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\right). \tag{3.27}$$

Aspects of this theory has been studied extensively in the literature. The Lagrangian can be reproduced by taking the following Horndeski functions [18]:

$$\begin{aligned} &G_2 = P + 8\xi^{(4)}X^2(3 - \ln X), \quad G_3 = 4\xi^{(3)}X(7 - 3\ln X), \\ &G_4 = f + 4\xi^{(2)}X(2 - \ln X), \quad G_5 = -4\xi^{(1)}\ln X, \end{aligned} \tag{3.28}$$

where $\xi^{(n)} = d^n\xi/d\phi^n$. Though this looks quite non-trivial, the corresponding ADM form is very simple:

$$A_2 = P, \quad A_3 = -2\frac{\dot{\phi}}{N}\frac{df}{d\phi}, \quad A_4 = -f, \quad A_5 = -\frac{4\xi^{(1)}}{3}\frac{\dot{\phi}}{N}, \quad B_4 = f, \quad B_5 = 8\xi^{(1)}\frac{\dot{\phi}}{N}. \tag{3.29}$$

Even this familiar theory admits self-anisotropizing inflationary solutions.

The theory (3.27) possesses a shift symmetry if $f = \text{const}$, $P = P(X)$, and $\xi \propto \phi$. In this case it is easy to find an inflationary solution with $H = \text{const}$, $\dot{\phi}/N = \text{const}$ retaining the nonvanishing shear

$$\frac{\sigma_\pm}{H} \sim \frac{f + 4H\xi^{(1)}\dot{\phi}/N}{H\xi^{(1)}\dot{\phi}/N}. \tag{3.30}$$

3.3 Axial symmetry of the anisotropic attractors

In addition to the action for the gravitational sector described in (2.35), we include the action for matter minimally coupled to gravity, $S_{\rm m}$. By the use of the residual gauge degrees of freedom one can further impose $N^i = 0$. Then, we obtain the evolution equations from (2.35) as

$$\begin{aligned} T^i_j =& \frac{1}{N\sqrt{g}}\partial_t\left[\sqrt{g}\left\{A_3\delta^i_j + 2A_4(K\delta^i_j - K^i_j) + 3A_5[(K^2 - K^k_lK^l_k)\delta^i_j - 2(KK^i_j - K^i_kK^k_j)]\right\}\right] \\ &- \delta^i_j\mathcal{L}_A + \left(2B_4 + \frac{\partial_t B_5}{N}\right)\left(R^i_j - \frac{1}{2}\delta^j_i R\right) + \Phi^i_j, \end{aligned} \tag{3.31}$$

where T_{ij} is the stress-energy tensor calculated from the matter action $S_{\rm m}$,

$$T_{ij} = -\frac{2}{N\sqrt{g}}\frac{\delta S_{\rm m}}{\delta g^{ij}}, \tag{3.32}$$

and $\mathcal{L}_A$ is the kinetic part of the Lagrangian,

$$\mathcal{L}_A = A_2 + A_3K + A_4(K^2 - K^i_jK^j_i) + A_5(K^3 - 3KK^i_jK^j_i + 2K^i_jK^j_kK^k_i). \tag{3.33}$$

We have collected the terms that vanish if the lapse function is homogeneous, $N(t,\vec{x}) = N(t)$, and written

$$\begin{aligned}
\Phi_{ij} &= \frac{2}{N}[\nabla^2(NB_4)g_{ij} - \nabla_i\nabla_j(NB_4)] \\
&+ g_{ij}K^{lm}\nabla_l\nabla_m B_5 + K\nabla_k\nabla_j B_5 - 2K^l_{(i}\nabla_{j)}\nabla_l B_5 + K_{ij}\nabla^2 B_5 - g_{ij}K\nabla^2 B_5 \\
&+ \frac{2}{N}\left[g_{ij}\nabla_l(NK^{lm})\nabla_m B_5 + \nabla_{(i}(NK)\nabla_{j)}B_5 - \nabla_l(NK^l_{(i})\nabla_{j)}B_5\right. \\
&\qquad \left. -\nabla_{(i}(NK^l_{j)})\nabla_l B_5 + \nabla_l(NK_{ij})\nabla^l B_5 - g_{ij}\nabla_l(NK)\nabla^l B_5\right]. \qquad (3.34)
\end{aligned}$$

The Hamiltonian constraint is given by

$$\begin{aligned}
&\partial_N(NA_2) + N\partial_N A_3 K + N^2\partial_N(N^{-1}A_4)(K^2 - K^i_j K^j_i) + \partial_N(NB_4)R \\
&+ N^3\partial_N(N^{-2}A_5)(K^3 - 3KK^i_jK^j_i + 2K^i_jK^j_kK^k_i) + N\partial_N B_5\left(R_{ij}K^{ij} - \frac{1}{2}RK\right) + \frac{1}{\sqrt{g}}\frac{\delta S_{\rm m}}{\delta N} = 0. \qquad (3.35)
\end{aligned}$$

In the following we will not use the momentum constraint equations.

We now show that even without any anisotropic matter sources the universe can exhibit anisotropic inflationary expansion as an attractor solution in the Horndeski theory.

Since we consider Bianchi cosmology, we may set $N^i = 0$. Thanks to the homogeneity, Φ_{ij} in the evolution equation (3.31) vanishes. To study anisotropic cosmological models it is convenient to decompose the extrinsic curvature K_{ij} into its trace K and trace-free part Σ_{ij} as

$$K_{ij} = \frac{1}{3}Kg_{ij} + \Sigma_{ij}, \qquad (3.36)$$

with $g^{ij}\Sigma_{ij} = 0$. The trace and trace-free parts of the evolution equation (3.31) read, respectively,

$$\frac{1}{N\sqrt{g}}\partial_t[\sqrt{g}(3A_3 + 4A_4K + A_5(2K^2 - 3\Sigma^i_j\Sigma^j_i))] - 3\mathcal{L}_A - \left(B_4 + \frac{\partial_t B_5}{2N}\right)R = T^i_i. \qquad (3.37)$$

and

$$\frac{2}{N\sqrt{g}}\partial_t\left[\sqrt{g}(-A_4\Sigma^i_j - A_5K\Sigma^i_j + 3A_5\{\Sigma^i_k\Sigma^k_j\}_{\rm TF})\right] + \left(2B_4 + \frac{\partial_t B_5}{N}\right)\{R^i_j\}_{\rm TF} = \{T^i_j\}_{\rm TF}, \qquad (3.38)$$

where $\{X^i_j\}_{\rm TF}$ stands for the trace-free part of a tensor X^i_j,

$$\{X^i_j\}_{\rm TF} = X^i_j - \frac{1}{3}X^k_k\delta^i_j. \qquad (3.39)$$

Let us look for slow-roll inflationary solutions in which $\sqrt{g}$ exponentially increases, while other functions remain either nearly constant or exponentially decrease. First, we focus on Eq. (3.38), assuming that the energy-momentum tensor consists of isotropic matter and hence $\{T^i_j\}_{\mathrm{TF}}$ vanishes. If the spatial curvature R^i_j decreases exponentially, the first term also decreases in the same way. As a result, we find, asymptotically,

$$-A_4\Sigma^i_j - A_5K\Sigma^i_j + 3A_5(\Sigma^i_k\Sigma^k_j - \frac{1}{3}\Sigma^k_l\Sigma^l_k\delta^i_j) = 0. \tag{3.40}$$

A trivial solution of Eq. (3.40) is that all components of Σ^i_j vanish. This solution corresponds to the isotropic attractor which we see in the conventional inflation models. The presence of the quadratic terms in Σ^i_j due to nonvanishing A_5 yields nontrivial solutions with $\Sigma^i_j \neq 0$ as well, which represent an expanding universe retaining finite anisotropies. We dub this anisotropic attractors as *self-anisotropizing* inflationary solutions, as this is *not* caused by an anisotropic energy-momentum tensor.

The self-anisotropizing attractors are distinct from the previous anisotropic inflationary solutions, because the anisotropic expansion of the previous scenarios are supported by some anisotropic energy-momentum source such as a vector field coupled with an inflaton field [15]. Such scenarios produce background anisotropies $\Sigma^i_j/H \approx \{T^i_j\}_{\mathrm{TF}}/(6A_4H^2) = (8\pi G/3H^2)\{T^i_j\}_{\mathrm{TF}}$, where H is the Hubble parameter. The trace-free part of the energy-momentum tensor, $\{T^i_j\}_{\mathrm{TF}}$, just displaces the terminal point from the isotropic one.

By contrast, here the self-anisotropizing inflationary solution is realized by the terms quadratic in Σ^i_j in Eq. (3.40), which is a consequence of modification of gravity. The magnitude of produced background anisotropies is estimated from (3.40) as $\Sigma^i_j/H \sim (A_4 + 3HA_5)/3HA_5$. We require neither an anisotropic energy-momentum tensor nor any fields other than the scalar ϕ built in the Horndeski theory. In this sense, the emerged anisotropic terminal points should be distinguished from those of previous anisotropic inflation models.

Let us evaluate the eigenvalues of the nontrivial solutions of Σ^i_j for given values of A_4, A_5 and K. We can prove that the root Σ of matrix equation (3.40) has two different eigenvalues at most as follows. First we define a polynomial $p(x)$ by substituting a real variable x for Σ in the left side of (3.40) as

$$p(x) = -A_4x - A_5Kx + 3A_5\left(x^2 - \frac{1}{3}\mathrm{tr}\left(\Sigma^2\right)\right), \tag{3.41}$$

where the remaining Σ in the trace is a root of (3.40). $p(\Sigma) = 0$ obviously follows from (3.40) and (3.41), and so $p(x)$ can be divided by the minimal polynomial $\phi_\Sigma(x)$ of Σ. In linear algebra, it is well-known that if λ is an eigenvalue of matrix Σ then λ is a root of $\phi_\Sigma(x) = 0$. Therefore, the eigenvalue λ is also a root of $p(x) = 0$. Since $p(x)$ is a quadratic polynomial of x, the number of different roots is equal to or less than two. This is the proof that Σ has two different eigenvalues, λ_1 and λ_2 at most. It induces that, *e.g.*, anisotropic attractors in Bianchi type-I model has axial symmetry in the order of background, which we show in Section 3.2. As one can see from (3.41), the different eigenvalues λ_1 and λ_2 satisfy

$$\lambda_1 + \lambda_2 = \frac{A_4 + A_5K}{3A_5}. \tag{3.42}$$

Being a three dimensional tensor, Σ has three eigenvalues. Without loss of generality, we set them as λ_1, λ_1 and λ_2, respectively. They also satisfy

$$2\lambda_1 + \lambda_2 = 0, \tag{3.43}$$

because Σ is trace-free. Therefore we have

$$\lambda_1 = -\frac{A_4 + A_5 K}{3A_5}, \qquad \lambda_2 = \frac{2(A_4 + A_5 K)}{3A_5}. \tag{3.44}$$

So far we have focused on the evolution equation for Σ^i_j (3.38) and its nontrivial solution under the assumption that the spatial volume element $\sqrt{g}$ increases exponentially and the spatial curvature R^i_j decreases accordingly. To determine all the components of the metric, we need to solve the Hamiltonian constraint (3.35) and the trace part of the evolution equations (3.37) consistently. On the anisotropic attractor where Σ^i_j's eigenvalues are given by (3.44), the rest of the field equations (3.35) and (3.37) are reduced to

$$\partial_N \left[N \left(A_2 - \frac{2A_4^3}{9A_5^2} \right) \right] + N\partial_N \left(A_3 - \frac{2A_4^2}{3A_5} \right) K = -\frac{1}{\sqrt{g}} \frac{\delta S_\mathrm{m}}{\delta N}, \tag{3.45}$$

$$\frac{1}{N}\frac{d}{dt} \left(A_3 - \frac{2A_4^2}{3A_5} \right) - \left(A_2 - \frac{2A_4^3}{9A_5^2} \right) = \frac{1}{3} T^i_i, \tag{3.46}$$

respectively. These two equations can be used to determine $K = K(t)$ and $N = N(t)$.

Let us ignore the matter field S_m for the moment and consider a theory with (approximate) shift symmetry. In this case, A_i's depend only on $\dot{\phi}/N$ and from Eq. (3.46) one obtains a solution $N \simeq \mathrm{const} \times \dot{\phi}$ satisfying $F(\dot{\phi}/N) \equiv A_2 - 2A_4^3/9A_5^2 \simeq 0$. Equation (3.45) is then solved to give $K \simeq -\partial_N(A_3 - 2A_4^2/3A_5)/\partial_N(A_2 - 2A_4^3/9A_5^2) \simeq \mathrm{const}$. One thus obtains an inflating solution with nonvanishing anisotropies.

3.4 Summary

Since $\mathcal{A}_\pm = \mathcal{O}(1)$ on the anisotropic attractors as is seen in Fig. 3.1, the magnitude of the resultant anisotropy is given by

$$\frac{\sigma_\pm}{H} \sim \frac{A_4 + 3HA_5}{3HA_5}, \tag{3.47}$$

which is typically of $\mathcal{O}(1)$ or larger. In theories with $A_4 \neq 0$ or $G_{5X} \neq 0$, initial anisotropies must be smaller than this value in order to realize an isotropic universe through inflation. Otherwise, the resultant universe would be unacceptably anisotropic. Another possibility is that one has $|A_4 + 3HA_5| \ll |3HA_5|$ via fine-tuning, leaving an observationally viable universe with only tiny anisotropies on the anisotropic attractor. This is a motivation to study higher-dimensional models in the context to show a new compactification mechanism of extra dimensions in the presence of the higher-order galileon terms in Chapter 6.

It is important to check the perturbative behavior around the anisotropic attractor as will be seen in Chapter 4. Since the mainly contributing term A_5 or G_{5X} changes the speed of gravitational waves, it has gotten the constraint as

$$\dot{\phi}^3 G_{5X} \sim \frac{|c_{\mathrm{GW}}^2 - 1|}{H_0} \lesssim 10^{45} M_{\mathrm{pl}}, \tag{3.48}$$

where H_0 is the present Hubble constant and M_{pl} is the Planck mass $M_{\mathrm{pl}} \equiv (8\pi G_{\mathrm{N}})^{-1/2}$. Note that since the energy scale of observation may be close to the cutoff scale of the model, the evaluation above should be taken into account carefully [64]. It is of great interest that the term G_{5X} gets constraint by other tests in which the strong field dynamics is seen. For this purpose, we develop the perturbations theory on static and spherically symmetric spacetime initiated by [21, 22], in Chapter 5.

Chapter 4

Perturbations on axially symmetric Bianchi type-I model

We have a good motivation to investigate perturbations in the anisotropic universe, since some class of generalized gravity theory induces anisotropic attractors, as discussed in the previous chapter, and anisotropic universe might realize in the cosmological history. In the present chapter, we aim to establish cosmological perturbation theory in the generalized Galileon or the Horndeski theory in a flat anisotropic background. The result can be used not only for the analysis of the perturbative behavior of the anisotropic attractors but also for the general analysis of the perturbations in any background of Bianchi type-I model. First we study perturbations with axial symmetry of the background because the symmetry enables us to decompose three degrees of freedom (two tensors and one scalar) into two decoupled system; One consists of the scalar mode and one of the polarizations of tensor modes; The other of the isolated remaining tensor mode. Next we apply it around the anisotropic attractors and explore the nature of the perturbations. Finally, we see that cosmic anisotropy in the generalized Galileon causes birefringence of gravitational waves, which is given rise to by the deviation between the speeds of the two polarization modes. Since our universe is very homogeneous, propagation over cosmic gravitational field cannot produce large birefringence, but it motivates us to work in strong gravitational fields around compact objects, such as black hole, which is investigated in the next chapter.

4.1 Classification of perturbations

Now we consider perturbations on the axially symmetric background

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -N^2(t)dt^2 + A^2(t)\left[dx^2 + dy^2\right] + B^2(t)dz^2, \tag{4.1}$$

We write perturbed metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ with perturbative variables:

$$h_{tt},\ h_{tz},\ h_{zz},\ h_{ta},\ h_{za},\ h_{ab},\ \delta\phi, \tag{4.2}$$

where $\delta\phi$ is deviation of the scalar field from $\phi(t)$ and the subscripts $a,\ b,\cdots$ means the directions x and y. h_{tt}, h_{tz}, h_{zz} and $\delta\phi$ have even parity under the inversion of x-y

plane:

$$x \to -x, \quad y \to -y. \tag{4.3}$$

We decompose the remaining variables h_{ta}, h_{za} and h_{ab} which have the subscripts $a, b, \cdots$ into:

$$h_{ta} = \partial_a h_0 + \epsilon_{ab}\partial_b \chi, \tag{4.4}$$

$$h_{za} = \partial_a h_1 + \epsilon_{ab}\partial_b \psi, \tag{4.5}$$

$$h_{ab} = h_2 \bar{g}_{ab} + \partial_a\partial_b h_3 + \frac{1}{2}\left(\epsilon_{ac}\partial_b\partial_c\gamma + \epsilon_{bc}\partial_a\partial_c\gamma\right), \tag{4.6}$$

where ϵ_{ab} is the totally antisymmetric symbol with $\epsilon_{xy} = -\epsilon_{yx} = 1$. Here h_0, h_1, h_2 and h_3 are even-parity variables and χ, ψ and γ are odd-parity ones.

Since we have decomposed the perturbative variables by the parity with respect to the inversion of the x-y plane, let us write gauge transformation with even/odd-parity infinitesimal transformation

$$t \to t + \delta t, \tag{4.7}$$

$$z \to z + \delta z, \tag{4.8}$$

$$x^a \to x^a + \bar{g}^{ab}(\partial_b \delta x + \epsilon_{bc}\partial_c \xi). \tag{4.9}$$

Then the perturbative variables transform as

$$\delta\phi \to \delta\phi - \dot{\phi}\delta t, \tag{4.10}$$

$$h_{tt} \to h_{tt} + 2\dot{\delta t} \tag{4.11}$$

$$h_{tz} \to h_{tz} - B^2\dot{\delta z} + \delta t', \tag{4.12}$$

$$h_{zz} \to h_{zz} - 2B^2\left(H_B\delta t + \delta z'\right), \tag{4.13}$$

$$h_0 \to h_0 + \delta t - A^2\dot{\delta x}, \tag{4.14}$$

$$\chi \to \chi - A^2\dot{\xi}, \tag{4.15}$$

$$h_1 \to h_1 - B^2\delta z + A^2\delta x', \tag{4.16}$$

$$\psi \to \psi - A^2\xi', \tag{4.17}$$

$$h_2 \to h_2 - 2H_A\delta t, \tag{4.18}$$

$$h_3 \to h_3 - 2\delta x, \tag{4.19}$$

$$\gamma \to \gamma - 2\xi, \tag{4.20}$$

where a prime denotes partial derivative with respect to z. Let us choose complete gauge fixing $\delta\phi = h_1 = h_3 = \gamma = 0$.

4.2 Odd-parity sector

Substituting the odd-parity variables to the Horndeski action (2.25), we get second-order action after redefinition $\psi \to B\psi$:

$$\mathcal{L}^{(2)} = \frac{\mathcal{G}_A}{4B}\left[(\partial\dot{\psi} - \partial\chi')^2 + 4H_A\partial\chi'\partial\psi\right] + \frac{1}{2A^2}\frac{d}{dt}\left[\frac{A^2H_A\mathcal{G}_A}{B}\right](\partial\psi)^2 + \frac{B\mathcal{G}_B}{4A^2}(\partial^2\chi)^2 - \frac{\mathcal{F}_{\rm T}}{4A^2B}(\partial^2\psi)^2, \tag{4.21}$$

where we define combinations of Horndeski's arbitrary functions

$$\mathcal{F}_{\mathrm{T}} = 2\left[G_4 - X\ddot{\phi}G_{5X} - XG_{5\phi}\right], \tag{4.22}$$

$$\mathcal{G}_A = 2\left[G_4 - 2XG_{4X} - XH_A\dot{\phi}G_{5X} + XG_{5\phi}\right], \tag{4.23}$$

$$\mathcal{G}_B = 2\left[G_4 - 2XG_{4X} - XH_B\dot{\phi}G_{5X} + XG_{5\phi}\right], \tag{4.24}$$

where $H_A \equiv \dot{A}/A$ and $H_B \equiv \dot{B}/B$. As can be seen in (4.21), χ has no time derivative and hence we find that the odd-parity sector has only one degree of freedom. To see this explicitly, we introduce an auxiliary variable Φ and rewrite (4.21) as

$$\mathcal{L}^{(2)} = \frac{1}{4A^2}\left[-\frac{B}{A^2\mathcal{G}_A}(\partial\Phi)^2 + 2\partial\Phi\left(\partial\dot{\psi} - \partial\chi' - 2H_A\partial\psi\right)\right] + \frac{B\mathcal{G}_B}{4A^2}(\partial^2\chi)^2 - \frac{\mathcal{F}_{\mathrm{T}}}{4A^2B}(\partial^2\psi)^2. \tag{4.25}$$

It is easy to confirm that (4.25) is equivalent to (4.21) after substituting solution of Φ in (4.25) [1]. From (4.25), we get

$$\partial^2\psi = \frac{B}{\mathcal{F}_T}\dot{\Phi}, \tag{4.26}$$

$$\partial^2\chi = \frac{1}{B\mathcal{G}_B}\Phi', \tag{4.27}$$

and after eliminating χ and ψ we finally get

$$\mathcal{L}^{(2)} = \frac{B}{4A^2\mathcal{F}_{\mathrm{T}}}\dot{\Phi}^2 + \frac{B}{4A^4\mathcal{G}_A}\Phi\partial^2\Phi + \frac{1}{4A^2B\mathcal{G}_B}\Phi\Phi''. \tag{4.28}$$

Propagation speed of the odd-parity sector is read immediately in (4.28) as:

$$c_\perp^2 = \frac{\mathcal{F}_{\mathrm{T}}}{\mathcal{G}_A}, \qquad c_z^2 = \frac{\mathcal{F}_{\mathrm{T}}}{\mathcal{G}_B}. \tag{4.29}$$

This shows that $\mathcal{G}_A \neq \mathcal{G}_B$ leads anisotropic propagation speed. As shown in (4.23) and (4.24), non-vanishing G_{5X} is necessary for $\mathcal{G}_A \neq \mathcal{G}_B$. Thus in anisotropically expanding universe with $G_{5X} \neq 0$, odd-parity perturbations, of which isotropic limit is tensor perturbations or gravitational waves, propagate with different speed along different spatial directions.

[1] The similar definition of Ψ is seen in [65], which has studied perturbations of black hole but provides systematic way available even in anisotropic spacetime.

4.3 Even-parity sector

We substitute the perturbative metric and the expressions of the even-parity perturbations (4.4), (4.5) and (4.6) to the Horndeski action (2.25), and we get

$$\begin{aligned}\mathcal{L}^{(2)}_{\text{even}} =& -\frac{A^2B}{2}\dot{\bar{h}}_{zz}\left[\mathcal{G}_A\left(\dot{h}_2-\frac{1}{A^2}\partial^2h_0\right)+\Theta_{\bar{B}}h_{tt}\right]+\frac{A^2B}{2}\left(H_A-H_B\right)\mathcal{G}_A\bar{h}_{zz}\left(\dot{h}_2-\frac{1}{A^2}\partial^2h_0\right)\\&+\frac{B}{4}\bar{h}_{zz}\left(\mathcal{G}_A\partial^2h_{tt}-\mathcal{F}_{\rm T}\partial^2h_2\right)+\frac{A^2}{B}h'_{tz}\left\{\mathcal{G}_A\left[\dot{h}_2+\left(H_A-H_B\right)h_2\right]+\Theta_{\bar{B}}h_{tt}\right\}\\&+\frac{\mathcal{G}_A}{4B}\left(\partial h_{tz}-\partial h'_0\right)^2+\frac{A^2B\Sigma}{4}h_{tt}^2-A^2B\Theta_{\bar{A}}h_{tt}\left(\dot{h}_2-\frac{1}{A^2}\partial^2h_0\right)-\frac{A^2\mathcal{G}_A}{2B}h'_{tt}h'_2\\&-\frac{A^2B\mathcal{G}_B}{4}\dot{h}_2^2+\frac{B\mathcal{G}_B}{2}\dot{h}_2\partial^2h_0+\frac{B\mathcal{G}_B}{4}h_{tt}\partial^2h_2+\frac{A^2\mathcal{F}_{\rm T}}{4B}{h'_2}^2,\end{aligned}\tag{4.30}$$

where $\bar{h}_{zz}=B^2h_{zz}$ and we have defined

$$\begin{aligned}\Sigma\equiv\,& X(G_{2X}+2XG_{2XX}-2G_{3\phi}-2XG_{3\phi X})\\&+2\left(2H_A+H_B\right)\dot{\phi}\left(2XG_{3X}+X^2G_{3XX}-G_{4\phi}-5XG_{4\phi X}-2X^2G_{4\phi XX}\right)\\&-2H_A\left(H_A+2H_B\right)\left(G_4-7XG_{4X}-16X^2G_{4XX}-4X^3G_{4XXX}\right)\\&-2H_A\left(H_A+2H_B\right)X\left(6G_{5\phi}+9XG_{5\phi X}+2X^2G_{5\phi XX}\right)\\&+2H_A^2H_B\dot{\phi}X\left(15G_{5X}+13XG_{5XX}+2X^2G_{5XXX}\right),\end{aligned}\tag{4.31}$$

$$\begin{aligned}\Theta_{\bar{A}}\equiv\,&-\dot{\phi}XG_{3X}+\dot{\phi}G_{4\phi}+2\dot{\phi}XG_{4\phi X}\\&+\left(H_A+H_B\right)\left(G_4-4XG_{4X}-4X^2G_{4XX}+3XG_{5\phi}+2X^2G_{5\phi X}\right)\\&-H_AH_B\dot{\phi}X\left(5G_{5X}+2XG_{5XX}\right),\end{aligned}\tag{4.32}$$

$$\begin{aligned}\Theta_{\bar{B}}\equiv\,&-\dot{\phi}XG_{3X}+\dot{\phi}G_{4\phi}+2\dot{\phi}XG_{4\phi X}\\&+2H_A\left(G_4-4XG_{4X}-4X^2G_{4XX}+3XG_{5\phi}+2X^2G_{5\phi X}\right)\\&-H_A^2\dot{\phi}X\left(5G_{5X}+2XG_{5XX}\right).\end{aligned}\tag{4.33}$$

Introducing an auxiliary variable Ψ and a Lagrange multiplier λ, we rewrite (4.30) as

$$\begin{aligned}\mathcal{L}^{(2)}_{\text{even}} =&\frac{1}{2}\bar{h}_{zz}\dot{\Psi}+\lambda\left[\Psi-A^2B\mathcal{G}_A\left(\dot{h}_2-\frac{1}{A^2}\partial^2h_0\right)-A^2B\Theta_Ah_{tt}\right]\\&+\frac{A^2B}{2}\left(H_A-H_B\right)\mathcal{G}_A\bar{h}_{zz}\left(\dot{h}_2-\frac{1}{A^2}\partial^2h_0\right)\\&+\frac{B}{4}\bar{h}_{zz}\left(\mathcal{G}_A\partial^2h_{tt}-\mathcal{F}_{\rm T}\partial^2h_2\right)+\frac{A^2}{B}h'_{tz}\left\{\mathcal{G}_A\left[\dot{h}_2+\left(H_A-H_B\right)h_2\right]+\Theta_{\bar{B}}h_{tt}\right\}\\&+\frac{\mathcal{G}_A}{4B}\left(\partial h_{tz}-\partial h'_0\right)^2+\frac{A^2B\Sigma}{4}h_{tt}^2-A^2B\Theta_{\bar{A}}h_{tt}\left(\dot{h}_2-\frac{1}{A^2}\partial^2h_0\right)-\frac{A^2\mathcal{G}_A}{2B}h'_{tt}h'_2\\&-\frac{A^2B\mathcal{G}_B}{4}\dot{h}_2^2+\frac{B\mathcal{G}_B}{2}\dot{h}_2\partial^2h_0+\frac{B\mathcal{G}_B}{4}h_{tt}\partial^2h_2+\frac{A^2\mathcal{F}_{\rm T}}{4B}{h'_2}^2.\end{aligned}\tag{4.34}$$

Now $\bar{h}_{zz}$ also acts as a Lagrange multiplier. Variations with respect to λ, $\bar{h}_{zz}$ and h_{tz} give

$$\Psi - A^2 B\mathcal{G}_A\left(\dot{h}_2 - \frac{1}{A^2}\partial^2 h_0\right) - A^2 B\Theta_{\bar{B}} h_{tt} = 0, \tag{4.35}$$

$$\dot{\Psi} + A^2 B\left(H_A - H_B\right)\mathcal{G}_A\left(\dot{h}_2 - \frac{1}{A^2}\partial^2 h_0\right) + \frac{B\mathcal{G}_A}{2}\partial^2 h_{tt} - \frac{B\mathcal{F}_{\rm T}}{2}\partial^2 h_2 = 0, \tag{4.36}$$

$$\mathcal{G}_A\left[\dot{h}_2' + \left(H_A - H_B\right)h_2'\right] + \Theta_{\bar{B}} h_{tt}' + \frac{\mathcal{G}_A}{2A^2}\left(\partial^2 h_{tz} - \partial^2 h_0'\right) = 0, \tag{4.37}$$

respectively. These three equations can be used to eliminate h_{tt}, h_{tz} and h_0 from the Lagrangian (4.34). We finally get the Lagrangian which depends only on Ψ and h_2. To write down the action, we substitute the Fourier modes $\tilde{\zeta} = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x} - ik_z z}\zeta$ for a perturbative variable ζ and we omit the tildes for simplicity. Here $\mathbf{k}$ and $\mathbf{x}$ denotes vectors (k_x, k_y) and (x, y), respectively. Let us represent final form of the Lagrangian as

$$\mathcal{L}^{(2)}_{\rm even} = \mathcal{K}_{ij}\dot{v}^i\dot{v}^j - A^{-2}\mathbf{k}^2\mathcal{O}_{ij}v^i v^j - B^{-2}k_z^2\mathcal{Z}_{ij}v^i v^j - B\mathcal{F}_T\mathcal{K}_{22}\dot{v}^2\partial^2 v^1, \tag{4.38}$$

where i and j run from 1 to 2, $v^1 \equiv h_2$ and $v^2 \equiv \Psi$. The coefficient matrices are given by

$$\mathcal{K}_{11} = \frac{A^2 B}{4}\mathcal{G}_B, \tag{4.39}$$

$$\mathcal{K}_{12} = -\frac{A^2\mathcal{G}_B\Theta_{\bar{B}}}{2\mathcal{D}^2}, \tag{4.40}$$

$$\mathcal{K}_{22} = \frac{A^2\mathcal{G}_A}{B\mathcal{D}^4}\left(\Sigma\mathcal{G}_A + 4\Theta_{\bar{A}}\Theta_{\bar{B}}\right), \tag{4.41}$$

$$\mathcal{Z}_{11} = \frac{A^2 B\mathcal{F}_{\rm T}}{4}, \tag{4.42}$$

$$\mathcal{Z}_{12} = -\frac{A^2\mathcal{F}_T\Theta_{\bar{B}}}{2\mathcal{D}^2}, \tag{4.43}$$

$$\mathcal{Z}_{22} = \frac{A^2\Theta_{\bar{B}}^2}{B^2\mathcal{D}^4}\frac{d}{dt}\left[\frac{B\mathcal{G}_A^2}{\Theta_{\bar{B}}}\right], \tag{4.44}$$

where $\mathcal{D}^2 \equiv \mathcal{G}_A\left[\mathcal{G}_A\mathbf{k}^2 - 2A^2\left(H_A - H_B\right)\Theta_{\bar{B}}\right]$. Ghost-free conditions are given by $\mathcal{K}_{11} > 0$ and $\det[\mathcal{K}_{ij}] > 0$. They are reduced to $\mathcal{G}_B > 0$ and

$$\det\mathcal{K}_{ij} = \frac{A^4\mathcal{G}_B\Theta_{\bar{B}}^2}{4\mathcal{D}^4}\left[\frac{\Sigma}{\Theta_{\bar{B}}^2}\mathcal{G}_A^2 + 4\mathcal{G}_A\frac{\Theta_{\bar{A}}}{\Theta_{\bar{B}}} - \mathcal{G}_B\right] > 0. \tag{4.45}$$

$\mathcal{O}_{ij}$ is relatively complicated but what we need here is dispersion relation. Now we take large $\mathbf{k}^2$ limit and we get

$$\mathcal{O}_{11} \approx \frac{A^2 B}{4} \frac{\mathcal{G}_B \mathcal{F}_{\mathrm{T}}}{\mathcal{G}_A}, \tag{4.46}$$

$$\mathcal{O}_{12} \approx \left\{ -\frac{A^2}{2} \frac{\Theta_{\bar{A}} \mathcal{F}_{\mathrm{T}}}{\mathcal{G}_A^2} + \frac{A^2}{4} \frac{d}{dt}\left[\frac{B\mathcal{G}_B}{A\mathcal{G}_A}\right] \right\} \frac{1}{\mathbf{k}^2}, \tag{4.47}$$

$$\mathcal{O}_{22} \approx \frac{\Theta_{\bar{A}}^2}{\mathcal{G}_A^4} \frac{d}{dt}\left[\frac{A^2 \mathcal{G}_A^2}{B\Theta_{\bar{A}}}\right] \frac{1}{\mathbf{k}^4}. \tag{4.48}$$

Dispersion relation of even-parity perturbations is given by eigenvalues of the matrix $\mathcal{K}^{-1}\left(A^{-2}\mathcal{O}\mathbf{k}^2 + B^{-2}\mathcal{Z}k_z^2\right)$. It is reduced to

$$\omega_{\mathrm{even}}^2 = \frac{1}{2}\left\{ \frac{k_z^2}{B^2}\left(\frac{\mathcal{F}_{\mathrm{T}}}{\mathcal{G}_B} + \frac{\mathcal{F}_{\mathrm{S}}^z}{\mathcal{G}_{\mathrm{S}}^z}\right) + \frac{\mathbf{k}^2}{A^2}\left(\frac{\mathcal{F}_{\mathrm{T}}}{\mathcal{G}_A} + c_\perp^2 + \mathcal{M}\right) \right.$$
$$\left. \pm \sqrt{\frac{\mathbf{k}^4}{A^4}\frac{\mathcal{G}_{\mathrm{S}}^z}{\mathcal{G}_B}\mathcal{M}^2 + \left[\frac{k_z^2}{B^2}\left(\frac{\mathcal{F}_{\mathrm{T}}}{\mathcal{G}_B} - \frac{\mathcal{F}_{\mathrm{S}}^z}{\mathcal{G}_{\mathrm{S}}^z}\right) + \frac{\mathbf{k}^2}{A^2}\left(\frac{\mathcal{F}_{\mathrm{T}}}{\mathcal{G}_A} - c_\perp^2 - \mathcal{M}\right)\right]^2}\right\}, \tag{4.49}$$

where

$$\mathcal{G}_{\mathrm{S}}^z \equiv \frac{\Sigma}{\Theta_{\bar{B}}^2}\mathcal{G}_A^2 + 4\mathcal{G}_A\frac{\Theta}{\Theta_{\bar{B}}} - \mathcal{G}_B \tag{4.50}$$

$$\mathcal{F}_{\mathrm{S}}^z \equiv \frac{1}{B}\frac{d}{dt}\left[\frac{B\mathcal{G}_A^2}{\Theta_{\bar{B}}}\right] - \mathcal{F}_{\mathrm{T}} \tag{4.51}$$

$$c_\perp^2 \equiv \frac{\Theta^2}{V}\left(\frac{B}{A^2}\frac{d}{dt}\left[\frac{A^2\mathcal{G}_A^2}{B\Theta}\right] - \frac{\mathcal{G}_B\Theta_{\bar{B}}^2}{\mathcal{G}_A\Theta^2}\mathcal{F}_{\mathrm{T}}\right) \tag{4.52}$$

$$\mathcal{M} \equiv \frac{\mathcal{G}_A\mathcal{G}_B\Theta_{\bar{B}}}{V}\left(\frac{2\mathcal{F}_{\mathrm{T}}}{\mathcal{G}_A}\left(\frac{\Theta_{\bar{B}}}{\mathcal{G}_A} - \frac{\Theta_{\bar{A}}}{\mathcal{G}_B}\right) - \left(H_A - H_B + \dot{\mathcal{G}}_A/\mathcal{G}_A - \dot{\mathcal{G}}_B/\mathcal{G}_B\right)\right) \tag{4.53}$$

$$V \equiv \Sigma\mathcal{G}_A^2 + 4\mathcal{G}_A\Theta_{\bar{A}}\Theta_{\bar{B}} - \mathcal{G}_B\Theta_{\bar{B}}^2 = \mathcal{G}_{\mathrm{S}}^z\Theta_{\bar{B}}^2. \tag{4.54}$$

When we set $G_4 = f(\phi)$ and $G_5 = 0$, then they lead

$$\mathcal{G}_{\mathrm{T}} \equiv \mathcal{G}_A = \mathcal{G}_B = \mathcal{F}_{\mathrm{T}} = 2f(\phi), \tag{4.55}$$

$$\mathcal{M} = \frac{\Theta_{\bar{B}}}{V}(2\mathcal{F}_{\mathrm{T}}(\Theta_{\bar{B}} - \Theta) - \mathcal{G}_{\mathrm{T}}^2(H_A - H_B)), \tag{4.56}$$

$$= \frac{\Theta_{\bar{B}}}{V}\left(2\mathcal{F}_{\mathrm{T}}(H_A - H_B)f(\phi) - \mathcal{G}_{\mathrm{T}}^2(H_A - H_B)\right) = 0. \tag{4.57}$$

When we set $G_{5X} = 0$, then we get

$$\mathcal{G}_\mathrm{T} \equiv \mathcal{G}_A = \mathcal{G}_B = 2[G_4 - 2XG_{4X} + XG_{5\phi}], \tag{4.58}$$

$$\mathcal{F}_\mathrm{T} = 2[G_4 - XG_{5\phi}], \tag{4.59}$$

$$\Gamma \equiv 2(\Theta_{\bar{B}} - \Theta)/(H_A - H_B) = 2[G_4 - 4XG_{4X} - 4X^2G_{4XX} + 3XG_{5\phi}], \tag{4.60}$$

$$\mathcal{M} = \frac{\Theta_{\bar{B}}}{V}(H_A - H_B)(\mathcal{F}_\mathrm{T}\Gamma - \mathcal{G}_\mathrm{T}^2) \tag{4.61}$$

$$= -16\frac{\Theta_{\bar{B}}}{V}(H_A - H_B)X^2\left[(G_4 - XG_{5\phi})G_{4XX} + G_{4X}^2 + G_{5\phi}^2\right]. \tag{4.62}$$

Note that the expression of Γ is similar to the definition in [66]. It suggests that coupling terms such as $\zeta\dot{h}^2$ contribute to the mixing of scalar and gravitaional waves.

4.4 Perturbations dependent only on z

We note that if perturbations does not depend on the directions x and y, the original second-order action (4.21) vanishes since all the terms have ∂. In that case, we cannot use (4.4), (4.5) and (4.6) to represent the perturbations, but we should use another representation for perturbations which is transverse and traceless in the directions x and y.

$$h_{ab} = A^2 h_{ab}^{(\mathrm{TT})}, \tag{4.63}$$

where $h_{aa}^{(\mathrm{TT})} = 0 = \partial_a h_{ab}^{(\mathrm{TT})}$ is satisfied. Its second-order action is

$$\mathcal{L}^{(2)} = A^2 B \frac{1}{8}\left[\mathcal{G}_B \dot{h}_{ab}^{(\mathrm{TT})2} - B^{-2}\mathcal{F}_\mathrm{T} h'^{(\mathrm{TT})2}_{ab}\right]. \tag{4.64}$$

As one can see, $h_{ab}^{(\mathrm{TT})}$ propagates only to the direction z and its speed is same as c_z in (4.29). Since (4.64) has the same dispersion relation as that of (4.28), we mainly use (4.28) afterward but attention is needed when we restore original expressions in terms of the metric.

4.5 Case without axial symmetry

Now we have the metric below for the background

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}dx^idx^j, \tag{4.65}$$

where we define the spatial metric

$$\gamma_{ij} = \mathrm{diag}(A^2, B^2, C^2). \tag{4.66}$$

The Hamiltonian constraint $\mathcal{E}_N = 0$ is given by variation with respect to N.

$$\begin{aligned}
\mathcal{E}_N =& G_2 - \dot{\phi}^2 G_{2X} + \dot{\phi}^2 G_{3\phi} - (H_A + H_B + H_C)\dot{\phi}^3 G_{3X} + 2(H_A H_B + H_B H_C + H_C H_A) G_4 \\
&+ 2(H_A + H_B + H_C)\dot{\phi} G_{4\phi} - 4(H_A H_B + H_B H_C + H_C H_A)\dot{\phi}^2 G_{4X} \\
&+ 2(H_A + H_B + H_C)\dot{\phi}^3 G_{4\phi X} - 2(H_A H_B + H_B H_C + H_C H_A)\dot{\phi}^4 G_{4XX} \\
&+ 3(H_A H_B + H_B H_C + H_C H_A)\dot{\phi}^2 G_{5\phi} - 5 H_A H_B H_C \dot{\phi}^3 G_{5X} \\
&+ (H_A H_B + H_B H_C + H_C H_A)\dot{\phi}^4 G_{5\phi X} - H_A H_B H_C \dot{\phi}^5 G_{5XX}.
\end{aligned} \tag{4.67}$$

Evolution equations $\mathcal{E}_A = 0$ is given by variation with respect to A.

$$\begin{aligned}
\mathcal{E}_A =& G_2 - \dot{\phi}^2 G_{3\phi} - \dot{\phi}^2 \ddot{\phi} G_{3X} + 2\left(H_B H_C + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) G_4 \\
&+ 2(\ddot{\phi} + (H_B + H_C)\dot{\phi}) G_{4\phi} - 2\left((H_B + H_C)\dot{\phi}\ddot{\phi} + \left(H_B H_C + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right)\dot{\phi}^2\right) G_{4X} \\
&+ 2\dot{\phi}^2 G_{4\phi\phi} + 2\dot{\phi}^2(\ddot{\phi} - (H_B + H_C)\dot{\phi}) G_{4\phi X} - 2(H_B + H_C)\dot{\phi}^3 \ddot{\phi} G_{4XX} \\
&+ \left(2(H_B + H_C)\dot{\phi}\ddot{\phi} + \left(H_B H_C + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right)\ddot{\phi}\right) G_{5\phi} \\
&- (3 H_B H_C \dot{\phi}^2 \ddot{\phi} + (H_C \frac{\ddot{B}}{B} + H_B \frac{\ddot{C}}{C})\dot{\phi}^3) G_{5X} + (H_B + H_C)\dot{\phi}^3 G_{5\phi\phi} \\
&+ ((H_B + H_C)\dot{\phi}^3 \ddot{\phi} - H_B H_C \dot{\phi}^4) G_{5\phi X} - H_B H_C \dot{\phi}^4 \ddot{\phi} G_{5XX}.
\end{aligned} \tag{4.68}$$

The other evolution equations of B and C are given by circulation of the expression of $\mathcal{E}_A$. Equation of motion of ϕ is given by linear combination of $\mathcal{E}_N$, $\mathcal{E}_A$, $\mathcal{E}_B$, and $\mathcal{E}_C$.

We define perturbative variables

$$\delta g_{\mu\nu} dx^\mu dx^\nu = \alpha^2 dt^2 + 2\beta_i dt dx^i + H_{ij} dx^i dx^j. \tag{4.69}$$

Thanks to the freedom of the gauge, we set $H_{12} = H_{23} = H_{31} = 0$. The second-order Lagrangian is written as

$$S^{(2)} = \int dt d^3x \mathcal{L}^{(2)}, \tag{4.70}$$

$$\begin{aligned}
\mathcal{L}^{(2)} =& -\frac{A\Sigma_A H_{22} H_{33}}{4BC} - \frac{B\Sigma_B H_{33} H_{11}}{4CA} - \frac{C\Sigma_C H_{11} H_{22}}{4AB} \\
&+ \frac{C H_B \mathcal{G}_C \dot{H}_{11} H_{22}}{4AB} + \frac{B H_C \mathcal{G}_B \dot{H}_{11} H_{33}}{4CA} + \frac{A H_C \mathcal{G}_A \dot{H}_{22} H_{33}}{4BC} \\
&+ \frac{C H_A \mathcal{G}_C \dot{H}_{22} H_{11}}{4AB} + \frac{B H_A \mathcal{G}_B \dot{H}_{33} H_{11}}{4CA} + \frac{A H_B \mathcal{G}_A \dot{H}_{33} H_{22}}{4BC} \\
&- \frac{A \mathcal{G}_A \dot{H}_{22} \dot{H}_{33}}{4BC} - \frac{B \mathcal{G}_B \dot{H}_{33} \dot{H}_{11}}{4CA} - \frac{C \mathcal{G}_C \dot{H}_{11} \dot{H}_{22}}{4AB} \\
&+ \frac{\mathcal{F}_{\mathrm{T}} \partial_1 H_{22} \partial_1 H_{33}}{4ABC} + \frac{\mathcal{F}_{\mathrm{T}} \partial_2 H_{33} \partial_2 H_{11}}{4ABC} + \frac{\mathcal{F}_{\mathrm{T}} \partial_3 H_{11} \partial_3 H_{22}}{4ABC}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C\mathcal{G}_C\partial_1\beta_1}{2AB}\left(\dot{H}_{22}-(H_A+H_B)H_{22}\right)+\frac{B\mathcal{G}_B\partial_1\beta_1}{2AC}\left(\dot{H}_{33}-(H_A+H_C)H_{33}\right)\\
&+\frac{A\mathcal{G}_A\partial_2\beta_2}{2BC}\left(\dot{H}_{33}-(H_B+H_C)H_{33}\right)+\frac{C\mathcal{G}_C\partial_2\beta_2}{2BA}\left(\dot{H}_{11}-(H_B+H_A)H_{11}\right)\\
&+\frac{B\mathcal{G}_B\partial_3\beta_3}{2CA}\left(\dot{H}_{11}-(H_C+H_A)H_{11}\right)+\frac{A\mathcal{G}_A\partial_3\beta_3}{2CB}\left(\dot{H}_{22}-(H_C+H_B)H_{22}\right)\\
&+\frac{A\mathcal{G}_A(\partial_2\beta_3-\partial_3\beta_1)^2}{4BC}+\frac{B\mathcal{G}_B(\partial_3\beta_1-\partial_1\beta_3)^2}{4CA}+\frac{C\mathcal{G}_C(\partial_1\beta_2-\partial_2\beta_1)^2}{4AB}\\
&+\frac{BCH_A\Theta_A\alpha H_{11}}{A}+\frac{CAH_B\Theta_B\alpha H_{22}}{B}+\frac{ABH_C\Theta_C\alpha H_{33}}{C}\\
&-\frac{BC\Theta_A\alpha\dot{H}_{11}}{2A}-\frac{CA\Theta_B\alpha\dot{H}_{22}}{2B}-\frac{AB\Theta_C\alpha\dot{H}_{33}}{2C}\\
&+\frac{A\mathcal{G}_A\alpha(\partial_2^2H_{33}+\partial_3^2H_{22})}{4BC}+\frac{B\mathcal{G}_B\alpha(\partial_3^2H_{11}+\partial_1^2H_{33})}{4CA}+\frac{C\mathcal{G}_C\alpha(\partial_3^2H_{11}+\partial_1^2H_{33})}{4AB}\\
&+\frac{BC\Theta_A\alpha\partial_1\beta_1}{A}+\frac{CA\Theta_B\alpha\partial_2\beta_2}{B}+\frac{AB\Theta_C\alpha\partial_3\beta_3}{C}+\frac{\Sigma}{4}ABC\alpha^2,
\end{aligned}
\tag{4.71}
$$

where we ignore total derivatives and we simplify the expressions with the background equations $\mathcal{E}_N=0$, $\mathcal{E}_A=0$, $\mathcal{E}_B=0$, $\mathcal{E}_C=0$. We have defined several functions

$$
\Sigma=\frac{1}{2}\dot{\phi}^2G_{2X}, \tag{4.72}
$$

$$
\begin{aligned}
\Sigma_A=&G_2-\dot{\phi}^2G_{3\phi}-\dot{\phi}^2\ddot{\phi}G_{3X}+2\left(\frac{\ddot{A}}{A}-H_BH_C\right)G_4+2(\ddot{\phi}+H_A\dot{\phi})G_{4\phi}\\
&-2\dot{\phi}\left(H_A\ddot{\phi}+\frac{\ddot{A}}{A}\dot{\phi}-H_BH_C\dot{\phi}\right)G_{4X}+2\dot{\phi}^2G_{4\phi\phi}+2\dot{\phi}^2(\ddot{\phi}-H_A\dot{\phi})G_{4\phi X}-2H_A\dot{\phi}^3\ddot{\phi}G_{4XX}\\
&+\dot{\phi}\left(2H_A\ddot{\phi}+\frac{\ddot{A}}{A}\dot{\phi}-H_BH_C\dot{\phi}\right)G_{5\phi}+H_AH_BH_C\dot{\phi}^3G_{5X}+H_A\dot{\phi}^3G_{5\phi\phi}+H_A\dot{\phi}^3\ddot{\phi}G_{5\phi X},
\end{aligned}
\tag{4.73}
$$

$$
\begin{aligned}
\Theta_A=&-\frac{1}{2}\dot{\phi}^3G_{3X}+(H_B+H_C)G_4+\dot{\phi}G_{4\phi}-2(H_B+H_C)\dot{\phi}^2G_{4X}+\dot{\phi}^3G_{4\phi X}-(H_B+H_C)\dot{\phi}^4G_{4\phi\phi}\\
&+\frac{3}{2}(H_B+H_C)\dot{\phi}^2G_{5\phi}-\frac{5}{2}H_BH_C\dot{\phi}^3G_{5X}+\frac{1}{2}(H_B+H_C)\dot{\phi}^4G_{5\phi X}-\frac{1}{2}H_BH_C\dot{\phi}^5G_{5XX},
\end{aligned}
\tag{4.74}
$$

$$
\mathcal{G}_A=2G_4-2\dot{\phi}^2G_{4X}+\dot{\phi}^2G_{5\phi}-H_A\dot{\phi}^3G_{5X}, \tag{4.75}
$$

$$
\mathcal{F}_{\mathrm{T}}=2G_4-\dot{\phi}^2G_{5\phi}-\dot{\phi}^2\ddot{\phi}G_{5X}. \tag{4.76}
$$

Σ_B, Σ_C, Θ_B, Θ_C, $\mathcal{G}_B$, and $\mathcal{G}_C$ are given by the circulation of their correspondence Σ_A, Θ_A, and $\mathcal{G}_A$, respectively. We define new variables as

$$
\delta_A=\frac{H_{11}}{A},\quad \delta_B=\frac{H_{22}}{B},\quad \delta_C=\frac{H_{33}}{C}, \tag{4.77}
$$

$$
\chi_A=\partial_1^{-1}\beta_1,\quad \chi_B=\partial_2^{-1}\beta_2,\quad \chi_C=\partial_3^{-1}\beta_3. \tag{4.78}
$$

Then the action with the new variables

$$
\begin{aligned}
\mathcal{L}^{(2)} =& -\frac{1}{4}A\Sigma_A\delta_B\delta_C - \frac{1}{4}B\Sigma_B\delta_C\delta_A - \frac{1}{4}C\Sigma_C\delta_A\delta_B \\
&- \frac{1}{4}A\mathcal{G}_A(\dot{\delta}_B\dot{\delta}_C - H_BH_C\delta_B\delta_C) - \frac{1}{4}B\mathcal{G}_C(\dot{\delta}_C\dot{\delta}_A - H_CH_A\delta_C\delta_A) \\
&- \frac{1}{4}C\mathcal{G}_C(\dot{\delta}_A\dot{\delta}_B - H_AH_B\delta_A\delta_B) \\
&+ \frac{\mathcal{F}_{\mathrm{T}}k_1^2}{4A}\delta_B\delta_C + \frac{\mathcal{F}_{\mathrm{T}}k_2^2}{4B}\delta_C\delta_A + \frac{\mathcal{F}_{\mathrm{T}}k_3^2}{4C}\delta_A\delta_B \\
&- \frac{k_1^2}{2A}\chi_A\left(C\mathcal{G}_C(\dot{\delta}_B - H_A\delta_B) + B\mathcal{G}_B(\dot{\delta}_C - H_A\delta_C)\right) \\
&- \frac{k_2^2}{2B}\chi_B\left(A\mathcal{G}_A(\dot{\delta}_C - H_B\delta_C) + C\mathcal{G}_C(\dot{\delta}_A - H_B\delta_A)\right) \\
&- \frac{k_3^2}{2C}\chi_C\left(B\mathcal{G}_B(\dot{\delta}_A - H_C\delta_A) + A\mathcal{G}_A(\dot{\delta}_B - H_C\delta_B)\right) \\
&+ \frac{k_2^2k_3^2}{4BC}A\mathcal{G}_A(\chi_B - \chi_C)^2 + \frac{k_3^2k_1^2}{4CA}B\mathcal{G}_B(\chi_C - \chi_A)^2 + \frac{k_1^2k_2^2}{4AB}C\mathcal{G}_C(\chi_A - \chi_B)^2 \\
&- \frac{k_1^2}{4A}\alpha(B\mathcal{G}_B\delta_C + C\mathcal{G}_C\delta_B) - \frac{k_2^2}{4B}\alpha(C\mathcal{G}_C\delta_A + A\mathcal{G}_A\delta_C) - \frac{k_3^2}{4C}\alpha(A\mathcal{G}_A\delta_B + B\mathcal{G}_B\delta_A) \\
&- \frac{1}{2}BC\Theta_A\alpha(\dot{\delta}_A - H_A\delta_A) - \frac{1}{2}CA\Theta_B\alpha(\dot{\delta}_B - H_B\delta_B) - \frac{1}{2}AB\Theta_C\alpha(\dot{\delta}_C - H_C\delta_C) \\
&- \frac{k_1^2}{A}BC\Theta_A\alpha\chi_A - \frac{k_2^2}{B}CA\Theta_B\alpha\chi_B - \frac{k_3^2}{C}AB\Theta_C\alpha\chi_C + \frac{\Sigma}{4}ABC\alpha^2,
\end{aligned}
\tag{4.79}
$$

where we have transformed all of the perturbative variables into their Fourier modes

$$
f(t, x^i) = \int \frac{d^3k}{(2\pi)^3} e^{ikx} f(t, k_i), \tag{4.80}
$$

The variation of the action with respect to χ_A and χ_B gives

$$
\frac{k_1^2}{2A}\Bigg((H_A\delta_B - \dot{\delta}_B)C\mathcal{G}_C + (H_A\delta_C - \dot{\delta}_C)B\mathcal{G}_B) + \frac{k_2^2}{B}C\mathcal{G}_C(\chi_A - \chi_B) \tag{4.81}
$$

$$
+\frac{k_3^2}{C}B\mathcal{G}_B(\chi_A - \chi_C) - 2\alpha BC\Theta_A\Bigg) = 0, \tag{4.82}
$$

$$
\frac{k_2^2}{2B}\Bigg((H_B\delta_C - \dot{\delta}_C)A\mathcal{G}_A + (H_B\delta_A - \dot{\delta}_A)C\mathcal{G}_C) + \frac{k_3^2}{C}A\mathcal{G}_A(\chi_B - \chi_C) \tag{4.83}
$$

$$
+\frac{k_1^2}{A}C\mathcal{G}_C(\chi_B - \chi_A) - 2\alpha CA\Theta_B\Bigg) = 0. \tag{4.84}
$$

Substituting these two into the action, we can eliminate χ_A and χ_B. Then χ_C act as a

Lagrange multiplier, and we get

$$\alpha = \Big[BCk_1^2(B\mathcal{G}_B(H_A\delta_C - \dot{\delta}_C) + C\mathcal{G}_C(H_A\delta_B - \dot{\delta}_B)) \tag{4.85}$$

$$CAk_2^2(C\mathcal{G}_C(H_B\delta_A - \dot{\delta}_A) + A\mathcal{G}_A(H_B\delta_C - \dot{\delta}_C)) \tag{4.86}$$

$$ABk_3^2(A\mathcal{G}_A(H_C\delta_B - \dot{\delta}_B) + B\mathcal{G}_B(H_C\delta_A - \dot{\delta}_A))\Big] \tag{4.87}$$

$$\left(2(B^2C^2\Theta_A k_1^2 + C^2A^2\Theta_B k_2^2 + A^2B^2\Theta_C k_3^2)\right)^{-1}. \tag{4.88}$$

After χ_C and α are eliminated, we finally get the action only with the variables δ_A, δ_B and δ_C.

We can write the action with matrices $\mathcal{K}$, $\mathcal{F}$, and $\mathcal{P}$ as

$$\mathcal{L}^{(2)} = \dot{\delta}_I\mathcal{K}_{IJ}\dot{\delta}_J + \delta_I\mathcal{F}_{IJ}\dot{\delta}_J - \delta_I\mathcal{P}_{IJ}\delta_J, \tag{4.89}$$

where I and J run A, B, and C. $\mathcal{K}_{IJ}$ and $\mathcal{P}_{IJ}$ are symmetric matrices and $\mathcal{F}_{IJ}$ is an antisymmetric matrix.

Variating the action with respect to δ_K, we get the equation of motion

$$2\frac{d}{dt}(\mathcal{K}_{KJ}\dot{\delta}_J) + \frac{d}{dt}(\delta_I\mathcal{F}_{IK}) = \mathcal{F}_{KJ}\dot{\delta}_J - 2\mathcal{P}_{KJ}\delta_J. \tag{4.90}$$

To get dispersion relation, we plug $\delta_I \propto e^{-i\omega t}$ into the equation of motion.

$$(\mathcal{K}_{KJ}\omega^2 - \mathcal{P}_{KJ})\delta_J = 0. \tag{4.91}$$

In order to obtain nontrivial δ_J, the matrix of coefficient must not be regular.

$$\det\left[\mathcal{K}_{IJ}\omega^2 - \mathcal{P}_{IJ}\right] = 0, \tag{4.92}$$

where we take the large wavenumber k_i limit, so that we can neglect $\mathcal{F}_{IJ}$ and time derivative of $\mathcal{K}_{IJ}$. The solutions for ω^2 are given by eigenvalues of $\mathcal{K}^{-1}\mathcal{P}$. The eigenvalues are given by the roots x of the cubic equation

$$a_0 + a_1x + a_2x^2 + x^3 = 0, \tag{4.93}$$

$$a_0 = \frac{a_1a_2}{3} - \frac{2a_2^3}{27} + \frac{4}{27}(8\mathcal{D}^2 + 9\mathcal{M}) \tag{4.94}$$

$$a_1 = \frac{a_2^2}{3} - \frac{1}{12}(4\mathcal{D}^2 + 3\mathcal{M}) \tag{4.95}$$

$$a_2 = -\mathcal{S} \tag{4.96}$$

where

$$\mathcal{S} = \frac{k_1^2}{A^2}\left(2\frac{\mathcal{F}_\mathrm{T}}{\mathcal{G}_A} + \frac{\mathcal{G}_A\Theta_A^2}{\Omega}(\mathcal{F}_{\mathrm{S}A} + M_{BC})\right) + \frac{k_2^2}{B^2}\left(2\frac{\mathcal{F}_\mathrm{T}}{\mathcal{G}_B} + \frac{\mathcal{G}_B\Theta_B^2}{\Omega}(\mathcal{F}_{\mathrm{S}B} + M_{CA})\right) + \frac{k_3^2}{C^2}\left(2\frac{\mathcal{F}_\mathrm{T}}{\mathcal{G}_C} + \frac{\mathcal{G}_C\Theta_C^2}{\Omega}(\mathcal{F}_{\mathrm{S}C} + M_{AB})\right), \tag{4.97}$$

$$\mathcal{D} = \frac{k_1^2}{A^2}\left(\frac{\mathcal{F}_\mathrm{T}}{\mathcal{G}_A} - \frac{\mathcal{G}_A\Theta_A^2}{\Omega}(\mathcal{F}_{\mathrm{S}A} + M_{BC})\right) + \frac{k_2^2}{B^2}\left(\frac{\mathcal{F}_\mathrm{T}}{\mathcal{G}_B} - \frac{\mathcal{G}_B\Theta_B^2}{\Omega}(\mathcal{F}_{\mathrm{S}B} + M_{CA})\right) + \frac{k_3^2}{C^2}\left(\frac{\mathcal{F}_\mathrm{T}}{\mathcal{G}_C} - \frac{\mathcal{G}_C\Theta_C^2}{\Omega}(\mathcal{F}_{\mathrm{S}C} + M_{AB})\right), \tag{4.98}$$

$$\mathcal{M}\Omega = \frac{k_1^4}{A^4}\Delta_{BC}^2 + \frac{k_2^4}{B^4}\Delta_{CA}^2 + \frac{k_3^4}{C^4}\Delta_{AB}^2 - 2\frac{k_1^2k_2^2}{A^2B^2}\Delta_{BC}\Delta_{CA} - 2\frac{k_2^2k_3^2}{B^2C^2}\Delta_{CA}\Delta_{AB} - 2\frac{k_3^2k_1^2}{C^2A^2}\Delta_{AB}\Delta_{BC}, \tag{4.99}$$

$$\Omega = \mathcal{G}_A\mathcal{G}_B\mathcal{G}_C\Sigma - \mathcal{G}_A^2\Theta_A^2 - \mathcal{G}_B^2\Theta_B^2 - \mathcal{G}_C^2\Theta_C^2 + 2\mathcal{G}_A\mathcal{G}_B\Theta_A\Theta_B + 2\mathcal{G}_B\mathcal{G}_C\Theta_B\Theta_C + 2\mathcal{G}_C\mathcal{G}_A\Theta_C\Theta_A, \tag{4.100}$$

$$\Delta_{AB} = 2(\mathcal{G}_A\Theta_A - \mathcal{G}_B\Theta_B)\frac{\mathcal{F}_\mathrm{T}}{\mathcal{G}_C} + \dot{\mathcal{G}}_A\mathcal{G}_B - \mathcal{G}_A\dot{\mathcal{G}}B + (H_A - H_B)\mathcal{G}_A\mathcal{G}_B, \tag{4.101}$$

$$M_{AB} = \frac{(\mathcal{G}_A\Theta_A - \mathcal{G}_B\Theta_B)}{\mathcal{G}_C\Theta_C^2}\left[(\mathcal{G}_A\Theta_A - \mathcal{G}_B\Theta_B)\frac{\mathcal{F}_\mathrm{T}}{\mathcal{G}_C} + \dot{\mathcal{G}}_A\mathcal{G}_B - \mathcal{G}_A\dot{\mathcal{G}}_B + (H_A - H_B)\mathcal{G}_A\mathcal{G}_B\right]. \tag{4.102}$$

Therefore, the dispersion relation is

$$\omega^2 = x = \frac{\mathcal{S}}{3} + \frac{y}{6} + \frac{4\mathcal{D}^2 + 3\mathcal{M}}{6y}, \tag{4.103}$$

$$y \equiv z\sqrt[3]{8\mathcal{D}^3 + 9\mathcal{D}\mathcal{M} - 3\sqrt{3}i\mathcal{M}\sqrt{\mathcal{D}^2 + \mathcal{M}}}, \tag{4.104}$$

where z is the root of $z^3 + 1 = 0$ and then

$$z = -1,\ \frac{1 \pm \sqrt{3}}{2}. \tag{4.105}$$

4.6 Perturbative behavior near anisotropic attractor

In this section, we see that it is not realistic that our Universe is in the vicinity of the anisotropic attractor, because near the anisotropic attractor, the even-parity sector suffers from a ghost unless we kill the scalar degree of freedom, and propagation speed of gravitational waves in the odd-parity sector easily increases. However, we argue the possibility that the behavior of the odd-parity sector can be used to make the universe homogeneous in its early stage.

Equation (3.16) with $\sigma_- = 0$ reads

$$\frac{d}{dt}\left[A^2 B(H_A - H_B)\mathcal{G}_A\right] = 0, \tag{4.106}$$

where we have used the relations from (2.40) to (2.45). This exhibits that if both A and B increase and anisotropy of the background expansion $H_A - H_B$ stays finite, then $\mathcal{G}_A$ approaches zero.

In the even-parity sector, we have no-ghost conditions $\mathcal{G}_B > 0$ and (4.45). Since $\mathcal{G}_A \to 0$ as the universe approaches the anisotropic attractor, the left-hand side of (4.45) converges at

$$\det \mathcal{K}_{ij} \to -\frac{A^4 \mathcal{G}_B^2 \Theta_{\bar{B}}^2}{4\mathcal{D}^4}. \tag{4.107}$$

This expression is inevitably negative and thus we suffer from a ghost. To avoid the ghost, one of the ways is to use a special class of model called the extended cuscuton which is referred in [67]. In this model, the scalar field behave as not a degree of freedom but a constraint, and we can consider the below discussion about the odd-parity sector in order to apply to the cuscuton model.

In the odd-parity sector, the propagation speed (4.29) along the directions x and y increases as $\mathcal{G}_A \to 0$. Hereinafter, we show that expansion of sound horizon forces perturbations to stay oscillating and it realizes a strongly homogenized universe in the two cases of exponential and power-law expansion.

Exponential expansion Here we assume that the scale factors grows exponentially as $A \propto e^{H_A t}$ and $B \propto e^{H_B t}$ and that H_A and H_B are nearly constant and positive, which justifies introduction of a new constant $g_A \equiv A^2 B \mathcal{G}_A$. We start with the final expression of the second-order action (4.28). First, we define a canonical variable $\zeta = (2A^2\mathcal{F}_T)^{-1/2}\Phi$ and conformal time η, which satisfies $dt = B d\eta$ and we fix the integration constant to have $B = (-H_B\eta)^{-1}$. Then we have a second-order action

$$S^{(2)} = \int d\eta \frac{1}{2}\left[\left(\frac{d\zeta}{d\eta}\right)^2 + \frac{H_A(H_A - H_B)}{H_B^2\eta^2}\zeta^2 + \left(-\frac{1}{H_B\eta}\right)^3 \frac{\mathcal{F}_T}{g_A}\zeta\partial^2\zeta + \frac{\mathcal{F}_T}{\mathcal{G}_B}\zeta\zeta''\right]. \tag{4.108}$$

We expand ζ into Fourier modes:

$$\zeta = \int \frac{d^3k}{(2\pi)^{3/2}}\left[\tilde{\zeta}_{\vec{k}} a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \tilde{\zeta}^*_{\vec{k}} a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}\right]. \tag{4.109}$$

From this we have an equation of motion for $\tilde{\zeta}_{\vec{k}}$,

$$\left[\frac{d^2}{d\eta^2} + \omega^2\right]\tilde{\zeta}_{\vec{k}} = 0, \tag{4.110}$$

$$\omega^2 = -|\eta|^{-2}\frac{H_A(H_A - H_B)}{H_B^2} + |\eta|^{-3}\frac{\mathcal{F}_T}{g_A H_B^3}\mathbf{k}^2 + \frac{\mathcal{F}_T}{\mathcal{G}_B}k_z^2, \tag{4.111}$$

where $\mathbf{k}$ denotes the comoving wavenumber vector in the directions x and y, and k_z is the comoving wavenumber in the direction z. When $|\eta|$ is large enough in a sufficiently past epoch, the second term in (4.111) can be neglected and we have a general solution for $\tilde{\zeta}_{\vec{k}}$ with the Hankel functions of the first and second kind $H_\nu^{(1,2)}$:

$$\tilde{\zeta}_{\vec{k}} = c_1\sqrt{-\eta}H_\nu^{(1)}[\sqrt{\kappa}(-\eta)] + c_2\sqrt{-\eta}H_\nu^{(2)}[\sqrt{\kappa}(-\eta)], \tag{4.112}$$

$$\nu \equiv (H_B - 2H_A)/2H_B, \tag{4.113}$$

$$\kappa \equiv (\mathcal{F}_\mathrm{T}/\mathcal{G}_B)k_z^2, \tag{4.114}$$

where the coefficients c_1 and c_2 are functions of $\mathbf{k}$ and k_z in general. Next, we define another canonical variable $\pi = B^{3/4}(2A^2\mathcal{F}_T)^{-1/2}\Phi$ and a time variable τ, which satisfies $dt = B^{-1/2}d\tau$ and we fix the integration constant to have $B = (H_B\tau/2)^2$. Then we have a second-order action

$$S^{(2)} = \int d\tau \frac{1}{2}\left[\left(\frac{d\pi}{d\tau}\right)^2 + \frac{16H_A^2 - 16H_AH_B + 3H_B^2}{4H_B^2\tau^2}\pi^2 + \frac{\mathcal{F}_T}{g_A}\pi\partial^2\pi + \left(\frac{H_B\tau}{2}\right)^{-6}\frac{\mathcal{F}_T}{\mathcal{G}_B}\pi\pi''\right], \tag{4.115}$$

and we have an equation of motion for the modes $\tilde{\pi}_{\vec{k}}$,

$$\left[\frac{d^2}{d\tau^2} + \Omega^2\right]\tilde{\pi}_{\vec{k}} = 0, \tag{4.116}$$

$$\Omega^2 = -\tau^{-2}\frac{(4H_A - H_B)(4H_A - 3H_B)}{4H_B^2} + \frac{\mathcal{F}_T}{g_A}\mathbf{k}^2 + \tau^{-6}\frac{64}{H_B^6}\frac{\mathcal{F}_T}{\mathcal{G}_B}k_z^2, \tag{4.117}$$

When τ is large enough in sufficient future, the third term in (4.117) can be neglected and we have a general solution for $\tilde{\pi}_{\vec{k}}$,

$$\tilde{\pi}_{\vec{k}} = d_1\sqrt{\tau}H_{\nu'}^{(1)}[\sqrt{\kappa'}\tau] + d_2\sqrt{\tau}H_{\nu'}^{(2)}[\sqrt{\kappa'}\tau], \tag{4.118}$$

$$\nu' \equiv (H_B - 2H_A)/H_B = 2\nu, \tag{4.119}$$

$$\kappa' \equiv (\mathcal{F}_\mathrm{T}/g_A)\mathbf{k}^2. \tag{4.120}$$

We connect two solutions (4.112) and (4.118) in their superhorizon period in which the first term in (4.111) and (4.117) is dominant. We use the properties of Hankel functions $H_\nu^{(1,2)}(z) = J_\nu(z) \pm iN_\nu(z)$ and $J_\nu(z) \propto \Gamma(\nu+1)^{-1}(z/2)^\nu$ for small z. Here we choose the vacuum in which only positive frequency modes exist with respect to k_z in the past and thus set $c_1 = \sqrt{\pi}/2$ and $c_2 = 0$. Then we find

$$d_{1,2} = \frac{c_1}{2\sqrt{2}i}\left[\frac{\Gamma(\nu)\Gamma(2\nu+1)}{\pi}R^\nu \mp \frac{\pi}{\Gamma(2\nu)\Gamma(\nu+1)}R^{-\nu}\left(1 + i\cot(\pi\nu)\right)\left(1 \mp i\cot(2\pi\nu)\right)\right], \tag{4.121}$$

$$\text{with } R \equiv \frac{2H_B^3}{\sqrt{\kappa\kappa'}} = \frac{2H_B^3}{|k_z|\mathbf{k}^2}\frac{g_A{\mathcal{G}_B}^{1/2}}{\mathcal{F}_\mathrm{T}^{3/2}}, \tag{4.122}$$

where the upper (lower) sign applies to d_1 (d_2). From this, number density $n_{\vec{k}}$ of the produced particles $\tilde{\pi}_{\vec{k}}$ is calculated and we get

$$n_{\vec{k}} = |d_1|^2 = \frac{|c_1|^2}{8\pi^2}\Gamma(\nu)^2\left[\Gamma(1+2\nu)^2R^{2\nu} - 2\Gamma(1-\nu)^2 + \frac{\Gamma(1-2\nu)^2\Gamma(1-\nu)^2}{\Gamma(1+\nu)^2}R^{-2\nu}\right], \tag{4.123}$$

$$= \frac{\pi}{32}\csc^2(\pi\nu)\left[\frac{\Gamma(1+2\nu)^2}{\Gamma(1-\nu)^2}R^{2\nu} + \frac{\Gamma(1-2\nu)^2}{\Gamma(1+\nu)^2}R^{-2\nu} - 2\right]. \tag{4.124}$$

We stress that the expression (4.124) is applicable only when the mode has the superhorizon period. If the first terms in (4.111) and (4.117) are always smaller than the sum of the other two, we can no longer justify the solutions with Hankel functions (4.112) and (4.118). We focus on the moderate case in which H_A and H_B are of the same order, and then the justifying condition can be written as $R \gg 1$. In other words, the mode satisfying $R \gg 1$ has its transient superhorizon period and its particle production is given by (4.124).

In late time, more modes which exit from horizon in z direction are still oscillating because they are still in larger sound horizon in x and y direction. We can treat such a mode with WKB approximation, which is expanded by a WKB parameter $\epsilon \equiv \dot{\omega}/\omega^2$, where the dot temporarily denotes derivative with respect to the referred time variable. In our case, the maximum of the WKB parameter in time evolution is of the order of $R^{1/3}$. Thus particle production of the mode satisfying $R \ll 1$ is suppressed by smallness of R.

Power-law expansion Next, we will discuss power-law inflation with self anisotropizing. We assume that the scale factors A and B grow as $A \propto t^{p_A}$ and $B \propto t^{p_B}$, where p_A and p_B are constant indices. It gives that $H_A \equiv \dot{A}/A = p_A/t$ and $H_B \equiv \dot{B}/B = p_B/t$. From this and (4.106), we can define a constant g_A as

$$g_A \equiv A^2B(H_A - H_B)\mathcal{G}_A = A^2B(p_A - p_B)t^{-1}\mathcal{G}_A, \tag{4.125}$$

and we assume that $\mathcal{G}_B$ and $\mathcal{F}_{\rm T}$ can be regard as constants. Substituting this into (4.28) and defining a canonical variable $\zeta = (2A^2\mathcal{F}_T)^{-1/2}\Phi$ and conformal time η which satisfies $dt = Bd\eta$ and of which integration constant we fix to have $t = (1-p_B)B\eta$, we have a second-order action

$$S^{(2)} = \int d\eta\frac{1}{2}\left[\left(\frac{d\zeta}{d\eta}\right)^2 + \frac{p_A(p_A-p_B+1)}{(1-p_B)^2\eta^2}\zeta^2 + \frac{B^2}{A^2}\frac{\mathcal{F}_T}{\mathcal{G}_A}\zeta\partial^2\zeta + \frac{\mathcal{F}_T}{\mathcal{G}_B}\zeta\zeta''\right]. \tag{4.126}$$

We expand ζ into Fourier modes as in (4.109) and then we have an equation of motion for $\tilde{\zeta}_{\vec{k}}$,

$$\left[\frac{d^2}{d\eta^2} + \omega^2\right]\tilde{\zeta}_{\vec{k}} = 0, \tag{4.127}$$

$$\omega^2 = -\frac{p_A(p_A-p_B+1)}{(1-p_B)^2\eta^2} + \frac{(p_A-p_B)B^2}{(1-p_B)\eta}\frac{\mathcal{F}_T}{g_A}\mathbf{k}^2 + \frac{\mathcal{F}_T}{\mathcal{G}_B}k_z^2. \tag{4.128}$$

If B grows faster than t, that is $p_B > 1$, then $|\eta|$ is a decreasing function and the second term becomes larger than the first term in the end. Thus we assume that $p_B > 1$ in this argument so that we can repeat similar study with that in the inflationary universe. We neglect the second term and get a general solution for $\tilde{\zeta}_{\vec{k}}$ with the Hankel functions $H_\nu^{(1,2)}$:

$$\tilde{\zeta}_{\vec{k}} = c_1\sqrt{-\eta}H_\nu^{(1)}[\sqrt{\kappa}(-\eta)] + c_2\sqrt{-\eta}H_\nu^{(2)}[\sqrt{\kappa}(-\eta)], \tag{4.129}$$

$$\nu \equiv (p_B - 2p_A - 1)/2(p_B - 1), \tag{4.130}$$

$$\kappa \equiv (\mathcal{F}_\mathrm{T}/\mathcal{G}_B)k_z^2. \tag{4.131}$$

Next we define another canonical variable $\pi = B^{3/4}|H_A - H_B|^{1/2}(2A^2\mathcal{F}_T)^{-1/2}\Phi$ and a time variable τ, which satisfies $dt = [B|H_A - H_B|]^{-1/2}d\tau$ and we fix the integration constant to have $t = [B|H_A - H_B|]^{-1/2}(1 + pB)\tau/2$. We have another second-order action with the canonical variable π:

$$S^{(2)} = \int d\tau \frac{1}{2}\left[\left(\frac{d\pi}{d\tau}\right)^2 + \frac{(4p_A - p_B + 3)(4p_A - 3p_B + 1)}{4(1 + p_B)^2\tau^2}\pi^2 + \frac{\mathcal{F}_T}{g_A}\pi\partial^2\pi + \frac{1}{B^3|H_A - H_B|}\frac{\mathcal{F}_T}{\mathcal{G}_B}\pi\pi''\right], \tag{4.132}$$

and we have an equation of motion for the modes $\tilde{\pi}_{\vec{k}}$,

$$\left[\frac{d^2}{d\tau^2} + \Omega^2\right]\tilde{\pi}_{\vec{k}} = 0, \tag{4.133}$$

$$\Omega^2 = -\frac{(4p_A - p_B + 3)(4p_A - 3p_B + 1)}{4(1 + p_B)^2\tau^2} + \frac{\mathcal{F}_T}{g_A}\mathbf{k}^2 + \frac{1}{B^3|H_A - H_B|}\frac{\mathcal{F}_T}{\mathcal{G}_B}k_z^2. \tag{4.134}$$

In the far future enough to neglect the third term in (4.134), we have a gengeral solution for $\tilde{\pi}_{\vec{k}}$,

$$\tilde{\pi}_{\vec{k}} = d_1\sqrt{\tau}H_{\nu'}^{(1)}[\sqrt{\kappa'}\tau] + d_2\sqrt{\tau}H_{\nu'}^{(2)}[\sqrt{\kappa'}\tau], \tag{4.135}$$

$$\nu' \equiv (p_B - 2p_A - 1)/(p_B + 1), \tag{4.136}$$

$$\kappa' \equiv (\mathcal{F}_\mathrm{T}/g_A)\mathbf{k}^2. \tag{4.137}$$

Then we connect the two solutions (4.129) and (4.135) in their superhorizon period $\sqrt{\kappa}(-\eta) \ll 1$ and $\sqrt{\kappa'}\tau \ll 1$ respectively. We finally gain number density

$$n_{\vec{k}} = \frac{\pi}{32}\frac{(p_B + 1)}{(p_B - 1)}\csc^2(\pi\nu)\left[\frac{\Gamma(1 + \nu')^2}{\Gamma(1 - \nu)^2}R^{2\nu} + \frac{\Gamma(1 - \nu')^2}{\Gamma(1 + \nu)^2}R^{-2\nu} - 2\frac{\nu'\sin(\pi\nu)}{\nu\sin(\pi\nu')}\cos[\pi(\nu - \nu')]\right], \tag{4.138}$$

where we have difined

$$R \equiv 2{B_0}^{2/(p_B+1)}\frac{(p_B - 1)}{\sqrt{\kappa}}\left[\frac{(p_B + 1)^2}{(p_A - p_B)\kappa'}\right]^{(p_B-1)/(p_B+1)}, \tag{4.139}$$

and $B_0 \equiv B/t^{p_B}$.

4.7 Summary

We have constructed the perturbation theory on axially symmetric Bianchi-type I universe. The perturbations can be decomposed into the odd-parity sector which contains one of the polarization modes of gravitational waves, and the even-parity sector which contains the other polarization mode of gravitational waves and scalar perturbations. We have calculated those perturbative actions and finally obtained the dispersion relations. We have also analyzed the perturbative behavior around the anisotropic attractor discussed in Chapter 3, especially for gravitational waves in the odd-parity sector. It has been shown that its propagation speed has a singular behavior on approaching the anisotropic attractor and thus it is unlikely for the anisotropic attractor in four dimensions to represent our universe. However it gives rise to a possibility that such a singular behavior can homogenize our universe in the early stage of the universe. It will be shown in Chapter 6 that the singular behavior might originate from the fact that the anisotropic attractor in four dimensions is surrounded by the singular plane in phase space.

Chapter 5

Perturbations on static and spherically symmetric spacetime

Since the first detection of gravitational waves from the binary black hole merger on September 14, 2015, advanced Laser Interferometer Gravitational-wave Observatory (aLIGO) has detected gravitational waves from three binary black hole mergers during the first observing run and gravitational waves from a binary neutron star inspiral and gravitational waves from seven binary black hole mergers during the second observing run [68]. Additionally, Event Horizon Telescope (EHT) has succeeded in imaging a supermassive black hole directly by the very long baseline interferometry of electromagnetic waves [69]. The development of observation is expected to become even faster in the near future, and we can regard compact objects such as black holes as more important testing grounds of the nature of gravity.

In the general relativity, the equation of motion and stability condition in the odd-parity sector have been derived in [70] and those in the even-parity sector in [71, 72]. In the Horndeski theory, which contains a scalar field, the equation of motion and the stability condition in the odd-parity sector have been obtained in [21] and the radial stability conditions in the even-parity sector are found in [22], which we review in the following. We calculate angular dispersion relations in the even-parity sector and discuss stability conditions of the angular perturbations.

5.1 Background equations

We consider a static and spherically symmetric spacetime as a background. The unperturbed metric $\bar{g}_{\mu\nu}$ is written as

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -A(r)dt^2 + \frac{dr^2}{B(r)} + C(r)r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5.1}$$

Let a scalar field of the generalized Galileon ϕ depend only on r and we regard $\phi = \phi(r)$ as an unperturbed variable rather than a field. Then $X = -B\phi'^2/2$, where a prime denotes partial derivative with respect to r. The set of three variables A, B, and C are redundant and we can fix one of them by using the freedom of coordinate transformation, and we fix $C(r) = 1$ after getting the equation of motion for C. Substituting the metric to the action (2.25) leads us to the background action

$$S^{(0)} = \int dtdr 4\pi r^2 \sqrt{\frac{A}{B}} C \mathcal{L}^{(0)}, \tag{5.2}$$

$$\begin{aligned}
\mathcal{L}^{(0)} =& G_2 - B\left(\phi'' + (\frac{2}{r} + \frac{A'}{2A} + \frac{B'}{2B} + \frac{C'}{C})\phi'\right) G_3 \\
&+ B\left(-\frac{2}{r^2} + \frac{2}{r^2 BC} - \frac{2A'}{rA} + \frac{A'^2}{2A^2} - \frac{2B'}{rB} - \frac{A'B'}{2AB}\right. \\
&\qquad \left. -\frac{6C'}{rC} - \frac{A'C'}{AC} - \frac{A'B'}{AB} + \frac{C'^2}{2C^2} - \frac{A''}{A} - \frac{2C''}{C}\right) G_4 \\
&+ B^2\phi'^2\left(\frac{2}{r^2} + \frac{2A'}{rA} + \frac{2B'}{rB} + \frac{A'B'}{2AB}\frac{2C'}{rC} + \frac{A'C'}{AC}\right. \\
&\qquad \left. + \frac{B'C'}{BC} + \frac{C'^2}{2C^2} + \frac{4\phi''}{r\phi'} + \frac{A'\phi''}{A\phi'} + \frac{2C'\phi''}{C\phi'}\right) G_{4X} \\
&+ B^2\left((\frac{1}{r^2} - \frac{1}{r^2 BC} + \frac{A'}{rA} + \frac{C'}{rC} + \frac{A'C'}{2AC} + \frac{C'^2}{4C^2})\phi''\right. \\
&\quad + (\frac{3A'}{2r^2 A} - \frac{A'}{2r^2 ABC} - \frac{A'^2}{2rA^2} + \frac{3B'}{2r^2 B} - \frac{B'}{2r^2 B^2 C} + \frac{2A'B'}{2rAB} + \frac{2C'}{r^2 C} \\
&\quad + \frac{5A'C'}{2AC} - \frac{A'^2 C'}{4A^2 C} + \frac{3B'C'}{2rBC} + \frac{3A'B'C'}{4ABC} + \frac{C'^2}{2rC^2} + \frac{A'C'^2}{2rC^2} + \frac{A'C'^2}{8AC^2} \\
&\quad \left. + \frac{3B'C'^2}{8BC^2} - \frac{C'^3}{4C^3} + \frac{A''}{rA} + \frac{A''C'}{2AC} + \frac{C''}{rC} + \frac{A'C''}{2AC} + \frac{C'C''}{2C^2})\phi'\right) G_5 \\
&- B^3\phi'^2\left((\frac{1}{r^2} + \frac{A'}{rA} + \frac{C'}{rC} + \frac{A'C'}{2AC}\frac{C'^2}{4C^2})\phi'' + (\frac{A'}{2r^2 A} + \frac{B'}{2r^2 B}\right. \\
&\quad \left. + \frac{A'B'}{2rAB} + \frac{A'C'}{2rAC} + \frac{B'C'}{2rBC} + \frac{A'B'C'}{4ABC} + \frac{A'C'^2}{8AC^2} + \frac{B'C'^2}{8BC^2})\phi'\right).
\end{aligned} \tag{5.3}$$

The variation of the action with respect to the metric variables A, B, and C and the scalar variable ϕ yield the equations of motion

$$\mathcal{E}_A = 0, \quad \mathcal{E}_B = 0, \quad \mathcal{E}_C = 0, \quad \mathcal{E}_\phi = \frac{1}{r^2}\sqrt{\frac{B}{A}}\frac{d}{dr}\left(r^2\sqrt{AB}\mathcal{J}\right) + \frac{\partial \mathcal{U}}{\partial \phi} = 0, \tag{5.4}$$

where

$$\begin{aligned}
\mathcal{E}_A =& G_2 + B\phi'^2 G_{3\phi} - \frac{1}{2}B\phi'^2(2B\phi'' + B'\phi')G_{3X} + \frac{2(1 - B - rB')}{r^2}G_4 \\
&- (B'\phi' + \frac{2B(r\phi'' + 2\phi')}{r})G_{4\phi} - \frac{2B\phi'(2rB\phi'' + (B + 2rB')\phi')}{r^2}G_{4X} - 2B\phi'^2 G_{4\phi\phi} \\
&+ \frac{B\phi'^2(2rB\phi'' + (rB' - 4B)\phi')}{r}G_{4\phi X} + \frac{2B^2\phi'^3(2B\phi'' + B'\phi')}{r}G_{4XX}
\end{aligned}$$

$$+\frac{B\phi'(4rB\phi''+(1+B+3rB')\phi')}{r^2}G_{5\phi}-\frac{B\phi'^2(2B(1-3B)\phi''+(1-5B)B'\phi')}{2r^2}G_{5X}$$
$$+\frac{2B^2\phi'^3}{r}G_{5\phi\phi}-\frac{B^2\phi'^3(2rB\phi''-(B-rB')\phi')}{r^2}G_{5\phi X}-\frac{B^3\phi'^4(2B\phi''+B'\phi')}{2r^2}G_{5XX}, \tag{5.5}$$

$$\mathcal{E}_B = G_2+B\phi'^2G_{2X}-B\phi'^2G_{3\phi}-\frac{B^2\phi'^3(4A+rA')}{2rA}G_{3X}+\frac{2(A-AB-rA'B)}{r^2A}G_4$$
$$-\frac{B(4A+rA')\phi'}{rA}G_{4\phi}+\frac{2B\phi'^2(A-2AB-2rA'B)}{r^2A}G_{4X}+\frac{B^2\phi'^3(4A+rA')^2}{rA}G_{4\phi X}$$
$$+\frac{2B^3\phi'^4(A+rA')}{r^2A}G_{4XX}-\frac{B\phi'^2(A-3AB-3rA'B)}{r^2A}G_{5\phi}-\frac{B^2\phi'^3(1-5B)A'}{2r^2A}G_{5X}$$
$$-\frac{B^3\phi'^4(A+rA')}{r^2A}G_{5\phi X}-\frac{B^4\phi'^5A'}{2r^2A}G_{5XX}, \tag{5.6}$$

$$\mathcal{E}_C = G_2+B\phi'^2G_{3\phi}-\frac{1}{2}B\phi'^2(2B\phi''+B'\phi')G_{3X}-(B'\phi'+B\phi''+B\phi'(\frac{2}{r}+\frac{A'}{A}))G_{4\phi}$$
$$-\frac{2AA'B-rA'^2B+2A^2B'+rAA'B'+2rAA''B}{2rA^2}G_4$$
$$-B^2\phi'((\frac{2}{r}+\frac{A'}{A})\phi''+(\frac{A'}{rA}-\frac{A'^2}{2A^2}+\frac{2B'}{rB}+\frac{A'B'}{AB}+\frac{A''}{A})\phi')G_{4X}-2B\phi'^2G_{4\phi\phi}$$
$$+B^2\phi'^2(2\phi''-(\frac{2}{r}+\frac{A'}{A}-\frac{B'}{B})\phi')G_{4\phi X}+\frac{B^2\phi'^3(2A+rA')(B'\phi'+2B\phi'')}{2rA}G_{4XX}$$
$$+B^2\phi'((\frac{A'}{2rA}-\frac{A'^2}{4A^2}+\frac{3B'}{2rB}+\frac{3A'B'}{4AB}+\frac{A''}{2A})\phi'+(\frac{2}{r}+\frac{A'}{A})\phi'')G_{5\phi}$$
$$-\frac{B^3\phi'^3}{r}((\frac{A'^2}{4A^2}-\frac{5A'B'}{4AB}-\frac{A''}{2A})\phi'-\frac{3A'}{2A}\phi'')G_{5X}+\frac{B^2\phi'^3(2A+rA')}{2rA}G_{5\phi\phi}$$
$$+B^3\phi'^3((\frac{A'}{2rA}-\frac{B'}{2rB}-\frac{A'B'}{4AB})\phi'-(\frac{1}{r}+\frac{A'}{2A})\phi'')G_{5\phi X}-\frac{B^3A'\phi'^4(2B\phi''+B'\phi')}{4rA}G_{5XX}, \tag{5.7}$$

$$\mathcal{J} = \phi'G_{2X}-\frac{B\phi'^2(4A+rA')}{2rA}G_{3X}+(\frac{4}{r}+\frac{A'}{A})G_{4\phi}+\frac{2\phi'}{r^2}(1-B-rB\frac{A'}{A})G_{4X}$$
$$+\frac{B\phi'^2(4A+rA')}{rA}G_{4\phi X}+\frac{2B^2\phi'^3(A+rA')}{r^2A}G_{4XX}-\frac{BA'\phi'^2(1-3B)}{2r^2A}G_{5X}$$
$$-\frac{B^2\phi'^3(A+rA')}{r^2A}G_{\phi X}-\frac{B^3A'\phi'^4}{2r^2A}G_{5XX}, \tag{5.8}$$

$$\mathcal{U} = G_2+B\phi'^2G_{3\phi}-\frac{1}{2}B\phi'^2(B'\phi'+2B\phi'')G_{3X}+\frac{2}{r^2}(1+B+rB\frac{A'}{A})G_4$$
$$+\frac{B\phi'(4A+rA')}{rA}G_{4\phi}+\frac{2B^2\phi'^2(A+rA')}{r^2A}G_{4X}+\frac{B\phi'^2}{r^2}(1-B-rB\frac{A'}{A})G_{5\phi}$$
$$-\frac{B\phi'^2(A'B^2+AB')\phi'+2AB\phi''}{2r^2A}G_{5X}, \tag{5.9}$$

where we have set $C=1$.

In the next section, we classify the perturbations with their parity in the spatial inversion with regard to the origin $r=0$. The perturbations can be analyzed with general configurations of A, B, C and ϕ. No particular configuration is considered but they must satisfy the background equations.

5.2 Classification of perturbations

For odd-parity mode, we write the perturbed metric $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ as

$$h_{tt} = 0, \tag{5.10}$$

$$h_{tr} = 0, \tag{5.11}$$

$$h_{rr} = 0, \tag{5.12}$$

$$h_{ta} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} h_{0,\ell m}(t,r) E_a{}^b \partial_b Y_{\ell m}(\theta, \varphi), \tag{5.13}$$

$$h_{ra} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} h_{1,\ell m}(t,r) E_a{}^b \partial_b Y_{\ell m}(\theta, \varphi), \tag{5.14}$$

$$h_{ab} = \frac{1}{2} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{2,\ell m}(t,r) \left[E_a{}^c \nabla_c \nabla_b Y_{\ell m}(\theta, \varphi) + E_b{}^c \nabla_c \nabla_a Y_{\ell m}(\theta, \varphi) \right], \tag{5.15}$$

where a and b denote the angular coordinates θ and φ, and E_{ab} is the antisymmetric tensor on the 2-sphere with $E_{\theta\phi} = -E_{\phi\theta} = \sin\theta$, $E_{\theta\theta} = E_{\phi\phi} = 0$. The odd parity is inherited from the antisymmetric tensor when the space is inversed with regard to the origin $r = 0$. $Y_{\ell m}$ is the spherical harmonics, which is the eigen function of the Laplacian on the 2-sphere

$$\Delta \equiv \frac{1}{\sin\theta} \partial_\theta \sin\theta \partial_\theta + \frac{1}{\sin^2\theta} \partial_\varphi^2, \tag{5.16}$$

and $\Delta Y_{\ell m} = -\ell(\ell+1) Y_{\ell m} \equiv -j^2 Y_{\ell m}$. The gauge transformation $x^\mu \to x^\mu + \xi^\mu$ with odd parity is given by

$$\xi_a = \sum_{\ell,m} \Lambda_{\ell m}(t,r) E_a^b \partial_b Y_{\ell m}(\theta, \varphi), \tag{5.17}$$

where only $\Lambda_{\ell m}$ can bring the gauge transformations to the variables h_0, h_1, and h_2

$$h_{0,\ell m}(t,r) \to h_{0,\ell m}(t,r) + \dot{\Lambda}_{\ell m}(t,r), \tag{5.18}$$

$$h_{1,\ell m}(t,r) \to h_{1,\ell m}(t,r) + \Lambda'_{\ell m}(t,r) - \frac{2}{r} \Lambda_{\ell m}(t,r), \tag{5.19}$$

$$h_{2,\ell m}(t,r) \to h_{2,\ell m}(t,r) + 2\Lambda_{\ell m}(t,r). \tag{5.20}$$

We can set $h_2 = 0$ for $\ell \geq 2$, which is called the Regge-Wheeler gauge. For the dipole perturbation $\ell = 1$, the spherical harmonics vanishes identically and we need another gauge condition, which is discussed later. Note that a perturbed scalar field does not contribute to the odd-parity mode because we cannot construct any odd-parity quantities with only the scalar field and its derivatives.

For even-parity mode, we write the metric perturbations $h_{\mu\nu}$ as

$$h_{tt} = A(r)\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} H_{0,\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \tag{5.21}$$

$$h_{tr} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} H_{1,\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \tag{5.22}$$

$$h_{rr} = \frac{1}{B(r)}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} H_{2,\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \tag{5.23}$$

$$h_{ta} = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} \beta_{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi), \tag{5.24}$$

$$h_{ra} = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} \alpha_{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi), \tag{5.25}$$

$$h_{ab} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} K_{\ell m}(t,r)g_{ab}Y_{\ell m}(\theta,\varphi) + \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} G_{\ell m}(t,r)\nabla_a\nabla_b Y_{\ell m}(\theta,\varphi). \tag{5.26}$$

The scalar field ϕ also provides an even-parity perturbation,

$$\phi(t,r,\theta,\varphi) = \phi(r) + \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \delta\phi_{\ell m}(t,r)Y_{\ell m}(\theta,\varphi). \tag{5.27}$$

With a gauge transformation $x^\mu \to x^\mu + \xi^\mu$, we can eliminate some of the variables. We decompose the even-parity transformation ξ^μ into orthogonal modes $T_{\ell m}(t,r)$, $R_{\ell m}(t,r)$ and $\Theta_{\ell m}(t,r)$ with the spherical harmonics

$$\xi_t = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} T_{\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \tag{5.28}$$

$$\xi_r = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} R_{\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \tag{5.29}$$

$$\xi_a = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} \Theta_{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi). \tag{5.30}$$

With the gauge transformation $T_{\ell m}(t,r)$, $R_{\ell m}(t,r)$ and $\Theta_{\ell m}(t,r)$, the even-parity perturbations transform as

$$H_{0,\ell m}(t,r) \to H_{0,\ell m}(t,r) + \frac{2}{A}\dot{T}_{\ell m}(t,r) - \frac{A'B}{A}R_{\ell m}(t,r), \tag{5.31}$$

$$H_{1,\ell m}(t,r) \to H_{1,\ell m}(t,r) + \dot{R}_{\ell m}(t,r) + T'_{\ell m}(t,r) - \frac{A'}{A}T_{\ell m}(t,r), \tag{5.32}$$

$$H_{2,\ell m}(t,r) \to H_{2,\ell m}(t,r) + 2BR'_{\ell m}(t,r) + B'R_{\ell m}(t,r), \tag{5.33}$$

$$\beta_{\ell m}(t,r) \to \beta_{\ell m}(t,r) + T_{\ell m}(t,r) + \dot{\Theta}_{\ell m}(t,r), \tag{5.34}$$

$$\alpha_{\ell m}(t,r) \to \alpha_{\ell m}(t,r) + R_{\ell m}(t,r) + \Theta'_{\ell m}(t,r) - \frac{2}{r}\Theta_{\ell m}(t,r), \tag{5.35}$$

$$K_{\ell m}(t,r) \to K_{\ell m}(t,r) + \frac{2B}{r}R_{\ell m}(t,r), \tag{5.36}$$

$$G_{\ell m}(t,r) \to G_{\ell m}(t,r) + 2\Theta_{\ell m}(t,r). \tag{5.37}$$

With these rules, we choose the gauge condition $\beta_{\ell m} = K_{\ell m} = G_{\ell m} = 0$, which can completely fix the gauge. For the monopole mode $\ell = 0$ and the dipole mode $\ell = 1$, some of the variables vanish identically, and we fix their gauges with other conditions. The monopole mode consists of $H_{0,\ell m}$, $H_{1,\ell m}$, $H_{2,\ell m}$ and $K_{\ell m}$, and $\Theta_{\ell m}$ does not contribute to the gauge transformation. In the dipole mode, $K_{\ell m}$ and $G_{\ell m}$ are not independent. The gauge fixing for these modes is given in the following description of calculation.

5.3 Odd-parity sector

The second-order action is written as a sum of the contribution from each multipole modes

$$S^{(2)} = 2\pi \sum_{\ell=0}^{\infty}(2\ell+1)\int dtdr\mathcal{L}_\ell^{(2)}, \tag{5.38}$$

where the angular summation $\sum_m$ has been already computed. Hereinafter we omit ℓ and m from the metric variables h_0, h_1, and h_2. For the multipole modes $\ell > 2$, the Lagrangian $\mathcal{L}_\ell^{(2)}$ is given by [21]

$$\mathcal{L}_\ell^{(2)} = a_1 h_0^2 + a_2 h_1^2 + a_3\left(\dot{h}_1^2 - 2\dot{h}_1 h_0' + h_0'^2 + \frac{4}{r}\dot{h}_1 h_0\right), \tag{5.39}$$

where a_i's are coefficients, which are written with Horndeski's functions,

$$a_1 = \frac{\ell(\ell+1)}{r^2}\left[\frac{d}{dr}\left(r\sqrt{\frac{B}{A}}\mathcal{H}\right) + \frac{\ell^2+\ell-2}{2\sqrt{AB}}\mathcal{F}\right], \tag{5.40}$$

$$a_2 = -\frac{(\ell-1)\ell(\ell+1)(\ell+2)}{2r^2}\sqrt{AB}\mathcal{G}, \tag{5.41}$$

$$a_3 = \frac{\ell(\ell+1)}{2}\sqrt{\frac{B}{A}}\mathcal{H}. \tag{5.42}$$

In the Lagrangian, Horndeski's functions are seen in specific combinations, and we define $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ as

$$\mathcal{F} \equiv 2\left(G_4 + \frac{1}{2}B\phi' X' G_{5X} - XG_{5\phi}\right), \tag{5.43}$$

$$\mathcal{G} \equiv 2\left[G_4 - 2XG_{4X} + X\left(\frac{A'}{2A}B\phi' G_{5X} + G_{5\phi}\right)\right], \tag{5.44}$$

$$\mathcal{H} \equiv 2\left[G_4 - 2XG_{4X} + X\left(\frac{B\phi'}{r}G_{5X} + G_{5\phi}\right)\right]. \tag{5.45}$$

In order to get a simpler action, we introduce an auxiliary field q

$$\mathcal{L}_\ell^{(2)} = \left(a_1 - \frac{2\left(ra_3\right)'}{r^2}\right) h_0^2 + a_2 h_1^2 + a_3 \left[-q^2 + 2q\left(\dot{h}_1 - h_0' + \frac{2}{r}h_0\right)\right]. \tag{5.46}$$

Substituting the solution for q into this expression, we get the action (5.39) again. We variate the action (5.46) with respect to h_0 and h_1 and then they can be easily solved for

$$h_0 = -\frac{r\left[ra_3' q + a_3\left(rq' + 2q\right)\right]}{r^2 a_1 - 2\left(ra_3' + a_3\right)}, \tag{5.47}$$

$$h_1 = \frac{a_3}{a_2}\dot{q}. \tag{5.48}$$

To obtain simpler form of the equation of motion, we write the action with Q instead of q

$$Q = \left(\frac{r^2 B^{3/2}\mathcal{H}^2}{\sqrt{A}\mathcal{F}}\right)^{1/2} q. \tag{5.49}$$

We finally get the simplest form of action

$$\mathcal{L}_\ell^{(2)} = \frac{\ell(\ell+1)}{2(\ell-1)(\ell+2)}\left[\frac{\mathcal{F}}{AB\mathcal{G}}\dot{Q}^2 - Q'^2 - \frac{\ell(\ell+1)\mathcal{F}}{r^2 B\mathcal{H}}Q^2 - V(r)Q^2\right]. \tag{5.50}$$

where the effective mass squared $V(r)$ is given by

$$\begin{aligned} V(r) = -\frac{1}{16}\Bigg[&\frac{A'^2}{A^2} + 4\frac{A'}{A}\left(\frac{\mathcal{F}'}{\mathcal{F}} + \frac{\mathcal{H}'}{\mathcal{H}} + \frac{3}{r}\right) + 3\frac{B'^2}{B^2} \\ &+4\left(\frac{B'}{B}\frac{\mathcal{F}'}{\mathcal{F}} + \frac{B'}{rB} - \frac{B''}{B} + \frac{10\mathcal{F}}{Br^2\mathcal{H}}\right) \\ &-4\left(3\frac{\mathcal{F}'^2}{\mathcal{F}^2} - \frac{2\mathcal{F}''}{\mathcal{F}} + \frac{4\mathcal{F}'}{r\mathcal{F}} + \frac{2\mathcal{H}'}{r\mathcal{H}} + \frac{10}{r^2}\right)\Bigg], \end{aligned} \tag{5.51}$$

which is regarded as an effective potential in the calculation of quasi-normal modes. The equation of motion for the master variable Q is given by

$$\frac{\mathcal{F}}{AB\mathcal{G}}\ddot{Q} - Q'' + \frac{\ell(\ell+1)\mathcal{F}}{r^2 B\mathcal{H}}Q + VQ = 0. \tag{5.52}$$

In a very small region the perturbations can freely propagate, which means that the potential term V is negligible compared to the other terms. We assume that its (t,r)-dependence is written as $Q \propto e^{-i\omega t + ikr}$ in a small patch. The dispersion relation of the odd-parity sector is given by

$$\frac{\omega^2}{A} = \frac{\mathcal{G}}{\mathcal{F}}Bk^2 + \frac{\mathcal{G}}{\mathcal{H}}\frac{\ell(\ell+1)}{r^2}, \tag{5.53}$$

where ω^2/A, Bk^2, and $\ell(\ell+1)/r^2$ are the physical frequency, radial wavenumber, and angular wavenumber, respectively. Therefore, the radial sound speed c_r and angular sound speed c_θ are given by

$$c_r^2 = \frac{\mathcal{G}}{\mathcal{F}}, \quad c_\theta^2 = \frac{\mathcal{G}}{\mathcal{H}}. \tag{5.54}$$

We now consider the stability of the perturbations. When Q is defined, positivity of $\mathcal{F}$ is implicitly assumed. Otherwise, as is seen in (5.50), $\mathcal{F} < 0$ would change the sign of the time derivative term and we have the gradient instability, in which the perturbations evolve exponentially at high wavenumber. From (5.54), one can tell that the ghost instability emerge when $\mathcal{G} < 0$, and then we see that $\mathcal{H} < 0$ also cause the gradient instability in the angular directions. Thus we require the conditions below to avoid those instabilities:

$$\mathcal{F} > 0, \qquad \mathcal{G} > 0, \qquad \mathcal{H} > 0. \tag{5.55}$$

Among the odd-parity modes, the dipole mode needs a special treatment, since it lacks one of the metric variables which represent the angular metric component h_{ab}. For the dipole mode $\ell = 1$, we can read the action from (5.39), where a_2 vanishes and $a_1 = \frac{2}{r^2}(ra_3)'$

$$\mathcal{L}^{(2)}_{\ell=1} = a_3 \frac{(2h_0 + r(\dot{h}_1 - h_0'))^2}{r^2}, \tag{5.56}$$

where we have not fixed the gauge yet. h_1 can be eliminated by the gauge freedom $\Lambda_{\ell m}$, and we get the equations of motion for h_0

$$\dot{h}_0' - \frac{2}{r}\dot{h}_0 = 0, \tag{5.57}$$

$$a_3 h_0'' + a_3' h_0' - \frac{2\,(ra_3)'}{r^2} h_0 = 0. \tag{5.58}$$

From the gauge transformation laws (5.18) and (5.19), one can tell that there is residual gauge freedom $\Lambda_{\ell m} = C(t) r^2$, where C is an arbitrary function of time. Therefore, we have the solution for h_0

$$h_0 = \frac{3Jr^2}{4\pi} \int^r \frac{\mathrm{d}\tilde{r}}{\tilde{r}^4 \mathcal{H}} \sqrt{\frac{A}{B}} + C(t) r^2, \tag{5.59}$$

where J is an integration constant and it physically represent the angular momentum of slightly rotating black hole.

5.4 Even-parity sector

The second-order action is given by the summation of the contribution from each mode ℓ

$$S^{(2)} = 2\pi \sum_{\ell=0}^{\infty} (2\ell+1) \int dt dr \mathcal{L}^{(2)}_\ell, \tag{5.60}$$

For $\ell > 2$, the second-order action is given by [22]

$$\begin{aligned}\mathcal{L}_\ell^{(2)} =& H_0\left[a_1\delta\phi'' + a_2\delta\phi' + a_3 H_2' + j^2 a_4\alpha' + \left(a_5 + j^2 a_6\right)\delta\phi + \left(a_7 + j^2 a_8\right)H_2 + j^2 a_9\alpha\right]\\ &+ j^2 b_1 H_1^2 + H_1\left(b_2\delta\dot{\phi}' + b_3\delta\dot{\phi} + b_4\dot{H}_2 + j^2 b_5\dot{\alpha}\right)\\ &+ c_1\dot{H}_2\delta\dot{\phi} + H_2\left[c_2\delta\phi' + \left(c_3 + j^2 c_4\right)\delta\phi + j^2 c_5\alpha\right] + c_6 H_2^2 + j^2 d_1\dot{\alpha}^2\\ &+ j^2\alpha\left(d_2\delta\phi' + d_3\delta\phi\right) + j^2 d_4\alpha^2\\ &+ e_1\delta\dot{\phi}^2 + e_2\delta\phi'^2 + \left(e_3 + j^2 e_4\right)\delta\phi^2,\end{aligned} \tag{5.61}$$

where we have omitted the subscripts ℓ and m and have defined the coefficients as

$$a_1 = r^2\sqrt{AB^3}\Xi\phi'^2, \tag{5.62}$$

$$a_2 = \frac{r\sqrt{AB}\mathcal{H}}{\phi'}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \sqrt{\frac{B}{A}}\frac{r\phi'(4AB\Xi\phi' + rB\Xi A'\phi' + 2rA\Xi B'\phi' + 2rAB\Xi'\phi' + 2rAB\Xi\phi'')}{2}, \tag{5.63}$$

$$a_3 = -r\sqrt{AB}\mathcal{H} - \frac{1}{2}r^2\sqrt{AB^3}\Xi\phi'^3, \tag{5.64}$$

$$a_4 = \sqrt{AB}\mathcal{H}, \tag{5.65}$$

$$a_5 = -r^2\sqrt{\frac{A}{B}}\frac{\partial\mathcal{E}_A}{\partial\phi}, \tag{5.66}$$

$$a_6 = \sqrt{\frac{A}{B}}\frac{\mathcal{F} - \mathcal{H} - r\mathcal{H}'}{r\phi'}, \tag{5.67}$$

$$a_7 = a_3', \tag{5.68}$$

$$a_8 = -\frac{1}{2}\sqrt{\frac{A}{B}}\mathcal{H}, \tag{5.69}$$

$$a_9 = \sqrt{AB}\left(\frac{\mathcal{H}}{r} + \frac{B'}{2B}\mathcal{H} + \mathcal{H}'\right), \tag{5.70}$$

$$b_1 = \frac{1}{2}\sqrt{\frac{B}{A}}\mathcal{H}, \tag{5.71}$$

$$b_2 = -2r^2\sqrt{\frac{B^3}{A}}\Xi\phi'^2, \tag{5.72}$$

$$b_3 = -2r\sqrt{\frac{B}{A}}\left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{\mathcal{H}}{\phi'} + r^2\sqrt{\frac{B}{A}}(B'\phi' + 2B\phi'')\Xi\phi', \tag{5.73}$$

$$b_4 = r\sqrt{\frac{B}{A}}(2\mathcal{H} + rB\Xi\phi'^3), \tag{5.74}$$

$$b_5 = -\sqrt{\frac{B}{A}}\mathcal{H}, \tag{5.75}$$

$$c_1 = -r^2\sqrt{\frac{B}{A}}\Xi\phi'^2, \tag{5.76}$$

$$c_2 = \frac{r}{2}\sqrt{\frac{B}{A}}\phi'(2rA\Sigma - 4AB\Theta\phi' - rA'B\Xi\phi'), \tag{5.77}$$

$$c_3 = r^2\sqrt{\frac{A}{B}}\frac{\partial\mathcal{E}_B}{\partial\phi}, \tag{5.78}$$

$$c_4 = \sqrt{AB}\Theta\phi'^2, \tag{5.79}$$

$$c_5 = -\sqrt{\frac{A}{B}}\frac{2A\mathcal{G} + rA'\mathcal{H} + 2rAB\Theta\phi'^3}{2r}, \tag{5.80}$$

$$c_6 = \frac{1}{4}\sqrt{\frac{A}{B}}(2A\mathcal{G} + 2rA'\mathcal{H} - r^2A\Sigma\phi'^2 + 4rAB\Theta\phi'^3 + r^2A'B\Xi\phi'^3), \tag{5.81}$$

$$d_1 = \frac{1}{2}\sqrt{\frac{A}{B}}\mathcal{H}, \tag{5.82}$$

$$\begin{aligned} d_2 = &-\frac{2A(\mathcal{G}-\mathcal{F}) + 2AB(\mathcal{H}-\mathcal{G}+r\mathcal{H}') - rA'B(2\mathcal{H}+r\mathcal{H}') + rAB'(\mathcal{G}+\mathcal{H})}{r^2\sqrt{AB}\phi'} \\ &-\sqrt{AB}\Theta\phi'(B'\phi' + 2B\phi''), \end{aligned} \tag{5.83}$$

$$d_3 = 2\sqrt{AB^3}\Theta\phi'^2, \tag{5.84}$$

$$d_4 = \sqrt{AB}\frac{\mathcal{G}}{r^2}, \tag{5.85}$$

$$e_1 = \frac{\tilde{e}_1}{2\sqrt{AB}}, \tag{5.86}$$

$$e_2 = -r^2\sqrt{AB}\Sigma, \tag{5.87}$$

$$e_3 = r^2\sqrt{\frac{A}{B}}\frac{\partial\mathcal{E}_\phi}{\partial\phi}, \tag{5.88}$$

$$\begin{aligned} e_4 = &\frac{2A(\mathcal{G}-\mathcal{F}) + 2AB(r\mathcal{H}-r\mathcal{G})' + rA'B(2\mathcal{F}-\mathcal{G}-3\mathcal{H}-2r\mathcal{H}'+rB\Theta\phi'^3)}{2r^2\sqrt{AB^3}\phi'^2}, \\ &+\frac{AB'(\mathcal{G}+\mathcal{H}+3rB\Theta\phi'^3) + 2rAB^2\phi'^2(\phi'\Theta' + 3\Theta\phi'')}{2r\sqrt{AB^3}\phi'^2}, \end{aligned} \tag{5.89}$$

where we have defined several functions

$$\begin{aligned} \Sigma = &G_{2X} - B\phi'^2G_{2XX} - 2G_{3\phi} - \frac{B\phi'(4A+rA')}{rA}G_{3X} + B\phi'^2G_{3\phi X} \\ &+\frac{B^2\phi'^3(4A+rA')}{2rA}G_{3XX} + \frac{2}{r^2}(1-B-rB\frac{A'}{A})G_{4X} + \frac{3B\phi'(4A+rA')}{rA}G_{4\phi X} \\ &-\frac{2B\phi'2}{r^2}(1-4B-4rB\frac{A'}{A})G_{4XX} - \frac{B^2\phi'^3(4A+rA')}{rA}G_{4\phi XX} \\ &-\frac{2B^3\phi'^4(A+rA')}{r^2A}G_{4XXX} - \frac{2}{r^2}(1-B-rB\frac{A'}{A})G_{5\phi} - \frac{A'B\phi'(1-3B)}{r^2A}G_{5X} \\ &+\frac{B\phi'^2}{r^2}(1-5B-5rB\frac{A'}{A})G_{5\phi X} - \frac{B^2(1-7B)A'\phi'^3}{2r^2A}G_{5XX} \\ &+\frac{B^3(A+rA')\phi'^4}{r^2A}G_{5\phi XX} + \frac{B^4A'\phi'^5}{2r^2A}G_{5XXX}, \end{aligned} \tag{5.90}$$

$$\Xi = G_{3X} + \frac{2}{B\phi'^2}G_{4\phi} + \frac{4}{r\phi'}G_{4X} - 2G_{4\phi X} - \frac{4B\phi'}{r}G_{4XX}$$

$$-\frac{4}{r\phi'}G_{5\phi}+\frac{1-3B}{r^2}G_{5X}+\frac{2B\phi'}{r}G_{5\phi X}+\frac{B^2\phi'^2}{r^2}G_{5XX}, \tag{5.91}$$

$$\begin{aligned}\Theta = {}& G_{3X}+\frac{2}{B\phi'^2}G_{4\phi}+\frac{2A+rA'}{rA\phi'}G_{4X}-2G_{4\phi X}-\frac{B\phi'(2A+rA')}{rA}G_{4XX}\\ &-\frac{2A+rA'}{rA\phi'}G_{5\phi}-\frac{3BA'}{2rA}G_{5X}+\frac{B\phi'(2A+rA')}{2rA}G_{5\phi X}+\frac{B^2A'\phi'^2}{2rA}G_{5XX}.\end{aligned} \tag{5.92}$$

In the Lagrangian (5.61), H_0 acts as a Lagrange multiplier. The variation with respect to H_0 gives

$$a_1\delta\phi''+a_2\delta\phi'+a_3H_2'+j^2a_4\alpha'+\left(a_5+j^2a_6\right)\delta\phi+\left(a_7+j^2a_8\right)H_2+j^2a_9\alpha=0. \tag{5.93}$$

We introduce a new variable ψ to descend the order of derivatives, which is defined as

$$H_2=\frac{1}{a_3}\left(\psi-a_1\delta\phi'-j^2a_4\alpha\right). \tag{5.94}$$

We use it to eliminate H_2 from (5.93)

$$\alpha=\frac{a_3\psi'+j^2a_8\psi+\left(a_3\left(a_2-a_1'\right)-j^2a_1a_8\right)\delta\phi'+a_3\left(a_5+j^2a_6\right)\delta\phi}{j^2\left[j^2a_4a_8+a_3(a_4'-a_9)\right]}. \tag{5.95}$$

The variation with respect to H_1 gives the solution for H_1 itself.

$$H_1=-\frac{1}{2j^2b_1}\left(b_2\delta\phi'+b_3\delta\phi+b_4H_2+j^2b_5\alpha\right)^{\cdot}. \tag{5.96}$$

With (5.93), (5.95) and (5.96), we reach the final expression of the action with two variables ψ and $\delta\phi$

$$\mathcal{L}_\ell^{(2)}=\frac{1}{2}\mathcal{K}_{ij}\dot{v}^i\dot{v}^j-\frac{1}{2}\mathcal{G}_{ij}v^{i'}v^{j'}-Q_{ij}v^iv^{j'}-\frac{1}{2}\mathcal{M}_{ij}v^iv^j, \tag{5.97}$$

where i and j run from 1 to 2, we take $v_1=\psi$ and $v_2=\delta\phi$, and we write the coefficients as the matrices $\mathcal{K}_{ij}$, $\mathcal{G}_{ij}$, Q_{ij} and $\mathcal{M}_{ij}$. We take that $\mathcal{K}_{ij}$, $\mathcal{G}_{ij}$, and $\mathcal{M}_{ij}$ are symmetric and Q_{ij} is antisymmetric.

To avoid ghost, which brings us negative kinetic energy, we require that all of the eigenvalues of the symmetric matrix $\mathcal{K}$ are positive. Since $\mathcal{K}$ is two-dimensional matrix, the requirement is reduced to the conditions

$$\mathrm{tr}(\mathcal{K})>0, \qquad \det(\mathcal{K})>0. \tag{5.98}$$

Furthermore the symmetric property can make the conditions simpler

$$\mathcal{K}_{11}>0, \qquad \det(\mathcal{K})>0. \tag{5.99}$$

The first condition reads

$$\mathcal{K}_{11}=\frac{8r^2\sqrt{AB}\left(2\mathcal{H}+rB\Xi\phi'^3\right)^2}{\ell(\ell+1)A^2\mathcal{H}^2}\frac{\ell(\ell+1)\mathcal{P}_1-\mathcal{F}}{\left(2r\mathcal{H}\ell(\ell+1)+\mathcal{P}_2\right)^2}>0, \tag{5.100}$$

where we have defined

$$\mathcal{P}_1 = \frac{B\left(2\mathcal{H} + rB\Xi\phi'^3\right)}{2rA\mathcal{H}^2}\left[\frac{r^2 A\mathcal{H}^4}{B\left(2\mathcal{H} + rB\Xi\phi'^3\right)^2}\right]', \tag{5.101}$$

$$\mathcal{P}_2 = -rB\left(2 - \frac{rA'}{A}\right)\left(2\mathcal{H} + rB\Xi\phi'^3\right). \tag{5.102}$$

The second condition of (5.99) reads

$$\det(\mathcal{K}) = \frac{16(\ell-1)(\ell+2)r^2\left(2\mathcal{H} + rB\Xi\phi'^3\right)^2\mathcal{F}\left(2\mathcal{P}_1 - \mathcal{F}\right)}{\ell(\ell+1)A^2\mathcal{H}^2\phi'^2\left(2r\mathcal{H}\ell(\ell+1) + \mathcal{P}_2\right)^2} > 0. \tag{5.103}$$

The condition (5.100) shows that we need

$$2\mathcal{P}_1 - \mathcal{F} > 0, \tag{5.104}$$

since we now assume $\mathcal{F} > 0$ in the odd-parity modes. If (5.104) is satisfied, (5.100) is also satisfied and thus only (5.104) is a new independent condition for avoiding ghost.

To derive the dispersion relation, we take the large ℓ limit such that lower-order terms of ℓ or j^2 in the equations of motion which are given by the variation of the action (5.4) are negligible. We also consider that the perturbations have high momentum in the radial direction. Then we can ignore Q_{ij} and $\mathcal{G}'_{ij}$ in the equations of motion and get

$$-\mathcal{K}_{ij}\ddot{v}^j + \mathcal{G}_{ij}v^{j\prime\prime} - \mathcal{M}_{ij}v^j = 0. \tag{5.105}$$

Plugging a plane wave solution, which is justified when we consider so small region that the perturbations cannot feel curvature, $v^j \propto e^{-i\omega t + ikr}$ into (5.105), we get

$$\mathcal{K}_{ij}\omega^2 v^j - \mathcal{G}_{ij}k^2 v^j - \mathcal{M}_{ij}v^j = 0. \tag{5.106}$$

In order for (5.106) to have nontrivial solution, the coefficient matrix must not have its inverse matrix and thus

$$\det(\mathcal{K}_{ij}\omega^2 - \mathcal{G}_{ij}k^2 - \mathcal{M}_{ij}) = 0. \tag{5.107}$$

This is what gives the dispersion relation. The matrices $\mathcal{K}_{ij}$, $\mathcal{G}_{ij}$ and $\mathcal{M}_{ij}$ reads

$$\mathcal{K}_{11} = \sqrt{\frac{B}{A^3}}\frac{2\mathcal{P}_1(2\mathcal{H} + rB\Xi\phi'^3)^2}{j^2\mathcal{H}^4}, \tag{5.108}$$

$$\mathcal{K}_{12} = \mathcal{K}_{21} = \frac{2\mathcal{F}(2\mathcal{H} + rB\Xi\phi'^3)}{j^2 A\mathcal{H}^2\phi'} - j^2\sqrt{\frac{A}{B}}\frac{\mathcal{H}}{\phi'}\mathcal{K}_{11}, \tag{5.109}$$

$$\mathcal{K}_{22} = -\frac{4\mathcal{F}(\mathcal{H} + rB\Xi\phi'^3)}{\sqrt{AB}\mathcal{H}\phi'^2} + \left(j^2\sqrt{\frac{A}{B}}\frac{\mathcal{H}}{\phi'}\right)^2\mathcal{K}_{11}, \tag{5.110}$$

$$\mathcal{G}_{11} = \sqrt{2\frac{B^3}{A}}\frac{1}{j^4\mathcal{H}^3}(2\mathcal{G}(\mathcal{H} + rB\Xi\phi'^3) + r^2\phi'^2(\mathcal{H}\Sigma + 2B^2\Theta\Xi\phi'^4)), \tag{5.111}$$

$$\mathcal{G}_{12} = -\mathcal{G}_{21} = \frac{2rB\phi'}{j^2\mathcal{H}^2}(r\mathcal{H}\Sigma + B\Xi\phi'(\mathcal{G} + 2rB\Theta\phi'^3)), \tag{5.112}$$

$$\mathcal{G}_{22} = \frac{r^2\sqrt{AB}(\mathcal{H}\Sigma + 2B^2\Theta\Xi\phi'^4)}{2\mathcal{H}}, \tag{5.113}$$

$$\mathcal{M}_{11} = -\sqrt{\frac{B}{A}}\frac{2y}{j^2r^2\mathcal{H}}, \tag{5.114}$$

$$\mathcal{M}_{12} = \mathcal{M}_{21} = \frac{r^2\mathcal{H}}{\sqrt{AB}\phi'}x + \frac{2\mathcal{G}}{r^2\mathcal{H}\phi'}\left(\frac{1}{B} - 2 + \frac{\mathcal{F}}{\mathcal{H}} + \frac{rB'}{2B} - \frac{r\mathcal{H}'}{\mathcal{H}} + \frac{r\mathcal{F}(A'\mathcal{H} + 2AB\Theta\phi'^3)}{2A\mathcal{G}\mathcal{H}}\right), \tag{5.115}$$

$$\mathcal{M}_{22} = -\frac{j^2r^2\mathcal{H}^2}{B\phi'2}x - \sqrt{\frac{A}{B}}\frac{2j^2\mathcal{G}}{r^2\phi'^2}\left(\frac{1}{B} - 1 - \frac{rA'}{2A} + \frac{rB'}{2B} - \frac{r\mathcal{G}'}{\mathcal{G}} + \frac{r\mathcal{F}(A'\mathcal{H} + 2AB\Theta\phi'^3)}{A\mathcal{G}\mathcal{H}}\right), \tag{5.116}$$

$$x \equiv \left[\sqrt{\frac{B}{A}}\frac{(A'\mathcal{H} + 2AB\Theta\phi'^3)}{r^2\mathcal{H}^2}\right]'. \tag{5.117}$$

Note that these expressions are given when we take the large k and ℓ limit. Substituting them into (5.107), we finally obtain

$$\left(\frac{\omega^2}{A} - \frac{\mathcal{G}}{\mathcal{F}}Bk^2 - \frac{\mathcal{G}}{\mathcal{H}}\frac{j^2}{r^2}\right)\left(\frac{\omega^2}{A} - \frac{\mathcal{G}_{\rm S}}{\mathcal{F}_{\rm S}}Bk^2 - \frac{\mathcal{G}_{\rm S}}{\mathcal{H}_{\rm S}}\frac{j^2}{r^2}\right) = M\frac{j^4}{r^4}, \tag{5.118}$$

where we have defined

$$\frac{\mathcal{G}_{\rm S}}{\mathcal{F}_{\rm S}} = \frac{r^2\phi'^2(2\mathcal{H}^2\Sigma + 4B^2\mathcal{H}\Theta\Xi\phi'^4 - B^2\mathcal{G}\Xi^2\phi'^4)}{(2\mathcal{P}_1 - \mathcal{F})\,(2\mathcal{H} + rB\Xi\phi'^3)^2}, \tag{5.119}$$

$$\frac{\mathcal{G}_{\rm S}}{\mathcal{H}_{\rm S}} = \delta - \frac{2\mathcal{H}^3y}{(2\mathcal{P}_1 - \mathcal{F})\,(2\mathcal{H} + rB\Xi\phi'^3)^2} - \frac{\mathcal{F}\mathcal{G}}{\mathcal{H}\,(2\mathcal{P}_1 - \mathcal{F})}, \tag{5.120}$$

$$M = \frac{\delta^2\,(2\mathcal{P}_1 - \mathcal{F})}{4\mathcal{F}}, \tag{5.121}$$

$$\delta = \frac{2A(\mathcal{F} - \mathcal{H})\mathcal{G}\mathcal{H} + r(\mathcal{G} - \mathcal{F})\mathcal{H}^2A' + 2rA\mathcal{H}\,(\mathcal{G}'\mathcal{H} - \mathcal{G}\mathcal{H}') + 2rAB\mathcal{F}(\mathcal{G}\Xi - \mathcal{H}\Theta)\phi'^3}{A\mathcal{H}\,(2\mathcal{P}_1 - \mathcal{F})\,(2\mathcal{H} + rB\Xi\phi'^3)}. \tag{5.122}$$

The dispersion relation (5.118) is the main result of the present chapter, and we explore it and stability conditions in the next section.

For the monopole mode $\ell = 0$, we need a different treatment of the gauge. Without any gauge transformations ξ^μ, $\alpha = \beta = G = 0$ holds identically. We have only two gauge freedom $T_{\ell m}$ and $R_{\ell m}$. We utilize $R_{\ell m}$ to kill K and then we can use the action (5.61) with $j^2 = 0$ and $K = 0$. Note that there is still a residual gauge freedom $T_{\ell m}$. We have the monopole Lagrangian

$$\mathcal{L}^{(2)}_{\ell=0} = \left(\frac{A}{2}H_0' - \dot{H}_1\right)(b_2\delta\phi' + b_3\delta\phi + b_4H_2) + c_1\dot{H}_2\delta\dot{\phi} + H_2\left[c_2\delta\phi' + c_3\delta\phi\right] + c_6H_2^2 + e_1\delta\dot{\phi}^2 + e_2\delta\phi'^2 + e_3\delta\phi^2. \tag{5.123}$$

With the variation with respect to H_0 and H_1, we get

$$H_2 = -\frac{1}{b_4}\left(b_2\delta\phi' + b_3\delta\phi\right) + \frac{2C_0}{Ab_4}, \tag{5.124}$$

where C_0 is an integration constant. It corresponds to the change of mass of the black hole [72] and is not a physical mode. Thus we set $C_0 = 0$. The action is finally written as

$$\mathcal{L}^{(2)}_{\ell=0} = \frac{1}{2}\mathcal{K}_0\delta\dot{\phi}^2 - \frac{1}{2}\mathcal{G}_0\delta\phi'^2 - \frac{1}{2}\mathcal{M}_0\delta\phi^2, \tag{5.125}$$

where the propagation speed c_s is given by $c_s^2 = (AB)^{-1}\mathcal{K}_0^{-1}\mathcal{G}_0$ and the coefficients $\mathcal{K}_0$, c_s^2, and $\mathcal{M}_0$ reads

$$\mathcal{K}_0 = \frac{4}{\sqrt{AB}\phi'^2}\left(2\mathcal{P}_1 - \mathcal{F}\right), \tag{5.126}$$

$$c_s^2 = \mathcal{G}_\mathrm{S}/\mathcal{F}_\mathrm{S}, \tag{5.127}$$

$$\mathcal{M}_0 = -\frac{1}{b_4^3}\left(2b_4^3e_3 + b_4^2\left(-2b_3c_3 + b_2'c_3 + b_2c_3' + b_3'c_2 + b_3c_2'\right)\right. \tag{5.128}$$

$$\left. + b_4\left(2b_3^2c_6 - 2(b_2'b_3c_6 + b_2b_3'c_6 + b_2b_3c_6') - b_4'(b_2c_3 + b_3c_2)\right) + 4b_4'b_2b_3c_6\right). \tag{5.129}$$

From (5.127), we can tell that the monopole mode can be identified as the scalar wave. The no-ghost condition $2\mathcal{P}_1 - \mathcal{F} > 0$ is the same as that for $\ell \geq 2$. We also get the condition for gradient stability as $c_s^2 = \mathcal{G}_\mathrm{S}/\mathcal{F}_\mathrm{S} > 0$.

For the dipole mode $\ell = 1$, K can be implicitly represented by G because the expressions written with the dipole spherical harmonics are no longer independent. Thus we can kill K without using any gauge freedoms ξ^μ. We use $T_{\ell m}$, $R_{\ell m}$, and $\Theta_{\ell m}$ to set $\beta = 0$, $\delta\phi = 0$, and $G = 0$, respectively. The analysis in the general multipole modes $\ell \geq 2$ is still available and we can utilize the expression with $\delta\phi = 0$. The action reads

$$\mathcal{L}^{(2)}_{\ell=1} = \frac{1}{2}\mathcal{K}_{11}\dot{\psi}^2 - \frac{1}{2}\mathcal{G}_{11}\psi'^2 - \frac{1}{2}\mathcal{M}_{11}\psi^2, \tag{5.130}$$

where $\mathcal{K}_{11}$, $\mathcal{G}_{11}$, and $\mathcal{M}_{11}$ are components of the matrices for $\ell \geq 2$. The action (5.130) gives the same no-ghost condition and propagation speed as in the monopole case $\ell = 0$. Thus the dipole mode is the scalar wave as well as the monopole mode.

5.5 Dispersion relation and stability conditions

Now we have derived the dispersion relation (5.118) including the propagation speed in axial directions. Suppose that $\delta = 0$ and consequently $M = 0$. Then the dispersion decouples to two independent dispersion relations

$$\frac{\omega^2}{A} = \frac{\mathcal{G}}{\mathcal{F}}Bk^2 + \frac{\mathcal{G}}{\mathcal{H}}\frac{j^2}{r^2}, \tag{5.131}$$

$$\frac{\omega^2}{A} = \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}}Bk^2 + \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\frac{j^2}{r^2}. \tag{5.132}$$

(5.131) and (5.132) are identified with the dispersion relation of gravitational waves and that of scalar waves, respectively. This shows that if $M = 0$ then gravitational waves and scalar waves propagate independently in the large momentum limit. In that case we can predicate that the propagation speeds of gravitational waves are given by $\mathcal{G}/\mathcal{F}$ and $\mathcal{G}/\mathcal{H}$ in radial direction and axial direction, respectively. It is the same result in the odd-parity sector (5.54). Likewise we can also predicate that the propagation speeds of scalar waves are given by $\mathcal{G}_\mathrm{S}/\mathcal{F}_\mathrm{S}$ and $\mathcal{G}_\mathrm{S}/\mathcal{H}_\mathrm{S}$ in radial and axial directions, respectively. The radial scalar propagation speed has been derived in [22], but the axial one has never been derived because of the complexity of calculation. In the even-parity sector, the perturbations of the scalar field and the metric are analyzed and in the general relativity, in which $\delta = M = 0$, they propagates independently.

Let $M \neq 0$ and observe how it affects the dispersions. The dispersion relation (5.118) can be regarded as a quadratic equation of ω^2 and we can express its roots as

$$\frac{\omega^2}{A} = \frac{1}{2}\Bigg\{ \left(\frac{\mathcal{G}}{\mathcal{F}} + \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}}\right) Bk^2 + \left(\frac{\mathcal{G}}{\mathcal{H}} + \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\right)\frac{j^2}{r^2} \tag{5.133}$$

$$\pm \sqrt{\left[\left(\frac{\mathcal{G}}{\mathcal{F}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}}\right) Bk^2 + \left(\frac{\mathcal{G}}{\mathcal{H}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\right)\frac{j^2}{r^2}\right]^2 + 4M\frac{j^4}{r^4}}\ \Bigg\}. \tag{5.134}$$

Here we observe that the roots contain both the propagation speeds of gravitational and scalar waves. Thus we regard M as the mixing term which causes the dispersion relations to mix between the two kinds of perturbative waves. The mixing term implies that it is difficult to construct the two independent equations of motion for purely scalar waves and for purely gravitational waves.

We investigate the condition for $M \neq 0$. With the stability conditions (5.55) and (5.104), $\delta \neq 0$ is the necessary and sufficient condition for $M \neq 0$. First, we set a model as $G_4 = f(\phi)$ and $G_5 = 0$, where f is an arbitrary function, we have

$$\mathcal{F} = \mathcal{G} = \mathcal{H} = 2f(\phi), \tag{5.135}$$

$$\mathcal{G}\Xi - \mathcal{H}\Theta = 0. \tag{5.136}$$

It causes that $\delta = M = 0$ and thus we have no mixing between gravitational and scalar waves. The general relativity is the model with $f(\phi) = (16\pi G_N)^{-1}$, and we also have no mixing. Next, we set a model as $G_{5X} = 0$, we have

$$\mathcal{F} = 2[G_4 - XG_{5\phi}], \tag{5.137}$$

$$\mathcal{G} = \mathcal{H} = 2[G_4 - 2XG_{4X} + XG_{5\phi}], \tag{5.138}$$

$$\mathcal{G}\Xi - \mathcal{H}\Theta = \frac{\mathcal{G}}{r\phi'}\left(2 - \frac{rA'}{A}\right)(G_{4X} - G_{5\phi} + 2XG_{4XX}). \tag{5.139}$$

They lead to $\delta \neq 0$ and $M \neq 0$ in general. From these considerations, one can notice that if the propagation speed of gravitational waves $\mathcal{G}/\mathcal{F}$ and $\mathcal{G}/\mathcal{H}$ are different from the speed of light $c_\gamma^2 = 1$, then the mixing term M does not vanish generally. The observation of the mixing between gravitational waves and scalar waves emitted by a binary of compact objects can provide a new point of view to test the nature of the gravity.

Finally we pursue the stability conditions to avoid gradient instabilities. Perturbations with any momentum should not cause such instabilities at any local point. Thus we impose the condition in which the squared frequency ω^2 is positive for any j and k at any spatial position r. For simplicity, we do not impose any ultraviolet cutoff. Then all of the three conditions below should be satisfied:

$$w_1 \equiv \left(\frac{\mathcal{G}}{\mathcal{F}} + \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}}\right) Bk^2 + \left(\frac{\mathcal{G}}{\mathcal{H}} + \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\right)\frac{j^2}{r^2} > 0, \tag{5.140}$$

$$w_2 \equiv \left[\left(\frac{\mathcal{G}}{\mathcal{F}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}}\right) Bk^2 + \left(\frac{\mathcal{G}}{\mathcal{H}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\right)\frac{j^2}{r^2}\right]^2 + 4M\frac{j^4}{r^4} \geq 0, \tag{5.141}$$

$$w_3 \equiv w_1^2 - w_2 > 0. \tag{5.142}$$

In order for each condition to be satisfied for all j and k, we get several inequalities for ratios of $\mathcal{G}$, $\mathcal{F}$, $\mathcal{H}$, $\mathcal{G}_\mathrm{S}$, $\mathcal{F}_\mathrm{S}$, $\mathcal{H}_\mathrm{S}$ and M. From the first condition, we get

$$\frac{\mathcal{G}}{\mathcal{F}} + \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}} > 0, \qquad \frac{\mathcal{G}}{\mathcal{H}} + \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}} > 0. \tag{5.143}$$

From the second condition, we obtain

$$\left(\frac{\mathcal{G}}{\mathcal{H}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\right)^2 + 4M \geq 0 \quad \text{if} \quad \left(\frac{\mathcal{G}}{\mathcal{H}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\right)\cdot\left(\frac{\mathcal{G}}{\mathcal{F}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}}\right) > 0, \tag{5.144}$$

$$M \geq 0 \quad \text{if} \quad \left(\frac{\mathcal{G}}{\mathcal{H}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\right)\cdot\left(\frac{\mathcal{G}}{\mathcal{F}} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}}\right) \leq 0. \tag{5.145}$$

From the third condition, we find

$$\frac{\mathcal{G}}{\mathcal{F}}\cdot\frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}} > 0, \qquad \frac{\mathcal{G}}{\mathcal{H}}\cdot\frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}} > 0, \tag{5.146}$$

and additionally we require

$$\frac{\mathcal{F}}{\mathcal{H}} + \frac{\mathcal{F}_\mathrm{S}}{\mathcal{H}_\mathrm{S}} > 0 \ \text{ or } \ \left(\frac{\mathcal{G}}{\mathcal{H}}\frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}} - \frac{\mathcal{G}}{\mathcal{F}}\frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\right)^2 + 4M\frac{\mathcal{G}}{\mathcal{F}}\cdot\frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}} < 0. \tag{5.147}$$

Taking all of those into account, we conclude that we need to require at any spatial point r

$$\frac{\mathcal{G}}{\mathcal{F}} > 0, \qquad \frac{\mathcal{G}}{\mathcal{H}} > 0, \qquad \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}} > 0, \qquad \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}} > 0, \tag{5.148}$$

and M must satisfy the conditional requirements (5.144) and (5.145). The conditions (5.147) are automatically satisfied, considering the ghost-free conditions (5.55) and (5.104) as well as (5.148). The conditions of (5.148) has been shown in [22], except for the fourth condition. We have shown that the fourth condition, which determines angular stability of the scalar waves, and the conditions for the mixing term (5.144) and (5.145) should be satisfied to perfect the stability conditions along any spatial direction.

5.6 Summary

We have reviewed the perturbation theory which is developed by [21, 22] in the first three sections. Since we have simplified some of expressions for the coefficients in the action, we have finally found the dispersion relation, including the angular directions, of the even-parity sector which represent the scalar wave and one polarization mode of gravitational waves. As a result, we have observed that their dispersion relations mix and it can cause the birefringence around a black hole. The analysis has revealed that the mixing between scalar and gravitational waves happens when the propagation speed of gravitational waves can be different from the speed of light. This may give suggestions on new ways to test the Horndeski theory around a compact object. We have also found the new stability conditions and it can be used to test the Horndeski theory through angular stability of the hairy solution of black holes.

Chapter 6

Freezing extra dimensions with anisotropic attractors

As seen in Chapter 3, the higher-order curvature term induces nontrivial attractor of the system, which we call anisotropic attractor. In the large limit of the higher-order term, that expansion of two dimensional space stopped. It motivates us to study it in higher-dimensional spacetime to think of a new mechanism to freeze extra dimensions. We require the two point that (i) expansion rate of the extra dimensions β is much smaller than that of the lower-dimensional universe α and (ii) dynamics of the lower-dimensional universe, or α, obeys the Friedmann equation in the general relativity in the whole of cosmic history as in Figure 6.1.

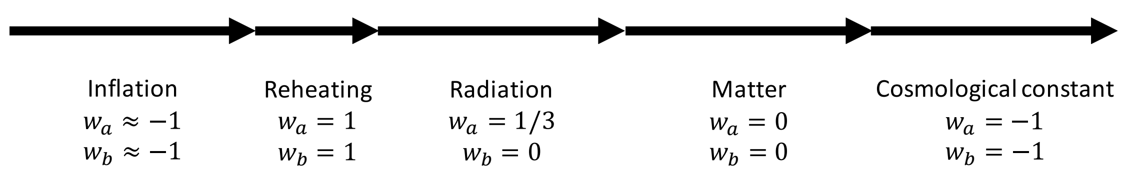

Figure 6.1: The standard view of the cosmic history. If extra dimensions exist, it is naturally deduced that the energy contents can have different pressure in the four-dimensional universe from that in the extra dimensions. w_a and w_b denotes the equation of state of energy contents in the universe dimensions and extra dimensions, respectively. The equation of state of reheating is relatively unknown and we assume that kinetic energy of inflaton is dominated.

In the first section of this chapter, we extend the concept of anisotropic attractor to higher dimensions by considering the Lovelock theory and part of the generalized Galileon. We also numerically calculate the evolution of expansion rate of each axis in the presence of a positive cosmological constant and show that inflation can make the system converge at the anisotropic attractor. This attractor realizes $\beta \ll \alpha$, so that the extra dimensions expand slowly and only the three-dimensional volume inflates. Finally, we obtain the large volume of the universe with the extra dimensions compactified by inflation in higher dimensions.

After the inflation, we need to keep the extra dimensions compactified, that is, $\beta \ll \alpha$ in the whole of cosmic history. We can still use the anisotropic attractor when energy contents with isotropic pressure are dominant, such as kinetic energy of the inflaton, cold matter and the cosmological constant. It means that in most of the cosmological period, as seen in Figure 6.1, all we need to freeze extra dimensions $\beta \ll \alpha$ is to impose a hierarchy condition. If energy contents with anisotropic pressure are dominant, the anisotropic attractor no longer exists and we cannot apply the same way to freezing extra dimensions. However, in the radiation-dominated era, the special condition is satisfied for the solution $\beta = \dot{\beta} = 0$ to exist. We also prove that given a general anisotropic pressure w_a and w_b, the acceleration of scale factor of the extra dimensions $\dot{\beta}$ is suppressed in comparison to that of the universe $\dot{\alpha}$ if $\beta \ll \alpha$ and the hierarchy condition are satisfied. At the end, we demonstrate the evolution of the scale factors in the presence of several energy contents, kination, radiation, cold matter and the cosmological constant.

6.1 Anisotropic attractors in higher dimensions

6.1.1 Lagrangian and equations of motion

We assume that the space is flat and thus we can write the metric down in the form of the Kasner metric, which has only diagonal components,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + a_{(i)}^2 \delta_{ij} dx^i dx^j, \tag{6.1}$$

where $a_{(i)} = a_{(i)}(t)$ is the scale factor in the direction of x^i, $N = N(t)$ is the lapse function and we take the synchronous gauge, in which the shift vector vanishes. We set topology of the space as a torus T^D, where D is the number of spatial dimensions. It does not prefer any special axis in the point that our model is invariant under exchange of any two axes. We start with the action of the generalized Galileon (2.19), which we show again:

$$S = \int d^{D+1}x \sqrt{-g} \sum_{n=0}^{D} \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \frac{4^{-p} n!}{(n-2p)!p!} \mathcal{L}_{n,p}\{f_n(\phi, X)\}, \tag{6.2}$$

$$\mathcal{L}_{n,p}\{f\} = g^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} (\partial_X)^{p_*} f \left[\prod_{i=1}^{p} R_{\mu_{2i-1}\mu_{2i}\nu_{2i-1}\nu_{2i}} \right] \left[\prod_{i=2p+1}^{n} \phi_{\mu_i \nu_i} \right], \tag{6.3}$$

$$p_* \equiv \left\lfloor \frac{n}{2} \right\rfloor - p, \tag{6.4}$$

where f_n are arbitrary functions, and we now redefine its integration (2.22) with derivatives. Substituting (6.1) into the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ and the second derivative of the scalar field $\phi_{\mu\nu}$, we get

$$R_{0i}{}^{0j} = \delta_i^j \frac{1}{Na_{(i)}} \frac{d}{dt} \left(\frac{\dot{a}_{(i)}}{N} \right), \tag{6.5}$$

$$R_{ij}{}^{kl} = (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) \frac{1}{N^2} \frac{\dot{a}_{(i)}}{a_{(i)}} \frac{\dot{a}_{(j)}}{a_{(j)}}, \tag{6.6}$$

$$\phi_0{}^0 = \frac{1}{N} \frac{d}{dt} \left(-\frac{\dot{\phi}}{N} \right), \tag{6.7}$$

$$\phi_i{}^j = \delta_i^j \frac{1}{N} \frac{\dot{a}_{(i)}}{a_{(i)}} \left(-\frac{\dot{\phi}}{N} \right), \tag{6.8}$$

where dots denote derivatives with respect to time t. With these, we find the final form of the Lagrangian which is equivalent to (6.3) up to total derivatives:

$$\begin{aligned} \mathcal{L}_{n,p}\{f\} =& 2^p \left(n! - 2p(n-2)!\right) \left[(\partial_X)^{p_*} f_n\right] \cdot s_n \left(-\frac{\dot{\phi}}{N} \right)^{n-2p} \\ &+ 2^p \cdot 2p(n-2)! \left[(\partial_X)^{p_*} \partial_\phi f_n\right] \cdot s_{n-1} \left(-\frac{\dot{\phi}}{N} \right)^{n-2p+1} . \end{aligned} \tag{6.9}$$

Here functions s_m's are the so-called symmetric polynomials of expansion rates $s_m = s_m(H_{(1)}, \cdots, H_{(D)})$, whose explicit form is given by

$$s_0 = 1, \tag{6.10}$$

$$s_1 = \sum_{1 \leq i \leq D} H_{(i)}, \tag{6.11}$$

$$s_2 = \sum_{1 \leq i < j \leq D} H_{(i)} H_{(j)}, \tag{6.12}$$

$$\vdots \tag{6.13}$$

$$s_D = H_{(1)} H_{(2)} \cdots H_{(D)}, \tag{6.14}$$

$$\text{otherwise } 0, \tag{6.15}$$

where $H_{(i)}$ is the expansion rate in the direction of x^i defined by

$$H_{(i)} = \frac{1}{N} \frac{\dot{a}_{(i)}}{a_{(i)}}. \tag{6.16}$$

Since we are now interested in aspects of the *gravity* which directly affects evolution of the spacetime, we desire that the system which we consider is not affected by evolution of the scalar field. For this reason, we restrict the action to that represented by the arbitrary functions f_n's below.

$$f_n = \text{const. for even } n, \tag{6.17}$$

$$\propto X^{-1/2} \text{ for odd } n. \tag{6.18}$$

Because of the shift symmetry of f_n, the second term in (6.9) vanishes. Moreover, the dependence on $\dot{\phi}$ in the first term in (6.9) totally disappears. We eliminate irrelevant factors and write such homogeneous action in a simple form with constant parameters c_m,

$$S = \int dt d^D x N \left[\prod_{i=1}^{D} a_{(i)}\right] \sum_{m=0}^{D} c_m s_m. \tag{6.19}$$

Note that the full action which gives us this homogeneous action has a scalar field in the odd-order terms $c_{m=2l+1}$, and we have to solve evolution of the scalar field in perturbative calculation. On the other hand, the even-order terms $c_{m=2l}$ are free from the scalar field and are equivalent to the Lovelock theory. The emergence of the odd-order terms provides us the way to analyze odd-order terms of the expansion rates $H_{(i)}$ in the homogeneous action.

$c_0 = -\Lambda/(16\pi G_{\mathrm{N}})$ and $c_2 = -1/(16\pi G_{\mathrm{N}})$ reduce the action to that of the general relativity with a positive cosmological constant in spatially flat spacetime. Throughout the present chapter, we assume $c_0 < 0$ and $c_2 < 0$ to recover the Friedmann equation in lower-dimensional universe.

Variation of the action (6.19) with respect to N gives the Hamiltonian constraint

$$\sum_{m=0}^{D} (1-m) c_m s_m = 0, \tag{6.20}$$

where we have set $N(t) = 1$ without loss of generality. The evolution equations of the scale factor $a_{(j)}$ are given by the variation of the action (6.19) with respect to $a_{(j)}$

$$\begin{aligned}
&\left[\prod_{i\neq j} a_{(i)}\right] \mathcal{L} + \left[\prod_{i} a_{(i)}\right] \frac{\partial \mathcal{L}}{\partial a_{(j)}} - \frac{d}{dt}\left\{\left[\prod_{i} a_{(i)}\right] \frac{\partial \mathcal{L}}{\partial \dot{a}_{(j)}}\right\} \\
&= \left[\prod_{i\neq j} a_{(i)}\right] \mathcal{L} - \left[\prod_{i\neq j} a_{(i)}\right] H_{(j)} \sum_{m=1}^{D} c_m \frac{\partial s_m}{\partial H_{(j)}} - \frac{d}{dt}\left\{\left[\prod_{i\neq j} a_{(i)}\right] \sum_{m=1}^{D} c_m \frac{\partial s_m}{\partial H_{(j)}}\right\} \\
&= \left[\prod_{i\neq j} a_{(i)}\right] \mathcal{L} - \frac{1}{a_{(j)}} \frac{d}{dt}\left\{\left[\prod_{i} a_{(i)}\right] \sum_{m=1}^{D} c_m \frac{\partial s_m}{\partial H_{(j)}}\right\} = 0 \\
&\quad \text{for } j = 1, \cdots, D,
\end{aligned} \tag{6.21}$$

where we have defined the Lagrangian density $\mathcal{L} \equiv \sum_{m=0}^{D} c_m s_m$. When we add perfect fluid to the system, which is characterized only by the energy density ρ, the pressure $p_{(i)}$, and the comoving flux vector $u^\mu = (-1, 0, \cdots, 0)$, we can treat such a energy content by adding the energy term and pressure term to (6.20) and (6.21). We obtain

$$\frac{\rho}{2} + \sum_{m=0}^{D}(m-1)c_m s_m = 0, \tag{6.22}$$

$$\mathcal{L} + \frac{p_{(j)}}{2} - \frac{1}{V}\frac{d}{dt}\left\{V\sum_{m=1}^{D} c_m \frac{\partial s_m}{\partial H_{(j)}}\right\} = 0 \tag{6.23}$$

for $j = 1, \cdots, D$,

where V is the volume factor $V = \prod_i a_{(i)}$. Note that we can eliminate c_1 from both the equations and choose $c_1 = 0$ without loss of generality. From these equations (6.22) and (6.23), we derive another equation: the continuity equation

$$d(\rho V) + \sum_{i=1}^{D} p_{(i)} V d\ln a_{(i)} = 0. \tag{6.24}$$

6.1.2 Attractors with isotropic energy contents

Here we argue that the system has fixed points or attractors if the spacetime is filled with isotropic energy content, which has isotropic pressure ($p_{(i)} = p$ for all i). We subtract (6.23) for j from that for $k \neq j$ to yield

$$\frac{d}{dt}\left\{V\left(H_{(j)} - H_{(k)}\right)\sum_{m=2}^{D} c_m \frac{\partial^2 s_m}{\partial H_{(j)}\partial H_{(k)}}\right\} = 0 \quad \text{for } 1 \leq j < k \leq D, \tag{6.25}$$

or we integrate them and get their equivalent expressions

$$\left(H_{(j)} - H_{(k)}\right)\sum_{m=2}^{D} c_m \frac{\partial^2 s_m}{\partial H_{(j)}\partial H_{(k)}} = \frac{\mathcal{A}_{jk}}{V} \quad \text{for } 1 \leq j < k \leq D, \tag{6.26}$$

where $\mathcal{A}_{jk}$ are integration constants. Only $(D-1)$ equations of them are independent and they can replace other equations of motion.

Note that the solution of the polynomial equations

$$\left(H_{(j)} - H_{(k)}\right)\sum_{m=2}^{D} c_m q_m^{(j,k)} = 0, \tag{6.27}$$

$$q_m^{(j,k)} \equiv \frac{\partial^2 s_m}{\partial H_{(j)}\partial H_{(k)}}, \tag{6.28}$$

constitute an invariant set of the system under the dynamics in the D-dimensional phase space $(H_{(1)}, H_{(2)}, \cdots, H_{(D)})$. A similar equation has been derived in the previous research [54], assuming constant expansion rates. As the volume factor $V = \prod_i a_{(i)}$ increases, the amplitude of the right-hand-side of (6.26) decreases. Thus we can regard the invariant set as an attractor if the total-volume expansion rate $\dot{V}/V = \sum_i H_{(i)}$ is positive. We call the invariant set "(an)isotropic attractors" although we must carefully analyze the system to see whether it really acts as an attractor. For this purpose, we have to define how to measure nearness.

6.1.3 Classification of attractors

Here we divide the root of the $(D-1)$ polynomial equations (6.27) into several types of attractors. Each attractor is a one-dimensional curve in the D-dimensional phase space $(H_{(1)}, H_{(2)}, \cdots, H_{(D)})$. On the attractors, some of the expansion rates $H_{(i)}$ have the same value with each other, and thus we label the attractors with N, which indicates the number of different values of expansion rates.

Isotropic case ($N = 1$) When all of $H_{(i)}$'s are the same, then all of the polynomial equations (6.27) are trivially satisfied. It means that the isotropic expansion is one of the attractors of the system, and it is just what we call the isotropic attractor.

Anisotropic case ($N = 2$) For the simplest departure of the isotropy, let us consider the case for $N = 2$ in which the expansion rates on attractors have two different values α and β.

$$H_{(i)} = \alpha \quad \text{for } 1 \leq i \leq d, \tag{6.29}$$

$$H_{(i)} = \beta \quad \text{for } d+1 \leq i \leq D, \tag{6.30}$$

where d is any integer which satisfies $1 \leq d \leq D-1$. For $1 \leq j \leq d$ and $d+1 \leq k \leq D$, (6.27) gives the relation between α and β

$$\sum_{m=2}^{D} c_m Q_m(\alpha, \beta) = 0, \tag{6.31}$$

where

$$Q_m(\alpha,\beta) \equiv q_m^{(j,k)}\big|_{H_{(1\leq j\leq d)}=\alpha, H_{(d+1\leq k\leq D)}=\beta} \tag{6.32}$$

$$= \sum_{l=0}^{m-2} \binom{d-1}{l} \binom{D-d-1}{m-2-l} \alpha^l \beta^{m-2-l}. \tag{6.33}$$

Evolution of the system on such anisotropic attractors is obtained by solving (6.31) with the Hamiltonian constraint and evolution equation in which the expansion rates $H_{(i)}$'s are replaced by (6.29) and (6.30).

$$\frac{\rho}{2} + \sum_{m=0}^{D} (m-1) c_m S_m = 0, \tag{6.34}$$

$$\frac{p}{2} + \sum_{m=0}^{D} c_m S_m - \frac{1}{d}\frac{1}{V}\frac{d}{dt}\left\{ V \sum_{m=1}^{D} c_m \frac{\partial S_m}{\partial \alpha} \right\} = 0, \tag{6.35}$$

$$\frac{p}{2} + \sum_{m=0}^{D} c_m S_m - \frac{1}{D-d}\frac{1}{V}\frac{d}{dt}\left\{ V \sum_{m=1}^{D} c_m \frac{\partial S_m}{\partial \beta} \right\} = 0, \tag{6.36}$$

where S_m is defined by

$$S_m(\alpha,\beta) \equiv s_m|_{H_{(1\le i\le d)}=\alpha, H_{(d+1\le i\le D)}=\beta} = \sum_{l=0}^{m} \binom{d}{l}\binom{D-d}{m-l}\alpha^l \beta^{m-l}, \tag{6.37}$$

for given number of dimensions D and d.

Anisotropic case ($N \geq 3$) Let us investigate the case with three different values of expansion rates on the attractor. Suppose that $H_{(j)} \neq H_{(k)}$ and $H_{(k)} \neq H_{(l)}$. Then (6.27) implies

$$\sum_{m=2}^{D} c_m q_m^{(j,k)} = 0, \tag{6.38}$$

$$\sum_{m=2}^{D} c_m q_m^{(k,l)} = 0, \tag{6.39}$$

For $q_m^{(j,k)}$, the following equation holds.

$$q_m^{(j,k)} = H_{(l)} \frac{\partial q_m^{(j,k)}}{\partial H_{(l)}} + \frac{\partial q_{m+1}^{(j,k)}}{\partial H_{(l)}}. \tag{6.40}$$

Using these equations, we get

$$H_{(l)} = -\left[\sum_{m=2}^{D} c_m \frac{\partial q_{m+1}^{(j,k)}}{\partial H_{(l)}}\right]\left[\sum_{m=3}^{D} c_m \frac{\partial q_m^{(j,k)}}{\partial H_{(l)}}\right]^{-1}, \tag{6.41}$$

$$H_{(j)} = -\left[\sum_{m=2}^{D} c_m \frac{\partial q_{m+1}^{(k,l)}}{\partial H_{(j)}}\right]\left[\sum_{m=3}^{D} c_m \frac{\partial q_m^{(k,l)}}{\partial H_{(j)}}\right]^{-1}. \tag{6.42}$$

Since

$$\frac{\partial q_m^{(j,k)}}{\partial H_{(l)}} = \frac{\partial q_m^{(k,l)}}{\partial H_{(j)}} = \frac{\partial^3 s_m}{\partial H_{(j)}\partial H_{(k)}\partial H_{(l)}}, \tag{6.43}$$

we inevitably find $H_{(j)} = H_{(l)}$ unless both of the insides of the square brackets vanish. In conclusion of the case for $N = 3$, we have to solve both equations

$$\sum_{m=3}^{D} c_m \left.\frac{\partial^3 s_m}{\partial H_{(j)}\partial H_{(k)}\partial H_{(l)}}\right|_{H_{(1\le j\le d_1)}=\alpha, H_{(d_1+1\le k\le d_2)}=\beta, H_{(d_2+1\le l\le D)}=\gamma} = 0, \tag{6.44}$$

$$\sum_{m=2}^{D} c_m \left.\frac{\partial^3 s_{m+1}}{\partial H_{(j)}\partial H_{(k)}\partial H_{(l)}}\right|_{H_{(1\le j\le d_1)}=\alpha, H_{(d_1+1\le k\le d_2)}=\beta, H_{(d_2+1\le l\le D)}=\gamma} = 0, \tag{6.45}$$

to find the position of such anisotropic attractors, where $1 \leq d_1 < d_2 \leq D-1$. In order to have a nontrivial solution for the first equation (6.44), we need $D > 3$. Otherwise, $D = 3$, we get a solution $c_2 = c_3 = 0$, but in this case the fixed-point equation (6.27) is always true and we no longer have anisotropic attractors.

The discussion above can be generalized straightforwardly to the cases with larger $N < D$. In order to show the essence of the anisotropic attractors, we focus on the case for $N = 2$ in the following.

6.1.4 Example of attractors for $D = 6$

Here we give the example for the system to converge on the anisotropic attractors. In seven-dimensional spacetime (or six-dimensional space $D = 6$), the Lagrangian has terms up to c_6

$$\mathcal{L} = c_0 + c_1 s_1 + c_2 s_2 + c_3 s_3 + c_4 s_4 + c_5 s_5 + c_6 s_6, \tag{6.46}$$

where the symmetric polynomials s_m are

$$\begin{aligned}
s_0 &= 1, \\
s_1 &= H_{(1)} + H_{(2)} + H_{(3)} + H_{(4)} + H_{(5)} + H_{(6)}, \\
s_2 &= H_{(1)}(H_{(2)} + H_{(3)} + H_{(4)} + H_{(5)} + H_{(6)}) + H_{(2)}(H_{(3)} + H_{(4)} + H_{(5)} + H_{(6)}) \\
&\quad + H_{(3)}H_{(4)} + H_{(3)}H_{(5)} + H_{(3)}H_{(6)} + H_{(4)}H_{(5)} + H_{(4)}H_{(6)} + H_{(5)}H_{(6)}, \\
s_3 &= H_{(1)}[H_{(2)}(H_{(3)} + H_{(4)} + H_{(5)} + H_{(6)}) + H_{(3)}(H_{(4)} + H_{(5)} + H_{(6)}) \\
&\qquad\quad + H_{(4)}H_{(5)} + H_{(4)}H_{(6)} + H_{(5)}H_{(6)}] \\
&\quad + H_{(2)}[H_{(3)}(H_{(4)} + H_{(5)} + H_{(6)}) + H_{(4)}(H_{(5)} + H_{(6)}) + H_{(5)}H_{(6)}] \\
&\quad + H_{(3)}H_{(4)}H_{(5)} + H_{(3)}H_{(4)}H_{(6)} + H_{(3)}H_{(5)}H_{(6)} + H_{(4)}H_{(5)}H_{(6)}, \\
s_4 &= H_{(1)}\{H_{(2)}[H_{(3)}(H_{(4)} + H_{(5)} + H_{(6)}) + H_{(4)}(H_{(5)} + H_{(6)}) + H_{(5)}H_{(6)}] \\
&\qquad\quad + H_{(3)}[H_{(4)}(H_{(5)} + H_{(6)}) + H_{(5)}H_{(6)}] + H_{(4)}H_{(5)}H_{(6)}\} \\
&\quad + H_{(2)}\{H_{(3)}[H_{(4)}(H_{(5)} + H_{(6)}) + H_{(5)}H_{(6)}] + H_{(4)}H_{(5)}H_{(6)}\} + H_{(3)}H_{(4)}H_{(5)}H_{(6)}, \\
s_5 &= H_{(1)}H_{(2)}\{H_{(3)}H_{(4)}H_{(5)} + H_{(3)}H_{(4)}H_{(6)} + H_{(3)}H_{(5)}H_{(6)} + H_{(4)}H_{(5)}H_{(6)}\} \\
&\quad + H_{(1)}H_{(3)}H_{(4)}H_{(5)}H_{(6)} + H_{(2)}H_{(3)}H_{(4)}H_{(5)}H_{(6)}, \\
s_6 &= H_{(1)}H_{(2)}H_{(3)}H_{(4)}H_{(5)}H_{(6)}.
\end{aligned}$$

We have the full set of the equations of motion:

$$\frac{\rho}{2} - c_0 + c_2 s_2 + 2c_3 s_3 + 3c_4 s_4 + 4c_5 s_5 + 5c_6 s_6 = 0, \tag{6.47}$$

$$\mathcal{L} + \frac{p_{(j)}}{2} - \frac{1}{V}\frac{d}{dt}\left\{V \sum_{m=1}^{6} c_m \frac{\partial s_m}{\partial H_{(j)}}\right\} = 0 \quad \text{for } j = 1, \cdots, 6, \tag{6.48}$$

$$d(\rho V) + \sum_{i=1}^{6} p_{(i)} V d\ln a_{(i)} = 0, \tag{6.49}$$

where one of these is redundant and $V = a_{(1)}a_{(2)}a_{(3)}a_{(4)}a_{(5)}a_{(6)}$. Let us consider the isotropic energy content and the anisotropic attractor which has two different expansion rate α and β.

$$H_{(i)} = \alpha \quad \text{for } 1 \le i \le d, \tag{6.50}$$

$$H_{(i)} = \beta \quad \text{for } d+1 \le i \le 6. \tag{6.51}$$

Here we regard α as the expansion rate of our universe and β as that of extra dimensions. The anisotropic attractors for $N = 2$ can be classified with d. For each d, the polynomial equation (6.31) reads

$$c_2 + 4c_3\beta + 6c_4\beta^2 + 4c_5\beta^3 + c_6\beta^4 = 0 \quad \text{for } d = 1, \tag{6.52}$$

$$c_2 + c_3(\alpha + 3\beta) + c_4\beta(\alpha+\beta)c_5\beta^2(3\alpha+\beta) + c_6\alpha\beta^3 = 0 \quad \text{for } d = 2, \tag{6.53}$$

$$c_2 + 2c_3(\alpha+\beta) + c_4(\alpha^2 + 4\alpha\beta + \beta^2) + 2c_5\alpha\beta(\alpha+\beta) + c_6\alpha^2\beta^2 = 0 \quad \text{for } d = 3, \tag{6.54}$$

$$c_2 + c_3(3\alpha + \beta) + c_4\alpha(\alpha+\beta)c_5\alpha^2(\alpha+3\beta) + c_6\alpha^3\beta = 0 \quad \text{for } d = 4, \tag{6.55}$$

$$c_2 + 4c_3\alpha + 6c_4\alpha^2 + 4c_5\alpha^3 + c_6\alpha^4 = 0 \quad \text{for } d = 5, \tag{6.56}$$

whereas the isotropic attractor is given by $H_{(1)} = H_{(2)} = H_{(3)} = H_{(4)} = H_{(5)} = H_{(6)}$.

For $d = 3$, which is the case of our most interest, solving (6.54) for β gives the roots $\beta = \tilde{\beta}(\alpha)$

$$\tilde{\beta}(\alpha) = \frac{-f_5 \pm \sqrt{f_5^2 - f_4 f_6}}{f_6}, \tag{6.57}$$

$$f_4 = c_2 + 2c_3\alpha + c_4\alpha^2, \tag{6.58}$$

$$f_5 = c_3 + 2c_4\alpha + c_5\alpha^2, \tag{6.59}$$

$$f_6 = c_4 + 2c_5\alpha + c_6\alpha^2, \tag{6.60}$$

when $f_6 \neq 0$ and $f_5^2 - f_4 f_6 \ge 0$. To emphasize that β is a function of α, we have put a tilde on β. The anisotropic attractor is one-dimensional curves expressed by the function $\tilde{\beta}$ in the phase space and we regard the function $\tilde{\beta}$ itself as the attractor.

The roots of (6.52–6.56) represent the one-dimensional curves in the six-dimensional phase space $(H_{(1)}, H_{(2)}, H_{(3)}, H_{(4)}, H_{(5)}, H_{(6)})$. Figure 6.2 shows an example of attractors in (α, β) space for $c_2 = -1$ and $c_6 = 1$ with the other parameters vanishing. There are five conservation equations given by (6.26), which restrict the position of the system to another one-dimensional curve. On the other hand, the Hamiltonian constraint (6.47) represents five-dimensional surfaces. Their intersection determines the evolution of the system. When we choose one of the attractor curves and it has no intersection with the Hamiltonian constraint surface for given c_m's, such an attractor is not the attractor of the system. Especially when the isotropic attractor curve has no interaction with the Hamiltonian constraint surface, the system cannot converge on the isotropic attractor and prefers to terminate in the anisotropic attractor. To see this, let us study the model in which $c_3 = c_4 = c_5 = 0$ and $\rho = p = 0$. On the isotropic attractor $H_{(1)} = H_{(2)} = H_{(3)} = H_{(4)} = H_{(5)} = H_{(6)} = \alpha$, the Hamiltonian constraint (6.47) reads

$$-c_0 + 15c_2\alpha^2 + 5c_6\alpha^6 = 0, \tag{6.61}$$

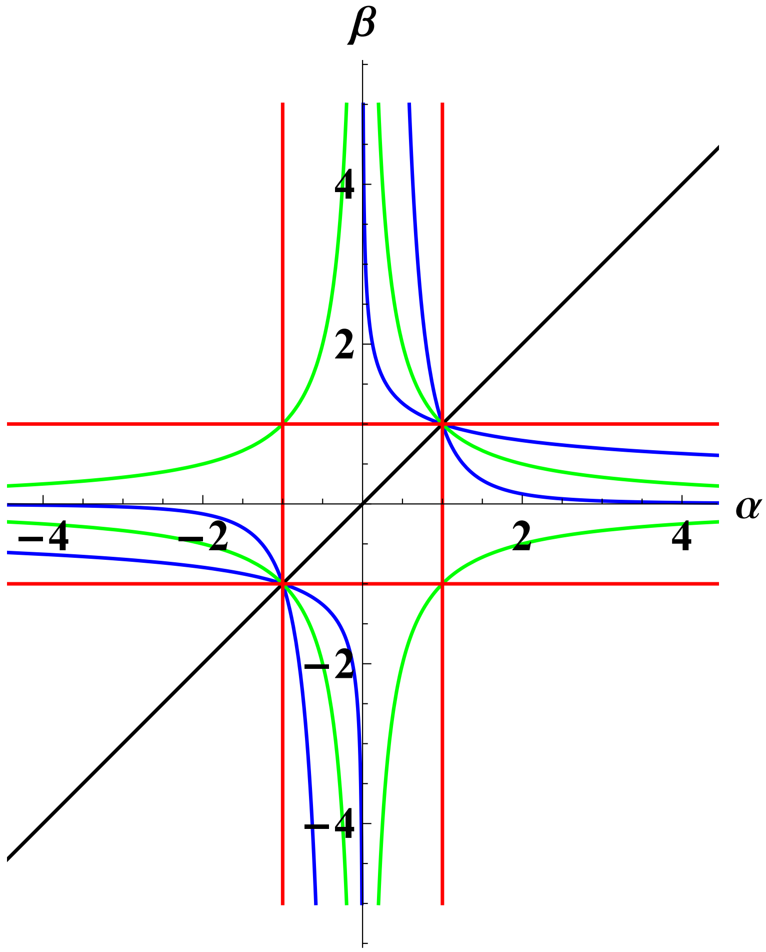

Figure 6.2: The isotropic attractors (black line) and anisotropic attractors (6.52–6.56) plotted in (α, β) plane. The parameters are taken as $c_2 = -1$, $c_6 = 1$, and $c_3 = c_4 = c_5 = 0$. The red, blue, and green curves corresponds to the cases for $d = 1$ or 5, $d = 2$ or 4, and $d = 3$, respectively.

of which the discriminant is given by

$$c_0 c_6^3 (100 c_2^3 + c_0^2 c_6)^2. \tag{6.62}$$

It shows that if $c_6 > 0$ and $c_6 > -100c_2^3/c_0^2$ the reduced Hamiltonian constraint has no real root, where we have imposed the assumption $c_0 < 0$ to get a positive cosmological constant. In the case for $d = 1$ and $d = 4$, we also observe that there is no intersection between the attractor curve and the Hamiltonian constraint surface. Thus the system is allowed to converge on the anisotropic attractor for $d = 2$, $d = 3$, and $d = 4$.

We demonstrate that the system actually converges to the anisotropic attractor for $d = 3$. Let $c_0 = -0.1$, $c_2 = -1$, $c_6 = 10^8$, $c_3 = c_4 = c_5 = 0$, and $\rho = p = 0$. Those parameters no longer allow the system to converge on the isotropic attractor. Solving the Hamiltonian constraint (6.47) with (6.27), we obtain one of the roots as $\alpha = 0.182$ and $\beta = 5.49 \times 10^{-4}$. With the cosmological constant c_0, the system expands nearly exponentially. Such an exponential volume expansion causes the scale factors to converge on specific values, as shown in Figure 6.3, where the initial values of the scale factors are chosen at random, but we have relabeled the scale factors in order of the amplitude of their expansion rates. We observe that three of the scale factors increase faster than the other three scale factors in Figure 6.4. We have started with the space of D-torus with $a_i(t = 0) = 1$ for $1 \le i \le 6$. We define the external volume of the universe $V_{\text{ex}} = a_{(1)}a_{(2)}a_{(3)}$ and the internal volume of the extra dimensions $V_{\text{in}} = a_{(4)}a_{(5)}a_{(6)}$. It is shown that V_{ex} reaches at 10^{24} but $V_{\text{in}} \approx 1.4$ at $t = 100$. As a result, the large volume of the universe has emerged and the volume of the extra dimensions is kept small. We stress that there is no difference among each spatial dimension except for the initial values of the expansion rates.

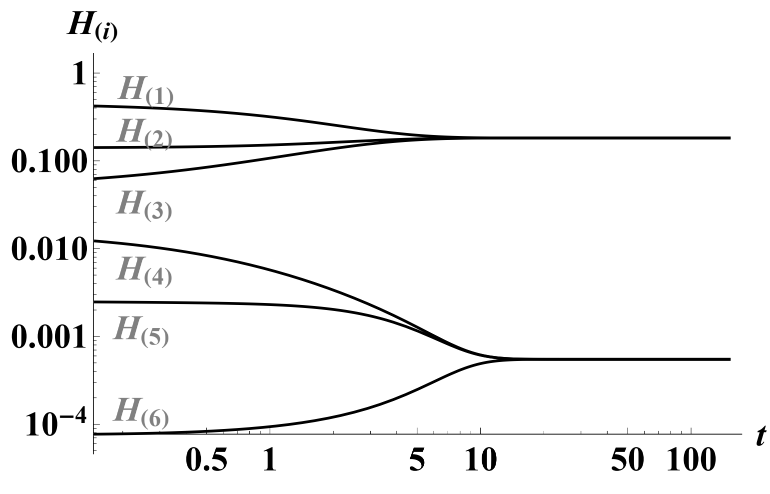

Figure 6.3: The evolution of the expansion rates $H_{(i)}$ with cosmological time t. The initial values of the six scale factors are determined at random. Three of them converge on a single value $\alpha = 0.182$ and the other three on $\beta = 5.49 \times 10^{-4}$.

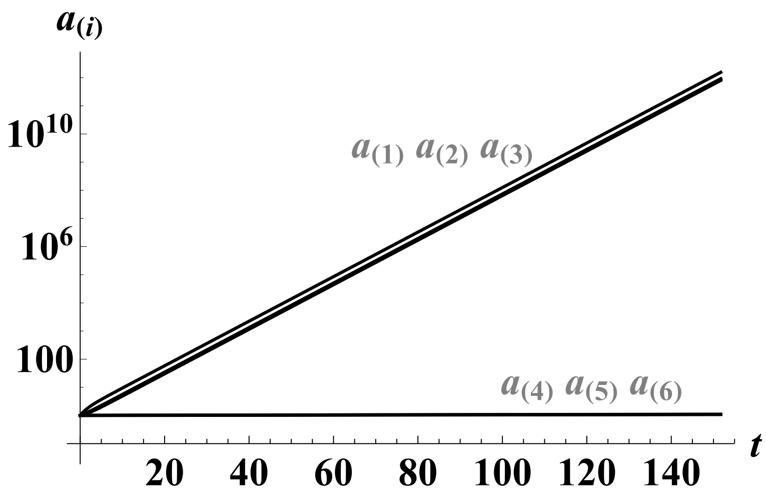

Figure 6.4: The evolution of the scale factors $a_{(i)}$ with cosmological time t. The initial values of the scale factors are set $a_{(i)}(t = 0) = 1$. Three of the scale factors expand rather faster than the other three. Since we start with a torus of the space, we finally observe large volume of the universe and small internal volume of the extra dimensions.

6.1.5 Analysis with two scale factors

In the standard isotropic cosmology, such as the Friedmann universe, we often analyze the dynamics of the universe with a single scale factor. It is only justified when the isotropic evolution is an attractor of the system, and most of the ordinary cases which we are interested in allow such a simple analysis. We also hope to study anisotropic evolution of the system around the anisotropic attractor in such a simplified way with two scale factors. In this subsection, we study the condition to justify it.

Suppose that the expansion rates are perturbed as

$$H_{(i)} = (1 + \delta_{(i)})\alpha \quad \text{for } 1 \le i \le d, \tag{6.63}$$
$$H_{(i)} = (1 + \delta_{(i)})\beta \quad \text{for } d + 1 \le i \le D, \tag{6.64}$$

where we define $\alpha \equiv H_{(1)}$ and $\beta \equiv H_{(d+1)}$ and so that $\delta_{(1)} = \delta_{(d+1)} = 0$. We now assume the equation of state [1]

$$p_{(i)} = w_a\rho \quad \text{for } 1 \le i \le d, \tag{6.65}$$
$$p_{(i)} = w_b\rho \quad \text{for } d + 1 \le i \le D. \tag{6.66}$$

It is no longer isotropic if $w_a \neq w_b$, but the same equation as (6.26) holds for $1 \le j < k \le d$ or $d + 1 \le j < k \le D$. If $\delta_{(i)} \ll 1$ for all i, then we neglect higher-order terms

[1] These type of equations of state holds for the energy contents generated after inflation. Figure 6.1 shows the equations of state for the energy contents which is often considered in the standard cosmic history.

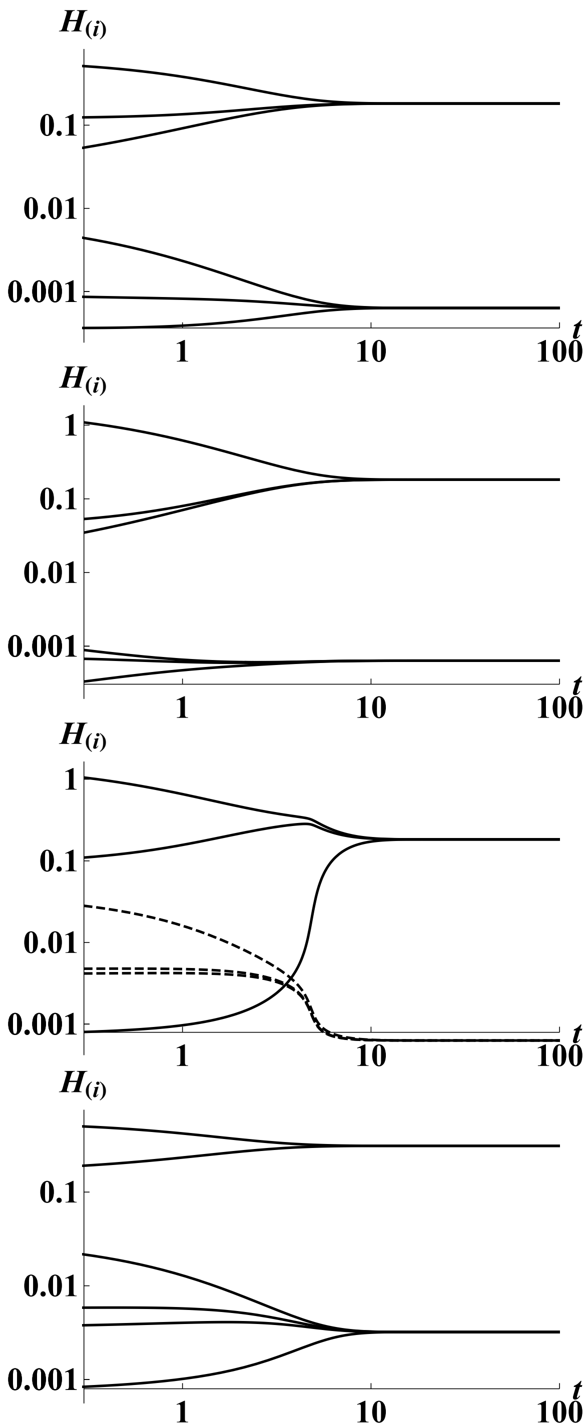

Figure 6.5: The evolution of the expansion rates with several initial conditions. Dashed lines means that their values are negative. The first two plots show the evolution to the anisotropic attractor of $d = 3$. The third plot also shows the evolution to the anisotropic attractor of $d = 3$ but one of the asymptotic values is negative. The forth plot shows the evolution to the anisotropic attractor for $d = 2$.

and get

$$\alpha\left(\delta_{(j)}-\delta_{(k)}\right)\frac{1}{d(d-1)}\sum_{m=2}^{D}c_m\frac{\partial^2 S_m}{\partial\alpha^2}=\frac{\mathcal{A}_{jk}}{V}\qquad\text{for } 1\le j<k\le d, \tag{6.67}$$

$$\beta\left(\delta_{(j)}-\delta_{(k)}\right)\frac{1}{(D-d)(D-d-1)}\sum_{m=2}^{D}c_m\frac{\partial^2 S_m}{\partial\beta^2}=\frac{\mathcal{A}_{jk}}{V}\qquad\text{for } d+1\le j<k\le D. \tag{6.68}$$

From them, we can calculate evolution of squared amplitude of the perturbations

$$\sum_{1\le j<k\le d}\left(\delta_{(j)}-\delta_{(k)}\right)^2\propto\left[\alpha V\sum_{m=2}^{D}c_m\frac{\partial^2 S_m}{\partial\alpha^2}\right]^{-2}, \tag{6.69}$$

$$\sum_{d+1\le j<k\le D}\left(\delta_{(j)}-\delta_{(k)}\right)^2\propto\left[\beta V\sum_{m=2}^{D}c_m\frac{\partial^2 S_m}{\partial\beta^2}\right]^{-2}. \tag{6.70}$$

When their right-hand sides are constants or decreasing functions, the expansion rates converge at the two values α and β. If we have the same expansion rates in some directions, we can rescale the scale factors for those directions and write them with a single variable. Therefore we can write all scale factors with two variables denoted by a and b

$$ds^2=-dt^2+a^2\left[\left(dx^1\right)^2\cdots\left(dx^d\right)^2\right]+b^2\left[\left(dx^{d+1}\right)^2+\left(dx^D\right)^2\right], \tag{6.71}$$

as long as the sum of deviations $\sum\left(\delta_{(j)}-\delta_{(k)}\right)^2$ decreases and the preceding period, such as inflation, realizes $\sum\left(\delta_{(j)}-\delta_{(k)}\right)^2\ll 1$ as is seen in the previous subsection. In the following, we analyze the dynamics of the system in the two-dimensional phase space $(\dot{a}/a,\dot{b}/b)$. We introduce the notation that (H_a,H_b) denotes the whole of the two-dimensional phase space, whereas (α,β) denotes the point of the system in the phase space which satisfies all of the equations of motion.

6.2 Freezing extra dimensions

In order for the system to be consistent with the fact that there is no observed time variation of the physical fundamental constants [58], the expansion rate of the extra dimensions β should be much smaller than α, the expansion rate of the universe, because changing size of extra dimensions causes variation of physical constants, such as the Newton constant $G_{\rm N}$ and the fine-structure constant $\alpha_{\rm EM}$. One of the most stringent constraints on those variance is, for example, that on the fine-structure constant by natural nuclear reactor at Oklo [73] as $|\dot{\alpha}_{\rm EM}/\alpha_{\rm EM}|<0.5\times10^{-16}$ year^{-1} at the redshift $z=0.15$, when we assume that the physical constant has changed constantly. If the $U(1)_{\rm EM}$ gauge field originates from a gauge field in higher dimensions, $\alpha_{\rm EM}$ is proportional to a power of b. Assuming this, the rate of change of physical constant is of the same order as that of expansion rate of extra dimensions β, and finally we estimate $|\beta| \lesssim 10^{-16}$ year^{-1} in the late universe. Today's Hubble parameter is $\alpha \sim 10^{-10}$ year^{-1} and thus we need $|\beta/\alpha| < 10^{-6}$ in the era when the universe is dominated by dark energy or the cosmological constant.

6.2.1 Energy contents with isotropic pressure

The Planck observation is consistent with the ΛCDM model, which describes the cosmological expansion of the space in the late universe. The cosmological constant Λ or c_0 has isotropic pressure even in higher dimensions. It is obvious from (6.23) that its equation of state is given by $w_{(i)} \equiv p_{(i)}/\rho = -1$. Cold (dark) matter has also isotropic (zero) pressure since it does not give its momentum to the outside and thus $w_{(i)} = 0$.

The Planck observation and the preceding observations of the cosmic microwave background have revealed the significant cosmological view that the universe is so flat that initial square amplitude of the curvature perturbations is of the order of 10^{-10}. One of the most convincing mechanism to provide such initial condition is inflation. The simplest model of inflation is caused by a slow-rolling scalar field, which has the equation of state $w \approx -1$. In higher dimensions, we also observe that a slow-rolling homogeneous scalar field provides $w_{(i)} \approx -1$. It means that we have isotropic pressure in inflationary period, and the system approach the anisotropic attractor exponentially. Hence we assign another role to inflation in higher dimensions: the inflation compactifies the higher-dimensional spacetime on approaching an anisotropic attractor. Then the large volume of three spatial dimensions of the universe appears and the extra dimensions freeze.

There are several attractors with different d, and roughly speaking, the system approaches the nearest attractor. If we find the probability distribution function of the initial condition, we can compute the probability that the higher dimensions are compactified to three large spatial dimensions. Otherwise we cannot find any other reasons of the four dimensionality of the universe than an accident.

In this section we see the model with isotropic energy contents, in which the anisotropic attractors can cause slow expansion of the extra dimensions $\beta \ll \alpha$ with the hierarchy between the Einstein-Hilbert term c_2 and higher-order terms $c_{m\geq d+2}$.

First, let us investigate the condition for the function $\tilde{\beta}$ to be finite in the limit $\alpha \to \infty$. This property is appropriate for keeping small extra dimensions even at high energy scale. To see the condition, we start with the special case for $D = 6$ and $d = 3$. From (6.57),

$$\tilde{\beta}(\alpha \to \infty) = \frac{-c_5 \pm \sqrt{c_5^2 - c_4 c_6}}{c_6}, \tag{6.72}$$

when $c_6 \neq 0$. In the limit $c_6 \to 0$, the asymptotic value is reduced to $\tilde{\beta}(\alpha \to \infty) = -c_4/(2c_5)$. Note we can no longer take the limit $c_5 \to 0$ without divergence of $\tilde{\beta}(\alpha \to \infty)$. It indicates that we need such nonvanishing higher-order terms as $c_{m\geq 5}$ to have a finite value of $\tilde{\beta}(\alpha \to \infty)$.

The polynomial equation (6.31) for general D tells that if we want three-dimensional space of the universe ($d = 3$), the highest-order terms of α in the equation should be quadratic. Vanishing coefficient of the quadratic terms gives the asymptotic value of $\tilde{\beta}$ in the limit $\alpha \to \infty$

$$\sum_{m=4}^{D} c_m \binom{D-4}{m-4} \left[\tilde{\beta}(\alpha \to \infty)\right]^{m-4} = 0. \tag{6.73}$$

From this, we can infer that $D-4 \geq 1$ should be satisfied in order to solve the equation for $\tilde{\beta}(\alpha \to \infty)$ so that the equation can contains its linear or higher-order term. This clarifies that we need six or higher-dimensional spacetime and a nonvanishing $c_{m\geq 5}$ term in the action to compactify the spacetime into a four-dimensional universe. For general d, we have the equation giving the asymptotic value $\tilde{\beta}(\alpha \to \infty)$

$$\sum_{m=d+1}^{D} c_m \binom{D-d-1}{m-d-1} \left[\tilde{\beta}(\alpha \to \infty)\right]^{m-d-1} = 0, \tag{6.74}$$

and we obtain the similar conclusion that we need $(d+3)$ or higher-dimensional spacetime at first to freeze the extra $(D-d)$ dimensions

For given values of c_m's, we can derive the asymptotic value $\tilde{\beta}(\alpha \to \infty)$. In large α limit, we easily find $\tilde{\beta}/\alpha \to 0$ if c_5 or c_6 is nonvanishing. Since the ratio $\tilde{\beta}/\alpha$ has the vanishing limit, we can say that for every $\epsilon > 0$, there is a minimum α_* such that $\left|\tilde{\beta}/\alpha\right| < \epsilon$ if $\alpha > \alpha_*$. Consequently, in order to utilize this property to *freeze*—which is a more appropriate word than *stabilize*—the extra dimensions, we set sufficient conditions such that the growth of extra dimensions can be suppressed enough for a small number ϵ:

(a) There is a minimum of expansion rate of the universe, which is denoted by α_*.

(b) For any $\alpha > \alpha_*$ the expansion rate of extra dimensions is suppressed $\tilde{\beta}(\alpha)/\alpha < \epsilon$.

If these two conditions are satisfied, we have frozen extra dimensions in the whole cosmic history. They may be too stringent to make the system be consistent with the observational constraints, because we do not need $\tilde{\beta}/\alpha < \epsilon$ in the very early universe where we cannot give a constraint. There is possibilities to relax the sufficient conditions and one can seek the values of c_m's which gives $\tilde{\beta}/\alpha < \epsilon$ only for $\alpha_{\text{min}} < \alpha < \alpha_{\text{max}}$. It is required when all of the $c_{m\geq d+2}$'s vanish, there is the maximum over which $\tilde{\beta}/\alpha < \epsilon$ no longer holds. However, it is beyond our scope to write down all of the possibilities and we focus on the case $c_{m\geq d+2} \neq 0$.

We again study the case for $d = 3$ but for a general D. Suppose that only the cosmological constant c_0, the Einstein-Hilbert term c_2 and a single higher-order term $c_{5\leq m\leq D}$ do not vanish. We get from (6.31)

$$c_2 + c_m \sum_{l=0}^{2} \binom{2}{l} \binom{D-4}{m-2-l} \alpha^l \tilde{\beta}^{m-2-l} = 0. \tag{6.75}$$

If we choose the root $\tilde{\beta}$ which converges to zero in the limit $\alpha \to 0$, an approximate value of $\tilde{\beta}$ in the large α limit is given by the roots of

$$\tilde{\beta}(\alpha)^{m-4} = -\frac{c_2}{c_m \alpha^2} \binom{D-4}{m-4}^{-1} . \tag{6.76}$$

The system which is just on the attractor $\beta = \tilde{\beta}(\alpha)$ obeys the effective Hamiltonian constraint

$$\frac{\rho}{2} - c_0 + 3c_2\alpha^2 + (m-1)c_m \binom{D-3}{m-3} \left[-\frac{c_2}{c_m} \binom{D-4}{m-4}^{-1} \right]^{\frac{m-3}{m-4}} \alpha^{\frac{m-6}{m-4}} \approx 0, \tag{6.77}$$

where we have chosen the positive root of (6.76) and we have neglected smaller terms with higher-order β. We observe that it recovers the Friedmann equation with the fractional-order term of α. It brings us a new effect in principle, which is expected to play a role of quintessence in the late universe. It is, however, out of our interest in this thesis, and we work in the hierarchy $c_m \gg c_2\alpha^{-m+2}$ which allows us to neglect the fractional-order term.

Let us consider the realistic system in the phase space (H_a, H_b). In the phase space, the attractor is one-dimensional curve represented by $H_b = \tilde{\beta}(H_a)$. Let the position of the system be $(H_a, H_b) = (\alpha, \beta)$, which initiates in the vicinity of the anisotropic attractor so that $\beta \approx \tilde{\beta}(\alpha)$. From (6.25) and the definition of $\tilde{\beta}(\alpha)$, we have

$$\sum_{m=2}^{D} c_m \left[Q_m(\alpha, \beta) - Q_m(\alpha, \tilde{\beta}(\alpha)) \right] = \frac{\mathcal{C}}{V(\alpha - \beta)}, \tag{6.78}$$

where $\mathcal{C}$ is an integration constant. Note that $V(\alpha - \beta)$ is nearly constant around the isotropic attractor. We define the ratio $\eta \equiv (\beta - \tilde{\beta}(\alpha))/\tilde{\beta}(\alpha)$ and we can say that the system is in the vicinity of the anisotropic attractor if $\eta \ll 1$. We finally find in the lowest order of η,

$$\eta = \mathcal{C} \left[V\left(\alpha - \tilde{\beta}(\alpha)\right) \tilde{\beta}(\alpha) \sum_{m=3}^{D} c_m \left. \frac{\partial Q_m}{\partial \beta} \right|_{\beta = \tilde{\beta}(\alpha)} \right]^{-1} , \tag{6.79}$$

for all dimension number D and d. As long as the inside of the square bracket increase, the expansion rate β approaches the attractor $\tilde{\beta}$.

6.2.2 Energy contents with anisotropic pressure

Radiation is one of the energy contents which apparently exists in our universe. The main origin of the radiation is relativistic particles. The relativistic particles propagate only in the d-dimensional universe and do not in the extra dimensions if the extra dimensions are small [2]. Pressure is caused by the interaction between the particles

[2] If particles propagate into the extra dimensions, we no longer regard them as relativistic ones since propagation in the extra dimensions causes the particles to obtain heavy mass.

giving their momentum to each other, and then in the non-propagating direction they behave as cold matter, which has no pressure. Therefore the radiation has anisotropic pressure:

$$w_{(i)} \equiv \frac{p_{(i)}}{\rho} = \frac{1}{d} \quad \text{for } 1 \le i \le d, \tag{6.80}$$

$$w_{(i)} \equiv \frac{p_{(i)}}{\rho} = 0 \quad \text{for } d+1 \le i \le D. \tag{6.81}$$

With anisotropic pressure, the polynomial equation (6.31) no longer holds, and we cannot use the same discussion as in the case of isotropic energy content. In the case that the anisotropic pressure is caused by the radiation, however, we see that the expansion rate of the extra dimensions can be damped if the Einstein-Hilbert term c_2 is dominant in the lower-order terms $c_{m \le d+1}$. To show it, let $c_0 = 0$ and $c_3 = c_4 = \cdots = c_{d+1} = 0$ and investigate the equation of state to hold the solution of static extra dimensions $\beta = \dot{\beta} = 0$. We assume the energy content has anisotropic pressure as below at first:

$$p_{(i)} = w_a \rho \quad \text{for } 1 \le i \le d, \tag{6.82}$$

$$p_{(i)} = w_b \rho \quad \text{for } d+1 \le i \le D. \tag{6.83}$$

It allows us to study with two expansions rates α and β. We define

$$L(\alpha, \beta) = \mathcal{L}|_{H_{(1\le j\le d)}=\alpha, H_{(d+1\le k\le D)}=\beta} = \sum_{m=0}^{D} c_m S_m(\alpha, \beta), \tag{6.84}$$

$$L_m(\alpha, \beta) = \sum_{m=0}^{D} (1-m) c_m S_m(\alpha, \beta), \tag{6.85}$$

to abbreviate the following expression. Substituting the Hamiltonian equation (6.22) into the evolution equations (6.23),

$$d(L + w_a L_m) - \frac{\dot{V}}{V} L_{,\alpha} - L_{,\alpha\alpha}\dot{\alpha} - L_{,\alpha\beta}\dot{\beta} = 0, \tag{6.86}$$

$$(D-d)(L + w_b L_m) - \frac{\dot{V}}{V} L_{,\beta} - L_{,\alpha\beta}\dot{\alpha} - L_{,\beta\beta}\dot{\beta} = 0, \tag{6.87}$$

where $L_{,x}$ denotes partial derivative $\frac{\partial L}{\partial x}$ and the variable x denotes α or β. Plugging $\dot{V}/V = d\alpha + (D-d)\beta$ into the two equations and solving them for $\dot{\alpha}$ and $\dot{\beta}$, we get

$$\dot{\alpha} = \frac{-d(L + w_a L_m)L_{,\beta\beta} + (D-d)(L + w_b L_m)L_{,\alpha\beta} + (d\alpha + (D-d)\beta)(L_{,\alpha}L_{,\beta\beta} - L_{,\beta}L_{,\alpha\beta})}{{L_{,\alpha\beta}}^2 - L_{,\alpha\alpha}L_{,\beta\beta}}, \tag{6.88}$$

$$\dot{\beta} = \frac{+d(L + w_a L_m)L_{,\alpha\beta} - (D-d)(L + w_b L_m)L_{,\alpha\alpha} + (d\alpha + (D-d)\beta)(L_{,\beta}L_{,\alpha\alpha} - L_{,\alpha}L_{,\alpha\beta})}{{L_{,\alpha\beta}}^2 - L_{,\alpha\alpha}L_{,\beta\beta}}. \tag{6.89}$$

To get the static solution $\beta = \dot{\beta} = 0$, we require the equation

$$d(L + w_a L_m)L_{,\alpha\beta} - (D-d)(L + w_b L_m)L_{,\alpha\alpha} + d\alpha(L_{,\beta}L_{,\alpha\alpha} - L_{,\alpha}L_{,\alpha\beta}) = 0 \tag{6.90}$$

to hold. The static condition $\beta = 0$ reduces L and its derivatives to

$$L|_{\beta=0} = \sum_{m=0}^{d} c_m \binom{d}{m} \alpha^m = \frac{1}{2}c_2 d(d-1)\alpha^2, \tag{6.91}$$

$$L_{,\alpha}|_{\beta=0} = \sum_{m=0}^{d} c_m \binom{d}{m} m\alpha^{m-1} = c_2 d(d-1)\alpha, \tag{6.92}$$

$$L_{,\beta}|_{\beta=0} = \sum_{m=0}^{d+1} c_m \binom{d}{m-1} (D-d)\alpha^{m-1} = c_2 d(D-d)\alpha, \tag{6.93}$$

$$L_{,\alpha\alpha}|_{\beta=0} = \sum_{m=0}^{d} c_m \binom{d}{m} m(m-1)\alpha^{m-2} = c_2 d(d-1), \tag{6.94}$$

$$L_{,\alpha\beta}|_{\beta=0} = \sum_{m=0}^{d+1} c_m \binom{d}{m-1} (D-d)(m-1)\alpha^{m-2} = c_2 d(D-d), \tag{6.95}$$

$$\begin{aligned} L_{,\beta\beta}|_{\beta=0} &= \sum_{m=0}^{d+2} c_m \binom{d}{m-2} (D-d)(D-d-1)\alpha^{m-2} \\ &= (D-d)(D-d-1)(c_2 + c_{d+2}\alpha^d), \end{aligned} \tag{6.96}$$

where we used $c_0 = c_3 = c_4 = \cdots = c_{d+1} = 0$ in the last equality in each equation. With $L_m = L - \alpha L_{,\alpha} - \beta L_{,\beta}$, the equation (6.90) leads to

$$\frac{1}{2}d^2(d-1)(D-d)(1 - dw_a + (d-1)w_b)c_2^2\alpha^2 = 0. \tag{6.97}$$

It means that if the equation of state satisfies $1 - dw_a + (d-1)w_b = 0$, $\beta = \dot{\beta} = 0$ is a solution. Thus the radiation ($w_a = 1/d$, $w_b = 0$) damps the expansion rate of the extra dimensions. In order for the static extra dimensions $\beta = \dot{\beta} = 0$ to be stable, we require that the partial derivative of the right-hand side in (6.89) with respect to β at $\beta = 0$ is negative. It is given by

$$-\frac{d\alpha((D-1)c_2 + \frac{1}{2}(D-d-1)(d-1)^2 c_{d+2}\alpha^d)}{(D-1)c_2 - (D-d-1)(d-1)c_{d+2}\alpha^d}, \tag{6.98}$$

where we have used the equation of state $w_a = 1/d$ and $w_b = 0$. It implies that the c_{d+2} term can induce the positive value of the expression. In the case with isotropic pressure, we have shown that the hierarchy condition between c_2 and any of $c_{m\geq d+2}$'s can freeze the extra dimensions, but we prefer $c_{m\geq d+3}$ to $c_{m=d+2}$ because we now have apprehension of the instability of the static solution $\beta = \dot{\beta} = 0$.

We now concentrate on the analysis for $d = 3$ but for a general D with the equation of state (6.82) and (6.83). We assume that only the Einstein-Hilbert term c_2 and a single higher-order term $c_{m\geq 6}$ are nonvanishing. We consider that the condition $\beta \sim \tilde{\beta}(\alpha) \ll \alpha$ holds initially due to the preceding isotropic period in which the system has approached anisotropic attractor. The assumptions reduce L and its derivative to

$$L = 3c_2\alpha^2 + c_m \binom{D-3}{m-3} \alpha^3\beta^{m-3}, \tag{6.99}$$

$$L_{,\alpha} = 6c_2\alpha + 3c_m \binom{D-3}{m-3} \alpha^2\beta^{m-3}, \tag{6.100}$$

$$L_{,\beta} = 3c_2(D-3)\alpha + c_m \binom{D-3}{m-3} (m-3)\alpha^3\beta^{m-4}, \tag{6.101}$$

$$L_{,\alpha\alpha} = 6c_2 + 6c_m \binom{D-3}{m-3} \alpha\beta^{m-3}, \tag{6.102}$$

$$L_{,\alpha\beta} = 3c_2(D-3) + 3c_m \binom{D-3}{m-3} (m-3)\alpha^2\beta^{m-4}, \tag{6.103}$$

$$L_{,\beta\beta} = c_2(D-3)(D-4) + c_m \binom{D-3}{m-3} (m-3)(m-4)\alpha^3\beta^{m-5}, \tag{6.104}$$

where we have neglected β compared to the same power of α. Suppose that the preceding isotropic era freeze extra dimensions with the anisotropic attractor and the equation (6.75) approximately holds initially. Then we observe that the c_2 term can be neglected in (6.104) and the c_m term in (6.99), (6.100), and (6.102), whereas both the terms in (6.101) and (6.103) are of the same order. We find the relations among their orders

$$L \sim \alpha L_{,\alpha} \sim \alpha L_{,\beta} \sim \alpha^2 L_{,\alpha\alpha} \sim \alpha^2 L_{,\alpha\beta} \sim \alpha\beta L_{,\beta\beta}. \tag{6.105}$$

With the order analysis in (6.88) and (6.89), the ratio $\dot{\beta}/\dot{\alpha}$ is suppressed as

$$\frac{\dot{\beta}}{\dot{\alpha}} \sim \frac{\beta}{\alpha}. \tag{6.106}$$

The extra dimensions restrain their acceleration with their own slow expansion. It is not the case in the Einstein gravity in which only the c_2 term does not vanish. If we put a nonvanishing c_2 in the evolution equation (6.88) and (6.89), we always find $\dot{\beta}/\dot{\alpha} \sim 1$. With the order relations (6.105), the evolution equation (6.88) is calculated in the leading order and it gives

$$\dot{\alpha} = \frac{-d(L + w_a L_m)L_{,\beta\beta} + d\alpha L_{,\alpha}L_{,\beta\beta}}{-L_{,\alpha\alpha}L_{,\beta\beta}} = -\frac{d}{2}(1+w_a)\alpha^2. \tag{6.107}$$

It is the same as the Friedmann equation in d-dimensional universe with the isotropic equation of state $w = p/\rho = w_a$. The next-leading order is suppressed by β/α and the Friedmann dynamics perfectly recovers in the lower-dimensional universe in the limit $\beta/\alpha \to 0$.

6.2.3 Examples of freezing extra dimensions for $D = 6$

Now we have a unified view to keep freezing the extra dimensions. In inflation, the anisotropic attractor divides the expansion rates into two hierarchical values so that only d-dimensional universe can inflate faster than the extra dimensions do. After inflation, the universe is thought to be dominated by several kinds of energy contents, kination, radiation, matter, and dark energy. Except for radiation, those energy contents have isotropic pressure and in their dominant era, the universe keeps converging on the anisotropic attractor. The extra dimensions are guaranteed to be frozen when only the Einstein-Hilbert term c_2 and a single higher-order term $c_{m\geq d+2}$ do not vanish and satisfy the hierarchy condition. Radiation has anisotropic pressure and does not allow the anisotropic attractor to be the terminal point of the system. Nevertheless the expansion rate of the extra dimensions has the static condition $\beta = \dot{\beta} = 0$ as the solution in radiation dominant era. Even when an anisotropic pressure with general w_a and w_b dominates the universe, the acceleration rate of the extra dimensions can be suppressed compared to that of the universe dimensions as is seen in the previous subsection.

Let us study below the simplest cases in which the two conditons can be satisfied.

$\mathbf{c_3 = c_4 = c_5 = 0}$ **:** First, we let $c_3 = c_4 = c_5 = 0$ and get the anisotropic attractor on which

$$\tilde{\beta}(\alpha) = \pm\sqrt{-\frac{c_2}{c_6}}\alpha^{-1}. \tag{6.108}$$

Substituting β into the Hamiltonian constraint (6.47), we get the equation below

$$\frac{\rho}{2} - c_0 + 3c_2\alpha^2 - \frac{3c_2^2}{c_6\alpha^2} \mp 4\sqrt{\frac{-c_2^3}{c_6}} = 0. \tag{6.109}$$

It can be regarded as the effective Hamiltonian constraint in the lower-dimensional universe. If $\alpha^4 \gg c_2/c_6$ is satisfied, we recover the Friedmann equation $\alpha^2 = (2c_0-\rho)/(6c_2)$. In order for the root of the Friedmann equation to satisfy $\alpha^4 \gg c_2/c_6$ recursively, we choose the parameter region which satisfy the hierarchy condition between parameters:

$$\left|\frac{c_2^3}{(\rho - 2c_0)^2 c_6}\right| \ll 1. \tag{6.110}$$

This hierarchy condition guarantees both neglecting the fourth and fifth term in (6.109) and freezing extra dimensions

$$\left|\tilde{\beta}/\alpha\right| = 6\sqrt{\frac{c_2^3}{(\rho - 2c_0)^2 c_6}} \ll 1. \tag{6.111}$$

With the ordinary energy contents of which the energy density decreases as the volume expands, we have $\rho \to 0$ in the very late universe and find that the minimum of expansion rate of the universe is given by $\alpha_* = \sqrt{c_0/(3c_2)}$. The expansion rate of the universe α is almost determined by c_0, c_2 and ρ, which means that its evolution is not affected by the higher-order term c_6. To freeze the extra dimensions compared to the universe for all the period, we have to set c_6 to a large number, which is parameterized by a small number ϵ,

$$c_6 = \frac{9c_2^3}{c_0^2}\epsilon^{-2}, \tag{6.112}$$

with which we always have $\left|\tilde{\beta}/\alpha\right| < \epsilon$. We have

$$\eta = \mathcal{C}\left[V\left(\alpha - \tilde{\beta}(\alpha)\right) 2c_6\alpha^2\tilde{\beta}(\alpha)^2\right]^{-1} \approx \mathcal{C}\left[-2c_2 V\alpha\right]^{-1}, \tag{6.113}$$

where we have ignored $\tilde{\beta}(\alpha)$ compared to α. It shows that if the total volume V expands faster than the decrease of α, the system approaches to the anisotropic attractor $\eta \to 0$. It is the case if we have the equation of state $w = p/\rho < 1$. If the system has approached the anisotropic attractor in the preceding period, $w = 1$ also allows the system to stay in the vicinity.

$\mathbf{c_3 = c_4 = c_6 = 0}$ **:** Next, let us repeat the same analysis in the case for $c_3 = c_4 = c_6 = 0$. The anisotropic attractors give

$$\tilde{\beta}(\alpha) = \frac{-c_5\alpha^2 \pm \sqrt{c_5^2\alpha^4 - 2c_2c_5\alpha}}{2c_5\alpha}. \tag{6.114}$$

As is seen here, one of the roots behaves as $\tilde{\beta}(\alpha) \propto \alpha$ as $\alpha \to \infty$, so that we can use the other root to freeze the extra dimensions, which converges to $\tilde{\beta} \to -c_2/(2c_5\alpha^2)$ if $c_5 > 0$. We substitute $\beta = \tilde{\beta}$ into the Hamiltonian constraint (6.25). Both of the roots lead to

$$\frac{\rho}{2} - c_0 + 3c_2\alpha^2 - \frac{3c_2^2}{2c_5\alpha} \approx 0, \tag{6.115}$$

which can be the effective Hamiltonian constraint in lower-dimensional universe. If $\alpha^3 \gg c_2/c_5$, then the fourth term in (6.115) can be neglected, so that we work in the hierarchy condition

$$\left|\frac{c_2^5}{(\rho - 2c_0)^3 c_5^2}\right| \ll 1, \tag{6.116}$$

with which the Friedmann equation is recovered and $-c_2/(2c_5\alpha^2)$ provides an good approximation of $\tilde{\beta}$. For a small number ϵ, if we take a large value of c_5 as

$$c_5 = \frac{3}{2}\sqrt{\frac{3c_2^5}{c_0^3}}\epsilon^{-1}, \tag{6.117}$$

we have such frozen extra dimensions that $\left|\tilde{\beta}/\alpha\right| < \epsilon$. From (6.79),

$$\eta = \mathcal{C}\left[V\left(\alpha - \tilde{\beta}\right) 2c_5\alpha\tilde{\beta}(\alpha + 2\tilde{\beta})\right]^{-1} \approx \mathcal{C}\left[-c_2 V\alpha\right]^{-1}, \tag{6.118}$$

and thus we need $w = p/\rho \leq 1$ to keep laying the system in such a vicinity of the anisotropic attractor that $\beta \approx \tilde{\beta}$.

Let us show the specific case for $D = 6$ and $d = 3$ in order to test our mechanism of compactification by hierarchy condition. To show it explicitly, we numerically calculate the evolution of the scale factors and energy density. Now we define the metric with two scale factors a and b

$$ds^2 = -dt^2 + a^2\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right] + b^2\left[\left(dx^4\right)^2 + \left(dx^5\right)^2 + \left(dx^6\right)^2\right]. \tag{6.119}$$

Now we set the constants of the model as

$$c_0 = -\frac{\lambda^2 M^5}{2},\ c_2 = -\frac{M^5}{2},\ c_3 = c_4 = c_5 = 0,\ c_6 = \frac{\lambda^{-4}M^5}{2}\epsilon^{-2}, \tag{6.120}$$

where we have mass-dimensional parameters M, λ but we have freedom to choose the over-all factor, so that we set M the unity. Note that ϵ is dimensionless. Energy density is given by

$$\rho = \rho_\phi a^{-6}b^{-6} + \rho_r a^{-4}b^{-3} + \rho_m a^{-3}b^{-3}, \tag{6.121}$$

where ρ_ϕ, ρ_r, and ρ_m denote the initial energy density of kination, radiation, and cold matter, respectively. In this parameterization, we have the value of β on the anisotropic attractor as

$$\tilde{\beta}(\alpha) = \pm\epsilon\frac{\lambda^2}{\alpha}. \tag{6.122}$$

Suppose the universe is around the anisotropic attractor. Plugging the expression of $\tilde{\beta}$ into the Hamiltonian constraint, we get

$$\rho + \lambda^2 - 3\alpha^2 - \epsilon^2\frac{3\lambda^4}{2\alpha^2} \mp \epsilon\lambda^2 = 0. \tag{6.123}$$

If $\epsilon \ll 1$ and $\alpha \gg \epsilon\lambda$ are satisfied, the fourth and fifth terms can be neglected compared to the second and third terms. It means that the Friedmann dynamics of the universe recovers around the anisotropic attractor.

$$\alpha^2 \approx \frac{\rho + \lambda^2}{3}. \tag{6.124}$$

If equations of state of energy contents are not exotic, the energy density ρ decreases as the universe volume a^3 increases and finally we find the asymptotic value $\alpha_* \equiv \lambda/\sqrt{3}$. Since α is a decreasing function in the universe without exotic contents, α_* gives the lowest value of α in a cosmic history. Thus the assumption $\alpha \gg \epsilon\lambda$ is automatically satisfied when $\epsilon \ll 1$ holds. We find the asymptotic value of $\beta_* \equiv \pm\sqrt{3}\epsilon\lambda$. If we always need such slow expansion of the extra dimensions as $\beta/\alpha < \gamma$, it is enough to take the value of ϵ as

$$\epsilon < \frac{\gamma}{3}. \tag{6.125}$$

Let us show a numerical calculation in the case of the parameters $\rho_\phi = 0.1$, $\rho_r = 10^{-2}$, $\rho_m = 10^{-4}$, $\epsilon = 10^{-2}$ and $\lambda^2 = 10^{-13}$. We show the evolution of each energy density in the figure 6.6, the evolution of the expansion rates α and β in the figure 6.7, and the evolution of the scale factors a and b in the figure 6.8. As is seen in

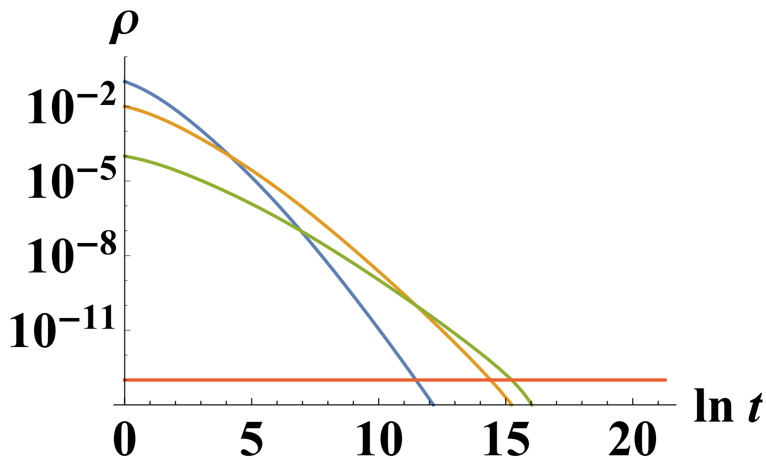

Figure 6.6: The energy densities of kination (blue), radiation (orange), matter (green), and the cosmological constant (red).

the figure 6.7, we have started the calculation around the anisotropic attractor in the kination dominated era, because we assume a preceding inflation converge the system on the anisotropic attractor. Except for in the radiation dominated era, since isotropic-pressure energy contents always dominate, we can observe that the system stays around the anisotropic attractor. In the radiation dominated era, the expansion rate seems slightly damped because our setup reduces the evolution equation (6.89) to a damping equation for β as

$$\dot{\beta} = -\xi(\alpha, \beta)\beta, \tag{6.126}$$

where ξ is a rational expression of α and β. After the era, the cold matter start to dominate the universe and we can see the expansion rate of the extra dimensions converges on that of the anisotropic attractor. When the cosmological constant dominates, both of the expansion rates cease varying. The ratio β/α can read $\sim 10^{-2}$ in the end in the figure 6.7, which is consistent with the expected value $\tilde{\beta}(\alpha_*)/\alpha_* = 3\epsilon$.

We have studied the model with $c_3 = c_4 = c_5 = 0$ above. Now we want to demonstrate the robustness of the model. Figure 6.9, 6.10, and 6.11 shows how the asymptotic

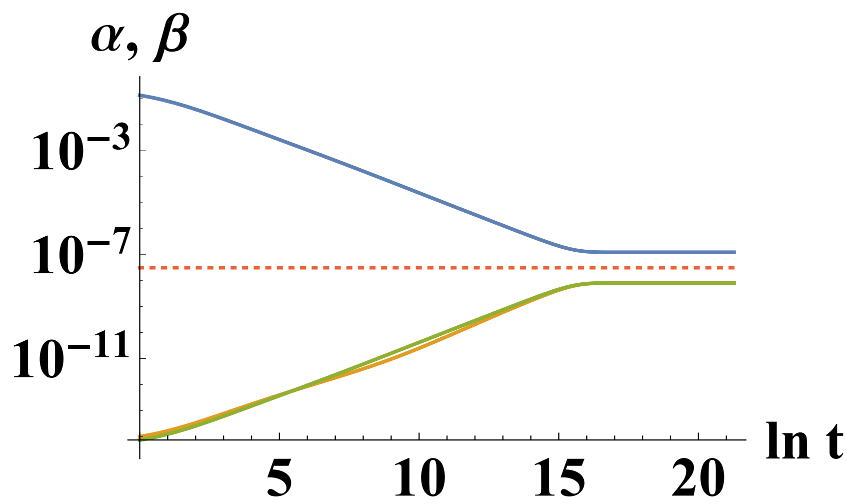

Figure 6.7: The time evolution of the expansion rate of the universe α (blue) and that of the extra dimensions β (orange). The expansion rate of the extra dimensions on the anisotropic attractor $\tilde{\beta}(\alpha)$ in (6.108) is also plotted as the green line. During the domination of the isotropic energy contents, the expansion rate β approaches the anisotropic attractor $\tilde{\beta}(\alpha)$. In the radiation domination around $\ln t \approx 7$, β departs from $\tilde{\beta}(\alpha)$ because of the damping equation (6.126). The blue and green lines are symmetric with respect to the red line which shows $\sqrt[4]{-c_2/c_6}$, the coefficient of α in (6.108).

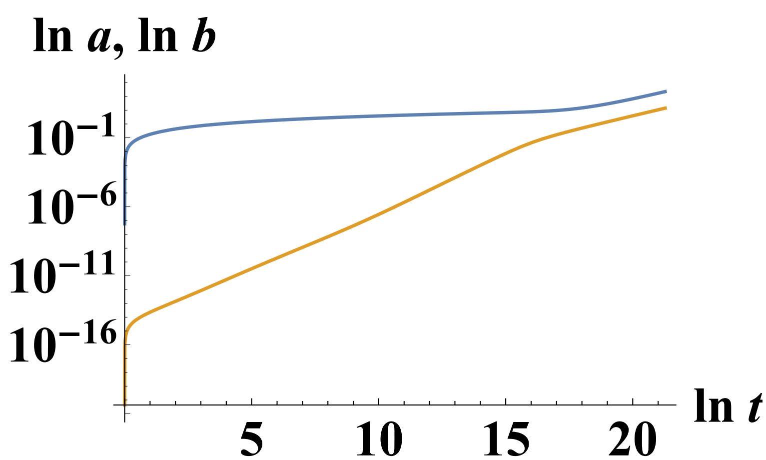

Figure 6.8: The time evolution of scale factors a (blue) and b (orange). We start the calculation with the initial conditions $a(t=1) = b(t=1) = 1$. The universe shown as the blue line always grows faster than the extra dimensions shown as the orange line.

value of $\tilde{\beta}(\alpha)/\alpha_*$ changes when c_3, c_4, and c_5 are added, respectively. They imply that the freezing mechanism breaks down when the added term is too large $c_n\lambda^{n-2}\epsilon^{(n-2)/2} \gtrsim 1$ $(n = 3, 4, \text{and } 5)$. c_5 is an exception as seen in Figure 6.11 where $\tilde{\beta}/\alpha_*$ decreases at $c_5 > 0$. It is consistent with the fact that we can freeze the extra dimensions with a single higher-order term not only c_6 but also c_5.

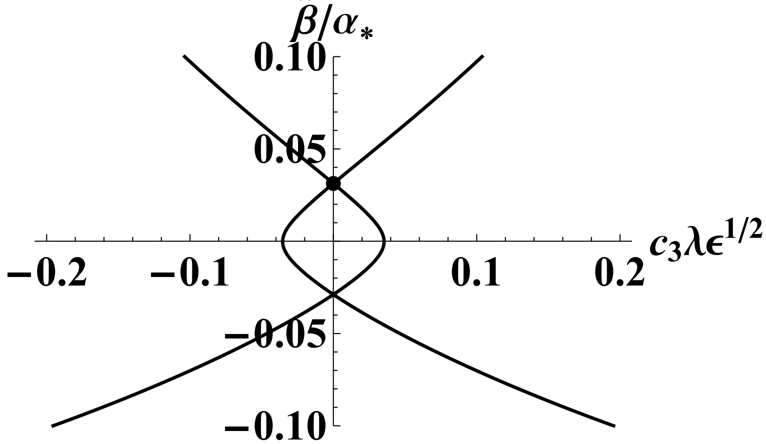

Figure 6.9: Change of asymptotic ratio $\tilde{\beta}(\alpha)/\alpha_*$ when we add c_3 to the model. The dot shows the original asymptotic ratio which realizes at the end of Figure 6.7 (in which $c_3 = 0$). Only the line is relevant which has a negative first derivative at the dot.

One may think that such a large higher-order term could decrease the energy scale of the unitarity bound of the model, but we show that it is not the case around the anisotropic attractor. To demonstrate this, we give a simple estimation of interaction of gravitons h_{ij} on the unperturbed metric $\bar{g}_{\mu\nu}$ which is equal to $g_{\mu\nu}$ in (6.1). Here the subscripts $i, j, \ldots$ denote the indices for the three spatial dimensions of the universe, and $M, N, \ldots$ for the extra dimensions. From the Einstein-Hilbert term c_2, we derive formal expression of perturbed action as

$$\delta\mathcal{L}_2 = -\frac{1}{2}c_2\delta R = -\frac{1}{4}c_2\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\delta R^{\nu_1\nu_2}_{\mu_1\mu_2} \tag{6.127}$$

where $\delta R^{\nu_1\nu_2}_{\mu_1\mu_2}$ is perturbed Riemann tensor. The higher-order term c_6 brings

$$c_6\delta^{\mu_1\mu_2\cdots\mu_6}_{\nu_1\nu_2\cdots\nu_6}\bar{R}^{\nu_3\nu_4}_{\mu_3\mu_4}\bar{R}^{\nu_5\nu_6}_{\mu_5\mu_6}\delta R^{\nu_1\nu_2}_{\mu_1\mu_2} \tag{6.128}$$

where $\bar{R}^{\nu_1\nu_2}_{\mu_1\mu_2}$ is the unperturbed Riemann tensor which is given by (6.5) and (6.6). If $c_6\delta^{\mu_1\mu_2\cdots\mu_6}_{\nu_1\nu_2\cdots\nu_6}\bar{R}^{\nu_3\nu_4}_{\mu_3\mu_4}\bar{R}^{\nu_5\nu_6}_{\mu_5\mu_6}$ is much larger than $c_2\delta^{\mu_1\mu_2}_{\nu_1\nu_2}$, then c_6 induces a huge couplings of the gravitons. $\delta R^{j_1j_2}_{i_1i_2}$ and δR^{0j}_{0i}, in which graviton h_{ij} appears, are contracted with $c_6\delta^{0ii_1i_2M_1M_2}_{0jj_1j_2N_1N_2}\bar{R}^{j_1j_2}_{i_1i_2}\bar{R}^{N_1N_2}_{M_1M_2}$ and $c_6\delta^{0ii_1i_2M_1M_2}_{0jj_1j_2N_1N_2}\bar{R}^{0j}_{0i}\bar{R}^{N_1N_2}_{M_1M_2}$, respectively. They are of the order of $c_6\alpha^2\beta^2$ and around the anisotropic attractor, it is of the same order of c_2. Therefore the self-couplings of the gravitons is not larger than that in the general relativity and the hierarchy condition does not cause the breakdown of the perturbation theory at much lower scale than the Planck scale.

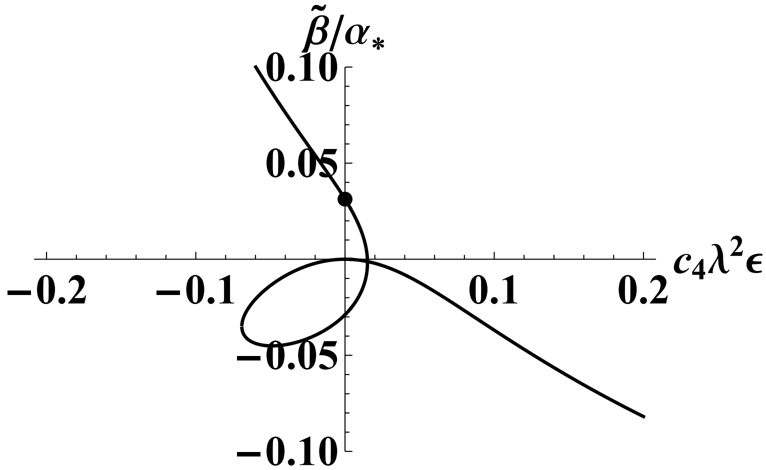

Figure 6.10: Change of asymptotic ratio $\tilde{\beta}(\alpha)/\alpha_*$ when we add c_4 to the model. The dot shows the original asymptotic ratio which realizes at the end of Figure 6.7 (in which $c_4 = 0$). At the origin, one might expect the solution could freeze the extra dimensions completely $\tilde{\beta}/\alpha_* = 0$. However, α_* diverges and thus the solution is not realistic.

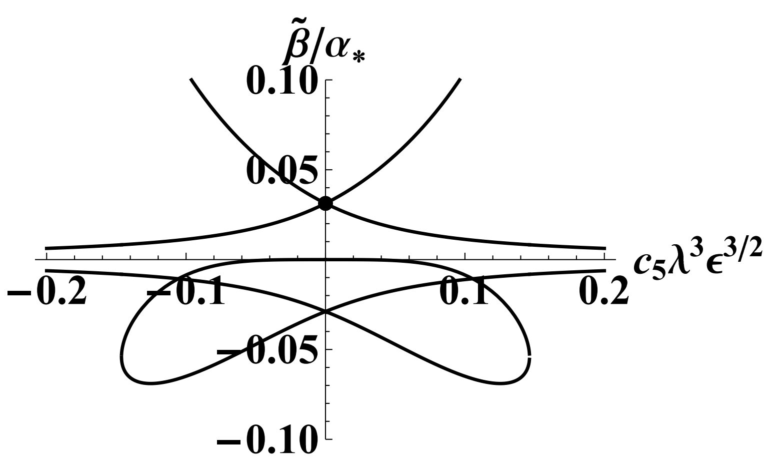

Figure 6.11: Change of asymptotic ratio $\tilde{\beta}(\alpha)/\alpha_*$ when we add c_5 to the model. The dot shows the original asymptotic ratio which realizes at the end of Figure 6.7 (in which $c_5 = 0$). Only the line is relevant which has a negative first derivative at the dot. At the origin, one might expect the solution could freeze the extra dimensions completely $\tilde{\beta}/\alpha_* = 0$. However, α_* diverges and thus the solution is not realistic.

6.3 Momentum density and Jacobian matrix

The expansion rate $H_{(i)}$ is the velocity of $\ln a_{(i)}$. We define (conjugate) momentum density of $\ln a_{(i)}$ as

$$\pi_{(i)} \equiv \frac{\partial \mathcal{L}}{\partial H_{(i)}} = \sum_{m=1}^{D} c_m \frac{\partial s_m}{\partial H_{(i)}}. \tag{6.129}$$

The time derivative of $\pi_{(i)}$ drives the system in the evolution equations (6.23). An infinitesimal change of the momentum density $d\pi_{(i)}$ is translated into the change in the phase space $dH_{(i)}$ as

$$dH_{(i)} = \left(\frac{\partial \pi_{(j)}}{\partial H_{(i)}}\right)^{-1} d\pi_{(j)}. \tag{6.130}$$

This shows that we observe infinite velocity if the Jacobian matrix $\partial\pi_{(j)}/\partial H_{(i)}$ is singular. We write down the (i,j)-components of Jacobian matrix J explicitly

$$(J)_{ij} \equiv \frac{\partial \pi_{(j)}}{\partial H_{(i)}} = \sum_{m=2}^{D} c_m \frac{\partial s_m}{\partial H_{(i)} \partial H_{(j)}}. \tag{6.131}$$

On the point where $\det(J) = 0$, some of the acceleration $dH_{(i)}/dt$ diverges and curvature invariants such as the Ricci scalar R become divergent. The basis $\partial_{\pi_{(i)}}$ is not complete when J is degenerate. Thus we name set of such points degeneracy surface, which is $(D-1)$-dimensional curved surface in the D-dimensional phase space $(H_{(1)}, H_{(2)}, \ldots, H_{(D)})$. If the system touches the degeneracy surface, we can no longer rely on subsequent calculation.

In the general relativity, we have $c_2 \neq 0$ and $c_{m>2} = 0$. The Jacobian matrix J reads

$$J = c_2 \times \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}, \tag{6.132}$$

anywhere in the phase space. Therefore the general relativity provides the constant Jacobian matrix and nonvanishing determinant $\det(J) \neq 0$. In other models, given nonvanishing $c_{m>2}$'s, J is usually a function of $H_{(i)}$'s. Such dependence causes anisotropic dynamics which we have seen in the preceding sections.

Now we explain how slow acceleration of the extra dimensions as seen in Sec. 6.2.2 become possible by using the Jacobian matrix J. With the ansatz of two scale factors,

J is reduced to

$$J = \begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\beta} \\ J_{\alpha\beta}^T & J_{\beta\beta} \end{pmatrix}, \tag{6.133}$$

$$J_{\alpha\alpha} \equiv L_{,\alpha\alpha} \times \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}, \tag{6.134}$$

$$J_{\beta\beta} \equiv L_{,\beta\beta} \times \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}, \tag{6.135}$$

$$J_{\alpha\beta} \equiv L_{,\alpha\beta} \times \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \tag{6.136}$$

where $J_{\alpha\alpha}$, $J_{\beta\beta}$, and $J_{\alpha\beta}$ are matrices of size $d \times d$, $(D-d) \times (D-d)$, and $d \times (D-d)$, respectively. In order to compute the evolution of the system, we calculate the acceleration $\dot{H}_{(i)}$ from the time derivative of momentum densities $\dot{\pi}_{(i)}$'s as

$$\frac{d\vec{H}}{dt} = J^{-1}\frac{d\vec{\pi}}{dt}, \tag{6.137}$$

where we have defined vectors $\vec{H} \equiv (H_{(1)}, \ldots, H_{(D)})^T$ and $\vec{\pi} \equiv (\pi_{(1)}, \ldots, \pi_{(D)})^T$. The inverse matrix J^{-1} is calculated as below

$$J^{-1} \equiv \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{6.138}$$

$$\begin{aligned} (A)_{ij} &= \left[\left(J_{\alpha\alpha} - J_{\alpha\beta}J_{\beta\beta}^{-1}J_{\alpha\beta}^T\right)^{-1}\right]_{ij} \\ &= \frac{1}{L_{,\alpha\alpha}}\left(\frac{(D-d)L_{,\alpha\beta}^2 - (D-d-1)L_{,\alpha\alpha}L_{,\beta\beta}}{d(D-d)L_{,\alpha\beta}^2 - (d-1)(D-d-1)L_{,\alpha\alpha}L_{,\beta\beta}} - \delta_{ij}\right), \end{aligned} \tag{6.139}$$

$$\begin{aligned} (B)_{ij} &= \left[\left(J_{\beta\beta} - J_{\alpha\beta}^TJ_{\alpha\alpha}^{-1}J_{\alpha\beta}\right)^{-1}\right]_{ij} \\ &= \frac{1}{L_{,\beta\beta}}\left(\frac{dL_{,\alpha\beta}^2 - (d-1)L_{,\alpha\alpha}L_{,\beta\beta}}{d(D-d)L_{,\alpha\beta}^2 - (d-1)(D-d-1)L_{,\alpha\alpha}L_{,\beta\beta}} - \delta_{ij}\right), \end{aligned} \tag{6.140}$$

$$(C)_{ij} = (-J_{\alpha\alpha}^{-1}J_{\alpha\beta}B)_{ij} = \frac{L_{,\alpha\beta}}{d(D-d)L_{,\alpha\beta}^2 - (d-1)(D-d-1)L_{,\alpha\alpha}L_{,\beta\beta}}, \tag{6.141}$$

where δ_{ij} denotes the Kronecker delta and $L_{,\alpha\alpha}$, $L_{,\alpha\beta}$, and $L_{,\beta\beta}$ are partial derivatives of (6.84). Note that the terms which do not contain indices i or j are independent of them. With $\epsilon \equiv \beta/\alpha \ll 1$ and the order relations (6.105), we obtain that

$$(A)_{ij} \approx \frac{1}{L_{,\alpha\alpha}}\left(\frac{1}{d-1} - \delta_{ij}\right) \sim \frac{1}{L_{,\alpha\alpha}}, \tag{6.142}$$

$$(B)_{ij} \approx \frac{1}{L_{,\beta\beta}}\left(\frac{1}{D-d-1} - \delta_{ij}\right) \sim \frac{\epsilon}{L_{,\alpha\alpha}}, \tag{6.143}$$

$$(C)_{ij} \approx -\frac{L_{,\alpha\beta}}{(d-1)(D-d-1)L_{,\alpha\alpha}L_{,\beta\beta}} \sim \frac{\epsilon}{L_{,\alpha\alpha}}, \tag{6.144}$$

Substituting these order estimations and (6.138) into (6.137), we can tell explicitly why the acceleration of the extra dimensions is suppressed compared to that of the lower-dimensional universe. On the other hand, the general relativity provides the inverse matrix $(J^{-1})_{ij} = c_2^{-1}(\frac{1}{D-1} - \delta_{ij})$ and does not yield such slow acceleration of extra dimensions.

Note that the expression of the Jacobian matrix (6.131) is related to the attractors. On the anisotropic attractors, the polynomial equations (6.27) are satisfied and J reads

$$J = \begin{pmatrix} J_{\alpha\alpha} & O \\ O & J_{\beta\beta} \end{pmatrix}, \tag{6.145}$$

where O is a null matrix. The Jacobian determinant is reduced to $\det(J) = \det(J_{\alpha\alpha}) \times \det(J_{\beta\beta})$. From this, especially in the case for $d = 1$, one can tell that $J_{\alpha\alpha} = (0)$ and $\det(J) = 0$ inevitably. It means that the anisotropic attractor discussed in Chapter 3 is embedded in the degeneracy surface, and implies that it induces the singular behavior seen in Chapter 4. For $d \neq 1$, the nondegeneracy condition $\det(J) \neq 0$ is satisfied as long as $L_{,\alpha\alpha} \neq 0$ and $L_{,\beta\beta} \neq 0$, and the anisotropic attractor is far from the degenerate surface. Therefore we expect moderate behavior around the anisotropic attractor for $d = 3$ which is of the most interest.

6.4 Summary

We have calculated the anisotropic attractor in a class of the generalized Galileon which contains the Lovelock theory. The expansion rate of each dimension converges on the attractors according to the total volume expansion. With the anisotropic attractor, a part of the spatial dimensions inflates faster than the other spatial dimensions, and at the final stage of inflation, a lower-dimensional universe with large volume and compactified extra dimensions have been observed. To freeze the extra dimensions, we have obtained the asymptotic equation in the large limit of expansion rate of the universe, and discussed the condition for making hierarchy between expansion rates of the universe and that of the extra dimensions. The asymptotic equation implies that if we want a four-dimensional universe, we have to start with at least six-dimensional spacetime. Even with anisotropic pressure, which does not allow us to use the anisotropic attractor, we have argued the conditions with which acceleration rate of the extra dimensions is suppressed compared to that of the dimensions of the universe.

Chapter 7

Conclusion

Cosmic inflation is a plausible mechanism to realize a homogeneous and isotropic universe with almost scale-invariant curvature fluctuations produced by quantum effects. The simplest way to realize inflation is to use the potential energy of a scalar field, and various models have been proposed. The generalized Galilean provides the most general interaction between gravity and a scalar field in which higher-order derivative terms do not appear in the equations of motion, and it can describe all the inflationary models of a single scalar field in a unified way. Furthermore, the generalized Galileon is a theory of the generalized gravity which includes general relativity and widely encompasses other known gravity theories, which may describe the gravity of our Universe.

In Einstein's general relativity, it has been proven that a spatially homogeneous anisotropic expanding universe is always isotropized in the presence of a cosmological constant unless the spatial curvature is positive. In other words, only isotropic attractors exist in the general relativity. This is called the cosmic no-hair theorem, which reinforces the expectation that inflation will make the universe isotropic. On the other hand, when we consider a spatially homogeneous anisotropic universe using the generalized Galilean theory, we have shown that anisotropic solutions can also be attractors. This anisotropic attractor appears in addition to the isotropic attractor which is present in the general relativity, and which attractor the system converges to depends on the initial condition. The implication is that when inflation occurs in a four-dimensional universe, it must be somewhat isotropic from the beginning to obtain a nearly homogeneous, isotropic universe in which we live.

Here we have considered three possibilities for how anisotropic attractors are related to our present Universe. One possibility is that the universe lies on an anisotropic attractor in four-dimensional spacetime, and in this thesis, we have computed perturbations around the anisotropic attractor. Another possibility is that the universe is on the isotropic attractor in four-dimensional spacetime, and the anisotropic attractor does not exist or has little effect. In this thesis, we have derived the stability conditions of perturbations which can be used even in a strong gravitational field, such as that around black holes, in order to constrain the theory which allows the existence of anisotropic attractors. The last is the possibility that an isotropic, lower-dimensional universe is generated on the anisotropic attractor in a higher-dimensional spacetime. With this naive idea, this thesis has examined the dynamics of higher-dimensional spacetime with the generalized gravity and found the solution which maintains a small extra-dimensional space.

In Chapter 3, we have calculated the evolution of the Bianchi-type I universe. It has been analytically shown that anisotropic attractors exist in some classes of the Horndeski theory for $G_{5X} \neq 0$ or $A_5 \neq 0$. Such terms are known to emerge after Kaluza-Klein compactification of higher-dimensional Lovelock gravity [74]. We have also numerically calculated that anisotropic inflation occurs in a few models. Previously known models of anisotropic inflation use a quantity having a special direction such as a vector field to maintain anisotropy during a period of inflation. On the other hand, this thesis has presented a novel paradigm in which anisotropic inflation is realized by the generalized Galileon including only a scalar field having no special direction. It has also been found that the typical magnitude of the resultant anisotropic expansion rate is about the same as the isotropic expansion rate, and that fine-tuning is necessary to create the 1%-level of anisotropy permitted by the CMB observations. On the other hand, the fact that it is possible to construct naturally a solution in which one direction expanded more slowly than the others has motivated us to study similar solutions in higher-dimensional spacetime to account for compactification of extra dimensions in Chapter 6.

In Chapter 4, in order to analyze the behavior of perturbations in an anisotropic attractor, we have calculated perturbations in an axisymmetric Bianchi type I universe. Axial symmetry allows us to classify perturbations by parity under the spatial inversion with respect to the symmetry axis. Since even-parity and odd-parity perturbations evolve independently of each other, they can be analyzed separately. First, we have investigated the dispersion relation of perturbations in a generic anisotropy state. As a result, we have found that the gravitational waves in the even-parity sector do not develop independently of the scalar waves and the propagation speeds are mixed. That is, gravitational waves passing through an anisotropic background in the Horndeski theory are allowed to have birefringence to occur. However, since the universe is very isotropic on the large scale, the effect during long-range propagation is considered to be small. For an observation of the birefringence, we investigate the behavior of perturbations in a strong gravitational field such as black holes in Chapter 5. Next, we have assumed that the universe converges at an anisotropic attractor. It has become clear that the propagation speed of gravitational waves increases as the universe approaches the anisotropic attractor, which conflicts with the constraint of the propagation speed by the observations of the neutron-star binary GW170817. This shows that anisotropic attractors in four-dimensional spacetime are not the attractor of our present universe. We have also given the dispersion relation in a nonaxisymmetric Bianchi-type I universe.

In Chapter 5, to study the behavior of perturbations in strong fields such as around black holes, we have investigated perturbation theory around a static and spherically symmetric spacetime and have obtained the dispersion relation. The spherical symmetry allows us to classify perturbations according to the parity under a spatial inversion with respect to the origin. The odd-parity sector represents one polarization mode of the gravitational waves, and the even-parity sector represents the other polarization mode and the scalar waves. The dispersion relation of odd-parity mode had been obtained in both radial and angular directions by the previous research [21]. For the even parity mode, only the radial dispersion relation had been obtained previously [22]. In this thesis, we have obtained the complete dispersion relation including the angular direction. It has been clarified that the expected mixing of dispersion relation occurred as seen in Chapter 4. It predicts the birefringence of gravitational waves near black holes and provides a new viewpoint to observationally limit models of generalized gravity. Using the obtained dispersion relation, new stability conditions for angular perturbations have been derived. This yields further theoretical constraints on extended gravity models using black-hole solutions.

In Chapter 6, in order to obtain the dynamics of higher-dimensional spacetime such that the excess dimensional space is kept small, we have calculated the evolution of a homogeneous spacetime of arbitrary dimensions with flat spatial curvature in the generalized Galileon including the Lovelock theory. For simplicity, however, we have limited the theory to the class of theory where the evolution of the homogeneous scalar field does not affect the dynamics of spacetime. As a result, there are anisotropic attractors as in the four-dimensional case. It has been observed that some spatial directions expand or contract extremely slowly compared to the other spatial directions, assuming hierarchical relations among the parameters. When these slowly evolving directions are regarded as an extra dimension, the ratio of the expansion rates of the extra dimensions to those of the dimensions of our universe can be arbitrarily reduced by enhancing the hierarchical relations. We have used these hierarchical relations to freeze extra dimensions whereas the previous researches, *e.g.* [49, 57], stabilize them by using spatial curvature.

Since the anisotropic attractor acts as an attractor when they are filled with energy contents having isotropic pressure, growth of the volume of extra dimensions can be suppressed during inflation, matter dominated era, and dark energy dominated era. When filled with energy contents having anisotropic pressure, such as radiation, the anisotropic attractor disappears generally. If the expansion rate of extra dimensions decreases in the preceding isotropic energy dominated period, we have found that the acceleration of extra dimensions is suppressed compared to that of the three spatial dimensions of the universe. In particular, it has been shown that the equation of state of radiation satisfies the special condition which allows the expansion rate of extra dimensions to be damped down. The numerical calculation has shown that in the seven-dimensional spacetime the anisotropic attractor makes expansion rates converge into two values in the presence of a cosmological constant and the expansion rate of extra dimensions stays small compared to that of the universe during the kination era, radiation dominated era, matter dominated era, and cosmological constant dominated era.

Therefore, we finally conclude that the use of the generalized Galileon and the Lovelock theory enables us to freeze the extra dimensions over the entire period of cosmological history.

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