Doctoral Dissertation
博士論文

Algebraic Proof of S-Duality Formula in Refined Topological Vertex

(位相的頂点におけるS双対性の代数的証明)

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Abstract

We deal with the five-dimensional $\mathcal{N}=1$ super Yang-Mills theories. Let $\mathcal{Z}_{\text{inst.}}^{(A_N,A_M)}$ be the instanton partition function of the A_M quiver gauge theory with A_N gauge group with $N_F=2(N+1)$ matters. We set all the Chern-Simons levels to be zero. Then, the S-duality claims the invariance of $\mathcal{Z}_{\text{inst.}}^{(A_N,A_M)}$ under the exchange between N and M. The main object of the present thesis is to prove this claim. By rewriting the equality in terms of the topological vertex, we obtain the duality formula under changing the preferred directions.

The key ingredient of the proof is the operator realization of the topological vertex. This is achieved by the intertwiners of the Ding-Iohara-Miki algebra. By gluing the intertwiners, we can realize what we call the Mukadé operator. The matrix elements of the Mukadé operator factorize as the products of the Nekrasov factors. This formula proves the claim. Moreover, the Mukadé operator reduces to the primary fields of the Virasoro algebra, under the $q,t\to 1$ limit. In the gauge theory terminology, this limit corresponds to the reduction to the four dimensions. Then, the matrix elements formula of the Mukadé operator can be interpreted as the proof of the five-dimensional analogue of the Alday-Gaiotto-Tachikawa correspondence.

Contents

Notation

General Notation

$\mathcal{U}_{q,t}$: Ding-Iohara-Miki Algebra (Definition 3.3.4)

$\Delta^{(N)}$: N-th coproduct of $\mathcal{U}_{q,t}$

$N\ (\in \mathbb{Z}_{\geq 1})$: Number of Fock tensor spaces

$\mathsf{U}(N)$: Algebra generated by $X_n^{(i)}$'s (Definition 4.3.6)

P : Set of all partitions

$\Lambda_n = \mathbb{Q}(q,t)[x_1,\dots,x_n]^{\mathfrak{S}_n}, \quad \Lambda = \varprojlim_n \Lambda_n$

$\gamma = (t/q)^{1/2}$

$\rho = (-1/2, -3/2, -5/2, \dots)$

$\delta = (-1, -2, -3, \dots)$

$f^{\mathfrak{gl}_n}, \widetilde{f}^{\mathfrak{gl}_n}, \varphi^{\mathfrak{gl}_n}$: Bispectral Macdonald functions (Definition 4.1.2)

$\mathcal{G}(z) = \prod_{i,j=0}^{\infty}(1 - zq^i t^{-j})$: $\mathcal{G}$-factor

$N_{\lambda,\mu}(u)$: Nekrasov factor (Definition 2.1.1)

$\widehat{\bigotimes}_{i=n}^{m} A_i := A_n \otimes \cdots \otimes A_m$

$\prod_{n\leq i\leq m}^{\curvearrowright} A_i := A_n \times A_{n+1} \times \cdots \times A_m$

Notations concerning Partitions ($\lambda \in \mathsf{P}$)

$|\lambda| = \sum_{i\geq 1} \lambda_i$

λ' : Transpose of λ

$\ell(\lambda) = \lambda'_1$: Length of λ

$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$

$\kappa(\lambda) = \sum_i \lambda_i(\lambda_i + 1 - 2i)$

$||\lambda||^2 = \sum_i \lambda_i^2$

$a_\lambda(i,j) = \lambda_i - j$: Arm length of (i,j)

$\ell_\lambda(i,j) = \lambda'_j - i$: Leg length of (i,j)

$a'_\lambda(i,j) = j - 1$: Co-arm length of (i,j)

$\ell'_\lambda(i,j) = i - 1$: Co-leg length of (i,j)

$A(\lambda)$ (resp. $R(\lambda)$) : Set of coordinates where we can add (resp. remove) a box

$f_\lambda = (-1)^{|\lambda|} q^{n(\lambda')+|\lambda|/2} t^{-n(\lambda)-|\lambda|/2}$: Taki's framing factor (Definition 4.2.27)

$g_\lambda = q^{n(\lambda')} t^{-n(\lambda)}$

$\mathcal{U}_{q,t}$-modules and Intertwiners

$\mathcal{F}_u^{(1,M)}$: $(1,M)$-module (Fact 3.4.3)

$\mathcal{F}_u^{(0,1)}$: $(0,1)$-module (Fact 3.4.4)

$\mathcal{F}_{\boldsymbol{u}}$: $(N,0)$-module in Chapter 4

$\Phi(x), \Phi^*(x), \Phi_\lambda(x), \Phi^*_\lambda(x)$: Intertwiners and its matrix elements(Fact 3.4.5)

States in Fock Tensor Spaces ($\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)}) \in \mathsf{P}^N$)

$|X_{\boldsymbol{\lambda}}\rangle\,(\langle X_{\boldsymbol{\lambda}}|)$: PBW basis (Definition 4.2.6)

$|P_{\boldsymbol{\lambda}}\rangle, \langle P_{\boldsymbol{\lambda}}|$: Generalized Macdonald functions (Fact 4.2.9)

$|Q_{\boldsymbol{\lambda}}\rangle = \prod_{i=1}^N \frac{c_{\lambda^{(i)}}}{c'_{\lambda^{(i)}}} |P_{\boldsymbol{\lambda}}\rangle$

$|K_{\boldsymbol{\lambda}}\rangle, \langle K_{\boldsymbol{\lambda}}|$: Integral form of generalized Macdonald functions (Definition 4.2.25)

Vertex Operators

$S^{(i)}(z)\ \ (i = 1, \ldots, N-1)$: Screening currents (Definition 4.3.1)

$\widetilde{S}^{(k)}(z) = S^{(k)}(\gamma^{-2k} t^{-1} z)$: Shifted screening currents

$\Phi^{(0)}(x)$: Top component (Definition 4.2.14)

$\Phi^{(k)}(x)\ \ (k = 1, \ldots, N-1)$: Screened vertex (Definition 4.2.16)

$g(x, y_1, \ldots, y_k) := \frac{\theta_q(tu_1 y_1/u_{k+1}x)}{\theta_q(ty_1/x)} \prod_{i=1}^{k-1} \frac{\theta_q(tu_{i+1}y_{i+1}/u_{k+1}y_i)}{\theta_q(ty_{i+1}/y_i)}$: Integral kernel in screening operator

$V^{(\boldsymbol{n})}(x_1, \ldots, x_{|\boldsymbol{n}|}) = \prod^{\curvearrowright}_{1\le i_1\le n_1} \Phi^{(0)}(x_{[1,i_1]_{\boldsymbol{n}}}) \cdot \prod^{\curvearrowright}_{1\le i_2\le n_2} \Phi^{(1)}(x_{[2,i_2]_{\boldsymbol{n}}}) \cdots \prod^{\curvearrowright}_{1\le i_N\le n_N} \Phi^{(N-1)}(x_{[N,i_N]_{\boldsymbol{n}}})$

: Composition of screened vertex operators (Definition 4.2.20)

$\mathcal{V}(x) = \mathcal{V}\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{u}\end{pmatrix};x)$: Mukadé operators (Definition 5.1.1)

$\mathcal{T}^V(\boldsymbol{u}, \boldsymbol{v}; w), \mathcal{T}^H(\boldsymbol{u}, \boldsymbol{v}; w)$: Definition 5.1.8

Multiples of Partitions and Parameters

$\boldsymbol{u} = (u_1, \dots, u_N)$, Spectral parameters of N-fold Fock tensor spaces

$t^{\pm\delta_i} \cdot \boldsymbol{u} := (u_1, \dots, u_{i-1}, t^{\pm 1} u_i, u_{i+1}, \dots, u_N)$

$t^{\alpha_i} \cdot \boldsymbol{u} := (u_1, \dots, u_{i-1}, t u_i, t^{-1} u_{i+1}, u_{i+2} \dots, u_N)$

$t^{\pm \boldsymbol{n}} \cdot \boldsymbol{u} := (t^{\pm n_1} u_1, \dots, t^{\pm n_N} u_N)$

$\boldsymbol{n} = (n_1, \dots, n_N)$: n_i stands for the number of $\Phi^{(i)}$'s in $V^{(\boldsymbol{n})}(x)$

$|\boldsymbol{n}| := \sum_{i=1}^N n_i$, Total number of the $\Phi^{(i)}$'s in $V^{(\boldsymbol{n})}(x)$

$[i,k] = [i,k]_{\boldsymbol{n}} := \sum_{s=1}^{i-1} n_s + k$

$\boldsymbol{m} = (m_1, \dots, m_N)$, m_i is the number of the $\Phi^{(i)}$'s in $|Q_{\boldsymbol{\lambda}}\rangle$

$\boldsymbol{m}| := \sum_{i=1}^N m_i$, Total number of the $\Phi^{(i)}$'s in $|Q_{\boldsymbol{\lambda}}\rangle$

$[i,k]_{\boldsymbol{m}} := \sum_{s=1}^{i-1} m_s + k$

$\boldsymbol{s} = (s_i)$, Generic parameters in Macdonald functions

Especially, in some propositions in Chapter 4 and 5, they are specialized to

$s_{[i,k]_{\boldsymbol{n}}} = q^{\lambda_k^{(i)}} t^{1-k} v_i \quad (1 \le k \le n_i,\ i = 1, \dots, N)$

$s_{|\boldsymbol{n}|+[i,k]_{\boldsymbol{m}}} = t^{1-n_i-k} v_i \quad (1 \le k \le m_i,\ i = 1, \dots, N)$

$\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)}) \in \mathsf{P}^N$

$[\boldsymbol{\lambda}]^{\boldsymbol{n}} = ([\boldsymbol{\lambda}]_i^{\boldsymbol{n}})_{1 \le i \le |\boldsymbol{n}|} := (\lambda_1^{(1)}, \dots, \lambda_{n_1}^{(1)}, \lambda_1^{(2)}, \dots, \lambda_{n_2}^{(2)}, \dots, \lambda_1^{(N)}, \dots, \lambda_{n_N}^{(N)})$

Chapter 1

Introduction

General Motivation

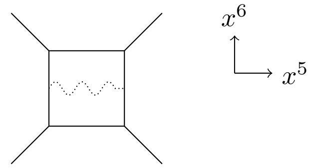
String theory is a well-known candidate of the theory which describes the quantum gravity. Moreover, apart from its original motivation, the existence of rich structures of various kinds makes the theory more intriguing. In the present thesis, we will study two such features of the string theory, that is, *duality* and *integrability.* Here let us make an excuse on this point. As we will see, our main interest is the quantum field theories (QFTs) which are obtained as the low-energy effective theories of the string theory. Especially, we are interested in the QFTs which live on the branes, the various dimensional dynamical objects in the string theory. Typically, the theories obtained in this way belong to a special class of QFTs, the class of supersymmetric QFTs. Thus, what we actually study in this thesis are duality and integrability in supersymmetric QFTs. In a word, the aim of the present thesis is to study the dualities (just conjectures) from the integrability in QFTs. Let us see this in more detail.

What is interesting about the string dualities is that the dualities which seem rather trivial in the string theory, end up with the highly non-trivial dualities in QFTs. As a consequence, those dualities produce the highly non-trivial identities among their physical observables, especially, their partition functions. In this thesis, we deal with the five-dimensional $\mathcal{N}=1$ (*i.e.* with eight supercharges) supersymmetric Yang-Mills theories. As we will see, this class of QFTs has nice property. These theories can be engineered by the (p,q)-webs [39, 2]. The (p,q)-web is the grid diagram, whose edges describe the fivebranes in the type IIB superstring theory. In order to explain these diagrams, we have a quick look at the type IIB superstring theory. The superstring theory is defined on the $9+1$ dimensional spacetime, and especially in type IIB string, the $(5+1)$ dimensional objects called the fivebranes exist and fill the six dimensions out of the ten dimensions. These fivebranes in the type IIB string are labelled by the two charges [94], that is, the magnetic charges of the R-R potential C_2 and the magnetic charges associated with the NS-NS 2-form gauge field B_2 in terms of the type IIB supergravity. Because of the Dirac quantization condition, these charges are restricted to integers. We denote them by (p,q). The fivebranes with one former charge are denoted by the $(0,1)$-fivebranes and those with one latter charge are by $(1,0)$-fivebranes. In most of the papers, the $(0,1)$-fivebranes are called the D5-branes, and the $(1,0)$-fivebranes are the NS5-branes. [1]

In order to see the 5d theories are actually produced, let $x^i, (i=0,\dots,9)$ be the coordinate of the ten-dimensional spacetime. When we put all the fivebrane from x^0 to x^4, we have one more dimension to fill. Then, we put the fivebrane with the charge (p,q) on the line with that slope in the two-dimensional (x^5,x^6) plane. Forgetting about $x^0,\dots,x^4$, we obtain the two-dimensional diagrams which indicate the positions and charges of the fivebranes. We refer to this diagram as the (p,q)-webs. For example, the D5-branes become the vertical lines in this diagram, while the NS5-branes the horizontal lines. Here, let us briefly see why these webs engineer the 5d $\mathcal{N}=1$ super Yang-Mills as the low energy effective theories. When we take the low energy limit, the fine structure of the brane-web is lost, and shrink to a point in the two-dimensional grid diagram. The branes on this point are five-dimensional, that is, they fill $(x^0,\dots,x^4)$. Qualitatively speaking, this is the reason why the IR theories are five-dimensional. Then, the information about gauge groups can be

[1] For later convenience, we use the unusual notation. In most of the references, the D5-branes are denoted by $(1,0)$-branes.

read off from the diagram as follows. As an example, consider the following (p, q)-web:



This diagram shows two parallel D5-branes are stretched between two NS5-branes. This web engineers the $\mathcal{N} = 1$ pure theory with $G = SU(2)$. This is because the strings stretched between the parallel D5-branes (the wavy line in the figure) behave as the gauge bosons. The number of parallel D5-branes is the rank of the gauge group. This is why the (p, q)-web contains enough information about QFTs. Let us remark that these webs have been studied extensively when the gauge group is of A-type ($SU(N)$). Except for that case, we know the construction of the C- or D-type gauge group using the orientifolds. See for example [58, 40],

The duality which we deal with, acts on the charges of branes and changes the brane configuration. This seems rather simple operation in the string theory, though for the supersymmetric QFTs in the IR, it shows very strange result. In general, it completely changes the structures of QFTs such as gauge groups, and matter contents. It sometimes changes the theory which admits the Lagrangian description to the theory which does not. For the type IIB superstring, one of the conjectural dualities is the $SL(2, \mathbb{Z})$-duality. Put $\tau = C_0 + i \exp(-\Phi)$ with C_0 the scalar potential and Φ the dilaton field in the type IIB supergravity. Roughly, this becomes the complexified coupling constant in the IR. At the level of the supergravity, the action is invariant under the $SL(2, \mathbb{Z})$ action on τ and (p, q),

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \qquad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Note that it is conjectured that this is the exact duality in the type IIB superstring. In what follows, we refer as the S-duality to the S-transformation in this $SL(2, \mathbb{Z})$. (Note that $SL(2, \mathbb{Z})$ is generated by S- and T-transformation which satisfy some relations.)

Let us again remark that what we call the string dualities are just conjectures, and there is no proof for them. Thus, the non-trivial identities in QFTs we obtain as a result, are also conjectures. The lack of those proofs drives us to study these dualities.

The key tool to attack this problem is integrability. The supersymmetric QFTs we deal with have at least eight supercharges. The theories in this class have been extensively studied, and many remarkable features have been revealed. One of them was found in [99, 100, 84] and numerous works that followed, and it says that they are integrable, especially in the sense that the partition functions of these theories (though the space-time manifolds where they live are not completely free,) can be computed exactly. In general, integrability is governed by an algebra behind it. For example, the R-matrix, the solution to the Yang-Baxter equation, governs the integrability of, say, the spin chains on lattices, and at the same time, reproduces the algebras like Yangian which ensure the existence of enough amount of integrals of motion.

When supersymmetric QFTs with eight supercharges live on the 4d space $\mathbb{C}^2_{\epsilon_1,\epsilon_2}$ called $\mathbb{C}^2$ with the omega background, the algebra which determines its integrability is known, and the answer is the $\mathcal{W}$-algebra or the affine Yangian of $\mathfrak{gl}_1$. This is the discovery by Alday, Gaiotto, and Tachikawa in [4, 121]. This proposal is quite surprising because the $\mathcal{W}$-algebra describes the symmetry in 2d conformal field theories (CFTs), while the theories we think of live on 4d.

Now, the situation we are interested in is the theories live on the 5d space $\mathbb{C}^2_{\epsilon_1,\epsilon_2} \times S^1$. In this case, the answer is given in [14, 15], and it is the Ding-Iohara-Miki (DIM) algebra. This algebra is the quantum deformation of the algebra which appears when we consider the 4d theories. The DIM algebra plays the central role throughout this thesis.

Combining all the above, our aim is to prove the non-trivial identity among the partition functions of 5d theories with eight supercharges, with the help of the Ding-Iohara-Miki algebra. This is our general motivation, and in what follows, we explain this goal more concretely and show the summary of what we prove.

Detailed Motivation and Summary

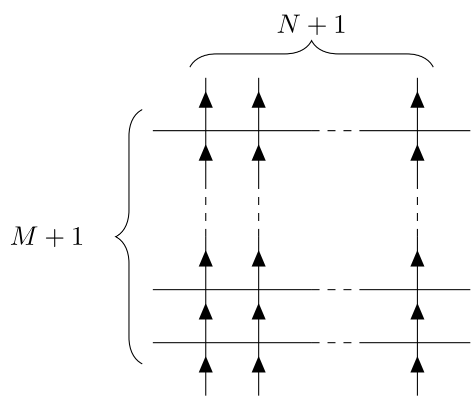
Among the string dualities, the most classical and engrossing one is the S-duality, which was originally introduced in order to avoid the UV divergence in the string amplitudes [94]. Let us see the consequence of the S-duality in the field theories. We can make the following important observation. As noted above, because the S-transform in $SL(2,\mathbb{Z})$ acts on the charges (p,q) in the fundamental rep., that is, acting on (p,q) by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : (p,q) \mapsto (-q,p)\,,$$

the S-transform flips the (p,q)-webs along the diagonal line (we neglect the overall minus sign).

The main example we deal with in this thesis of the five-dimensional theory is the A_M quiver gauge theory with A_N gauge group ($G = A_N^{\otimes M}$) with $N_F = 2(N+1)$ matters and the Chern-Simons levels $\vec{\kappa} = 0$. We call this theory the (A_N, A_M)-theory and denote its the partition function of this theory by $\mathcal{Z}^{(A_N,A_M)}$. We concentrate on this theory because it is the most general one as far as we consider the A-type gauge groups. For the other types of gauge groups, there still exists a problem that the closed forms (which do not include integrals) of the instanton partition functions are not known, and thus those theories are not good objects to deal with.

This theory is engineered by the following (p,q)-web:



That is, we have $N+1$ parallel D5-branes, and intersecting $M+1$ parallel NS5-branes. The marked lines indicate the $(0,1)$ *i.e.* the D5-branes' direction. As noted above, because the S-transform flips the diagram along the diagonal line, the S-duality exchange N and M in $\mathcal{Z}^{(A_N,A_M)}$. Thus we have the following conjecture.

Conjecture. *As the consequence of the S-duality, we have*

$$\mathcal{Z}^{(A_N,A_M)} \sim \mathcal{Z}^{(A_M,A_N)}\,, \tag{1.0.1}$$

where $\sim$ means both sides are identical up to some overall factor. This extra factor has a natural explanation when we consider the topological string theory. See the discussion below.

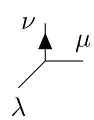
The main goal of this thesis is to prove the claim above. Note that the proof of this claim using the K-theory is almost done in [23] for $N=1$ case, and in [80] for generic N. We give another proof using the Ding-Iohara-Miki algebra and the Macdonald functions. Later we will discuss the difference between those previous works and our result. See Remark below.

One more important fact is we can rephrase the fact above in terms of the topological string theory. Using the string dualities (see Appendix A.2 and A.2.1 for more details), we can engineer the same theories using the A-type open topological string theory. The topological string theories are obtained by coupling the gravity to the two-dimensional topological sigma models, whose target spaces are typically chosen to be the Calabi-Yau threefolds (*i.e.* three complex dimensions). Especially, we are interested in the case where the Calabi-Yau threefolds are toric. In this case, the Calabi-Yau threefolds are uniquely determined by the web diagrams called the toric diagrams, which indicate the fixed loci of the torus actions. The concept which connects the topological string and the 5d $\mathcal{N}=1$ SYM, is called *the geometric engineering*. It ensures that it is possible to compute the partition function of the 5-dimensional QFT which is engineered by a (p,q)-web, by the topological string on the toric Calabi-Yau manifold whose toric diagram is of the same form as the (p,q)-web. Note that from this point of view, the duality formula (1) is called the fibre-base duality [74], because the flip of the toric diagram corresponds to exchanging the base manifolds and fibres.

The contribution for the topological string amplitudes can be regarded to come only from the strings localized on the toric diagram, and this motivates us to develop the diagrammatic technique to compute the partition functions of the topological string. This technique is called the refined topological vertex [1, 47], and it provides the systematic way to compute the topological string partition functions. The refined topological vertex is defined by

$$C^{(\mathrm{IKV})}_{\lambda\mu\nu}(q,t) = (q/t)^{(||\mu||^2+||\nu||^2)/2}\, t^{\kappa(\mu)/2} P_{\nu'}(t^{-\rho};q,t) \times \sum_{\eta} (q/t)^{(|\eta|+|\lambda|-|\mu|)/2}\, s_{\lambda'/\eta}(t^{-\rho}q^{-\nu};q) s_{\mu/\eta}(t^{-\nu'}q^{-\rho};q)\,,$$

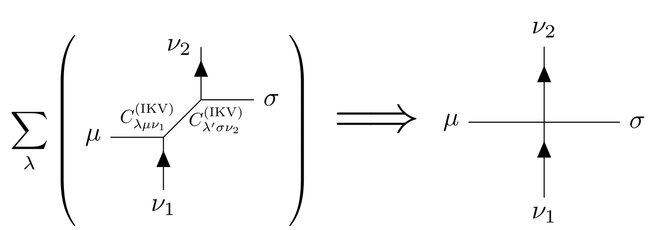
with P_ν the Macdonald polynomial and $s_{\lambda/\eta}$ the skew Schur polynomial, and $C^{(\mathrm{IKV})}_{\lambda\mu\nu}$ can be represented by the trivalent diagram.



$: C^{(\mathrm{IKV})}_{\lambda\mu\nu}(q,t)\,.$

As is obvious from its form, one direction corresponding to ν is special. This direction is called the preferred direction. In the diagram, the preferred direction is denoted by the marked edge. Under the string dualities, $(0,1)$-fivebranes corresponds to the preferred directions. The gluing of two vertices means to take summation over all the partitions about the partition associated with the glued edges.

Using the refined topological vertex, we can also rephrase the conjecture on $\mathcal{Z}^{(A_N,A_M)}$. By the geometric engineering, $\mathcal{Z}^{(A_N,A_M)}$ is obtained by arranging them in parallel crosses like the (p,q)-web above. We assign the emptyset to each external edge. Note that to represent the 4-valent crosses in the diagram, we glue two topological vertices as follows:



We denote the topological string partition function obtained by this way by $\mathcal{Z}^{(A_N,A_M)}_{\mathrm{top.}}$. Here, let us remark the relation between $\mathcal{Z}^{(A_N,A_M)}$ and $\mathcal{Z}^{(A_N,A_M)}_{\mathrm{top.}}$. They are identical up to an overall factor, and we denote the factor by $\mathcal{Z}^{(A_N,A_M)}_{\mathrm{extra}}$. That is,

$$\mathcal{Z}^{(A_N,A_M)}_{\mathrm{top.}} = \mathcal{Z}^{(A_N,A_M)}_{\mathrm{extra}} \cdot \mathcal{Z}^{(A_N,A_M)}\,.$$

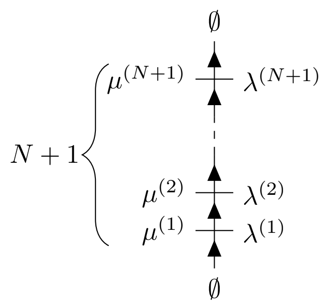
The concrete form of $\mathcal{Z}^{(A_N,A_M)}_{\mathrm{extra}}$ will be given in Section 2.1. The main conjecture above is actually the invariance of this function $\mathcal{Z}^{(A_N,A_M)}_{\mathrm{top.}}$ under the S-duality.

Conjecture (See Conjecture 2.2.1, Proposition 5.2.6 and Conjecture 5.2.9)**.** *From the S-duality of the type IIB superstring, we conjecture*

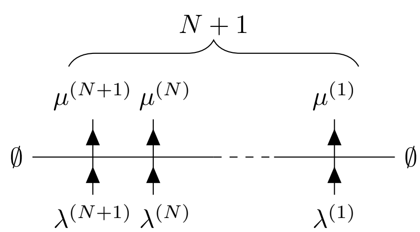
$$\mathcal{Z}^{(A_N,A_M)}_{top.} = \mathcal{Z}^{(A_M,A_N)}_{top.} \,. \tag{1}$$

Let us remark that when we take $q \to t$, which is called the unrefined limit, there are no preferred directions, and the unrefined topological vertex is invariant under any permutation of the partitions associated with each edge. This was proved in [91]. Thus, in this limit, the conjecture becomes trivial. In this limit, the vertex counts the number of the plane partitions (the three-dimensional partitions) with the fixed asymptotic states.

Now, to simplify the problem, we decompose the toric diagrams to the following ladder diagrams:



$\mu^{(i)}$ and $\lambda^{(i)}$ are the partitions associated with each edge. We denote the gluing of the topological vertex in this form by $\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ (Definition 2.2.2). Then, changing the preferred directions in $\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ ends up with the following diagram:



We denote the gluing of the topological vertex in this horizontal ladder form by $\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ (Definition 2.2.2). Then, it is easily shown that the main claim (1) is equivalent to the following equation (Claim 2.2.3):

$$\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}} \sim \sum_{\substack{\boldsymbol{\sigma},\boldsymbol{\nu}\in \mathrm{P}^{N+1} \\ |\boldsymbol{\sigma}|=|\boldsymbol{\lambda}|,|\boldsymbol{\nu}|=|\boldsymbol{\mu}|}} T^*_{\boldsymbol{\lambda}',\boldsymbol{\sigma}} T_{\boldsymbol{\mu}',\boldsymbol{\nu}}\, \mathcal{C}^V_{\boldsymbol{\sigma},\boldsymbol{\nu}} \,, \tag{2}$$

where $T^*_{\boldsymbol{\lambda}',\boldsymbol{\sigma}}$ and $T_{\boldsymbol{\mu}',\boldsymbol{\nu}}$ are some matrices which satisfy

$$\sum_{\boldsymbol{\lambda}} T_{\boldsymbol{\lambda},\boldsymbol{\mu}} T^*_{\boldsymbol{\lambda},\boldsymbol{\nu}} = \delta_{\boldsymbol{\mu},\boldsymbol{\nu}} \,.$$

$\sim$ means the both sides are equal up to the trivial monomial factors. Moreover, $\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ can be computed directly, and we obtain

$$\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}} \sim \prod_{i,j=1}^{N+1} N_{\lambda^{(i)},\mu^{(j)}}(q v_i / t u_j) \,, \tag{3}$$

where $N_{\lambda,\mu}(u)$ is the Nekrasov factor, and $\sim$ means we omit the monomial factors.

It is easy to restore the original claim from this equation. That is, when we compose $M+1$ $\mathcal{C}^H$'s and $M+1$ $\mathcal{C}^V$'s, the former becomes $\mathcal{Z}^{(A_M,A_N)}_{\text{top.}}$ and the latter becomes $\mathcal{Z}^{(A_N,A_M)}_{\text{top.}}$. As is explained in Definition 5.2.12, unfortunately, it is too much to expect $T_{\boldsymbol{\lambda},\boldsymbol{\mu}} = T^*_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. They are given as the transition matrices from the generalized Macdonald functions to the tensor of Schur functions.

Now we summarize the strategy to prove the main claim above. In order to prove, we construct the operators whose matrix elements become $\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. In order to construct them, we make use of the intertwiners of the modules of the Ding-Iohara-Miki (DIM) algebra $\mathcal{U}_{q,t}$ (Fact 3.4.5). As noted above, the DIM algebra (Definition 3.3.4) describes the infinite symmetry of the 5d $\mathcal{N}=1$ supersymmetric QFTs. As we see in Chapter 3, this algebra acts on the space which is spanned by the Macdonald functions. Because the space spanned by the Macdonald functions is isomorphic to the Fock space of the Heisenberg algebra (3.2.1)

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}\,,$$

the Fock space is endowed with the $\mathcal{U}_{q,t}$-module structure. The module structure is not unique. They are labelled by the two integers (n,m) corresponding to the images of the two centers in $\mathcal{U}_{q,t}$. At this point, the $(0,1)$-modules and $(1,M)$-modules ($M \in \mathbb{Z}$) are known, and they are enough for our purpose. Then, in [9], the intertwiner from one $(1,1)$-module to the tensor of one $(1,0)$-module and one $(0,1)$-module (and its inverse) was introduced. Because these intertwiners connects three $\mathcal{U}_{q,t}$-modules, they can be represented by the trivalent vertices. Surprisingly, the matrix elements of these intertwiners with respect to the Schur functions agree with the refined topological vertex (Proposition 3.4.9). From this proposition, we can see the preferred directions correspond to the $(0,1)$-modules, thus the charges of the fivebranes are identified with the labels of the $\mathcal{U}_{q,t}$-modules. By this observation, we use the same diagram as the refined topological vertex to indicates the intertwiners. Then, combining these intertwiners like the ladder diagram of $\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, we construct the operator $\mathcal{T}^V$ (Definition 5.1.8). Let $\boldsymbol{\lambda} = (\lambda^{(1)},\ldots,\lambda^{(N)})$ be an N-tuple of partitions, and $|s_{\boldsymbol{\mu}}\rangle$ be the tensor of the Schur functions (Notation 5.2.10). In the end, it is easy to check that the matrix element $\langle S_{\boldsymbol{\lambda}}|\,\mathcal{T}^V\,|s_{\boldsymbol{\mu}}\rangle$ is identical to $\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ (Lemma 5.2.11). This $\mathcal{T}^V$ ensures the existence of what we call the Mukadé operator, defined below. Let $\mathcal{F}_{\boldsymbol{u}} = \widehat{\bigotimes}_{j=1}^N \mathcal{F}_{u_j}$ be the N-fold tensor space of the Fock spaces.

Definition (See Definition 5.1.1)**.** *Let $\mathcal{V}(x) : \mathcal{F}_{\boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{v}}$ be a linear map satisfying the commutation relations*

$$\left(1-\frac{x}{z}\right)X^{(i)}(z)\mathcal{V}(x) = \left(1-(t/q)^i\frac{x}{z}\right)\mathcal{V}(x)X^{(i)}(z) \qquad (i=1,\ldots,N)\,, \tag{4}$$

and the normalization condition $\langle \mathbf{0}|\,\mathcal{V}(x)\,|\mathbf{0}\rangle = 1$. We refer to this operator as the Mukadé operator. Here $|\mathbf{0}\rangle$ (resp. $\langle\mathbf{0}|$) is the vacuum (resp. dual vacuum) state.

Roughly speaking, $X^{(i)}(z)$'s (Definition 4.2.3) are the generating currents of q-deformed $\mathcal{W}$-algebra for $\mathfrak{g} = A_{N-1}$. Then, we compute the matrix elements of the Mukadé operator. As easily seen, it is not smart to compute the matrix elements directly with respect to the Schur functions. Instead of that, we introduce the better basis of the Fock tensor space $\mathcal{F}_{\boldsymbol{u}}$, called the generalized Macdonald functions. The introduction and construction of this basis are the main subjects in Chapter 4. We denote by $|K_{\boldsymbol{\lambda}}\rangle = |K_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle$, the canonical form of the generalized Macdonald function on $\mathcal{F}_{\boldsymbol{u}}$ (Definition 4.2.25). These states are called the integral forms of the generalized Macdonald functions. Then, the following theorem, which was conjectured in [8], is our final result.

Theorem (See Theorem 5.2.1)**.** *We have*

$$\langle K_{\boldsymbol{\lambda}}(\boldsymbol{v})|\,\mathcal{V}(x)\,|K_{\boldsymbol{\mu}}(\boldsymbol{u})\rangle = \frac{\left((-\gamma^2)^N e_N(\boldsymbol{u})x\right)^{|\boldsymbol{\lambda}|}}{(\gamma^2 x)^{|\boldsymbol{\mu}|}}\prod_{i=1}^N \frac{u_i^{|\mu^{(i)}|}g_{\mu^{(i)}}}{\left(v_i^{|\lambda^{(i)}|}g_{\lambda^{(i)}}\right)^{N-1}}\cdot\prod_{i,j=1}^N N_{\lambda^{(i)},\mu^{(j)}}(qv_i/tu_j)\,. \tag{5}$$

Here, $\gamma = (t/q)^{1/2}$, $e_N(\boldsymbol{u}) = u_1 \cdots u_N$ and $N_{\lambda,\mu}$ is the Nekrasov factor. For the definition of g_λ, see Definition 4.2.27.

This becomes the proof of the main claim above. Denote by $T_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, the transition matrix from the generalized Macdonald functions $|P_{\boldsymbol{\lambda}}\rangle$ to the Schur functions $|s_{\boldsymbol{\mu}}\rangle$. Then, combining this $T_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, (2) and (3), we can conclude the proof of the equation (5), and thus that of the main claim (1). For more details, see Section 5.2.

Now what is left is to prove the main theorem (5). Chapter 5 is devoted to this proof. Recall the Mukadé operator $\mathcal{V}(x) : \mathcal{F}_{\boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{v}}$ depends on the two sets of parameters, $\boldsymbol{u}$ and $\boldsymbol{v}$. Then the proof is by the analytic continuation on the parameters $\boldsymbol{v}$. That is, we first tune the parameters as

$$v_i \to t^{n_i} u_i \,, \quad \text{for} \quad \forall n_i \in \mathbb{Z}_{\geq \ell(\lambda_i)} \,.$$

($\boldsymbol{\lambda}$ are the Young diagrams in the bra vector.) Under this specialization, the Mukadé operator has a drastically simplified realization $\widetilde{V}^{(\boldsymbol{n})}(x)$. Then the matrix elements of $\widetilde{V}^{(\boldsymbol{n})}(x)$ become the Macdonald functions. Next, we apply the Kajihara-Noumi transformation formula (Section 5.3) to the Macdonald functions, and we obtain the Nekrasov factors with the variables specialized. Finally, we analytically continue them to the generic parameters $\boldsymbol{v}$ by the identity theorem.

Remark. *Here, we explain the difference between the previous work [80] and our result. (Note that the result in [23] corresponds to the case $N = 1$ in [80].) Roughly speaking, in [80], they follow the top-down approach, while our approach is bottom-up.*

Let us refine this statement. Let $K_{\boldsymbol{u}}$ be what is called the equivariant K-group of the instanton moduli space, which is isomorphic to $\mathcal{F}_{\boldsymbol{u}}$ as a vector space. They firstly introduce what is called the Ext*-operator $\Phi_m(z) : K_{\boldsymbol{u}} \to K_{\boldsymbol{v}}$ (with m a parameter)*[2]*, whose matrix elements are given by the Nekrasov factor.(Note that this is the definition of the* Ext*-operator.) Then, they prove that the definition is equivalent to the fact that $\Phi_m(z)$ satisfies the following commutation relations with the generating currents $W_k(y)$ $(k = 1, \ldots, N)$ of the q-$\mathcal{W}_N$ algebra:*

$$\prod_{i=1}^{k} \left(1 - \frac{m^k e_N(\boldsymbol{u}) z}{q^{N-i} e_N(\boldsymbol{v}) y}\right) \cdot \left(\Phi_m(z) W_k(y) - m^k W_k(y) \Phi_m(z)\right) = 0 \,.$$

Note that $W_k(y)$ is essentially same as $X^{(k)}(y)$ in our definition of $\mathcal{V}(z)$.

However, once we forget the original definition of the Ext*-operator, these commutation relations do not define the operator uniquely. More concretely, for $k \geq 2$, the matrix elements $\langle \boldsymbol{0}| (W_k)_l \Phi_m(z) |\boldsymbol{0}\rangle$ $(1 < l < k)$ (with $(W_k)_l$ the l-th mode of $W_k(y)$) cannot be computed from the defining relations. Thus once we adopt these defining relations, we have some degrees of freedom to add some operators to the Mukadé operator. (Note that the Mukadé operator also satisfies these relations in addition to its defining relations.) By recalling that the Mukadé operator can be identified with the glued topological vertices in the shape of the ladder (that is, $\mathcal{T}^V$), it is obvious that this situation (where there are degrees of freedom to add some extra operators to the refined topological vertex,) is not good for our purpose to see the duality formula for the refined topological vertex. For short, because of these extra degrees of freedom, the* Ext*-operator cannot be uniquely identified with $\mathcal{T}^V$. This is why we take the bottom-up approach in our work.*

Moreover, the equation (5) has one more interesting application. In order to explain it, we first briefly recall the Alday-Gaiotto-Tachikawa (AGT) correspondence [4]. The AGT correspondence claims the equivalence between the instanton partition function and the Virasoro conformal block. This conjecture has been proved in [71], [97] and [3]. The idea of the proof is to confirm the equivariant cohomology of the instanton moduli space admits the Virasoro module structure. Especially, [3] proves the correspondence by showing the matrix elements of the Virasoro primary field with respect to the generalized Jack functions, are equal to the 4d version of the Nekrasov factors.

We can also state the five-dimensional analogue of the AGT correspondence [3], that is, the equivariant K-group of the instanton moduli space admits the q-Virasoro module structure [76].

[2]More precisely, the Ext-operator and $\Phi_m(z)$ in [80] differ by a simple factor. We omit the difference for simplicity.
[3]Although the q-deformation of the conformal field theory is not defined, from this point of view, this does not matter.

Now we come back to the Mukadé operator. We can show that when $N=2$, under the limit $q,t\to 1$, the defining relation of the Mukadé operator (4) reduces to

$$[L_n, V(z)] = z^n \left(z\frac{\partial}{\partial z} + h(n+1) \right) V(z)\,.$$

This is the defining relation of the Virasoro primary fields. Therefore, the Mukadé operator can be regarded as the q-analogue of the primary field. Then, the main theorem (5) means the matrix elements of the q-primary field with respect to the generalized Macdonald functions are equal to the (5d) Nekrasov factors, and thus becomes the five-dimensional analogue of the proof *á la* Alba-Fateev-Litvinov-Tarnopolsky [3] of the Alday-Gaiotto-Tachikawa correspondence [4].

Structure of the Thesis

This thesis is organized as follows. In Chapter 2, we briefly summarize the results concerning the refined topological vertex and the concept of the geometric engineering. After introducing them, we show the main claim which is the consequence of the S-duality. That is, the equality between $\mathcal{Z}_{\mathrm{top.}}^{(A_N,A_M)}$ and $\mathcal{Z}_{\mathrm{top.}}^{(A_M,A_N)}$ as noted above.

Chapter 3 and Chapter 4 are irrelevant to Chapter 2. These two chapters are devoted to the explanation of the Macdonald functions. In Chapter 3, we review the Macdonald functions, realized on the Fock space, and the algebra associated with this symmetric functions. In this chapter, we see the Macdonald functions on the Fock space can be regarded as the limit of the Macdonald polynomials where the number of the variables goes to infinity. We introduce the vertex operator $\eta(z)$ whose zero mode is intertwined to the Macdonald operator in the case of the finite variables under the appropriate projection. After that, we review the explicit algorithm to construct the Macdonald functions on the Fock space. For this purpose, we prepare two tools, the Macdonald polynomials and the vertex operator called the top component. By multiplying the Macdonald polynomial to some products of the top components, taking the constant term, and applying the resulted operator to the vacuum, we obtain the Macdonald function on the Fock space. Next, we study the algebra associated with this current $\eta(z)$. In the end, this algebra turns out to be the Ding-Iohara-Miki (DIM) algebra $\mathcal{U}_{q,t}$. We summarize the known results about the representation theory of the DIM algebra. One of the key ingredients in this thesis, the intertwiner of $\mathcal{U}_{q,t}$, is introduced in this chapter.

Chapter 4 is devoted to the extension of Chapter 3 to the Fock tensor spaces, that is, the introduction of the generalized Macdonald functions. The story goes almost the same way as in Chapter 3. The generalized Macdonald functions are defined as the eigenstates of the zero mode of $\Delta(\eta(z))$, where Δ is the coproduct in $\mathcal{U}_{q,t}$. From the lesson we learned in the previous chapter, we know that in order to construct such states, we first have to know about the "polynomials". We see that the answer is what is called the bispectral Macdonald functions. Moreover, the analogous vertex operator of the top component requires the screening currents of q-deformed $\mathcal{W}$-algebra. In this chapter, we mainly study these two ingredients, the bispectral Macdonald functions, and the screening currents.

Preparations in all the former chapters come to fruition in Chapter 5. We give the algebraic proof of the main claim stated in Chapter 2, using theorems in Chapter 3 and 4. More precisely, we prove the formula for the matrix elements of the Mukadé operator with respect to the generalized Macdonald functions. The key tool for the proof is the Kajihara-Noumi identity, which is roughly the Euler transformation formula for the multiple hypergeometric series.

In Chapter 6, we revisit the bispectral Macdonald functions. We realize the Macdonald functions as the compositions of the Mukadé operators (with parameters specialized).

This thesis is based on the following paper [35]:

- M. Fukuda, Y. Ohkubo and J. Shiraishi, "Generalized Macdonald Functions on Fock Tensor Spaces and Duality Formula for Changing Preferred Direction", arXiv:1903.05905[math.QA].

The original part in the present thesis is provided in some part of Chapter 4, Chapter 5 and Chapter 6. More precisely, in Chapter 4, Chapter 5 and Chapter 6, all the lemmas, propositions and theorems which have no citations in their labels, are the original works.

Chapter 2

Main Claim: Duality under S-transformation

In this chapter, we summarize the facts known in the string theory and show some claims which the string theory conjectures. The proofs of those claims are the main object in this thesis.

In Section 2.1, we review the concept called *the geometric engineering*. It relates the five-dimensional super Yang-Mills (SYM) theory to the topological string theory. More precisely, through this correspondence, the partition functions of the 5d SYM can be computed as the partition functions of the topological string. Thus we first summarize the exact form of the 5d SYM partition functions, especially their non-perturbative part called *the instanton partition functions*. Then, we see the idea of *the topological vertex*, which gives us the technique to compute the (limit of) topological string partition functions. The geometric engineering suggests these functions agree with each other. We deal with two examples. First, the simplest one, the pure A_N gauge theory. The second one is the A_M quiver gauge theory with A_N gauge groups. We refer to the latter theory as the (A_N, A_M)-theory.

In Section 2.2, we see the string duality called *S-duality* claims the highly non-trivial identity between the partition functions of (A_N, A_M)-theory and (A_M, A_N)-theory. Moreover, this identity suggests the duality formula under changing the preferred directions of the refined topological vertex. Again, note that these formulas are consequences of the string duality, and they are merely conjectures which have to be proved. The proofs are delivered in Chapter 5.

2.1 Geometric Engineering

2.1.1 Instanton Counting

Our main concern is the 5d $\mathcal{N}=1$ (*i.e.* eight supercharges) super Yang-Mills (SYM for short) theory. This class of QFTs has been extensively studied since the original work [98] by Seiberg. Throughout the thesis, we only consider the case that the 5-dimensional theories live on $\mathbb{C}^2_{\epsilon_1,\epsilon_2}\times S^1_R$, where R stands for the circumference of S^1. This geometry is defined by the identification

$$(z,w,y)\sim(e^{-\epsilon_1}z,e^{-\epsilon_2}w,y+R)\,,\quad \text{for}\quad (z,w)\in\mathbb{C}^2\,,\ y\in S^1_R\,. \tag{2.1.1}$$

We refer to this geometry as the Ω-background. In what follows, we put $q=e^{-\epsilon_1}$ and $t^{-1}=e^{-\epsilon_2}$.

One of the salient features of this class of QFTs is the partition functions can be computed exactly. The partition function is defined by the the path integral of the unity, that is,

$$\mathcal{Z}^{5D}=\int[\mathcal{D}(\text{fields})]\ e^{-S[(\text{fields})]}\,, \tag{2.1.2}$$

where S is the action of the theory we consider. (We always consider the Euclidean theories.) First, let us see what kind of parameters this function depends on. Because $\mathbb{C}^2$ is not compact, we have to specify the boundary conditions at infinity (*i.e.* the values of each field at infinity) to make the partition function finite. We have two sets of such parameters $\boldsymbol{u}$ and $\boldsymbol{m}$, specifying the boundary conditions. ($\boldsymbol{u}$ is the shorthand notation for the multiple of parameters $(u_1, u_2, \ldots)$.) u_i's are parameters corresponding to the exponential of the values of $A_t + i\varphi$ at infinity, where A_t is the 5d gauge field along S^1 direction, and φ is the real scalar field in the vector multiplet in 5d. In the limit where the theory reduces to 4d, they become the complex scalar in the vector multiplet, we call them the Coulomb branch parameters. Similarly, in 5d, the real mass parameters become the scalar field in a vector multiplet, and the component along S^1 direction of the vector field in this vector multiplet and the mass are combined into a complex field. m_j's are parameters corresponding to the exponential of the values of this complex field at infinity. In the 4d limit, this combination becomes the complex mass parameters, and thus we just refer to them as the mass parameters in what follows.

Besides, the partition function depends on the coupling constant. In 5d, the coupling constant $\sim 1/g^2$ becomes the real scalar field in the vector multiplet whose vector component couples to the conserved current corresponding to the instanton number. We have to specify the value of this field at infinity, and we denote the exponential of the parameter by q. q is called the instanton fugacity.

In the end, noting we also have q, t parameters, the partition function is the function of all these parameters, that is,

$$\mathcal{Z}^{5D} = \mathcal{Z}^{5D}(\mathsf{q}, \boldsymbol{u}, \boldsymbol{m} | q, t)\,. \tag{2.1.3}$$

Moreover, the partition function is decomposed into the perturbative part $\mathcal{Z}_{\text{1-loop}}$ and non-perturbative part $\mathcal{Z}_{\text{inst.}}$, as

$$\mathcal{Z}^{5D} = \mathcal{Z}_{\text{1-loop}} \cdot \mathcal{Z}_{\text{inst.}}\,. \tag{2.1.4}$$

We put the subscript "1-loop" to the perturbative part because it is one-loop exact thanks to the supersymmetry.

First, we take a look at the non-perturbative part $\mathcal{Z}_{\text{inst.}}$, which is also called the instanton partition function. (On the 1-loop part, we will give some comments later.) In the weak coupling limit, we can integrate out the non-compact 4d space, and the instanton partition function of the theory reduces to the partition function of the supersymmetric quantum mechanics on S^1, that is, the following index (Section 4 in [84]):

$$\mathcal{Z}_{\text{inst.}} = \sum_{k=0}^{\infty} \mathsf{q}^k \; Z_k\,, \tag{2.1.5}$$

with $\mathsf{q} = e^{-4\pi^2 R/g^2}$ and

$$Z_k = \mathrm{tr}_{\mathcal{H}_k} \left[(-1)^F q^{J_1} t^{-J_2} \boldsymbol{u}^{\boldsymbol{\Pi}} \boldsymbol{m}^{\boldsymbol{K}} \right]\,, \tag{2.1.6}$$

where $\mathcal{H}_k$ is the Hilbert space of the supersymmetric quantum mechanics whose target is k-instanton moduli space and the meanings of the other symbols are as follows. We use the notations like $\boldsymbol{u}^{\boldsymbol{\Pi}} = \prod_i u_i^{\Pi_i}$. F is the fermion number operator, $J_{1,2}$ are the generators of the Cartan algebra of $SO(4)$, and Π_i's (resp. K_j's) are the generators of the Cartan algebra of the gauge group (resp. the flavor group). The rough idea of the derivation of the results in this section is explained in Appendix A.1, and for more details, see the references cited there.

Using the localization technique, we can compute $\mathcal{Z}_{\text{inst.}}$ as the summation over the Young diagrams, which represent the fixed points of the instanton moduli space. In order to show the closed form of $\mathcal{Z}_{\text{inst.}}$, we introduce the function, called *the Nekrasov function.*

Definition 2.1.1 (Nekrasov Factor). *Let $\lambda, \mu \in \mathsf{P}$, define the function*

$$N_{\lambda\mu}(u) := \prod_{(i,j)\in\lambda} \left(1 - u q^{a_\lambda(i,j)} t^{\ell_\mu(i,j)+1}\right) \prod_{(i,j)\in\mu} \left(1 - u q^{-a_\mu(i,j)-1} t^{-\ell_\lambda(i,j)}\right)\,. \tag{2.1.7}$$

$\mathcal{Z}_{\text{inst.}}$ has the contributions from several multiplets. We summarize some of them in the following fact.

Fact 2.1.2. *We concentrate on the case where the gauge group is $G = A_N$, and every vectors which appear below have $(N+1)$ components.*

1. *The vector multiplets:*
$$\mathcal{Z}_{vec.}(\boldsymbol{u},\boldsymbol{\lambda}) = \prod_{i,j=1}^{N+1} \frac{1}{N_{\lambda^{(i)}\lambda^{(j)}}(qu_i/tu_j)} \,. \tag{2.1.8}$$

2. *The bifundamental matter with the mass parameter m:*
$$\mathcal{Z}_{bifund.}(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{u},\boldsymbol{\mu}|m) = \prod_{i,j=1}^{N+1} N_{\lambda^{(i)}\mu^{(j)}}(m\,qv_i/tu_j) \,. \tag{2.1.9}$$

3. *The Chern-Simons term with level κ:*
$$\mathcal{Z}_{CS}(\boldsymbol{u},\boldsymbol{\lambda}|\kappa) = \prod_{\ell=1}^{N+1} \prod_{(i,j)\in\lambda^{(\ell)}} \left(u_\ell q^{j-1}t^{1-i}\right)^{\kappa} = e_{N+1}(\boldsymbol{u}) \left(\prod_{i=1}^{N+1} g_{\lambda^{(i)}}\right)^{\kappa} \,. \tag{2.1.10}$$

4. *The fundamental and anti-fundamental matter with the mass parameter m*[1]*:*
$$\mathcal{Z}_{fund.}(\boldsymbol{u},\boldsymbol{\lambda}|\boldsymbol{m}) = \prod_{i,j=1}^{N+1} N_{\lambda^{(i)}\emptyset}(qu_i/tm_j) \,, \quad \mathcal{Z}_{af.}(\boldsymbol{u},\boldsymbol{\lambda}|\boldsymbol{m}) = \prod_{i,j=1}^{N+1} N_{\emptyset\lambda^{(i)}}(qm_j/tu_i) \,. \tag{2.1.11}$$

Here, $\boldsymbol{u}$ are the exponentiated Coulomb branch parameters.

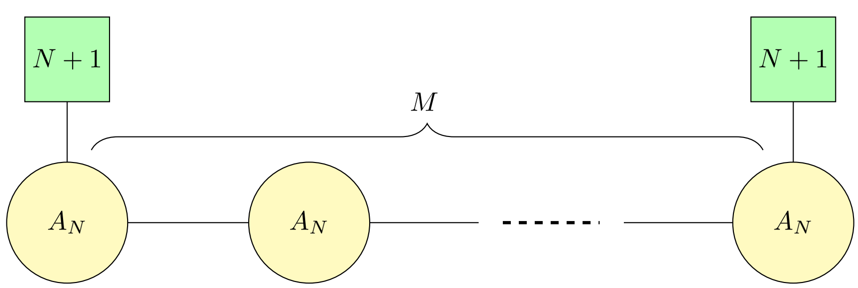
Let us see two examples. The first one is the pure theory, and the second one is the A-type linear quiver gauge theory with the A-type gauge groups. The latter example is out main target, and its exact form is used repeatedly throughout this thesis.

Example 1: Pure gauge theory with gauge group A_N with CS level κ

Fact 2.1.3. *For $\boldsymbol{u} \in \mathbb{C}^{N+1}$, the following function gives the instanton partition function of the 5d $\mathcal{N}=1$ pure SYM with the gauge group $G = A_N$:*
$$\mathcal{Z}^{A_N}_{inst.}(\mathfrak{q}|\boldsymbol{u}|\kappa) = \sum_{\lambda^{(1)},\ldots,\lambda^{(N+1)}\in\mathsf{P}} \mathfrak{q}^{|\boldsymbol{\lambda}|}\, \mathcal{Z}_{CS}(\boldsymbol{u},\boldsymbol{\lambda}|\kappa)\, \mathcal{Z}_{vec.}(\boldsymbol{u},\boldsymbol{\lambda}) \,. \tag{2.1.12}$$

Example 2: A_M quiver gauge theory with A_N gauge group with $N_F = 2(N+1)$

We consider the following quiver diagram:



We denote the partition function of this theory by $\mathcal{Z}^{(A_N,A_M)}$.

[1]We use the slightly different notation for the mass parameters than conventional one for later convenience.

Fact 2.1.4. *For $\boldsymbol{u}^{(i)} \in \mathbb{C}^{N+1}$ $(i = 1, \dots, M)$, $\boldsymbol{m}_{f,a} \in \mathbb{C}^{N+1}$, the following function gives the instanton partition function of this theory:*

$$\begin{aligned}
&\mathcal{Z}^{(A_N,A_M)}_{\textit{inst.}}((\mathfrak{q}_i)_{i=1}^M | \boldsymbol{u}^{(1)}, \dots, \boldsymbol{u}^{(M)} | \boldsymbol{\kappa} | \boldsymbol{m}_f, \boldsymbol{m}_a, (m_i)_{i=1}^{M-1}) \\
&\quad = \sum_{\substack{\boldsymbol{\lambda}^{(i)} \in \mathsf{P}^{N+1} \\ 1 \le i \le M}} \prod_{k=1}^{M} \mathfrak{q}_k^{|\boldsymbol{\lambda}^{(k)}|} \cdot \mathcal{Z}_{CS}(\boldsymbol{u}^{(k)}, \boldsymbol{\lambda}^{(i)} | \kappa_k) \cdot \mathcal{Z}_{vec.}(\boldsymbol{u}^{(k)}, \boldsymbol{\lambda}^{(k)}) \\
&\qquad\quad \times \mathcal{Z}_{fund.}(\boldsymbol{u}^{(M)}, \boldsymbol{\lambda}^{(M)} | \boldsymbol{m}_f) \cdot \mathcal{Z}_{af.}(\boldsymbol{u}^{(1)}, \boldsymbol{\lambda}^{(1)} | \boldsymbol{m}_a) \prod_{k=1}^{M-1} \mathcal{Z}_{bifund.}(\boldsymbol{u}^{(k)}, \boldsymbol{\lambda}^{(k)} | \boldsymbol{u}^{(k+1)}, \boldsymbol{\lambda}^{(k+1)} | m_k)\,.
\end{aligned} \tag{2.1.13}$$

Roughly speaking, the contributions from the vector multiplets come out of each node, and the bifundamental contributions are associated with the edges connecting nodes. As we will see in Chapter 5, the parameters m_i $(i = 1, \dots, M-1)$, which are related to the masses of the bifundamental matters, are fixed to $\gamma = (t/q)^{1/2}$. When the Chern-Simons level $\boldsymbol{\kappa} = (0, \dots, 0)$, we refer to this theory as the (A_N, A_M)-theory.

Comments on 1-loop Part

Here, let us give some comments on the 1-loop part $\mathcal{Z}_{\text{1-loop}}$ of the partition function. The closed formula for $\mathcal{Z}_{\text{1-loop}}$ is known (see [61] for the derivation), and the result is summarized as follows.

Fact 2.1.5. *Put*

$$\mathcal{G}(z) := \prod_{i,j=0}^{\infty} (1 - z q^i t^{-j}) = \prod_{i,j=0}^{\infty} \frac{1}{1 - z q^i t^{j+1}}\,. \tag{2.1.14}$$

For the vector multiplet, the contribution to $\mathcal{Z}_{\textit{1-loop}}$ is given by

$$\mathcal{Z}^{\textit{vec.}}_{\textit{1-loop}}(\boldsymbol{u}) = \prod_{\alpha \in \Delta^+} \mathcal{G}(\boldsymbol{u}^{\alpha}) \cdot \mathcal{G}(q \boldsymbol{u}^{\alpha}/t)\,, \tag{2.1.15}$$

where Δ^+ is the positive root of the gauge group (that is, A_N in this thesis), and $\boldsymbol{u}$ is the exponentiated Coulomb branch parameters. $\boldsymbol{u}^{\alpha}$ stands for $\prod_{i=1}^{N+1} u_i^{\alpha_i}$ when we represent α as the $N+1$-vector.

For the hypermultiplet, the contribution is given by

$$\mathcal{Z}^{\textit{hyper.}}_{\textit{1-loop}}(\boldsymbol{u}, m) = \left(\prod_{w \in \boldsymbol{R}} \mathcal{G}(m \boldsymbol{u}^{w}/\gamma) \right)^{-1}\,, \tag{2.1.16}$$

where $\boldsymbol{R}$ is the weight of the representation of A_N, which is associated with the matter.

Let us see two examples. The first example is the pure gauge theory with gauge group A_N (see Example 1 above). We denote it by $\mathcal{Z}^{A_N}_{\text{1-loop}}$. The contribution only comes from the vector multiplets, and we obtain the following fact.

Fact 2.1.6. *For $\boldsymbol{u} \in \mathbb{C}^{N+1}$, the following identity holds:*

$$\mathcal{Z}^{A_N}_{\textit{1-loop}}(\boldsymbol{u}) = \prod_{1 \le i < j \le N+1} \mathcal{G}(u_j/u_i) \cdot \mathcal{G}(q u_j / t u_i)\,. \tag{2.1.17}$$

The second example is the (A_N, A_M)-theory (see Example 2 above). In this case, the contribution comes from the fundamental and anti-fundamental matters and the $M-1$ bifundamental matters, in addition to the vector multiplets at each nodes. The result is given by the following fact.

Fact 2.1.7. *The following function gives the 1-loop contribution of (A_N, A_M)-theory:*

$$\mathcal{Z}^{(A_N,A_M)}_{\textit{1-loop}}(\boldsymbol{u}^{(1)},\ldots,\boldsymbol{u}^{(M)}|\boldsymbol{u}^{(0)},\boldsymbol{u}^{(M+1)}) = \mathcal{Z}^{(A_N,A_M)}_{\textit{1-loop, bifund.}} \cdot \mathcal{Z}^{(A_N,A_M)}_{\textit{1-loop, f,a}} \cdot \mathcal{Z}^{(A_N,A_M)}_{\textit{1-loop, vec.}}\,, \tag{2.1.18}$$

with

$$\begin{aligned}
\mathcal{Z}^{(A_N,A_M)}_{\textit{1-loop, bifund.}} &= \prod_{k=2}^{M}\left(\prod_{1\le j<i\le N}\mathcal{G}(u_i^{(k)}/\gamma u_j^{(k-1)})\prod_{1\le i\le j\le N}\mathcal{G}(u_j^{(k-1)}/\gamma u_i^{(k)})\right)^{-1},\\
\mathcal{Z}^{(A_N,A_M)}_{\textit{1-loop, f,a}} &= \left(\prod_{1\le j<i\le N}\mathcal{G}(u_i^{(1)}/\gamma u_j^{(0)})\prod_{1\le i\le j\le N}\mathcal{G}(u_j^{(0)}/\gamma u_i^{(1)})\right)^{-1}\\
&\quad\times\left(\prod_{1\le j<i\le N}\mathcal{G}(u_i^{(M+1)}/\gamma u_j^{(M)})\prod_{1\le i\le j\le N}\mathcal{G}(u_j^{(M)}/\gamma u_i^{(M+1)})\right)^{-1}\\
\mathcal{Z}^{(A_N,A_M)}_{\textit{1-loop, vec.}} &= \prod_{k=1}^{M}\left(\prod_{1\le i<j\le N+1}\mathcal{G}(u_j^{(k)}/u_i^{(k)})\cdot\mathcal{G}(qu_j^{(k)}/tu_i^{(k)})\right).
\end{aligned} \tag{2.1.19}$$

Here, $\boldsymbol{u}^{(i)} \in \mathbb{C}^{N+1}$ $(i = 1,\ldots,M)$ is the Coulomb branch parameters at i-th node, and $\boldsymbol{u}^{(0)}, \boldsymbol{u}^{(M+1)} \in \mathbb{C}^{N+1}$ are associated with the parameters $\boldsymbol{m}_{f,a}$ above, the fundamental and anti-fundamental matters.

When $M = 1$ (note that there is no bifundamental matter in this case), the fact that $\mathcal{Z}^{(A_N,A_1)}_{\text{1-loop}}$ is of this form was confirmed in [111] (see also Appendix C in [41]). The generalization of to generic M is straightforward because the bifundamental matter can be regarded as just the combination of the two fundamental matters.

2.1.2 Topological Vertex and its Refinement

Next, we introduce the topological vertex and its refinement. Again, the rough sketch of the derivation of them is in Appendix A.2, and for more details, see the references therein. First, we introduce the unrefined topological vertex.

Definition 2.1.8 (Topological Vertex [1])**.** *For $\lambda, \mu, \nu \in \mathsf{P}$, define the topological vertex $C_{\lambda\mu\nu}(q)$ by*

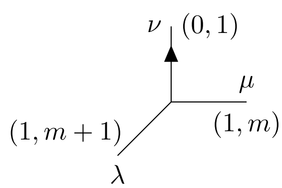
$$C_{\lambda\mu\nu}(q) = q^{\kappa(\mu)/2}s_{\nu'}(q^{-\rho};q,t)\sum_{\eta}s_{\lambda'/\eta}(q^{-\nu-\rho};q)s_{\mu/\eta}(q^{-\nu'-\rho};q)\,. \tag{2.1.20}$$

The refinement is proposed in [47], and the result is summarized by the following definition.

Definition 2.1.9 (Refined Topological Vertex [47])**.** *Define the refined topological vertex $C^{(\mathrm{IKV})}_{\lambda\mu\nu}(q,t)$ by*

$$\begin{aligned}
C^{(\mathrm{IKV})}_{\lambda\mu\nu}(q,t) &= (q/t)^{(||\mu||^2+||\nu||^2)/2}\,t^{\kappa(\mu)/2}P_{\nu'}(t^{-\rho};q,t)\\
&\quad\times\sum_{\eta}(q/t)^{(|\eta|+|\lambda|-|\mu|)/2}\,s_{\lambda'/\eta}(t^{-\rho}q^{-\nu};q)s_{\mu/\eta}(t^{-\nu'}q^{-\rho};q)\,.
\end{aligned} \tag{2.1.21}$$

We assign the following diagram to the refined topological vertex:



$$C^{(\mathrm{IKV})}_{\lambda\mu\nu}(q,t)\,.$$

For $v = (v_1, v_2)$ and $w = (w_1, w_2)$, define the symplectic product $\wedge$ by

$$v \wedge w = v_1 w_2 - v_2 w_1 \,. \tag{2.1.22}$$

Then, the triple of 2-vectors (v_1, v_2, v_3) $(v_i \in \mathbb{Z}^2)$ which satisfy

$$v_1 + v_2 + v_3 = 0\,, \quad v_1 \wedge v_2 = v_2 \wedge v_3 = v_3 \wedge v_1 = 1\,, \tag{2.1.23}$$

specify the slopes of each legs of the trivalent vertex. (The graph above corresponds to $m = 0$ case.) We refer to the direction corresponding to the partition ν as *the preferred direction*. We also denote the direction to μ (resp. λ) as t-direction (resp. q-direction).

As is obvious from its form, in $q \to t$ (self-dual background) limit, the refined topological vertex reduces to the (unrefined) topological vertex.

Now in order to define the gluing rules of the refined topological vertices, we first introduce the framing factor, which was introduced in [47] and slightly modified in Taki's paper [111].

Definition 2.1.10 (Taki's framing factor)**.** *Define the framing factor by*

$$f'_\nu(q,t) := (-1)^{|\nu|+|\nu|/2} t^{n(\nu)} q^{-n(\nu')}\,, \tag{2.1.24}$$

with ν' the transposition of ν.

Note that we put $'$ on $f_\nu(q,t)$ because it differs by $(-1)^{|\nu|/2}$ from $f_\nu(q,t) = (-1)^{|\nu|} t^{n(\nu)} q^{-n(\nu')}$, which was introduced in [47] and will be used in the succeeding chapters.

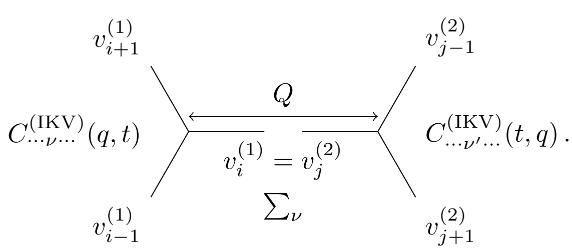
Now we define the gluing rules of two refined topological vertices.

Definition 2.1.11 (Gluing Rules)**.** *Let $(v_1^{(1)}, v_2^{(1)}, v_3^{(1)})$ and $(v_1^{(2)}, v_2^{(2)}, v_3^{(2)})$ be the triple of 2-vectors which specify two glued vertices.*

Define the gluing rules for two vertices associated with $v_i^{(1)}$ and $v_j^{(2)} (= v_i^{(1)})$, by

$$\sum_\nu Q^{|\nu|} C^{(\mathrm{IKV})}_{\cdots\nu\cdots}(q,t) \cdot (f'_\nu(q,t))^n \cdot C^{(\mathrm{IKV})}_{\cdots\nu'\cdots}(t,q)\,, \tag{2.1.25}$$

with $n = v_{i+1}^{(1)} \wedge v_{j+1}^{(2)} = v_{i-1}^{(1)} \wedge v_{j-1}^{(2)}$ (identifying $i+3$ with i if needed) the framing number, which is determined by the slope of the two vertices. The corresponding diagram is given by



These are all the possible gluing of the vertices.

We introduce the toric Calabi-Yau threefold called A_N geometry. As is well-known, the toric variety is uniquely identified once the two dimensional diagram, called the toric diagram is specified.

Definition 2.1.12 (A_N Geometry)**.** *Define the toric Calabi-Yau threefold whose toric diagram is given by the following diagram:*

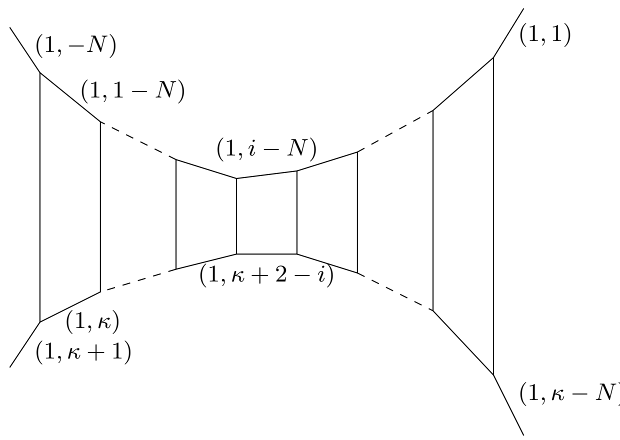


Figure 2.1: Toric Diagram for A_N Geometry

The double of integers (n, m) denotes the slopes of each lines. We refer to this CY as the A_N-geometry.

This geometry is the $\mathbb{C}^2/\mathbb{Z}_{N+1}$ fibration over $\mathbb{P}^1$. The way to recover the toric diagram above is explained in the appendix in [46].

The purpose of this subsection is to introduce the following refined topological string partition function.

Definition 2.1.13. *Define the topological string partition function $\mathcal{Z}^{A_N}_{top.}(q,t)$ on A_N geometry by the gluing of the refined topological vertices in the form of the toric diagram of the A_N geometry. We regard the vertical lines as the preferred directions and assign the empty partitions to all external legs.*

2.1.3 Geometric Engineering

The concept of the geometric engineering was introduced in [59], and further studied in numerous papers. The most basic idea is summarized into the following fact.

Fact 2.1.14. *We have the following identity:*

$$\mathcal{Z}^{A_N}_{top.}(q,t) = \mathcal{Z}^{A_N}_{\textit{1-loop}} \cdot \mathcal{Z}^{A_N}_{inst.}(q,t)\,. \tag{2.1.26}$$

Here, the Chern-Simons level corresponds to κ in Figure 2.1.

The proof is by direct computation and is given in [111].

This correspondence can be extended to the wider class of theories. Let X be the toric diagram of some CY threefold, and $\bar{X}$ be the brane web diagram[2] which is equal to the diagram X. Then we have

$$\text{type IIB string with } (p,q)\text{-fivebrane web } \bar{X} \quad \simeq \quad \text{Type-A topological string on X}\,. \tag{2.1.27}$$

For the toric diagram X, let $\mathcal{Z}^X_{\text{top.}}$ be the partition function which is obtained by gluing the refined topological vertices in the form of X. For every 5D $\mathcal{N}=1$ SYM theory T_X which can be realized by the (p,q)-web $\bar{X}$, the partition function of T_X is equal to $\mathcal{Z}^X_{\text{top.}}$. For more details of this idea, see Appendix A.2.1.

Another example we deal with is the A_M quiver gauge theory with A_N gauge group at each nodes. In the present thesis, we concentrate on the case where the Chern-Simons levels at each nodes are zero. This theory is engineered by the following toric diagram:

[2] Note that we use the unusual convention, with which the D5-branes lie in the vertical direction, while in most of the papers, they lie in the horizontal direction.

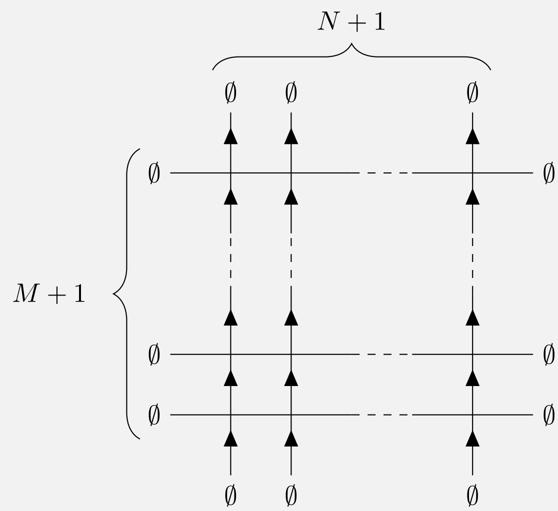
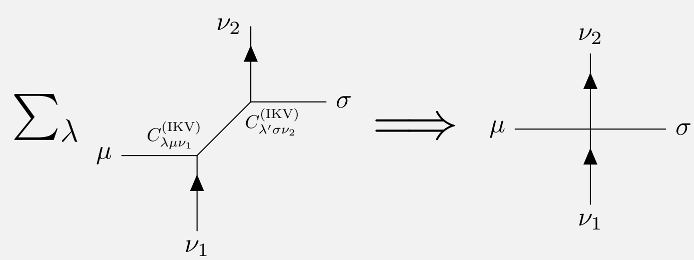


Figure 2.2: Toric Diagram for (A_N, A_M) Theory

Here, we use the simplified notation,



Definition 2.1.15. *Define the topological string partition function* $\mathcal{Z}^{(A_N,A_M)}_{top.}$ *by the gluing of the refined topological vertices in the form of the toric diagram of Figure 2.2.*

Then, the geometric engineering claims the following statement.

Fact 2.1.16. *Under the appropriate identification of parameters, we have the following identity:*

$$\mathcal{Z}^{(A_N,A_M)}_{top.}(q,t) = \mathcal{Z}^{(A_N,A_M)}_{extra} \cdot \mathcal{Z}^{(A_N,A_M)}_{1\text{-}loop} \cdot \mathcal{Z}^{(A_N,A_M)}_{inst.}(q,t)\,. \tag{2.1.28}$$

The RHS is defined in Fact 2.1.4 and Fact 2.1.7. $\mathcal{Z}^{(A_N,A_M)}_{extra}$ *is some normalization factor.*

Here, the normalization factor $\mathcal{Z}^{(A_N,A_M)}_{\mathrm{extra}}$ is the contribution from the string stretch between parallel D5-branes in the external legs. For more details, see [41, 42]. Following these references, for the (A_N, A_M)-theory, we can compute the concrete form of $\mathcal{Z}^{(A_N,A_M)}_{\mathrm{extra}}$, see especially Appendix C in [41]. This will be used later in Chapter 5.

Fact 2.1.17. *For* $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^{N+1}$*, the form of* $\mathcal{Z}^{(A_N,A_M)}_{extra}$ *is given by*

$$\mathcal{Z}^{(A_N,A_M)}_{extra}(\boldsymbol{u}|\boldsymbol{v}) = \prod_{1\le i<j\le N+1} \mathcal{G}(u_j/u_i)\cdot \mathcal{G}(qv_j/tv_i)\,. \tag{2.1.29}$$

The ratios of the parameters $\boldsymbol{u}, \boldsymbol{v}$ *corresponds to the Kähler parameters among the parallel external legs.*

2.2 S-duality and Main Claim

On the LHS of the duality (2.1.27), there exists a natural action of $SL(2,\mathbb{Z})$ on the brane web, which is induced from that on τ, (which roughly corresponds to the coupling constant of the gauge theory in the IR,)

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}\,, \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{2.2.1}$$

At the same time, $SL(2,\mathbb{Z})$ acts on the double of integers (p,q), which labels the fivebrane charge. In this thesis, we are especially interested in

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{2.2.2}$$

Through the string duality above, this action is intertwined to that on the toric diagram. The S-action on the diagram above ends up with the exchange between A_N and A_M. This is due to the fact that S-transform exchanges the $(0,1)$-branes and $(1,0)$-branes, and thus in terms of the toric diagrams, it changes the preferred direction. (We neglect overall negative sign.) As a result, the S-duality in the string theory conjectures the following equality.

Conjecture 2.2.1. *Under the appropriate identification of parameters, we have*

$$\mathcal{Z}^{(A_N,A_M)}_{top.}(q,t) = \mathcal{Z}^{(A_M,A_N)}_{top.}(q,t)\,. \tag{2.2.3}$$

For the details, see Proposition 5.2.6 (and Conjecture 5.2.9).

The proof is presented in Section 5.2.

2.2.1 Main Claim in terms of Refined Topological Vertex

Now we rephrase the above claim in terms of the refined topological vertex. First we define the gluing corresponding to Figure 2.3.

Definition 2.2.2. *For the $(N-1)$-tuple of parameters $(Q_1,\ldots,Q_{N-1})$ and the N-tuple of parameters $(Q'_1,\ldots,Q'_N)$, define*

$$\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}((Q_i),(Q'_i)) := \sum_{\boldsymbol{\nu}\in\mathsf{P}^N}\sum_{\boldsymbol{\sigma}\in\mathsf{P}^{N-1}}\prod_{i=1}^{N-1} Q_i^{|\sigma^{(i)}|}\prod_{j=1}^{N}(Q'_j)^{|\nu^{(j)}|}\prod_{i=1}^{N} C^{(\mathrm{IKV})}_{\nu^{(i)}(\sigma^{(i)})'\lambda^{(i)}}(q,t)\cdot C^{(\mathrm{IKV})}_{(\nu^{(i)})'\sigma^{(i-1)}\mu^{(i)}}(t,q)\,, \tag{2.2.4}$$

$$\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}((Q_i),(Q'_i)) := \sum_{\boldsymbol{\nu}\in\mathsf{P}^N}\sum_{\boldsymbol{\sigma}\in\mathsf{P}^{N-1}}\prod_{i=1}^{N-1} Q_i^{|\sigma^{(i)}|}\prod_{j=1}^{N}(Q'_j)^{|\nu^{(j)}|}\prod_{i=1}^{N} C^{(\mathrm{IKV})}_{\nu^{(i)}\mu^{(i)}(\sigma^{(i-1)})'}(q,t)\cdot C^{(\mathrm{IKV})}_{(\nu^{(i)})'\lambda^{(i)}\sigma^{(i)}}(t,q)\,. \tag{2.2.5}$$

We put $\sigma^{(0)}=\emptyset$.

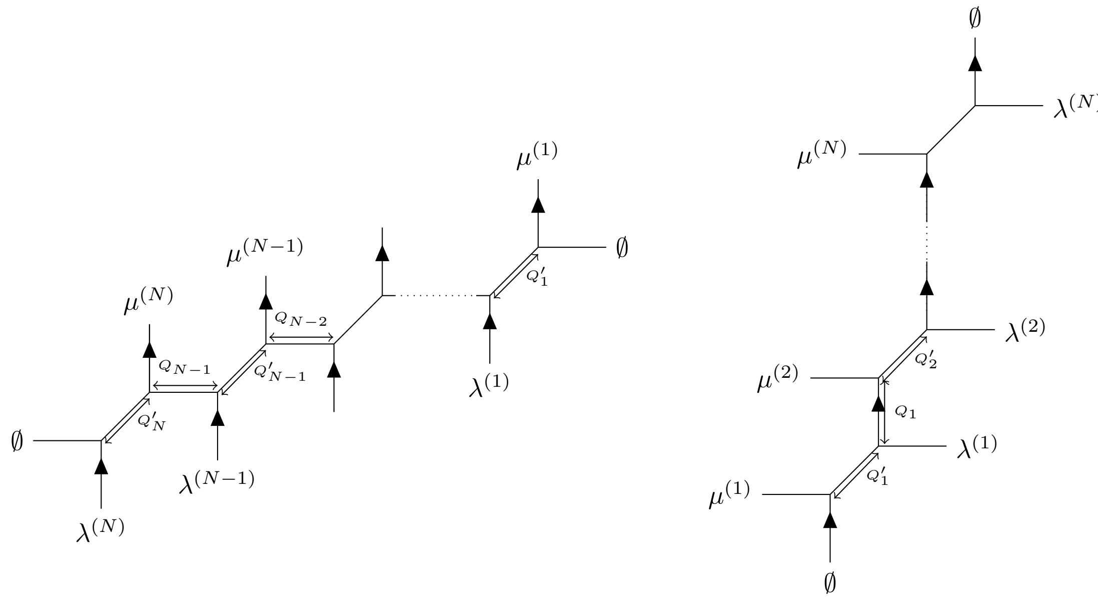


Figure 2.3: Left:$\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}((Q_i),(Q'_i))$ and Right:$\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}((Q_i),(Q'_i))$

Then, Conjecture 2.2.1 is translated into the duality formula under changing the preferred directions of the refined topological vertex.

Claim 2.2.3. *Under changing the preferred directions, $\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and $\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ are related (schematically) with each other by*

$$\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}((Q_i),(Q'_i)) \sim \sum_{\substack{\boldsymbol{\sigma},\boldsymbol{\nu}\in \mathrm{P}^N \\ |\boldsymbol{\sigma}|=|\boldsymbol{\lambda}|,|\boldsymbol{\nu}|=|\boldsymbol{\mu}|}} T^*_{\boldsymbol{\lambda}',\boldsymbol{\sigma}} T_{\boldsymbol{\mu}',\boldsymbol{\nu}} \mathcal{C}^V_{\boldsymbol{\sigma},\boldsymbol{\nu}}((Q_i),(Q'_i))\,, \tag{2.2.6}$$

where $T^*_{\boldsymbol{\lambda}',\boldsymbol{\sigma}}$ and $T_{\boldsymbol{\mu}',\boldsymbol{\nu}}$ are some matrices which satisfy

$$\sum_{\boldsymbol{\lambda}} T_{\boldsymbol{\lambda},\boldsymbol{\mu}} T^*_{\boldsymbol{\lambda},\boldsymbol{\nu}} = \delta_{\boldsymbol{\mu},\boldsymbol{\nu}}\,. \tag{2.2.7}$$

The algorithm to compute the exact proportional coefficients and the proof are presented in Corollary 5.2.13.

Chapter 3

Macdonald Symmetric Functions and Ding-Iohara-Miki Algebra

The Macdonald symmetric polynomials were introduced in the 2nd edition of the textbook "Symmetric Functions and Hall Polynomials" [69] by Ian Macdonald. They give the two-parameter generalization of the Schur polynomials and are defined once the root systems of Lie algebra are specified. In this thesis, we concentrate on the Macdonald polynomials associated with the root systems of A-type. The basic definition and important properties are reviewed in Section 3.1.

Next, in Section 3.2, we review the realization of the Macdonald polynomials on the Fock space of some Heisenberg algebra. We refer to such states on the Fock space as *the Macdonald functions*. They can be regarded as the limit of Macdonald polynomials where the number of variables goes to infinity because under the appropriate projection (Definition 3.2.3), they reduce to the Macdonald polynomials. Then, Theorem 3.2.6 gives the explicit algorithm to construct the Macdonald functions. Through the construction, we get the important lesson that in order to construct the Macdonald functions labelled by the partition λ, we first need to prepare the Macdonald polynomials labelled by λ. This concept we learned here gives us the strong guiding principle to construct the generalized Macdonald functions on the Fock tensor spaces, which is the main subject in Chapter 4.

Section 3.3 and 3.4 are devoted to review the algebra $\mathcal{U}_{q,t}$ called *the Ding-Iohara-Miki algebra* (DIM for short). We show the DIM algebra naturally emerges from the consideration of the Macdonald functions discussed in the previous section. After introducing $\mathcal{U}_{q,t}$, we summarize its representation theory. In a word, the $\mathcal{U}_{q,t}$-modules are labelled by two integers (n, m), corresponding to the images of two centers in $\mathcal{U}_{q,t}$. Then, the key fact is there exist two types of intertwiners that intertwine three such $\mathcal{U}_{q,t}$-modules (Fact 3.4.5). The salient property of these intertwiners is the matrix elements of them become the refined topological vertex in Definition 2.1.9. In the proof of the main claim, we make use of these intertwiners instead of the topological vertex itself.

Then because in the topological vertex side we have the S-transformation (which corresponds to the change of preferred directions), we have a similar action on the $\mathcal{U}_{q,t}$-modules which appear in the definition of the intertwiners. In the end, we will see the S action actually exists as the automorphism of $\mathcal{U}_{q,t}$, and the labels (n, m) of $\mathcal{U}_{q,t}$-modules are acted by the S-transform matrix (2.2.2). This automorphism is called *the Miki automorphism* [72], and reviewed in Section 3.5.

Section 3.6 is the complement of Section 3.2. As we see in Section 3.1, the Macdonald functions are the joint eigenstates of commuting Hamiltonians. On the Fock space, we have to show this property holds, that is, we have to construct the infinitely many commuting operators. This problem is miraculously solved by introducing *the Feigin-Odesskii algebra*.

3.1 Macdonald Symmetric Polynomials

We briefly review the basic facts about the Macdonald polynomials, following [69, Chapter 6], but with infinite variables. Though we deal with q and t as complex parameters and thus the base field $\mathbb{F} = \mathbb{C}$, we sometimes use the notation $\mathbb{Q}(q,t)$ to indicate the dependence on q,t.

Before that, we introduce some notations, which will be used through the following chapters. Let Λ_n be the space of symmetric polynomials with coefficients in $\mathbb{Q}(q,t)$, that is,

$$\Lambda_n := \Lambda_n(q,t) = \mathbb{Q}(q,t)[x_1,\ldots,x_n]^{\mathfrak{S}_n} .$$

We have a natural surjective homomorphism,

$$\Lambda_{n+1} \to \Lambda_n ,$$

defined by setting $x_{n+1} = 0$. This map allows us to take the inverse limit, and define

$$\Lambda := \varprojlim_n \Lambda_n .$$

The Macdonald polynomials are labelled by the partition λ, a sequence of integers,

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) , \quad \text{with} \quad \lambda_i \in \mathbb{Z}_{\geq 0} , \quad \lambda_i \geq \lambda_{i+1} \quad (i = 1,2,3,\ldots) .$$

Partitions which only differ by the sequence of zeros are identified. The partition in which all elements are 0 is denoted by $\emptyset$. The transpose of λ is denoted by λ'. We denote the set of all partitions as P. We also use some notations listed in the top of this thesis.

We also introduce the multi-index notation. For $\boldsymbol{x} = (x_1, x_2, \ldots)$,

$$\boldsymbol{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots .$$

3.1.1 Definition for A-type

We concentrate on the Macdonald polynomials with the A-type root systems. (For the Macdonald polynomials associated with BC-type root systems, see [68].)

Definition 3.1.1 (Dominance Ordering)**.** *Define the partial ordering in* P *by*

$$\lambda \geq \mu \quad \textit{if and only if} \quad |\lambda| = |\mu|, \quad \textit{and} \quad \sum_{i=1}^r \lambda_i \geq \sum_{i=1}^r \mu_i, \quad (\forall r \geq 1) . \tag{3.1.1}$$

This is compatible with the usual ordering of the root systems.

We introduce some basic symmetric polynomials which will be used repeatedly throughout this thesis.

Definition 3.1.2. *For* $\lambda \in \mathsf{P}$*, Define*

1. *monomial symmetric functions :* $m_\lambda(\boldsymbol{x})$ *defined by*

$$m_\lambda(\boldsymbol{x}) = \sum_{(i_1,\ldots,i_{\ell(\lambda)}) \in I_\lambda} x_{i_1}^{\lambda_1} \cdots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}} , \quad \textit{with} \quad I_\lambda = \left\{ (i_1,\ldots,i_{\ell(\lambda)}) \in \mathbb{Z}_{>0}^{\ell(\lambda)} \,\middle|\, i_j \neq i_k, \ \textit{if} \ j \neq k \right\} . \tag{3.1.2}$$

2. *power sum :* $\mathsf{p}_\lambda(\boldsymbol{x}) = \mathsf{p}_{\lambda_1} \mathsf{p}_{\lambda_2} \cdots$ *with*

$$\mathsf{p}_n(\boldsymbol{x}) = m_{(n)}(\boldsymbol{x}) = \sum_i x_i^n . \tag{3.1.3}$$

3. *elementary symmetric functions :* $e_\lambda(\boldsymbol{x}) = e_{\lambda_1} e_{\lambda_2} \cdots$*, where* e_n *is defined by*

$$\sum_n e_n(\boldsymbol{x}) y^n := \exp\left(- \sum_{n>0} \frac{1}{n} \mathsf{p}_n(\boldsymbol{x}) (-y)^n \right) . \tag{3.1.4}$$

4. *complete symmetric functions :* $h_\lambda(\boldsymbol{x}) = h_{\lambda_1} h_{\lambda_2}\cdots$, *where* h_n *is defined by*

$$\sum_n h_n(\boldsymbol{x}) y^n := \exp\left(\sum_{n>0} \frac{1}{n}\mathsf{p}_n(\boldsymbol{x}) y^n\right). \tag{3.1.5}$$

5. $g_\lambda(\boldsymbol{x}) = g_{\lambda_1} g_{\lambda_2}\cdots$, *where* g_n *is defined by*

$$\sum_n g_n(\boldsymbol{x})\, y^n = \exp\left(\sum_{n>0} \frac{1}{n}\frac{1-t^n}{1-q^n}\mathsf{p}_n(\boldsymbol{x}) y^n\right). \tag{3.1.6}$$

Note that when the number of variables is finite (say n), the more appropriate definition of the monomial symmetric function is

$$m_\lambda(\boldsymbol{x}) = \frac{1}{|\mathrm{Stab}(\lambda)|} \sum_{\mu \in W_{A_{n-1}}\cdot\lambda} \prod_i x_i^{\mu_i}. \tag{3.1.7}$$

$W_{A_{n-1}}$ is the Weyl group of $A_{n-1} = SU(n)$ and $\mathrm{Stab}(\lambda)$ is the stabilizer of λ in $W_{A_{n-1}}$. As is obvious from its form, this definition can be generalized to all types of root systems.

Definition 3.1.3. *Define the bilinear form* $\langle -,- \rangle_{q,t} : \Lambda \otimes \Lambda \to \mathbb{C}$ *by*

$$\langle \mathsf{p}_\lambda, \mathsf{p}_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \quad z_\lambda := \prod_{i\geq 1} i^{m_i}\cdot m_i!, \tag{3.1.8}$$

where m_i *is the number of entries in* λ *equal to* i.

Now we introduce a set of difference operators, which determines the Macdonald polynomials.

Definition 3.1.4. *Let* n *be a positive integer (the number of variables). Define the set of difference operators* $D_n^{(k)}(\boldsymbol{x})(k = 0, 1, \dots, n)$ *by*

$$D_n^{(k)}(\boldsymbol{x}|q,t) = t^{k(k-1)/2} \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=k}} \prod_{i\in I; j\in I^c} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}, \tag{3.1.9}$$

where T_{q,x_i} *is the* q*-shift operator with respect to* x_i*. We refer to these difference operators as the Macdonald difference operators.* n *and* $\boldsymbol{x}$ *will be omitted when it is obvious.*

Fact 3.1.5. *The Macdonald operators are self-adjoint with respect to the scalar products in Definition 3.1.3. That is, for any* $f, g \in \Lambda$*, we have*

$$\left\langle g, D^{(k)} f\right\rangle_{q,t} = \left\langle D^{(k)} g, f\right\rangle_{q,t} \quad (k = 0, 1, 2, \dots). \tag{3.1.10}$$

Furthermore, by direct computation, we can prove the following fact, which states the Macdonald difference operators commute with each other.

Fact 3.1.6. *Let* N *be the number of variables. For arbitrary* k *and* $l \in \{1, 2, \dots, N\}$*, we have*

$$\left[D_n^{(k)}(\boldsymbol{x}|q,t), D_n^{(l)}(\boldsymbol{x}|q,t)\right] = 0. \tag{3.1.11}$$

This means it is meaningful to consider the joint eigenfunctions of these Macdonald operators. The next fact shows that those eigenfunctions actually exist uniquely, and this is the very fundamental theorem in the theory of Macdonald polynomials.

Fact 3.1.7. *There exists the unique polynomial* $P_\lambda(\boldsymbol{x}|q,t) \in \Lambda_n$ *such that*

$$P_\lambda(\boldsymbol{x}|q,t) = m_\lambda(\boldsymbol{x}) + \sum_{\mu<\lambda} \alpha_{\lambda,\mu} m_\mu(\boldsymbol{x}) \quad (\alpha_{\lambda,\mu} \in \mathbb{Q}(q,t)) , \tag{3.1.12}$$

$$D_n^{(k)}(\boldsymbol{x}|q,t) P_\lambda(\boldsymbol{x}|q,t) = \epsilon_\lambda^{(k)}(q,t) P_\lambda(\boldsymbol{x}|q,t) \quad (k = 0, 1, \ldots, n) , \tag{3.1.13}$$

with the eigenvalue

$$\epsilon_\lambda^{(k)}(q,t) = e_k(t^{n+\delta} q^\lambda) . \tag{3.1.14}$$

e_1 *is the first elementary symmetric polynomial defined in Definition 3.1.2,* $\delta = (-1, -2, -3, \ldots)$, *and* $t^{n+\delta}q^\lambda$ *stands for* $(t^{n-1}q^{\lambda_1}, t^{n-2}q^{\lambda_2}, \ldots)$.

This definition says $P_\lambda(\boldsymbol{x}|q,t)$ is the joint eigenstate of $D_n^{(k)}(\boldsymbol{x})$. The equation (3.1.12) fixes the normalization. With this fact, we refer to the polynomial $P_\lambda(\boldsymbol{x}|q,t)$ as the Macdonald polynomial.

Remark 3.1.8. *If we follow the original discussion by Macdonald, we should replace the condition (3.1.13) with the following:*

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad (\lambda \neq \mu) . \tag{3.1.15}$$

However, we adopt the definition above for convenience. Actually, this follows from (3.1.13) and Fact 3.1.5.

Remark 3.1.9. *Note that*

$$D_n^{(k)}(\boldsymbol{x}|q,t) = (const.) \times D_n^{(k)}(\boldsymbol{x}^{-1}|1/q, 1/t) , \tag{3.1.16}$$

with the constant $t^{-k(n-1)}$. *This means that when the variables are inverted, they are still the Macdonald polynomials with* q *and* t *inverted. In other words, the inversion of* q *and* t *means to convert the polynomials which are defined on the negative root lattice to those on the positive root lattice. Moreover, we can check*

$$\left[D_n^{(k)}(\boldsymbol{x}|q,t), D_n^{(l)}(\boldsymbol{x}|1/q, 1/t) \right] = 0 , \quad \textit{for} \quad \forall\, k, l \in \mathbb{Z}_{>0} . \tag{3.1.17}$$

This property becomes important later in Section 3.3.1.

Remark 3.1.10. *When* λ *is one-column or one-row, the corresponding Macdonald polynomials are*

$$P_{(1^r)}(\boldsymbol{x}|q,t) = e_r(\boldsymbol{x}) , \tag{3.1.18}$$

$$P_{(r)}(\boldsymbol{x}|q,t) = \frac{(q;q)_r}{(t;q)_r} g_r(\boldsymbol{x}) . \tag{3.1.19}$$

We introduce the dual Macdonald polynomials, which has the different normalization.

Fact 3.1.11. *Define the dual Macdonald polynomial* $Q_\lambda(\boldsymbol{x}|q,t)$ *by*

$$Q_\lambda(\boldsymbol{x}|q,t) = \frac{c_\lambda(q,t)}{c'_\lambda(q,t)} P_\lambda(\boldsymbol{x}|q,t) , \tag{3.1.20}$$

with

$$\begin{aligned} c_\lambda(q,t) &= \prod_{(i,j)\in\lambda} \left(1 - q^{a_\lambda(i,j)} t^{\ell_\lambda(i,j)+1}\right) = \prod_{1\le i\le j} (q^{\lambda_i-\lambda_j} t^{j-i+1}; q)_{\lambda_j-\lambda_{j+1}} , \\ c'_\lambda(q,t) &= \prod_{(i,j)\in\lambda} \left(1 - q^{a_\lambda(i,j)+1} t^{\ell_\lambda(i,j)}\right) = \prod_{1\le i\le j} (q^{\lambda_i-\lambda_j+1} t^{j-i}; q)_{\lambda_j-\lambda_{j+1}} . \end{aligned} \tag{3.1.21}$$

Then, we have

$$\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda,\mu} . \tag{3.1.22}$$

3.1.2 Kernel Function

Now, we introduce the kernel function, which plays the central role when we move on to the Fock representation of the Macdonald polynomials. As it is obvious from its name, it actually behaves as the reproducing kernel with the scalar products. Let us check this below.

Definition 3.1.12. *For two sets of variables $\boldsymbol{x}, \boldsymbol{y}$, define the kernel function $\Pi(\boldsymbol{x}, \boldsymbol{y}|q,t)$ by*

$$\Pi(\boldsymbol{x}, \boldsymbol{y}|q,t) = \exp\left(\sum_{n>0} \frac{1}{n}\frac{1-t^n}{1-q^n} \mathsf{p}_n(\boldsymbol{x})\mathsf{p}_n(\boldsymbol{y})\right) = \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}\,. \tag{3.1.23}$$

Then the crucial properties are stated in the following fact.

Fact 3.1.13. *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$. The kernel function satisfies*

$$D_n^{(k)}(\boldsymbol{x}|q,t)\,\Pi(\boldsymbol{x}, \boldsymbol{y}|q,t) = D_m^{(k)}(\boldsymbol{y}|q,t)\,\Pi(\boldsymbol{x}, \boldsymbol{y}|q,t) \quad (k = 0, 1, \ldots, \min(n,m))\,. \tag{3.1.24}$$

From this, we can expand

$$\Pi(\boldsymbol{x}, \boldsymbol{y}|q,t) = \sum_\lambda P_\lambda(\boldsymbol{x}|q,t)Q_\lambda(\boldsymbol{y}|q,t) = \sum_\lambda Q_\lambda(\boldsymbol{x}|q,t)P_\lambda(\boldsymbol{y}|q,t)\,. \tag{3.1.25}$$

Using this fact and the orthogonality (3.1.15), we can prove

$$P_\lambda(\boldsymbol{x}|q,t) = \big\langle P_\lambda(-|q,t), \Pi(\boldsymbol{x}, -|q,t)\big\rangle_{q,t}\,. \tag{3.1.26}$$

This is why we call $\Pi(\boldsymbol{x}, \boldsymbol{y}|q,t)$ the kernel function.

For later use, we also show the different expansion of the Cauchy kernel.

Fact 3.1.14. *We have*

$$\Pi(\boldsymbol{x}, \boldsymbol{y}|q,t) = \sum_\lambda g_\lambda(\boldsymbol{x})m_\lambda(\boldsymbol{y})\,. \tag{3.1.27}$$

The proof is easy once we note the

$$\Pi(\boldsymbol{x}, \boldsymbol{y}|q,t) = \prod_i \left(\sum_{n\in\mathbb{Z}_{\geq 0}} g_n(\boldsymbol{x})y_i\right)\,. \tag{3.1.28}$$

3.1.3 Pieri Rules and Skew Macdonald Polynomials

It is no exaggeration to say that the outstanding feature which makes the Macdonald symmetric polynomials interesting, is that they are closed under multiplications and divisions. The former corresponds to the Pieri rules, and the latter to the skew Macdonald polynomials.

Pieri Rule

For a partition λ and a coordinate $s = (i,j)$, we write

$$b_\lambda(s) = \begin{cases} \frac{c_\lambda(q,t)}{c'_\lambda(q,t)} \dfrac{1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}}{1-q^{a_\lambda(s)+1}t^{l_\lambda(s)}}, & s \in \lambda; \\ 1, & \text{otherwise.} \end{cases} \tag{3.1.29}$$

Note that $\prod_{s\in\lambda} b_\lambda(s) = c_\lambda(q,t)/c'_\lambda(q,t)$. We introduce what we call the Pieri coefficients as follows.

Definition 3.1.15 (Pieri Coefficient). *For $\mu \subset \lambda \in \mathsf{P}$, define*

$$\varphi_{\lambda/\mu} = \prod_{s \in C_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)}\,, \tag{3.1.30}$$

$$\psi_{\lambda/\mu} = \prod_{s \in R_{\lambda/\mu} - C_{\lambda/\mu}} \frac{b_\mu(s)}{b_\lambda(s)}\,, \tag{3.1.31}$$

$$\varphi'_{\lambda/\mu} = \prod_{s \in R_{\lambda/\mu}} \frac{b_\mu(s)}{b_\lambda(s)}\,, \tag{3.1.32}$$

$$\psi'_{\lambda/\mu} = \prod_{s \in C_{\lambda/\mu} - R_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)}\,. \tag{3.1.33}$$

Here $R_{\lambda/\mu}$ (resp. $C_{\lambda/\mu}$) is the union of the rows (resp. columns) that intersect $\lambda - \mu$.

Fact 3.1.16 ([69]). *We have the Pieri rules:*

$$g_r P_\mu = \sum_\lambda \varphi_{\lambda/\mu} P_\lambda\,, \tag{3.1.34}$$

$$g_r Q_\mu = \sum_\lambda \psi_{\lambda/\mu} Q_\lambda\,, \tag{3.1.35}$$

$$e_r P_\mu = \sum_\lambda \varphi'_{\lambda/\mu} P_\lambda\,, \tag{3.1.36}$$

$$e_r Q_\mu = \sum_\lambda \psi'_{\lambda/\mu} Q_\lambda\,. \tag{3.1.37}$$

Here, the summations in (3.1.34) and (3.1.35) are over the partitions λ such that λ/μ is a horizontal r-strip, i.e., λ/μ has at most one box in each column. Those in (3.1.36) and (3.1.37) are over the partition such that λ/μ is a vertical r-strip.

Note that by this formula, we can see that Q_μ ($\mu \not\leq \lambda$) does not appear in the expansion of the product $\prod_{i\geq 1} g_{\lambda_i}$ in the basis of Macdonald polynomials.

One interesting remark is that the Pieri rules are invertible. Once we rewrite the Pieri rules in the form of matrices, these matrices are of the Bressoud's matrix, and thus are the matrices with each element are factorized. That is, we can expand the Macdonald polynomials with respect to the products like $g_n P_\mu$ with the factorized coefficients. This technique is called Krattenthaler's matrix inversion. For the details, see [67].

Skew Macdonald Polynomials

Next, we see the "division" of the polynomials.

Definition 3.1.17. *Let $\lambda, \mu \in \mathsf{P}$ be the partitions such that $\mu \subset \lambda$. Define $P_{\lambda/\mu} \in \Lambda$ by*

$$P_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} P_\nu\,, \tag{3.1.38}$$

where the coefficient $f^\lambda_{\mu\nu} \in \mathbb{C}(q,t)$ is given by [1]

$$f^\lambda_{\mu\nu} = \langle P_\lambda\,,\, Q_\mu Q_\nu \rangle\,. \tag{3.1.39}$$

We refer to this as the skew Macdonald polynomials.

[1] The notation here is different from that in [69]. In order to make them agree, we need to take the transposition of λ, μ and ν, or multiply $b_\lambda / b_\mu b_\nu$ to $f^\lambda_{\mu\nu}$ here.

As it is clear from the form of $f^{\lambda}_{\mu\nu}$, ν in (3.1.38) runs all partitions such that $\nu \subset \lambda$. Also note that, from (3.1.39), we have

$$Q_\mu Q_\nu = \sum_{\lambda} f^{\lambda}_{\mu\nu} Q_\lambda \,. \tag{3.1.40}$$

One of the most important formula for the skew Macdonald polynomials is following.

Fact 3.1.18. *We have*

$$P_\lambda(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\mu} P_{\lambda/\mu}(\boldsymbol{x}) P_\mu(\boldsymbol{y}) \,. \tag{3.1.41}$$

Proof. We have

$$\begin{aligned} \sum_{\lambda,\mu} P_{\lambda/\mu}(\boldsymbol{x}) Q_\lambda(\boldsymbol{z}) P_\mu(\boldsymbol{y}) &= \sum_{\lambda,\mu,\nu} f^{\lambda}_{\mu\nu} Q_\lambda(\boldsymbol{z}) P_\nu(\boldsymbol{x}) P_\mu(\boldsymbol{y}) \\ &= \sum_{\mu,\nu} Q_\mu(\boldsymbol{z}) Q_\nu(\boldsymbol{z}) P_\nu(\boldsymbol{x}) P_\mu(\boldsymbol{y}) \\ &= \Pi(\boldsymbol{x}, \boldsymbol{z}) \Pi(\boldsymbol{y}, \boldsymbol{z}) = \sum_{\lambda} P_\lambda(\boldsymbol{x}, \boldsymbol{y}) Q_\lambda(\boldsymbol{z}) \,, \end{aligned} \tag{3.1.42}$$

and comparing the both sides proves the claim. □

3.1.4 Tableaux Sum Formula

By virtue of the Pieri rules, introduced in 3.1.3, we can derive the tableaux sum formula for the A-type Macdonald polynomials. To write down the concrete formula, we concentrate on the case of finite number variables.

We first define the tableaux, a refinement of the Young diagram.

Definition 3.1.19. *For the Young diagram $\lambda \in \mathsf{P}$ of length $n \in \mathbb{Z}_{>0}$, a tableaux of shape λ is a set of integers $\theta = \{\theta_{i,j} \in \mathbb{Z}_{\geq 0} | 1 \leq i < j \leq n\}$ which is defined as follows. First we decompose the partition λ to a set of integers $\{\lambda_{i,j} | 1 \leq i \leq j \leq n\}$ such that*

$$\sum_{j=i}^{n} \lambda_{i,j} = \lambda_i \,, \tag{3.1.43}$$

$$\textit{and} \qquad \sum_{j=i}^{i+k} \lambda_{i,j} \geq \sum_{j=i+1}^{i+k+1} \lambda_{i+1,j} \quad (k \in \mathbb{Z}_{\geq 0}) \,. \tag{3.1.44}$$

From this set, we define the $\{\theta_{i,j}\}$ by

$$\theta_{n-i-k,n-k} := \sum_{j=i}^{i+k} \lambda_{i,j} - \sum_{j=i+1}^{i+k+1} \lambda_{i+1,j} \quad (k \in \mathbb{Z}_{\geq 0}) \,. \tag{3.1.45}$$

Note that by definition, $\theta_{i,j} \geq 0$.

We denote the all set of tableaux of shape λ by $\mathrm{T}(\lambda)$.

Then, we have the following result.

Fact 3.1.20 (Tableaux Sum Formula)**.** *For $n \in \mathbb{Z}_{>0}$ and $\lambda \in \mathsf{P}$ with $\ell(\lambda) \leq n$, the n-variable Macdonald polynomial $P_\lambda(x_1, \ldots, x_n | q, t)$ is of the form,*

$$P_\lambda(x_1, \ldots, x_n | q, t) = \boldsymbol{x}^\lambda \sum_{\{\theta_{i,j} | 1 \leq i < j \leq n\} \in \mathrm{T}(\lambda)} \tilde{c}(\theta; \lambda | q, t) \prod_{1 \leq i < j \leq n} (x_j / x_i)^{\theta_{i,j}} \,, \tag{3.1.46}$$

where the coefficient $\tilde{c}(\theta;\lambda|q,t)$ is given by some products of the Pieri coefficients (Definition 3.1.15), as

$$\begin{aligned}\tilde{c}(\theta;\lambda|q,t) &:= \prod_{i=1}^{n} \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t) \\ &= \prod_{k=2}^{n} \left(\prod_{1\le i<j\le k} \frac{(t^{1+i-j} q^{\lambda_j-\lambda_i+\sum_{a>k}(\theta_{i,a}-\theta_{j,a})};q)_{\theta_{i,k}}}{(t^{i-j} q^{\lambda_j-\lambda_i+\sum_{a>k}(\theta_{i,a}-\theta_{j,a})+1};q)_{\theta_{i,k}}} \prod_{1\le i\le j<k} \frac{(t^{-1+i-j} q^{\lambda_j-\lambda_i-\theta_{j,k}+\sum_{a>k}(\theta_{i,a}-\theta_{j,a})+1};q)_{\theta_{i,k}}}{(t^{i-j} q^{\lambda_j-\lambda_i-\theta_{j,k}+\sum_{a>k}(\theta_{i,a}-\theta_{j,a})};q)_{\theta_{i,k}}} \right),\end{aligned} \tag{3.1.47}$$

with $\sum_{a>k} = \sum_{a=k+1}^{n}$. Here $\{\lambda^{(0)},\dots,\lambda^{(n)}\}$ is a sequence of Young diagrams which is in one-to-one correspondence to the set of θ as follows, with the conditions $\lambda^{(0)}=\emptyset$ and $\lambda^{(n)}=\lambda$. As in Definition 3.1.19, for θ, we can fix the decomposition of λ as $\{\lambda_{i,j}|1\le i\le j\le n\}$. $\lambda^{(i)}$ is obtained by removing $\{\lambda_{j,i+1}|1\le j\le i+1\}$ from $\lambda^{(i+1)}$ (that is, removing the horizontal strip of length $\lambda_{n-i}-\sum_{k=i}^{n}\theta_{n-i,k}$).

We will see an example later in this subsection.

Derivation of Fact 3.1.20

Using Fact 3.1.18, we can decompose the Macdonald polynomial with n variables into those with $n-1$ variables and 1 variable, as

$$P_\lambda(x_1,x_2,\dots,x_n|q,t) = \sum_{\lambda^{(n-1)}} P_{\lambda/\lambda^{(n-1)}}(x_1) P_{\lambda^{(n-1)}}(x_2,\dots,x_n)\,. \tag{3.1.48}$$

Then we compute $P_{\lambda/\lambda^{(n-1)}}(x_1)$. Because g_ν and m_ν are dual with respect to the scalar product (Fact 3.1.14), we expand as

$$\begin{aligned} P_{\lambda^{(i)}/\lambda^{(i-1)}}(x_1|q,t) &= \sum_\nu \left\langle P_{\lambda^{(i)}/\lambda^{(i-1)}}, g_\nu \right\rangle_{q,t} m_\nu(x_1) = \sum_\nu \left\langle P_{\lambda^{(i)}}, g_\nu Q_{\lambda^{(i-1)}} \right\rangle_{q,t} m_\nu(x_1) \\ &= \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t)\; x_i^{|\lambda^{(i)}-\lambda^{(i-1)}|} \\ &= \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t)\; x_i^{\lambda_i-\sum_{j=i+1}^{n}\theta_{i,j}+\sum_{j=1}^{i-1}\theta_{j,i}}\,. \end{aligned} \tag{3.1.49}$$

From the first to second line, we use the fact that $P_{\lambda^{(i)}/\lambda^{(i-1)}}$ with one variable becomes zero unless $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip. This is because $m_\nu(x)=0$ if $\ell(\nu)\ge 2$. From the second to third line, we carefully replace $\lambda^{(i)}$ with $\{\theta_{i,j}\}$, following Definition 3.1.19.

By repeating this procedure until the Young diagram reduces to $\lambda^{(0)}=\emptyset\,(i=1)$, we obtain the final result, Fact 3.1.20.

Example: $n=3$ Case

In order to get some feels, we see the example of A_2 Macdonald polynomials.

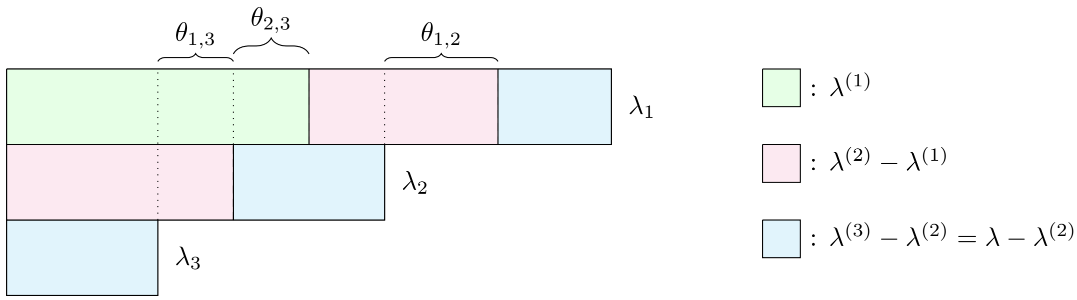


Figure 3.1: Example of Tableaux and $\theta_{i,j}$ for $\lambda=(\lambda_1,\lambda_2,\lambda_3)$

For later convenience, we change the sub-indices of x-variables as $x_i \to x_{n+1-i} (i = 1, \dots, n)$. First, we reduce one variable by going from $\lambda = \lambda^{(3)}$ to $\lambda^{(3)}$ as

$$\begin{aligned}
&\psi_{\lambda^{(3)}/\lambda^{(2)}}(q,t)\ x_1^{|\lambda^{(3)}-\lambda^{(2)}|} \\
&= \frac{\frac{(t;q)_{\theta_{1,2}}}{(q;q)_{\theta_{1,2}}}}{\frac{(tq^{\lambda_1-\lambda_2-\theta_{1,2}-1};q)_{\theta_{1,2}}}{(q^{\lambda_1-\lambda_2-\theta_{1,2}};q)_{\theta_{1,2}}}} \cdot \frac{\frac{(t;q)_{\theta_{1,3}}}{(q;q)_{\theta_{1,3}}}}{\frac{(tq^{\lambda_2-\lambda_3-\theta_{1,3}-1};q)_{\theta_{1,3}}}{(q^{\lambda_2-\lambda_3-\theta_{1,3}-1};q)_{\theta_{1,3}}}} \cdot \frac{\frac{(t^2q^{\lambda_2-\lambda_3+\theta_{1,2}-\theta_{1,3}};q)_{\theta_{1,3}}}{(tq^{\lambda_2-\lambda_3+\theta_{1,2}-\theta_{1,3}+1};q)_{\theta_{1,3}}}}{\frac{(t^2q^{\lambda_1-\lambda_3-\theta_{1,3}};q)_{\theta_{1,3}}}{(tq^{\lambda_1-\lambda_3-\theta_{1,3}+1};q)_{\theta_{1,3}}}} \cdot x_1^{\lambda_1-\theta_{1,2}-\theta_{1,3}} \, .
\end{aligned} \tag{3.1.50}$$

Next, going down to $\lambda^{(1)}$, we have

$$\psi_{\lambda^{(2)}/\lambda^{(1)}}(q,t)\ x_2^{|\lambda^{(2)}-\lambda^{(1)}|} = \frac{\frac{(t;q)_{\theta_{2,3}}}{(q;q)_{\theta_{2,3}}}}{\frac{(tq^{\lambda_2-\lambda_3+\theta_{1,2}-\theta_{1,3}-\theta_{2,3}};q)_{\theta_{2,3}}}{(q^{\lambda_2-\lambda_3+\theta_{1,2}-\theta_{1,3}-\theta_{2,3}+1};q)_{\theta_{2,3}}}} \cdot x_2^{\lambda_2+\theta_{1,2}-\theta_{2,3}} \, . \tag{3.1.51}$$

Finally, because we have

$$m_{\lambda^{(1)}}(x_3) = x_3^{\lambda_3+\theta_{1,3}+\theta_{2,3}} \, , \tag{3.1.52}$$

by combining all above and massaging them, we can confirm the coefficient agrees with (3.1.47).

3.1.5 Some Important Properties

For completeness of the review of Macdonald polynomials, we just name some important properties.

- **Stability**

For any two integers $n > m \geq 1$, the following diagram commutes:

$$\begin{array}{ccc}
\Lambda_n & \xrightarrow{\pi_n \circ \cdots \circ \pi_{m+1}} & \Lambda_m \\
{\scriptstyle D_n^{(k)}} \downarrow & & \downarrow {\scriptstyle D_m^{(k)}} \\
\Lambda_n & \xrightarrow{\pi_n \circ \cdots \circ \pi_{m+1}} & \Lambda_m \, .
\end{array}$$

This property is called *the stability.* This property plays an important role when we construct the infinite variables limit, that is, the Macdonald functions on the Fock space. When the number of variables goes to infinity, the difference operator seems not to be well-defined because we need the infinite sum. Thus, in order to see the constructed operator is the good one, we have to project it to the finite number cases. See Theorem 3.2.4 for more details.

This property only holds for the Macdonald polynomials of type-A. No such property for B, C and D-type is known. The stability-like condition in inverse direction for C and D-type Macdonald polynomials labelled by one-column partitions was studied in [45].

- **Another Scalar Product**

We introduce what is called *the Macdonald's another scalar product.* This is the different bilinear form on $\Lambda_n \times \Lambda_n$ than that defined in Definition 3.1.3.

Definition 3.1.21 (Macdonald's another scalar product)**.** *Define the bilinear form $\langle , \rangle'_{q,t}$ on Λ_N by*

$$\langle f, g \rangle'_{q,t} = \frac{1}{N!} \int_{\mathbb{T}^N} \frac{dx}{2\pi i x} \, \Delta(x) \, f(x^{-1}) g(x) \, , \tag{3.1.53}$$

with

$$\Delta(x) := \prod_{i \neq j} \frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty} \,. \tag{3.1.54}$$

Fact 3.1.22.

$$\langle P_\lambda(\boldsymbol{x}|q,t), P_\mu(\boldsymbol{x}|q,t)\rangle'_{q,t} = n_\lambda \delta_{\lambda,\mu} \,. \tag{3.1.55}$$

Proof. The orthogonality can be proven by the following fact.

Define the first Macdonald operator,

$$D_x^1 = \sum_{i=1}^N \prod_{j \neq i} \frac{1 - tx_i/x_j}{1 - x_i/x_j} \, T_i \,, \tag{3.1.56}$$

where $T_i = T_{q,x_i}$, the shift operator. Also define

$$\widetilde{D}_x^1 = \sum_{i=1}^N \prod_{j \neq i} \frac{1 - tx_i/qx_j}{1 - x_i/qx_j} \, T_i^{-1} \,. \tag{3.1.57}$$

Note that

$$\widetilde{D}_x^1 \, \left(\Delta(x) f(x^{-1})\right) = \Delta(x) \left(D_{x^{-1}}^1 f(x^{-1})\right) \,, \tag{3.1.58}$$

where we used

$$\frac{T_i^{-1}\,\Delta(x)}{\Delta(x)} = \prod_{j \neq i} \frac{1 - tx_j/x_i}{1 - x_j/x_i} \frac{1 - x_i/qx_j}{1 - tx_i/qx_j} \,. \tag{3.1.59}$$

Inserting D_x^1, we have

$$\epsilon_\lambda(q,t) \langle P_\mu(\boldsymbol{x}|q,t), P_\lambda(\boldsymbol{x}|q,t)\rangle'_{q,t} = \int [dx] (D_x^1 P_\lambda(X)) \Delta(x) P_\mu(x^{-1}) \tag{3.1.60}$$

$$= \int [dx] (\widetilde{D}_x^1 \Delta(x) P_\mu(X^{-1})) P_\lambda(x) = \epsilon_\mu(q,t) \langle P_\mu(\boldsymbol{x}|q,t), P_\lambda(\boldsymbol{x}|q,t)\rangle'_{q,t} \,, \tag{3.1.61}$$

where we assumed the boundary term vanishes. The normalization can be determined by the following fact. □

The coefficient n_λ can be deduced from the Pieri rule and the famous Macdonald's constant term conjecture.

Fact 3.1.23 (Macdonald's constant term conjecture)**.**

$$\langle 1, 1\rangle'_{q,t} = \prod_{i=1}^N \frac{(t;q)_\infty (qt^{i-1};q)_\infty}{(q;q)_\infty (t^i;q)_\infty} \tag{3.1.62}$$

This conjecture was proved by several methods. The constant term conjecture is generalized for the Macdonald polynomials associated with the arbitrary (classical) Lie algebra, and its proof can be uniformly achieved by making use of the Cherednik algebra [62, 68].

We have the following result.

Fact 3.1.24.

$$n_\lambda = \prod_{i=1}^N \frac{(t;q)_\infty (qt^{i-1};q)_\infty}{(q;q)_\infty (t^i;q)_\infty} \cdot \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{\ell(s)}}{1 - q^{a(s)} t^{\ell(s)+1}} \frac{1 - q^{a'(s)} t^{N-\ell'(s)}}{1 - q^{a'(s)+1} t^{N-\ell'(s)-1}} \,. \tag{3.1.63}$$

3.1.6 Relation to Other Symmetric Functions

In some appropriate limits of the parameters q, t, the Macdonald polynomials reduce to the various known symmetric polynomials. We name just some of them.

1. $q = t$ limit : Schur polynomial $s_\lambda(\boldsymbol{x})$:

$$s_\lambda(\boldsymbol{x}) = a_{\lambda+\delta}(\boldsymbol{x})/a_\delta(\boldsymbol{x}) , \tag{3.1.64}$$

where $\delta = (N-1, N-2, \cdots)$ and

$$a_\lambda(\boldsymbol{x}) = \det \left(x_i^{\lambda_j} \right)_{1 \le i,j \le N} . \tag{3.1.65}$$

For the boson and fermion realization of the Schur polynomials, see [50].

2. $q \to 0$ limit : Hall-Littlewood polynomial $h_\lambda(\boldsymbol{x}|q)$: The realization by vertex operators was studied in [52].

3. $t \to 0$ limit with shift of variables : This limit is called *the Toda limit*,

$$\lim_{t \to 0} P_\lambda(t^\delta \boldsymbol{x}|q, t) . \tag{3.1.66}$$

Under this limit, the Bamp-Stade formula are known [21], and admit the simpler expressions.

4. $t \to 0$ limit : q-Whittaker polynomial. This limit has been enthusiastically studied in the context of the stochastic process. See [16] for example.

5. $q \to 1, \ t = q^\beta \to 1$ limit (with β fixed) : Jack polynomial $J_\lambda(\boldsymbol{x}|\beta)$. See [107] for example.

6. $q, t \to e^{2\pi\sqrt{-1}/k}$(root of unity) limit : Uglov polynomial. See [115].

7. $q \to 1$ limit : Elementary symmetric polynomial

$$\lim_{q \to 1} P_\lambda(\boldsymbol{x}|q, t) = e_{\lambda'}(\boldsymbol{x}) \tag{3.1.67}$$

8. $t \to 1$ limit : Monomial symmetric polynomial

$$\lim_{t \to 1} P_\lambda(\boldsymbol{x}|q, t) = m_\lambda(\boldsymbol{x}) \tag{3.1.68}$$

3.2 Macdonald polynomials on Fock Space

Now we realize the Macdonald polynomials on the Fock space. This work was first done in [13], and has been extended in successive works. In this section, we quickly summarize those results.

3.2.1 Macdonald Operator and Stability

Definition 3.2.1. *Let $\{a_n | n \in \mathbb{Z}\}$ be the Heisenberg algebra with the relation*

$$[a_m, a_n] = m \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n,0} \, a_0 . \tag{3.2.1}$$

Using this Heisenberg algebra, define the Fock space $\mathcal{F}$ with the vacuum $|0\rangle$, defined by $a_n |0\rangle = 0$ for $n \in \mathbb{Z}_{>0}$. That is, $\mathcal{F} = \mathbb{C}[a_{-1}, a_{-2}, \dots] |0\rangle$ as the vector space. Similarly, define the dual Fock space $\mathcal{F}^*$ with the dual vacuum $\langle 0|$, defined by $\langle 0| a_n = 0$ $(n \in \mathbb{Z}_{<0})$. The basis of $\mathcal{F}$ (resp. $\mathcal{F}^*$) is given by $|a_\lambda\rangle = a_{-\lambda_1} a_{-\lambda_2} \cdots |0\rangle$ (resp. $\langle a_\lambda| = \langle 0| \cdots a_{\lambda_2} a_{\lambda_1}$) with a partition $\lambda = (\lambda_1, \lambda_2, \dots)$. The bilinear form $\mathcal{F}^* \otimes \mathcal{F} \to \mathbb{C}$ is given by $\langle 0|0\rangle = 1$. We also define the normal ordering $: - :$ as usual.

Note that this definition of the Heisenberg algebra is motivated by the inner product of the power sum (Definition 3.1.3),

$$\langle \mathsf{p}_m, \mathsf{p}_n \rangle_{q,t} = m \frac{1-q^m}{1-t^m} \delta_{m,n} \,. \tag{3.2.2}$$

Therefore, the map $\Lambda \to \mathcal{F}$

$$\mathsf{p}_n \mapsto a_{-n} \left|0\right\rangle \,, \tag{3.2.3}$$

gives the isomorphism as the graded vector space.

Now we define the Macdonald operator acting on the Fock space.

Definition 3.2.2. *Define the vertex operator in* $\mathrm{End}(\mathcal{F})$ *by*

$$\eta(z) := \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} a_{-n} z^n \Big) \exp\Big(- \sum_{n=1}^{\infty} \frac{1-t^{n}}{n} a_{n} z^{-n} \Big) \,. \tag{3.2.4}$$

We denote its Fourier components as $\eta(z) = \sum_{n\in\mathbb{Z}} \eta_n z^{-n}$. *Especially, we denote its zero mode by* η_0.

As we will see in below, this plays the role of the Macdonald operator on $\mathcal{F}$.

Next, we define the projector from the Fock space $\mathcal{F}$ to Λ_N. The definition seems quite natural because the bosons must be projected to the power sum.

Definition 3.2.3. *For* $|u\rangle \in \mathcal{F}$, *we define the projector* $\pi_N : \mathcal{F} \to \mathbb{C}[[x_1, \ldots, x_N]]$ *by*

$$\pi_N(|u\rangle) := \langle 0| \, \phi(x_1) \cdots \phi(x_N) \, |u\rangle \,, \tag{3.2.5}$$

where

$$\phi(x) := \exp\left(\sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} a_n x^n \right) \,. \tag{3.2.6}$$

The following theorem states that η_0 is actually intertwined to the Macdonald operator under the projection.

Theorem 3.2.4 ([13])**.** *For* $|u\rangle \in \mathcal{F}$, *the following equality holds:*

$$\pi_N(\eta_0 \, |u\rangle) = (D_N^{(1)})' \pi_N(|u\rangle) \,, \tag{3.2.7}$$

with

$$(D_N^{(1)})' = (t-1) t^{-N} D_N^{(1)} + t^{-N} \,. \tag{3.2.8}$$

Proof. Let C_0 be the circle around 0, excluding $1/x_i$ $(i = 1, \ldots, N)$ and C_∞ be the circle which includes all poles except for the infinity. The following computation gives the proof:

$$\begin{aligned} \oint_{C_0} \frac{dz}{z} \langle 0| \, \phi(x_1) \cdots \phi(x_N) \eta(z) \, |u\rangle &= \oint_{C_1} \frac{dz}{z} \prod_{i=1}^{N} \frac{1 - z x_i/t}{1 - z x_i} \langle 0| : \eta(z) \phi(x_1) \cdots \phi(x_N) : |u\rangle \\ = (1 - t^{-1}) \sum_{i=1}^{N} \prod_{j \neq i} \frac{1 - x_j/t x_i}{1 - x_j/x_i} T_{q,x_i} \pi_N(|u\rangle) &+ \oint_{C_\infty} \frac{dz}{z} \prod_{i=1}^{N} \frac{1 - z x_i/t}{1 - z x_i} \langle 0| : \eta(z) \phi(x_1) \cdots \phi(x_N) : |u\rangle \,. \end{aligned} \tag{3.2.9}$$

The last term becomes just constant t^{-N}. Note that we use the fact,

$$\langle 0| : \eta(1/x) \phi(x) : = \langle 0| \, \phi(qx) \,. \tag{3.2.10}$$

□

In other words, Theorem 3.2.4 states that once we construct the states $|P_\lambda\rangle$ in $\mathcal{F}$ such that

$$\eta_0 |P_\lambda\rangle = \left((t-1)t^{-N}\epsilon_\lambda^{(1)}(q,t) + t^{-N}\right)|P_\lambda\rangle \ , \tag{3.2.11}$$

$\pi_N(|P_\lambda\rangle)$ is the Macdonald polynomials with N-variables. In this sense, $|P_\lambda\rangle$ can be regarded as the projective limit of the Macdonald polynomials with finite variables. Then the problem is how to construct such the states $|P_\lambda\rangle$. This is the main subject in the next subsection.

3.2.2 Explicit Construction

Definition 3.2.5. *Define the vertex operator*

$$\varphi(z) = \exp\left(\sum \frac{1}{n}\frac{1-t^n}{1-q^n}a_{-n}z^n\right) \ . \tag{3.2.12}$$

Note that this is the operator analogue of the generating function of g_n, which is introduced in Definition 3.1.2. That is, we have the Fock realization of g_n as

$$|g_n\rangle = \left[z^{-n}\varphi(z)\,|0\rangle\right]_{z,1} \ , \tag{3.2.13}$$

where $[\cdots]_{z,1}$ means the constant term in $\cdots$ with respect to z. Then, piling up g_n's gives the Macdonald function on the Fock space.

The problem left is to determine the coefficients, which is analogous to the inversion of the Pieri coefficients. The next theorem gives the answer to this problem.

Theorem 3.2.6. *Fix $\lambda \in \mathsf{P}$. Let $n \geq \ell(\lambda)(\in \mathbb{Z})$.*

$$|Q_\lambda\rangle = \left[\prod_{1\leq i<j\leq n}(1-x_j/x_i)\cdot P_\lambda(\boldsymbol{x}^{-1}|q,q/t)\ \varphi(x_1)\cdots\varphi(x_n)\,|0\rangle\right]_{\boldsymbol{x},1} \ . \tag{3.2.14}$$

By analogy of the Macdonald polynomials, we define

$$|P_\lambda\rangle = \frac{c'_\lambda(q,t)}{c_\lambda(q,t)}\,|Q_\lambda\rangle \ . \tag{3.2.15}$$

Proof. The proof is same as that of Theorem 4.2.21, which will be given in the next chapter. □

This gives the explicit algorithm to construct the Macdonald functions on the Fock space. In below, we show some examples of this construction.

Examples

We just show some examples with $|\lambda| \leq 3$.

$$
\begin{aligned}
&|P_{\emptyset}\rangle = |0\rangle\ ,\\
&|P_{\square}\rangle = a_{-1}|0\rangle\ ,\\
&|P_{\square\square}\rangle = \frac{1}{2}\left(\frac{(1-q)(1+t)}{1-qt}a_{-2} + \frac{(1-t)(1+q)}{1-qt}a_{-1}^2\right)|0\rangle\ ,\\
&|P_{\substack{\square\\ \square}}\rangle = -\frac{1}{2}\left(a_{-2} - a_{-1}^2\right)|0\rangle\ ,\\
&|P_{\square\square\square}\rangle = \frac{1}{6}\left(2\frac{1-t^3}{1-t}\frac{(1-q)(1-q^2)}{(1-qt)(1-q^2t)}a_{-3} + 3\frac{(1-q^3)(1-t^2)}{(1-qt)(1-q^2t)}a_{-2}a_{-1} + \frac{(1+q)(1-q^3)}{(1-q)}\frac{(1-t)^2}{(1-qt)(1-q^2t)}a_{-1}^3\right)|0\rangle\ ,\\
&|P_{\substack{\square\square\\ \square}}\rangle = -\frac{1}{6}\left(2\frac{(1-q)(1-t^3)}{(1-t)(1-qt^2)}a_{-3} + 3q\frac{(1+t)(1-t/q)}{1-qt^2}a_{-2}a_{-1} - (2+q+t+2qt)\frac{1-t}{1-qt^2}a_{-1}^3\right)|0\rangle\ ,\\
&|P_{\substack{\square\\ \square\\ \square}}\rangle = \frac{1}{6}\left(2a_{-3} - 3a_{-2}a_{-1} + a_{-1}^3\right)|0\rangle\ .
\end{aligned}
\tag{3.2.16}
$$

3.3 Ding-Iohara-Miki Algebra $\mathcal{U}_{q,t}$

The Ding-Iohara-Miki algebra $\mathcal{U}_{q,t}$ was introduced independently in [24] and [72]. In [24], it was introduced as the generalization of the Drinfeld's new realization of the quantum affine algebra to the wider class of algebra. Actually, from this point of view, $\mathcal{U}_{q,t}$ is also called the quantum toroidal algebra of $\widehat{\mathfrak{gl}_1}$. As we will see below, from its construction, $\mathcal{U}_{q,t}$ admits the Drinfeld coproduct, and furthermore, the Hopf algebra structure. On the other hand, in [72], $\mathcal{U}_{q,t}$ was introduced as the one-parameter deformation of q-deformed W-algebra. Because these both algebras agree with each other, the algebra is called Ding-Iohara-Miki algebra. In this section and Section 4.3, we will see both sides of this algebra.

Instead of introducing $\mathcal{U}_{q,t}$ by the top-down approach, we do so by the bottom-up approach. That is, we extract the commutation relations of $\mathcal{U}_{q,t}$ from those among the operators which are introduced to construct the Macdonald polynomials on the Fock space.

3.3.1 From Macdonald to DIM

We can show one of the "derivation" of $\mathcal{U}_{q,t}$ by 4 steps. The final result will be summarized in the next section.

Step1 : $\eta(z)\eta(w)$

We first see the commutation relation between two η's, introduced in (3.2.2). Because $\eta(z)\eta(w) = f^{-1}(w/z) :\eta(z)\eta(w):$, this is given by

$$f(w/z)\eta(z)\eta(w) = f(z/w)\eta(w)\eta(z)\,, \tag{3.3.1}$$

with

$$f(z) := \frac{(1-qz)(1-z/t)}{(1-z)(1-qz/t)}\,. \tag{3.3.2}$$

Step2 : $\xi(z)\xi(w)$

Now, we recall Remark 3.1.9. Then, we need to introduce a new current, whose zero mode is intertwined to $D^{(1)}(\boldsymbol{x}|1/q, 1/t)$ under the projection (Definition 3.2.3). In the end, we have the following answer:

$$\xi(z) = \exp\Big(-\sum_{n=1}^{\infty}\frac{1-t^{-n}}{n}q^{-n/2}t^{n/2}a_{-n}z^n\Big)\exp\Big(\sum_{n=1}^{\infty}\frac{1-t^n}{n}q^{-n/2}t^{n/2}a_nz^{-n}\Big)\,. \tag{3.3.3}$$

Then this current satisfies the following commutation relation:

$$f(tw/qz)\xi(z)\xi(w) = f(tz/qw)\xi(w)\xi(z)\,. \tag{3.3.4}$$

Step3 : $\eta(z)\xi(w)$

Now we compute the commutation relation between these two currents $\eta(z)$ and $\xi(w)$. Put $\gamma = (t/q)^{1/2}$. First, note that

$$\begin{aligned}
\eta(z)\xi(w) &= \frac{(1-q^{1/2}t^{1/2}w/z)(1-q^{-1/2}t^{-1/2}w/z)}{(1-q^{-1/2}t^{1/2}w/z)(1-q^{1/2}t^{-1/2}w/z)} : \eta(z)\xi(w) := f(\gamma w/z) : \eta(z)\xi(w) :\,,\\
\xi(w)\eta(z) &= \frac{(1-q^{1/2}t^{1/2}z/w)(1-q^{-1/2}t^{-1/2}z/w)}{(1-q^{-1/2}t^{1/2}z/w)(1-q^{1/2}t^{-1/2}z/w)} : \xi(w)\eta(z) := f(\gamma z/w) : \xi(w)\eta(z) :\,,
\end{aligned} \tag{3.3.5}$$

and the coefficients are identical as a rational function. (Note $f(z) = f(\gamma^2/z)$ as a rational function.)

Before computing the commutation relation, we introduce some useful tools.

Definition 3.3.1. *Define the delta function by*

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n\,. \tag{3.3.6}$$

We call $\delta(z)$ the delta function, because for arbitrary $f(z) \in \mathbb{C}[[z]]$, we have

$$\delta(w/z)f(z) = f(w)\,. \tag{3.3.7}$$

The following lemma is most fundamental and will be used repeatedly in this thesis.

Lemma 3.3.2.

$$\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1-z/b_j}{1-z/a_i} - \frac{\prod_{i=1}^{N}(-a_i/z)}{\prod_{j=1}^{M}(-b_j/z)}\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1-b_j/z}{1-a_i/z} = \sum_{i=1}^{N}\delta(z/a_i)\prod_{k\neq i}\frac{1-a_i/b_j}{1-a_i/a_k}\,. \tag{3.3.8}$$

Now we can compute the commutation relation,

$$\begin{aligned}
&\eta(z)\xi(w) - \xi(w)\eta(z)\\
&\quad = \frac{(1-q)(1-t^{-1})}{1-q/t}\Big(\delta(\gamma w/z)\varphi^+(\gamma^{1/2}w) - \delta(\gamma^{-1}w/z)\varphi^-(\gamma^{-1/2}w)\Big)\,.
\end{aligned} \tag{3.3.9}$$

Here we introduced two new currents,

$$\begin{aligned}
\varphi^+(z) &=: \eta(\gamma^{1/2}z)\xi(z/\gamma^{1/2}) := \exp\Big(-\sum_{n=1}^{\infty}\frac{1-t^n}{n}(1-t^nq^{-n})q^{n/4}t^{-n/4}a_nz^{-n}\Big),\\
\varphi^-(z) &=: \eta(z/\gamma^{1/2})\xi(\gamma^{1/2}z) := \exp\Big(\sum_{n=1}^{\infty}\frac{1-t^{-n}}{n}(1-t^nq^{-n})q^{n/4}t^{-n/4}a_{-n}z^n\Big).
\end{aligned} \tag{3.3.10}$$

Remark 3.3.3. *From the relation (3.3.9), we can show*

$$[\eta_0, \xi_0] = 0\,, \tag{3.3.11}$$

and thus reproduce the result in Remark 3.1.9.

Step4 : Other Commutation Relations

The remaining task is to compute the commutation relations among $\varphi^\pm(z)$ and $\eta(z), \xi(z)$. This is just computation, and in the next section, we just show the results.

3.3.2 Definition of Algebra

Now it is easy to put these results in order. The algebra $\mathcal{U}_{q,t}$ is defined as follows.

Definition 3.3.4. *Let $\mathcal{U}_{q,t}$ be the unital associative algebra over $\mathbb{C}$ generated by the Drinfeld currents*

$$x^{\pm}(z) = \sum_{n\in\mathbb{Z}} x_n^{\pm} z^{-n}\,, \quad \psi^{\pm}(z) = \sum_{n\in\mathbb{Z}_{\geq 0}} \psi_n^{\pm} z^{\mp n}\,,$$

and the invertible central element $c^{1/2}$, satisfying the following defining relations:

$$\begin{aligned}
&\psi^+(z)x^{\pm}(w) = g(c^{\mp 1/2}w/z)^{\mp 1}x^{\pm}(w)\psi^+(z), \qquad \psi^-(z)x^{\pm}(w) = g(c^{\mp 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^-(z),\\
&\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z), \qquad \psi^+(z)\psi^-(w) = \frac{g(c^{+1}w/z)}{g(c^{-1}w/z)}\psi^-(w)\psi^+(z),\\
&[x^+(z), x^-(w)] = \frac{(1-q)(1-1/t)}{1-q/t}\Big(\delta(c^{-1}z/w)\psi^+(c^{1/2}w) - \delta(cz/w)\psi^-(c^{-1/2}w)\Big),\\
&G^{\mp}(z/w)x^{\pm}(z)x^{\pm}(w) = G^{\pm}(z/w)x^{\pm}(w)x^{\pm}(z)\,,
\end{aligned} \tag{3.3.12}$$

where

$$g(z) = \frac{G^+(z)}{G^-(z)}\,, \qquad G^{\pm}(z) = (1-q^{\pm 1}z)(1-t^{\mp 1}z)(1-q^{\mp 1}t^{\pm 1}z)\,, \qquad \delta(z) = \sum_{n\in\mathbb{Z}} z^n\,. \tag{3.3.13}$$

Remark 3.3.5. *In some papers, the following Serre relations are further imposed in the definition of $\mathcal{U}_{q,t}$:*

$$[x_0^+, [x_1^+, x_{-1}^+]] = 0\,, \quad [x_0^-, [x_1^-, x_{-1}^-]] = 0\,. \tag{3.3.14}$$

We omit these relations, because in the modules that we deal with in this thesis, the Serre relations are automatically satisfied. See also Example 3.6.3.

Note that by multiplying $(1-w/z)(1-qw/tz)(1-tw/qz)$ to the relation (3.3.1), we obtain the last relation in (3.3.12). The other relations can be also recovered from the computations in Section 3.3.1.

Remark 3.3.6. *One non-trivial step from Section 3.3.1 to the definition above, is the choice of the centers, that is, how the central elements $c^{\pm 1/2}$ enter in the relations. It is chosen so that the relations are compatible with the Hopf algebra structure, especially with the coproduct.*

3.3.3 Hopf Algebra Structure

The algebra which is equipped with the coalgebra structure, is called the Hopf algebra. As is shown in [24], $\mathcal{U}_{q,t}$ admits the Hopf algebra structure. Actually, the algebras of the form of "the new realization" always admit the Drinfeld coproduct (and also counit and antipode) [24]. In below, we summarize the Hopf algebra structure of $\mathcal{U}_{q,t}$.

Coproduct

Fact 3.3.7. *The $\mathcal{U}$ admits the (topological) Hopf algebra structure with the Drindeld coprocuct Δ:*

$$\begin{aligned}
\Delta(c^{\pm 1/2}) &= c^{\pm 1/2} \otimes c^{\pm 1/2}\,,\\
\Delta(x^+(z)) &= x^+(z) \otimes 1 + \psi^-(c_{(1)}^{1/2}z) \otimes x^+(c_{(1)}z)\,,\\
\Delta(x^-(z)) &= x^-(c_{(2)}z) \otimes \psi^+(c_{(2)}^{1/2}z) + 1 \otimes x^-(z)\,,\\
\Delta(\psi^{\pm}(z)) &= \psi^{\pm}(c_{(2)}^{\pm 1/2}z) \otimes \psi^{\pm}(c_{(1)}^{\mp 1/2}z)\,,
\end{aligned} \tag{3.3.15}$$

where $c_{(1)}^{\pm 1/2} = c^{\pm 1/2} \otimes 1$ and $c_{(2)}^{\pm 1/2} = 1 \otimes c^{\pm 1/2}$.

This coproduct plays the central role throughout this thesis.

Antipode

The antipode $a : \mathcal{U}_{q,t} \to \mathcal{U}_{q,t}$ is defined by

$$\begin{aligned} &a(c^{\pm 1/2}) = c^{\mp 1/2}\,, \\ &a(x^+(z)) = -\left(\psi^-(z/c^{1/2})\right)^{-1} x^+(z/c)\,, \\ &a(x^-(z)) = -x^-(z/c)\left(\psi^+(z/c^{1/2})\right)^{-1}\,, \\ &a(\psi^\pm(z)) = \left(\psi^\pm(z)\right)^{-1}\,. \end{aligned} \tag{3.3.16}$$

This antipode plays some role when we construct the $\mathcal{U}_{q,t}$-module in Section 3.4.3.

Counit

Although the counit $\epsilon : \mathcal{U}_{q,t} \to \mathbb{C}$ plays no role in this thesis, we present it for completeness.

$$\epsilon(c^{\pm 1/2}) = 1\,, \quad \epsilon(\psi^\pm(z)) = 1\,, \quad \epsilon(x^\pm(z)) = 0\,. \tag{3.3.17}$$

3.4 $\mathcal{U}_{q,t}$-modules and Intertwiners

There are two centers in $\mathcal{U}_{q,t}$, c and $(\psi_0^+/\psi_0^-)^{1/2}$. Thus, we label the $\mathcal{U}_{q,t}$-modules by the image of these centers.

We define the (n,m)-modules as the $\mathcal{U}_{q,t}$-modules where the two centers act as

$$c = \gamma^n\,, \quad (\psi_0^+/\psi_0^-)^{1/2} = \gamma^{-m}\,, \tag{3.4.1}$$

with $\gamma = (t/q)^{1/2}$. In what follows, we deal with four modules, two $(0,0)$, $(0,1)$, and $(1,M)$-modules. ($M \in \mathbb{Z}$.)

3.4.1 Two Types of $(0,0)$-modules

First, we introduce the $(0,0)$-modules. There are two different types of them, and the relation among them will be clear in Section 3.5.

1. $(0,0)$-modules (Level-0 modules)

The first $(0,0)$-module was introduced in [29], and they are tightly related to the Macdonald difference operators. Sometimes, we refer to this module as the level-0 module.

The following fact defines the representation, and ensures the map actually defines the representation of $\mathcal{U}_{q,t}$.

Fact 3.4.1. *The map $\pi_{x,d} : \mathcal{U}_{q,t} \to \mathrm{End}(V_x)$ with $V_x = \mathbb{Q}(q^{1/2}, t^{1/2})[x^{\pm 1}]$, and $d \in \mathbb{Q}(q^{1/2}, t^{1/2})^\times$, defined by*

$$\begin{aligned} &\pi_{x,d}(c^{\pm 1/2}) = 1\,, \\ &\pi_{x,d}(x^\pm(z)) = d^{\pm 1}(1 - t^{\mp 1})\delta(q^{\mp 1/2}x/z)T_{q,x}^{\mp 1}\,, \\ &\pi_{x,d}(\psi^+(z)) = \frac{(1 - q^{1/2}t^{-1}x/z)(1 - q^{-1/2}tx/z)}{(1 - q^{1/2}x/z)(1 - q^{-1/2}x/z)}\,, \\ &\pi_{x,d}(\psi^-(z)) = \frac{(1 - q^{1/2}t^{-1}z/x)(1 - q^{-1/2}tz/x)}{(1 - q^{1/2}z/x)(1 - q^{-1/2}z/x)}\,, \end{aligned} \tag{3.4.2}$$

endows the $\mathcal{U}_{q,t}$-module structure with the space of Laurent polynomials V_x.

Tensor Modules The coproduct of $\mathcal{U}_{q,t}$ helps us to define the tensor modules on $V_{x_1} \otimes \cdots \otimes V_{x_N}$. We use the following notation:

$$\begin{aligned}
&\Delta^{(N)} = (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta)\Delta^{(N-1)} \,, \\
&\Delta^{(2)} = \Delta \,.
\end{aligned} \tag{3.4.3}$$

To enjoy the nice properties of these modules, first we concentrate on $N = 2$ case. Let us consider the operator $\bar{D}$ on $V_{x_1} \otimes V_{x_2}$,

$$\bar{D} = \left[\pi_{x_1,t^{-1}} \otimes \pi_{x_2,1}\left(\Delta(x^-(z))\right)\right]_{z,1} \,. \tag{3.4.4}$$

$[\cdots]_{z,1}$ stands for the constant term of $\cdots$ with respect to z. Then we have

$$\begin{aligned}
\bar{D} &= t(1-t)\frac{(1-t^{-1}x_2/x_1)(1-tx_2/qx_1)}{(1-x_2/x_1)(1-x_2/qx_1)}T_{q,x_1} + (1-t)T_{q,x_2} \\
&= (1-t)\frac{(x_2/x_1;q)_\infty}{(tx_2/x_1;q)_\infty}\left(\frac{tx_1-x_2}{x_1-x_2}T_{q,x_1} + \frac{tx_2-x_1}{x_2-x_1}T_{q,x_2}\right)\frac{(tx_2/x_1;q)_\infty}{(x_2/x_1;q)_\infty} \,,
\end{aligned} \tag{3.4.5}$$

and thus $\bar{D}$ is the gauge transformation of the first Macdonald operator $D_2^{(1)}$, defined in Definition 3.1.4.

This fact can be generalized to the arbitrary A-type Macdonald operators, including the higher operators.

Theorem 3.4.2. *Fix $N \in \{2,3,\ldots\}$, and set $d_i = t^{i-N}$. For $1 \le r \le N$, we have*

$$\frac{t^{r(r-1)/2}}{(1-t)^r r!}\left[\frac{\epsilon_r(z;q)}{\prod_{1\le i<j\le r} w(z_i,z_j)}\pi_{x_1,d_1} \otimes \cdots \otimes \pi_{x_N,d_N}\Delta^{(N)}\left(x^-(z_1)\cdots x^-(z_r)\right)\right]_{z_i,1} = g^{-1}D_N^{(r)}g \,, \tag{3.4.6}$$

with

$$g = \prod_{1\le i<j\le N}\frac{(tx_j/x_i;q)_\infty}{(x_j/x_i;q)_\infty} \,. \tag{3.4.7}$$

2. $(0,0)$**-modules (Vector Representation)**

The other $(0,0)$-module of $\mathcal{U}_{q,t}$ and its tensor representations were introduced in [27]. This module is also called the vector representation.

$$\begin{aligned}
c^{\pm 1/2}[u]_i &= [u]_i \,, \\
x^+(z)[u]_i &= \frac{1}{1-q_1}\delta(q_1^i u/z)[u]_{i+1} \,, \\
x^-(z)[u]_i &= -\frac{1}{1-q_1^{-1}}\delta(q_1^{i-1}u/z)[u]_{i-1} \,, \\
\psi^+(z)[u]_i &= \frac{(1-q_1^i q_3 u/z)(1-q_1^i q_2 u/z)}{(1-q_1^i u/z)(1-q_1^{i-1}u/z)}[u]_i \,, \\
\psi^-(z)[u]_i &= \frac{(1-q_1^{-i}q_3^{-1}z/u)(1-q_1^{-i}q_2^{-1}z/u)}{(1-q_1^{-i}z/u)(1-q_1^{-i+1}z/u)}[u]_i \,.
\end{aligned} \tag{3.4.8}$$

The tensor modules of this module allow us to construct the $(0,1)$-modules. We will see this soon in below.

3.4.2 $(0,1)$-modules & $(1,M)$-modules

Now, we construct the $(0,1)$ and $(1,M)$-modules ($M \in \mathbb{Z}$). Roughly speaking, $(0,1)$-modules are constructed through piling up the vector representations, while $(1,0)$-modules can be regarded as the infinitely many tensor products of the level-0 representations.

Again, as the two $(0,0)$-modules, they are related under some automorphism of $\mathcal{U}_{q,t}$. This will be explained in Section 3.5.

• $(1, M)$**-modules (Horizontal representation)**

The $(1, M)$-modules are almost the same as the bosonic realization of $\mathcal{U}_{q,t}$ which is used to construct the Macdonald operator on the Fock space. Recall that the zero mode of $x^+(z)$ is intertwined to the Macdonald operator, and this is the Fock analogue of Theorem 3.4.2. In this sense, this module can be thought of as the infinitely many tensor products of the level-0 modules.

Fact 3.4.3 ([29])**.** *Let u be a nonzero complex parameter. The following algebra homomorphism $\rho_u^{(1,M)}$: $\mathcal{U}_{q,t} \to \mathrm{End}(\mathcal{F})$ endows the $\mathcal{U}_{q,t}$-module structure on $\mathcal{F}$:*

$$\begin{aligned} &c^{1/2} \mapsto (t/q)^{1/4}, \quad x^+(z) \mapsto u z^{-M} q^{-M/2} t^{M/2} \eta(z), \quad x^-(z) \mapsto u^{-1} z^M q^{M/2} t^{-M/2} \xi(z), \\ &\psi^+(z) \mapsto q^{M/2} t^{-M/2} \varphi^+(z), \quad \psi^-(z) \mapsto q^{-M/2} t^{M/2} \varphi^-(z)\,, \end{aligned} \tag{3.4.9}$$

where

$$\begin{aligned} \eta(z) &= \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} a_{-n} z^n\Big) \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} a_{n} z^{-n}\Big), \\ \xi(z) &= \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} q^{-n/2} t^{n/2} a_{-n} z^n\Big) \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} q^{-n/2} t^{n/2} a_{n} z^{-n}\Big), \\ \varphi^+(z) &= \exp\Big(-\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} (1-t^n q^{-n}) q^{n/4} t^{-n/4} a_{n} z^{-n}\Big), \\ \varphi^-(z) &= \exp\Big(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (1-t^n q^{-n}) q^{n/4} t^{-n/4} a_{-n} z^{n}\Big). \end{aligned} \tag{3.4.10}$$

We call u the spectral parameter, and denote this $\mathcal{U}_{q,t}$-module by $\mathcal{F}_u^{(1,M)}$.

We can also define the dual $\mathcal{U}_{q,t}$-module structure $\mathcal{F}_u^{(1,M)*}$ on $\mathcal{F}^*$ through the same ρ_u by regarding its image as in $\mathrm{End}(\mathcal{F}^*)$.

• $(0, 1)$**-modules (Vertical representation)**

As introduced in [27], by tensoring the vector representations, we obtain the $(0, 1)$-representation. The representation space is again the space of the Macdonald functions. To avoid confusion, we use $|\lambda\rangle$ to indicate the Macdonald function with the partition λ.

Through piling up the vector representations, this state is constructed as follows. For the partition $\lambda = (\lambda_1, \ldots, \lambda_N \geq 0)$, define

$$|\lambda\rangle_N = [u]_{\lambda_1} \otimes [(t/q)u]_{\lambda_2 - 1} \otimes \cdots \otimes [(t/q)^{N-1} u]_{\lambda_N - N + 1}\,, \tag{3.4.11}$$

and denote the space spanned by these states by $W^N(u)$. Then the state $|\lambda\rangle$ of the $(0, 1)$-modules is the element in $\varprojlim_N W^N(u)$.

Fact 3.4.4 ([32, 27])**.** *Let u be an indeterminate. We can endow a $\mathcal{U}_{q,t}$-module structure to $\mathcal{F}$ by setting*

$$c^{1/2} |\lambda\rangle = |\lambda\rangle\,, \tag{3.4.12}$$

$$x^+(z) |\lambda\rangle = \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}\, \delta(q^{\lambda_i} t^{-i+1} u/z) |\lambda + \mathbf{1}_i\rangle\,, \tag{3.4.13}$$

$$x^-(z) |\lambda\rangle = q^{1/2} t^{-1/2} \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}\, \delta(q^{\lambda_i - 1} t^{-i+1} u/z) |\lambda - \mathbf{1}_i\rangle\,, \tag{3.4.14}$$

$$\psi^+(z) |\lambda\rangle = q^{1/2} t^{-1/2} B^+_\lambda(u/z) |\lambda\rangle\,, \tag{3.4.15}$$

$$\psi^-(z) |\lambda\rangle = q^{-1/2} t^{1/2} B^-_\lambda(z/u) |\lambda\rangle\,, \tag{3.4.16}$$

with

$$A^{+}_{\lambda,i} = (1-t)\prod_{j=1}^{i-1}\frac{(1-q^{\lambda_i-\lambda_j}t^{-i+j+1})(1-q^{\lambda_i-\lambda_j+1}t^{-i+j-1})}{(1-q^{\lambda_i-\lambda_j}t^{-i+j})(1-q^{\lambda_i-\lambda_j+1}t^{-i+j})}, \tag{3.4.17}$$

$$A^{-}_{\lambda,i} = (1-t^{-1})\frac{1-q^{\lambda_{i+1}-\lambda_i}}{1-q^{\lambda_{i+1}-\lambda_i+1}t^{-1}}\prod_{j=i+1}^{\infty}\frac{(1-q^{\lambda_j-\lambda_i+1}t^{-j+i-1})(1-q^{\lambda_{j+1}-\lambda_i}t^{-j+i})}{(1-q^{\lambda_{j+1}-\lambda_i+1}t^{-j+i-1})(1-q^{\lambda_j-\lambda_i}t^{-j+i})}, \tag{3.4.18}$$

$$B^{+}_{\lambda}(z) = \frac{1-q^{\lambda_1-1}tz}{1-q^{\lambda_1}z}\prod_{i=1}^{\infty}\frac{(1-q^{\lambda_i}t^{-i}z)(1-q^{\lambda_{i+1}-1}t^{-i+1}z)}{(1-q^{\lambda_{i+1}}t^{-i}z)(1-q^{\lambda_i-1}t^{-i+1}z)}, \tag{3.4.19}$$

$$B^{-}_{\lambda}(z) = \frac{1-q^{-\lambda_1+1}t^{-1}z}{1-q^{-\lambda_1}z}\prod_{i=1}^{\infty}\frac{(1-q^{-\lambda_i}t^{i}z)(1-q^{-\lambda_{i+1}+1}t^{i-1}z)}{(1-q^{-\lambda_{i+1}}t^{i}z)(1-q^{-\lambda_i+1}t^{i-1}z)}. \tag{3.4.20}$$

We refer to this module as $\mathcal{F}^{(0,1)}$-module. We denote the basis P_λ of $(0,1)$-module by $|\lambda\rangle$ to distinguish from those of $\mathcal{F}^{(1,M)}$. To represent this module, we sometimes use the notation $\rho^{(0,1)}: \mathcal{U}_{q,t}\to \mathrm{End}(\mathcal{F})$.

3.4.3 Intertwiners and Refined Topological Vertex

Now we introduce the intertwiners which intertwine the tensor product of two modules to one module or its inverse. These intertwiners are introduced in the beautiful paper [9]. As we see below, because these intertwiners connect three modules, they can be diagrammatically represented by trivalent vertices.

Fact 3.4.5 ([9]). *When $w=-uv$, there exists a unique intertwiner which satisfies*

$$\Phi\begin{bmatrix}(1,M+1),w\\(0,1),v;(1,M),u\end{bmatrix}: \mathcal{F}^{(0,1)}_v\otimes\mathcal{F}^{(1,M)}_u \longrightarrow \mathcal{F}^{(1,M+1)}_w; \qquad a\Phi=\Phi\Delta(a) \qquad (\forall a\in\mathcal{U}_{q,t}) \tag{3.4.21}$$

and the normalization condition $\langle 0|\,\Phi(|\emptyset\rangle\otimes|0\rangle)=1$. Moreover, its component Φ_λ, defined by

$$\Phi_\lambda(\alpha)=\Phi(|\lambda\rangle\otimes\alpha) \qquad (\forall\,|\lambda\rangle\otimes\alpha\in\mathcal{F}^{(0,1)}_v\otimes\mathcal{F}^{(1,M)}_u)\,, \tag{3.4.22}$$

has the following realization,

$$\Phi_\lambda\begin{bmatrix}(1,M+1),-vu\\(0,1),v;(1,M),u\end{bmatrix} = \hat{t}(\lambda,u,v,M)\widehat{\Phi}_\lambda(v)\,, \tag{3.4.23}$$

where

$$\begin{aligned}
&\hat{t}(\lambda,u,v,M) = (-vu)^{|\lambda|}(-v)^{-(M+1)|\lambda|}f_\lambda^{-M-1}q^{n(\lambda')}/c_\lambda\,,\\
&\widehat{\Phi}_\lambda(v) =: \Phi_\emptyset(v)\eta_\lambda(v):\,,\\
&\Phi_\emptyset(v) = \exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{1-q^n}a_{-n}v^n\Big)\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}a_n v^{-n}\Big),\\
&\eta_\lambda(v) =: \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\eta(q^{j-1}t^{-i+1}v):\,.
\end{aligned} \tag{3.4.24}$$

Similarly, the following intertwiner exists uniquely,

$$\Phi^*\begin{bmatrix}(1,M),u;(0,1),v\\(1,M+1),-vu\end{bmatrix}: \mathcal{F}^{(1,M+1)}_{-uv}\longrightarrow\mathcal{F}^{(1,M)}_u\otimes\mathcal{F}^{(0,1)}_v, \qquad \Delta(a)\Phi^*=\Phi^*a \qquad (\forall a\in\mathcal{U}_{q,t})\,, \tag{3.4.25}$$

with normalization $\Phi^*(|0\rangle) = |0\rangle \otimes 1 + \cdots$, *and its component, defined by*

$$\Phi^*(\alpha) = \sum_\lambda \Phi^*_\lambda(\alpha) \otimes \left(\frac{c_\lambda}{c'_\lambda}\,|\lambda\rangle\right) \qquad (\forall \alpha \in \mathcal{F}^{(1,M+1)}_{-uv})\,, \tag{3.4.26}$$

is realized by

$$\Phi^*_\lambda \begin{bmatrix} (1,M), v; (0,1), u \\ (1,M+1), -vu \end{bmatrix} = \hat{t}^*(\lambda, u, v, M)\widehat{\Phi}^*_\lambda(u)\,, \tag{3.4.27}$$

where

$$\begin{aligned}
&\hat{t}^*(\lambda, u, v, M) = (q^{-1}v)^{-|\lambda|}(-u)^{M|\lambda|} f^M_\lambda q^{n(\lambda')}/c_\lambda\,,\\
&\widehat{\Phi}^*_\lambda(u) =: \Phi^*_\emptyset(u)\xi_\lambda(u) :\,,\\
&\Phi^*_\emptyset(u) = \exp\Big(\sum_{n=1}^\infty \frac{1}{n}\frac{1}{1-q^n} q^{-n/2}t^{n/2}a_{-n}u^n\Big)\exp\Big(\sum_{n=1}^\infty \frac{1}{n}\frac{q^n}{1-q^n} q^{-n/2}t^{n/2}a_{n}u^{-n}\Big)\,, \qquad (3.4.28)\\
&\xi_\lambda(u) =: \prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i} \xi(q^{j-1}t^{-i+1}u) :\,.
\end{aligned}$$

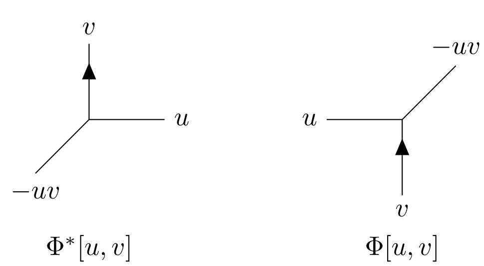
Notation 3.4.6. *In what follows, we mainly consider the* $M = 0$ *case, and we introduce the simplified notations for the intertwiners.*

$$\Phi[u,v] := \Phi\begin{bmatrix} (1,1), -vu \\ (0,1), v; (1,0), u \end{bmatrix}\,, \qquad \Phi^*[u,v] := \Phi^*\begin{bmatrix} (1,0), u; (0,1), v \\ (1,1), -vu \end{bmatrix}\,, \tag{3.4.29}$$

and their components,

$$\Phi_\lambda[u,v] := \Phi_\lambda\begin{bmatrix} (1,1), -vu \\ (0,1), v; (1,0), u \end{bmatrix}\,, \qquad \Phi^*_\lambda[u,v] := \Phi^*_\lambda\begin{bmatrix} (1,0), u; (0,1), v \\ (1,1), -vu \end{bmatrix}\,. \tag{3.4.30}$$

We also assign the trivalent diagrams to each intertwiner as follows. The arrows stand for the $\mathcal{F}^{(0,1)}$-*modules, and we refer to this direction as the preferred direction, following the terminology of the refined topological vertex in [47].*



The proof is by direct computation. For example, we can check

$$x^+(z)\Phi_\lambda - q^{-1/2}t^{1/2}B^-_\lambda(z/v)\Phi_\lambda x^+(z) = \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}\,\delta(q^{\lambda_i}t^{-i+1}v/z)\Phi_{\lambda+\mathbf{1}_i}\,. \tag{3.4.31}$$

Remark 3.4.7. *Define the adjoint action of* $\forall x \in \mathcal{U}_{q,t}$ *on* $\Phi \in \mathrm{End}(\mathcal{F})$ *by*

$$\mathrm{adj}(x)\,\Phi := \sum_i x_i^{(1)}\,\Phi\, a(x_i^{(2)})\,, \tag{3.4.32}$$

with $\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$ *and* a *the antipode. Then, (3.4.31) can be rewritten as*

$$\mathrm{adj}(x^+(z))\,\Phi_\lambda = \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}\,\delta(q^{\lambda_i} t^{-i+1} v/z)\Phi_{\lambda+\mathbf{1}_i}\,. \tag{3.4.33}$$

This means the $(0,1)$ *representation can be regarded as the adjoint representation.*

Let us see these intertwiners more carefully. Then we notice

$$\Phi_\emptyset(v) =: \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \frac{1}{\eta(vq^{j-1}t^{1-i})} : \,. \tag{3.4.34}$$

(For $\Phi^*_\emptyset$, replace η with ξ.) That is, diagrammatically, $\Phi_\emptyset$ means the $1/\eta$'s are spread all over the infinitely large partition. Then in Φ_λ, since some $1/\eta$'s are cancelled, they are spread except for λ part. This situation is summarized in the next figure.

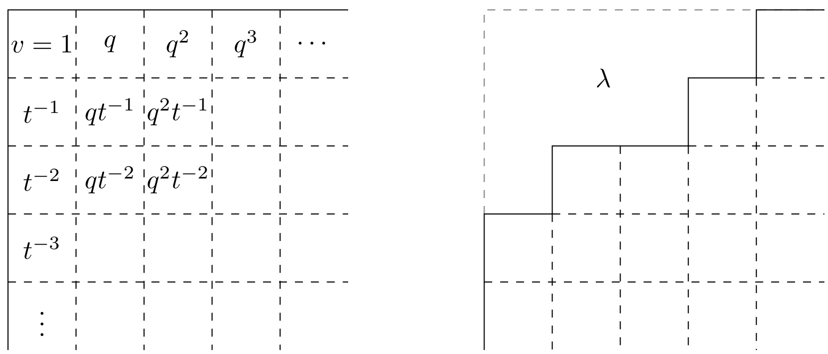


Figure 3.2: Left:$\Phi_\emptyset$ and Right:Φ_λ

This consideration becomes important in Chapter 6.

Identification with Refined Topological Vertex

The resemblance between the diagram of intertwiners and that of the refined topological vertex is not a coincidence. They are actually related with each other. In order to show this relation, we first introduce the Schur functions on the Fock space and their dual.

Notation 3.4.8. *Denote by* $|s_\lambda\rangle$*, the Schur functions on the Fock space, and by* $\langle S_\lambda(q,t)|$*, the dual states, that is,*

$$\langle S_\mu(q,t)\,|s_\lambda\rangle = \delta_{\mu,\lambda}\,. \tag{3.4.35}$$

This means the polynomial S_λ is dual to the Schur polynomial with respect to the kernel function (Definition 3.1.12). This can be constructed as follows. Let $w_{u,v}$ and ι be the endomorphisms of Λ,

$$w_{u,v}(\mathsf{p}_n) = -(-1)^n \frac{1-u^n}{1-v^n}\mathsf{p}_n\,, \tag{3.4.36}$$

$$\iota(\mathsf{p}_n) = -\mathsf{p}_n\,, \tag{3.4.37}$$

with p_n the n-th power sum. Then, S_λ can be represented as

$$S_\lambda(\boldsymbol{x};q,t) := \iota w_{t,q} s_\lambda(-\boldsymbol{x})\,. \tag{3.4.38}$$

Now the following fact reveals the relation between the $\mathcal{U}_{q,t}$ intertwiners and the refined topological vertex.

Proposition 3.4.9 ([9]).

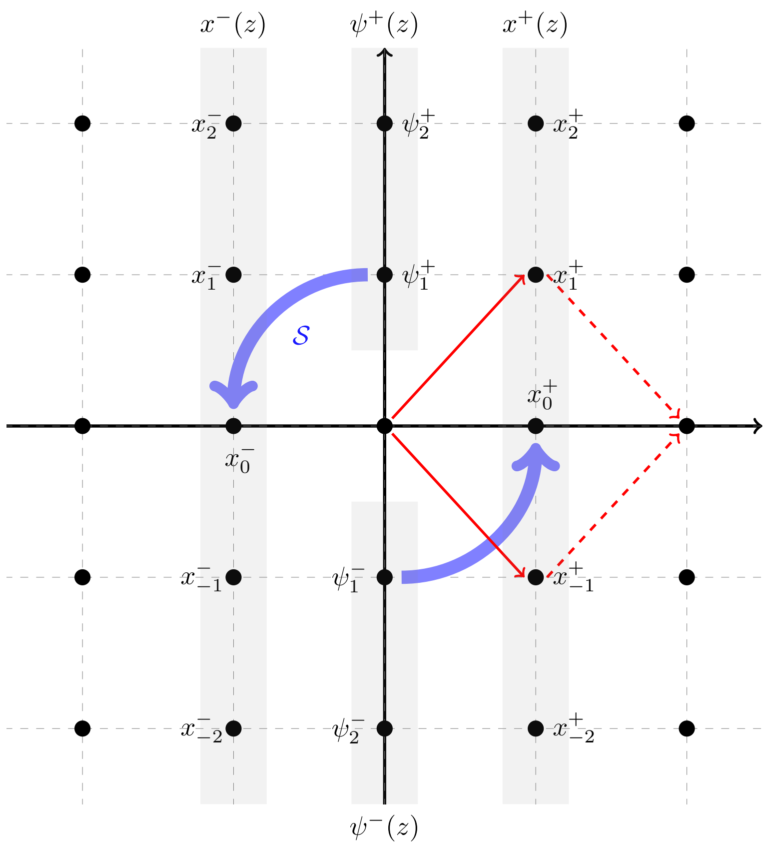
$$\frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}} \langle S_\mu(q,t)|\, \Phi_\lambda \begin{bmatrix} (1, M+1), -vu \\ (0,1), v; (1,M), u \end{bmatrix} |s_\nu\rangle \tag{3.4.39}$$
$$= \left(\frac{q^{-1/2}u}{(-v)^M}\right)^{|\lambda|} f_\lambda^{-M} \cdot \ (-q^{-1/2}v)^{-|\nu|} f_\nu \cdot (t^{-1/2}v)^{|\mu|} \cdot (-1)^{|\mu|+|\nu|+|\lambda|}\ C^{(\mathrm{IKV})}_{\mu\nu'\lambda'}(q,t),$$

$$\langle S_\nu(q,t)|\, \Phi^*_\lambda \begin{bmatrix} (1,M), v; (0,1), u \\ (1, M+1), -vu \end{bmatrix} |s_\mu\rangle \tag{3.4.40}$$
$$= \left(\frac{(-u)^M}{q^{-1/2}v}\right)^{|\lambda|} f_\lambda^{M} \cdot \ (-q^{-1/2}u)^{|\nu|} f_\nu^{-1} \cdot (t^{-1/2}u)^{-|\mu|} \cdot C^{(\mathrm{IKV})}_{\mu'\nu\lambda}(t,q).$$

3.5 Miki Automorphism (S-duality)

3.5.1 Structure of $\mathcal{U}_{q,t}$

One important fact about $\mathcal{U}_{q,t}$ is, it is $\mathbb{Z}^2$-graded [27].[2] Especially, $\deg(x_n^\pm) = (\pm 1, n)$ and $\deg(\psi_i^\pm) = (0, \pm i)$. The following picture shows this situation.



All the elements in $\mathcal{U}_{q,t}$ are generated by $x_n^\pm (n \in \mathbb{Z})$, and $\psi_i^\pm (i \in \mathbb{Z}_{\geq 0})$ so that the degrees are preserved. For example, the element with a degree $(2,0)$ is generated by $[x_1^+, x_{-1}^+]$, see the red lines in the figure. It is easy to check the degree is conserved $(1,1) + (1,-1) = (2,0)$. Repeating this procedure, we can generate all the elements in $\mathcal{U}_{q,t}$.

[2] In this section, we do not care about the centers.

From this consideration, it is natural to expect the whole algebra to be generated by only 4 elements in each orthants in $\mathbb{Z}^2$, and this is correct. Actually, with the defining relations (3.3.12), $x_0^\pm$ and $\psi_1^\pm$ generate all the other generators. For example, we can construct $x_n^\pm$, $(n \in \mathbb{Z}_{>0})$ from $x_0^\pm$, ψ_1^+ and the first relation in (3.3.12),

$$\psi_1^+ x_n^\pm = \tilde{g}_1^\pm x_{n+1}^\pm \psi_0^+ + \tilde{g}_0^\pm x_n^\pm \psi_1^+ \,, \tag{3.5.1}$$

where we put

$$g(c^{\mp 1/2} w/z)^{\mp 1} = \sum_{i \in \mathbb{Z}_{\geq 0}} \tilde{g}_i^\pm \cdot (w/z)^i \,. \tag{3.5.2}$$

By solving the equation in terms of $x_{n+1}^\pm$, we obtain the expression for $x_{n+1}^\pm$ from $x_n^\pm$ and ψ_1^+. (Note that ψ_0^+ is the invertible center.) Similarly, we can construct $x_n^\pm$, $(n \in \mathbb{Z}_{<0})$ from $x_0^\pm$, ψ_1^-.

Definition 3.5.1 ([72])**.** *Define the automorphism $\mathcal{S} : \mathcal{U}_{q,t} \to \mathcal{U}_{q,t}$ by*

$$\psi_1^\pm \mapsto x_0^\mp \,, \qquad x_0^\pm \mapsto \psi_1^\pm \,, \qquad (\psi_0^+/\psi_0^-)^{-1/2} \mapsto c \,, \qquad c \mapsto (\psi_0^+/\psi_0^-)^{-1/2} \,. \tag{3.5.3}$$

We omit the constant coefficient in the image of this map.

We indicate this map by the blue line in the figure above.

In the following, we see the representations introduced above are related to each other under this automorphism. By the consideration above, we only need to check the automorphism is compatible with the representations of $x_0^\pm$ and $\psi_1^\pm$.

3.5.2 Duality between $(0,1)$ and $(1,0)$ Representations

First, we write down the $(0,1)$-representation for each mode as follows.

$$\begin{aligned}
&\rho^{(0,1)}(x_0^+) \left|\lambda\right\rangle = \sum_{i=1}^{\ell(\lambda)+1} A_{\lambda,i}^+ \left|\lambda + \mathbf{1}_i\right\rangle \,, \qquad \rho^{(0,1)}(x_0^-) \left|\lambda\right\rangle = \gamma^{-1} \sum_{i=1}^{\ell(\lambda)} A_{\lambda,i}^- \left|\lambda - \mathbf{1}_i\right\rangle \,, \\
&\rho^{(0,1)}(\psi_1^+) \left|\lambda\right\rangle = \gamma^{-1} u (1-t/q)(1-t^{-1}) \sum_{i=1}^{\infty} q^{\lambda_i} t^{1-i} \left|\lambda\right\rangle \,, \\
&\rho^{(0,1)}(\psi_1^-) \left|\lambda\right\rangle = \gamma u^{-1} (1-q/t)(1-t) \sum_{i=1}^{\infty} q^{-\lambda_i} t^{-1+i} \left|\lambda\right\rangle \,.
\end{aligned} \tag{3.5.4}$$

Next, we write the $(1,0)$-representation.

$$\begin{aligned}
&\rho^{(1,0)}(x_0^+) \left|P_\lambda\right\rangle = u(1-t^{-1}) \sum_{i=1}^{\infty} q^{\lambda_i} t^{1-i} \left|P_\lambda\right\rangle \,, \\
&\rho^{(1,0)}(x_0^-) \left|P_\lambda\right\rangle = u^{-1}(1-t) \sum_{i=1}^{\infty} q^{-\lambda_i} t^{i-1} \left|P_\lambda\right\rangle \,, \\
&\rho^{(1,0)}(\psi_1^+) \left|P_\lambda\right\rangle = -\gamma^{-1/2}(1-t)(1-t/q) a_1 \left|P_\lambda\right\rangle = q\gamma^{-1/2}(1-t/q) \sum_{i=1}^{\infty} A_{\lambda,i}^- \left|P_{\lambda-\mathbf{1}_i}\right\rangle \,, \\
&\rho^{(1,0)}(\psi_1^-) \left|P_\lambda\right\rangle = \gamma^{-1/2}(1-t^{-1})(1-t/q) a_{-1} \left|P_\lambda\right\rangle = -q^{-1}\gamma^{-1/2}(1-q/t) \sum_{i=1}^{\infty} A_{\lambda,i}^+ \left|P_{\lambda+\mathbf{1}_i}\right\rangle \,.
\end{aligned} \tag{3.5.5}$$

In the last two lines, we used the Pieri rule. Again, note that $A_{\lambda,i}^\pm$ are the Pieri coefficients, defined in Definition 3.1.15, up to constant. Now it is obvious that these two modules are related via the Miki automorphism, that is,

$$\gamma^{\mp 1}(1-(t/q)^{\pm 1}) \rho^{(1,0)}(x_0^\pm) \leftrightarrow \rho^{(0,1)}(\psi_1^\pm) \,, \qquad \pm \frac{q^{\mp 1}\gamma^{\pm 1/2}}{1-(t/q)^{\pm 1}} \rho^{(1,0)}(x_0^\pm) \leftrightarrow \rho^{(0,1)}(x_0^\mp) \,. \tag{3.5.6}$$

It is also possible to observe a similar correspondence between two kinds of $(0,0)$-modules.

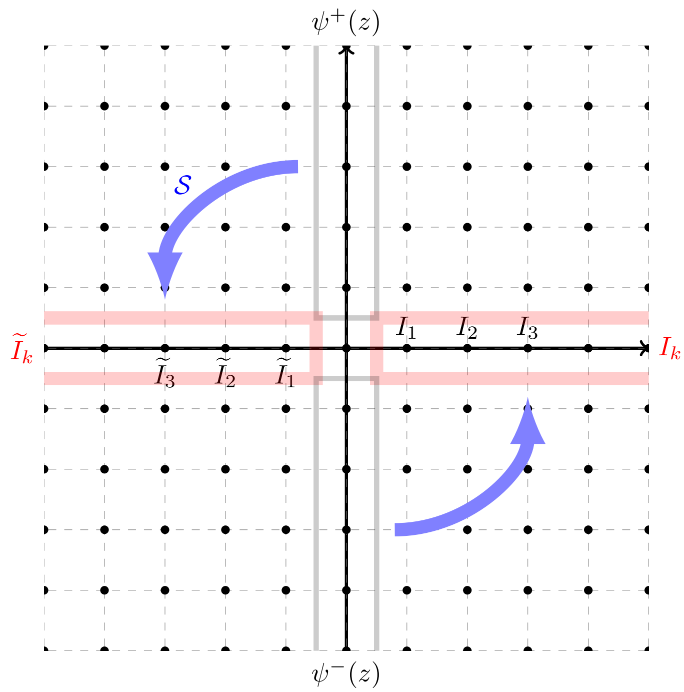
3.5.3 Higher Hamiltonians

As the Macdonald polynomials are joint eigenfunctions of commuting Hamiltonians (see Fact 3.1.7), the Macdonald functions on the Fock space, (which correspond to infinitely many variables,) also must be characterized by infinitely many commuting difference operator. In Section 3.2, we show the zero mode of the $\eta(z)$ currents in $\mathcal{U}_{q,t}$ is intertwined to the Macdonald operator. In order to make ends meet, we have to construct the higher operators which commute with η_0.

Because for $n, m \in \mathbb{Z}_{\geq 0}$ we have

$$[\psi_n^+, \psi_m^+] = 0\,, \qquad [\psi_n^-, \psi_m^-] = 0\,, \tag{3.5.7}$$

the images of $\psi_n^\pm$ under the Miki automorphism $\mathcal{S}$ commutes with $x_0^\pm$. This situation is summarized in the following figure:



In [72], the concrete forms of I_k and $\widetilde{I}_k (k \in \mathbb{Z}_{\geq 2})$ are given by

$$I_k = [x_{-1}^+, \overbrace{[x_0^+, \cdots [x_0^+}^{k-2}, x_1^+] \cdots]]\,, \tag{3.5.8}$$

$$\widetilde{I}_k = [x_1^-, \underbrace{[x_0^-, \cdots [x_0^-}_{k-2}, x_{-1}^-] \cdots]]\,. \tag{3.5.9}$$

We can easily find these forms from the consideration of their degrees. Note that the Serre relation ensures the commutation relation between $I_1 = x_0^+$ and $I_2 = [x_1^+, x_{-1}^+]$ vanishes.

3.6 Shuffle Algebra and Feigin–Odesskii Algebra

In the previous section, we observed there actually exist infinitely many commuting Hamiltonians. In this section, we present an explicit way to construct them.

There is an ad hoc way to construct the commuting integrals of motion, associated with the A-type Macdonald operators. This is the Feigin–Odesskii(FO) algebra, which was introduced in [87, 31]. There is a beautiful paper about the FO algebra [29], and we follow their arguments.

Throughout this section, we put the base field $\mathbb{F} = \mathbb{Q}(q, t)$.

3.6.1 Definition

Notation 3.6.1. *Denote the space of the m-variable symmetric rational functions with coefficients in* $\mathbb{F}$ *as* $\bar{\mathcal{A}}_m$ $(m > 0)$*. We also write* $\bar{\mathcal{A}} = \bigoplus_{n\geq 0} \bar{\mathcal{A}}_n$*.*

In order to introduce the FO algebra, we first define the star product.

Definition 3.6.2. *We put* $\bar{\mathcal{A}}_0 = \mathbb{F}$*. For* $f \in \bar{\mathcal{A}}_m$ *and* $g \in \bar{\mathcal{A}}_n$*, define the map* $* : \bar{\mathcal{A}}_m \times \bar{\mathcal{A}}_n \to \bar{\mathcal{A}}_{m+n}$ *by*

$$(f * g)(x_1, \ldots, x_{m+n}) = \mathrm{Sym}\left[f(x_1, \ldots, x_m)g(x_1, \ldots, x_n) \prod_{1\leq\alpha\leq m} \prod_{m+1\leq\beta\leq m+n} w(x_\alpha, x_\beta)\right], \tag{3.6.1}$$

with

$$w(x, y) := \frac{(x - q^{-1}y)(x - ty)(x - qy/t)}{(x - y)^3}, \tag{3.6.2}$$

and the symmetrizer

$$\mathrm{Sym}\,(f(x_1, \ldots, x_m)) = \frac{1}{m!} \sum_{\sigma\in\mathfrak{S}_m} f(x_{\sigma(1)}, \ldots, x_{\sigma(m)}). \tag{3.6.3}$$

We refer to this bilinear map as the star product.

Remark 3.6.3. *Note that this is the degenerate case of the original Feigin-Odesskii algebra. In the original case, instead of* $w(x, y)$*, they use*

$$\lambda(x, y) = \frac{\Theta_p(y/qx)\Theta_p(ty/x)\Theta_p(qy/tx)}{\Theta_p(y/x)^3}. \tag{3.6.4}$$

We introduce two operators.

Definition 3.6.4. *For* $f \in \bar{\mathcal{A}}_n$ *and* $k = 1, \ldots, n$*, define*

$$\partial^{(0,k)} \;:\; f \mapsto \frac{n!}{(n-k)!} \lim_{\zeta\to 0} f(x_1, \ldots, x_{n-k}, \zeta x_{n-k+1}, \ldots, \zeta x_n), \tag{3.6.5}$$

$$\partial^{(\infty,k)} \;:\; f \mapsto \frac{n!}{(n-k)!} \lim_{\zeta\to \infty} f(x_1, \ldots, x_{n-k}, \zeta x_{n-k+1}, \ldots, \zeta x_n), \tag{3.6.6}$$

whenever the limits exist. For $n = 0$*, we put the actions to be* 0*. We also set* $\partial^{(0,0)}, \partial^{(\infty,0)}$ *to be the identity operators.*

We define the subset of $\bar{\mathcal{A}} = \bigoplus_{n\geq 0} \bar{\mathcal{A}}_n$.

Definition 3.6.5. *The subspace* $\mathcal{A}_n \subset \bar{\mathcal{A}}_n$ *is defined by the following three conditions:*

(i) *For* $f \in \mathcal{A}_n$ *and* $0 \leq k \leq n$*,* $\partial^{(0,k)} f$ *and* $\partial^{(\infty,k)} f$ *exist and* $\partial^{(0,k)} f = \partial^{(\infty,k)} f$*.*

(ii) *The poles of* $f \in \mathcal{A}_n$ *locate only on* $\{(x_1, \ldots, x_n) | \exists (i, j), i \neq j, x_i = x_j\}$*, and the orders of the poles are at most two.*

(iii) *For* $n \geq 3$*,* $f \in \mathcal{A}_n$*,* $f(x, q^{-1}x, qx/t, x_4, \ldots) = f(x, tx, qx/t, x_4, \ldots) = 0$*.*

We also denote $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$*.*

The following theorem is highly non-trivial and essential.

Theorem 3.6.6. *The vector space* $\mathcal{A}$ *is closed with respect to the star product* $*$*, and this defines the unital associative algebra* $(\mathcal{A}, *)$*. Moreover, the algebra* $(\mathcal{A}, *)$ *is commutative. We refer to the commuting algebra* $(\mathcal{A}, *)$ *as the Feigin–Odesskii algebra.*

Example

We just show one family of examples which are the elements in $\mathcal{A}$, and thus commute with each other with respect to the star product.

Definition 3.6.7. *For $\boldsymbol{x} = (x_1, \ldots, x_n)$, set*

$$g_n(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \tilde{g}(x_i, x_j)\,, \tag{3.6.7}$$

$$\tilde{g}(x_i, x_j) = \frac{1 - x_j/tx_i}{1 - x_j/x_i} \frac{1 - x_i/tx_j}{1 - x_i/x_j}\,. \tag{3.6.8}$$

Then, it is shown that

$$g_n(\boldsymbol{x}) * g_m(\boldsymbol{y}) - g_m(\boldsymbol{y}) * g_n(\boldsymbol{x}) = 0\,. \tag{3.6.9}$$

The proof is done through applying the Liouville theorem to

$$\frac{g_n * g_m - g_m * g_n}{g_{n+m}}\,. \tag{3.6.10}$$

3.6.2 Family of Commuting Operators

Using this FO algebra, we can construct the family of commuting operators.

First, from (3.3.1), we know

$$f(x_2/x_1)\eta(x_1)\eta(x_2) = f(x_1/x_2)\eta(x_2)\eta(x_1)\,, \tag{3.6.11}$$

with

$$f(z) = \frac{(1-qz)(1-z/t)}{(1-z)(1-qz/t)}\,. \tag{3.6.12}$$

Then it it easy to show the following lemma.

Lemma 3.6.8. *Define the operator*

$$\mathcal{O}_n(z_1, \ldots, z_n) = \prod_{1 \le i < j \le n} f(x_j/x_i) \cdot \eta(z_1) \cdots \eta(z_n)\,. \tag{3.6.13}$$

Then, the operator $\mathcal{O}_n$ is $\mathfrak{S}_n$-invariant. That is, the combinations of f and η's are operator-valued symmetric Laurent polynomials.

Motivated by this lemma, we define the following map $\mathcal{O} : \mathcal{A} \to \mathrm{End}(\mathcal{F})$.

Definition 3.6.9. *Define the map $\mathcal{O} : \mathcal{A} \to \mathrm{End}(\mathcal{F})$,*

$$I_n := \left[\prod_{1 \le i < j \le n} S(z_i, z_j) \cdot \mathcal{O}_n(z_1, \ldots, z_n) \right]_1\,, \tag{3.6.14}$$

with

$$S(z, w) = \frac{\tilde{g}(z, w)}{f(w/z) w(z, w)}\,. \tag{3.6.15}$$

For $\tilde{g}$, see Definition 3.6.7. $[\cdots]_1$ means taking the constant term in $\cdots$ as before.

We can also easily prove the following lemma by direct computation.

Lemma 3.6.10.

$$\begin{aligned}&\left(\prod_{1\le i<j\le n} S(z_i,z_j)\cdot\mathcal{O}_n(z_1,\dots,z_n)\right)\cdot\left(\prod_{1\le i<j\le m} S(z_{n+i},z_{n+j})\cdot\mathcal{O}_m(z_{n+1},\dots,z_{n+m})\right)\\&\qquad=\prod_{1\le i\le n}\prod_{1\le j\le m}\frac{w(z_i,z_{n+j})}{\tilde g(z_i,z_{n+j})}\cdot\prod_{1\le i<j\le n+m} S(z_i,z_j)\cdot\mathcal{O}_{n+m}(z_1,\dots,z_{n+m})\,.\end{aligned}\tag{3.6.16}$$

Remark 3.6.11. *In Definition 3.6.9, we use the special element $g_n\in\mathcal{A}_n$. $\mathcal{A}$ admits some Gordon filtration, and g_n's are top components with respect to that filtration. Thus we can generalize I_n to*

$$I_n(f)=\left[f(z_1,\dots,z_n)\prod_{1\le i<j\le n}\frac{1}{w(z_i,z_j)f(z_j/z_i)}\cdot\mathcal{O}_n(z_1,\dots,z_n)\right]_1\,,\tag{3.6.17}$$

for $f\in\mathcal{A}_n$.

Combining all above, we have the following main theorem.

Theorem 3.6.12. *For arbitrary n and $m\in\mathbb{Z}_{>0}$, we have*

$$[I_n,I_m]=0\,.\tag{3.6.18}$$

Proof. From the lemma above,

$$\begin{aligned}I_n\cdot I_m&=\left[\prod_{1\le i\le n}\prod_{1\le j\le m}\frac{w(z_i,z_{n+j})}{\tilde g(z_i,z_{n+j})}\cdot\prod_{1\le i<j\le n+m} S(z_i,z_j)\cdot\mathcal{O}_{n+m}(z_1,\dots,z_{n+m})\right]_1\\&=\left[g_n(z_1,\dots,z_n)g_m(z_{n+1},\dots,z_{n+m})\prod_{1\le i\le n}\prod_{1\le j\le m} w(z_i,z_{n+j})\cdot\prod_{1\le i<j\le n+m}\frac{S(z_i,z_j)}{\tilde g(z_i,z_j)}\cdot\mathcal{O}_{n+m}(z_1,\dots,z_{n+m})\right]_1\\&=\left[g_n * g_m\cdot\prod_{1\le i<j\le n+m}\frac{S(z_i,z_j)}{\tilde g(z_i,z_j)}\cdot\mathcal{O}_{n+m}(z_1,\dots,z_{n+m})\right]\end{aligned}\tag{3.6.19}$$

In the last line, we symmetrize the integration variables. Note that $(\prod S/\tilde g)\mathcal{O}_{n+m}$ is $\mathfrak{S}_{n+m}$-invariant. The commutativity of g_n completes the proof. □

3.6.3 Examples

By the Serre relation, we know $[\eta_1,\eta_{-1}]$ and η_0 commute with each other,

$$[\eta_0,[\eta_1,\eta_{-1}]]=0\,.\tag{3.6.20}$$

Let us show this fact using the FO algebra. We first rewrite $[\eta_1,\eta_{-1}]$ (plus something which trivially commutes with η_0) as

$$[\eta_1,\eta_{-1}]-(q-q^{-1})\,\eta_0^2=\oint\frac{dz}{2\pi iz}\frac{dw}{2\pi iw}\frac{(1-w/z)(1-qw/tz)}{(1-qw/z)(1-w/tz)}\left(\frac{w}{z}-\frac{z}{w}-(q-q^{-1})\right):\eta(z)\eta(w):\,.\tag{3.6.21}$$

Then, by symmetrizing the integral contours, we can perform the following transformation.

$$\begin{aligned}&\oint\frac{dz}{2\pi iz}\frac{dw}{2\pi iw}\frac{(1-w/z)(1-qw/tz)}{(1-qw/z)(1-w/tz)}\left(\frac{w}{z}-\frac{z}{w}-(q-q^{-1})\right):\eta(z)\eta(w):\\&\quad=\frac{1}{2}\frac{(1-q)(q+t)}{qt}\oint\frac{dz}{2\pi iz}\frac{dw}{2\pi iw}(1+t)\frac{(w-z)^2}{(tw-z)(w-tz)}:\eta(z)\eta(w):\\&\quad=\frac{(1-q)(q+t)}{qt}\oint\frac{dz}{2\pi iz}\frac{dw}{2\pi iw}\frac{1-w/z}{1-t^{-1}w/z}:\eta(z)\eta(w):\,:=\oint\frac{dz}{2\pi iz}\frac{dw}{2\pi iw}\frac{f(z,w)}{w(z,w)}\eta(z)\eta(w)\,,\end{aligned}\tag{3.6.22}$$

with

$$\frac{f(z,w)}{\omega(z,w)} = \frac{1-qw/z}{1-qw/tz}\,. \tag{3.6.23}$$

In this transformation, the most RHS can be seen as the map from the FO algebra. It is easy to check f in (3.6.23) satisfies the conditions in $\mathcal{A}$.

Chapter 4

Generalized Macdonald Functions on N-Fock Tensor Space

This chapter is devoted to the explanation of the generalized Macdonald functions on the N-fold tensor space of the Fock spaces $\mathcal{F}_{\boldsymbol{u}} = \widetilde{\bigotimes}_{j=1}^{N} \mathcal{F}_{u_j}$. These states are defined as the eigenstates of the zero mode of the N-th coproduct of $\eta(z)$. Our goal is to extend Theorem 3.2.6 to the Fock tensor spaces. The lesson we learned in Section 3.2, is that in order to construct $|P_\lambda\rangle$, we need the Macdonald polynomials P_λ and the vertex operator called the top component.

Thus, we need to prepare the extension of the Macdonald polynomials which are associated with the N-tuples of partitions. This is *the bispectral Macdonald functions*, introduced in [86]. This is the main object in Section 4.1. This function has very nice properties, such as the bispectral duality, the Poincaré duality, and the factorization formula.

Next, we need to prepare the vertex operators $\Phi^{(i)}$ $(i = 0, \ldots, N-1)$. First, we define the N-th tensor analogue of the top component, denoted by $\Phi^{(0)}$. Then, $\Phi^{(i)}$ $(i \neq 0)$'s are constructed by combining the top component with the screening currents of the q-deformed $\mathcal{W}$-algebra (q-$\mathcal{W}$ for short). Then, with these tools, we obtain the explicit construction of the generalized Macdonald functions. See Section 4.2 for more details.

Section 4.3 is in some sense, a supplement of Section 4.2. In the previous section, we use the screening currents without mentioning the q-deformed $\mathcal{W}$-algebra itself, and thus we review it in this section. Moreover, we prove the screening operators, which are the integral of the screening currents, are well-defined. Using these facts, we can complete the proof of the Kac determinant formula for q-$\mathcal{W}$ algebra.

4.1 Bispectral Macdonald Functions

In the previous chapter, we introduced the Macdonald "polynomials". We now extend these polynomials to the basic hypergeometric series with multi-variables. The biggest reason for this extension is that we would like to treat the main variables and the eigenvalue of the Macdonald operator on equal footing. More concretely, in the previous chapter, the eigenvalues of the Macdonald operators are expressed in terms of

$$s_i = t^{n-i} q^{\lambda_i} \,, \tag{4.1.1}$$

and they are just numbers. Here, we would like to deal with them as another set of variables $(s_1, \ldots)$, that is, as the parameters which have nothing to do with the partitions. This will be clear in Definition 4.1.2. We refer to these hypergeometric series as *the bispectral Macdonald functions*.

After this extension, the various known transformation formulas of the basic hypergeometric functions becomes applicable to the Macdonald functions. It is the Kajihara-Noumi identity that is the master formula for such transformation. (See Section 5.3.) We also observe the good analytic properties of these functions, and these become important in the proof of the main claim (see Proposition 5.3.7). Moreover, we obtain two important duality formulas, *the bispectral duality* and *the Poincaré duality*.

To be fair, we have to note that because now the s-variables are generic, the salient features of Macdonald polynomials, such as the Pieri rules and the skew analogue, are lost in the bispectral Macdonald functions.

Now, let us begin with the definition of bispectral Macdonald functions.

4.1.1 Definition

We introduce the formal Macdonald difference operators.

Definition 4.1.1. *For $1 \leq k \leq n$, define the operators on $\mathbb{C}[[x_2/x_1, x_3/x_2 \ldots, x_n/x_{n-1}]]$ by*

$$D_n^{(k)}(\boldsymbol{s}; q, t) := \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=k}} s^{\epsilon_I} \prod_{\substack{i<j \\ i\in I, j\notin I}} \frac{1 - x_j/tx_i}{1 - x_j/x_i} \prod_{\substack{j<i \\ i\in I, j\notin I}} \frac{1 - tx_i/x_j}{1 - x_i/x_j} T_{q,x}^{\epsilon_I}, \tag{4.1.2}$$

where $\boldsymbol{s} = (s_1, \ldots, s_n)$ are indeterminates, and

$$s^{\epsilon_I} = \prod_{i\in I} s_i \,, \qquad T_{q,x}^{\epsilon_I} = \prod_{i\in I} T_{q,x_i} \,. \tag{4.1.3}$$

For later use, we also introduce

$$\widetilde{D_n^{(k)}}(\boldsymbol{s}; q, t) := \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=k}} s^{\epsilon_I} \prod_{\substack{i<j \\ i\in I, j\notin I}} \frac{1 - tx_j/qx_i}{1 - x_j/qx_i} \prod_{\substack{j<i \\ i\in I, j\notin I}} \frac{1 - qx_i/tx_j}{1 - qx_i/x_j} T_{q^{-1},x}^{\epsilon_I} \,. \tag{4.1.4}$$

We use the almost same notation as Definition 3.1.4. In what follows, we forget about the original operators, and use the notation $D_n^{(k)}$ in the sense of this definition.

Note that when we rewrite the factors $\frac{1-x_j/tx_i}{1-x_j/x_i}$ to the form of Definition 3.1.4, some extra powers of t is multiplied to s^{ϵ_I}. Then conjugating by $\boldsymbol{x}^{-\lambda}$, we can go from $D_n^{(k)}((t^{n-i}q^{\lambda_i}); q, t)$ back to the original Macdonald operators.

We now define the bispectral Macdonald functions, following [86, 103].

Definition 4.1.2. *Let $\boldsymbol{x} = (x_i)_{1\leq i\leq n}$ be indeterminates and $\boldsymbol{s} = (s_i)_{1\leq i\leq n}$ be generic parameters. Define $f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q, t)$ and $\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q, t) \in \mathbb{C}[[x_2/x_1, x_3/x_2, \ldots, x_n/x_{n-1}]]$ by*

$$f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q, t) = \sum_{\theta\in M_n} c_n(\theta; \boldsymbol{s}|q, t) \prod_{1\leq i<j\leq n} (x_j/x_i)^{\theta_{i,j}} \,, \tag{4.1.5}$$

$$\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q, t) = \prod_{1\leq k<\ell\leq n} (1 - x_\ell/x_k) \cdot f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q^{-1}, t^{-1}). \tag{4.1.6}$$

We put $f^{\mathfrak{gl}_0} = \widetilde{f}^{\mathfrak{gl}_0} = 1$. Here M_n is the set of all $n \times n$ upper triangular matrices with non-negative integers, whose diagonal elements are 0. $c_n(\theta; \boldsymbol{s}|q, t)$ are coefficients defined by the following recurrence relations:

$$\begin{aligned} &c_1(-; s_1, q, t) = 1 \,, \\ &c_n((\theta_{i,j})_{1\leq i<j\leq n}; (s_i)_{1\leq i\leq n}|q, t) \\ &\quad = d_n\left((\theta_{i,n})_{1\leq i\leq n-1}; (s_i)_{1\leq i\leq n}|q, t\right) \cdot c_{n-1}((\theta_{i,j})_{1\leq i<j\leq n-1}; (q^{-\theta_{i,n}} s_i)_{1\leq i\leq n}|q, t) \,, \end{aligned} \tag{4.1.7}$$

with

$$d_n((\theta_i)_{1\leq i\leq n-1}; (s_i)_{1\leq i\leq n}|q, t) = \prod_{1\leq i<j\leq n} \frac{(ts_j/s_i; q)_{\theta_i}}{(qs_j/s_i; q)_{\theta_i}} \prod_{1\leq i\leq j\leq n-1} \frac{(q^{-\theta_j} qs_j/ts_i; q)_{\theta_i}}{(q^{-\theta_j} s_j/s_i; q)_{\theta_i}}. \tag{4.1.8}$$

From the recurrence relations, the explicit form of c_n can be computed as

$$
\begin{aligned}
&c_n((\theta_{i,j})_{1\le i<j\le n};(s_i)_{1\le i\le n}|q,t)\\
&=\prod_{k=2}^{n}\left(\prod_{1\le i<j\le k}\frac{(q^{\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j,a})}ts_j/s_i;q)_{\theta_{i,k}}}{(q^{\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j,a})}qs_j/s_i;q)_{\theta_{i,k}}}\prod_{1\le i\le j<k}\frac{(q^{-\theta_{j,k}+\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j,a})}qs_j/ts_i;q)_{\theta_{i,k}}}{(q^{-\theta_{j,k}+\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j,a})}s_j/s_i;q)_{\theta_{i,k}}}\right). \qquad (4.1.9)
\end{aligned}
$$

Let us see some examples.

- $n=2,3$ **Examples**

$$
\begin{aligned}
f^{\mathfrak{gl}_2}(\boldsymbol{x}|\boldsymbol{s}|q,t)&=\sum_{\theta\in\mathbb{Z}_{\ge0}}\frac{(t;q)_\theta}{(q;q)_\theta}\frac{(ts_2/s_1;q)_\theta}{(qs_2/s_1;q)_\theta}(qx_2/tx_1)^\theta\,,\\
f^{\mathfrak{gl}_3}(\boldsymbol{x}|\boldsymbol{s}|q,t)&=\sum_{\theta_{1,2},\theta_{1,3},\theta_{2,3}\in\mathbb{Z}_{\ge0}}\frac{(t;q)_{\theta_{1,2}}}{(q;q)_{\theta_{1,2}}}\frac{(q^{\theta_{1,3}-\theta_{2,3}}ts_2/s_1;q)_{\theta_{1,2}}}{(q^{\theta_{1,3}-\theta_{2,3}}qs_2/s_1;q)_{\theta_{1,2}}}\\
&\quad\times\frac{(t;q)_{\theta_{1,3}}}{(q;q)_{\theta_{1,3}}}\frac{(ts_2/s_1;q)_{\theta_{1,3}}}{(qs_2/s_1;q)_{\theta_{1,3}}}\frac{(ts_3/s_1;q)_{\theta_{1,3}}}{(qs_3/s_1;q)_{\theta_{1,3}}}\frac{(q^{-\theta_{2,3}}qs_2/ts_1;q)_{\theta_{1,3}}}{(q^{-\theta_{2,3}}qs_2/s_1;q)_{\theta_{1,3}}}\\
&\quad\times\frac{(t;q)_{\theta_{2,3}}}{(q;q)_{\theta_{2,3}}}\frac{(ts_3/s_2;q)_{\theta_{2,3}}}{(qs_3/s_2;q)_{\theta_{2,3}}}\cdot\prod_{1\le i<j\le3}(qx_j/tx_i)^{\theta_{i,j}}\,.
\end{aligned}
\qquad (4.1.10)
$$

Then, the following fact was conjectured in [103] and proved in [86].

Fact 4.1.3 ([86, 103])**.** *The function $f^{\mathfrak{gl}_n}(\boldsymbol{x};\boldsymbol{s}|q,t)$ is a unique formal solution to the 1st order difference equation*

$$D_n^{(k)}(\boldsymbol{s};q,t)f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t)=e_k(\boldsymbol{s})\cdot f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t)\,, \qquad (4.1.11)$$

up to some constant. e_k is the k-th elementary symmetric polynomial. Further, $\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x};\boldsymbol{s}|q,t)$ is a unique function such that

$$\widetilde{D_n^{(k)}}(\boldsymbol{s};q,t)\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t)=e_k(\boldsymbol{s})\cdot\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t)\,. \qquad (4.1.12)$$

Remark 4.1.4. *Put $\delta=(-1,-2,\dots,-n)$, and $t^{n+\delta}q^\lambda=(t^{n-1}q^{\lambda_1},\dots,q^{\lambda_n})$. It is easy to show*

$$\boldsymbol{x}^\lambda f^{\mathfrak{gl}_n}(\boldsymbol{x}|t^{n+\delta}q^\lambda|q,t)=P_\lambda(\boldsymbol{x}|q,t)\,. \qquad (4.1.13)$$

Note that under this specialization, there exists $n\in\mathbb{N}$ such that for any $m\ge n$, c_m becomes zero.

The proof of Fact 4.1.3 goes as follows. For an arbitrary partition $\lambda\in\mathsf{P}$, the specialization $\boldsymbol{s}=t^{n+\delta}q^\lambda$ degenerates $f^{\mathfrak{gl}_n}$ to the Macdonald polynomial P_λ. Under this specialization, the equation (3.1.13) gives the set of recurrence relations on the coefficients $c_n(\theta;\boldsymbol{s}|q,t)$. Because these recurrence relations hold for any $\boldsymbol{s}=t^{n+\delta}q^\lambda$ with $\lambda\in\mathsf{P}$, they hold as the relations on the rational functions in $\boldsymbol{s}$. Thus the equation (4.1.11) holds.

These functions $f^{\mathfrak{gl}_n}$ and $\widetilde{f}^{\mathfrak{gl}_n}$, are dual to each other. We can easily show the following lemma.

Lemma 4.1.5. *Let λ and μ satisfy $\ell(\lambda),\ell(\mu)\le n$.*

$$\left[x^{-\lambda}x^\mu\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}(\lambda)|q,q/t)f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}(\mu)|q,q/t)\right]_{x,1}=\delta_{\lambda,\mu}\,. \qquad (4.1.14)$$

Here, $\boldsymbol{s}(\lambda)=(s_j(\lambda))_{1\le j\le n}$, $s_k(\lambda)=q^{\lambda_k}t^{1-k}u$.

Proof. We denote the LHS by $F(\lambda|\mu)$. Inserting the Macdonald operator $D_n^1(\boldsymbol{s};q,q/t)$ and integrating by parts, we obtain the equality

$$\begin{aligned}&\left[\boldsymbol{x}^{-\lambda}\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}(\lambda)|q,q/t)\left(D_n^1(\bar{\boldsymbol{s}};q,q/t)\boldsymbol{x}^{\mu}f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}(\mu)|q,q/t)\right)\right]_{\boldsymbol{x},1}\\&=\left[\left(\widetilde{D_n^1}(\bar{\boldsymbol{s}};q,q/t)\boldsymbol{x}^{-\lambda}\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}(\lambda)|q,q/t)\right)\boldsymbol{x}^{\mu}f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}(\mu)|q,q/t)\right]_{\boldsymbol{x},1},\end{aligned}\tag{4.1.15}$$

with $\bar{s}_k = ut^{1-k}$. The LHS becomes $\epsilon_\mu F(\lambda|\mu)$ while the RHS $\epsilon_\lambda F(\lambda|\mu)$. Thus, we have $F(\lambda|\mu) = C(\lambda)\delta_{\lambda,\mu}$ with $C(\lambda) = F(\lambda|\lambda)$. It is easy to show $C(\lambda) = 1$ by noting and both $\widetilde{f}^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}(\lambda)|q,q/t)$ and $f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}(\mu)|q,q/t)$ are in $\mathbb{C}(s_1,\ldots,s_n)[[x_2/x_1,x_3/x_2,\ldots,x_n/x_{n-1}]]$ and so is their product. □

4.1.2 Duality

At this point, in $f^{\mathfrak{gl}_n}$, x-variables and s-variables are still not equal partners. When $x_{i+1}/x_i = 0$ for all i, $f^{\mathfrak{gl}_n} = 1$. On the other hand, $s_{i+1}/s_i = 0$, $f^{\mathfrak{gl}_n}(\boldsymbol{x};0|q,t) \neq 1$. Actually, it is easy to check

$$f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t)\big|_{s_{i+1}/s_i=0} = \sum_{\theta\in M_n}\prod_{1\le i<j\le n}\frac{(t;q)_{\theta_{i,j}}}{(q;q)_{\theta_{i,j}}}(qx_j/tx_i)^{\theta_{i,j}} = \prod_{1\le i<j\le n}\frac{(qx_j/x_i;q)_\infty}{(qx_j/tx_i;q)_\infty}\,.\tag{4.1.16}$$

In the last equality, we use the q-binomial formula,

$$\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n\,.\tag{4.1.17}$$

Thus, by dividing by this factor we obtain the better version of the bispectral Macdonald functions.

Definition 4.1.6. *Define the function* $\varphi^{\mathfrak{gl}_n}(\boldsymbol{x};\boldsymbol{s}|q,t)\in\mathbb{C}[[s^{-Q_+}]][[x^{-Q_+}]]$ *by*

$$\varphi^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t) = \prod_{1\le i<j\le n}\frac{(qx_j/tx_i;q)_\infty}{(qx_j/x_i;q)_\infty}\cdot f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t)\,.\tag{4.1.18}$$

Then, the following fact shows the equivalence between $\boldsymbol{x}$ and $\boldsymbol{s}$.

Fact 4.1.7 ([86])**.** *We have the following dualities:*

$$\varphi^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t) = \varphi^{\mathfrak{gl}_n}(\boldsymbol{s}|\boldsymbol{x}|q,t)\,,\quad \textit{(The bispectral duality)}\,,\tag{4.1.19}$$

$$\varphi^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t) = \varphi^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,q/t)\,,\quad \textit{(The Poincaré duality)}\,.\tag{4.1.20}$$

The proof of the bispectral duality is straightforward. It is easy to see the action of $D_n^{(k)}(\boldsymbol{x};q,t)$ (note that this is the operator which shift the $\boldsymbol{s}$-varibales!) on $\varphi^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t)$ is controllable, and we can show they act diagonally on $\varphi^{\mathfrak{gl}_n}$. This means in $\varphi^{\mathfrak{gl}_n}$, we can regard $\boldsymbol{s}$-variables as the main variables while $\boldsymbol{x}$-variables as the accessory parameters.

The Poincaré duality is also easy to check, once we note the prefactor $\prod_{1\le i<j\le n}\frac{(qx_j/tx_i;q)_\infty}{(qx_j/x_i;q)_\infty}$ interpolate $D_n^{(k)}(\boldsymbol{s};q,t)$ to $D_n^{(k)}(\boldsymbol{s};q,q/t)$, and $\prod_{1\le i<j\le n}\frac{(tx_j/x_i;q)_\infty}{(qx_j/x_i;q)_\infty}$ does its inverse. The reason we call this duality the Poincaré duality is following. As studied in [19], the Macdonald functions are the generating functions of the Euler characteristics of the Laumon space, that is, the generating functions of some cohomology class of the (twisted) de Rham complex. Then, there exists the Poincaré duality in the usual sense, and in terms of the generating functions, this duality is interpreted as (4.1.19).

Remark 4.1.8. *The operators $D_n^{(k)}$ act on $\mathbb{C}[[x^{-Q_+}]]$, where $-Q_+$ stands for the negative root lattice of type A_n. The coefficients of each $x^\mu = \prod_{i=1}^n x_i^{\mu_i}$ $(\mu\in -Q_+)$ in $f^{\mathfrak{gl}_n}(\boldsymbol{x};\boldsymbol{s}|q,t)$ are in $\mathbb{C}(s^{-Q_+})$.*

By Fact 4.1.3, we know that the dependence of s-variables should be associated with the coroot lattice. (The x-variables are associated with the eigenfunctions while s-variables are associated with their eigenvalues.) Thus, for general root systems, the coefficients of each monomials are in $\mathbb{C}(s^{-Q_+^\vee})$. This implies that under the bispectral duality, the Macdonald functions transform to those associated with the Langlands dual root system.

4.1.3 Analytic Property

Definition 4.1.9. *For $a = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$ define the projection for the canonical coordinates $\pi_n : (\mathbb{C}^*)^n \to \mathbb{C}^{n-1}$ by*

$$\pi_n(a) = (a_2/a_1, \dots, a_n/a_{n-1}) \in \mathbb{C}^{n-1}\,. \tag{4.1.21}$$

Notation 4.1.10. *For later use in Section 5.4, we prepare some notations.*

$$\begin{aligned}
z &:= (z_1, \dots, z_{|\boldsymbol{n}|-1}) = \pi_{|\boldsymbol{n}|}((x_1, \dots, x_{|\boldsymbol{n}|}))\,,\\
\tilde{z} &:= (\tilde{z}_1, \dots, \tilde{z}_{|\boldsymbol{n}|+|\boldsymbol{m}|-1}) = \pi_{|\boldsymbol{n}|+|\boldsymbol{m}|}((x_1, \dots, y_1, \dots))\,,\\
w &:= (w_1, \dots, w_{|\boldsymbol{n}|-1}) = \pi_{|\boldsymbol{n}|}((s_1, \dots, s_{|\boldsymbol{n}|}))\,,\\
\tilde{w} &:= (\tilde{w}_1, \dots, \tilde{w}_{|\boldsymbol{n}|+|\boldsymbol{m}|-1}) = \pi_{|\boldsymbol{n}|+|\boldsymbol{m}|}((s_1, \dots, s_{|\boldsymbol{n}|+|\boldsymbol{m}|}))\,.
\end{aligned}$$

Definition 4.1.11. *Define the open subset $D_w \subset \mathbb{C}^{n-1}$ by*

$$D_w^n = \{w = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1} \big|\ w_i \cdots w_{j-1} \notin q^{-\mathbb{Z}} \cup \{0\},\ 1 \le i < j \le n\}, \tag{4.1.22}$$

so that

$$\pi^{-1}(D_w^n(r)) = \{s = (s_1, \dots, s_n) \in (\mathbb{C}^*)^n \big|\ s_j/s_i \notin q^{-\mathbb{Z}},\ 1 \le i < j \le n\}\,. \tag{4.1.23}$$

Definition 4.1.12. *Define the subsets $U_z^n(r), B_z^n(r) \subset \mathbb{C}^{n-1}$ by*

$$U_z^n(r) = \{z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1} \big|\ |z_i| < r,\ i = 1, \dots, n-1\}\,, \tag{4.1.24}$$

$$B_z^n(r) = \{z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1} \big|\ |z_i| \le r,\ i = 1, \dots, n-1\}\,, \tag{4.1.25}$$

so that

$$\pi^{-1}(B_z^n(r)) = \{x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n \big|\ |x_j/x_i| \le r^{j-i},\ 1 \le i < j \le n\}\,. \tag{4.1.26}$$

We make use of the following fact, proved in [86]. This plays the central role in the first step of the proof of the main theorem. (See Step 1 of Section 5.4.)

Fact 4.1.13 ([86])**.** *Let τ be a generic complex parameter, and let $\mathcal{O}(D_w^n)$ be the ring of holomorphic functions on D_w^n For $n = 2, 3, \dots$, we regard $f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q, \tau)$ as a formal power series in $z = (z_1, \dots, z_{n-1})$ with coefficients in $\mathcal{O}(D_w^n)$, that is,*

$$f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q, \tau) = \sum_{\theta \in M_n} c_n(\theta; \boldsymbol{s}|q, \tau) \prod_{1 \le i < j \le n} (x_j/x_i)^{\theta_{i,j}} \ \in \mathcal{O}(D_w^n)[[z]]. \tag{4.1.27}$$

Put $r_0 = |q/\tau|^{\frac{n-2}{n-1}}$ if $|q/\tau| \le 1$, and $r_0 = |\tau/q|$ if $|q/\tau| \ge 1$. Then for any compact subset $K \subset D_w$ and for any $r < r_0$, this series 4.1.27 is absolutely convergent, uniformly on $B_z^n(r) \times K$. Thus $f^{\mathfrak{gl}_n}(\boldsymbol{x}; \boldsymbol{s}|q, \tau)$ is a holomorphic function on $U_z^n(r_0) \times D_w^n$.

4.1.4 Factorization Formula

For completeness, we see the factorization of $f^{\mathfrak{gl}_n}$ under the specialization,

$$x_i \to t^{n-i}\,. \tag{4.1.28}$$

We refer to this specialization as the principle specialization. Then, we have the following fact.

Fact 4.1.14 (Principle Specialization [86])**.** *Let $|t| > |q|^{-(n-2)}$. Then, $f^{\mathfrak{gl}_n}$ factorizes as*

$$f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q, t)\Big|_{x_i \to t^{n-i}} = \prod_{i=1}^{n} \frac{(q/t; q)_\infty}{(q/t^i; q)_\infty} \cdot \prod_{1 \le i < j \le n} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty}. \tag{4.1.29}$$

Note that this is the natural generalization of

$$t^{-||\nu'||/2} P_\nu(t^{-\rho}; q, t) = \frac{1}{c_\nu}\,, \tag{4.1.30}$$

in the usual Macdonald polynomials [69].

4.2 Generalized Macdonald Functions on Fock Tensor Space

In this section, we introduce what is called the generalized Macdonald functions on the Fock tensor spaces. Roughly speaking, these states are the boson realization of the bispectral Macdonald functions on the Fock tensor spaces. Throughout this section, N stands for the number of the Fock tensor. The story goes completely in parallel with that in one Fock space (Section 3.2).

We first introduce some notation to simplify the symbols to express the Fock tensor space.

Definition 4.2.1. *Let $N \in \mathbb{Z}_{\geq 1}$, and let $\boldsymbol{u} = (u_1, \dots, u_N) \in (\mathbb{C}^\times)^N$ be an N-tuple of parameters. Define the $(N,0)$-representation by*

$$\rho_{\boldsymbol{u}}^{(N,0)} := (\rho_{u_1} \otimes \rho_{u_2} \otimes \cdots \otimes \rho_{u_N}) \circ \Delta^{(N)}, \tag{4.2.1}$$
$$\text{where } \Delta^{(1)} := \mathrm{id},\ \Delta^{(N)} := (\underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{N-2} \otimes \Delta) \circ \cdots \circ (\mathrm{id} \otimes \Delta) \circ \Delta \ (N \geq 2).$$

The $\rho_{\boldsymbol{u}}^{(N,0)}$ endows the $(N,0)$-module structure to the N-th tensor space of $\mathcal{F}$. We denote it by $\mathcal{F}_{u_1}^{(1,0)} \otimes \cdots \otimes \mathcal{F}_{u_N}^{(1,0)}$. For brevity of notation, we introduce the notation $\mathcal{F}_{\boldsymbol{u}} = \mathcal{F}_{u_1}^{(1,0)} \otimes \cdots \otimes \mathcal{F}_{u_N}^{(1,0)}$ for simplicity. We also denote $\mathcal{F}_{\boldsymbol{u}}^* = \mathcal{F}_{u_1}^{(1,0)*} \otimes \cdots \otimes \mathcal{F}_{u_N}^{(1,0)*}$. We denote the vacuum (resp. dual vacuum) states of $\mathcal{F}_{\boldsymbol{u}}$ (resp. $\mathcal{F}_{\boldsymbol{u}}^*$) by $|\mathbf{0}\rangle$ (resp. $\langle \mathbf{0}|$), that is, the N-the tensor of the vacuum $|\mathbf{0}\rangle = |0\rangle^{\otimes N}$ (resp. $\langle \mathbf{0}| = \langle 0|^{\otimes N}$). We also introduce the notation,

$$a_n^{(i)} = \overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes a_n \otimes \overbrace{1 \otimes \cdots \otimes 1}^{N-i}. \tag{4.2.2}$$

Now we introduce some facts necessary to define the Macdonald operator on the Fock tensor space.

Fact 4.2.2 ([30])**.** *On $\mathcal{F}_{\boldsymbol{u}}$, we have*

$$\rho_{\boldsymbol{u}}^{(N,0)}(x^+(z)) = \sum_{i=1}^{N} u_i \Lambda^{(i)}(z)\,, \tag{4.2.3}$$

where, for $k = 1, 2, \dots, N$, we put

$$\Lambda^{(i)}(z) := \varphi^-(\gamma^{1/2} z) \otimes \cdots \otimes \varphi^-(\gamma^{i-3/2} z) \otimes \overbrace{\eta(\gamma^{i-1} z)}^{i\text{-th Fock space}} \otimes 1 \otimes \cdots \otimes 1\,, \tag{4.2.4}$$

with $\gamma = (t/q)^{1/2}$.

Definition 4.2.3. *Set $X^{(1)}(z) = \rho_{\boldsymbol{u}}^{(N,0)}(x^+(z))$. Define the set of generators $X_n^{(k)}$ ($k = 1, \dots, N, n \in \mathbb{Z}$) by the fusion of several $X^{(1)}$'s as*

$$X^{(k)}(z) = \sum_{n \in \mathbb{Z}} X_n^{(k)} z^{-n} = X^{(1)}(\gamma^{2(1-k)} z) X^{(1)}(\gamma^{2(2-k)} z) \cdots X^{(1)}(z) \in \mathrm{End}(\mathcal{F}_{\boldsymbol{u}})[[z^{\pm 1}]]\,. \tag{4.2.5}$$

We can massage the currents $X^{(k)}(z)$ to simpler forms. This is summarized in the following fact.

Fact 4.2.4 ([8])**.** *We have*

$$X^{(k)}(z) = \sum_{1 \leq j_1 < \cdots < j_k \leq N} :\Lambda^{(j_1)}(z) \cdots \Lambda^{(j_k)}((q/t)^{k-1} z): u_{j_1} \cdots u_{j_k}. \tag{4.2.6}$$

Definition 4.2.5. *We denote by $\mathsf{U}(N)$ (the completion in the sense of the adic topology, of) the algebra $\langle X_n^{(i)} | n \in \mathbb{Z}, i = 1, \dots N \rangle$ in $\mathrm{End}(\mathcal{F}_{\boldsymbol{u}})$. That is, $\mathsf{U}(N)$ is the completion of the algebra generated by the set of operators $\left\{ X_n^{(i)} \right\}$.*

This algebra $\mathsf{U}(N)$ can be thought of as the q-deformed $\mathcal{W}_N$ algebra, combined with some Heisenberg algebra [30]. This will be reviewed in Section 4.3.

Now we introduce a candidate for the basis of the Fock tensor space.

Definition 4.2.6. *For an N-tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(N)}) \in \mathsf{P}^N$, we define $|X_{\boldsymbol{\lambda}}\rangle = |X_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle \in \mathcal{F}_{\boldsymbol{u}}$ and $\langle X_{\boldsymbol{\lambda}}| = \langle X_{\boldsymbol{\lambda}}(\boldsymbol{u})| \in \mathcal{F}^*_{\boldsymbol{u}}$ by*

$$|X_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle := X^{(1)}_{-\lambda^{(1)}_1} X^{(1)}_{-\lambda^{(1)}_2} \cdots X^{(2)}_{-\lambda^{(2)}_1} X^{(2)}_{-\lambda^{(2)}_2} \cdots X^{(N)}_{-\lambda^{(N)}_1} X^{(N)}_{-\lambda^{(N)}_2} \cdots |\mathbf{0}\rangle \,, \tag{4.2.7}$$

$$\langle X_{\boldsymbol{\lambda}}(\boldsymbol{u})| := \langle \mathbf{0}| \cdots X^{(N)}_{\lambda^{(N)}_2} X^{(N)}_{\lambda^{(N)}_1} \cdots X^{(2)}_{\lambda^{(2)}_2} X^{(2)}_{\lambda^{(2)}_1} \cdots X^{(1)}_{\lambda^{(1)}_2} X^{(1)}_{\lambda^{(1)}_1} . \tag{4.2.8}$$

In what follows, we omit the spectral parameter $\boldsymbol{u}$, unless there is any confusion.

The next fact shows they form the basis of the Fock tensor space.

Fact 4.2.7 ([89])**.** *The set $(|X_{\boldsymbol{\lambda}}\rangle)$ (resp. $(\langle X_{\boldsymbol{\lambda}}|)$) forms a PBW-type basis of $\mathcal{F}_{\boldsymbol{u}}$ (resp. $\mathcal{F}^*_{\boldsymbol{u}}$), if $u_i \neq q^s t^{-r} u_j$ and $u_i \neq 0$ for all i, j and $r, s \in \mathbb{Z}$.*

The proof of this fact owes to the Kac determinant formula. See Proposition 4.3.19 for more detail.

4.2.1 Definition

To define the generalized Macdonald functions, we first generalize the dominance ordering to the N-tuple of partition.

Definition 4.2.8. *The partial ordering $\overset{*}{>}$ on the set of N-tuples of partitions is defined by*

$$\begin{aligned} \boldsymbol{\lambda} \overset{*}{>} \boldsymbol{\mu} \quad \overset{\text{def}}{\Longleftrightarrow} \quad & |\boldsymbol{\lambda}| = |\boldsymbol{\mu}|, \quad \sum_{i=k}^{N} |\lambda^{(i)}| \geq \sum_{i=k}^{N} |\mu^{(i)}| \quad (\forall k) \quad \text{and} \\ & (|\lambda^{(1)}|, |\lambda^{(2)}|, \dots, |\lambda^{(N)}|) \neq (|\mu^{(1)}|, |\mu^{(2)}|, \dots, |\mu^{(N)}|). \end{aligned} \tag{4.2.9}$$

Once we adapt this ordering, when we move boxes in a right Young diagram to one in its left, it gets smaller. Inside one Young diagram, the ordering is compatible with the dominance ordering.

Fact 4.2.9 (Existence and Uniqueness [90])**.** *For an N-tuple of partitions $\boldsymbol{\lambda}$, there exists a unique vector $|P_{\boldsymbol{\lambda}}\rangle = |P_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle \in \mathcal{F}_{\boldsymbol{u}}$ such that*

$$|P_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle = \prod_{i=1}^{N} P_{\lambda^{(i)}}(a^{(i)}_{-n}) \, |\mathbf{0}\rangle + \sum_{\boldsymbol{\mu} \overset{*}{<} \boldsymbol{\lambda}} u_{\boldsymbol{\lambda},\boldsymbol{\mu}} \prod_{i=1}^{N} P_{\mu^{(i)}}(a^{(i)}_{-n}) \, |\mathbf{0}\rangle \,, \quad u_{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \mathbb{C}; \tag{4.2.10}$$

$$X^{(1)}_0 \, |P_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle = \epsilon_{\boldsymbol{\lambda}}(\boldsymbol{u}) \, |P_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle \,, \quad \epsilon_{\boldsymbol{\lambda}}(\boldsymbol{u}) \in \mathbb{C}, , \tag{4.2.11}$$

with the eigenvalues

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{u}) = \sum_{k=1}^{N} u_k \epsilon_{\lambda^{(k)}} \,, \quad \epsilon_{\lambda} := (1 - t^{-1}) \sum_{i=1}^{\ell(\lambda)} q^{\lambda_i} t^{1-i} + t^{-\ell(\lambda)} \,. \tag{4.2.12}$$

Similarly, there exists a unique vector $\langle P_{\boldsymbol{\lambda}}| = \langle P_{\boldsymbol{\lambda}}(\boldsymbol{u})| \in \mathcal{F}^*_{\boldsymbol{u}}$ such that

$$\langle P_{\boldsymbol{\lambda}}(\boldsymbol{u})| = \langle \mathbf{0}| \prod_{i=1}^{N} P_{\lambda^{(i)}}(a^{(i)}_{n}) + \sum_{\boldsymbol{\mu} \overset{*}{>} \boldsymbol{\lambda}} u^*_{\boldsymbol{\lambda},\boldsymbol{\mu}} \, \langle \mathbf{0}| \prod_{i=1}^{N} P_{\mu^{(i)}}(a^{(i)}_{n}), \quad u^*_{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \mathbb{C}; \tag{4.2.13}$$

$$\langle P_{\boldsymbol{\lambda}}(\boldsymbol{u})| \, X^{(1)}_0 = \epsilon^*_{\boldsymbol{\lambda}}(\boldsymbol{u}) \, \langle P_{\boldsymbol{\lambda}}(\boldsymbol{u})| \,, \quad \epsilon^*_{\boldsymbol{\lambda}}(\boldsymbol{u}) \in \mathbb{C} \,, \tag{4.2.14}$$

with $\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{u}) = \epsilon^*_{\boldsymbol{\lambda}}(\boldsymbol{u})$. Again, we omit $\boldsymbol{u}$ unless mentioned otherwise.

Remark 4.2.10. *Throughout this thesis, we assume that $q, t \in \mathbb{C}^\times$ and the spectral parameters $\boldsymbol{u}$ are generic in the following sense:*

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{u}) \neq 0 \quad (\forall \boldsymbol{\lambda}); \tag{4.2.15}$$

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{u}) \neq \epsilon_{\boldsymbol{\mu}}(\boldsymbol{u}) \quad (\boldsymbol{\lambda} \neq \boldsymbol{\mu}); \tag{4.2.16}$$

$$u_i \neq q^n t^m u_j \quad (n, m \in \mathbb{Z},\, i, j = 1, \dots, N). \tag{4.2.17}$$

Fact 4.2.11 ([8])**.** *$|P_{\boldsymbol{\lambda}}\rangle$ and $\langle P_{\boldsymbol{\lambda}}|$ satisfy*

$$\langle P_{\boldsymbol{\lambda}} | P_{\boldsymbol{\mu}} \rangle = \prod_{i=1}^{N} \frac{c'_{\lambda^{(i)}}}{c_{\lambda^{(i)}}} \delta_{\boldsymbol{\lambda}, \boldsymbol{\mu}}. \tag{4.2.18}$$

That is, they form the orthogonal basis. c_λ, c'_λ are defined in (3.1.21).

Definition 4.2.12. *Define $|Q_{\boldsymbol{\lambda}}\rangle$ as the states in $\mathcal{F}_{\boldsymbol{u}}$ which satisfy $\langle P_{\boldsymbol{\lambda}} | Q_{\boldsymbol{\mu}} \rangle = \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, i.e., $|Q_{\boldsymbol{\lambda}}\rangle := \prod_{i=1}^{N} \frac{c_{\lambda^{(i)}}}{c'_{\lambda^{(i)}}} |P_{\boldsymbol{\lambda}}\rangle$.*

4.2.2 Explicit Formula for Generalized Macdonald Functions

Now we give the explicit construction of the generalized Macdonald functions, using some vertex operators. Unlike the one Fock space case in Section 3.2, we need to put "colors" to each vertex operators to distinguish the Fock space on which the operator acts. This is achieved by combining the bare vertex operators with the screening currents of q-deformed $\mathcal{W}$-algebra. In this section, we use some results in the next section (Section 4.3), where the screening currents and the algebra themselves are reviewed.

Notation 4.2.13. *For an N-tuple of parameters $\boldsymbol{u} = (u_1, \dots, u_N)$, we prepare the following notations:*

$$t^{\pm\delta_i} \cdot \boldsymbol{u} := (u_1, \dots, u_{i-1}, t^{\pm 1} u_i, u_{i+1}, \dots, u_N)\,, \tag{4.2.19}$$

$$t^{\alpha_i} \cdot \boldsymbol{u} := (u_1, \dots, u_{i-1}, t u_i, t^{-1} u_{i+1}, u_{i+2} \dots, u_N)\,. \tag{4.2.20}$$

For $\boldsymbol{n} = (n_1, \dots, n_N) \in (\mathbb{Z}_{\geq 0})^N$, we use the following notation:

$$|\boldsymbol{n}| := \sum_{s=1}^{N} n_s, \qquad [i,k] = [i,k]_{\boldsymbol{n}} := \sum_{s=1}^{i-1} n_s + k \quad (1 \leq i \leq N, \leq k \leq n_i), \tag{4.2.21}$$

$$t^{\pm \boldsymbol{n}} \cdot \boldsymbol{u} := (t^{\pm n_1} u_1, \dots, t^{\pm n_N} u_N). \tag{4.2.22}$$

We first define the analogue of the top component in Definition 3.2.5. Note that when $N = 1$, the following top component reduces to the operator in Definition 3.2.5.

Definition 4.2.14. *Define the vertex operators $\Phi^{(0)}(x) : \mathcal{F}_{t^{-\delta_1} \cdot \boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{u}}$ by*

$$\begin{aligned} \Phi^{(0)}(x) = \, : \exp\left(\sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} a_{-n}^{(1)} x^n \right) \exp\left(\sum_{n>0} \frac{1}{n} \frac{1-\gamma^{2n} t^n}{1-q^{-n}} t^{-n} a_n^{(1)} x^{-n} \right) \\ \times \exp\left(\sum_{n>0} \frac{1}{n} \frac{1-\gamma^{2n}}{1-q^{-n}} \sum_{j=2}^{N} \gamma^{(j-1)n} a_n^{(j)} x^{-n} \right) : \, . \end{aligned} \tag{4.2.23}$$

We refer to $\Phi^{(0)}(z)$ as the top component.

Next, we introduce the screening currents.

Definition 4.2.15. *Define the screening currents $S^{(i)}(y) : \mathcal{F}_{t^{\alpha_i} \cdot \boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{u}}$ by*

$$S^{(i)}(z) := \overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes \phi^{\mathrm{sc}}(\gamma^{i-1} z) \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-i-1}, \qquad i = 1, \dots, N-1\,, \tag{4.2.24}$$

with

$$\begin{aligned}\phi^{\mathrm{sc}}(z) := &\exp\left(-\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}\gamma^{2n}a_{-n}z^n\right)\exp\left(\sum_{n>0}\frac{1}{n}\frac{1-t^{-n}}{1-q^{-n}}a_n z^{-n}\right)\\ &\otimes\exp\left(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}\gamma^{n}a_{-n}z^n\right)\exp\left(-\sum_{n>0}\frac{1}{n}\frac{1-t^{-n}}{1-q^{-n}}\gamma^{-n}a_n z^{-n}\right).\end{aligned} \tag{4.2.25}$$

The screening operators which define the q-$\mathcal{W}$ algebra, are defined by the integral of these screening currents. The existence of such screening operators is proved in the next section.

Now, we define the screened vertex operators, using these screening currents.

Definition 4.2.16. *Define the screened vertex operators $\Phi^{(k)}(x) : \mathcal{F}_{t^{-\delta_{k+1}}\cdot\boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{u}}$ $(k = 1,\dots,N-1)$ by*

$$\Phi^{(k)}(x) := \prod_{i=1}^{k}\frac{(q;q)_\infty(q/t;q)_\infty}{(\frac{qu_i}{u_{k+1}};q)_\infty(\frac{qu_{k+1}}{tu_i};q)_\infty}\cdot\oint_C\prod_{i=1}^{k}\frac{dy_i}{2\pi\sqrt{-1}y_i}\Phi^{(0)}(x)S^{(1)}(y_1)\cdots S^{(k)}(y_k)g(x,y_1,\dots,y_k), \tag{4.2.26}$$

with an integral kernel

$$g(x,y_1,\dots,y_k) = \frac{\theta_q(tu_1y_1/u_{k+1}x)}{\theta_q(ty_1/x)}\prod_{i=1}^{k-1}\frac{\theta_q(tu_{i+1}y_{i+1}/u_{k+1}y_i)}{\theta_q(ty_{i+1}/y_i)}. \tag{4.2.27}$$

Here, the contour of the integration C is chosen so that $|t^{-1}| < |y_j/y_i| < |q|$ for $1 \le i < j \le k$, and $|q/t| < |y_i/x| < 1$ for $i \ge 1$.

The first remark is that the integration contour is well-defined if $|t^{-1}| < |q^{N-2}|$. The second remark is following. As we will see in Proposition 4.3.4 below, we originally have infinite possible choices of $g(x,y_1,\dots,y_k)$ because the only requirement on g is it satisfies

$$T_{q,y_i}\,g(x,y_1,\dots,y_k) = \frac{u_{i+1}}{tu_i}g(x,y_1,\dots,y_k). \tag{4.2.28}$$

The reason we make the special choice of g as in (4.2.27), is that we have to make the screened vertex operators $\Phi^{(k)}$ well-defined. That is, the action of these operators on the vacuum in the Fock tensor space must be well-defined. We make this state more precisely in the following remark.

Remark 4.2.17. *The screened vertex operator $\Phi^{(k)}(x)$ is normalized such that $\oint\frac{dx}{2\pi\sqrt{-1}x}\Phi^{(k)}(x)\,|\mathbf{0}\rangle = |\mathbf{0}\rangle$. Indeed, $\Phi^{(k)}(x)$ can be expanded as*

$$\Phi^{(k)}(y_0) = \oint_C\prod_{i=1}^{k}\frac{dy_i}{2\pi\sqrt{-1}y_i}\sum_{r_1,\dots,r_k\in\mathbb{Z}}\prod_{i=1}^{k}\frac{(tu_i/u_{k+1};q)_{r_i}}{(qu_i/u_{k+1};q)_{r_i}}(y_i/y_{i-1})^{r_i} : \Phi^{(0)}(y_0)S^{(1)}(y_1)\cdots S^{(k)}(y_k):. \tag{4.2.29}$$

This expression follows from the normal ordering formulas (B.1.14)-(B.1.22) and Ramanujan's ${}_1\psi_1$ summation formula (Fact 4.2.18 below). By (4.2.29), we can show that the coefficient of $y_0^{r_1}$ in the expansion of $\Phi^{(0)}(y_0)\,|\mathbf{0}\rangle$ is given by a finite sum and the constant term with respect to y_0 is 1.

Fact 4.2.18 ([36], Section 5)**.**

$${}_1\psi_1(a;b;q;z) := \sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n = \frac{(q;q)_\infty(b/a;q)_\infty(az;q)_\infty(q/az;q)_\infty}{(b;q)_\infty(q/a;q)_\infty(z;q)_\infty(b/az;q)_\infty}\quad(|b/a| < z < 1). \tag{4.2.30}$$

For later use, we introduce a useful notation and the composition of these screened vertices.

Notation 4.2.19.

$$\overset{\curvearrowright}{\prod_{n\le i\le m}} A_i := A_n \times A_{n+1}\times\cdots\times A_m. \tag{4.2.31}$$

Definition 4.2.20. *Let $\boldsymbol{n} = (n_1, \dots, n_N)$ be an N-tuple of non-negative integers. Define the operator $V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) : \mathcal{F}_{t^{-\boldsymbol{n}}\cdot\boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{u}}$ by*

$$V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) = \prod^{\curvearrowright}_{1 \le i_1 \le n_1} \Phi^{(0)}(x_{[1,i_1]_{\boldsymbol{n}}}) \cdot \prod^{\curvearrowright}_{1 \le i_2 \le n_2} \Phi^{(1)}(x_{[2,i_2]_{\boldsymbol{n}}}) \cdots \prod^{\curvearrowright}_{1 \le i_N \le n_N} \Phi^{(N-1)}(x_{[N,i_N]_{\boldsymbol{n}}}) \,. \tag{4.2.32}$$

When it is clear, we omit the dependence on the spectral parameters as $V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) = V^{(\boldsymbol{n})}\left(\begin{matrix} \boldsymbol{u} \\ t^{-\boldsymbol{n}} \cdot \boldsymbol{u} \end{matrix} ; x_1, \dots, x_{|\boldsymbol{n}|} \right)$.

Now, we are equipped enough to state the algorithm to construct the generalized Macdonald functions on the Fock tensor space. The basic idea is same as Theorem 3.2.6. The only difference is that we use $\Phi^{(k-1)}$, instead of the top component, to add a horizontal strip to the Young diagram in the k-th Fock space.

Theorem 4.2.21. *Let $\boldsymbol{n} = (n_1, \dots, n_N)$ be an N-tuple of integers satisfying $n_i \ge \ell(\lambda^{(i)})$ for all i. Once we put $s_{[i,k]_{\boldsymbol{n}}} = q^{\lambda^{(i)}_k} t^{1-k} u_i$ $(1 \le k < n_i,\ i = 1, \dots, N)$, we have the following formula:*

$$\left[\boldsymbol{x}^{-\lambda} \widetilde{f}^{\mathfrak{gl}_{|\boldsymbol{n}|}}(\boldsymbol{x}|\boldsymbol{s}|q, q/t) V^{(\boldsymbol{n})}\left(\begin{matrix} \boldsymbol{u} \\ t^{-\boldsymbol{n}} \cdot \boldsymbol{u} \end{matrix} ; x_1, \dots, x_{|\boldsymbol{n}|} \right) |\boldsymbol{0}\rangle \right]_{\boldsymbol{x},1} = \mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{u}) \, |Q_{\boldsymbol{\lambda}}\rangle \ , \tag{4.2.33}$$

with

$$\boldsymbol{x}^{-\boldsymbol{\lambda}} := \prod_{i=1}^{N} \prod_{k=1}^{n_i} x_{[i,k]_{\boldsymbol{n}}}^{-\lambda^{(i)}_k} . \tag{4.2.34}$$

$[\cdots]_{\boldsymbol{x},1}$ means to take the constant term in $\cdots$ with respect to $\boldsymbol{x}$, and $\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{u}) \in \mathbb{C}(u_1, \dots, u_N)$ is some coefficient.

Equivalently, we have

$$\boldsymbol{x}^{-\boldsymbol{\lambda}} \langle P_{\boldsymbol{\lambda}} | \, V^{(\boldsymbol{n})}\left(\begin{matrix} \boldsymbol{u} \\ t^{-\boldsymbol{n}} \cdot \boldsymbol{u} \end{matrix} ; x_1, \dots, x_{|\boldsymbol{n}|} \right) |\boldsymbol{0}\rangle = \mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{u}) \, f^{\mathfrak{gl}_{|\boldsymbol{n}|}}(\boldsymbol{x}|\boldsymbol{s}|q, q/t) \,. \tag{4.2.35}$$

The following proposition gives the explicit form of $\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{u})$.

Proposition 4.2.22. *The coefficient $\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{u})$ is of the form,*

$$\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{u}) = \gamma^{\sum_{i=1}^{N}(i-1)|\lambda^{(i)}|} \prod_{k=2}^{N} \prod_{i=1}^{n_k} \prod_{l=1}^{k-1} \frac{(t^{-n_l+i} u_l/u_k; q)_{-\lambda^{(k)}_i}}{(q t^{-n_l+i-1} u_l/u_k; q)_{-\lambda^{(k)}_i}} \,. \tag{4.2.36}$$

Proof. By directly expanding $V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) |\boldsymbol{0}\rangle$ with respect to x_i, and taking the constant terms, we can prove the formula. Note that we only need to compare the coefficient of the term $\prod_i P_{\lambda^{(i)}}(a^{(i)}_{-n}) |\boldsymbol{0}\rangle$, which is the top term with respect to the ordering Definition 4.2.8, because by definition, the coefficient of the top term must be 1 if we get rid of $\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{u})$. See [35] for the details. ☐

Examples

We show the examples with $N = 2$ and $|\boldsymbol{\lambda}| \le 2$.

$$\begin{aligned} |P_{(\square,\emptyset)}\rangle &= |P_{\square}\rangle \otimes |P_{\emptyset}\rangle \ , \\ |P_{(\emptyset,\square)}\rangle &= |P_{\emptyset}\rangle \otimes |P_{\square}\rangle - \frac{1}{1 - u_1/u_2} |P_{\square}\rangle \otimes |P_{\emptyset}\rangle \ , \end{aligned} \tag{4.2.37}$$

$$
\begin{aligned}
|P_{(\square\square,\emptyset)}\rangle &= |P_{\square\square}\rangle \otimes |P_\emptyset\rangle \ , \\
|P_{(\boxminus,\emptyset)}\rangle &= |P_{\boxminus}\rangle \otimes |P_\emptyset\rangle \ , \\
|P_{(\square,\square)}\rangle &= |P_\square\rangle \otimes |P_\square\rangle - |P_{\square\square}\rangle \otimes |P_\emptyset\rangle - \frac{1+t}{1-u_1/tu_2} |P_{\boxminus}\rangle \otimes |P_\emptyset\rangle \ , \\
|P_{(\emptyset,\square\square)}\rangle &= |P_\emptyset\rangle \otimes |P_{\square\square}\rangle - \frac{1}{1-u_1/u_2} |P_{\square\square}\rangle \otimes |P_\emptyset\rangle \ , \\
|P_{(\emptyset,\boxminus)}\rangle &= |P_\emptyset\rangle \otimes |P_{\boxminus}\rangle - \frac{1}{1-tu_1/u_2} |P_\square\rangle \otimes |P_\square\rangle + \frac{1}{1-tu_1/u_2} |P_{\square\square}\rangle \otimes |P_\emptyset\rangle \\
&\quad + \frac{t}{(1-u_1/u_2)(1-tu_1/u_2)} |P_{\boxminus}\rangle \otimes |P_\emptyset\rangle \ .
\end{aligned}
\tag{4.2.38}
$$

Proof of Theorem 4.2.21

For the proof, we prepare the following proposition.

Proposition 4.2.23. *For $k = 0, 1, \dots, N-1$ and $r = 1, \dots, N$,*

$$
X^{(r)}(z)\Phi^{(k)}(x) - \frac{1-(q/t)^r z/tx}{1-z/tx}\Phi^{(k)}(x)X^{(r)}(z) = u_{k+1}(1-t^{-1})\delta(tx/z)Y^{(r)}(x)\Phi^{(k)}(qx)\Psi^+(x), \tag{4.2.39}
$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ is the formal delta function, and we defined

$$
Y^{(r)}(x) := \sum_{2\le i_2<\cdots<i_r\le N} :\Lambda^{(i_2)}((q/t)tx)\cdots\Lambda^{(i_r)}((q/t)^{r-1}tx): u_{i_2}\cdots u_{i_r}, \tag{4.2.40}
$$

$$
\Psi^+(z) := \exp\left(\sum \frac{1}{n}(1-\gamma^{2n})\sum_{j=1}^N \gamma^{(j-1)n} a_n^{(j)} z^{-n}\right) = \prod_{k=0}^\infty \frac{1}{\rho^{(N)}(\psi^+(\gamma^{-1}t^{-k}z))}. \tag{4.2.41}
$$

Note that in particular $Y^{(1)}(z) = 1$.

Proof. By direct calculation. See Appendix 4.A.1. □

By Proposition 4.2.23 and the operator product formulas in Section B.1.1, we can get the relation

$$
\begin{aligned}
X^{(1)}(z)V^{(\boldsymbol{n})}\left({\boldsymbol{u} \atop t^{-\boldsymbol{n}}\cdot\boldsymbol{u}}; x_1,\dots,x_{|\boldsymbol{n}|}\right) &= \prod_{k=1}^{|\boldsymbol{n}|} \frac{1-qz/t^2x_k}{1-z/tx_k} \cdot V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})X^{(1)}(z) \\
+ (1-t^{-1})\sum_{i=1}^N\sum_{k=1}^{n_i} u_i t^{1-k}\delta(tx_{[i,k]}/z) &\prod_{1\le\ell<[i,k]} \frac{1-qx_{[i,k]}/tx_\ell}{1-x_{[i,k]}/x_\ell} \cdot \prod_{[i,k]<\ell\le|\boldsymbol{n}|} \frac{1-tx_\ell/qx_{[i,k]}}{1-x_\ell/x_{[i,k]}} \\
&\times T_{q,x_{[i,k]}}\left(V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})\right)\Psi^+(x_{[i,k]}).
\end{aligned}
\tag{4.2.42}
$$

By noting $X_0^{(k)}|\mathbf{0}\rangle = \left(\sum_{i=1}^N t^{-n_i}u_i\right)|\mathbf{0}\rangle$ and by taking the constant term of z, we obtain

$$
\begin{aligned}
X_0^{(1)}V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})|\mathbf{0}\rangle &= V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})|\mathbf{0}\rangle\left(\sum_{i=1}^N t^{-n_i}u_i\right) \\
&+ (1-t^{-1})D^1_{|\boldsymbol{n}|}(\boldsymbol{s}|_{\lambda_k^{(i)}=0}; q, q/t)V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})|\mathbf{0}\rangle \ .
\end{aligned}
\tag{4.2.43}
$$

Once we note the integrals of the total difference vanish, integrating by parts, we can show

$$
\left[\boldsymbol{x}^{-\boldsymbol{\lambda}}\widetilde{f}^{\mathfrak{gl}_{|\boldsymbol{n}|}}(\boldsymbol{x}|\boldsymbol{s}|q,t)D^1_n(\boldsymbol{s}|_{\lambda_k^{(i)}=0}; q, q/t)V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})|\mathbf{0}\rangle\right]_{\boldsymbol{x},1}
$$

$$
\begin{aligned}
&= \left[\left(\widetilde{D^1_{|\boldsymbol{n}|}}(\boldsymbol{s}|_{\lambda^{(i)}_k=0}; q, q/t) \boldsymbol{x}^{-\boldsymbol{\lambda}} \widetilde{f}^{\mathfrak{gl}_{|n|}}(\boldsymbol{x}|\boldsymbol{s}|q, q/t) \right) V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) \left|\boldsymbol{0}\right\rangle \right]_{\boldsymbol{x},1} \\
&= \left[\left(\boldsymbol{x}^{-\boldsymbol{\lambda}} \widetilde{D^1_{|\boldsymbol{n}|}}(\boldsymbol{s}; q, q/t) \widetilde{f}^{\mathfrak{gl}_{|n|}}(\boldsymbol{x}|\boldsymbol{s}|q, q/t) \right) V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) \left|\boldsymbol{0}\right\rangle \right]_{\boldsymbol{x},1} \\
&= \left(\sum_{i=1}^{N} \sum_{k=1}^{n_i} s_{[i,k]_{\boldsymbol{n}}} \right) \left[\boldsymbol{x}^{-\boldsymbol{\lambda}} \widetilde{f}^{\mathfrak{gl}_{|n|}}(\boldsymbol{x}|\boldsymbol{s}|q, q/t) V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) \left|\boldsymbol{0}\right\rangle \right]_{\boldsymbol{x},1} . \qquad (4.2.44)
\end{aligned}
$$

Here, in the third line, we made use of Fact 4.1.3. This equality identifies the LHS of (4.2.33) with the generalized Macdonald functions up to the normalization.

For the formula (4.2.35), from (4.2.43), we have

$$
D^1_{|\boldsymbol{n}|}(\boldsymbol{s}; q, q/t) \boldsymbol{x}^{-\boldsymbol{\lambda}} \left\langle P_{\boldsymbol{\lambda}} \right| V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) \left|\boldsymbol{0}\right\rangle = (s_1 + \cdots + s_{|\boldsymbol{n}|}) \boldsymbol{x}^{-\boldsymbol{\lambda}} \left\langle P_{\boldsymbol{\lambda}} \right| V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) \left|\boldsymbol{0}\right\rangle . \qquad (4.2.45)
$$

This equality says $\boldsymbol{x}^{-\boldsymbol{\lambda}} \left\langle P_{\boldsymbol{\lambda}} \right| V^{(\boldsymbol{n})} \left|\boldsymbol{0}\right\rangle$ is an eigenfunction of $D^1_{|\boldsymbol{n}|}$, and implies (4.2.35). Since the constant terms in $f^{\mathfrak{gl}_n}$ and $\widetilde{f}^{\mathfrak{gl}_n}$ are $c_1(-; s_1|q,t) = 1$, the proportionality constant is same as $\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{u})$.

4.2.3 Integral forms

In this subsection, we introduce what is called *the integral form*, which has a different normalization than that of the generalized Macdonald functions. In $N = 1$ case, the integral form has the property that the coefficient of $X^{(1)}_{1^{|\lambda|}}$ is 1, that is, the integral form is expanded in the PBW-type basis as

$$
\left|K_\lambda\right\rangle = \left|X^{(1)}_{1^{|\lambda|}}\right\rangle + \sum_{\mu > (1^{|\lambda|})} c_{\lambda\mu}(u) \left|X^{(1)}_\mu\right\rangle . \qquad (4.2.46)
$$

The biggest reason to introduce the integral form is the main result looks better when written in those basis.

First, we recall the Nekrasov factor, introduced in Chapter 2,

$$
N_{\lambda\mu}(u) := \prod_{(i,j)\in\lambda} \left(1 - u q^{a_\lambda(i,j)} t^{\ell_\mu(i,j)+1}\right) \prod_{(i,j)\in\mu} \left(1 - u q^{-a_\mu(i,j)-1} t^{-\ell_\lambda(i,j)}\right) . \qquad (4.2.47)
$$

We define the normalization factors to define the integral forms.

Definition 4.2.24. *Set*

$$
\mathcal{C}^{(+)}_{\boldsymbol{\lambda}}(\boldsymbol{u}) := \xi^{(+)}_{\boldsymbol{\lambda}}(\boldsymbol{u}) \times \prod_{1\le i<j\le N} N_{\lambda^{(i)},\lambda^{(j)}}(qu_i/tu_j) \cdot \prod_{k=1}^{N} c_{\lambda^{(k)}}, \qquad (4.2.48)
$$

$$
\mathcal{C}^{(-)}_{\boldsymbol{\lambda}}(\boldsymbol{u}) := \xi^{(-)}_{\boldsymbol{\lambda}}(\boldsymbol{u}) \times \prod_{1\le i<j\le N} N_{\lambda^{(j)},\lambda^{(i)}}(qu_j/tu_i) \cdot \prod_{k=1}^{N} c_{\lambda^{(k)}}, \qquad (4.2.49)
$$

$$
\begin{aligned}
\xi^{(+)}_{\boldsymbol{\lambda}}(\boldsymbol{u}) :=& \prod_{i=1}^{N} (-1)^{(N-i+1)|\lambda^{(i)}|} u_i^{(-N+i)|\lambda^{(i)}| + \sum_{k=1}^{i} |\lambda^{(k)}|} \\
&\times \prod_{i=1}^{N} (q/t)^{\left(\frac{1-i}{2}\right)|\lambda^{(i)}|} q^{(i-N)\left(n(\lambda^{(i)\prime}) + |\lambda^{(i)}|\right)} t^{(N-i-1)\left(n(\lambda^{(i)}) + |\lambda^{(i)}|\right)}, \qquad (4.2.50)
\end{aligned}
$$

$$
\begin{aligned}
\xi^{(-)}_{\boldsymbol{\lambda}}(\boldsymbol{u}) :=& \prod_{i=1}^{N} (-1)^{i|\lambda^{(i)}|} u_i^{(-i+1)|\lambda^{(i)}| + \sum_{k=i}^{N} |\lambda^{(k)}|} \\
&\times \prod_{i=1}^{N} (q/t)^{\left(\frac{i-1}{2}\right)|\lambda^{(i)}|} t^{|\lambda^{(i)}|} q^{(1-i)\left(n(\lambda^{(i)\prime}) + |\lambda^{(i)}|\right)} t^{(i-2)\left(n(\lambda^{(i)}) + |\lambda^{(i)}|\right)}, \qquad (4.2.51)
\end{aligned}
$$

where c_λ is defined in (3.1.21) and $n(\lambda) = \sum_{j\ge1}(j-1)\lambda_j$.

Definition 4.2.25. *Define* $|K_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle \in \mathcal{F}_{\boldsymbol{u}}$ *and* $\langle K_{\boldsymbol{\lambda}}(\boldsymbol{u})| \in \mathcal{F}_{\boldsymbol{u}}^*$ *by*

$$|K_{\boldsymbol{\lambda}}\rangle = |K_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle := \mathcal{C}_{\boldsymbol{\lambda}}^{(+)}(\boldsymbol{u})\, |P_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle\,, \quad \langle K_{\boldsymbol{\lambda}}| = \langle K_{\boldsymbol{\lambda}}(\boldsymbol{u})| := \mathcal{C}_{\boldsymbol{\lambda}}^{(-)}(\boldsymbol{u})\, \langle P_{\boldsymbol{\lambda}}(\boldsymbol{u})|\,. \tag{4.2.52}$$

This normalization arises from the following conjecture with respect to the expansion coefficient in the PBW-type basis $|X_{\boldsymbol{\lambda}}\rangle$ and $\langle X_{\boldsymbol{\lambda}}|$.

Conjecture 4.2.26.

$$|K_{\boldsymbol{\lambda}}\rangle = \sum_{\boldsymbol{\mu}} \alpha_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{(+)}\, |X_{\boldsymbol{\mu}}\rangle\,, \quad \alpha^{(+)}_{\boldsymbol{\lambda},((1^{|\boldsymbol{\lambda}|}),\emptyset,\dots,\emptyset)} = 1, \tag{4.2.53}$$

$$\langle K_{\boldsymbol{\lambda}}| = \sum_{\boldsymbol{\mu}} \alpha_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{(-)}\, \langle X_{\boldsymbol{\mu}}|\,, \quad \alpha^{(-)}_{\boldsymbol{\lambda},((1^{|\boldsymbol{\lambda}|}),\emptyset,\dots,\emptyset)} = 1. \tag{4.2.54}$$

Definition 4.2.27 (Taki's flaming factors)**.** *Define*

$$f_\lambda := (-1)^{|\lambda|} q^{n(\lambda')+|\lambda|/2} t^{-n(\lambda)-|\lambda|/2} = (-\gamma)^{-|\lambda|} g_\lambda\,, \tag{4.2.55}$$

$$g_\lambda := q^{n(\lambda')} t^{-n(\lambda)}. \tag{4.2.56}$$

By definition, we can compute the following inner product of the integral forms, and we have the following proposition in the end.

Proposition 4.2.28.

$$\langle K_{\boldsymbol{\lambda}} | K_{\boldsymbol{\lambda}}\rangle = \left((-1)^N \gamma^2 e_N(\boldsymbol{u})\right)^{|\boldsymbol{\lambda}|} \prod_{i=1}^{N} \left(u_i^{|\lambda^{(i)}|} \gamma^{-2|\lambda^{(i)}|} g_{\lambda^{(i)}}\right)^{(2-N)} \cdot \prod_{i,j=1}^{N} N_{\lambda^{(i)},\lambda^{(j)}}(qu_i/tu_j), \tag{4.2.57}$$

where we put $e_N(\boldsymbol{u}) := \prod_{i=1}^N u_i$.

4.3 Generalized Macdonald Functions and Singular Vectors of q-Deformed $\mathcal{W}$-Algebra

The $\mathcal{W}$-algebras are defined as the commutants of the set of the screening operators [28, 26, 33]. They are associated with the Lie algebra $\mathfrak{g}$, and in the case $\mathfrak{g} = \mathfrak{sl}_2$, the corresponding $\mathcal{W}$-algebra is the Virasoro algebra. When $\mathfrak{g} = \mathfrak{sl}_N$, the $\mathcal{W}$-algebra, denoted by $\mathcal{W}_N$, contains the higher spin operators, including the Virasoro generators. Interestingly, the singular vectors in the Verma modules of $\mathcal{W}_N$-algebra, are represented by the Jack polynomials [73].

The quantum deformation of the $\mathcal{W}_N$-algebra (q-deformed $\mathcal{W}_N$-algebra) was introduced in [105, 12]. One guiding principle to define the "good" deformation is that the singular vectors of the deformed algebra must be related to the Macdonald polynomials, which are the q-deformation of the Jack polynomials. Another principle is the solvable model, called the RSOS model, see remarks in [51, 66, 102].

Later, the quantum deformation of $\mathcal{W}$-algebras associated with any simple Lie algebra was defined in [34].

In this section, we define the q-deformed $\mathcal{W}_N$-algebra as the commutant of the screening operators, and then show the formula for the Kac determinant.

In this section, we change the region of t so that $|t| < |q|$, for later convenience.

4.3.1 Screening Operators and q-Deformed $\mathcal{W}_N$-Algebra

First, we define the screening currents, the integration of which becomes the screening operators. We have already used these operators to give the explicit formula for the generalized Macdonald functions in Section 4.2.2.

Definition 4.3.1. *Define the screening currents $S^{(i)}(y) : \mathcal{F}_{t^{\alpha_i}\cdot \boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{u}}$ by the following,*

$$S^{(i)}(z) := \overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes \phi^{\rm sc}(\gamma^{i-1} z) \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-i-1}, \qquad i = 1, \dots, N-1, \tag{4.3.1}$$

with

$$\begin{aligned} \phi^{\rm sc}(z) := &\exp\left(-\sum_{n>0} \frac{1}{n}\frac{1-t^n}{1-q^n}\gamma^{2n} a_{-n} z^n\right) \exp\left(\sum_{n>0} \frac{1}{n}\frac{1-t^{-n}}{1-q^{-n}} a_n z^{-n}\right) \\ &\otimes \exp\left(\sum_{n>0} \frac{1}{n}\frac{1-t^n}{1-q^n}\gamma^{n} a_{-n} z^n\right) \exp\left(-\sum_{n>0} \frac{1}{n}\frac{1-t^{-n}}{1-q^{-n}}\gamma^{-n} a_n z^{-n}\right). \end{aligned} \tag{4.3.2}$$

Remark 4.3.2. *In the original papers [105, 12], the zero modes are attached to the screening currents. In this thesis, instead of zero modes, we use the pseudo-constant functions in order to make it easy to prove the existence of the integration contours. See Proposition 4.3.7 below.*

Remark 4.3.3. *The main problem here is, of course, how to find the screening currents. Once a candidate for the currents of an algebra is given, there is a general strategy to check whether there are the screening currents. See the next paragraph.*

Here we begin with the Drinfeld current $X^{(1)}(z)$ of $\mathcal{U}_{q,t}$ in mind. The current is of the form,

$$X^{(1)}(z) = u_1 \Lambda^{(1)}(z) + u_2 \Lambda^{(2)}(z) + \cdots + u_N \Lambda^{(N)}(z). \tag{4.3.3}$$

Then, we define the (candidate of) first screening current by the following difference equation:

$$: \frac{\Lambda^{(2)}(z)}{\Lambda^{(1)}(z)} :=: \frac{S_1(\alpha z)}{S_1(z)} :, \tag{4.3.4}$$

α is either of $q^{\pm 1}, t^{\pm 1}$. The idea of the difference equation is as follows. With c, c_1, c_2 some constants, we can compute from the normal ordering as $\Lambda^{(1)}(z) S_1(z) - c S_1(z) \Lambda^{(1)}(z) = c_1 : \Lambda^{(1)}(z) S_1(z) :$, and $\Lambda^{(2)}(z) S_1(z) - c S_1(z) \Lambda^{(2)}(z) = c_2 : \Lambda^{(2)}(z) S_1(z) :$. Thus if the equation like $: \Lambda^{(1)}(z) S_1(z) : \sim : \Lambda^{(2)}(z) S_1(z) :$ does not hold, there is no chance for the two terms to sum up to 0, up to the total $q^{\pm 1}$- (or $t^{\pm 1}$-) shift.

Note that the LHS is chosen so that it respects the structure of the root system. That is, the ratio u_2/u_1 is an element in negative simple roots, and thus the screening current S_1 is associated with this negative simple root. The other screening currents also can be defined by considering the other simple roots.

Actually, in $\mathcal{U}_{q,t}$ case, and $\alpha = q$, the following relation holds:

$$: \frac{\Lambda^{(2)}(tz)}{\Lambda^{(1)}(tz)} :=: \frac{\varphi^-(\gamma^{1/2} z) \otimes \eta(\gamma z)}{\eta(z) \otimes 1} :=: \frac{S^{(1)}(qz)}{S^{(1)}(z)} : . \tag{4.3.5}$$

In the definition above, we use $\alpha = q$ for all $i = 1, \dots, N-1$. We can make another set of the screening currents if we put $\alpha = t$. This is compatible with the fact that in $\mathcal{W}$-algebra, there are two sets of the screening currents $S_i^+(z)$ and $S_i^-(z)$ (see [34], for example).

Proposition 4.3.4. *Let $g_i(w)$ $(i = 1, \dots, N-1)$ be a function such that $g_i(qw) = \frac{u_{i+1}}{t u_i} g_i(w)$. Then, the commutation relation between $S^{(i)}(w) g_i(w)$ and the generating currents of the algebra $\mathsf{U}(N)$ is a total q-difference, that is,*

$$\left[S^{(i)}(w) g_i(w), X_n^{(k)} \right] = (1 - T_{q,w}) \, (\textit{some operators}) \quad (\forall i, k, n). \tag{4.3.6}$$

Proof. By the normal ordering formulas (B.1.8)-(B.1.13), the operator $\Lambda^{(j)}(z)$ with $j \neq i, i+1$ just commute with each other and we can show

$$\left[: \Lambda^{(i)}(z) \Lambda^{(i+1)}(\gamma^{-2} z) :, S^{(i)}(w) \right] = 0. \tag{4.3.7}$$

Thus, all we have to compute is the relation with $u_i\Lambda^{(i)}(z)+u_{i+1}\Lambda^{(i+1)}(z)$.

We have

$$\Lambda^{(i)}(z)S^{(i)}(w)-S^{(i)}(w)\Lambda^{(i)}(z)t=(1-t)\delta\left(\frac{tw}{qz}\right):\Lambda^{(i)}(tw/q)S^{(i)}(w):, \tag{4.3.8}$$

and

$$\begin{aligned}\Lambda^{(i+1)}(z)S^{(i)}(w)-S^{(i)}(w)\Lambda^{(i+1)}(z)t^{-1}&=(1-t^{-1})\delta\left(\frac{tw}{z}\right):\Lambda^{(i+1)}(tw)S^{(i)}(w):\\&=(1-t^{-1})\delta\left(\frac{tw}{z}\right):\Lambda^{(i)}(tw)S^{(i)}(qw):.\end{aligned} \tag{4.3.9}$$

Then, by the defining property $g_i(qz)=\frac{u_{i+1}}{tu_i}g_i(z)$ under the q-difference, we obtain

$$\begin{aligned}&\left(u_i\Lambda^{(i)}(z)+u_{i+1}\Lambda^{(i+1)}(z)\right)S^{(i)}(w)g_i(w)-S^{(i)}(w)\left(tu_i\Lambda^{(i)}(z)+t^{-1}u_{i+1}\Lambda^{(i+1)}(z)\right)g_i(w)\\&\qquad=(t-1)u_i(T_{q,w}-1)\delta\left(\frac{tw}{qz}\right):\Lambda^{(i)}(tw/q)S^{(i)}(w):g(w)\,.\end{aligned} \tag{4.3.10}$$

□

Remark 4.3.5. *The function $g_i(w)$ can be realized as $g_i(w)=\frac{\theta_q(t^2u_iw/u_{i+1})}{\theta_q(tw)}$.*

We again write the definition of $\mathsf{U}(N)$.

Definition 4.3.6. *We denote by $\mathsf{U}(N)$ (the completion in the sence of the adic topology, of) the algebra $\langle X_n^{(i)}|n\in\mathbb{Z},i=1,\dots N\rangle$ in $\mathrm{End}(\mathcal{F}_{\boldsymbol{u}})$. Namely, $\mathsf{U}(N)$ is the completion of the algebra generated by the set of operators $\left\{X_n^{(i)}\right\}$.*

Now we show $\mathsf{U}(N)$ is actually in the commutant of these screening operators. The proof is similar to the case corresponding to the Minimal model, given in [49]. Note that, as is same as the Virasoro algebra, the integration contour does not exist for the generic spectral parameters. Let us remark that in the Virasoro case, the existence of such a contour is studied in [56].

Proposition 4.3.7. *Let $r,s\in\mathbb{Z}_{>0}$ and $k,j\in\{1,\dots,r\}$. We also assume $|t|<|q|$. Then the generating currents of $\mathsf{U}(N)$ commute with the screening operators, that is,*

$$\left[X^{(j)}(z),\oint\frac{dw}{w}S^{(k)}(w_1)\cdots S^{(k)}(w_r)\prod_{i=1}^{r}\frac{\theta_q(t^{2i}\frac{u_k}{u_{k+1}}w_i)}{\theta_q(tw_i)}\right]=0, \tag{4.3.11}$$

where the spectral parameter of the codomain of $S^{(k)}(w_1)$ is chosen to be $\boldsymbol{u}$ with $u_k=q^st^{-r}u_{k+1}$. Here and hereafter, we use the shorthand notation

$$\oint\frac{dw}{w}:=\oint_T\prod_{i=1}^{r}\frac{dw_i}{2\pi\sqrt{-1}w_i}, \tag{4.3.12}$$

where the cycle is the r-dimensional torus $T:|w_1|=\cdots=|w_r|=1$.

What we have to prove is there are no poles between the contour T and $q^{-1}\cdot T$. $q^{-1}\cdot T$ denotes the contour with one of w is deformed to $|w_i|=q^{-1}$. Once we can prove this, we can complete the proof by using Proposition 4.3.4 and shifting the integration variables by q. For the detail, see below.

Proof of Proposition 4.3.7

We first prepare a lemma concerning the symmetrization of theta functions.

Lemma 4.3.8. *Define*

$$\widehat{F}_{r,s}(z_1,\dots,z_r) := \frac{1}{r!}\sum_{\sigma\in\mathfrak{S}_r}\prod_{i=1}^{r}\theta_q(q^s t^{2i-r} z_{\sigma(i)})\cdot\prod_{\substack{i<j\\ \sigma(i)>\sigma(j)}} t^{-1}\frac{\theta_q(tz_{\sigma(i)}/z_{\sigma(j)})}{\theta_q(t^{-1}z_{\sigma(i)}/z_{\sigma(j)})}\,. \tag{4.3.13}$$

Then, we have

$$\widehat{F}_{r,s}(z_1,\dots,z_r) = \frac{1}{r!}\prod_{i=2}^{r}\frac{\theta_q(t^i)}{\theta_q(t)}\cdot\prod_{1\le i<j\le r}\frac{\theta_q(z_i/z_j)}{\theta_q(tz_i/z_j)}\cdot\prod_{i=1}^{r}\theta_q(q^s t z_i)\,. \tag{4.3.14}$$

For the proof, see the proof of Lemma 4 in [49].

As commented in the proof of Proposition 4.3.4, it is sufficient to consider the relation with $\Lambda^{(k)}(z)+\Lambda^{(k+1)}(z)$. With the help of (4.3.10) used in the proposition above, we can show

$$\begin{aligned}
&\left[\Lambda^{(k)}(z)+\Lambda^{(k+1)}(z),\oint\frac{dw}{w}S^{(k)}(w_1)\cdots S^{(k)}(w_r)\cdot\prod_{i=1}^{r}\frac{\theta_q(t^{2i}\frac{u_k}{u_{k+1}}w_i)}{\theta_q(tw_i)}\right]\\
&=\sum_{m=1}^{r}\oint\frac{dw}{w}(t-1)t^{m-1}u_k\left(T_{q,w_m}-1\right)\delta\left(\frac{tw_m}{qz}\right)\Delta(w)\prod_{1\le i<j\le r}\frac{\theta_q(tw_i/w_j)}{\theta_q(w_i/w_j)}\cdot\prod_{i=1}^{r}\frac{\theta_q(q^s t^{2i-r}w_i)}{\theta_q(tw_i)}\\
&\quad\times\prod_{i=1}^{m-1}\frac{1-t^{-1}w_m/w_i}{1-w_m/w_i}\cdot\prod_{i=m+1}^{r}\frac{1-tw_i/w_m}{1-w_i/w_m}\cdot :\Lambda^{(k)}(tw_m/q)\prod_{i=1}^{r}S^{(k)}(z_i):,
\end{aligned}\tag{4.3.15}$$

with

$$\Delta(x)=\prod_{i<j}\frac{(x_i/x_j;q)_\infty(x_j/x_i;q)_\infty}{(tx_i/x_j;q)_\infty(tx_j/x_i;q)_\infty}\,, \tag{4.3.16}$$

which is introduced in Definition 3.1.21, the integral kernel of the scalar product. When we symmetrize the variables w_i's, the expression reduces as follows:

$$\begin{aligned}
&\frac{1}{r!}\sum_{m=1}^{r}\sum_{\sigma\in\mathfrak{S}_r}\oint\frac{dw}{w}(t-1)u_k\left(T_{q,w_{\sigma(m)}}-1\right)\delta\left(\frac{tw_{\sigma(m)}}{qz}\right)\Delta(w)\\
&\quad\times\prod_{1\le i<j\le r}\frac{\theta_q(tw_{\sigma(i)}/w_{\sigma(j)})}{\theta_q(w_{\sigma(i)}/w_{\sigma(j)})}\cdot\prod_{i=1}^{r}\frac{\theta_q(q^s t^{2i-r}w_{\sigma(i)})}{\theta_q(tw_{\sigma(i)})}\times\prod_{i\ne m}\frac{1-tw_{\sigma(i)}/w_{\sigma(m)}}{1-w_{\sigma(i)}/w_{\sigma(m)}}\cdot :\Lambda^{(k)}(tw_{\sigma(m)}/q)\prod_{i=1}^{r}S^{(k)}(z_i):\\
&=\frac{1}{r!}\sum_{l=1}^{r}\sum_{m=1}^{r}\sum_{\substack{\sigma\in\mathfrak{S}_r\\ \sigma(m)=l}}\oint\frac{dw}{w}(t-1)u_k\left(T_{q,w_l}-1\right)\delta\left(\frac{tw_l}{qz}\right)\Delta(w)\\
&\quad\times\prod_{1\le i<j\le r}\frac{\theta_q(tw_{\sigma(i)}/w_{\sigma(j)})}{\theta_q(w_{\sigma(i)}/w_{\sigma(j)})}\cdot\prod_{i=1}^{r}\frac{\theta_q(q^s t^{2i-r}w_{\sigma(i)})}{\theta_q(tw_{\sigma(i)})}\cdot\prod_{\sigma(i)\ne l}\frac{1-tw_{\sigma(i)}/w_l}{1-w_{\sigma(i)}/w_l}\cdot :\Lambda^{(k)}(tw_l/q)\prod_{i=1}^{r}S^{(k)}(z_i):\\
&=\frac{1}{r!}\sum_{l=1}^{r}\oint\frac{dw}{w}(t-1)u_k\left(T_{q,w_l}-1\right)\delta\left(\frac{tw_l}{qz}\right)\Delta(w)\\
&\quad\times\left(\sum_{\sigma\in\mathfrak{S}_r}\prod_{1\le i<j\le r}\frac{\theta_q(tw_{\sigma(i)}/w_{\sigma(j)})}{\theta_q(w_{\sigma(i)}/w_{\sigma(j)})}\cdot\prod_{i=1}^{r}\frac{\theta_q(q^s t^{2i-r}w_{\sigma(i)})}{\theta_q(tw_{\sigma(i)})}\right)\prod_{i\ne l}\frac{1-tw_i/w_l}{1-w_i/w_l}\cdot :\Lambda^{(k)}(tw_l/q)\prod_{i=1}^{r}S^{(k)}(z_i):\,.
\end{aligned}\tag{4.3.17}$$

Then, we can apply Lemma 4.3.8 and we obtain

$$\frac{1}{r!}\sum_{l=1}^{r}\oint\frac{dw}{w}(t-1)u_k\left(T_{q,w_l}-1\right)\delta\left(\frac{tw_l}{qz}\right)\Delta(w)$$
$$\times\prod_{i=2}^{r}\frac{\theta_q(t^i)}{\theta_q(t)}\cdot\prod_{i=1}^{r}q^{\frac{s(1-s)}{2}}(tz_i)^{-s}\cdot\prod_{i\neq l}\frac{1-tw_i/w_l}{1-w_i/w_l}\cdot:\Lambda^{(k)}(tw_l/q)\prod_{i=1}^{r}S^{(k)}(z_i):. \tag{4.3.18}$$

In this expression, we have poles at $w_l=0$, $w_l=q^ntw_i$ and $w_l=q^{-n+1}t^{-1}w_i$ ($i\neq l$, $n=1,2,\ldots$). Since $|t|<|q|$, the q-shift of the cycle $w_l\to qw_l$ does not pick up any poles and thus the integral (4.3.18) is zero. This completes the proof. □

4.3.2 Example: q-Virasoro Algebra

Now let us see the example, q-deformed Virasoro (q-Virasoro for short), that is, $N=2$ case in more detail.

As noted above, in $\mathsf{U}(N)$, the Heisenberg algebra is contained. In order to strip it off, we introduce the following currents [30].

Definition 4.3.9. *Define two currents $\alpha(z),\beta(z)$ by*

$$\alpha(z):=\exp\left(-\sum_{n=1}^{\infty}\frac{1}{c^n-c^{-n}}b_{-n}z^n\right),\quad\beta(z):=\exp\left(\sum_{n=1}^{\infty}\frac{1}{c^n-c^{-n}}b_nz^{-n}\right) \tag{4.3.19}$$

with the boson

$$\begin{aligned}&[b_m,b_n]=\frac{1}{m}(1-q^{-m})(1-t^m)(1-\gamma^{2m})(c^m-c^{-m})c^{-|m|}\delta_{m+n,0}\,,\\&\Delta(b_n)=b_n\otimes c^{-|n|}+1\otimes b_n\,.\end{aligned} \tag{4.3.20}$$

We define the image of this boson under the horizontal representation $\rho^{(1,0)}$, as

$$b_n\mapsto\frac{1-t^n}{|n|}(\gamma^{|n|}-\gamma^{-|n|})a_n\,. \tag{4.3.21}$$

We define the q-Virasoro generator.

Definition 4.3.10.

$$T(z):=\rho^{(2,0)}_{u_1,u_2}(\Delta(t(z)))\,, \tag{4.3.22}$$

with

$$t(z)=\alpha(z)x^+(z)\beta(z)\,. \tag{4.3.23}$$

Remark 4.3.11. *Note that the currents $\alpha(z),\beta(z)$ are chosen so that*

$$\rho^{(1,0)}(t(z))=1\,. \tag{4.3.24}$$

This means when $N=1$, $\mathsf{U}(N)$ is just the Heisenberg algebra. When $N>1$, the non-trivial algebra, q-$\mathcal{W}_N$ algebra appears.

Proposition 4.3.12 ([105]). *We have the commutation relation*

$$f(w/z)T(z)T(w)-f(z/w)T(w)T(z)=\frac{(1-q)(1-t^{-1})}{1-q/t}\left(\delta(\gamma^2w/z)-\delta(w/\gamma^2z)\right)\,, \tag{4.3.25}$$

with

$$f(z)=\frac{(1-qz)(1-z/t)}{(1-z)(1-qz/t)}\,. \tag{4.3.26}$$

We refer to this algebra as the q-Virasoro algebra.

Let us give three remarks on this algebra.

Remark 4.3.13. *When $N \geq 3$, on the RHS of (4.3.25), the new currents appear in the coefficient of the delta functions. Thus unlike the Virasoro algebra, q-Virasoro is not subalgebra of q-$\mathcal{W}_N$-algebra.*

Remark 4.3.14. *Recall that the Virasoro algebra has the coset construction [37, 6],*

$$\frac{\widehat{\mathfrak{sl}}_{2,k} \otimes \widehat{\mathfrak{sl}}_{2,1}}{\widehat{\mathfrak{sl}}_{2,k+1}} \simeq \mathrm{Vir}_k \,. \tag{4.3.27}$$

The similar construction for the q-Virasoro is studied in [51].

Remark 4.3.15. *Moreover, as noted in [51, 66], the structure function $f(z)$ of q-Virasoro and the generating current $T(z)$ appear from the integrable model called the restricted solid-on-solid (RSOS) model. Because the way to deform the Virasoro algebra is not unique, the relation with the integrable model ensures this is the "good" q-deformation.*

4.3.3 Kac Determinant Formula

The Kac determinant formula of the q-deformed $\mathcal{W}_N$-algebra is discussed in [105]. In order to show Fact 4.2.7, that is, the fact that $|X_{\boldsymbol{\lambda}}\rangle$ form a basis on the Fock space, the Kac determinant formula with respect to $|X_{\boldsymbol{\lambda}}\rangle$ is proposed in [89]. While in [89], the proof goes without clarifying the choice of the integral cycles, now we know the existence of the integration cycle (see the proof of Proposition 4.3.7).

Definition 4.3.16. *Let $1 \leq k \leq N-1$ and $u_k = q^s t^{-r} u_{k+1}$ ($r, s \in \mathbb{Z}_{>0}$). Define the vector $|\chi^{(k)}_{r,s}\rangle \in \mathcal{F}_{\boldsymbol{u}}$ by the integral*

$$|\chi^{(k)}_{r,s}\rangle := \oint \frac{dz}{z} S^{(k)}(z_1) \cdots S^{(k)}(z_r) \prod_{i=1}^{r} \frac{\theta_q(t^{2i} u_k z_i / u_{k+1})}{\theta_q(t z_i)} \, |\mathbf{0}\rangle \,. \tag{4.3.28}$$

Again, note that $\boldsymbol{u}$ is the spectral parameter of the codomain of $S^{(k)}(z_1)$.

Proposition 4.3.17. *The vector $|\chi^{(k)}_{r,s}\rangle$ exists at level rs.*

Proof of Proposition 4.3.17. By the normal ordering formulas among screening currents and Lemma 4.3.8, we can show

$$\begin{aligned}
|\chi^{(k)}_{r,s}\rangle &= \oint \frac{dz}{z} \Delta(z) \prod_{1 \leq i < j \leq r} \frac{\theta_q(t z_i/z_j)}{\theta_q(z_i/z_j)} \cdot \prod_{i=1}^{r} \frac{\theta_q(q^s t^{2i-r} z_i)}{\theta_q(t z_i)} : S^{(k)}(z_1) \cdots S^{(k)}(z_r) : |\mathbf{0}\rangle \\
&= \oint \frac{dz}{z} \Delta(z) \prod_{1 \leq i < j \leq r} \frac{\theta_q(t z_i/z_j)}{\theta_q(z_i/z_j)} \cdot \prod_{i=1}^{r} \frac{1}{\theta_q(t z_i)} \widehat{F}_{r,s}(z_1, \ldots, z_r) : S^{(k)}(z_1) \cdots S^{(k)}(z_r) : |\mathbf{0}\rangle \\
&= \frac{q^{\frac{r}{2}s(1-s)}}{r!} \prod_{i=2}^{r} \frac{\theta_q(t^i)}{\theta_q(t)} \cdot \oint \frac{dz}{z} \Delta(z) \prod_{i=1}^{r} (t z_i)^{-s} \; : \prod_{i=1}^{r} S^{(k)}(z_i) : |\mathbf{0}\rangle \,,
\end{aligned} \tag{4.3.29}$$

with $\Delta(z)$ defined in (4.3.16). Note that $: \prod_{i=1}^{r} S^{(k)}(z_i) : |\mathbf{0}\rangle$ agrees with the kernel function Definition 3.1.12, because we can neglect the annihilation part of $S^{(k)}(z)$ and the creation part is of the form of $\varphi(z)$ in Definition 3.2.5. Therefore, it is expanded in terms of the Macdonald functions. Noting that $\prod_{i=1}^{r} z_i^{-s}$ is the Macdonald polynomial labelled by the rectangular Young diagram (s^r) in r variables, we can rewrite (4.3.29) as

$$\begin{aligned}
&\frac{q^{rs(1-s)/2} t^{-rs}}{r!} \prod_{i=2}^{r} \frac{\theta_q(t^i)}{\theta_q(t)} \cdot \sum_{\lambda} P_\lambda(\alpha^{(k)}_{-n}) \, |\mathbf{0}\rangle \, \langle P^{(r)}_{(s^r)}(z), Q^{(r)}_\lambda(z) \rangle'_{q,t} \\
&= \frac{q^{rs(1-s)/2} t^{-rs}}{r!} \prod_{i=2}^{r} \frac{\theta_q(t^i)}{\theta_q(t)} \cdot P_{(s^r)}(\alpha^{(k)}_{-n}) \, |\mathbf{0}\rangle \, \langle P^{(r)}_{(s^r)}(z), Q^{(r)}_{(s^r)}(z) \rangle'_{q,t} \,,
\end{aligned} \tag{4.3.30}$$

where

$$\alpha_{-n}^{(k)} := \gamma^{kn}(-\gamma^n a_{-n}^{(k)} + a_{-n}^{(k+1)}), \tag{4.3.31}$$

and $\langle -,-\rangle'_{q,t}$ denotes the Macdonald's another scalar product defined in Definition 3.1.21. Since the Macdonald polynomials are orthogonal with respect to $\langle -,-\rangle'_{q,t}$, the inner product $\langle P_{(s^r)}^{(r)}, Q_{(s^r)}^{(r)}\rangle'_{q,t}$ does not vanish. Thus, $|\chi_{r,s}^{(k)}\rangle \neq 0$ and of level rs. □

By these two propositions (Proposition 4.3.17 and Proposition 4.3.7), we can show the existence of the singular vectors of the algebra $\mathsf{U}(N)$.

Corollary 4.3.18. *The vector $|\chi_{r,s}^{(k)}\rangle$ is a singular vector of level rs, i.e.,*

$$X_n^{(i)}\,|\chi_{r,s}^{(k)}\rangle = 0 \tag{4.3.32}$$

for all $n > 0$ and $i = 1, \ldots, N$.

We revisit the proof of the following formula for the Kac determinant $\det_n := \det\left(\langle X_{\boldsymbol{\lambda}}(\boldsymbol{u})\,|\,X_{\boldsymbol{\mu}}(\boldsymbol{u})\rangle\right)_{\boldsymbol{\lambda},\boldsymbol{\mu}\vdash n}$, where $\boldsymbol{\mu}\vdash n$ means $\boldsymbol{\mu}$ is a N-tuple of partitions with $|\boldsymbol{\mu}| = n$.

Proposition 4.3.19 ([89])**.** *We have*

$$\begin{aligned}\det_n = &\prod_{\boldsymbol{\lambda}\vdash n}\prod_{k=1}^{N} b_{\lambda^{(k)}}(q)b'_{\lambda^{(k)}}(t^{-1})\\ &\times \prod_{\substack{1\le r,s\\ rs\le n}}\left((u_1u_2\cdots u_N)^2\prod_{1\le i<j\le N}(u_i - q^s t^{-r}u_j)(u_i - q^{-r}t^s u_j)\right)^{P^{(N)}(n-rs)},\end{aligned} \tag{4.3.33}$$

with

$$\begin{aligned} &b_\lambda(q) := \prod_{i\ge 1}(q;q)_{m_i}\,, \qquad b'_\lambda(q) := (-1)^{\sum_i m_i}\prod_{i\ge 1}(q;q)_{m_i}\,.\\ &P^{(N)}(n) = \#\{\boldsymbol{\lambda} = (\lambda^{(1)},\ldots,\lambda^{(N)})\,|\,|\boldsymbol{\lambda}| = n\}\,. \end{aligned} \tag{4.3.34}$$

(Recall m_i is the number of entries in λ equal to i.) In particular, if $N = 1$,

$$\det_n = \prod_{\lambda\vdash n} b_\lambda(q)b'_\lambda(t^{-1}) \times u_1^{2\sum_{\lambda\vdash n}\ell(\lambda)}. \tag{4.3.35}$$

Proof. We can compute the inner product $\langle X_{\boldsymbol{\lambda}}|X_{\boldsymbol{\lambda}}\rangle$ by commutation relations of $X_n^{(i)}$, and the parameters $u_1,\ldots,u_N$ appear as the eigenvalues of $X_0^{(i)}$. Thus, $\langle X_{\boldsymbol{\lambda}}|X_{\boldsymbol{\lambda}}\rangle$ turns out to be a polynomial in

$$m_{(1^i)}(u_1,\ldots,u_N) = \sum_{j_1<\cdots<j_i} u_{j_1}\cdots u_{j_i} \tag{4.3.36}$$

over $\mathbb{Q}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$, and thus so is $\det_n$. Because we have the action of the Weyl group of type A_{N-1} on polynomials in u_j, we can define the action of the symmetric group $\mathfrak{S}_N$ on them in the usual way. Note that the Weyl group fixes the symmetric polynomials $m_{(1^i)}(u_1,\ldots,u_N)$. Therefore, $\det_n$ is also a symmetric polynomial in u_j.

Let us introduce the new parameters u'_i and u'' by

$$\prod_{i=1}^{N} u'_i = 1, \qquad u_i = u'_i u''. \tag{4.3.37}$$

Then $\langle X_{\boldsymbol{\lambda}} | X_{\boldsymbol{\mu}} \rangle$ can be decomposed as

$$\langle X_{\boldsymbol{\lambda}} | X_{\boldsymbol{\mu}} \rangle = (u'')^{\sum_{k=1}^{N} k(\ell(\lambda^{(k)})+\ell(\mu^{(k)}))} \times (\text{polynomial in } u'_i)\,, \tag{4.3.38}$$

and thus $\det_n$ is also of the form

$$\det_n = (u'')^{2\sum_{|\boldsymbol{\lambda}|=n} \sum_{k=1}^{N} k\ell(\lambda^{(k)})} \times F(u'_1, \dots, u'_N), \tag{4.3.39}$$

where $F(u'_1, \dots, u'_N)$ is some symmetric polynomial in u'_i. Note that the degree of $F(u'_1, \dots, u'_N)$ is $2\sum_{|\boldsymbol{\lambda}|=n} \sum_{k=1}^{N} \ell(\lambda^{(k)})$.

By Corollary 4.3.18, for $r, s \in \mathbb{Z}_{>0}$ with $rs \le n$, we can show that $\det_n$ contains the following factor:

$$(u_k - q^s t^{-r} u_{k+1})^{P^{(N)}(n-rs)} = \left(u''(u'_k - q^s t^{-r} u'_{k+1})\right)^{P^{(N)}(n-rs)}\,. \tag{4.3.40}$$

This is the same discussion as that of the Virasoro Kac determinant. Thanks to the $\mathfrak{S}_N$ invariance, $\det_n$ must also contain the factors

$$(u_i - q^{\pm s} t^{\mp r} u_j)^{P^{(N)}(n-rs)} = \left(u''(u'_i - q^{\pm s} t^{\mp r} u'_j)\right)^{P^{(N)}(n-rs)} \tag{4.3.41}$$

for $i \neq j$. Recalling the degree of $F(u'_1, \dots, u'_N)$, we can show

$$\begin{aligned}
\det_n &= g_{N,n}(q,t) \times (u'')^{2\sum_{\boldsymbol{\lambda}\vdash n} \sum_{i=1}^{N} i\,\ell(\lambda^{(i)})} \\
&\quad \times \prod_{1 \le i < j \le N} \prod_{\substack{1 \le r,s \\ rs \le n}} \left((u'_i - q^s t^{-r} u'_j)(u'_i - q^{-r} t^s u'_j)\right)^{P^{(N)}(n-rs)} \\
&= g_{N,n}(q,t) \\
&\quad \times \prod_{\substack{1 \le r,s \\ rs \le n}} \left((u_1 u_2 \cdots u_N)^2 \prod_{1 \le i < j \le N} (u_i - q^s t^{-r} u_j)(u_i - q^{-r} t^s u_j) \right)^{P^{(N)}(n-rs)}\,,
\end{aligned} \tag{4.3.42}$$

with $g_{N,n}(q,t) \in \mathbb{Q}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. The prefactor $g_{N,n}(q,t)$ has been derived in [89]. □

Similarly to the Virasoro case, the form of the Kac determinant ensures the linear independence of the basis $|X_{\boldsymbol{\lambda}}\rangle$, because for $u_k \neq q^s t^{-r} u_{k+1}$ $(r, s \in \mathbb{Z}_{>0}$), the Kac determinant does not become zero, and thus there is no zero eigenvalue. As the corollary, Fact 4.2.7 follows.

Appendix 4.A Proof of Propositions

4.A.1 Proof of Proposition 4.2.23

First, we prove the relation for $k=0$. Note that it is this computation that becomes crucial in the proof. For $i_1 < \cdots < i_r$, we introduce the notation

$$\Lambda^{(i_1,\dots,i_r)}(z) =: \Lambda^{(i_1)}(z)\cdots\Lambda^{(i_r)}((q/t)^{r-1}z) : . \tag{4.A.1}$$

By virtue of the normal ordering formulas (B.1.23)-(B.1.26) and the identity of operators $:\Phi^{(0)}(w)\Lambda^{(1)}(tw):= \Phi^{(0)}(qw)\Psi^+(w)$, we can show that when $i_1=1$,

$$\begin{aligned}&\Lambda^{(i_1,\dots,i_r)}(z)\Phi^{(0)}(x) - t^{-1}\frac{1-(q/t)^r z/tx}{1-z/tx}\Phi^{(0)}(x)\Lambda^{(i_1,\dots,i_r)}(z)\\ &= (1-t^{-1})\delta(tx/z) : \Lambda^{(i_2)}((q/t)tx)\cdots\Lambda^{(i_r)}((q/t)^{r-1}tx)\Phi^{(0)}(qx)\Psi^+(x) :,\end{aligned} \tag{4.A.2}$$

and when $i_1 \geq 2$,

$$\Lambda^{(i_1,\dots,i_r)}(z)\Phi^{(0)}(x) - \frac{1-(q/t)^r z/tx}{1-z/tx}\Phi^{(0)}(x)\Lambda^{(i_1,\dots,i_r)}(z) = 0. \tag{4.A.3}$$

Combining these equations, we obtain the result for the case $k=0$:

$$X^{(r)}(z)\Phi^{(0)}(x) - \frac{1-(q/t)^r z/tx}{1-z/tx}\Phi^{(0)}(x)X^{(r)}(z) = u_1(1-t^{-1})\delta(tx/z)Y^{(r)}(x)\Phi^{(0)}(qx)\Psi^+(x)\,. \tag{4.A.4}$$

Next, for the case $k \neq 0$, we multiply the screening currents, which almost commute with the generating currents $X^{(i)}(z)$, to this relation from the right side. Note that $\Psi^+(x)$ also commutes with $S^{(i)}(y)$, that is,

$$\Psi^+(z)S^{(i)}(y) = S^{(i)}(y)\Psi^+(z)\,. \tag{4.A.5}$$

Combining the defining property of g (4.2.28), (4.A.5), and commutativity of the screening currents, we can derive the relation for general k.

Thus, what is left is to show the commutativity of the screening currents. This goes as follows. First, from the Leibniz rule of the commutator, we have

$$\begin{aligned}&\Phi^{(0)}(x)\cdot\left[X^{(r)}(z), S^{(1)}(y_1)\cdots S^{(k)}(y_k)g(x,y_1,\dots,y_k)\right]\\ &= \sum_{i=1}^{k}\Phi^{(0)}(x)S^{(1)}(y_1)\cdots\left[X^{(r)}(z),S^{(i)}(y_i)\right]\cdots S^{(k)}(y_k)g(x,y_1,\dots,y_k)\,.\end{aligned} \tag{4.A.6}$$

By the same argument as the proof of Proposition 4.3.4, the term which survives in the RHS of (4.A.6) can be written as

$$\begin{aligned}&(1-T_{q,y_i})\delta\left(\frac{t\gamma^{2\ell}y_i}{qz}\right)\Phi^{(0)}(x)S^{(1)}(y_1)\cdots\\ &\quad\cdots : \Lambda^{(j_1,\dots,j_\ell,i,j_{\ell+2}\dots,j_r)}(t\gamma^{2\ell}y_i/q)\,S^{(i)}(y_i) : \cdots S^{(k)}(y_k)g(x,y_1,\dots,y_k)\,.\end{aligned} \tag{4.A.7}$$

Note that $j_{\ell+2} \geq i+2$.

Next, let us name the positions of poles in y_i, because we have to be careful about the integral cycle. Combining the θ-functions containing y_i in $g(x,y_1,\dots,y_k)$ and theta's coming from the normal orderings among screening currents and $\Phi^{(0)}(x)$, we have the factor

$$\frac{1}{\theta_q(ty_i/y_{i-1})\theta_q(ty_{i+1}/y_i)}\frac{(ty_i/y_{i-1};q)_\infty(ty_{i+1}/y_i;q)_\infty}{(y_i/y_{i-1};q)_\infty(y_{i+1}/y_i;q)_\infty}$$

$$= \frac{1}{(qy_{i-1}/ty_i;q)_\infty (qy_i/ty_{i+1};q)_\infty} \frac{1}{(y_i/y_{i-1};q)_\infty (y_{i+1}/y_i;q)_\infty}, \quad y_0 := x. \tag{4.A.8}$$

From the normal ordering between $\Phi^{(0)}(x)$ and $\Lambda^{(i)}(ty_1/q)$ in $\Lambda^{(j_1,\dots,j_\ell,i,j_{\ell+2}\dots,j_r)}$, we also have

$$\frac{1 - t^{\delta_{i,1}-1} ty_i/qx}{1 - y_i/tx}. \tag{4.A.9}$$

Note that for $i \geq 2$, from the normal ordering between $S^{(i-1)}(y_{i-1})$ and $\Lambda^{(i)}(ty_i/q)$, we have

$$S^{(i-1)}(y_{i-1})\Lambda^{(i)}(ty_i/q) = \frac{1 - ty_i/qy_{i-1}}{1 - y_i/qy_{i-1}} : S^{(i-1)}(y_{i-1})\Lambda^{(i)}(ty_i/q) : . \tag{4.A.10}$$

Then as a result, we have the following set of poles of (4.A.6) in y_i:

$$y_i = tx, \tag{4.A.11}$$
$$y_i = t^{-1}q^{2+n}y_{i-1}, \quad y_i = q^n y_{i+1}, \tag{4.A.12}$$
$$y_i = t\,q^{-1-n}y_{i+1} \quad (i \geq 1,\, n = 0, 1, 2, \dots), \tag{4.A.13}$$

and

$$y_i = q^{-n+1}y_{i-1} \quad \text{for } i \geq 2, \tag{4.A.14}$$
$$y_i = q^{-n}x \quad \text{for } i = 1 \quad (n = 0, 1, 2 \dots). \tag{4.A.15}$$

When $r = 1$, these are all of the poles, while for general r, this list does not exhaust the possible poles. In case $r \geq 2$, from $S^{(j)}(y_j)$ and $\Lambda^{(j_m)}$ in $\Lambda^{(j_1,\dots,j_\ell,i,j_{\ell+2}\dots,j_r)}$ with $m \neq \ell+1$, and from $\Phi^{(0)}(x)$ and $\Lambda^{(j_m)}$, we have additional poles. That is, we have the following factors:

$$\prod_{m=1}^{\ell} \frac{1 - t^{-1}(q/t)^{-\ell+m-1}y_i/y_{j_m}}{1 - (q/t)^{-\ell+m-1}y_i/y_{j_m}} \frac{1 - (q/t)^{-\ell+m-2}y_i/y_{j_m-1}}{1 - q^{-1}(q/t)^{-\ell+m-1}y_i/y_{j_m-1}}$$
$$\times \prod_{m=\ell+2}^{r} \frac{1 - t(q/t)^{\ell-m+1}y_{j_m}/y_i}{1 - (q/t)^{\ell-m+1}y_{j_m}/y_i} \frac{1 - (q/t)^{\ell-m+2}y_{j_m-1}/y_i}{1 - q(q/t)^{\ell-m+1}y_{j_m-1}/y_i}, \tag{4.A.16}$$

and

$$\prod_{m\neq \ell+1} \frac{1 - t^{\delta_{j_m,1}-1}(q/t)^{-\ell+m-2}y_i/x}{1 - t^{-1}(q/t)^{-\ell+m-1}y_i/x}. \tag{4.A.17}$$

In the end, combining all above, we can show that the following set of the points contains all the poles in y_i are in the following positions: For $i \geq 1$,

$$y_i = (q/t)^{-n-1}y_{j+1}, \quad y_i = q(q/t)^{-n-1}y_j \quad (j > i), \tag{4.A.18}$$
$$y_i = q(q/t)^{-n}x, \quad (n = 0, 1, 2 \dots), \tag{4.A.19}$$

and for $i \geq 2$,

$$y_i = (q/t)^{n+1}y_j \quad (1 \leq j < i), \quad y_i = q(q/t)^{n+1}y_{j-1} \quad (2 \leq j < i), \tag{4.A.20}$$
$$y_i = q(q/t)^{n+1}x \quad (n = 0, 1, 2 \dots). \tag{4.A.21}$$

For the given integration contour, the poles (4.A.12), (4.A.20) and (4.A.21) are inside the disk $\{z; |z| < |qy_i|\}$, while the poles (4.A.14), (4.A.15), (4.A.11), (4.A.13), (4.A.18) and (4.A.19) are outside $\{z; |z| > |y_i|\}$. Therefore, the shift $y_i \to qy_i$ is not affected by these poles, and we can show the commutativity between the screening currents and $X^{(r)}$. □

Chapter 5

Main Theorem: Mukadé Operator and its Matrix Elements

We are now ready to prove (a part of) the main claim (Conjecture 2.2.1). The key idea is to realize the instanton partition function $\mathcal{Z}^{(A_N,A_M)}_{\text{inst.}}$ by the expectation value of the composition of intertwiners. As we soon see below, we introduce *the Mukadé operator* $\mathcal{V}(x)$ (defined in Section 5.1), and prove the formula for the matrix elements of this operator with respect to the generalized Macdonald functions. Roughly, this ends with the products of the Nekrasov factors, that is,

$$\langle K_{\boldsymbol{\lambda}}(\boldsymbol{v})|\, \mathcal{V}(x)\, |K_{\boldsymbol{\mu}}(\boldsymbol{u})\rangle \sim \prod_{i,j=1}^{N} N_{\lambda^{(i)},\mu^{(j)}}(qv_i/tu_j)\,. \tag{5.0.1}$$

(See Theorem 5.2.1.)

For our final purpose, we prepare two facts. One of them is the M-th compositions of the Mukadé operators give the partition function of the (A_{M-1}, A_{N-1}) theory, that is, schematically,

$$\langle \emptyset|\, \mathcal{V}(x_1)\cdots\mathcal{V}(x_M)\, |\emptyset\rangle \sim \mathcal{Z}^{(A_{M-1},A_{N-1})}_{\text{inst.}}\,. \tag{5.0.2}$$

This is because the Mukadé operator has the realization $\mathcal{T}^V$ (see Proposition 5.1.9).

The other fact is the M-th compositions of the RHS of (5.0.1) (with some appropriate instanton fugacities $\mathfrak{q}_i$) give the $\mathcal{Z}^{(A_{N-1},A_{M-1})}_{\text{inst.}}$ (see Proposition 5.2.3), that is,

$$\sum_{\substack{\boldsymbol{\lambda}^{[i]}\in\mathsf{P}^N,\\ i=2,\dots,M}} \prod_{k=2}^{M} \mathfrak{q}_k^{|\boldsymbol{\lambda}^{[k]}|}\cdot(\text{RHS of (5.0.1)}) \sim \mathcal{Z}^{(A_{N-1},A_{M-1})}_{\text{inst.}}\,. \tag{5.0.3}$$

Then, combining these three relations, we (almost) completes the proof of Conjecture 2.2.1. All these facts are summarized in Section 5.2.

Because the latter two facts are easy to prove, the most difficult part is the proof of (5.0.1). Section 5.4 is devoted to this proof. The proof itself is straightforward, though we have to make use of the Kajihara-Noumi identity for the multiple hypergeometric series, which we explain in Section 5.3.

5.1 Mukadé Operator

In this section, we introduce the Mukadé Operator $\mathcal{V}(x)$. Its uniqueness is easy to see, though its existence is not trivial. In order to prove the existence, we concretely construct the Mukadé operator using the intertwiners of DIM algebra. The term "Mukadé" means the centipede, and the reason for its name is that the diagram of this realization looks like a centipede.

5.1.1 Definition

We define the Mukadé operator.

Definition 5.1.1. *Define the linear operator* $\mathcal{V}(x) = \mathcal{V}\binom{\boldsymbol{v}}{\boldsymbol{u}}; x) : \mathcal{F}_{\boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{v}}$ *by the following relations:*

$$\left(1 - \frac{x}{z}\right) X^{(i)}(z)\mathcal{V}(x) = \left(1 - (t/q)^i \frac{x}{z}\right) \mathcal{V}(x)X^{(i)}(z), \qquad i \in \{1, 2, \ldots, N\} \tag{5.1.1}$$

and $\langle \mathbf{0} | \mathcal{V}(x) | \mathbf{0} \rangle = 1$.

Remark 5.1.2. *Under the* $q, t \to 1$ *limit, the defining relation of the Mukadé operator reduces to*

$$[L_n, V(z)] = z^n \left(z\frac{\partial}{\partial z} + h(n+1) \right) V(z)\,. \tag{5.1.2}$$

Thus, we can regard the Mukadé operator as the q-analogue of the Virasoro primary field.

Then we have the following proposition.

Proposition 5.1.3. *The* $\mathcal{V}(x)$ *exists uniquely.*

Proof. If the operator $\mathcal{V}(x)$ exists, the uniqueness is clear by definition of $\mathcal{V}(w)$ and the fact that the vectors $|X_{\boldsymbol{\lambda}}\rangle$ form a basis (Fact 4.2.7). The existence of $\mathcal{V}(w)$ is shown in Proposition 5.1.9 in the next section. □

5.1.2 Existence

We now show the existence of the Mukadé operator, composing the intertwiners of DIM algebra as the following diagrams.

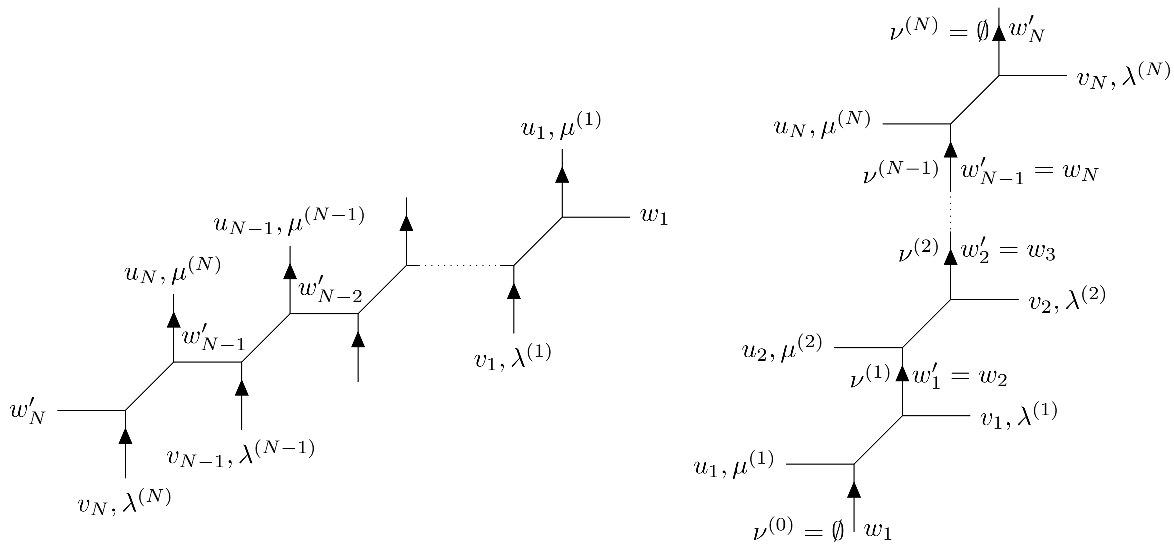


Figure 5.1: Left:$\mathcal{T}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\boldsymbol{u}, \boldsymbol{v}; w)$ and Right:$\mathcal{T}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\boldsymbol{u}, \boldsymbol{v}; w)$ (up to the overall factor)

For the preparation, we introduce some notations.

Notation 5.1.4. *We introduce the notations* w_i *and* w'_i *related to the spectral parameters* $\boldsymbol{u}, \boldsymbol{v}$ *of the modules.*

$$\begin{aligned} w_1 &= w, \\ w'_i &= \frac{u_i}{v_i} w_i, \quad \textit{for} \quad i = 1, \ldots, N, \\ w_{i+1} &= w'_i, \quad \textit{for} \quad i = 1, \ldots, N-1. \end{aligned} \tag{5.1.3}$$

Notation 5.1.5. *For integers $n \leq m$, we write*

$$\widehat{\bigotimes}_{i=n}^{m} A_i := A_n \otimes \cdots \otimes A_m. \tag{5.1.4}$$

For later use, we have to construct the basis for $(0,N)$-modules. These modules are studied in [17].

Fact 5.1.6 ([17])**.** *The following vectors span $\mathcal{F}_{\boldsymbol{u}}^{(0,N)}$ and its dual space:*

$$\begin{aligned} |\boldsymbol{\lambda}\rangle &= \widehat{\bigotimes}_{i=1}^{N} |\lambda^{(i)}\rangle\ , \quad \left(|\lambda^{(i)}\rangle \in \mathcal{F}_{u_i}^{(0,1)}\right) \\ \langle\boldsymbol{\lambda}| &= \widehat{\bigotimes}_{i=1}^{N} \langle\lambda^{(i)}|\ . \end{aligned} \tag{5.1.5}$$

They are normalized as

$$\langle\boldsymbol{\lambda}|\boldsymbol{\mu}\rangle = \prod_{i=1}^{N} \frac{c'_{\lambda^{(i)}}}{c_{\lambda^{(i)}}}\, \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}\,. \tag{5.1.6}$$

Especially, we denote the vacuum state by $|\emptyset\rangle$ and $\langle\emptyset|$.

Remark 5.1.7. *Note that the basis for $(0,N)$-modules are just the tensor product of Macdonald functions, because they are the eigenfunctions of $\mathcal{S}(\Delta^{(N)}(x_0^+)) = \Delta^{(N)}(\psi_1^+) = \psi_1^+\otimes\cdots\otimes\psi_1^+$ with $\mathcal{S}$ the Miki automorphism (Section 3.5). As in the $N=1$ case, $(0,N)$-modules and $(N,0)$-modules are also exchanged under $\mathcal{S}$. For example, we can easily see the following correspondence of the eigenvalues:*

$$\rho^{(N,0)}(x_0^+)\,|P_{\boldsymbol{\lambda}}\rangle = \sum_{k=1}^{N} u_k \epsilon_{\lambda^{(k)}}\,|P_{\boldsymbol{\lambda}}\rangle\ , \quad \rho^{(0,N)}(\psi_1^-)\,|\boldsymbol{\lambda}\rangle = (1-t/q)\sum_{k=1}^{N} u_k \epsilon_{\lambda^{(k)}}\,|P_{\boldsymbol{\lambda}}\rangle\ . \tag{5.1.7}$$

(Recall that $\epsilon_\lambda := (1-t^{-1})\sum_{i=1}^{\ell(\lambda)} q^{\lambda_i} t^{1-i} + t^{-\ell(\lambda)}$.) For more detail, see [88].

Now we define the compositions of the intertwiners, one of which gives the realization of the Mukadé operator.

Definition 5.1.8. *Define the map $T^H(\boldsymbol{u},\boldsymbol{v};w) : \left(\widehat{\bigotimes}_{i=1}^{N}\mathcal{F}_{v_i}^{(0,1)}\right) \to \left(\widehat{\bigotimes}_{i=1}^{N}\mathcal{F}_{u_i}^{(0,1)}\right)$ by the following composition*

$$\begin{aligned} &\mathcal{F}_{v_1}^{(0,1)}\otimes\cdots\otimes\mathcal{F}_{v_{N-1}}^{(0,1)}\otimes\mathcal{F}_{v_N}^{(0,1)}\otimes|0\rangle \xrightarrow{\mathrm{id}\otimes\cdots\otimes\Phi} \mathcal{F}_{v_1}^{(0,1)}\otimes\cdots\otimes\mathcal{F}_{v_{N-1}}^{(0,1)}\otimes\mathcal{F}_{-v_N w'_N}^{(1,1)} \\ &\xrightarrow{\mathrm{id}\otimes\cdots\otimes\Phi^*}\mathcal{F}_{v_1}^{(0,1)}\otimes\cdots\otimes\mathcal{F}_{v_{N-1}}^{(0,1)}\otimes\mathcal{F}_{w_N}^{(1,0)}\otimes\mathcal{F}_{u_N}^{(0,1)} \xrightarrow{\mathrm{id}\otimes\cdots\otimes\Phi\otimes\mathrm{id}} \mathcal{F}_{v_1}^{(0,1)}\otimes\cdots\otimes\mathcal{F}_{-v_{N-1}w'_{N-1}}^{(1,1)}\otimes\mathcal{F}_{u_N}^{(0,1)} \\ &\xrightarrow{\mathrm{id}\otimes\cdots\otimes\Phi^*\otimes\mathrm{id}}\cdots\cdots \xrightarrow{\Phi^*\otimes\cdots\otimes\mathrm{id}} |0\rangle\otimes\mathcal{F}_{u_1}^{(0,1)}\otimes\cdots\otimes\mathcal{F}_{u_{N-1}}^{(0,1)}\otimes\mathcal{F}_{u_N}^{(0,1)}\,. \end{aligned}$$

Here, $\cdots\otimes|0\rangle$ and $|0\rangle\otimes\cdots$ mean taking the vacuum expectation value at the level (1,0) modules. For simplicity, we introduce the following notation,

$$T^H(\boldsymbol{u},\boldsymbol{v};w) = \langle 0|\,\Phi^*[w_1,u_1]\Phi[w'_1,v_1]\Phi^*[w_2,u_2]\Phi[w'_2,v_2]\cdots\Phi^*[w_N,u_N]\Phi[w'_N,v_N]\,|0\rangle\ . \tag{5.1.8}$$

For later use, we introduce the map $\mathcal{T}^H(\boldsymbol{u},\boldsymbol{v};w) : \left(\widehat{\bigotimes}_{i=1}^{N}\mathcal{F}_{v_i}^{(0,1)}\right) \to \left(\widehat{\bigotimes}_{i=1}^{N}\mathcal{F}_{u_i}^{(0,1)}\right)$ by

$$\mathcal{T}^H(\boldsymbol{u},\boldsymbol{v};w) := \frac{T^H(\boldsymbol{u},\boldsymbol{v};w)}{\langle\emptyset|\,T^H(\boldsymbol{u},\boldsymbol{v};w)\,|\emptyset\rangle}\,. \tag{5.1.9}$$

We denote its matrix elements (with respect to the $\mathcal{F}^{(0,N)}$-basis) by

$$\mathcal{T}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\boldsymbol{u},\boldsymbol{v};w) = \langle\boldsymbol{\mu}|\,\mathcal{T}^H(\boldsymbol{u},\boldsymbol{v};w)\,|\boldsymbol{\lambda}\rangle\ . \tag{5.1.10}$$

$|\boldsymbol{\lambda}\rangle$ is the basis for the $(0,N)$-module defined in Fact 5.1.6. Note that we have the expression for the matrix elements of $T^H(\boldsymbol{u},\boldsymbol{v};w)$,

$$\langle\boldsymbol{\mu}|\,T^H(\boldsymbol{u},\boldsymbol{v};w),|\boldsymbol{\lambda}\rangle = \langle 0|\,\Phi^*_{\mu^{(1)}}[w_1,u_1]\Phi_{\lambda^{(1)}}[w'_1,v_1]\cdots\Phi^*_{\mu^{(N)}}[w_N,u_N]\Phi_{\lambda^{(N)}}[w'_N,v_N]\,|0\rangle\ . \tag{5.1.11}$$

Also, define the vertex operator $\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w) : \mathcal{F}_{\boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{v}}$ by

$$\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w) := \frac{T^V(\boldsymbol{u},\boldsymbol{v};w)}{\langle \boldsymbol{0}| T^V(\boldsymbol{u},\boldsymbol{v};w) |\boldsymbol{0}\rangle}\,, \tag{5.1.12}$$

with

$$T^V(\boldsymbol{u},\boldsymbol{v};w) := \sum_{\nu^{(1)},\dots,\nu^{(N-1)}} \prod_{i=1}^{N-1} \frac{c_{\nu^{(i)}}}{c'_{\nu^{(i)}}}\ \Phi^*_{\nu^{(1)}}[v_1,w'_1]\Phi_{\emptyset}[u_1,w_1] \otimes \Phi^*_{\nu^{(2)}}[v_2,w'_2]\Phi_{\nu^{(1)}}[u_2,w_2] \otimes \cdots \\ \cdots \otimes \Phi^*_{\emptyset}[v_N,w'_N]\Phi_{\nu^{(N-1)}}[u_N,w_N]\,, \tag{5.1.13}$$

and its matrix elements (with respect to the $\mathcal{F}^{(N,0)}$-basis, that is, the generalized Macdonald functions) are denoted by

$$\mathcal{T}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\boldsymbol{u},\boldsymbol{v};w) = \langle P_{\boldsymbol{\lambda}}| \mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w) |P_{\boldsymbol{\mu}}\rangle\ . \tag{5.1.14}$$

This operator $\mathcal{T}^V$ guarantees the existence of the Mukadé operator, defined in Def. 5.1.1, and the following proposition completes the proof of Proposition 5.1.3.

Proposition 5.1.9. *For arbitrary $i \in \{1,2,\dots,N\}$, the operator $\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w)$ satisfies*

$$\left(1-\frac{w}{z}\right) X^{(i)}(z)\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w) = \gamma^{-i}\left(1-\gamma^{2i}\frac{w}{z}\right)\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w)X^{(i)}(z)\,. \tag{5.1.15}$$

Proof. The proof is done by direct calculation. For more details, see Section 5.B.2. □

Remark 5.1.10. *The factor γ^{-i} in the RHS of (5.1.15) can be compensated by redefining the spectral parameters as $\gamma^{-1}u_i$. Under this redefinition, the matrix elements of $\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w)$ agree with those of $\mathcal{V}(w)$.*

5.2 Main Claim

Now we state the main claim in this thesis. The matrix elements of the Mukadé operator with respect to the integral forms of the generalized Macdonald functions factorize as the product of the Nekrasov factors.

Theorem 5.2.1.

$$\langle K_{\boldsymbol{\lambda}}(\boldsymbol{v})| \mathcal{V}(x) |K_{\boldsymbol{\mu}}(\boldsymbol{u})\rangle = \frac{\left((-\gamma^2)^N e_N(\boldsymbol{u})x\right)^{|\boldsymbol{\lambda}|}}{(\gamma^2 x)^{|\boldsymbol{\mu}|}} \prod_{i=1}^{N} \frac{u_i^{|\mu^{(i)}|} g_{\mu^{(i)}}}{\left(v_i^{|\lambda^{(i)}|} g_{\lambda^{(i)}}\right)^{N-1}} \cdot \prod_{i,j=1}^{N} N_{\lambda^{(i)},\mu^{(j)}}(qv_i/tu_j)\,. \tag{5.2.1}$$

The proof is given in the next section.

5.2.1 Application to Physics

Before going into the proof, we name some implication of the theorem for physics. Here we name two applications. The first application is for proving Conjecture 2.2.1. The composition of the Mukadé operators gives the instanton partition function $\mathcal{Z}^{(A_{N-1},A_{M-1})}_{\text{inst.}}$ (defined in Fact 2.1.4). Then, Theorem 5.2.1, which shows the equivalence between the Mukadé operator ($=\mathcal{T}^V$) and $\mathcal{T}^H$, gives the proof of the conjecture. As a corollary, we obtain the duality formula for the refined topological vertex under changing preferred directions.

The second application is for proving the 5d version of the Alday-Gaiotto-Tachikawa (AGT) conjecture [4]. In the original AGT, the 4d instanton partition functions are tied to 2d CFT. When we lift the 4d theories to 5d, the counterparts of 2d theories have not been well-formulated, because the VOA structure gets lost via this lift. Theorem 5.2.1 strongly insists that the Mukadé operators are regarded as the q-deformation of the primary fields in CFT.

S-duality (Conjecture 2.2.1)

First, we show the composition of $\mathcal{T}^H$ gives $\mathcal{Z}_{\text{inst.}}^{(A_{N-1},A_{M-1})}$. For that purpose, we prepare the following lemma.

Lemma 5.2.2. *By taking normal ordering, we can compute $\mathcal{T}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}$, defined in (5.1.10), and we obtain*

$$\mathcal{T}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\boldsymbol{u},\boldsymbol{v};w) = \mathcal{T}^H_{mono.} \cdot \mathcal{Z}'_{\boldsymbol{\mu},\boldsymbol{\lambda}}(\boldsymbol{u},\boldsymbol{v})\,, \tag{5.2.2}$$

with

$$\begin{aligned}\mathcal{T}^H_{mono.} = \prod_{i=1}^{N}(-1)^{|\mu^{(i)}|}&\left(\frac{e_N(\boldsymbol{u})}{e_N(\boldsymbol{v})}w\prod_{j=i+1}^{N}v_j\right)^{|\lambda^{(i)}|-|\mu^{(i)}|}\left(\frac{u_i}{\gamma v_i}\right)^{\sum_{j=i}^{N}|\mu^{(j)}|}v_i^{-(N-i)|\lambda^{(i)}|}u_i^{(N-i)|\mu^{(i)}|}\\ &\times\left(\frac{f_{\mu^{(i)}}}{f_{\lambda^{(i)}}}\right)^{N-i+1}q^{n((\lambda^{(i)})')}t^{n(\mu^{(i)})+|\mu^{(i)}|}\gamma^{-(N-i)|\mu^{(i)}|}\,,\end{aligned} \tag{5.2.3}$$

$$\mathcal{Z}'_{\boldsymbol{\mu},\boldsymbol{\lambda}}(\boldsymbol{u},\boldsymbol{v}) = \frac{\prod_{i,j=1}^{N}N_{\lambda^{(i)},\mu^{(j)}}(v_i/\gamma u_j)}{\prod_{1\le i<j\le N}N_{\mu^{(i)},\mu^{(j)}}(qu_i/tu_j)\,N_{\lambda^{(j)},\lambda^{(i)}}(qv_j/tv_i)\prod_{k=1}^{N}c_{\mu^{(k)}}\,c_{\lambda^{(k)}}}\,. \tag{5.2.4}$$

Proof. By direct computation, using some formula in Appendix B.1.1. □

We show the following fact that the M-th composition of $\mathcal{T}^H$ gives the instanton partition function $\mathcal{Z}_{\text{inst.}}^{(A_{N-1},A_{M-1})}(q,t)$.

Proposition 5.2.3. *For $\boldsymbol{s}^{(i)}\in\mathbb{C}^N$ $(i=1,\dots,M)$, we have*

$$\langle\boldsymbol{\emptyset}|\overset{\curvearrowleft}{\prod_{1\le i\le M}}\mathcal{T}^H(\boldsymbol{s}^{(i+1)},\boldsymbol{s}^{(i)};x_i)\,|\boldsymbol{\emptyset}\rangle = \mathcal{Z}_{inst.}^{(A_{N-1},A_{M-1})}((\mathfrak{q}_k)|\boldsymbol{s}^{(2)},\dots,\boldsymbol{s}^{(M)}|\boldsymbol{0}|\gamma\boldsymbol{s}^{(M+1)},\gamma\boldsymbol{s}^{(1)},\boldsymbol{\gamma})\,, \tag{5.2.5}$$

with the following identification of the instanton fugacities:

$$\mathfrak{q}_k \quad\longleftrightarrow\quad \frac{e_N(\boldsymbol{s}^{(k+1)})}{\gamma^N e_N(\boldsymbol{s}^{(k)})}\frac{x_k}{x_{k-1}}\,. \tag{5.2.6}$$

We use the notations $\boldsymbol{0}=(0,\dots,0),\boldsymbol{\gamma}=(\gamma,\dots,\gamma)$.

Proof. We insert the complete system $\mathrm{id}=\sum_{\boldsymbol{\lambda}\in\mathsf{P}^N}\frac{|\boldsymbol{\lambda}\rangle\langle\boldsymbol{\lambda}|}{\langle\boldsymbol{\lambda}|\boldsymbol{\lambda}\rangle}$, and use Lemma 5.2.2. We use the simplified notations $c_{\boldsymbol{\lambda}}=\prod_{i=1}^{N}c_{\lambda^{(i)}}$, and $\lambda^{[i,j]}=(\boldsymbol{\lambda}^{[i]})^{(j)}\in\mathsf{P}$, the j-th element of N-tuple of partitions $\boldsymbol{\lambda}^{[i]}$. With $\boldsymbol{\lambda}^{[1]}=\boldsymbol{\lambda}^{[M+1]}=\boldsymbol{\emptyset}$, we have

$$\begin{aligned}\langle\boldsymbol{\emptyset}|\overset{\curvearrowleft}{\prod_{1\le i\le M}}\mathcal{T}^H(\boldsymbol{s}^{(i+1)},\boldsymbol{s}^{(i)};x_i)\,|\boldsymbol{\emptyset}\rangle &= \sum_{\substack{\boldsymbol{\lambda}^{[i]}\in\mathsf{P}^N,\\ i=2,\dots,M}}\prod_{i=2}^{M}\frac{c_{\boldsymbol{\lambda}^{[i]}}}{c'_{\boldsymbol{\lambda}^{[i]}}}\prod_{i=1}^{M}\langle\boldsymbol{\lambda}^{[i+1]}|\,\mathcal{T}^H(\boldsymbol{s}^{(i+1)},\boldsymbol{s}^{(i)};x_i)\,|\boldsymbol{\lambda}^{[i]}\rangle\\ &= \sum_{\substack{\boldsymbol{\lambda}^{[i]}\in\mathsf{P}^N,\\ i=2,\dots,M}}\prod_{k=2}^{M}\left(\frac{e_N(\boldsymbol{s}^{(k+1)})}{\gamma^N e_N(\boldsymbol{s}^{(k)})}\frac{x_k}{x_{k-1}}\right)^{|\boldsymbol{\lambda}^{[k]}|}\frac{\prod_{k=1}^{M}\prod_{i,j=1}^{N}N_{\lambda^{[k,i]},\lambda^{[k+1,j]}}(s_i^{(k)}/\gamma s_j^{(k+1)})}{\prod_{k=2}^{M}\prod_{i,j=1}^{N}N_{\lambda^{[k,i]},\lambda^{[k,j]}}(qs_i^{(k)}/ts_j^{(k)})}\,.\end{aligned} \tag{5.2.7}$$

□

Although, in this case, the Chern-Simons levels are fixed to zero, we can change them by replacing the intertwiners with the other type of intertwiners ($M\neq 0$ intertwiners in Fact 3.4.5).

We also have the following proposition which says the M-th composition of $\mathcal{T}^V$ essentially gives $\mathcal{Z}_{\text{inst.}}^{(A_{M-1},A_{N-1})}(q,t)$.

Proposition 5.2.4. *For $\boldsymbol{s}^{(i)} \in \mathbb{C}^N$ $(i=1,\ldots,M+1)$ and $\boldsymbol{x} \in \mathbb{C}^M$, let $\boldsymbol{x}^{(i)} \in \mathbb{C}^M$ $(i=1,\ldots,N+1)$ be the vectors whose elements are determined by the relations*

$$\begin{aligned} x_j^{(1)} &:= x_j\,, \quad (j=1,\ldots,M) \\ x_j^{(i)} &:= \frac{s_{i-1}^{(j+1)}}{s_{i-1}^{(j)}} x_j^{(i-1)}\,, \quad (i=2,\ldots,N+1, \text{and } j=1,\ldots,M)\,. \end{aligned} \tag{5.2.8}$$

Using this notation, we have

$$\langle \mathbf{0}| \prod_{1\le i\le M}^{\curvearrowright} \mathcal{T}^V(\boldsymbol{s}^{(i+1)}, \boldsymbol{s}^{(i)}; x_i)\,|\mathbf{0}\rangle = \mathcal{M}\cdot \mathcal{Z}_{inst.}^{(A_{M-1},A_{N-1})}((\mathfrak{q}'_k)|\boldsymbol{x}^{(2)},\ldots,\boldsymbol{x}^{(N)}|\mathbf{0}|\gamma\boldsymbol{x}^{(N+1)},\gamma\boldsymbol{x}^{(1)},\boldsymbol{\gamma})\,, \tag{5.2.9}$$

with

$$\mathcal{M} = \frac{\prod_{i=1}^N \langle\emptyset|\,T^{H,M}(\boldsymbol{x}^{(i+1)},\boldsymbol{x}^{(i)}; s_i^{(1)})\,|\emptyset\rangle}{\prod_{i=1}^M \langle\mathbf{0}|\,T^V(\boldsymbol{s}^{(i+1)},\boldsymbol{s}^{(i)};x_i)\,|\mathbf{0}\rangle}\,, \tag{5.2.10}$$

(where $T^{H,M}(\boldsymbol{u},\boldsymbol{v};x)$'s are same as T^H except that they are maps from $\mathcal{F}_{\boldsymbol{v}}^{(0,M)}$ to $\mathcal{F}_{\boldsymbol{u}}^{(0,M)}$) and the identifications

$$\mathfrak{q}'_k \quad\longleftrightarrow\quad \frac{e_M(\boldsymbol{x}^{(k+1)})}{\gamma^M e_M(\boldsymbol{x}^{(k)})}\frac{s_k^{(1)}}{s_{k-1}^{(1)}}\,. \tag{5.2.11}$$

Proof. First, we have

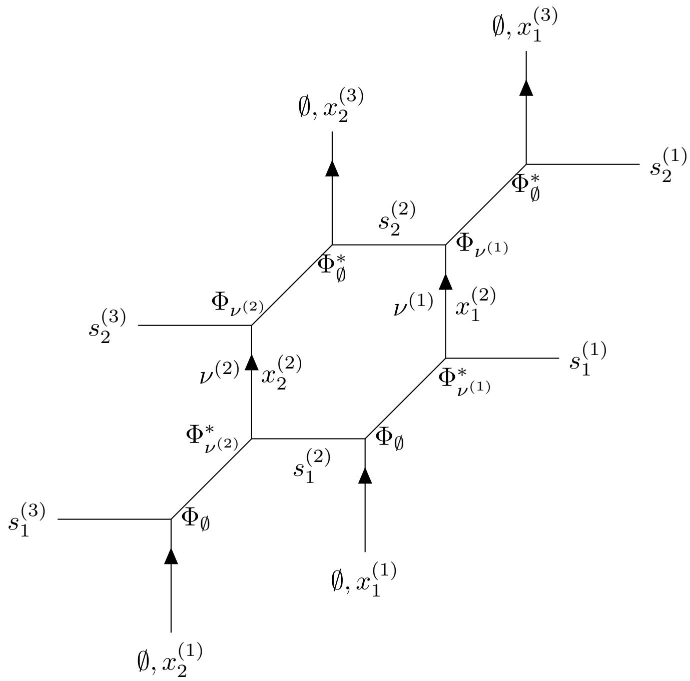
$$\langle\mathbf{0}| \prod_{1\le i\le M}^{\curvearrowright} T^V(\boldsymbol{s}^{(i+1)},\boldsymbol{s}^{(i)};x_i)\,|\mathbf{0}\rangle = \langle\emptyset| \prod_{1\le i\le N}^{\curvearrowleft} T^{H,M}(\boldsymbol{x}^{(i+1)},\boldsymbol{x}^{(i)};s_i^{(1)})\,|\emptyset\rangle\,, \tag{5.2.12}$$

where $T^{H,M}$'s are same as T^H except that they are operators acting on M-th Fock-tensor spaces as explained above. The equality above is trivial when we write down both sides using the intertwiners Φ's and Φ^*'s. Note that both sides are schematically the vacuum expectation value of the map from $\mathcal{F}_{\boldsymbol{x}^{(1)}}^{(0,M)} \otimes \mathcal{F}_{\boldsymbol{s}^{(M+1)}}^{(N,0)}$ to $\mathcal{F}_{\boldsymbol{s}^{(1)}}^{(N,0)} \otimes \mathcal{F}_{\boldsymbol{x}^{(N+1)}}^{(0,M)}$.

Taking the case $N=M=2$ as an example, let us check this identity. In this case, the LHS of (5.2.12) becomes

$$\begin{aligned} &\langle\mathbf{0}|\,T^V(\boldsymbol{s}^{(2)},\boldsymbol{s}^{(1)};x_1)T^V(\boldsymbol{s}^{(3)},\boldsymbol{s}^{(2)};x_2)\,|\mathbf{0}\rangle \\ &= \sum_{\nu^{(1)},\nu^{(2)}\in\mathsf{P}} \prod_{i=1}^2 \frac{c_{\nu^{(i)}}}{c'_{\nu^{(i)}}}\ \langle 0|\,\Phi^*_{\nu^{(1)}}[s_1^{(1)},x_1^{(2)}]\Phi_\emptyset[s_1^{(2)},x_1^{(1)}]\Phi^*_{\nu^{(2)}}[s_1^{(2)},x_2^{(2)}]\Phi_\emptyset[s_1^{(3)},x_2^{(1)}]\,|0\rangle \\ &\qquad\times \langle 0|\,\Phi^*_\emptyset[s_2^{(1)},x_1^{(3)}]\Phi_{\nu^{(1)}}[s_2^{(2)},x_1^{(2)}]\Phi^*_\emptyset[s_2^{(2)},x_2^{(3)}]\Phi_{\nu^{(2)}}[s_2^{(3)},x_2^{(2)}]\,|0\rangle\,, \end{aligned} \tag{5.2.13}$$

which corresponds to the vacuum expectation value of the following diagram, obtained by gluing two vertical ladders:



Then, we can decompose this diagram to two horizontal ladders. That is, once we recall the definition of $\langle \boldsymbol{\mu}| T^H |\boldsymbol{\lambda}\rangle$ in Definition 5.1.8, the RHS of (5.2.13) can be written as

$$\sum_{\nu^{(1)},\nu^{(2)}\in\mathsf{P}} \prod_{i=1}^{2} \frac{c_{\nu^{(i)}}}{c'_{\nu^{(i)}}} \ \langle \boldsymbol{\nu}| T^H(\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(1)}; s_1^{(1)}) |\boldsymbol{\emptyset}\rangle \cdot \langle \boldsymbol{\emptyset}| T^H(\boldsymbol{x}^{(3)}, \boldsymbol{x}^{(2)}; s_2^{(1)}) |\boldsymbol{\nu}\rangle$$
$$= \langle \boldsymbol{\emptyset}| T^H(\boldsymbol{x}^{(3)}, \boldsymbol{x}^{(2)}; s_2^{(1)})\ T^H(\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(1)}; s_1^{(1)}) |\boldsymbol{\emptyset}\rangle\ . \tag{5.2.14}$$

Here, we use the fact $\mathrm{id} = \sum_{\boldsymbol{\nu}\in\mathsf{P}^2} \prod_{i=1}^2 \frac{c_{\nu^{(i)}}}{c'_{\nu^{(i)}}} \ |\boldsymbol{\nu}\rangle \langle\boldsymbol{\nu}|$. Thus, we have the expected identity. For the generic N and M, we can do the same calculation.

We also have to note that we have extra factor $\mathcal{M}$ which appears from the normalizations of $\mathcal{T}^V$'s and $\mathcal{T}^H$'s. Then, by Proposition Proposition 5.2.3, we obtain the result. □

As the direct consequence of Theorem 5.2.1, we have the following theorem.

Theorem 5.2.5. *The following equality between the two matrix elements holds:*

$$\mathcal{T}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\boldsymbol{u}, \boldsymbol{v}; w) = (-1)^{|\boldsymbol{\lambda}|+|\boldsymbol{\mu}|} \mathcal{T}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\boldsymbol{u}, \boldsymbol{v}; w)\,. \tag{5.2.15}$$

Proof. We can show, up to the monomial factors, the both sides are equal to

$$\frac{\prod_{i,j=1}^N N_{\lambda^{(i)},\mu^{(j)}}(v_i/\gamma u_j)}{\prod_{1\le i<j\le N} N_{\mu^{(i)},\mu^{(j)}}(qu_i/tu_j) \prod_{k=1}^N c_{\mu^{(k)}} \prod_{1\le i<j\le N} N_{\lambda^{(j)},\lambda^{(i)}}(qv_j/tv_i) \prod_{k=1}^N c_{\lambda^{(k)}}}\,. \tag{5.2.16}$$

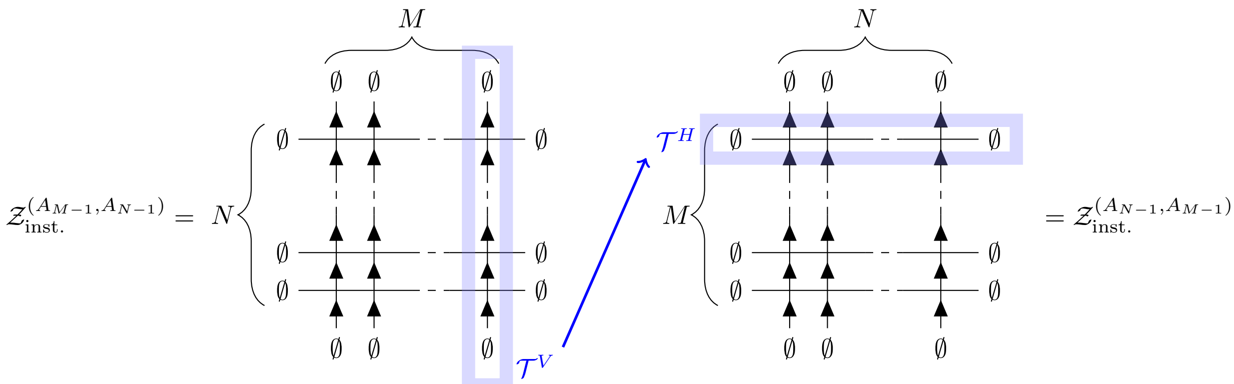
For the RHS, it is obvious from Theorem 5.2.1 and Definition 4.2.25. For the LHS, it can be shown by a direct computation, using some formulas in Appendix B.1.1. The monomial factor $(-1)^{|\boldsymbol{\lambda}|+|\boldsymbol{\mu}|}$ is also determined by direct calculation. □

Then, combining Proposition 5.2.3, Proposition 5.2.4 and Theorem 5.2.5, we have the following proposition.

Proposition 5.2.6. *Using the same notations in Proposition 5.2.3 and Proposition 5.2.4, we have*

$$\begin{aligned}&\mathcal{Z}^{(A_{N-1},A_{M-1})}_{\textit{inst.}}((\mathfrak{q}_k)|\boldsymbol{s}^{(2)},\dots,\boldsymbol{s}^{(M)}|\boldsymbol{0}|\gamma\boldsymbol{s}^{(M+1)},\gamma\boldsymbol{s}^{(1)},\boldsymbol{\gamma})\\ &\quad=\mathcal{M}\cdot\mathcal{Z}^{(A_{M-1},A_{N-1})}_{\textit{inst.}}((\mathfrak{q}'_k)|\boldsymbol{x}^{(2)},\dots,\boldsymbol{x}^{(N)}|\boldsymbol{0}|\gamma\boldsymbol{x}^{(N+1)},\gamma\boldsymbol{x}^{(1)},\boldsymbol{\gamma})\,.\end{aligned} \tag{5.2.17}$$

The following figure shows what this proposition says. The arrows in the figure means they are identical up to some scalar multiplication.



Here, let us consider the proportional factor $\mathcal{M}$. We first compute the vacuum expectation value of T^H.

Lemma 5.2.7. *We have*

$$\langle\boldsymbol{\emptyset}|\,T^H(\boldsymbol{u},\boldsymbol{v};x)\,|\boldsymbol{\emptyset}\rangle=G^{(N)}(\boldsymbol{u}|\boldsymbol{v})\,, \tag{5.2.18}$$

with

$$G^{(N)}(\boldsymbol{u}|\boldsymbol{v}):=\frac{\prod_{1\le i<j\le N}\mathcal{G}(u_j/u_i)\cdot\mathcal{G}(qv_j/tv_i)}{\prod_{1\le j<i\le N}\mathcal{G}(u_i/\gamma v_j)\ \prod_{1\le i\le j\le N}\mathcal{G}(v_j/\gamma u_i)}\,. \tag{5.2.19}$$

The proof only requires the normal ordering formulas in Appendix B.1.1.

As a result, using the functions in Fact 2.1.7 and Fact 2.1.17, we have

$$\prod_{i=1}^{M}\langle\boldsymbol{\emptyset}|\,T^H(\boldsymbol{s}^{(i+1)},\boldsymbol{s}^{(i)};x_i)\,|\boldsymbol{\emptyset}\rangle=\mathcal{Z}^{(A_{N-1},A_{M-1})}_{\textit{extra}}(\boldsymbol{s}^{(1)}|\boldsymbol{s}^{(M+1)})\cdot\mathcal{Z}^{(A_{N-1},A_{M-1})}_{\textit{1-loop}}(\boldsymbol{s}^{(1)},\dots,\boldsymbol{s}^{(M)}|\boldsymbol{s}^{(1)},\boldsymbol{s}^{(M+1)})\,. \tag{5.2.20}$$

Now, we state a conjecture about the vacuum expectation value of T^V.

Conjecture 5.2.8. *We conjecture*

$$\langle\boldsymbol{0}|\,T^V(\boldsymbol{u},\boldsymbol{v};x)\,|\boldsymbol{0}\rangle=\langle\boldsymbol{\emptyset}|\,T^H(\boldsymbol{u},\boldsymbol{v};x)\,|\boldsymbol{\emptyset}\rangle\ . \tag{5.2.21}$$

Using the mathematica, we have checked this identity for small N by expanding the both sides up to some powers of the spectral parameters $\boldsymbol{u},\boldsymbol{v}$.

Note that the LHS of the conjecture above is the infinite series,

$$\begin{aligned}&\langle\boldsymbol{0}|\,T^V(\boldsymbol{u},\boldsymbol{v};x)\,|\boldsymbol{0}\rangle\\ &=\sum_{\nu^{(1)},\dots,\nu^{(N-1)}}\prod_{i=1}^{N-1}\frac{c_{\nu^{(i)}}}{c'_{\nu^{(i)}}}\ \langle 0|\,\Phi^*_{\nu^{(1)}}[v_1,w'_1]\Phi_{\emptyset}[u_1,w_1]\,|0\rangle\otimes\langle 0|\,\Phi^*_{\nu^{(2)}}[v_2,w'_2]\Phi_{\nu^{(1)}}[u_2,w_2]\,|0\rangle\otimes\cdots\\ &\qquad\cdots\otimes\langle 0|\,\Phi^*_{\emptyset}[v_N,w'_N]\Phi_{\nu^{(N-1)}}[u_N,w_N]\,|0\rangle\ ,\end{aligned} \tag{5.2.22}$$

while the RHS factorizes as the products of the double infinite product, $\mathcal{G}$. Thus, the conjecture is highly non-trivial. At this point, we do not know the proof, and this is our future work. Then, as a corollary, we obtain the following conjecture.

Conjecture 5.2.9. *We use the same notations as in Proposition 5.2.3 and Proposition 5.2.4. For* $\boldsymbol{s}^{(i)} \in \mathbb{C}^N$ $(i = 1, \ldots, M+1)$ *and* $\boldsymbol{x} \in \mathbb{C}^M$*, put*

$$\begin{aligned} &\mathcal{Z}^{(A_{N-1},A_{M-1})}_{top.}(\boldsymbol{s}^{(1)}, \ldots, \boldsymbol{s}^{(M+1)}|\boldsymbol{x}) := \mathcal{Z}^{(A_{N-1},A_{M-1})}_{extra}(\boldsymbol{s}^{(1)}|\boldsymbol{s}^{(M+1)}) \\ &\quad \times \mathcal{Z}^{(A_{N-1},A_{M-1})}_{1\text{-}loop}(\boldsymbol{s}^{(1)}, \ldots, \boldsymbol{s}^{(M)}|\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(M+1)}) \cdot \mathcal{Z}^{(A_{N-1},A_{M-1})}_{inst.}((\mathfrak{q}_k)|\boldsymbol{s}^{(2)}, \ldots, \boldsymbol{s}^{(M)}|\boldsymbol{0}|\gamma\boldsymbol{s}^{(M+1)}, \gamma\boldsymbol{s}^{(1)}, \boldsymbol{\gamma}), \end{aligned} \tag{5.2.23}$$

that is, the explicit form is

$$\begin{aligned} &\mathcal{Z}^{(A_{N-1},A_{M-1})}_{top.}(\boldsymbol{s}^{(1)}, \ldots, \boldsymbol{s}^{(M+1)}|\boldsymbol{x}) \\ &= \prod_{i=1}^{M} G^{(N)}(\boldsymbol{s}^{(i+1)}|\boldsymbol{s}^{(i)}) \\ &\quad \times \sum_{\substack{\boldsymbol{\lambda}^{[i]} \in \mathsf{P}^N, \\ i=2,\ldots,M}} \prod_{k=2}^{M} \left(\frac{e_N(\boldsymbol{s}^{(k+1)})}{\gamma^N e_N(\boldsymbol{s}^{(k)})} \frac{x_k}{x_{k-1}} \right)^{|\boldsymbol{\lambda}^{[k]}|} \frac{\prod_{k=1}^{M} \prod_{i,j=1}^{N} N_{\lambda^{[k,i]},\lambda^{[k+1,j]}}(s_i^{(k)}/\gamma s_j^{(k+1)})}{\prod_{k=2}^{M} \prod_{i,j=1}^{N} N_{\lambda^{[k,i]},\lambda^{[k,j]}}(q s_i^{(k)}/t s_j^{(k)})}. \end{aligned} \tag{5.2.24}$$

Then, combining Proposition 5.2.6 and Conjecture 5.2.8, we conjecture

$$\mathcal{Z}^{(A_{N-1},A_{M-1})}_{top.}(\boldsymbol{s}^{(1)}, \ldots, \boldsymbol{s}^{(M+1)}|\boldsymbol{x}) = \mathcal{Z}^{(A_{M-1},A_{N-1})}_{top.}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N+1)}|\boldsymbol{s}^{(1)}). \tag{5.2.25}$$

Rephrasing in Terms of Refined Topological Vertex Now let us rephrase Theorem 5.2.5 in terms of the refined topological vertex. We first introduce some notation.

Notation 5.2.10. *Denote by* $|s_{\boldsymbol{\lambda}}\rangle$*, the* N*-tensor product of the Schur functions, defined in Section 3.4.3, that is,*

$$\begin{aligned} |s_{\boldsymbol{\lambda}}\rangle &= \overset{\curvearrowright}{\bigotimes}{}_{i=1}^{N} |s_{\lambda^{(i)}}\rangle, \\ \langle S_{\boldsymbol{\lambda}}| &= \overset{\curvearrowright}{\bigotimes}{}_{i=1}^{N} \langle S_{\lambda^{(i)}}|. \end{aligned} \tag{5.2.26}$$

Note that $\langle S_{\boldsymbol{\mu}} | s_{\boldsymbol{\lambda}}\rangle = \delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}$*.*

The following lemma is the direct consequence of Proposition 3.4.9.

Lemma 5.2.11. *We have*

$$\langle \boldsymbol{\mu}| T^H(\boldsymbol{u}, \boldsymbol{v}; w) |\boldsymbol{\lambda}\rangle = \prod_{i=1}^{N} \frac{c'_{\lambda^{(i)}}}{c_{\lambda^{(i)}}} (-q^{-1/2} w'_i)^{|\lambda^{(i)}|} (q^{-1/2} w_i)^{-|\mu^{(i)}|} \cdot \mathcal{C}^H_{\boldsymbol{\lambda}',\boldsymbol{\mu}}((-u_{i+1}/v_i), (-v_i/u_i)), \tag{5.2.27}$$

$$\langle S_{\boldsymbol{\lambda}}| T^V(\boldsymbol{u}, \boldsymbol{v}; w) |s_{\boldsymbol{\mu}}\rangle = \prod_{i=1}^{N} \frac{f_{\mu^{(i)}}}{f_{\lambda^{(i)}}} (-q^{-1/2} w'_i)^{|\lambda^{(i)}|} (q^{-1/2} w_i)^{-|\mu^{(i)}|} \cdot \mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}'}((-u_{i+1}/v_i), (-v_i/u_i)). \tag{5.2.28}$$

$\mathcal{C}^H_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *and* $\mathcal{C}^V_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *are defined in Definition 2.2.2, and we use the notation in Notation 5.1.4.*

Definition 5.2.12. *Because both* $|P_{\boldsymbol{\lambda}}\rangle$ *and* $|s_{\boldsymbol{\lambda}}\rangle$ *form the basis of* $\mathcal{F}^{(N,0)}$*, there exists the transition matrix which connects the two bases. Denote the transition matrix by*

$$|P_{\boldsymbol{\lambda}}\rangle = \sum_{\substack{\boldsymbol{\mu} \in \mathsf{P}^N, \\ |\boldsymbol{\mu}|=|\boldsymbol{\lambda}|}} T_{\boldsymbol{\lambda},\boldsymbol{\mu}} |s_{\boldsymbol{\mu}}\rangle, \qquad \langle Q_{\boldsymbol{\lambda}}| = \sum_{\substack{\boldsymbol{\mu} \in \mathsf{P}^N, \\ |\boldsymbol{\mu}|=|\boldsymbol{\lambda}|}} T^*_{\boldsymbol{\lambda},\boldsymbol{\mu}} \langle S_{\boldsymbol{\mu}}|. \tag{5.2.29}$$

Note that $T_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and $T^*_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ are inverse to each other, that is,

$$\sum_{\boldsymbol{\lambda}} T_{\boldsymbol{\lambda},\boldsymbol{\mu}} T^*_{\boldsymbol{\lambda},\boldsymbol{\nu}} = \delta_{\boldsymbol{\mu},\boldsymbol{\nu}}. \tag{5.2.30}$$

$T_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ can be computed as follows. First, recall (Definition 4.2.9) the generalized Macdonald functions can be expanded in the tensor products of the usual Macdonald functions, as

$$|P_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle = \prod_{i=1}^{N} P_{\lambda^{(i)}}(a_{-n}^{(i)})\,|\mathbf{0}\rangle + \sum_{\boldsymbol{\mu}\overset{*}{<}\boldsymbol{\lambda}} u_{\boldsymbol{\lambda},\boldsymbol{\mu}} \prod_{i=1}^{N} P_{\mu^{(i)}}(a_{-n}^{(i)})\,|\mathbf{0}\rangle\,, \quad u_{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \mathbb{C}\,. \tag{5.2.31}$$

From the explicit formula (Theorem 4.2.21), we can compute $u_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ from the bottom. Second, the Macdonald functions can be expanded in the Schur functions, as

$$|P_{\mu}\rangle = \sum_{\nu} C_{\nu\mu}(q,t)\,|s_{\nu}\rangle\;, \tag{5.2.32}$$

because both functions span $\mathcal{F}$. Combining these two facts, we have

$$T_{\boldsymbol{\lambda},\boldsymbol{\nu}} = \sum_{\boldsymbol{\mu}} u_{\boldsymbol{\lambda},\boldsymbol{\mu}} \prod_{i=1}^{N} C_{\nu^{(i)}\mu^{(i)}}(q,t)\,. \tag{5.2.33}$$

The closed formula for the generalized Jack functions (that is, the 4d case) is studied in [106]. They use the stable envelope to compute the coefficients, and thus we conjecture the Macdonald version is obtained by exponentiating every factors like the Nekrasov factors. This is our future work.

Then, as a corollary of Theorem 5.2.5 and Lemma 5.2.11, we have the following identity of the refined topological vertex. Again, we use the notation like $f_{\boldsymbol{\lambda}} = \prod_{i=1}^{N} f_{\lambda^{(i)}}$.

Corollary 5.2.13. *We have*

$$\begin{aligned}
&(-1)^{|\boldsymbol{\lambda}|+|\boldsymbol{\mu}|} \prod_{i=1}^{N} (w_i')^{|\lambda^{(i)}|} w_i^{-|\mu^{(i)}|} \cdot \frac{\mathcal{C}^{H}_{\boldsymbol{\lambda}',\boldsymbol{\mu}}\left((-u_{i+1}/v_i),(-v_i/u_i)\right)}{G^{(N)}(\boldsymbol{u}|\boldsymbol{v})} \\
&= \sum_{\substack{\boldsymbol{\sigma},\boldsymbol{\nu}\in\mathsf{P}^N \\ |\boldsymbol{\sigma}|=|\boldsymbol{\lambda}|,|\boldsymbol{\nu}|=|\boldsymbol{\mu}|}} \frac{f_{\boldsymbol{\nu}}}{f_{\boldsymbol{\sigma}}} \prod_{i=1}^{N} (w_i')^{|\sigma^{(i)}|} w_i^{-|\nu^{(i)}|} \cdot T^{*}_{\boldsymbol{\lambda},\boldsymbol{\sigma}} T_{\boldsymbol{\mu},\boldsymbol{\nu}} \frac{\mathcal{C}^{V}_{\boldsymbol{\sigma},\boldsymbol{\nu}'}\left((-u_{i+1}/v_i),(-v_i/u_i)\right)}{\mathcal{C}^{V}_{\boldsymbol{\emptyset},\boldsymbol{\emptyset}}\left((-u_{i+1}/v_i),(-v_i/u_i)\right)}\,,
\end{aligned} \tag{5.2.34}$$

where $G^{(N)}(\boldsymbol{u}|\boldsymbol{v})$ is defined in Lemma 5.2.7.

AGT Proof

Second, Theorem 5.2.1 is the proof of the five-dimensional analogue of the Alday-Gaiotto-Tachikawa (AGT) conjecture [4]. One proof of the 4d AGT conjecture is given in [3]. The basic idea of [3] is to compute the matrix elements of the Virasoro primary fields with respect to the generalized Jack functions, and the result shows they factorize as the bifundamental contribution, that is, the product of 4d Nekrasov factors.

Because as remarked in the previous subsection, the Mukadé operator is the q-analogue of the primary field, Theorem 5.2.1 is the direct generalization of the AGT proof *á la* [3] to the five dimensions.

We can confirm the correspondence more concretely. The 4-point function of the Mukadé operators gives

$$\begin{aligned}
\langle\mathbf{0}|\,\mathcal{V}\binom{\boldsymbol{w}}{\boldsymbol{v}};z_1\Big)\mathcal{V}\Big(\binom{\boldsymbol{v}}{\boldsymbol{u}};z_2\Big)\,|\mathbf{0}\rangle &= \sum_{\boldsymbol{\lambda}} \frac{\langle\mathbf{0}|\,\mathcal{V}\Big(\binom{\boldsymbol{w}}{\boldsymbol{v}};z_1\Big)\,|K_{\boldsymbol{\lambda}}\rangle\,\langle K_{\boldsymbol{\lambda}}|\,\mathcal{V}\Big(\binom{\boldsymbol{v}}{\boldsymbol{u}};z_2\Big)\,|\mathbf{0}\rangle}{\langle K_{\boldsymbol{\lambda}}|K_{\boldsymbol{\lambda}}\rangle} \\
&= \sum_{\boldsymbol{\lambda}} \left(\frac{e_N(\boldsymbol{u})z_2}{e_N(\boldsymbol{v})z_1}\right)^{|\boldsymbol{\lambda}|} \prod_{i,j=1}^{N} \frac{N_{\emptyset,\lambda^{(j)}}(qw_i/tv_j)N_{\lambda^{(i)},\emptyset}(qv_i/tu_j)}{N_{\lambda^{(i)},\lambda^{(j)}}(qv_i/tv_j)}\,.
\end{aligned} \tag{5.2.35}$$

This is the instanton partition function of $U(N)$ gauge theory with $2N$ flavors. The similar equality in 4d was the original statement of the AGT correspondence.

5.3 Kajihara-Noumi Identity

First, we state the Kajihara-Noumi identity in the most general form. The formula holds for the rational, trigonometric, and elliptic hypergeometric series, and thus we introduce the unified notation.

Notation 5.3.1. *We denote by $[x]$, a non-zero holomorphic function on $\mathbb{C}$, satisfying*

$$\begin{aligned} &[-x] = -[x]\,, \\ &[x \pm y][u \pm v] + [x \pm v][y \pm u] = [x \pm u][y \pm v]\,, \end{aligned} \tag{5.3.1}$$

where $[x \pm y] = [x+y][x-y]$. The second relation is called the recurrence relation of Hirota-type.

We also use

$$[x]_k := [x][x+\delta]\cdots[x+(k-1)\delta]\,. \tag{5.3.2}$$

By the Hermite's theorem (see [117, Exercise 38 of Chapter 20]), the function which satisfies the above conditions is any of following three functions up to constant:

$$\begin{array}{lcc} [x] = & x\,, & \text{(rational)} \\ [x] = & \sin(x)\,, & \text{(trigonometric)} \\ [x] = & \sigma(x|\mathbb{Z}w_1 \oplus \mathbb{Z}w_2)\,, & \text{(elliptic)} \end{array} \tag{5.3.3}$$

where $\sigma(x|\Omega)$ is the Weierstrass sigma function.

Definition 5.3.2. *For $(a_1,\dots,a_m),(x_1,\dots,x_m) \in \mathbb{C}^m$ and $(b_1,\dots,b_n),(c_1,\dots,c_n) \in \mathbb{C}^n$, define*

$$\Phi_L^{m,n}\left(\begin{matrix} a_1,\dots,a_m \\ x_1,\dots,x_m \end{matrix}\middle| \begin{matrix} b_1,\dots,b_n \\ c_1,\dots,c_n \end{matrix}\right) = \sum_{\mu \in \mathbb{Z}_{\geq 0}^m, |\mu| = L} \frac{\Delta(x+\mu\delta)}{\Delta(x)} \prod_{1 \leq i,j \leq m} \frac{[x_i - x_j + a_j]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}} \cdot \prod_{\substack{1 \leq k \leq n \\ 1 \leq i \leq m}} \frac{[x_i + b_k]_{\mu_i}}{[x_i + c_k]_{\mu_i}}\,, \tag{5.3.4}$$

with

$$\Delta(x) := \prod_{1 \leq i < j \leq m} [x_i - x_j]\,, \quad \Delta(x+\mu\delta) = \prod_{1 \leq i < j \leq m} [x_i - x_j + (\mu_i - \mu_j)\delta]\,. \tag{5.3.5}$$

Note that this series $\Phi_L^{m,n}$ is of the form of very well-poised hypergeometric series.

Now we state the Kajihara-Noumi identity, which is the duality formula for exchanging the two integers m and n in $\Phi_L^{m,n}$. After [55], Rosengren also found the similar transformation formula from the study of the elliptic kernel functions in [95].

Fact 5.3.3 (Kajihara-Noumi identity [55])**.** *Let $(a_1,\dots,a_m) \in \mathbb{C}^m$ and $(b_1,\dots,b_n) \in \mathbb{C}^n$ two sets of parameters, satisfying the balancing condition,*

$$a_1 + \cdots + a_m = b_1 + \cdots + b_n\,. \tag{5.3.6}$$

Then for two sets of variables $(x_1,\dots,x_m)$ and $(y_1,\dots,y_n)$, we have

$$\Phi_L^{m,n}\left(\begin{matrix} a_1,\dots,a_m \\ x_1,\dots,x_m \end{matrix}\middle| \begin{matrix} y_1-b_1,\dots,y_n-b_n \\ y_1,\dots,y_n \end{matrix}\right) = \Phi_L^{n,m}\left(\begin{matrix} b_1,\dots,b_n \\ y_1,\dots,y_n \end{matrix}\middle| \begin{matrix} x_1-a_1,\dots,x_m-a_m \\ x_1,\dots,x_m \end{matrix}\right). \tag{5.3.7}$$

This identity contains, in the various limits, most of the known transformation formulas of the multiple hypergeometric functions. For example, when $n = 1$ and $m = 2$, the identity reduces to the Frenkel-Turaev formula [36, Section 11]. When $n = m = 2$, it reduces to the elliptic Bailey identity.

The proof is very beautiful, and we just show the idea of proof. Let $\boldsymbol{z} = (z_1,\dots,z_M)$, $\boldsymbol{w} = (w_1,\dots,w_M)$, and $\lambda \in \mathbb{C}$. Then, by the Frobenius determinant formula, we have

$$D(\boldsymbol{z};\boldsymbol{w}|\lambda) := \det\left(\frac{[\lambda + z_i + w_j]}{[\lambda][z_i + w_j]}\right)_{i,j=1}^M = \frac{[\lambda + \sum_i z_i + \sum_i w_i]\Delta(\boldsymbol{z})\Delta(\boldsymbol{w})}{[\lambda]\prod_{i,j}[z_i + w_j]}\,. \tag{5.3.8}$$

Next, we define the generating currents of the shift operators,

$$E(T_z;u) := \prod_{i=1}^{M}(1+uT^{\delta}_{z_i})\,, \tag{5.3.9}$$

with T^{δ}_z is the shift operator in the additive notation. Then, we have the Cauchy identity,

$$\frac{E(T_z;u)D(\boldsymbol{z};\boldsymbol{w}|\lambda)}{D(\boldsymbol{z};\boldsymbol{w}|\lambda)} = \frac{E(T_w;u)D(\boldsymbol{z};\boldsymbol{w}|\lambda)}{D(\boldsymbol{z};\boldsymbol{w}|\lambda)}\,. \tag{5.3.10}$$

Next, for $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{N}^m$ with $|\alpha| = M$, we put

$$(z_1,\ldots,z_M) = (x_1, x_1+\delta, x_1+(\alpha_1-1)\delta, \ldots\ldots, x_m, x_m+\delta,\ldots,x_m+(\alpha_m-1)\delta)\,. \tag{5.3.11}$$

We do the similar substitution for $\boldsymbol{w} \to (y_1,\ldots,y_n)$. Then by massaging (5.3.10) and comparing the coefficient of u^N, we obtain Fact 5.3.3.

Application to Macdonald Functions

In the trigonometric case, which is our main target, the balancing condition can be removed. In order to simplify the notation, we introduce the generating current of $\Phi^{m,n}_L$'s, and the multiplicative notation.

Definition 5.3.4. *Define*

$$\phi^{m,n}\left(\begin{matrix}a_1,\ldots,a_m\\x_1,\ldots,x_m\end{matrix}\middle|\begin{matrix}b_1,\ldots,b_n\\c_1,\ldots,c_n\end{matrix};u\right) = \sum_{\mu\in\mathbb{Z}^m_{\geq 0}} u^{\sum_{i=1}^m \mu_i}\phi^{m,n}_{\mu}\left(\begin{matrix}a_1,\ldots,a_m\\x_1,\ldots,x_m\end{matrix}\middle|\begin{matrix}b_1y_1,\ldots,b_ny_n\\c_1y_1,\ldots,c_ny_n\end{matrix}\right) \tag{5.3.12}$$

with

$$\phi^{m,n}_{\mu}\left(\begin{matrix}a_1,\ldots,a_m\\x_1,\ldots,x_m\end{matrix}\middle|\begin{matrix}b_1,\ldots,b_n\\c_1,\ldots,c_n\end{matrix}\right) = \prod_{i<j}\frac{q^{\mu_i}x_i - q^{\mu_j}x_j}{x_i-x_j}\prod_{i,j}\frac{(a_jx_i/x_j;q)_{\mu_i}}{(qx_i/x_j;q)_{\mu_i}}\prod_{i,k}\frac{(b_kx_i;q)_{\mu_i}}{(c_kx_i;q)_{\mu_i}}. \tag{5.3.13}$$

Note that $\phi^{m,n}_{\mu}$ is essentially same as one term in $\Phi^{m,n}_L$ in Definition 5.3.2. Then, as a corollary of Fact 5.3.3, we can prove the following identity. This equality was proved in [54], just before its elliptic generalization Fact 5.3.3.

Corollary 5.3.5 ([54, 55])**.** *We have*

$$\begin{aligned}&\phi^{m,n}\left(\begin{matrix}a_1,\ldots,a_m\\x_1,\ldots,x_m\end{matrix}\middle|\begin{matrix}b_1y_1,\ldots,b_ny_n\\cy_1,\ldots,cy_n\end{matrix};u\right)\\&=\frac{(a_1\cdots a_mb_1\cdots b_nu/c^n;q)_\infty}{(u;q)_\infty}\cdot\phi^{n,m}\left(\begin{matrix}c/b_1,\ldots,c/b_n\\y_1,\ldots,y_n\end{matrix}\middle|\begin{matrix}cx_1/a_1,\ldots,cx_m/a_m\\cx_1,\ldots,cx_m\end{matrix};a_1\cdots a_mb_1\cdots b_nu/c^n\right).\end{aligned} \tag{5.3.14}$$

In the main proof, we would like to apply the Kajihara-Noumi identity to the bispectral Macdonald functions with some variables specialized. For that purpose, we prepare the following notation.

Definition 5.3.6. *For non-negative integers n, m and $\mu = (\mu_i)_{1\leq i\leq m} \in \mathbb{Z}^m$, we introduce*

$$\mathsf{N}^{n,m}_{\mu}(s_1,\ldots,s_{n+m}) := \prod_{k=1}^{m}\left(\prod_{i=1}^{n+k}\frac{(qs_{n+k}/ts_i;q)_{\mu_k}}{(qs_{n+k}/s_i;q)_{\mu_k}}\right)\cdot\prod_{1\leq i<j\leq m}\frac{(t\,q^{-\mu_i}s_{n+j}/s_{n+i};q)_{\mu_j}}{(q^{-\mu_i}s_{n+j}/s_{n+i};q)_{\mu_j}}. \tag{5.3.15}$$

By using Fact 5.3.3, we can show the following formula which expands the Macdonald function $f^{\mathfrak{gl}_{n+m}}(\boldsymbol{x}|\boldsymbol{s}|q,t)$ with respect to its "sub"-Macdonald functions $f^{\mathfrak{gl}_m}$.

Proposition 5.3.7. *Let s_i $(i=1,\dots,n+m)$ be generic complex parameters and $|t|>|q|^{-(n-2)}$. We put $x_i = xt^{n-i}$ for $i=1,\dots,n$ and $x_{n+k}=y_k$ for $k=1,\dots,m$. Then*

$$\begin{aligned}
&\prod_{k=1}^{m}\frac{(qy_k/t^n x;q)_\infty}{(ty_k/x;q)_\infty}\cdot f^{\mathfrak{gl}_{n+m}}(x_1,\dots,x_{n+m}|s_1,\dots,s_{n+m}|q,t)\\
&=\prod_{i=1}^{n}\frac{(q/t;q)_\infty}{(q/t^i;q)_\infty}\cdot\prod_{1\le i<j\le n}\frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty}\\
&\quad\times\sum_{\mu\in\mathbb{Z}^m_{\ge 0}}\mathsf{N}^{n,m}_{\mu}(s_1,\dots,s_{n+m})\,f^{\mathfrak{gl}_m}(y_1,\dots,y_m|q^{\mu_1}s_{n+1},\dots,q^{\mu_m}s_{n+m}|q,t)\prod_{k=1}^{m}(ty_k/x)^{\mu_k}.
\end{aligned}\tag{5.3.16}$$

The proof is presented in Section 5.B. In particular, if $m=0$, this proposition is nothing but the specialization formula Fact 4.1.14.

Again let us remark that throughout this chapter, s_i are treated as generic parameters with $s_i\neq 0$ and $s_i\neq q^{r_1}t^{r_2}s_j$ $(r_1\in\mathbb{Z},\ r_2=0,\pm1,\ \forall i,j)$.

5.4 Proof of Main Theorem

The proof consists of three steps.

Step 1. Specialize the spectral parameters as

$$v_i\to t^{n_i}u_i\,,\quad \text{for}\quad \forall n_i\in\mathbb{Z}_{\ge\ell(\lambda_i)}\,. \tag{5.4.1}$$

($\boldsymbol{\lambda}$ are the Young diagrams in the bra vector.) Under this specialization, the operator $\mathcal{V}\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{u}\end{pmatrix};x\Big)$ has the drastically simplified realization $\widetilde{V}^{(\boldsymbol{n})}(x)$. Then the matrix elements of $\widetilde{V}^{(\boldsymbol{n})}(x)$ become the bispectral Macdonald functions.

Step 2. Apply the Kajihara-Noumi transformation formula to the bispectral Macdonald functions, and we obtain the Nekrasov factors with the variables specialized.

Step 3. Use the identity theorem for the holomorphic functions.

Step 1.

Firstly, we give a realization of $\mathcal{V}(x)$ in the case $\boldsymbol{v}=t^{\boldsymbol{n}}\cdot\boldsymbol{u}$. The strategy is to specialize the variables in the operator $V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})$ (defined in Definition 4.2.20) as

$$x_i\to t^{|\boldsymbol{n}|-i}x\,. \tag{5.4.2}$$

Under this specialization, the resulted operator satisfies the relation (5.1.1) with $\boldsymbol{v}=t^{\boldsymbol{n}}\cdot\boldsymbol{u}$ (See Proposition 5.4.7 below).

However, it is non-trivial that we can actually take the limit (5.4.2) in the operator. In order to confirm this is a well-defined limit, we first show the analytic property of the matrix elements of $V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})$ as the rational function with respect to $(x_1,\dots,x_{|\boldsymbol{n}|})$.

The following theorem achieves our purpose.

Theorem 5.4.1. *The operator $V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|})$ is well-defined on $\pi^{-1}_{|\boldsymbol{n}|}(U_z^{|\boldsymbol{n}|}(r_0))$ with $r_0=|t^{-1}|$, in the sense that its matrix elements are the holomorphic functions there.*

Before the proof, we prepare the following lemma.

Lemma 5.4.2. *Let $\boldsymbol{u} = t^{-\boldsymbol{n}} \cdot \boldsymbol{v}$, $\boldsymbol{w} = t^{-\boldsymbol{m}} \cdot \boldsymbol{u}$ and $n_i \geq \ell(\lambda^{(i)})$ $(\forall i)$. Then*

$$\boldsymbol{x}^{-\boldsymbol{\lambda}} \langle P_{\boldsymbol{\lambda}}| \, V^{(\boldsymbol{n})}\binom{\boldsymbol{v}}{\boldsymbol{u}}; x_1, \dots, x_{|\boldsymbol{n}|}\Big) V^{(\boldsymbol{m})}\binom{\boldsymbol{u}}{\boldsymbol{w}}; y_1, \dots, y_{|\boldsymbol{m}|}\Big) |\boldsymbol{0}\rangle = \mathcal{R}_{\boldsymbol{\lambda}}(\boldsymbol{v}) \, f^{\mathfrak{gl}_{|\boldsymbol{n}|+|\boldsymbol{m}|}}((\boldsymbol{x}, \boldsymbol{y})|\boldsymbol{s}|q, q/t) \tag{5.4.3}$$

under the identification

$$s_{[i,k]_{\boldsymbol{n}}} = q^{\lambda_k^{(i)}} t^{1-k} v_i \quad (1 \leq k \leq n_i, \, i = 1, \dots, N), \tag{5.4.4}$$

$$s_{|\boldsymbol{n}|+[i,k]_{\boldsymbol{m}}} = t^{1-n_i-k} v_i \quad (1 \leq k \leq m_i, \, i = 1, \dots, N). \tag{5.4.5}$$

This lemma is the corollary of Theorem 4.2.21 by noting the normalization of the screened vertex operators (Remark 4.2.17).

Proof of Theorem 5.4.1.
First, by Fact 4.1.13, the LHS of (5.4.3) is a holomorphic function on $U_{\tilde{z}}^{|\boldsymbol{n}|+|\boldsymbol{m}|}(r_0)$. Recall the notation in Section 4.1.3. Through the pull-back $\pi^*_{|\boldsymbol{n}|+|\boldsymbol{m}|}$, it is the holomorphic function in $\pi^{-1}_{|\boldsymbol{n}|+|\boldsymbol{m}|}(U_{\tilde{z}}^{|\boldsymbol{n}|+|\boldsymbol{m}|}(r_0))$. Thus, once we fix $y \in (\mathbb{C}^*)^{|\boldsymbol{m}|}$ such that $|y_j/y_i| < r_0^{j-i}$ $(1 \leq i < j \leq |\boldsymbol{m}|)$, it can be regarded a holomorphic function of $x \in \widetilde{\pi}^{-1}_{|\boldsymbol{n}|,y_1}(U_z^{|\boldsymbol{n}|}(r_0))$, where

$$\widetilde{\pi}^{-1}_{|\boldsymbol{n}|,y_1}(U_z^{|\boldsymbol{n}|}(r_0)) := \left\{(x_1, \dots, x_{|\boldsymbol{n}|}) \in (\mathbb{C}^*)^{|\boldsymbol{n}|} \,\middle|\, |x_j/x_i| < r_0^{j-i} \, (1 \leq i < j \leq |\boldsymbol{n}|), \; |y_1/x_{|\boldsymbol{n}|}| < r_0 \right\}. \tag{5.4.6}$$

Then, by multiplying both sides of (5.4.3) by $\boldsymbol{y}^{-\mu} \widetilde{f}^{\mathfrak{gl}_{|\boldsymbol{m}|}}(\boldsymbol{y}|\boldsymbol{s}|q, q/t)$ and taking the constant term in $\boldsymbol{y}$, we complete the proof. □

Now we can safely take the limit (5.4.2), and define the specialized operators.

Definition 5.4.3. *Let $|t| > |q|^{-(n-2)}$. Define $\widetilde{V}^{(\boldsymbol{n})}(x) = \widetilde{V}^{(\boldsymbol{n})}\binom{\boldsymbol{v}}{\boldsymbol{u}}; x\Big) : \mathcal{F}_{\boldsymbol{u}} \to \mathcal{F}_{\boldsymbol{v}}$ with $\boldsymbol{u} = t^{-\boldsymbol{n}} \cdot \boldsymbol{v}$ by*

$$\widetilde{V}^{(\boldsymbol{n})}(x) = \lim_{x_i \to t^{|\boldsymbol{n}|-i} x} \prod_{1 \leq i < j \leq |\boldsymbol{n}|} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \cdot V^{(\boldsymbol{n})}(x_1, \dots, x_{|\boldsymbol{n}|}) A^{-1}_{(|\boldsymbol{n}|)}(x), \tag{5.4.7}$$

where

$$A_{(r)}(x) = \exp\left(\sum_{n>0} \frac{(1-(q/t)^r)(1-t^{(1-r)n})t^{2r}}{n(1-q^n)(1-t^{-n})} \sum_{i=1}^{N} \gamma^{(i-1)n} a_n^{(i)} x^{-n} \right). \tag{5.4.8}$$

Remark 5.4.4. *Note that $\sum_{i=1}^N \gamma^{(i-1)n} a_n^{(i)}$ is the boson corresponding to the Cartan part $\Delta^{(N)}(\psi^+(z))$.*

Proposition 5.4.5. *$\widetilde{V}^{(\boldsymbol{n})}(x)$ is well-defined on $\mathbb{C}^*$, i.e., its arbitrary matrix elements are holomorphic functions there.*

Before the proof, we prepare the following fact. This fact tells us the duality of the Macdonald functions under exchanging t and q/t. Also recall the bispectral Macdonald functions $\varphi^{\mathfrak{gl}_n}$ in the previous chapter.

Fact 5.4.6 ([86])**.** *The formal series $f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t)$ with the leading coefficient 1 satisfies the symmetry relation*

$$f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q,t) = \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \cdot f^{\mathfrak{gl}_n}(\boldsymbol{x}|\boldsymbol{s}|q, q/t) \, . \tag{5.4.9}$$

Proof. For the proof, we only have to note

$$D_n^{(k)}(\boldsymbol{s}; q, t) \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} = \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} D_n^{(k)}(\boldsymbol{s}; q, q/t) \, . \tag{5.4.10}$$

□

Proof of Proposition 5.4.5. We introduce the following product of currents,

$$\widetilde{A}_{(r)}(x_1,\dots,x_r) := \prod_{k=1}^{r} \exp\left(\sum_{n>0} \frac{(1-(q/t)^r)t^{2r}}{n(1-q^n)} \sum_{i=1}^{N} \gamma^{(i-1)n} a_n^{(i)} x_k^{-n} \right). \tag{5.4.11}$$

We have the following equality.

$$\begin{aligned} &\prod_{1\le i<j\le |\boldsymbol{n}|} \frac{(tx_j/x_i;q)_\infty}{(qx_j/tx_i;q)_\infty} \cdot \prod_{1\le i<j\le |\boldsymbol{m}|} \frac{(ty_j/y_i;q)_\infty}{(qy_j/ty_i;q)_\infty} \\ &\times \boldsymbol{x}^{-\boldsymbol{\lambda}} \langle P_{\boldsymbol{\lambda}}|\, V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|}) \widetilde{A}^{-1}_{(|\boldsymbol{n}|)}(x_1,\dots,x_{|\boldsymbol{n}|}) V^{(\boldsymbol{m})}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{pmatrix}; y_1,\dots,y_{|\boldsymbol{m}|}\Big) |\boldsymbol{0}\rangle \\ &= \prod_{1\le i<j\le |\boldsymbol{n}|+|\boldsymbol{m}|} \frac{(tx_j/x_i;q)_\infty}{(qx_j/tx_i;q)_\infty} \cdot \mathcal{R}_{\boldsymbol{\lambda}}(\boldsymbol{v})\, f^{\mathfrak{gl}_{|\boldsymbol{n}|+|\boldsymbol{m}|}}((\boldsymbol{x},\boldsymbol{y})|\boldsymbol{s}|q,q/t) = \mathcal{R}_{\boldsymbol{\lambda}}(\boldsymbol{v})\, f^{\mathfrak{gl}_{|\boldsymbol{n}|+|\boldsymbol{m}|}}((\boldsymbol{x},\boldsymbol{y})|\boldsymbol{s}|q,t). \end{aligned} \tag{5.4.12}$$

Here, for simplicity, we set $x_{|\boldsymbol{n}|+i} = y_i$, and used Fact 5.4.6 and Lemma 5.4.2.

Then, by the same argument as the proof of Theorem 5.4.1, we can show the matrix elements of

$$\prod_{1\le i<j\le |\boldsymbol{n}|} \frac{(tx_j/x_i;q)_\infty}{(qx_j/tx_i;q)_\infty} \cdot V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|}) \widetilde{A}^{-1}_{(|\boldsymbol{n}|)}(x_1,\dots,x_{|\boldsymbol{n}|})$$

are the holomorphic functions on $\pi^{-1}_{|\boldsymbol{n}|}(U_z^{|\boldsymbol{n}|}(\bar{r}_0))$ with $\bar{r}_0 = |q/t|^{\frac{|\boldsymbol{n}|-2}{|\boldsymbol{n}|-1}}$. In this case, $\boldsymbol{y}^{-\boldsymbol{\mu}} \widetilde{f}^{\mathfrak{gl}_{|\boldsymbol{m}|}}(\boldsymbol{y}|\boldsymbol{s}|q,t)$ is multiplied before the integration in $\boldsymbol{y}$. Under the assumption $|t| > |q|^{-(n-2)}$, it is clear $|t^{-1}| < \bar{r}_0$, and thus we can safely take the limit $x_i \to t^{|\boldsymbol{n}|-i}x$. By taking limit, we prove the claim. □

As noted, this operator $\widetilde{V}^{(\boldsymbol{n})}(x)$ is a realization of $\mathcal{V}$ in the case $\boldsymbol{v} = t^{\boldsymbol{n}} \cdot \boldsymbol{u}$. This follows from the following relation which is essentially the same as (5.1.1).

Proposition 5.4.7. *For $r = 1,\dots,N$, the $\widetilde{V}^{(\boldsymbol{n})}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{pmatrix}; x\Big)$ satisfies*

$$\left(1 - t^{|\boldsymbol{n}|}\frac{x}{z}\right) X^{(r)}(z) \widetilde{V}^{(\boldsymbol{n})}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{pmatrix}; x\Big) = (q/t)^r \left(1 - (t/q)^r t^{|\boldsymbol{n}|}\frac{x}{z}\right) \widetilde{V}^{(\boldsymbol{n})}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{pmatrix}; x\Big) X^{(r)}(z). \tag{5.4.13}$$

Proof. By Proposition 4.2.23, we obtain

$$\begin{aligned} X^{(r)}(z) V^{(\boldsymbol{n})}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{pmatrix}; x_1,\dots,x_{|\boldsymbol{n}|}\Big) &= \prod_{k=1}^{|\boldsymbol{n}|} \frac{1-(q/t)^r z/tx_k}{1-z/tx_k} \cdot V^{(\boldsymbol{n})}(x_1,\dots,x_{|\boldsymbol{n}|}) X^{(r)}(z) \\ &+ (1-t^{-1}) \sum_{i=1}^{N}\sum_{k=1}^{n_i} v_i t^{1-k} \delta(tx_{[i,k]_{\boldsymbol{n}}}/z) \prod_{1\le \ell<[i,k]_{\boldsymbol{n}}} \frac{1-(q/t)^r x_{[i,k]_{\boldsymbol{n}}}/x_\ell}{1-x_{[i,k]_{\boldsymbol{n}}}/x_\ell} \cdot \prod_{[i,k]_{\boldsymbol{n}}<\ell\le |\boldsymbol{n}|} \frac{1-tx_\ell/qx_{[i,k]_{\boldsymbol{n}}}}{1-x_\ell/x_{[i,k]_{\boldsymbol{n}}}} \\ &\qquad \times \widetilde{U}^{(\boldsymbol{n})}_{[i,k]_{\boldsymbol{n}}}(x_1,\dots,x_{|\boldsymbol{n}|}) \Psi^+(x_{[i,k]_{\boldsymbol{n}}}), \end{aligned} \tag{5.4.14}$$

where $\widetilde{U}^{(\boldsymbol{n})}_{[i,k]_{\boldsymbol{n}}}$ is the operator which is obtained by replacing $\Phi^{(i)}(x_{[i,k]_{\boldsymbol{n}}})$ in $V^{(\boldsymbol{n})}$ with $Y^{(r)}(x_{[i,k]_{\boldsymbol{n}}})\Phi^{(i)}(qx_{[i,k]_{\boldsymbol{n}}})$, that is,

$$\widetilde{U}^{(\boldsymbol{n})}_{[i,k]_{\boldsymbol{n}}}(x_1,\dots,x_{|\boldsymbol{n}|}) := \Phi^{(0)}(x_1)\cdots \Phi^{(i)}(x_{[i,k]_{\boldsymbol{n}}-1}) Y^{(r)}(x_{[i,k]_{\boldsymbol{n}}}) \Phi^{(i)}(qx_{[i,k]_{\boldsymbol{n}}}) \Phi^{(i)}(x_{[i,k]_{\boldsymbol{n}}+1}) \cdots \Phi^{(N-1)}(x_{|\boldsymbol{n}|}). \tag{5.4.15}$$

Under the principle specialization, only the term with $[i,k]_{\boldsymbol{n}} = 1$ survives. This is because we can show

$$\lim_{x_i\to t^{|\boldsymbol{n}|-i}x} \prod_{1\le i<j\le |\boldsymbol{n}|} \frac{(tx_j/x_i;q)_\infty}{(qx_j/tx_i;q)_\infty} \cdot \widetilde{U}^{(\boldsymbol{n})}_{[i,k]_{\boldsymbol{n}}}(x_1,\dots,x_{|\boldsymbol{n}|}) = 0 \quad ([i,k]_{\boldsymbol{n}} \ne 1). \tag{5.4.16}$$

Note that when $[i,k]_{\boldsymbol{n}} = 1$, the normal ordering inside $\widetilde{U}_1^{(\boldsymbol{n})}$ cancels the vanishing factors. By the operator product (B.1.29), it is easy to see

$$A_{(s)}(x)X^{(r)}(z) = \prod_{k=1}^{r-1} \frac{1-t^{-k}z/x}{1-t^{-k}(q/t)^r z/x} \cdot X^{(r)}(z)A_{(s)}(x). \tag{5.4.17}$$

Combining (5.4.14), (5.4.16) and (5.4.17), we obtain

$$X^{(r)}(z)\widetilde{V}^{(\boldsymbol{n})}(x) = \frac{1-(q/t)^r z/t^{|\boldsymbol{n}|}x}{1-z/t^{|\boldsymbol{n}|}x}\widetilde{V}^{(\boldsymbol{n})}(x)X^{(r)}(z) \tag{5.4.18}$$

$$+ (1-t^{-1})v_1 t^{1-k} \lim_{x_i \to t^{|\boldsymbol{n}|-i}x} \delta(tx_1/z) \prod_{1<\ell\le|\boldsymbol{n}|} \frac{1-tx_\ell/qx_1}{1-x_\ell/x_1} \prod_{1\le i<j\le|\boldsymbol{n}|} \frac{(tx_j/x_i;q)_\infty}{(qx_j/tx_i;q)_\infty} \cdot \widetilde{U}_1^{(\boldsymbol{n})}\Psi^+(x_1). \tag{5.4.19}$$

By multiplying the both sides by $\left(1-t^{|\boldsymbol{n}|}\frac{x}{z}\right)$, we obtain the expected result,

$$\left(1-t^{|\boldsymbol{n}|}\frac{x}{z}\right) X^{(r)}(z)\widetilde{V}^{(\boldsymbol{n})}\left(\begin{matrix}\boldsymbol{v}\\ \boldsymbol{u}\end{matrix};x\right) = (q/t)^r \left(1-(t/q)^r t^{|\boldsymbol{n}|}\frac{x}{z}\right) \widetilde{V}^{(\boldsymbol{n})}\left(\begin{matrix}\boldsymbol{v}\\ \boldsymbol{u}\end{matrix};x\right)X^{(r)}(z)\,. \tag{5.4.20}$$

□

Step 2.

Next, we compute the matrix elements of $\widetilde{V}^{(\boldsymbol{n})}(x)$ with respect to the generalized Macdonald functions. In the end, we obtain the following result.

Proposition 5.4.8. *The matrix elements*

$$\frac{\langle K_{\boldsymbol{\lambda}}|\widetilde{V}^{(\boldsymbol{n})}(x)|K_{\boldsymbol{\mu}}\rangle}{\langle \boldsymbol{0}|\,\widetilde{V}^{(\boldsymbol{n})}(x)\,|\boldsymbol{0}\rangle} = \left((-1)^N e_N(\boldsymbol{v})x\right)^{|\boldsymbol{\lambda}|}\left((t/q)t^{|\boldsymbol{n}|}x\right)^{-|\boldsymbol{\mu}|}\prod_{i=1}^N v_i^{-(N-1)|\lambda^{(i)}|}((q/t)u_i)^{|\mu^{(i)}|}g_{\lambda^{(i)}}^{-N+1}g_{\mu^{(i)}}$$
$$\times \prod_{i=1}^N (q/t)^{(N-i)|\mu^{(i)}|-\sum_{k=1}^i|\mu^{(k)}|}\prod_{i,j=1}^N N_{\lambda^{(i)},\mu^{(j)}}(t^{n_j}v_i/v_j)\,. \tag{5.4.21}$$

Below in this step, we prove this proposition.

Proof of Proposition 5.4.8.

Put $\boldsymbol{s}' := (s'_i)_{1\le i\le|\boldsymbol{m}|}$ with

$$s'_{[i,k]_{\boldsymbol{m}}} := q^{\mu_k^{(i)}}t^{1-n_i-k}v_i \quad (1\le k\le m_i, i=1,\dots,N) \tag{5.4.22}$$

and $\boldsymbol{s} = (s_i)_{1\le i\le|\boldsymbol{n}|+|\boldsymbol{m}|}$ be the same one given in (5.4.4), (5.4.5), *i.e.*,

$$s_{[i,k]_{\boldsymbol{n}}} = q^{\lambda_k^{(i)}}t^{1-k}v_i \quad (1\le k\le n_i,\ i=1,\dots,N), \tag{5.4.23}$$

$$s_{|\boldsymbol{n}|+[i,k]_{\boldsymbol{m}}} = t^{1-n_i-k}v_i \quad (1\le k\le m_i,\ i=1,\dots,N). \tag{5.4.24}$$

We also put $x_{|\boldsymbol{n}|+i} = y_i$. By the explicit algorithm to construct $|Q_{\boldsymbol{\lambda}}\rangle$ (Theorem 5.2.1) and Lemma 5.4.2, we can rewrite the matrix elements as

$$\langle P_{\boldsymbol{\lambda}}|\widetilde{V}^{(\boldsymbol{n})}(x)|Q_{\boldsymbol{\mu}}\rangle = \frac{1}{\mathcal{R}_{\boldsymbol{\mu}}^{\boldsymbol{m}}(\boldsymbol{u})}\left[\boldsymbol{y}^{-\boldsymbol{\mu}}\widetilde{f}^{\mathfrak{gl}_{|\boldsymbol{m}|}}(\boldsymbol{y}|\boldsymbol{s}'|q,q/t)\,\langle P_{\boldsymbol{\lambda}}|\,\widetilde{V}^{(\boldsymbol{n})}(x)V^{(\boldsymbol{m})}(y_1,\dots,y_{|\boldsymbol{m}|})\,|\boldsymbol{0}\rangle\right]_{\boldsymbol{y},1}$$
$$= \frac{\mathcal{R}_{\boldsymbol{\lambda}}^{\boldsymbol{n}}(\boldsymbol{v})}{\mathcal{R}_{\boldsymbol{\mu}}^{\boldsymbol{m}}(\boldsymbol{u})}\Bigg[\lim_{\substack{x_i\to t^{|\boldsymbol{n}|-i}x\\(1\le i\le|\boldsymbol{n}|)}}\boldsymbol{y}^{-\boldsymbol{\mu}}\boldsymbol{x}^{\boldsymbol{\lambda}}\widetilde{f}^{\mathfrak{gl}_{|\boldsymbol{m}|}}(\boldsymbol{y}|\boldsymbol{s}'|q,q/t)\prod_{1\le i<j\le|\boldsymbol{m}|}\frac{(qy_j/ty_i;q)_\infty}{(ty_j/y_i;q)_\infty}$$

$$\times \prod_{k=1}^{|\boldsymbol{m}|} \frac{(qy_k/t^{|\boldsymbol{n}|}x;q)_\infty}{(ty_k/x;q)_\infty} \cdot f^{\mathfrak{gl}_{|\boldsymbol{n}|+|\boldsymbol{m}|}}((x_i)_{1\le i\le |\boldsymbol{n}|+|\boldsymbol{m}|}|\boldsymbol{s}|q,t)\Bigg]_{\boldsymbol{y},1} . \tag{5.4.25}$$

The next step is crucial. By taking advantage of Proposition 5.3.7, the Macdonald function $f^{\mathfrak{gl}_{|\boldsymbol{n}|+|\boldsymbol{m}|}}$ in (5.4.25) can be transformed to the summation of its "sub"-Macdonald functions $f^{\mathfrak{gl}_{|\boldsymbol{m}|}}$ as

$$\begin{aligned}(\text{LHS of } (5.4.25)) = \frac{\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{v})}{\mathcal{R}^{\boldsymbol{m}}_{\boldsymbol{\mu}}(\boldsymbol{u})} \prod_{i=1}^{|\boldsymbol{n}|} \frac{(q/t;q)_\infty}{(q/t^i;q)_\infty} \cdot \prod_{1\le i<j\le |\boldsymbol{n}|} \frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty} \cdot \lim_{\substack{x_i\to t^{|\boldsymbol{n}|-i}x\\(1\le i\le |\boldsymbol{n}|)}} \boldsymbol{x}^{\boldsymbol{\lambda}} \\ \times \sum_{\nu\in\mathbb{Z}^{|\boldsymbol{m}|}_{\ge 0}} \mathsf{N}^{|\boldsymbol{n}|,|\boldsymbol{m}|}_{\nu}(s_1,\dots,s_{|\boldsymbol{n}|+|\boldsymbol{m}|}) \left[\boldsymbol{y}^{-\boldsymbol{\mu}} \widetilde{f}^{\mathfrak{gl}_{|\boldsymbol{m}|}}(\boldsymbol{y}|\boldsymbol{s}'|q,q/t) f^{\mathfrak{gl}_{|\boldsymbol{m}|}}(\boldsymbol{y}|(q^{\nu_i}s_{|\boldsymbol{n}|+i})|q,q/t) \prod_{k=1}^{|\boldsymbol{m}|}(ty_k/x)^{\nu_k}\right]_{\boldsymbol{y},1} .\end{aligned} \tag{5.4.26}$$

Fact 5.4.6 is also used through the computation. Note that since $s_{|\boldsymbol{n}|+[i,k]_{\boldsymbol{m}}} = t^{1-n_i-k}v_i$, $\mathsf{N}^{|\boldsymbol{n}|,|\boldsymbol{m}|}_{\nu} = 0$ if ν cannot be regarded as an N-tuple of partitions. By virtue of Lemma 4.1.5 and

$$\boldsymbol{x}^{\boldsymbol{\lambda}}\Big|_{x_i\to t^{|\boldsymbol{n}|-i}x} = x^{|\boldsymbol{\lambda}|} t^{(|\boldsymbol{n}|-1)|\boldsymbol{\lambda}|} \prod_{i=1}^{N} t^{-n(\lambda^{(i)})-|\lambda^{(i)}|\sum_{k=1}^{i-1} n_k} , \tag{5.4.27}$$

it is shown that

$$\begin{aligned}\langle P_{\boldsymbol{\lambda}}|\widetilde{V}^{(\boldsymbol{n})}(x)|Q_{\boldsymbol{\mu}}\rangle =& x^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|} t^{|\boldsymbol{\mu}|+(|\boldsymbol{n}|-1)|\boldsymbol{\lambda}|} \prod_{i=1}^{N} t^{-n(\lambda^{(i)})-|\lambda^{(i)}|\sum_{k=1}^{i-1} n_k} \\ &\times \frac{\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{v})}{\mathcal{R}^{\boldsymbol{m}}_{\boldsymbol{\mu}}(\boldsymbol{u})} \prod_{i=1}^{|\boldsymbol{n}|} \frac{(q/t;q)_\infty}{(q/t^i;q)_\infty} \cdot \prod_{1\le i<j\le |\boldsymbol{n}|} \frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty} \cdot \mathsf{N}^{|\boldsymbol{n}|,|\boldsymbol{m}|}_{[\boldsymbol{\mu}]^{\boldsymbol{m}}}(s_1,\dots,s_{|\boldsymbol{n}|+|\boldsymbol{m}|}) ,\end{aligned} \tag{5.4.28}$$

where we use the notation

$$[\boldsymbol{\mu}]^{\boldsymbol{m}} = ([\boldsymbol{\mu}]^{\boldsymbol{m}}_i)_{1\le i\le |\boldsymbol{m}|} := (\mu^{(1)}_1,\dots,\mu^{(1)}_{m_1},\mu^{(2)}_1,\dots,\mu^{(2)}_{m_2},\dots,\mu^{(N)}_1,\dots,\mu^{(N)}_{m_N}) . \tag{5.4.29}$$

By massaging (5.4.28), we obtain the matrix elements with respect to the integral forms of the generalized Macdonald functions,

$$\begin{aligned}\frac{\langle K_{\boldsymbol{\lambda}}|\widetilde{V}^{(\boldsymbol{n})}(x)|K_{\boldsymbol{\mu}}\rangle}{\langle \boldsymbol{0}|\,\widetilde{V}^{(\boldsymbol{n})}(x)\,|\boldsymbol{0}\rangle} =& \mathcal{C}^{(-)}_{\boldsymbol{\lambda}}\,\mathcal{C}^{(+)}_{\boldsymbol{\mu}} \prod_{i=1}^{N} \frac{c'_{\mu^{(i)}}}{c_{\mu^{(i)}}} \cdot x^{|\boldsymbol{\lambda}|-|\boldsymbol{\mu}|} t^{|\boldsymbol{\mu}|+(|\boldsymbol{n}|-1)|\boldsymbol{\lambda}|} \prod_{i=1}^{N} t^{-n(\lambda^{(i)})-|\lambda^{(i)}|\sum_{k=1}^{i-1} n_k} \\ &\times \frac{\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}(\boldsymbol{v})}{\mathcal{R}^{\boldsymbol{m}}_{\boldsymbol{\mu}}(\boldsymbol{u})} \prod_{1\le i<j\le |\boldsymbol{n}|} \frac{(qs_j/ts_i;q)_{[\boldsymbol{\lambda}]^{\boldsymbol{n}}_i-[\boldsymbol{\lambda}]^{\boldsymbol{n}}_j}}{(qs_j/s_i;q)_{[\boldsymbol{\lambda}]^{\boldsymbol{n}}_i-[\boldsymbol{\lambda}]^{\boldsymbol{n}}_j}} \cdot \mathsf{N}^{|\boldsymbol{n}|,|\boldsymbol{m}|}_{[\boldsymbol{\mu}]^{\boldsymbol{m}}}(s_1,\dots,s_{|\boldsymbol{n}|+|\boldsymbol{m}|}) .\end{aligned} \tag{5.4.30}$$

Here for the simplicity of notation, we put

$$\begin{aligned}\prod_{1\le i<j\le |\boldsymbol{n}|} \frac{(qs_j/ts_i;q)_{[\boldsymbol{\lambda}]^{\boldsymbol{n}}_i-[\boldsymbol{\lambda}]^{\boldsymbol{n}}_j}}{(qs_j/s_i;q)_{[\boldsymbol{\lambda}]^{\boldsymbol{n}}_i-[\boldsymbol{\lambda}]^{\boldsymbol{n}}_j}} =& \prod_{k=1}^{N} \prod_{1\le i<j\le n_k} \frac{(q^{\lambda^{(k)}_j-\lambda^{(k)}_i+1}t^{-j+i-1};q)_{-\lambda^{(k)}_j+\lambda^{(k)}_i}}{(q^{\lambda^{(k)}_j-\lambda^{(k)}_i+1}t^{-j+i};q)_{-\lambda^{(k)}_j+\lambda^{(k)}_i}} \\ &\times \prod_{1\le k<l\le N} \prod_{\substack{1\le i\le n_k\\1\le j\le n_l}} \frac{(q^{\lambda^{(l)}_j-\lambda^{(k)}_i+1}t^{-j+i-1}v_l/v_k;q)_{-\lambda^{(l)}_j+\lambda^{(k)}_i}}{(q^{\lambda^{(l)}_j-\lambda^{(k)}_i+1}t^{-j+i}v_l/v_k;q)_{-\lambda^{(l)}_j+\lambda^{(k)}_i}} .\end{aligned} \tag{5.4.31}$$

Finally, we have to show that the expression (5.4.30) coincides with the Nekrasov factors. The following proposition achieves this goal.

Proposition 5.4.9. *We have*

$$\begin{aligned}
&\frac{\mathcal{R}^{\boldsymbol{n}}_{\boldsymbol{\lambda}}}{\mathcal{R}^{\boldsymbol{m}}_{\boldsymbol{\mu}}} \mathsf{N}^{|\boldsymbol{n}|,|\boldsymbol{m}|}_{[\boldsymbol{\mu}]}(s_1,\dots,s_{|\boldsymbol{n}|+|\boldsymbol{m}|}) \prod_{1\le i<j\le|\boldsymbol{n}|} \frac{(qs_j/ts_i)_{[\boldsymbol{\lambda}]_i-[\boldsymbol{\lambda}]_j}}{(qs_j/s_i)_{[\boldsymbol{\lambda}]_i-[\boldsymbol{\lambda}]_j}} \\
&= \prod_{i=1}^{N} (-1)^{|\mu^{(i)}|} t^{-(|\boldsymbol{n}|+n_i)|\lambda^{(i)}|} t^{-(N-i)|\mu^{(i)}|n_i} t^{(|\lambda^{(i)}|-|\mu^{(i)}|)\sum_{s=1}^{i} n_s} t^{2n(\lambda^{(i)})+|\lambda^{(i)}|} \\
&\times \prod_{i=1}^{N} \gamma^{-(N-1)(|\lambda^{(i)}|+|\mu^{(i)}|)-2|\mu^{(i)}|} q^{n(\mu^{(i)\prime})} \left(\frac{f_{\mu^{(i)}}}{f_{\lambda^{(i)}}}\right)^{N-i} \prod_{1\le i<j\le N} (v_i/v_j)^{-|\lambda^{(i)}|+|\mu^{(i)}|} \\
&\times \prod_{i=1}^{N} \frac{1}{c_{\lambda^{(i)}} c'_{\mu^{(i)}}} \frac{\prod_{i,j=1}^{N} N_{\lambda^{(i)},\mu^{(j)}}(t^{n_j} v_i/v_j)}{\prod_{1\le i<j\le N} N_{\lambda^{(j)}\lambda^{(i)}}(qv_j/tv_i) \prod_{1\le i<j\le N} N_{\mu^{(i)}\mu^{(j)}}(qt^{-n_i+n_j} v_i/tv_j)} .
\end{aligned} \tag{5.4.32}$$

The proof is by direct computation using recursion relations, and we omit it because it is tedious and not essential. In a word, the recursion relations with respect to the length of the partitions, coincide on both sides. For the detail, see [35].

With the help of this proposition, we complete the proof of Proposition 5.4.8. □

Step 3.

Now we finalize the main proof. First, we note the difference between (5.1.1) and (5.4.13) can be modified once we make the transformation

$$x \to t^{|\boldsymbol{n}|}x, \quad u_i \to (q/t)u_i \quad (i=1,\dots,N). \tag{5.4.33}$$

Note that this transformation also cures the difference of the normalization constant (4.2.48). Then we can see that the equation (5.4.21) shows Theorem 5.2.1 in the case $\boldsymbol{v} = t^{\boldsymbol{n}} \cdot \boldsymbol{u}$. Because the equation (5.4.21) holds for arbitrary $n_i \ge \ell(\lambda^{(i)})$, the identity theorem, (which says the two rational functions which agree at infinitely many points are identical on the whole complex space,) guarantees the main theorem.

Remark 5.4.10. *One may think we have to show that the matrix elements are rational functions. This is trivial from the definition of the Mukadé operator (Definition 5.1.1), because once we fix $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, the matrix elements are the finite product of factors like $(1-\bullet x/z)$.*

Let us refine this statement. If $\boldsymbol{\lambda} \ne (\emptyset,\dots,\emptyset)$, let $j = \min\{i | \lambda^{(i)} \ne \emptyset\}$. The defining relation of $\mathcal{V}(z)$ gives

$$\begin{aligned}
\langle X_{\boldsymbol{\lambda}} | \mathcal{V}(w) | X_{\boldsymbol{\mu}} \rangle =& w \left\langle X_{(\emptyset,\dots,(\lambda_2^{(j)},\lambda_3^{(j)},\dots),\lambda^{(j+1)},\dots,\lambda^{(N)})} \middle| X^{(j)}_{\lambda_1^{(j)}-1} \mathcal{V}(w) \middle| X_{\boldsymbol{\mu}} \right\rangle \\
&+ \left\langle X_{(\emptyset,\dots,(\lambda_2^{(j)},\lambda_3^{(j)},\dots),\lambda^{(j+1)},\dots,\lambda^{(N)})} \middle| \mathcal{V}(w) \left(X^{(j)}_{\lambda_1^{(j)}} - (q/t)^j w X^{(j)}_{\lambda_1^{(j)}-1} \right) \middle| X^{(1)}_{\boldsymbol{\mu}} \right\rangle .
\end{aligned} \tag{5.4.34}$$

The first term in RHS can be rewritten by the matrix elements $\langle X_{\boldsymbol{\nu}} | \mathcal{V}(w) | X_{\boldsymbol{\mu}} \rangle$ with the condition $|\boldsymbol{\nu}| = |\boldsymbol{\lambda}| - 1$ (particularly, in the case $\lambda_1^{(j)} - 1 < \lambda_2^{(j)}$), and the second term can be expanded by vectors $|X_{\boldsymbol{\rho}}\rangle$ of level $|\boldsymbol{\rho}| = |\boldsymbol{\mu}| - \lambda_1^{(j)}$ or $|\boldsymbol{\mu}| - \lambda_1^{(j)} + 1$. When $\boldsymbol{\lambda} = (\emptyset,\dots,\emptyset)$, we can move the negative modes $X^{(i)}_{-n}$, (which annihilate the bra vacuum) to the left side of $\mathcal{V}(w)$ by the defining relation, and as a result, it makes the size of Young diagrams in the bra state smaller. Therefore, starting from the case where all Young diagrams are empty, the matrix elements $\langle X_{\boldsymbol{\nu}} | \mathcal{V}(w) | X_{\boldsymbol{\mu}} \rangle$ can be inductively and uniquely determined. Because this procedure to compute some specific matrix element is achieved by the finite number of iterations, we can show that the matrix elements are the rational functions.

Appendix 5.A Idea of Another Proof for $N = 1$ Case

In this section, we give the sketch of another proof of Theorem 5.2.1 for $N = 1$ case. Let us again see the main claim for $N = 1$,

$$\langle P_\lambda | \mathcal{V}(u,v;z) | P_\nu \rangle = \widetilde{N}_{\lambda,\nu}(qv/tu) \,, \tag{5.A.1}$$

with

$$\widetilde{N}_{\lambda,\nu}(qv/tu) := \frac{(-tux/q)^{|\lambda|}}{(tx/q)^{|\nu|}} \frac{g_\nu}{c_\lambda c_\nu} N_{\lambda,\nu}(qv/tu) \,. \tag{5.A.2}$$

Note that when $N = 1$, the Mukadé operator becomes (up to the extra $\mathbb{G}$-factor)

$$\mathcal{V}(u,v;z) = \exp\left(-\sum_n \frac{1}{n} \frac{1-(tu/qv)^n}{1-q^n} a_{-n} z^n \right) \exp\left(\sum_n \frac{1}{n} \frac{1-(v/u)^{-n}}{1-q^{-n}} a_n z^{-n} \right) . \tag{5.A.3}$$

When $N = 1$, we know the Pieri rules (Fact 3.1.16), and thus we can add some boxes (a vertical or horizontal strip) in the ket state, and move it to the bra state. Then, this erases the strip from the ket state. Using this fact, we can give another proof of the claim by the induction on ν, the partition in the ket state.

Let us see more in detail. As we learned in Chapter 2, we have the generating current of g_n (Definition 3.2.5),

$$\varphi(w) = \exp\left(\sum_n \frac{1}{n} \frac{1-t^n}{1-q^n} a_{-n} w^n \right) . \tag{5.A.4}$$

Then, by taking normal ordering, we have the following identity

$$\exp\left(\sum_n \frac{1}{n} \frac{1-(v/u)^{-n}}{1-q^{-n}} (w/z)^n \right) \langle P_\lambda | \varphi(w) \mathcal{V}(u,v;z) | P_\nu \rangle = \langle P_\lambda | \mathcal{V}(u,v;z) \varphi(w) | P_\nu \rangle \,, \tag{5.A.5}$$

and when we take the coefficient of w^1, we have

$$\begin{aligned} &\frac{1-(v/u)^{-1}}{1-q^{-1}} \frac{1}{z} \langle P_\lambda | \mathcal{V}(u,v;z) | P_\nu \rangle + \sum_{i \in R(\lambda)} \varphi_{\lambda/\lambda-i} \langle P_{\lambda-i} | \mathcal{V}(u,v;z) | P_\nu \rangle \\ &\quad = \sum_{i \in A(\nu)} \varphi_{\nu+i/\nu} \langle P_\lambda | \mathcal{V}(u,v;z) | P_{\nu+i} \rangle \,. \end{aligned} \tag{5.A.6}$$

Note that we use the Pieri rule Fact 3.1.16 and its dual. Though we also have to take the coefficients of w^n for the general n, to simplify the discussion, we concentrate on the equation above. The discussion in general n goes in the similar way.

Then, we fix λ and ν, and assume the main claim (5.A.1) holds for arbitrary partition in the bra state (respectively in the ket state) which has the smaller (or equal) number of boxes than λ (resp. ν). Then, by dividing both sides by $\widetilde{N}_{\lambda,\nu}(qv/tu)$, the LHS of (5.A.6) can be written as

$$\frac{1-v^{-1}}{1-q^{-1}} \frac{1}{z} + \sum_{i \in R(\lambda)} \varphi_{\lambda/\lambda-i} \frac{\widetilde{N}_{\lambda-i,\nu}(qv/tu)}{\widetilde{N}_{\lambda,\nu}(qv/tu)} \,. \tag{5.A.7}$$

Then, if the RHS agrees with

$$\sum_{i \in A(\nu)} \varphi_{\nu+i/\nu} \frac{\widetilde{N}_{\lambda,\nu+i}(qv/tu)}{\widetilde{N}_{\lambda,\nu}(qv/tu)} \,, \tag{5.A.8}$$

by induction, we prove the main claim.[1]

We can show this actually holds.

[1] Again, note that we have only the condition for one linear combination of the Nekrasov factors, and for the complete proof, we also need the other equations which are obtained from (5.A.5) by taking the coefficients of w^n for the general n, though we omit them here.

Proposition 5.A.1.

$$\frac{1-v^{-1}}{1-q^{-1}}\frac{1}{z}+\sum_{i\in R(\lambda)}\varphi_{\lambda/\lambda-i}\frac{\widetilde{N}_{\lambda-i,\nu}(qv/tu)}{\widetilde{N}_{\lambda,\nu}(qv/tu)}=\sum_{i\in A(\nu)}\varphi_{\nu+i/\nu}\frac{\widetilde{N}_{\lambda,\nu+i}(qv/tu)}{\widetilde{N}_{\lambda,\nu}(qv/tu)}\,. \tag{5.A.9}$$

The proof is done by using Fact 5.3.3. Let us see how to apply the identity to this situation. For convenience, we introduce the notation, for $x=(i,j)\in\lambda$, $\chi_x=q^{j-1}t^{i-1}$. We also use the additive and multiplicative notations interchangeably. Then, the following identification of parameters allows us to apply the Kajihara-Noumi identity Fact 5.3.3 to (5.A.8).

$$\begin{aligned}
&\delta\to 0\,,\\
&m=|A(\nu)|\,,\\
&n=|A(\lambda)|\,,\\
&L=1\,,\\
&x_i\quad\longleftrightarrow\quad v\chi_x\,,\quad (x\in A(\nu))\,,\\
&x_j-a_j\quad\longleftrightarrow\quad vt\chi_y/q\,,\quad (y\in R(\nu))\,,\\
&y_k\quad\longleftrightarrow\quad q/ut\chi_y\,,\quad (y\in R(\lambda))\,,\\
&y_k-b_k\quad\longleftrightarrow\quad 1/u\chi_y\,,\quad (y\in A(\lambda))\,,\\
&x_i-a_i\quad\longleftrightarrow\quad\infty\,,\\
&y_{|A(\lambda)|}\quad\longleftrightarrow\quad\infty\,,\quad\text{with}\quad\frac{[x_i-x_i+a_i]}{[x_i+y_{|A(\lambda)|}]}\to\frac{v}{u}\,.
\end{aligned} \tag{5.A.10}$$

($\longleftrightarrow$ means we interpret the additive symbols as the multiplicative ones.) Note that because we take the limit $\delta\to\infty$, both sides diverges because of the factor $1/[\delta]$. Thus, we first multiply $[\delta]$ to both sides and take the limit. As we can see from the identification of parameters above, one of a_i's and one of b_i's go to infinity. Then, the balancing condition holds as the both sides go to infinity.

As mentioned above, instead of inserting g_1, we can insert g_L, and then the similar equality holds by the Kajihara-Noumi identity for $\Phi_L^{n,m}$.

Appendix 5.B Proof of Propositions

5.B.1 Proof of Proposition 5.3.7

We prepare two lemmas. They transform the LHS of (5.3.16) to the form to which we can apply the Euler transformation formula for the multiple trigonometric basic hypergeometric series. The proofs of these lemmas are by direct calculation, though a little tedious.

Lemma 5.B.1. *Let $\sigma=(\sigma_k)_{1\le k\le m-1}\in\mathbb{Z}^{m-1}$ and $\theta=(\theta_i)_{1\le i\le n+m-1}\in\mathbb{Z}_{\ge 0}^{n+m-1}$. Let h be the parameter, satisfying $q^{r_1}t^{r_2}h\neq 1$ ($\forall r_1,r_2\in\mathbb{Z}$). Under the change of variables $\rho_k=\sigma_k-\theta_{n+k}$, we can show*

$$\begin{aligned}
&\prod_{1\le i<j\le n}\frac{(qq^{-\theta_j}s_j/tq^{-\theta_i}s_i;q)_\infty}{(qq^{-\theta_j}s_j/q^{-\theta_i}s_i;q)_\infty}\cdot d_{n+m}(\theta;s|q,t)\mathsf{N}_\sigma^{n,m-1}(q^{-\theta_1}s_1,\dots,q^{-\theta_{n+m-1}}s_{n+m-1})\\
&\quad=\prod_{1\le i<j\le n}\frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty}\prod_{i=1}^n q^{\theta_i}t^{-i\theta_i}\prod_{i=n+1}^{m-1}q^{\theta_i}t^{-n\theta_i}\cdot\lim_{h\to 1}\widetilde{\mathsf{N}}_\rho^{n,m-1}(h;s_1,\dots,s_{n+m-1})\\
&\qquad\times\phi_\theta^{n+m-1,m}\left(\begin{matrix}t & ,\dots, & t\\ hs_1^{-1} & ,\dots, & hs_{n+m-1}^{-1}\end{matrix}\middle|\begin{matrix}qq^{\rho_1}s_{n+1}/t & ,\dots, & qq^{\rho_{m-1}}s_{n+m-1}/t & ,ts_{n+m}\\ qq^{\rho_1}s_{n+1} & ,\dots, & qq^{\rho_{m-1}}s_{n+m-1} & ,qs_{n+m}\end{matrix}\right),
\end{aligned} \tag{5.B.1}$$

where

$$\widetilde{\mathsf{N}}_\mu^{n,m}(h;s_1,\dots,s_{n+m}):=\prod_{k=1}^m\left(\prod_{i=1}^{n+k}\frac{(qs_{n+k}/ts_i;q)_{\mu_k}}{(hqs_{n+k}/s_i;q)_{\mu_k}}\right)\cdot\prod_{1\le i<j\le m}\frac{(t\,q^{-\mu_i}s_{n+j}/s_{n+i};q)_{\mu_j}}{(q^{-\mu_i}s_{n+j}/s_{n+i};q)_{\mu_j}}\,. \tag{5.B.2}$$

Proof. As noted, the proof is by brute-force computation. We have

$$\prod_{1\le i<j\le n+m} \frac{(q^{-\theta_j} q s_j/ts_i;q)_{\theta_i}}{(q^{-\theta_j} s_j/s_i;q)_{\theta_i}} = \prod_{1\le i<j\le n+m} \frac{q^{\theta_i} s_i^{-1} - q^{\theta_j} s_j^{-1}}{s_i^{-1} - s_j^{-1}} \frac{(ts_i/s_j;q)_{\theta_j}}{(qs_i/s_j;q)_{\theta_j}} t^{-\theta_j} \frac{(qs_j/ts_i;q)_{\theta_i-\theta_j}}{(qs_j/s_i;q)_{\theta_i-\theta_j}} \tag{5.B.3}$$

and

$$\prod_{1\le i<j\le n} \frac{(qq^{-\theta_j} s_j/tq^{-\theta_i} s_i;q)_\infty}{(qq^{-\theta_j} s_j/q^{-\theta_i} s_i;q)_\infty} = \prod_{1\le i<j\le n} \frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty} \frac{(qs_j/s_i;q)_{\theta_i-\theta_j}}{(qs_j/ts_i;q)_{\theta_i-\theta_j}}. \tag{5.B.4}$$

Combining (5.B.3) and (5.B.4), we have

$$\begin{aligned}\prod_{1\le i<j\le n} \frac{(qq^{-\theta_j} s_j/tq^{-\theta_i} s_i;q)_\infty}{(qq^{-\theta_j} s_j/q^{-\theta_i} s_i;q)_\infty} \cdot d_{n+m}(\theta;s|q,t) &= \prod_{1\le i<j\le n} \frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty} \cdot \phi_\theta^{n+m-1,1}\left(\begin{matrix} t & ,\dots, & t \\ s_1^{-1} & ,\dots, & s_{n+m-1}^{-1}\end{matrix}\middle| \begin{matrix} ts_{n+m} \\ qs_{n+m}\end{matrix}\right) \\ &\quad \times \prod_{i=1}^{n+m-1} q^{\theta_i} t^{-i\theta_i} \cdot \prod_{j=n+1}^{n+m-1}\prod_{i=1}^{j-1} \frac{(qs_j/ts_i;q)_{\theta_i-\theta_j}}{(qs_j/s_i;q)_{\theta_i-\theta_j}}.\end{aligned} \tag{5.B.5}$$

It is also easy computation to show

$$\begin{aligned}&\prod_{k=1}^{m-1}\prod_{i=1}^{n+k} \frac{(qq^{-\theta_{n+k}} s_{n+k}/tq^{-\theta_i} s_i;q)_{\sigma_k}}{(qq^{-\theta_{n+k}} s_{n+k}/q^{-\theta_i} s_i;q)_{\sigma_k}} \\ &= \lim_{h\to 1} \prod_{k=1}^{m-1}\prod_{i=1}^{n+k} \frac{(qq^{\rho_k} s_{n+k}/ts_i;q)_{\theta_i}}{(hqq^{\rho_k} s_{n+k}/s_i;q)_{\theta_i}} \frac{(qs_{n+k}/ts_i;q)_{\rho_k}}{(hqs_{n+k}/s_i;q)_{\rho_k}} \frac{(qs_{n+k}/s_i;q)_{\theta_i-\theta_{n+k}}}{(qs_{n+k}/ts_i;q)_{\theta_i-\theta_{n+k}}}\end{aligned} \tag{5.B.6}$$

and

$$\begin{aligned}&\prod_{1\le k<l\le m-1} \frac{(tq^{-\sigma_k} q^{-\theta_{n+l}} s_{n+l}/q^{-\theta_{n+k}} s_{n+k};q)_{\sigma_l}}{(q^{-\sigma_k} q^{-\theta_{n+l}} s_{n+l}/q^{-\theta_{n+k}} s_{n+k};q)_{\sigma_l}} \\ &= \prod_{1\le k<l\le m-1} \frac{(tq^{-\rho_k} s_{n+l}/s_{n+k};q)_{\rho_l}}{(q^{-\rho_k} s_{n+l}/s_{n+k};q)_{\rho_l}} \frac{(qq^{\rho_k} s_{n+k}/ts_{n+l};q)_{\theta_{n+l}}}{(q^{\rho_k} s_{n+k}/s_{n+l};q)_{\theta_{n+l}}} \times t^{\theta_{n+l}}.\end{aligned} \tag{5.B.7}$$

Combining (5.B.5), (5.B.6) and (5.B.7), we obtain the claim. □

Note that h is the parameter which goes to 1 in the main proof. The necessity of the introduction of such parameter is to avoid some divergence. See the remark below.

Remark 5.B.2. *If $\rho_k < 0$ and $\rho_k + \theta_{n+k} > 0$ for some k, then $\widetilde{\mathsf{N}}_\rho^{n,m-1}$ is 0 and $\phi_\theta^{n+m-1,m}$ diverges when $h \to 1$. Because (5.B.1) must converge as a result. we inserted the parameter h. For the detail of this convergence, see the main proof of Proposition 5.3.7.*

We prepare one more lemma.

Lemma 5.B.3. *Let $\rho = (\rho_k)_{1\le k\le m-1} \in \mathbb{Z}^{m-1}$ and $\nu = (\nu_k)_{1\le k\le m} \in \mathbb{Z}_{\ge 0}^m$. Then*

$$\begin{aligned}&\lim_{h\to 1} \widetilde{\mathsf{N}}_\rho^{n,m-1}(h;s_1,\dots,s_{n+m-1}) \\ &\times \phi_\nu^{m,n+m-1}\left(\begin{matrix} t & ,\dots, & t, & q/t \\ q^{\rho_1} s_{n+1} & ,\dots, & q^{\rho_{m-1}} s_{n+m-1}, & s_{n+m}\end{matrix}\middle| \begin{matrix} hq/ts_1 & ,\dots, & hq/ts_{n+m-1} \\ hq/s_1 & ,\dots, & hq/s_{n+m-1}\end{matrix}\right) \\ &= \mathsf{N}_\mu^{n,m}(s_1,\dots,s_{n+m}) \times d_m((\theta_i);(q^{\mu_i} s_{n+i})|q,t)\,,\end{aligned} \tag{5.B.8}$$

under the identification of running variables

$$\rho_k = \mu_k - \theta_k \quad (k=1,\dots,m-1)\,, \tag{5.B.9}$$

$$\nu_k = \theta_k \quad (k=1,\dots,m-1)\,, \tag{5.B.10}$$

$$\nu_m = \mu_m\,. \tag{5.B.11}$$

Proof. First, put

$$A = \prod_{k=1}^{m-1} \frac{q^{\nu_k+\rho_k} s_{n+k} - q^{\nu_m} s_{n+m}}{q^{\rho_k} s_{n+k} - s_{n+k}}$$
$$\times \prod_{k=1}^{m-1} \frac{(t;q)_{\nu_k}}{(q;q)_{\nu_k}} \frac{(qq^{\rho_k} s_{n+k}/ts_{n+m};q)_{\nu_k}}{(qq^{\rho_k} s_{n+k}/s_{n+m};q)_{\nu_k}} \frac{(tq^{-\rho_k} s_{n+m}/s_{n+k};q)_{\nu_m}}{(qq^{-\rho_k} s_{n+m}/s_{n+k};q)_{\nu_m}}, \tag{5.B.12}$$

$$B = \prod_{1\le k<l\le m-1} \frac{q^{\nu_k+\rho_k} s_{n+k} - q^{\nu_l+\rho_l} s_{n+l}}{q^{\rho_k} s_{n+k} - q^{\rho_l} s_{n+l}} \cdot \prod_{k\neq l} \frac{(tq^{\rho_k-\rho_l} s_{n+k}/s_{n+l};q)_{\nu_k}}{(qq^{\rho_k-\rho_l} s_{n+k}/s_{n+l};q)_{\nu_k}}, \tag{5.B.13}$$

$$C = \prod_{k=1}^{m-1} \prod_{i=1}^{n+m-1} \frac{(hqq^{\rho_k} s_{n+k}/ts_i;q)_{\nu_k}}{(hqq^{\rho_k} s_{n+k}/s_i;q)_{\nu_k}}, \tag{5.B.14}$$

$$D = \frac{(q/t;q)_{\nu_m}}{(q;q)_{\nu_m}} \prod_{i=1}^{n+m-1} \frac{(hqs_{n+m}/ts_i;q)_{\nu_m}}{(hqs_{n+m}/s_i;q)_{\nu_m}}, \tag{5.B.15}$$

to simplify the notation as $\phi_\nu^{m,n+m-1} = A \cdot B \cdot C \cdot D$. First, we have a look at

$$A = \prod_{k=1}^{m-1} (q/t)^{\theta_k} \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(tq^{-\mu_k+\mu_m} s_{n+m}/s_{n+k};q)_{\theta_k}}{(qq^{-\mu_k+\mu_m} s_{n+m}/s_{n+k};q)_{\theta_k}} \prod_{k=1}^{m-1} \frac{(tq^{-\mu_k} s_{n+m}/s_{n+k};q)_{\mu_m}}{(q^{-\mu_k} s_{n+m}/s_{n+k};q)_{\mu_m}}. \tag{5.B.16}$$

We can see the first part in (5.B.16) is equal to the factors in $d_m((\theta_i);(q^{\mu_i} s_{n+i})|q,t)$, *i.e.*, the factors in the first product in (4.1.8). Next, we can show

$$\lim_{h\to 1} \widetilde{\mathsf{N}}_\rho^{n,m-1}(h;s_1,\dots,s_{n+m-1}) \cdot C = \mathsf{N}_\mu^{n,m-1}(s_1,\dots,s_{n+m-1}) \cdot E, \tag{5.B.17}$$

where E is defined by

$$E \equiv \prod_{1\le k<l\le m-1} t^{-\theta_k} \frac{(tq^{-\mu_k+\mu_l} s_{n+l}/s_{n+k};q)_{\theta_k-\theta_l}}{(q^{-\mu_k+\mu_l} s_{n+l}/s_{n+k};q)_{\theta_k-\theta_l}}. \tag{5.B.18}$$

Note some factors in $d_m((\theta_i);(q^{\mu_i} s_{n+i})|q,t)$ are remaining, and these factors can be reproduced by the product BE as follows:

$$B \cdot E = \prod_{1\le i<j\le m} \frac{(tq^{-\mu_k+\mu_l} s_{n+l}/s_{n+k};q)_{\theta_k}}{(qq^{-\mu_k+\mu_l} s_{n+l}/s_{n+k};q)_{\theta_k}} \frac{(qq^{-\mu_k+\mu_l-\theta_l} s_{n+l}/ts_{n+k};q)_{\theta_k}}{(q^{-\mu_k+\mu_l-\theta_l} s_{n+l}/s_{n+k};q)_{\theta_k}}. \tag{5.B.19}$$

This corresponds to the second part in (4.1.8). Now, what are left are the product of $\mathsf{N}_\mu^{n,m-1}$, D and the remaining factors (the factors in the second product) in (5.B.16). They can be simplified, and we obtain the claim as follows:

$$\mathsf{N}_\mu^{n,m-1}(s_1,\dots,s_{n+m-1}) \cdot \prod_{k=1}^{m-1} \frac{(tq^{-\mu_k} s_{n+m}/s_{n+k};q)_{\mu_m}}{(q^{-\mu_k} s_{n+m}/s_{n+k};q)_{\mu_m}} \cdot \lim_{h\to 1} D = \mathsf{N}_\mu^{n,m}(s_1,\dots,s_{n+m}). \tag{5.B.20}$$

□

Proof of Proposition 5.3.7

Now, we prove Proposition 5.3.7 by using these lemmas and the Kajihara-Noumi transformation formula.

The proof is done by induction on m. If $m = 0$, (5.3.16) follows from Fact 4.1.14. Roughly speaking, in $f^{\mathfrak{gl}_{n+m}}$ in the LHS of (5.3.16), we have the summation over $n+m-1$-variables associated with the variable x_{n+m}. On the other hand, in the RHS of (5.3.16), we only have the summation over $m-1$-variables in $f^{\mathfrak{gl}_m}$. There is no choice but the Kajihara-Noumi identity for the formula which is applicable to such situation.

Assuming the claim holds for $m-1$, it can be shown that the LHS of (5.3.16) is

$$\begin{aligned}
&\sum_{\theta\in\mathbb{Z}_{\geq 0}^{n+m-1}} \frac{(qy_m/t^n x;q)_\infty}{(ty_m/x;q)_\infty} d_{n+m}(\theta,\boldsymbol{s}|q,t) \prod_{i=1}^{n+m-1}(x_{n+m}/x_i)^{\theta_i}\\
&\qquad \times \prod_{k=1}^{m-1}\frac{(qy_k/t^n x;q)_\infty}{(ty_k/x;q)_\infty}\cdot f^{\mathfrak{gl}_{n+m-1}}(x_1,\dots,x_{n+m-1}|q^{-\theta_1}s_1,\dots,q^{-\theta_{n+m-1}}s_{n+m-1}|q,t)\\
&=\sum_{\substack{\theta\in\mathbb{Z}_{\geq 0}^{n+m-1}\\ \sigma\in\mathbb{Z}_{\geq 0}^{m-1}}} \prod_{i=1}^{n}\frac{(q/t;q)_\infty}{(q/t^i;q)_\infty}\cdot\frac{(qy_m/t^n x;q)_\infty}{(ty_m/x;q)_\infty}\prod_{1\leq i<j\leq n}\frac{(qq^{-\theta_j}s_j/tq^{-\theta_i}s_i;q)_\infty}{(qq^{-\theta_j}s_j/q^{-\theta_i}s_i;q)_\infty}d_{n+m}(\theta,\boldsymbol{s}|q,t)\\
&\qquad\times \mathsf{N}^{n,m-1}_{\sigma}(q^{-\theta_1}s_1,\dots,q^{-\theta_{n+m}}s_{n+m})f^{\mathfrak{gl}_{m-1}}((y_i)|(q^{\sigma_i}q^{-\theta_{n+i}}s_{n+i})|q,t)\prod_{k=1}^{m-1}(ty_k/x)^{\sigma_k}\,.
\end{aligned}\tag{5.B.21}$$

Thanks to the contribution of the factor $\prod_{k=1}^{m-1}1/(q;q)_{\sigma_k}$ in $\mathsf{N}^{n,m-1}_{\sigma}$, we can extend the range which θ and σ ran over to

$$\theta\in\mathbb{Z}_{\geq 0}^{n+m-1},\quad \sigma\in\mathbb{Z}^{m-1}\,,\tag{5.B.22}$$

because for $\sigma\in\mathbb{Z}_{<0}$, this factor becomes zero. Under this range, by applying Lemma 5.B.1, we can rewrite (5.B.21) as

$$\begin{aligned}
&\lim_{h\to 1}\sum_{\substack{\theta\in\mathbb{Z}_{\geq 0}^{n+m-1}\\ \rho\in\mathbb{Z}^{m-1}}}\prod_{i=1}^{n}\frac{(q/t;q)_\infty}{(q/t^i;q)_\infty}\cdot\frac{(qy_m/t^n x;q)_\infty}{(ty_m/x;q)_\infty}\prod_{1\leq i<j\leq n}\frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty}\\
&\times\widetilde{\mathsf{N}}^{n,m-1}_{\rho}(h;s_1,\dots,s_{n+m-1})f^{\mathfrak{gl}_{m-1}}(y|(q^{\rho_i}s_{n+i})|q,t)\prod_{k=1}^{m-1}(ty_k/x)^{\rho_k}\\
&\times\phi^{n+m-1,m}_{\theta}\left(\begin{matrix} t & ,\dots, & t\\ hs_1^{-1} & ,\dots, & hs_{n+m-1}^{-1}\end{matrix}\,\middle|\,\begin{matrix} qq^{\rho_1}s_{n+1}/t\,,\dots, & qq^{\rho_{m-1}}s_{n+m-1}/t & ,ts_{n+m}\\ qq^{\rho_1}s_{n+1}\quad,\dots, & qq^{\rho_{m-1}}s_{n+m-1} & ,qs_{n+m}\end{matrix}\right)\\
&\times\prod_{i=1}^{n+m-1}(qy_m/t^n x)^{\theta_i}.
\end{aligned}\tag{5.B.23}$$

By using the Kajihara-Noumi identity for the trigonometric multiple hypergeometric series (Fact 5.3.3), the term (5.B.23) can be written as

$$\begin{aligned}
&\lim_{h\to 1}\sum_{\substack{\nu\in\mathbb{Z}_{\geq 0}^{m}\\ \rho\in\mathbb{Z}^{m-1}}}\prod_{i=1}^{n}\frac{(q/t;q)_\infty}{(q/t^i;q)_\infty}\cdot\prod_{1\leq i<j\leq n}\frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty}\\
&\times\widetilde{\mathsf{N}}^{n,m-1}_{\rho}(h;s_1,\dots,s_{n+m-1})p_{m-1}(y;(q^{\rho_i}s_{n+i})|q,t)\prod_{k=1}^{m-1}(ty_k/x)^{\rho_k}\\
&\times\phi^{m,n+m-1}_{\nu}\left(\begin{matrix} t & ,\dots, & t, & q/t\\ q^{\rho_1}s_{n+1} & ,\dots, & q^{\rho_{m-1}}s_{n+m-1}, & s_{n+m}\end{matrix}\,\middle|\,\begin{matrix} hq/ts_1\,,\dots, & hq/ts_{n+m-1}\\ hq/s_1\;\,,\dots, & hq/s_{n+m-1}\end{matrix}\right)\\
&\times\prod_{k=1}^{m}(ty_m/x)^{\nu_k}.
\end{aligned}\tag{5.B.24}$$

Finally, Lemma 5.B.3 shows that (5.B.24) is equal to

$$\prod_{i=1}^{n}\frac{(q/t;q)_\infty}{(q/t^i;q)_\infty}\cdot\prod_{1\leq i<j\leq n}\frac{(qs_j/ts_i;q)_\infty}{(qs_j/s_i;q)_\infty}\tag{5.B.25}$$

$$\times \sum_{\substack{(\mu_i)_{i=1}^{m-1}\in\mathbb{Z}^{m-1}\\ \mu_m\geq 0}} \mathsf{N}^{n,m}_{\mu}(s_1,\dots,s_{n+m})\, f^{\mathfrak{gl}_m}(y_1,\dots,y_m|q^{\mu_1}s_{n+1},\dots,q^{\mu_m}s_{n+m}|q,t)\prod_{k=1}^{m}(ty_k/x)^{\mu_k}.$$

Note that the summation is restricted to $\mu\in\mathbb{Z}^m_{\geq 0}$ by the contribution of the factor in $\mathsf{N}^{n,m}_{\mu}$, Thus we complete the proof. □

5.B.2 Proof of Proposition 5.1.9

As noted, the proof is by direct computation. First, we give the proof for $X^{(1)}(z)$. For $X^{(k)}(z)$ with $k>1$, the proof goes completely in parallel to that of $k=1$ case.

$k=1$ **case** We first introduce the new notation which we use only in this appendix. We write the operator $\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w)$ as

$$\begin{aligned}\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w) &= \sum_{\nu^{(1)},\dots,\nu^{(N-1)}} \mathcal{C}_{\boldsymbol{\nu}}\ :\widehat{\Phi}_{\emptyset}(w_1)\widehat{\Phi}^*_{\nu^{(1)}}(w_1'):\otimes:\widehat{\Phi}_{\nu^{(1)}}(w_2)\widehat{\Phi}^*_{\nu^{(2)}}(w_2'):\\ &\qquad\qquad\otimes\cdots\otimes:\widehat{\Phi}_{\nu^{(N-1)}}(w_N)\widehat{\Phi}^*_{\emptyset}(w_N'):\\ &=: \sum_{\nu^{(1)},\dots,\nu^{(N-1)}} \mathcal{T}^V_{\boldsymbol{\nu}}(\boldsymbol{u},\boldsymbol{v};w),\end{aligned} \tag{5.B.26}$$

with

$$\begin{aligned}\mathcal{C}_{\boldsymbol{\nu}} := \prod_{i=1}^{N}\frac{c_{\nu^{(i)}}}{c'_{\nu^{(i)}}}\ \mathcal{G}(w_i/\gamma w_i')^{-1}\\ \times\hat{t}(\nu^{(i-1)},u_i,w_i,0)\ \hat{t}^*(\nu^{(i)},w_i',v_i,0)\ N_{\nu^{(i-1)}\nu^{(i)}}(w_i/\gamma w_i')/\left\langle \mathbf{0}\right|\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w)\left|\mathbf{0}\right\rangle .\end{aligned} \tag{5.B.27}$$

Note that the factor $\mathcal{C}_{\boldsymbol{\nu}}$ is the products of the coefficients of the intertwiners and the factors appearing from the normal orderings. Here, we put $\nu^{(0)}=\nu^{(N)}=\emptyset$ for convenience. In the following, we omit the parameters $\boldsymbol{u},\boldsymbol{v}$ in $\mathcal{T}^V(\boldsymbol{u},\boldsymbol{v};w)$ unless there are any confusion.

As a preparation, we first compute the commutation relations between $\Lambda^{(j)}, j=1,\dots,N$ and $\mathcal{T}^V(w)$. Putting $z_j=\gamma^{j-1}z$, we have

$$\begin{aligned}&(w_j/\gamma w_j')\ v_j\Lambda^{(j)}(z)\ \mathcal{T}^V_{\boldsymbol{\nu}}(w) - \frac{1-z/\gamma^2 w_1}{1-z/w_1}\mathcal{T}^V_{\boldsymbol{\nu}}(w)\ u_j\Lambda^{(j)}(z)\\ &= -u_j\left(\text{delta functions from } A(\nu^{(j-1)})\right)\\ &- u_j\sum_{y\in R(\nu^{(j)})}\delta(\gamma z_j/w_j'\chi_y)\ \mathcal{C}_{\boldsymbol{\nu}}\ \frac{\prod_{x\in R(\nu^{(j-1)})}1-\gamma w_j'\chi_y/w_j\chi_x}{\prod_{x\in A(\nu^{(j-1)})}1-w_j'\chi_y/\gamma w_j\chi_x}\frac{\prod_{x\in A(\nu^{(j)})}1-\chi_y/\gamma^2\chi_x}{\prod_{x\in R(\nu^{(j)}),x\neq y}1-\chi_y/\chi_x}\\ &\times\cdots\otimes:\eta(\gamma^{-1}w_j'\chi_y)\widehat{\Phi}_{\nu^{(j-1)}}\widehat{\Phi}^*_{\nu^{(j)}}:\otimes:\widehat{\Phi}_{\nu^{(j)}}\widehat{\Phi}^*_{\nu^{(j+1)}}:\otimes\cdots\\ &= -u_j\left(\text{delta functions from } A(\nu^{(j-1)})\right)\\ &- u_j\sum_{y\in R(\nu^{(j)})}\delta(\gamma z_j/w_j'\chi_y)\ \mathcal{C}_{\boldsymbol{\nu}}\ \frac{\prod_{x\in R(\nu^{(j-1)})}1-\gamma w_j'\chi_y/w_j\chi_x}{\prod_{x\in A(\nu^{(j-1)})}1-w_j'\chi_y/\gamma w_j\chi_x}\frac{\prod_{x\in A(\nu^{(j)})}1-\chi_y/\gamma^2\chi_x}{\prod_{x\in R(\nu^{(j)}),x\neq y}1-\chi_y/\chi_x}\\ &\times\cdots\otimes:\varphi^-(\gamma^{-1/2}w_j'\chi_y)\widehat{\Phi}_{\nu^{(j-1)}}(w_j)\widehat{\Phi}^*_{\nu^{(j)}-y}(w_j'):\otimes:\widehat{\Phi}_{\nu^{(j)}}(w_{j+1})\widehat{\Phi}^*_{\nu^{(j+1)}}(w_{j+1}'):\otimes\cdots .\end{aligned} \tag{5.B.28}$$

Note that the following identity of the operators is crucial in the computation:

$$\eta(\gamma^{-1}u) = :\varphi^-(\gamma^{-1/2}u)\xi^{-1}(u): . \tag{5.B.29}$$

Now, for $\nu^{(j)} \neq \emptyset$, fix a $y \in R(\nu^{(j)})$ and take a look at the corresponding term in the equation above. Then, we put $\nu^{(j)} - y$ as $\bar{\nu}^{(j)}$, and rewrite the term using $\bar{\nu}^{(j)}$. In the end, we obtain the following equality:

$$\begin{aligned}
&- u_j \delta(\gamma z_j / w'_j \chi_y)\ \mathcal{C}_{\boldsymbol{\nu}}\ \frac{\prod_{x \in R(\nu^{(j-1)})} 1 - \gamma w'_j \chi_y / w_j \chi_x}{\prod_{x \in A(\nu^{(j-1)})} 1 - w'_j \chi_y / \gamma w_j \chi_x} \frac{\prod_{x \in A(\nu^{(j)})} 1 - \chi_y / \gamma^2 \chi_x}{\prod_{x \in R(\nu^{(j)}), x \neq y} 1 - \chi_y / \chi_x} \\
&\times \cdots \otimes : \varphi^-(\gamma^{-1/2} w'_j \chi_y) \widehat{\Phi}_{\nu^{(j-1)}}(w_j) \widehat{\Phi}^*_{\nu^{(j)} - y}(w'_j) : \otimes : \widehat{\Phi}_{\nu^{(j)}}(w_{j+1}) \widehat{\Phi}^*_{\nu^{(j+1)}}(w'_{j+1}) : \otimes \cdots \\
&\quad = u_{j+1} \delta(\gamma z_j / w'_j \chi_y)\ \mathcal{C}_{(\dots, \bar{\nu}^{(j)}, \dots)}\ \frac{\prod_{x \in R(\nu^{(j)})} 1 - \gamma^2 \chi_y / \chi_x}{\prod_{x \in A(\nu^{(j)}), x \neq y} 1 - \chi_y / \chi_x} \frac{\prod_{x \in A(\nu^{(j+1)})} 1 - w_{j+1} \chi_y / \gamma w'_{j+1} \chi_x}{\prod_{x \in R(\nu^{(j+1)})} 1 - \gamma w_{j+1} \chi_y / w'_{j+1} \chi_x} \\
&\quad \times \cdots \otimes : \varphi^-(\gamma^{-1/2} w'_j \chi_y) \widehat{\Phi}_{\nu^{(j-1)}}(w_j) \widehat{\Phi}^*_{\bar{\nu}^{(j)}}(w'_j) : \otimes : \widehat{\Phi}_{\bar{\nu}^{(j)} + y}(w_{j+1}) \widehat{\Phi}^*_{\nu^{(j+1)}}(w'_{j+1}) : \otimes \cdots . \qquad (5.B.30)
\end{aligned}$$

When we note that in the end, we take summation over $\boldsymbol{\nu}$, this term is canceled by the term appearing from the commutation relation between $\Lambda^{(j+1)}$ and $\mathcal{T}^V_{\boldsymbol{\nu}'}(w)$ with $\nu^{(j)\prime} = \bar{\nu}^{(j)}$. This situation is summarized in the following identity:

$$\begin{aligned}
&(w_{j+1} / \gamma w'_{j+1})\ v_{j+1} \Lambda^{(j+1)}(z)\ \mathcal{T}^V_{\boldsymbol{\nu}'}(w) - \frac{1 - z/\gamma^2 w_1}{1 - z/w_1} \mathcal{T}^V_{\boldsymbol{\nu}'}(w)\ u_{j+1} \Lambda^{(j+1)}(z) \\
&\quad = -u_{j+1} \left(\text{delta functions from } R(\nu^{(j+1)\prime}) \right) \\
&\quad - u_{j+1} \sum_{y \in A(\nu^{(j)\prime})} \delta(z_{j+1} / w_{j+1} \chi_y) \frac{\prod_{x \in R(\nu^{(j)\prime})} 1 - \gamma^2 \chi_y / \chi_x}{\prod_{x \in A(\nu^{(j)\prime}), x \neq y} 1 - \chi_y / \chi_x} \times \frac{\prod_{x \in A(\nu^{(j+1)\prime})} 1 - w_{j+1} \chi_y / \gamma w'_{j+1} \chi_x}{\prod_{x \in R(\nu^{(j+1)\prime})} 1 - \gamma w_{j+1} \chi_y / w'_{j+1} \chi_x} \\
&\quad \times \cdots \otimes : \varphi(\gamma^{-1/2} w_{j+1} \chi_y) \widehat{\Phi}_{\nu^{(j-1)\prime}}(w_j) \widehat{\Phi}^*_{\nu^{(j)\prime}}(w'_j) : \otimes : \widehat{\Phi}_{\nu^{(j)\prime} + y}(w_{j+1}) \widehat{\Phi}^*_{\nu^{(j+1)\prime}}(w'_{j+1}) : \otimes \cdots . \qquad (5.B.31)
\end{aligned}$$

This cancellation mechanism deletes almost all the terms which appear in the commutation relations. While summing up for j from 1 to N, the term which is not cancelled by this mechanism exists at $A(\nu^{(0)}) = A(\emptyset)$. This term proportional to the delta function at z_1/z, vanishes when we multiply both sides by $1 - w_1/z$. Thus we obtain the main claim of Proposition 5.1.9 for the $k = 1$ case.

$k > 1$ **case** We use the following simplified notation,

$$\Lambda^{(i_1, \dots, i_k)}(z) :=: \Lambda^{(i_1)}(z) \cdots \Lambda^{(i_k)}(\gamma^{2(1-k)} z) : . \qquad (5.B.32)$$

The commutation relations between $\Lambda^{(i_1, \dots, i_k)}(z)$ and $\mathcal{T}^V_{\boldsymbol{\nu}}(w)$ schematically lead to the following form:

$$\begin{aligned}
&\prod_{j=1}^{k} (w_{i_j} / \gamma w'_{i_j})\ v_{i_1} \cdots v_{i_k} \Lambda^{(i_1, \dots, i_k)}(z)\ \mathcal{T}^V_{\boldsymbol{\nu}}(w) - \frac{1 - z/\gamma^{2k} w_1}{1 - z/w_1} \mathcal{T}^V_{\boldsymbol{\nu}}(w)\ u_{i_1} \cdots u_{i_k} \Lambda^{(i_1, \dots, i_k)}(z) \\
&\quad = -u_{i_1} \cdots u_{i_k} \Big(\text{delta functions from } A(\nu^{(i_1 - 1)}), R(\nu^{(i_1)}), A(\nu^{(i_2 - 1)}), R(\nu^{(i_2)}), \\
&\qquad \dots, A(\nu^{(i_k - 1)}) \text{ and } R(\nu^{(i_k)}) \Big) \qquad (5.B.33)
\end{aligned}$$

The delta functions related to $R(\nu^{(i_j)})$ (for $j = 1, \dots, k$) cancel those related to $A(\nu^{(i_j)})$ (recall these terms appear in the commutation relations between $\Lambda^{(\dots, i_j + 1, \dots)}(z)$ and $\mathcal{T}^V_{\boldsymbol{\nu}}(w)$). This sequence of the cancellation begins when $i_j = i_{j-1} + 1$ and terminates when $i_j = i_{j+1} - 1$, because in those cases, the poles and zeros related to $\nu^{(i_j)}$ cancel each other, and no delta functions related to them appear.

As a result, the only delta function related to $A(\nu^{(0)})$ survives this cancellation. Again, it vanishes when we multiply $1 - w_1/z$, and we complete the proof of the expected commutation relation between $X^{(k)}(z)$ and $\mathcal{T}^V(w)$.

Chapter 6

Macdonald Functions Revisited

Now we revisit the bispectral Macdonald functions.

Recalling Proposition 5.1.9 and the proof of Theorem 4.2.21, $\mathcal{T}^V$ and $\Phi^{(k)}$ satisfy almost the same commutation relation with $X^{(i)}$. From this observation, we expect we can construct the Macdonald functions $f^{\mathfrak{gl}_N}$ by gluing $\mathcal{T}^V$'s. This expectation turns out to be true with some slight tuning of parameters.

The most difficulty is that inside $\mathcal{T}^V$ there is the summation over $N-1$ partitions, while $f^{\mathfrak{gl}_N}$ is expressed as the summation over $N(N-1)/2$ non-negative integers. There is a mismatch between the numbers of summations.

To cure this mismatch, we tune the spectral parameters in $\mathcal{T}^V$ to reduce the summation over the partitions to that over some integers. After this kind of tuning, by composing N such operators and taking its vacuum expectation value, we obtain $f^{\mathfrak{gl}_n}$. This is summarized in 6.1.

Note that in particular, $N_{\emptyset,\mu}(t) \neq 0$ if and only if $\mu = (m)$ for some $m \in \mathbb{Z}_{\geq 0}$.

6.1 Bispectral Macdonald Functions from Topological Vertex

In order to get rid of the extra $\mathcal{G}$-factors, we set

$$\widetilde{T}^V(\boldsymbol{u}, \boldsymbol{v}; w) := \prod_{k=1}^{N} \mathcal{G}(u_k/\gamma v_k) \cdot T^V(\boldsymbol{u}, \boldsymbol{v}; w)\,, \tag{6.1.1}$$

where T^V is defined in Definition 5.1.8.

Now we tune the spectral parameters.

Definition 6.1.1.

$$\widetilde{\mathcal{T}}_i(x) := \widetilde{T}^V\big(\boldsymbol{v}, \boldsymbol{u}; x\big)\bigg|_{\substack{v_k \to \gamma^{-1} t^{-\delta_{k,i}} u_k \\ (1 \leq k \leq N)}}\,. \tag{6.1.2}$$

Note that the overall factor (the products of $\mathcal{G}$-factors) goes to zero under the specialization, though the operator $\widetilde{\mathcal{T}}_i(x)$'s are well-defined. This is because the zeros are cancelled by the divergence from the $\mathcal{G}$-factors appearing from the normal orderings among Φ's and Φ^*'s in T^V.

To see under these tuning of parameters, the summation under partitions actually reduces, we prepare the following notation and lemma.

Notation 6.1.2. *For a partitions λ and non-negative integers $r, s \in \mathbb{Z}_{\geq 0}$, we denote by $\mathsf{B}_{r,s}(\lambda)$ the partition obtained by removing s-rows and r-columns from the top left of the original partition, that is,*

$$\mathsf{B}_{r,s}(\lambda) := (\mathsf{P}(\lambda_{s+i} - r))_{i \geq 1}, \quad \mathsf{P}(n) = \begin{cases} n\,, & n \geq 0; \\ 0\,, & n < 0\,. \end{cases} \tag{6.1.3}$$

For example, if $\lambda = (5,5,4,4,4,1,1)$, then $\mathsf{B}_{2,1}(\lambda) = (3,2,2,2)$.

Lemma 6.1.3. *For $m \geq 0$ and $n \leq 0$, we have*

$$N_{\lambda,\mu}(q^n t^m) \neq 0 \quad \Longleftrightarrow \quad \mu \supset \mathsf{B}_{-n,m}(\lambda)\,. \tag{6.1.4}$$

For $m \leq -1$ and $n \geq 1$, we have

$$N_{\lambda,\mu}(q^n t^m) \neq 0 \quad \Longleftrightarrow \quad \lambda \supset \mathsf{B}_{n-1,-m-1}(\mu)\,. \tag{6.1.5}$$

By the formula (B.1.33) and Lemma 6.1.3, we can show the Young diagrams associated with the glued vertices, are restricted to one row as (Fig. 6.1). That is, we have the following lemma.

Lemma 6.1.4. *We have*

$$\begin{aligned}\widetilde{\mathcal{T}}_i(x) = & \sum_{0 \leq m_1 \leq m_2 \leq \cdots \leq m_{i-1} < \infty} t^{-m_{i-1}} \prod_{k=1}^{i-1} \frac{q^{2n((1^{m_k}))}}{c'_{(m_k)} c_{(m_k)}} \left(\frac{q u_{k+1}}{\gamma u_k}\right)^{m_k} f^{-1}_{(m_k)} \\ & \times N_{\emptyset,(m_{i-1})}(t^{-1}) \prod_{k=1}^{i-1} N_{(m_{k-1}),(m_k)}(1) \\ & \times : \widehat{\Phi}^*_{(m_1)}(\gamma^{-1}x) \widehat{\Phi}_{(\emptyset)}(x) : \otimes \widehat{\bigotimes}_{k=2}^{i-1} : \widehat{\Phi}^*_{(m_k)}(\gamma^{-k}x) \widehat{\Phi}_{(m_{k-1})}(\gamma^{-k+1}x) : \\ & \otimes : \widehat{\Phi}^*_{\emptyset}(\gamma^{-i}t^{-1}x) \widehat{\Phi}_{(m_{i-1})}(\gamma^{-i+1}x) : \otimes \widehat{\bigotimes}_{k=i+1}^{N} : \widehat{\Phi}^*_{\emptyset}(\gamma^{-k}t^{-1}x) \widehat{\Phi}_{\emptyset}(\gamma^{-k+1}t^{-1}x) :,\end{aligned} \tag{6.1.6}$$

where we put $m_0 = 0$. Here, $\widehat{\bigotimes}$ is introduced in Notation 5.1.5.

Nota bene! The operators $\Phi^{(k)}(x)$ $(k = 0, \ldots, N-1)$ in this chapter slightly differ from those in Section 4.2.2. In this chapter, $\Phi^{(k)}(x)$ is a map $\mathcal{F}_{t^{-\delta_{k+1}} \cdot \tilde{\boldsymbol{u}}} \to \mathcal{F}_{\boldsymbol{u}}$ with $\tilde{\boldsymbol{u}} = (\gamma^{-1}u_1, \ldots, \gamma^{-1}u_N)$.

Remark 6.1.5. *We have $\widetilde{\mathcal{T}}_1(x) = \Phi^{(0)}(t^{-1}x)$.*

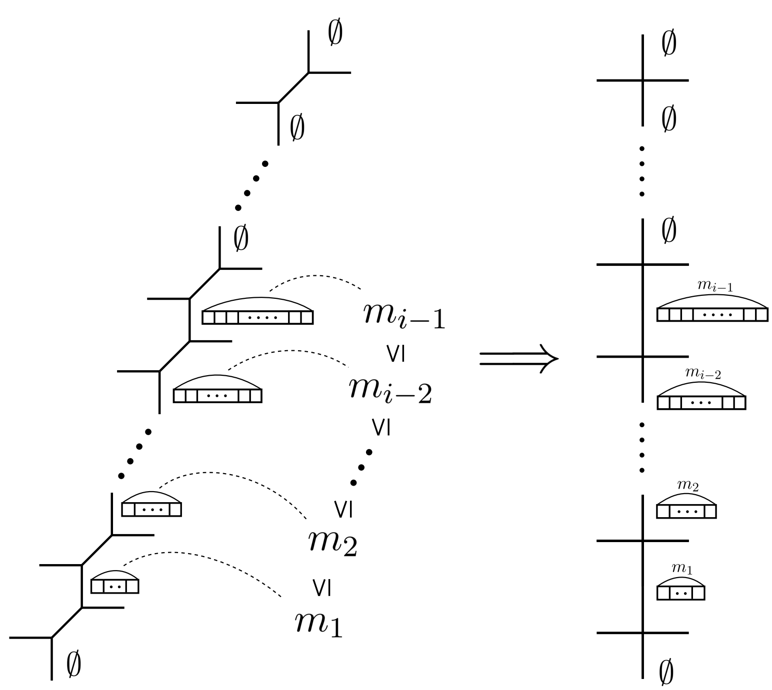


Figure 6.1: The operator $\widetilde{\mathcal{T}}_i(x)$. ($\Longrightarrow$ stands for a simplification of the diagram for convenience)

Now by gluing these operators, we obtain the expression of Macdonald functions. The following proposition says the vacuum expectation value of Fig. 6.2 gives $f^{\mathfrak{gl}_N}$.

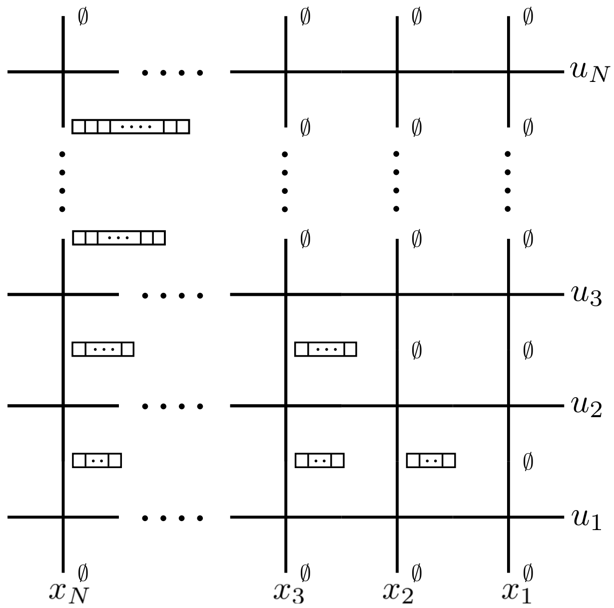


Figure 6.2: The diagram for the Macdonald function $f^{\mathfrak{gl}_N}(\boldsymbol{x};\boldsymbol{u}|q,q/t)$.

Proposition 6.1.6. *The vacuum expectation value of the composition of $\widetilde{\mathcal{T}}_i(x_i)$'s gives the Macdonald function, that is,*

$$\langle \mathbf{0}|\, \widetilde{\mathcal{T}}_1(\boldsymbol{u};x_1)\widetilde{\mathcal{T}}_2(x_2)\cdots\widetilde{\mathcal{T}}_N(x_N)\,|\mathbf{0}\rangle = \prod_{1\le i<j\le N}\frac{(qu_j/tu_i;q)_\infty}{(u_j/u_i;q)_\infty}\cdot f^{\mathfrak{gl}_N}(\boldsymbol{x};\boldsymbol{u}|q,q/t)\,. \tag{6.1.7}$$

Proof. The basic idea of the proof is the same as that of Theorem 4.2.21. That is, we insert $X_0^{(1)}$ in the most left, bring it to the most right, and then confirm the resulting operator is the Macdonald difference operator. Thus, we only need to compute the commutation relation among $X^{(1)}(z)$ and $\widehat{\Phi}_{(m)}(z)$'s.

The operators $\widehat{\Phi}_{(m)}(z)$ and $\widehat{\Phi}^*_{(m)}(z)$ can be formally decomposed as

$$\widehat{\Phi}_{(m)}(z) =: \widehat{\Phi}_\emptyset(t^{-1}z)\mathcal{A}(q^m z) :,\quad \widehat{\Phi}^*_{(m)}(z) =: \widehat{\Phi}^*_\emptyset(t^{-1}z)\mathcal{A}^*(q^m z) :, \tag{6.1.8}$$

(recall Figure 3.2) where

$$\mathcal{A}(z) = \exp\left(-\sum_{n>0}\frac{1-t^{-n}}{n(1-q^n)}a_{-n}z^n\right)\exp\left(\sum_{n>0}\frac{1-t^n}{n(1-q^{-n})}a_n z^{-n}\right), \tag{6.1.9}$$

$$\mathcal{A}^*(z) = \exp\left(\sum_{n>0}\frac{1-t^{-n}}{n(1-q^n)}\gamma^n a_{-n}z^n\right)\exp\left(-\sum_{n>0}\frac{1-t^n}{n(1-q^{-n})}\gamma^n a_n z^{-n}\right). \tag{6.1.10}$$

Then, $\mathcal{A}(z)$ and $\mathcal{A}^*(z)$ agrees with the screening current (Definition 4.3.1):

$$\mathcal{A}^*(z)\otimes\mathcal{A}(z) = \phi^{sc}(\gamma^{-1}t^{-1}z). \tag{6.1.11}$$

Thus, we have

$$\widetilde{\mathcal{T}}_i(\boldsymbol{u};x) = \left(\frac{(q/t;q)_\infty}{(q;q)_\infty}\right)^{i-1} \tag{6.1.12}$$
$$\times \sum_{0\le m_1\le m_2\le\cdots\le m_{i-1}<\infty} \Phi^{(0)}(t^{-1}x)\widetilde{S}^{(1)}(q^{m_1}x)\cdots\widetilde{S}^{(i-1)}(q^{m_{i-1}}x)\prod_{k=1}^{i-1}(u_{k+1}/u_k)^{m_k}.$$

Here, we use the screening currents of the different normalization,

$$\widetilde{S}^{(k)}(z) = S^{(k)}(\gamma^{-2k}t^{-1}z)\,. \tag{6.1.13}$$

Same as the proof of Proposition 4.3.4, $X^{(1)}(z)$ commutes with $\widetilde{S}^{(k)}(w)$ up to q-difference:

$$\left[X^{(1)}(z),\widetilde{S}^{(k)}(w)\right] = (t-1)(u_{k+1}T_{q,w}-u_k)\left(\delta\left(\frac{\gamma^{-2k}w}{qz}\right):\Lambda^{(k)}(\gamma^{-2k}w/q)\widetilde{S}^{(k)}(w):\right). \tag{6.1.14}$$

By the normal ordering, we also have

$$\widetilde{S}^{(k-1)}(x)\Lambda^{(k)}(\gamma^{-2k}x/q) = \Phi^{(0)}(t^{-1}x)\Lambda^{(1)}(\gamma^{-2}x/q) = 0\,. \tag{6.1.15}$$

Then it is easy to show that

$$\Phi^{(0)}(t^{-1}x)\cdot\left[X_0^{(1)},\sum_{m=0}^{\infty}\widetilde{S}^{(1)}(q^m x)(u_2/u_1)^m\right] = 0, \tag{6.1.16}$$

$$\widetilde{S}^{(k-1)}(x)\cdot\left[X_0^{(1)},\sum_{m=0}^{\infty}\widetilde{S}^{(k)}(q^m x)(u_{k+1}/u_k)^m\right] = 0 \quad (k\ge 2). \tag{6.1.17}$$

Combining all above, we obtain

$$X^{(1)}(z)\widetilde{\mathcal{T}}_i(\boldsymbol{u};w) - \gamma\frac{1-qz/tw}{1-z/w}\widetilde{\mathcal{T}}_i(\boldsymbol{u};w)X^{(1)}(z) = u_i(1-t^{-1})\widetilde{\mathcal{T}}_i(\boldsymbol{u};qw)\Psi^+(t^{-1}w)\delta(w/z)\,. \tag{6.1.18}$$

Finally, with this relation, we get

$$\langle\mathbf{0}|\,X^{(1)}(z)\widetilde{\mathcal{T}}_1(x_1)\cdots\widetilde{\mathcal{T}}_N(x_N)\,|\mathbf{0}\rangle = \gamma^N\prod_{k=1}^{N}\frac{1-qz/tx_k}{1-z/x_k}\cdot\langle\mathbf{0}|\,\widetilde{\mathcal{T}}_1(x_1)\cdots\widetilde{\mathcal{T}}_N(x_N)X^{(1)}(z)\,|\mathbf{0}\rangle$$
$$+(1-t^{-1})\sum_{i=1}^{N}\delta(x_i/z)u_i\prod_{k=1}^{i-1}\frac{1-qx_i/tx_k}{1-x_i/x_k}\prod_{k=1}^{i-1}\frac{1-tx_k/qx_i}{1-x_k/x_i}T_{q,x_i}\,\langle\mathbf{0}|\,\widetilde{\mathcal{T}}_1(x_1)\cdots\widetilde{\mathcal{T}}_N(x_N)\,|\mathbf{0}\rangle\,. \tag{6.1.19}$$

Thus, the LHS of the claim can be identified with the eigenfunction of the Macdonald operator D_N^1:

$$D_N^1(u;q,q/t)\,\langle\mathbf{0}|\,\widetilde{\mathcal{T}}_1(x_1)\cdots\widetilde{\mathcal{T}}_N(x_N)\,|\mathbf{0}\rangle = (u_1+\cdots+u_N)\,\langle\mathbf{0}|\,\widetilde{\mathcal{T}}_1(x_1)\cdots\widetilde{\mathcal{T}}_N(x_N)\,|\mathbf{0}\rangle\ . \tag{6.1.20}$$

□

Remark 6.1.7. *As studied in [112, 48], the diagram in Figure 6.2 seems to be the puncture corresponding to the full surface defect, that is, the defect which breaks the gauge group completely to $U(1)^{\mathrm{rank}(G)}$. The moduli space with the full surface defect is called the Laumon space, and we can regard the partition function of the theory with the defect as that of the 2d sigma model whose target space is the Laumon space. Then, the fact that the defect partition function becomes the Macdonald function, is compatible with the main claim in [19].*

Chapter 7

Conclusion

7.1 Recapitulation of Main Results

We now conclude the thesis. The main results in the present thesis consist of two parts and one by-product. These results resolve some open problems about the representation theory of the quantum algebra called the Ding-Iohara-Miki algebra $\mathcal{U}_{q,t}$. Some applications to the physics are stated in the next subsection.

1. The first result is the explicit algorithm to construct the generalized Macdonald functions $|P_{\boldsymbol{\lambda}}(\boldsymbol{u})\rangle$. This is summarized in Theorem 4.2.21 in Chapter 4, that is, schematically,

$$|P_{\boldsymbol{\lambda}}\rangle = (\text{Constant}) \cdot \left[\boldsymbol{x}^{-\lambda} \prod_{i<j} (1 - x_j/x_i) \cdot f^{\mathfrak{gl}_{|\boldsymbol{n}|}}(\boldsymbol{x}|\boldsymbol{s}|q, q/t) V^{(\boldsymbol{n})}\left(\begin{matrix} \boldsymbol{u} \\ t^{-\boldsymbol{n}} \cdot \boldsymbol{u} \end{matrix} ; x_1, \ldots, x_{|\boldsymbol{n}|} \right) |\boldsymbol{0}\rangle \right]_{\boldsymbol{x},1},$$

The key ingredients are the bispectral Macdonald functions $f^{\mathfrak{gl}_{|\boldsymbol{n}|}}$ (Definition 4.1.2) and the screened vertex $V^{(\boldsymbol{n})}(z)$ (Definition 4.2.20) constructed from the screening currents of the q-deformed $\mathcal{W}$-algebra. This result can be seen as the natural generalization of the known result on the Macdonald functions on the Fock space to the multi Fock tensor spaces.

2. The second result consists of the introduction of the Mukadé operator $\mathcal{V}(x)$ (Definition 5.1.1), and the explicit computation of the matrix elements of $\mathcal{V}(x)$ with respect to the generalized Macdonald functions. The existence of the Mukadé operator is ensured by the explicit construction of the operator $\mathcal{T}^V(z)$ (Definition 5.1.8) using the intertwiners of $\mathcal{U}_{q,t}$. As noted in Chapter 5, those matrix elements factorize as the products of the Nekrasov factors. In the end, we obtain, in Theorem 5.2.1,

$$\langle K_{\boldsymbol{\lambda}}(\boldsymbol{v})| \mathcal{V}(x) |K_{\boldsymbol{\mu}}(\boldsymbol{u})\rangle = \frac{\left((-\gamma^2)^N e_N(\boldsymbol{u}) x\right)^{|\boldsymbol{\lambda}|}}{(\gamma^2 x)^{|\boldsymbol{\mu}|}} \prod_{i=1}^{N} \frac{u_i^{|\mu^{(i)}|} g_{\mu^{(i)}}}{\left(v_i^{|\lambda^{(i)}|} g_{\lambda^{(i)}}\right)^{N-1}} \cdot \prod_{i,j=1}^{N} N_{\lambda^{(i)},\mu^{(j)}}(qv_i/tu_j),$$

where $|K_{\boldsymbol{\mu}}(\boldsymbol{u})\rangle$ is the integral form of the generalized Macdonald functions (Definition 4.2.25). These two facts are our main results.

3. Moreover, as a by-product, we obtain the way to construct the bispectral Macdonald functions using the intertwiners of $\mathcal{U}_{q,t}$. This is the result of Chapter 6. More precisely, in the first step, we glue the intertwiners in the shape of the toric diagram which corresponds to the A_M quiver gauge theory with $G = A_N$ in 5D ((A_N, A_M)-theory). In the second step, we specialize some parameters to specific values. Then, in the end, by taking the vacuum expectation value of those operators, we obtain the bispectral Macdonald functions. This means that by tuning parameters in the instanton partition functions of the (A_N, A_M)-theory, we obtain the bispectral Macdonald functions.

7.1.1 Application to Physics

As summarized in Section 5.2, we have two applications of the results above. The first one is the S-duality formula for the partition functions of (A_N, A_M)-theories. Let $\mathcal{Z}_{\text{top.}}^{(A_N,A_M)}$ be the topological partition functions of the (A_N, A_M)-theory. The S-duality in the type IIB superstring theory exchanges N and M in the (A_N, A_M)-theory. In the end, in Proposition 5.2.6, we obtain

$$\mathcal{Z}_{\text{top.}}^{(A_N,A_M)} \sim \mathcal{Z}_{\text{top.}}^{(A_M,A_N)},$$

where $\sim$ means the both sides are identical up to some overall factor. Moreover, once we admit Conjecture 5.2.8, we show the overall factor is one, and both sides becomes exactly identical.

The second application is the proof of the 5D analogue of the Alday-Gaiotto-Tachikawa (AGT) correspondence. The formula for the matrix elements of $\mathcal{V}(z)$ is nothing but the claim of the 5D AGT correspondence. Note that one of the defining relations of the Mukadé operator in Definition 5.1.1,

$$\left(1 - \frac{x}{z}\right) X^{(2)}(z)\mathcal{V}(x) = \left(1 - (t/q)^2 \frac{x}{z}\right) \mathcal{V}(x)X^{(2)}(z)$$

reduces to the defining relation of the primary field of the Virasoro algebra, under the limit $q, t \to 1$. We will give some comments on this point soon in the next section.

7.2 Future Directions

There are many possible extensions and applications of these results. In this section, we discuss four of such future directions. By pursuing these directions, we may acquire a deeper insight into the string theory from integrability.

1. The first one is to prove Conjecture 5.2.8 to complete the proof of the S-duality formula Conjecture 5.2.9. The strategy of the proof may be the same as that of the main theorem Theorem 5.2.1. That is, we first specialize the spectral parameters so that the inner Young diagrams are restricted to ℓ rows, and prove the identity at that value of the spectral parameters, using the Kajihara-Noumi identity. Because such specialization is not unique, we may carry out the analytic continuation. This strategy is just a guess, though this is the most convincing strategy.

2. The second one is the extension of the result of Chapter 6 (see **3.** in Section 7.1). By making loops in the diagram (Figure 6.2), we obtain a new function. This function turns out to be the special case of the non-stationary Ruijsenaars function, which was introduced in [104]. As proved in [104], this function is the generating function (the Hirzebruch χ_y-genus) of the Euler characteristics of the affine Laumon space. As pointed out in [57], this space can be identified with the instanton moduli space of the theories with the full surface defect (*i.e.* the surface defect which completely breaks the gauge group to the products of $U(1)$). Because such the generating function is the partition function of the 5D $\mathcal{N} = 1^*$ theory (see [116] for example), the function we obtain by making loops in Figure 6.2, is the specialization of the 5D $\mathcal{N} = 1^*$ theory with the full surface defect.

The most intriguing feature of the non-stationary Ruijsenaars function is the bispectral duality (see [104]), physically which exchanges one of the Ω-background parameters and the adjoint mass parameter in the $\mathcal{N} = 1^*$ theory. This is a highly non-trivial duality, and we have to clarify this duality from the string theory point of view.

3. The third one is the generalization of the generalized Macdonald functions (see **1.** in Section 7.1) to the Koornwinder functions. Recall that the algebra $\mathcal{U}_{q,t}$ and the generalized Macdonald functions are associated with the A-type root system. In order to deal with the other types of root systems, we have to introduce the Macdonald functions associated with the other root systems, and these are called the Koornwinder functions [64]. These functions are associated with the BC-type root system, and in the various limit, they contains the A-, B-, C-, D-type Macdonald functions. Actually, it is possible to give the Koornwinder analogue of Theorem 3.2.4, and from that, we can construct the associated algebra. Though many problems about this algebra are still open, through the study of this algebra, we may consider the AGT correspondence with other types of gauge groups.

4. The last one is quite challenging. As noted in the previous section, we now identify the defining relation of the q-analogue of the primary fields. Thus, following the standard textbooks of CFT, the next step we have to take, is to define a q-analogue of the degenerate fields. This can be regarded as the first step to construct the field theory which has the q-Virasoro algebra as its symmetry.

Acknowledgement

As Kurt Vonnegut (or Mark Twain) said, what helps us is the "good opinion of our neighbors". The author thanks from the bottom of his heart to all his good neighbors. Unfortunately, because there is not enough space, the author just names some of his good neighbors below.

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Appendix A

Remarks on Physical Facts

In this appendix, we give the sketch of the derivation of some facts stated in Chapter 2. Especially, we give a brief review on the instanton counting and the topological vertex.

A.1 Super Yang-Mills and Instanton Counting

First, we summarize just the idea of the instanton counting because there already exist many beautiful reviews on this subject. We begin with the 4d $\mathcal{N}=2$ (*i.e.* with the eight supercharges) super Yang-Mills theories with A-type gauge groups, and later we uplift them to the 5d theories.

We name some good review articles on these subjects. The good review on these theories themselves is [108] and on their various aspects is [113]. The reviews on the instanton counting are [109, 77, 96, 101, 76]. The stringy realization of the instantons is as the bound states of $D0$-$D4$ branes. For more details, see [114]. For the theories with other types of gauge groups, see [83, 60].

A.1.1 Instanton Counting in 4D

For simplicity, we deal with the 4d $\mathcal{N}=2$ theory without any matters on the 4d flat space $\mathbb{C}^2$, the basic setup of the Seiberg-Witten theories [99, 100]. The Lagrangian of the theory is given by

$$\mathcal{L} = \int d\theta_1^2 d\theta_2^2 \, \mathrm{Tr}\, \mathcal{F}(\Phi) \,, \tag{A.1.1}$$

with Φ the vector multiplet. $\mathcal{F}$ is the rational function of Φ, and called *the prepotential.* Because there exist the $SU(2)_{\mathcal{R}}$ R-symmetry and the isometric symmetry of the space $SO(4) \simeq SU(2)_L \times SU(2)_R$, we can make the topological twist to preserve the diagonal of $SU(2)_L \times SU(2)_{\mathcal{R}}$. The resulting theory belongs to the class of the cohomological field theories [119], and especially it becomes the example of the Donaldson-Witten theory. Then, the action can be written as

$$S = (Q\text{-exact terms}) + \frac{1}{8\pi^2} \int \mathrm{tr} F \wedge F \,, \tag{A.1.2}$$

and thus is minimized if F is the anti-self-dual connection, that is, the gauge connection satisfies the instanton equation,

$$F + *F = 0 \,. \tag{A.1.3}$$

The solution (called the instanton solution) is labelled by the integer, called the instanton number. We denote by $\mathcal{M}_k$, the moduli space of k-instanton solutions. Then, the partition function of the theory becomes

$$\mathcal{Z} = \int e^{-S} = \mathcal{Z}_{\text{class.}} \cdot \mathcal{Z}_{\text{1-loop}} \cdot \left(\sum_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{q}^k \mathcal{Z}_k \right) \,, \quad \mathcal{Z}_k = \int_{\mathcal{M}_k} 1 \,, \tag{A.1.4}$$

where $\mathfrak{q}$ is the exponentiated coupling (contained in the normalization of F). We also used the fact that the super Yang-Mills theories are one loop exact. Nekrasov conjectured in [84] that the prepotential is obtained from the limit of the instanton partition function, that is,

$$\mathcal{F} = \lim_{\epsilon_{1,2}\to 0} \epsilon_1\epsilon_2 \log \mathcal{Z}_{\text{inst.}} . \tag{A.1.5}$$

This conjecture was proved in three independent way in [82], [78] and [18].

Then, the next step is to compute the integral of 1 over $\mathcal{M}_k$. As studied in [75], $\mathcal{M}_k$ is realized to be the hyper-Kähler quotient, and the same moduli space admits the ADHM construction [1] [7] (for the reviews, see [20, 25]). Though the integration is not well-defined because $\mathcal{M}_k$ is not compact, we can make it well-defined once we make use of the equivariant localization. See [78] and reviews cited above. We also cite [93] as the review. Physically, this procedure corresponds to the introduction of the Ω-background to $\mathbb{C}^2$. That is, we add one virtual dimension S^1 with the circumference R, and make the following identification:

$$(z, w, 0) \sim (e^{-\epsilon_1}z, e^{-\epsilon_2}w, R) , \tag{A.1.6}$$

and after the identification, we take $R \to 0$.

Finally, the computation was achieved in the legendary paper [84] by Nekrasov, and using the Ω-background parameters, the result is given by

$$\mathcal{Z}_k = \sum_{\vec{\lambda}\in \mathsf{P}^{\times \mathrm{rk}G}, |\vec{\lambda}|=k} \mathcal{Z}^{4d}_{\text{vec.}}(\vec{a}, \vec{\lambda}) , \qquad \mathcal{Z}^{4d}_{\text{vec.}}(\vec{a}, \vec{\lambda}) = \prod_{i,j} \prod_{x\in\lambda^{(i)}\cup\lambda^{(j)}} \frac{1}{E(x,i,j)(\epsilon_+ - E(x,i,j))} \tag{A.1.7}$$

with $E(x,i,j) := a_i - a_j + \epsilon_1((\lambda^{(m)})' - l + 1) - \epsilon_2(\lambda^{(l)} - m)$ for $x = (l, m)$. When we have matters, the k-instanton partition functions become

$$\int_{\mathcal{M}_k} \mathcal{T} , \tag{A.1.8}$$

where $\mathcal{T}$ is the corresponding matter bundle. After all, the contributions are from the fixed points of the localization, and thus we only need to evaluate the matter bundles at those fixed points. The final results for the case with matter contents are summarized in Chapter 2.

5D Lift

Now we lift the theory to 5d, $\mathbb{C}^2 \times S^1$. Mathematically, this lift means we go from the equivariant cohomology to the equivariant K-theory [79]. As studied in [81], the action is given by

$$S = \int_{\mathbb{C}^2\times S^1} \theta \wedge \mathrm{tr}(F\wedge F) + (Q\text{-exact term}) , \tag{A.1.9}$$

where θ is the $U(1)$ gauge field which gauges the conserved current $\mathrm{tr}(F\wedge F)$ [98]. We consider the S^1 direction as the time direction, and instantons as the particles move around the S^1 as time passes. In the weak coupling limit, [81] shows we can integrate out the $\mathbb{C}^2$ direction, and the theory reduces to the supersymmetric quantum mechanics on S^1. This discussion goes as follows. In (Q-exact term), we have the term proportional to $F^+_{\mu\nu}$ (+ means the self-dual part) where μ, ν are the indices running only the four-dimensional space directions, (not the S^1 direction). Because the term is Q-exact, we can take the limit where the coupling constant term in front of this term goes to infinity. Then, we obtain the equation which define the moduli space, as

$$F^+|_{4d} = 0 . \tag{A.1.10}$$

This is nothing but the instanton equation and thus the moduli space is the same as that of the four-dimensional theory above. The action reduces to the supersymmetric version of the sigma model whose target space is the

[1] For the exceptional Lie group, we do not have the ADHM construction.

instanton moduli. Then, the partition function becomes the index associated with the Dirac operator $\mathcal{D}$, and it ends with

$$\mathcal{Z}^{5D}_{\text{inst.}} = \text{Ind}\mathcal{D} = \sum_k \mathfrak{q}^k \int_{\mathcal{M}_k} \hat{A}(T\mathcal{M}_k)\,, \tag{A.1.11}$$

with $\hat{A}(T\mathcal{M}_k)$ the A-roof genus. See [43] for more details. Roughly speaking, when the Chern class of the tangent bundle of the instanton moduli space is $\prod_i e^{x_i}$, the A-roof genus is given by the $\prod_i 1/\sinh(x_i)$. Thus, practically, we just need to exponentiate each factor in the 4d instanton partition functions to obtain the 5d version. The final result is summarized in Section 2.1.

Because the Atiyah-Singer index of the Dirac operator is related to the Witten index, the partition function also admits the following expression:

$$\mathcal{Z}^{5D}_{\text{inst.}} = \text{tr}_{\mathcal{H}} \left[(-1)^F \mathfrak{q}^I q^{J_1} t^{-J_2} \boldsymbol{u}^{\boldsymbol{\Pi}} \boldsymbol{m}^{\boldsymbol{K}} \right]\,, \tag{A.1.12}$$

where I is the conserved charge corresponding to the instanton number. For the other parameters, see Section 2.1.

A.2 Topological Vertex

We now introduce the topological vertex, which is the technique to compute the topological string partition functions. It was introduced in [1]. The review of the derivation of the topological vertex has already been in [70, 85, 110], and thus we just show the idea of the derivation.

The open topological string theory is defined on the Calabi-Yau threefolds. We concentrate on the toric Calabi-Yau threefolds, characterized by the two-dimensional grid diagram, called the toric diagram. For the details of this class of the Calabi-Yau manifolds, see [22, 44]. Then, by the localization argument in [63], the contribution to the topological string amplitudes only comes from the strings stretched on edges in the toric diagrams.

Thus, the key idea is to decompose the toric diagram to the simpler patches, compute each patch, and finally glue them.

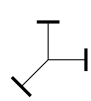
Step 1. Decompose the toric Calabi-Yau threefolds to local patches each of which is isomorphic to $\mathbb{C}^3$.

Step 2. Compute the partition functions on those $\mathbb{C}^3$ by using the conifold transition and the Chern-Simons theory.

Step 3. Glue them.

Step 3 requires the careful treatment of the framing, and the result is summarized in Definition 2.1.11. For more details, see the review articles above.

Step 1. The first step is done by inserting the A-brane/anti-brane pair to the internal legs of the toric diagrams. A-branes are the analogue of the D-branes in the open topological string. By the same argument as the pair of pants decomposition, we can decompose the toric diagram to the following trivalent diagram:

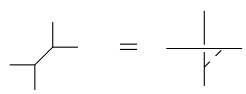


The thick lines represent the inserted branes. Once we forget the branes on the boundary, this is the toric diagram for $\mathbb{C}^3$.

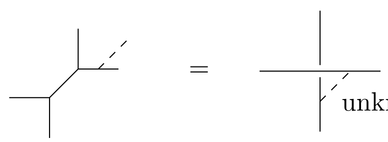
In the next step, we compute the topological string amplitudes on $\mathbb{C}^3$ with three branes inserted.

Step 2. Before starting the computation, we have to note some important facts. We enumerate them in the following list.

1. As pointed out in [118], A-branes wrap on the Lagrangian sub-manifolds in the Calabi-Yau, and the action on those A-branes is given by the Chern-Simons action.
2. By the Gopakumar-Vafa duality [38], the open topological string partition function on the deformed conifold is identical to the closed topological string partition function on the resolved conifold. That is, the open string on the right of the following toric diagram is equal to the closed string on the left. The dashed line means the Lagrangian S^3.



3. By combining the proposals by [92] and by [65], once we insert the Wilson loops in the Lagrangian S^3, the open string contribution appears in the closed string partition functions on the resolved conifolds. For later use, we realize S^3 as $|z|^2 + |w|^2 = 1$ with $z, w \in \mathbb{C}$. Then, when we insert the unknot to S^3 along the coordinate z, we have the insertion of A-branes on the one of the external edges. This is summarized in the following diagram. If along the coordinate w, it is on the vertical edge. We refer to this duality by the Ooguri-Vafa duality.



Now we are ready to go ahead. First, we note that by taking the appropriate limit (the length of the diagonal line to infinity), the toric diagram of the resolved conifold reduces to that of $\mathbb{C}^3$.

We also have to note the A-branes wrapping the Lagrangian sub-manifolds on the boundaries have the labels to represent their winding numbers. As noted in [1], these winding numbers are labelled by the partitions. Under the Ooguri-Vafa duality, these partitions turn out to be the labels of the representations of $U(\infty)$ which are associated with the Wilson lines in the deformed conifold side.

Then at the trivalent vertex, we represent the partition functions on the $\mathbb{C}^3$ patch by

$$\mathcal{Z}^{\mathbb{C}^3}(\{U_i, \lambda_i\}_{i=1\sim 3}) = C'_{\lambda_1\lambda_2\lambda_3}(q) \prod_{i=1}^{3} \mathrm{tr}_{\lambda_i} U_i \,. \tag{A.2.1}$$

Again, λ_i's are the labels to represent the winding numbers, and $\mathrm{tr}_{\lambda_i} U_i$ is the expectation value of the Wilson loop which forms the unknot with the representation λ_i. Actually the expectation value of the Wilson loop which forms the unknot is given by the Schur function, that is, the character of the representations of $U(\infty)$ [120]. Thus this is just the expansion of the partition functions in the basis of Schur functions.

In the end, we obtain the following result.

Claim A.2.1.

$$C'_{\lambda_1\lambda_2\lambda_3}(q) = \sum_{\rho,\sigma} c^{\lambda_1\lambda_3'}_{\rho\sigma'} q^{(\kappa(\lambda_2)+\kappa(\lambda_3))/2} \frac{W_{\lambda_2'\rho} W_{\lambda_2\sigma'}}{W_{\lambda_2\emptyset}} \,, \tag{A.2.2}$$

where

$$W_\mu(q) := (-1)^{|\mu|} q^{\kappa(\mu)} s_\mu(q^{-\rho}) = s_\mu(q^\rho) \,, \tag{A.2.3}$$

$$\begin{aligned} W_{\mu\nu}(q) &:= W_\mu(q) s_\nu(q^{\mu+\rho}) = s_\mu(q^\rho) s_\nu(q^{\mu+\rho}) \\ &= (-1)^{|\mu|+|\nu|} q^{(\kappa(\mu)+\kappa(\nu))/2} \sum_\eta s_{\mu/\eta}(q^{-\rho}) s_{\nu/\eta}(q^{-\rho}) \,, \end{aligned} \tag{A.2.4}$$

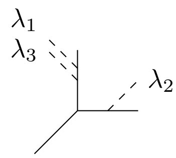
and

$$c^{\lambda_1\lambda_3'}_{\rho\sigma'} = \sum_{\eta} c^{\lambda_1}_{\eta\rho} c^{\lambda_3'}_{\eta\sigma'}\,, \tag{A.2.5}$$

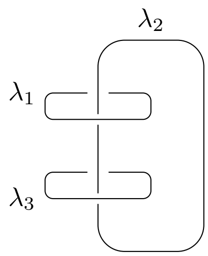
with $c^{\nu}_{\lambda\mu}$ *the Littlewood-Richardson coefficients of the skew Schur functions,*

$$s_{\nu/\lambda} = \sum_{\mu} c^{\nu}_{\lambda\mu} s_{\mu}\,. \tag{A.2.6}$$

The strategy is as follows. We compute the partition function corresponding to the following toric diagram:



That is, we have two insertions of Lagrangian sub-manifolds on one edge. Then, by the Ooguri-Vafa duality, this reduces to computing the expectation value of the following knot:



Then by using some results in [120], we can compute it with the expectation values of the Hopf links and the unknot as

$$\langle \text{Knot in figure above} \rangle = \frac{\langle (\text{Hopf link} : (\lambda_2, \lambda_1)) \rangle \langle (\text{Hopf link} : (\lambda_2, \lambda_3)) \rangle}{\langle (\text{unknot} : (\lambda_2)) \rangle}\,. \tag{A.2.7}$$

We also have $\langle (\text{Hopf link} : (\lambda_1, \lambda_2)) \rangle = W_{\lambda_1,\lambda_2}$, and $\langle (\text{unknot} : (\lambda)) \rangle = W_{\lambda,\emptyset}$. After this computation, in order to restore the original situation in Step 1, we have to move one Lagrangian branes to the other side of the trivalent vertex. This procedure ends with the above result.

By the careful computation, we can rewrite (A.2.2) as the following form.

Proposition A.2.2.

$$\begin{aligned} C'_{\lambda_1\lambda_2\lambda_3}(q) &= (-1)^{|\lambda_2|} q^{\kappa(\lambda_3)/2} s_{\lambda_2'}(q^{-\rho}) \sum_{\eta} s_{\lambda_1/\eta}(q^{\lambda_2'+\rho}) s_{\lambda_3'/\eta}(q^{\lambda_2+\rho}) \\ &= (-1)^{\sum_{i=1}^{3}|\lambda_i|} q^{\kappa(\lambda_3)/2} s_{\lambda_2'}(q^{-\rho}) \sum_{\eta} s_{\lambda_1/\eta}(q^{-\lambda_2'-\rho}) s_{\lambda_3/\eta}(q^{-\lambda_2-\rho})\,. \end{aligned} \tag{A.2.8}$$

We use some formulas in [122]. With some trivial correction such as the redefinition of the partitions, this agree with the topological vertex defined in Definition 2.1.8.

Refinement

The problem of the topological vertex is that it has only one parameter q, while the instanton partition functions contain two parameters q, t. The topological vertex is the tool to compute the partition functions for the self-dual background $q = t$. Thus, we need to "refine" the topological vertex to include one more parameter.

The guiding principle for the refinement is that the resulted vertex actually computes the correct instanton partition functions with two parameters. There seem to be two solutions to this question.

One solution is invented in [47], the other is in [10, 11]. The former is summarized in Chapter 2, and thus we show the result in the latter.

The solution in [10, 11] is quite natural, because they replace the Schur functions with the Macdonald functions. That is, they introduced the following vertices:

$$\begin{aligned} C^{\nu}_{\mu\lambda}(q,t) &= P_\lambda(t^\rho;q,t)\sum_\sigma \iota P_{\mu'/\sigma'}(-t^{\lambda'}q^\rho;t,q)P_{\nu/\sigma}(q^\lambda t^\rho;q,t)(q^{1/2}/t^{1/2})^{|\sigma|-|\nu|}f_\nu(q,t)^{-1}\,,\\ C^{\mu\lambda}_{\nu}(q,t) &= (-1)^{|\lambda|+|\mu|+|\nu|}C_{\mu'\lambda'}{}^{\nu'}(t,q)\,. \end{aligned} \tag{A.2.9}$$

Recall that ι is the endomorphism of Λ, $\iota(\mathsf{p}_n) = -\mathsf{p}_n$.

Then, by the same argument as Proposition 3.4.9, we have the following fact.

Fact A.2.3 ([9]). *We have*

$$\begin{aligned} \frac{1}{\langle P_\lambda, P_\lambda\rangle}\langle \iota P_\mu|\,\Phi_\lambda\begin{bmatrix}(1,N+1),-vu\\(0,1),v;(1,N),u\end{bmatrix}|\iota Q_\nu\rangle &= \left(\frac{-t^{1/2}u}{q(-v)^N}\right)^{|\lambda|} f_\lambda^{-N}\ (t^{-1/2}v)^{|\mu|-|\nu|}f_\nu^{-1}\ C^{\mu\lambda}{}_\nu(q,t)\,,\\ \langle \iota P_\nu|\,\Phi^*_\lambda\begin{bmatrix}(1,N),v;(0,1),u\\(1,N+1),-vu\end{bmatrix}|\iota Q_\mu\rangle &= \left(\frac{q(-u)^N}{-t^{1/2}v}\right)^{|\lambda|} f_\lambda^{N}\ (t^{-1/2}u)^{-|\mu|+|\nu|}f_\nu\ C_{\mu\lambda}{}^\nu(q,t)\,. \end{aligned} \tag{A.2.10}$$

A.2.1 From (p,q)-web to Topological Vertex

As studied in [2], the 5D $\mathcal{N}=1$ SYM can be engineered through the brane web of type IIB superstring. Now we connect the partition function of the theory to the partition function computed by the topological vertex. This can be accomplished by following the sequence of the string duality. Schematically, the sequence is the following:

$$\begin{aligned} \text{5D } \mathcal{N}=1 \text{ SYM} &\longleftarrow \text{type IIB string on } M_9\times S^1 \text{ with } (p,q)\text{-Web } \bar{X}\\ &\overset{①}{=\!=} \text{type IIA string on } M_9\times \tilde{S}^1\\ &\overset{②}{=\!=} \text{M-theory on } M_9\times \mathbb{T}^2 \text{ with the M5-brane wrapping with } \Sigma_X\\ &\overset{③}{=\!=} \text{A-type topological string on } X\\ &\longrightarrow \text{topological vertex partition function}\,. \end{aligned} \tag{A.2.11}$$

Here X is the toric Calabi-Yau threefold, and $\bar{X}$ is the (p,q)-Web which is obtained by regarding the toric diagram of X as the fivebrane web. The both arrows at the beginning and the end mean the gravity decoupling limits in the appropriate sense. In the following, we will sketch the idea of each duality steps. The circled numbers over the equalsigns correspond to those of the paragraphs below.

① Using the T-duality, we can go from the type IIB string on $\mathbb{R}^2\times M_5\times\mathbb{R}^2\times S^1_R$ with (p,q)-web to the type IIA string on $\mathbb{R}^2\times M_5\times\mathbb{R}^2\times S^1_{1/R}$ with some D6-branes. Here, we put the fivebranes, forming (p,q)-web, on $\bar{X}\times M_5$ in the first $\mathbb{R}^2\times M_5$, and the T-dual is taken with respect to the last S^1_R. Let us see more detail of the D6-branes whose origins are D5-branes in the (p,q)-web. As explained in [53], the NS5-branes become the Kaluza-Klein monopoles when T-dualize w.r.t. the normal direction and the total geometry becomes S^1 fibration over $\mathbb{R}^2\times M_5\times\mathbb{R}^2$ with the fibre degenerating at the loci where there were NS5-branes. The $D6$-branes wrap $\mathbb{R}^2\times M_5$ times the fibred S^1.

② Now we lift the type IIA theory to the M-theory on $M_5\times X$. Lifting the type IIA string to the M-theory, the total geometry becomes the torus fibration over $\mathbb{R}^2\times M_5\times\mathbb{R}^2$ with the fibre degenerating at some loci. The A-cycle of the fibre degenerates at the place where the S^1 fibre in the previous geometry degenerates, and the B-cycle where there were D6-branes in the type IIA string. These facts show the (p,q)-web has all the information about the degenerating loci of the torus fibration, that is, at the position (p,q)-fivebrane existing, the (p,q)-cycle degenerates. By direct comparison of the momentum map, we can identify this geometry with the toric Calabi-Yau threefold X.

③ We finally identify the partition functions of the field theories which are obtained by integrating out the Calabi-Yau directions from type IIA string, agree with the topological string partition functions. This was confirmed in [5].

Thus, combining all we can identify the topological vertex partition functions and the instanton partition functions.

Appendix B

Some Useful Formulas

B.1 Some Formulas for Nekrasov Factor

$$N_{\lambda\mu}(u) = \prod_{(i,j)\in\lambda} \left(1 - uq^{a_\lambda(i,j)}t^{\ell_\mu(i,j)+1}\right) \prod_{(i,j)\in\mu} \left(1 - uq^{-a_\mu(i,j)-1}t^{-\ell_\lambda(i,j)}\right) . \tag{B.1.1}$$

$$N_{\lambda\mu}(u) = \prod_{j\geq i\geq 1} (uq^{-\mu_i+\lambda_{j+1}}\kappa^{-i+j};q)_{\lambda_j-\lambda_{j+1}} \cdot \prod_{\beta\geq\alpha\geq 1} (uq^{\lambda_\alpha-\mu_\beta}\kappa^{\alpha-\beta-1};q)_{\mu_\beta-\mu_{\beta+1}} . \tag{B.1.2}$$

$$N_{\lambda\mu}(u) = \frac{\Pi_0(\gamma ut^{-\lambda'}q^{-\rho}, q^{-\mu}t^{-\rho})}{\Pi_0(-\gamma uq^{-\rho}, t^{-\rho})} , \quad \text{with} \quad \Pi_0(x|y) = \prod_{i,j}(1 - x_iy_j) . \tag{B.1.3}$$

$$N_{\lambda\mu}(u) = \frac{\Pi(uq^\lambda t^\rho, q^{-\mu}t^{-\rho};q,t)}{\Pi(ut^\rho, t^{-\rho};q,t)} , \quad \text{with} \quad \Pi(x|y;q) = \prod_{i,j} \frac{(tx_iy_j/\gamma;q)_\infty}{(x_iy_j/\gamma;q)_\infty} . \tag{B.1.4}$$

For $x = (i,j) \in \lambda$, put $\chi_x = t^{1-i}q^{j-1}$. With $f(z) = \frac{(1-qz)(1-z/t)}{(1-z)(1-qz/t)}$,

$$N_{\lambda\mu}(u) = \prod_{x\in\lambda}(1 - u\chi_x) \prod_{y\in\mu}(1 - ut/q\chi_y) \cdot \prod_{x\in\lambda, y\in\mu} f\left(ut\chi_x/q\chi_y\right) . \tag{B.1.5}$$

$$c_\lambda c'_\lambda = (-1)^{(|\lambda|)}q^{n(\lambda')+|\lambda|}t^{n(\lambda)}N_{\lambda,\lambda}(1), \tag{B.1.6}$$

$$N_{\lambda,\mu}(\gamma^{-1}x) = N_{\mu,\lambda}(\gamma^{-1}x^{-1})x^{|\lambda|+|\mu|}\frac{f_\lambda}{f_\mu}. \tag{B.1.7}$$

B.1.1 List of operator products

In this subsection, we list some formulas for the normal ordering among the various operators appeared in the main text. We have

$$\Lambda^{(i)}(z)S^{(i)}(w) = \frac{1 - t^2w/qz}{1 - tw/qz} : \Lambda^{(i)}(z)S^{(i)}(w) : , \tag{B.1.8}$$

$$\Lambda^{(i+1)}(z)S^{(i)}(w) = \frac{1 - w/z}{1 - tw/z} : \Lambda^{(i+1)}(z)S^{(i)}(w) : , \tag{B.1.9}$$

$$\Lambda^{(j)}(z)S^{(i)}(w) =: \Lambda^{(j)}(z)S^{(i)}(w) : \quad \text{for} \quad j < i \quad \text{and} \quad j > i+1 , \tag{B.1.10}$$

$$S^{(i)}(w)\Lambda^{(i)}(z) = \frac{1 - qz/t^2w}{1 - qz/tw} : S^{(i)}(w)\Lambda^{(i)}(z) : , \tag{B.1.11}$$

$$S^{(i)}(w)\Lambda^{(i+1)}(z) = \frac{1-z/w}{1-z/tw} : S^{(i)}(w)\Lambda^{(i+1)}(z) : , \tag{B.1.12}$$

$$S^{(i)}(w)\Lambda^{(j)}(z) =: S^{(i)}(w)\Lambda^{(j)}(z) : \quad \text{for} \quad j < i \quad \text{and} \quad j > i+1 , \tag{B.1.13}$$

$$\Phi^{(0)}(z)S^{(1)}(w) = \frac{(tw/z;q)_\infty}{(w/z;q)_\infty} : \Phi^{(0)}(z)S^{(1)}(w) :, \tag{B.1.14}$$

$$\Phi^{(0)}(z)S^{(i)}(w) =: \Phi^{(0)}(z)S^{(i)}(w) : \quad (i \geq 2), \tag{B.1.15}$$

$$S^{(1)}(z)\Phi^{(0)}(w) = \frac{(qw/z;q)_\infty}{(qw/tz;q)_\infty} : S^{(1)}(z)\Phi^{(0)}(w) :, \tag{B.1.16}$$

$$S^{(i)}(z)\Phi^{(0)}(w) =: S^{(i)}(z)\Phi^{(0)}(w) : \quad (i \geq 2), \tag{B.1.17}$$

$$\Phi^{(0)}(z)\Phi^{(0)}(w) = \frac{(qw/tz;q)_\infty}{(tw/z;q)_\infty} : \Phi^{(0)}(z)\Phi^{(0)}(w) :, \tag{B.1.18}$$

$$S^{(i)}(z)S^{(i)}(w) = (1-w/z)\frac{(qw/tz;q)_\infty}{(tw/z;q)_\infty} : S^{(i)}(z)S^{(i)}(w) : , \tag{B.1.19}$$

$$S^{(i)}(z)S^{(i+1)}(w) = \frac{(tw/z;q)_\infty}{(w/z;q)_\infty} : S^{(i)}(z)S^{(i+1)}(w) :, \tag{B.1.20}$$

$$S^{(i+1)}(z)S^{(i)}(w) = \frac{(qw/z;q)_\infty}{(qw/tz;q)_\infty} : S^{(i+1)}(z)S^{(i)}(w) : \quad (\forall i), \tag{B.1.21}$$

$$S^{(i)}(z)S^{(j)}(w) =: S^{(i)}(z)S^{(j)}(w) : \quad \text{for} \quad |i-j| > 2, \tag{B.1.22}$$

$$\Lambda^{(1)}(z)\Phi^{(0)}(x) = \frac{1-x/z}{1-tx/z} : \Lambda^{(1)}(z)\Phi^{(0)}(x) : , \tag{B.1.23}$$

$$\Phi^{(0)}(x)\Lambda^{(1)}(z) = \frac{1-z/x}{1-qz/t^2x} : \Phi^{(0)}(x)\Lambda^{(1)}(z) : , \tag{B.1.24}$$

$$\Lambda^{(i)}(z)\Phi^{(0)}(x) =: \Lambda^{(i)}(z)\Phi^{(0)}(x) : , \tag{B.1.25}$$

$$\Phi^{(0)}(x)\Lambda^{(i)}(z) = \frac{1-z/tx}{1-qz/t^2x} : \Phi^{(0)}(x)\Lambda^{(i)}(z) : \quad (i > 1) , \tag{B.1.26}$$

$$\Psi^+(z)S^{(i)}(w) = S^{(i)}(w)\Psi^+(z) =: \Psi^+(z)S^{(i)}(w) : \quad (\forall i), \tag{B.1.27}$$

$$\Psi^+(z)\Phi^{(0)}(w) = \frac{1-tw/qz}{1-w/z} : \Psi^+(z)\Phi^{(0)}(w) : , \tag{B.1.28}$$

$$A_{(s)}(x)\Lambda^{(i)}(z) = \prod_{k=1}^{r-1} \frac{1-t^{-k}z/x}{1-t^{-k-1}qz/x} : \Lambda^{(i)}(z)A_{(s)}(x) :, \tag{B.1.29}$$

$$A^{(r)}(x)\Phi^{(0)}(y) = \prod_{k=0}^{r-2} \frac{(t^{-k}qy/tx;q)_\infty}{(t^{-k}y/x;q)_\infty} : A^{(r)}(x)\Phi^{(0)}(y) :, \tag{B.1.30}$$

$$A^{(r)}(x)S^{(i)}(y) =: A^{(r)}(x)S^{(i)}(y) : . \tag{B.1.31}$$

With $\mathcal{G}(z) = \prod_{i,j=0}^{\infty}(1 - zq^i t^{-j})$,

$$\Phi_\lambda(v_i)\Phi^*_\mu(u_j) = \mathcal{G}(u_j/\gamma v_i)^{-1} N_{\mu\lambda}(u_j/\gamma v_i) : \Phi_\lambda(v_i)\Phi^*_\mu(u_j) :, \tag{B.1.32}$$

$$\Phi^*_\mu(u_j)\Phi_\lambda(v_i) = \mathcal{G}(v_i/\gamma u_j)^{-1} N_{\lambda\mu}(v_i/\gamma u_j) : \Phi^*_\mu(u_j)\Phi_\lambda(v_i) :, \tag{B.1.33}$$

$$\Phi_{\lambda^{(i)}}(v_i)\Phi_{\lambda^{(j)}}(v_j) = \frac{\mathcal{G}(v_j/\gamma^2 v_i)}{N_{\lambda^{(j)}\lambda^{(i)}}(v_j/\gamma^2 v_i)} : \Phi_{\lambda^{(i)}}(v_i)\Phi_{\lambda^{(j)}}(v_j) :, \tag{B.1.34}$$

$$\Phi^*_{\mu^{(i)}}(u_i)\Phi^*_{\mu^{(j)}}(u_j) = \frac{\mathcal{G}(u_j/u_i)}{N_{\mu^{(j)}\mu^{(i)}}(u_j/u_i)} : \Phi^*_{\mu^{(i)}}(u_i)\Phi^*_{\mu^{(j)}}(u_j) : . \tag{B.1.35}$$

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