博士論文

論文題目 Intersection and displacement energy of rational Lagrangian immersions via sheaf quantization

（層量子化による有理的ラグランジュはめ込みの交叉と分離エネルギーの研究）

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Intersection and displacement energy of rational Lagrangian immersions via sheaf quantization

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1 Introduction

1.1 Microlocal sheaf theory and symplectic geometry

The microlocal sheaf theory due to Kashiwara–Schapira [KS85, KS90] has been applied to symplectic geometry for about a decade. After the pioneering works by Nadler-Zaslow [NZ09] and Tamarkin [Tam18], many statements in symplectic geometry are (re-)proved via microlocal sheaf theory.

Guillermou [Gui12, Gui19] and Viterbo [Vit19] independently associtated an object of the derived category of sheaves on $M \times \mathbb{R}$ to each compact exact Lagrangian submanifold $L \subset T^*M$ so that the microsupport of the object is contained in a conification of L. Using this object, Guillermou gave an alternative proof to the homotopy equivalence of $\pi|_L \colon L \to M$ (which was originally proved by Abouzaid and Kragh [AK18] in a stronger form). Guillermou's construction does not depends on Floer theory while Viterbo's construction is based on Floer theory.

1.2 Rational Lagrangian immersions and our main results

In [AI17], we gave a sheaf theoretic method to estimate the displacement energy of compact subsets in cotangent bundles. See also Zhang [Zha18]. The main theorem of [AI17] asserts that an estimate of displacement energy is given by two sheaves whose microsupports are contained in the conification of the subsets. However the main theorem claims nothing about construction or existence of sheaves which give a good estimate. In this paper, we give an explicit estimate of the displacement energies for Lagrangian immersions with intersection estimates based on the method by [AI17].

Let M be a connected manifold without boundary and denote by T^*M its cotangent bundle and by ω the symplectic form on T^*M. A Lagrangian immersion $\iota \colon L \to T^*M$ is said to be *rational* if there exists $\sigma_\iota \in \mathbb{R}_{\geq 0}$ such that

$$\left\{ \int_{D^2} v^*\omega \;\middle|\; (v, \bar{v}) \in \Sigma_\iota \right\} = \sigma_\iota \mathbb{Z}, \tag{1.1}$$

where

$$\Sigma_\iota := \left\{ (v, \bar{v}) \;\middle|\; \begin{array}{l} v \colon D^2 \to T^*M, \bar{v} \colon \partial D^2 \to L, \\ v|_{\partial D^2} = \iota \circ \bar{v} \end{array} \right\}. \tag{1.2}$$

Let I be an open interval containing $[0, 1]$. A compactly supported C^∞-function $H = (H_s)_{s \in I} \colon T^*M \times I \to \mathbb{R}$ defines a time-dependent Hamiltonian vector field $X_H = (X_{H_s})_s$ on T^*M. By the compactness of the support, X_H generates a Hamiltonian isotopy $\phi^H =$

$(\phi_s^H)_s \colon T^*M \times I \to T^*M$. Following Hofer [Hof90], for a compactly supported function $H \colon T^*M \times I \to \mathbb{R}$, we define

$$\|H\| := \int_0^1 \left(\max_p H_s(p) - \min_p H_s(p) \right) ds. \tag{1.3}$$

Let $\iota \colon L \to T^*M$ be a rational Lagrangian immersion. We give a bound of $\#\{(y, y') \in L \times L \mid \iota(y) = \phi_1^H \circ \iota(y')\}$ and condition for $\|H\|$. One defines

$$K_\iota := \left\{ (v, \bar{v}) \;\middle|\; \begin{array}{l} v \colon D^2 \to T^*M, \bar{v} \colon [0,1] \to L, \\ \bar{v}(0) \neq \bar{v}(1), \iota \circ \bar{v}(0) = \iota \circ \bar{v}(1), \\ v|_{\partial D^2} \circ \exp(2\pi i -) = \iota \circ \bar{v} \end{array} \right\}. \tag{1.4}$$

and

$$e_\iota := \inf \left(\left(\left\{ \int_{D^2} v^*\omega \;\middle|\; (v, \bar{v}) \in K_\iota \right\} \cup \{\sigma_\iota\} \right) \cap \mathbb{R}_{>0} \right). \tag{1.5}$$

Before stating our main result, we put an assumption on rational Lagrangian immersions.

Assumption 1.1. *There exists no $(v, \bar{v}) \in K_\iota$ with $\int_{D^2} v^*\omega = 0$.*

Our main theorems are the following.

Theorem 1.2 (cf. Chekanov [Che98] and Akaho [Aka15])**.** *Let $\iota \colon L \to T^*M$ be a compact rational Lagrangian immersion satisfying Assumption 1.1. If $\|H\| < e_\iota$ and $\iota \colon L \to T^*M$ intersects $\phi_1^H \circ \iota \colon L \to T^*M$ transversally, then*

$$\#\left\{(y, y') \in L \times L \;\middle|\; \iota(y) = \phi_1^H \circ \iota(y')\right\} \geq \sum_{i=0}^{\dim L} b_i(L). \tag{1.6}$$

As a corollary, we obtain an estimate for the displacement energy of $\iota(L)$ since the empty intersection is transversal.

Corollary 1.3. *Let $\iota \colon L \to T^*M$ be a compact rational Lagrangian immersion satisfying Assumption 1.1. For any Hamiltonian function $H \colon T^*M \times I \to \mathbb{R}$ with $\|H\| < e_\iota$, $\iota(L) \cap \phi_1^H(\iota(L)) \neq \emptyset$.*

Theorem 1.4 (cf. Liu [Liu05])**.** *Let $\iota \colon L \to T^*M$ be a compact rational Lagrangian immersion satisfying Assumption 1.1. If $\|H\| < \min(\{e_\iota\} \cup (\{\sigma_\iota/2\} \cap \mathbb{R}_{>0}))$, then*

$$\#\left\{(y, y') \in L \times L \;\middle|\; \iota(y) = \phi_1^H \circ \iota(y')\right\} \geq \mathrm{cl}(L) + 1, \tag{1.7}$$

where $\mathrm{cl}(L)$ is the cup-length of L over $\mathbb{F}_2$.

Not only the proofs of the theorems above but also their statements are new. Our proofs can be regarded as a refinement of the arguments in [AI17]. Additionally, we use local property of *μhom*'s, what is called "microlocal property" of sheaves, directly and essentially to prove Theorem 1.4. This part is essentially new compared to [AI17] and more straightforward compared to Floer theoretic arguments.

1.3 Related works

It seems to be possible to prove our main theorems via Floer theory. Indeed, Chekanov[Che98], Liu[Liu05], and Akaho[Aka15] respectively proved Theorem 1.2 for rational embeddings, Theorem 1.4 for rational embeddings, and Theorem 1.2 for exact immersions (for more general symplectic manifolds). Moreover, Floer theory would give better estimates in the cases that bounding cochains exist [FOOO09, FOOO13].

1.4 Organization

This paper is organized as follows. In Section 2, we recall some results of the microlocal sheaf theory. In Section 3, we define parametrized and circled Tamarkin category and give a sheaf-theoretic energy estimate, which is a generalization of the result obtained by Asano–Ike [AI17]. In Section 4, we prove the existence of the sheaf quantizations of rational Lagrangian immersions in cotangent bundles. Finally in Section 5, we prove Theorems 1.2 and 1.4.

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2 Preliminaries on microlocal sheaf theory

In this paper, we assume that all manifolds are real manifolds of class C^∞ without boundary. Throughout this paper, let $\mathbf{k}$ be a field, which we specialize $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$ later.

In this section, we recall some definitions and results from [KS90]. We mainly follow the notation in [KS90]. Until the end of this section, let X be a C^∞-manifold without boundary.

2.1 Geometric notions

See [KS90, §4.3, §A.2] for details concerning this subsection. For a locally closed subset A of X, we denote by $\overline{A}$ its closure and by $\mathrm{Int}(A)$ its interior. We also denote by Δ_X the diagonal of $X \times X$. We denote by $\tau_X \colon TX \to X$ the tangent bundle of X, and by $\pi_X \colon T^*X \to X$ the cotangent bundle of X. If there is no risk of confusion, we simply write τ and π instead of τ_X and π_X, respectively. For a submanifold M of X, we denote by $T_M X$ the normal bundle to M in X, and by $T^*_M X$ the conormal bundle to M in X. In particular, $T^*_X X$ denotes the zero-section of T^*X. We set $\mathring{T}^*X := T^*X \setminus T^*_X X$. For two subsets S_1 and S_2 of X, we denote by $C(S_1, S_2) \subset TX$ the normal cone of the pair (S_1, S_2).

Let $f \colon X \to Y$ be a morphism of manifolds. With f we associate the following morphisms and commutative diagram:

$$\begin{array}{ccccc} T^*X & \xleftarrow{f_d} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Y} \\ X & = & X & \xrightarrow[f]{} & Y, \end{array} \tag{2.1}$$

where f_π is the projection and f_d is induced by the transpose of the tangent map $f' \colon TX \to X \times_Y TY$.

We denote by $(x;\xi)$ a local homogeneous coordinate system on T^*X. The cotangent bundle T^*X is an exact symplectic manifold with the Liouville 1-form $\alpha = \langle \xi, dx \rangle$. We denote by $a_X \colon T^*X \to T^*X, (x;\xi) \mapsto (x;-\xi)$ the antipodal map. For a subset A of T^*X, we denote by A^a its image under the antipodal map a_X. We also denote by $\mathbf{h} \colon T^*T^*X \xrightarrow{\sim} TT^*X$ the Hamiltonian isomorphism given in local coordinates by $\mathbf{h}(dx_i) = -\partial/\partial\xi_i$ and $\mathbf{h}(d\xi_i) = \partial/\partial x_i$. We will identify T^*T^*X and TT^*X by $-\mathbf{h}$.

2.2 Microsupports of sheaves

See [KS90, §5.1, §5.4, §6.1] for details concerning this subsection. Note that our notation is the same as in [Gui12, Gui19] and slightly differs from that of [KS90].

We denote by $\mathbf{k}_X$ the constant sheaf with stalk $\mathbf{k}$ and by $\mathrm{Mod}(\mathbf{k}_X)$ the abelian category of sheaves of $\mathbf{k}$-vector spaces on X. Moreover, we denote by $\mathbf{D}^{\mathrm{b}}(X)$ or $\mathbf{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of sheaves of $\mathbf{k}$-vector spaces. One can define Grothendieck's six operations $R\mathcal{H}om, \otimes, Rf_*, f^{-1}, Rf_!, f^!$ for a morphism of manifolds $f \colon X \to Y$. For a locally closed subset Z of X, we denote by $\mathbf{k}_Z$ the zero-extension of the constant sheaf with stalk $\mathbf{k}$ on Z to X, extended by 0 on $X \setminus Z$. Moreover, for a locally closed subset Z of X and $F \in \mathbf{D}^{\mathrm{b}}(X)$, we define $F_Z, R\Gamma_Z(F) \in \mathbf{D}^{\mathrm{b}}(X)$ by

$$F_Z := F \otimes \mathbf{k}_Z, \quad R\Gamma_Z(F) := R\mathcal{H}om(\mathbf{k}_Z, F). \tag{2.2}$$

One denotes by $\omega_X \in \mathbf{D}^{\mathrm{b}}(X)$ the dualizing complex on X, that is, $\omega_X := a_X^! \mathbf{k}$, where $a_X \colon X \to \mathrm{pt}$ is the natural morphism. Note that ω_X is isomorphic to $\mathrm{or}_X[\dim X]$, where or_X is the orientation sheaf on X. More generally, for a morphism of manifolds $f \colon X \to Y$, we denote by $\omega_f = \omega_{X/Y} := f^! \mathbf{k}_Y \simeq \omega_X \otimes f^{-1}\omega_Y^{\otimes -1}$ the relative dualizing complex.

Let us recall the definition of the *microsupport* $\mathrm{SS}(F)$ of an object $F \in \mathbf{D}^{\mathrm{b}}(X)$.

Definition 2.1 ([KS90, Def. 5.1.2])**.** Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $p \in T^*X$. One says that $p \notin \mathrm{SS}(F)$ if there is a neighborhood U of p in T^*X such that for any $x_0 \in X$ and any C^∞-function φ on X (defined on a neighborhood of x_0) satisfying $d\varphi(x_0) \in U$, one has $R\Gamma_{\{\varphi \geq \varphi(x_0)\}}(F)_{x_0} \simeq 0$.

One can check the following properties:

(i) The microsupport of an object in $\mathbf{D}^{\mathrm{b}}(X)$ is a conic (i.e., invariant under the action of $\mathbb{R}_{>0}$ on T^*X) closed subset of T^*X.

(ii) For an object $F \in \mathbf{D}^{\mathrm{b}}(X)$, one has $\mathrm{SS}(F) \cap T^*_X X = \pi(\mathrm{SS}(F)) = \mathrm{Supp}(F)$.

(iii) The microsupports satisfy the triangle inequality: if $F_1 \longrightarrow F_2 \longrightarrow F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathbf{D}^{\mathrm{b}}(X)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for $j \neq k$.

We also set $\mathring{\mathrm{SS}}(F) := \mathrm{SS}(F) \cap \mathring{T}^*X = \mathrm{SS}(F) \setminus T^*_X X$.

The following proposition is called (a particular case of) the microlocal Morse lemma. See [KS90, Prop. 5.4.17 and Corollary 5.4.19] for more details. The classical theory corresponds to the case F is the constant sheaf $\mathbf{k}_X$.

Proposition 2.2. *Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $\varphi\colon X \to \mathbb{R}$ be a C^∞-function. Let moreover $a, b \in \mathbb{R}$ with $a < b$ or $a \in \mathbb{R}, b = +\infty$. Assume*

(1) *φ is proper on $\mathrm{Supp}(F)$,*

(2) *$d\varphi(x) \notin \mathrm{SS}(F)$ for any $x \in \varphi^{-1}([a, b))$.*

Then the canonical morphism

$$R\Gamma(\varphi^{-1}((-\infty, b)); F) \longrightarrow R\Gamma(\varphi^{-1}((-\infty, a)); F) \tag{2.3}$$

is an isomorphism.

By using microsupports, we can microlocalize the category $\mathbf{D}^{\mathrm{b}}(X)$ as follows. For a subset $A \subset \mathring{T}^*X$, we denote by $\mathbf{D}^{\mathrm{b}}_A(X)$ the subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ consisting of F with $\mathring{\mathrm{SS}}(F) \subset A$. Note that $\mathbf{D}^{\mathrm{b}}_A(X)$ contains locally constant sheaves on X. By the triangle inequality, the subcategory $\mathbf{D}^{\mathrm{b}}_A(X)$ is a triangulated subcategory. For a subset Ω of T^*X, we define $\mathbf{D}^{\mathrm{b}}(X; \Omega)$ as the categorical localization of $\mathbf{D}^{\mathrm{b}}(X)$ by $\mathbf{D}^{\mathrm{b}}_{T^*X \setminus \Omega}(X)$: $\mathbf{D}^{\mathrm{b}}(X; \Omega) := \mathbf{D}^{\mathrm{b}}(X)/\mathbf{D}^{\mathrm{b}}_{T^*X \setminus \Omega}(X)$. For a subset A of Ω, $\mathbf{D}^{\mathrm{b}}_A(X; \Omega)$ denotes the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X; \Omega)$ consisting of F with $\mathring{\mathrm{SS}}(F) \cap \Omega \subset A$.

For a subset A of $\mathring{T}^*X$, $\mathbf{D}^{\mathrm{b}}_{(A)}(X)$ denotes the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ consisting of F for which there exists a neighborhood U of A such that $\mathring{\mathrm{SS}}(F) \cap U \subset A$.

2.3 Functorial operations

We consider behavior of the microsupports with respect to functorial operations. See [KS90, §5.4] for details concerning this subsection.

Definition 2.3 ([KS90, Def. 5.4.12])**.** Let $f\colon X \to Y$ be a morphism of manifolds and A be a closed conic subset of T^*Y. The morphism f is said to be *non-characteristic* for A if

$$f_\pi^{-1}(A) \cap f_d^{-1}(T^*_X X) \subset X \times_Y T^*_Y Y. \tag{2.4}$$

See (2.1) for the notation f_π and f_d. In particular, any submersion from X to Y is non-characteristic for any closed conic subset of T^*Y. Note that submersions are called smooth morphisms in [KS90]. One can show that if $f\colon X \to Y$ is non-characteristic for a closed conic subset A of T^*Y, then $f_d f_\pi^{-1}(A)$ is a closed conic subset of T^*X.

Proposition 2.4 ([KS90, Prop. 5.4.4, Prop. 5.4.13, and Prop. 5.4.5])**.** *Let $f\colon X \to Y$ be a morphism of manifolds, $F \in \mathbf{D}^{\mathrm{b}}(X)$, and $G \in \mathbf{D}^{\mathrm{b}}(Y)$.*

(i) *Assume that f is proper on $\mathrm{Supp}(F)$. Then $\mathrm{SS}(f_*F) \subset f_\pi f_d^{-1}(\mathrm{SS}(F))$.*

(ii) *Assume that f is non-characteristic for $\mathrm{SS}(G)$. Then the canonical morphism $f^{-1}G \otimes \omega_f \to f^!G$ is an isomorphism and $\mathrm{SS}(f^{-1}G) \cup \mathrm{SS}(f^!G) \subset f_d f_\pi^{-1}(\mathrm{SS}(G))$.*

(iii) *Assume that f is a submersion. Then $\mathrm{SS}(F) \subset f_d(X \times_Y T^*Y)$ if and only if locally on X there exists $H \in \mathbf{D}^{\mathrm{b}}(Y)$ satisfying $F \simeq f^{-1}H$. Moreover if $f\colon X \to Y$ is a vector bundle over Y, then $\mathrm{SS}(F) \subset f_d(X \times_Y T^*Y)$ if and only if the counit morphism $f^{-1}Rf_*F \to F$ is isomorphic.*

For closed conic subsets A and B of T^*X, let us denote by $A+B$ the fiberwise sum of A and B, that is,

$$A+B := \{(x; a+b) \mid x \in \pi(A) \cap \pi(B), a \in A \cap \pi^{-1}(x), b \in B \cap \pi^{-1}(x)\} \subset T^*X. \quad (2.5)$$

Proposition 2.5 ([KS90, Prop. 5.4.14]). *Let $F, G \in \mathbf{D}^{\mathrm{b}}(X)$.*

(i) *If $\mathrm{SS}(F) \cap \mathrm{SS}(G)^a \subset T^*_X X$, then $\mathrm{SS}(F \otimes G) \subset \mathrm{SS}(F) + \mathrm{SS}(G)$.*

(ii) *If $\mathrm{SS}(F) \cap \mathrm{SS}(G) \subset T^*_X X$, then $\mathrm{SS}(\mathcal{H}om(F, G)) \subset \mathrm{SS}(F)^a + \mathrm{SS}(G)$.*

Let $\varphi \colon X \to \mathbb{R}$ be a C^∞-function and assume that $d\varphi(x) \neq 0$ for any $x \in \varphi^{-1}(0)$. Set $U := \{x \in X \mid \varphi(x) < 0\}$. For such $U \subset X$, define

$$N^*(U) := \mathrm{SS}(\mathbf{k}_U)^a = T^*_X X|_U \cup \{(x; \lambda d\varphi(x)) \mid \varphi(x) = 0, \lambda \leq 0\}. \quad (2.6)$$

Proposition 2.6 ([KS90, Prop. 6.3.1]). *Let U be an open subset of X as above and $j \colon U \to X$ be the open embedding. For $G \in \mathbf{D}^{\mathrm{b}}(U)$, let $\overline{\mathrm{SS}(G)}$ denote the closure of $\mathrm{SS}(G)$ in T^*X.*

(i) *If $\overline{\mathrm{SS}(G)} \cap N^*(U)^a \subset T^*_X X$, then $\mathrm{SS}(Rj_*G) \subset \overline{\mathrm{SS}(G)} + N^*(U)$.*

(ii) *If $\overline{\mathrm{SS}(G)} \cap N^*(U) \subset T^*_X X$, then $\mathrm{SS}(Rj_!G) \subset \overline{\mathrm{SS}(G)} + N^*(U)^a$.*

Lemma 2.7. *Let $\varphi \colon X \to \mathbb{R}$ be a C^∞-function and assume that $d\varphi(x) \neq 0$ for any $x \in \varphi^{-1}(0)$. Set $U := \{x \in X \mid \varphi(x) < 0\}, Z := \{x \in X \mid \varphi(x) \leq 0\}$ and let $j \colon U \to X$ be the inclusion.*

(i) *If $\mathrm{Supp}(F) \subset Z$ and $\mathrm{SS}(F) \cap N^*(U) \subset T^*_X X$, then there exists a natural isomorphism $F_U \simeq F$.*

(ii) *If $\mathrm{SS}(F) \cap N^*(U)^a \subset T^*_X X$, then there exists a natural isomorphism $F_Z \simeq Rj_*j^{-1}F$.*

Proof. (i) Consider the distinguished triangle $F_U \longrightarrow F \longrightarrow F_{\varphi^{-1}(0)} \xrightarrow{+1}$. By $\mathrm{SS}(F_U) \subset N^*(U)^a + \mathrm{SS}(F)$, $(N^*(U)^a + \mathrm{SS}(F)) \cap N^*(U) \subset T^*_X X$, and the triangle inequality, we have $F_{\varphi^{-1}(0)} \cap N^*(U) \subset T^*_X X$. However, for any $G \in \mathbf{D}^{\mathrm{b}}(X)$ supported on a closed submanifold $N \subset X$, $\mathrm{SS}(G)$ contains $T^*_N X|_{\mathrm{Supp}(G)}$, which intersects with $N^*(U)$ outside the zero-section unless $\mathrm{Supp}(G) = \emptyset$. Hence, we obtain $F_{\varphi^{-1}(0)} \simeq 0$.
(ii) This morphism is obtained by applying Ri_* to the unit morphism $i^{-1}F \to Rj'_*j'^{-1}i^{-1}F$ where $i \colon Z \to X$ and $j' \colon U \to Z$ are the inclusions. The cone of $F_Z \to Rj_*j^{-1}F$ is supported on $\varphi^{-1}(0)$. By [KS90, 5.4.8], $\mathrm{SS}(F_Z) \cup \mathrm{SS}(Rj_*j^{-1}F) \subset N^*(U) + \mathrm{SS}(F)$ and hence the cone is 0 as in (i). □

2.4 Kernels

See [KS90, §3.6] for details concerning this subsection. For $i = 1, 2, 3$, let X_i be a manifold. We write $X_{ij} := X_i \times X_j$ and $X_{123} := X_1 \times X_2 \times X_3$ for short. We use the same symbol q_i for the projections $X_{ij} \to X_i$ and $X_{123} \to X_i$. We also denote by q_{ij} the projection $X_{123} \to X_{ij}$. Similarly, we denote by p_{ij} the projection $T^*X_{123} \to T^*X_{ij}$. One denotes by p_{12^a} the composite of p_{12} and the antipodal map on T^*X_2.

Let $A \subset T^*X_{12}$ and $B \subset T^*X_{23}$. We set

$$A \circ B := p_{13}(p_{12^a}^{-1} A \cap p_{23}^{-1} B) \subset T^*X_{13}. \quad (2.7)$$

We define the operation of composition of kernels as follows:

$$\begin{aligned} \underset{X_2}{\circ} \colon \mathbf{D}^{\mathrm{b}}(X_{12}) \times \mathbf{D}^{\mathrm{b}}(X_{23}) &\to \mathbf{D}^{\mathrm{b}}(X_{13}) \\ (K_{12}, K_{23}) &\mapsto K_{12} \underset{X_2}{\circ} K_{23} := Rq_{13!}\,(q_{12}^{-1}K_{12} \otimes q_{23}^{-1}K_{23}). \end{aligned} \tag{2.8}$$

If there is no risk of confusion, we simply write $\circ$ instead of $\underset{X_2}{\circ}$. By Proposition 2.4(i)–(ii) and Proposition 2.5, we have the following.

Proposition 2.8. *Let $K_{ij} \in \mathbf{D}^{\mathrm{b}}(X_{ij})$ and set $\Lambda_{ij} := \mathrm{SS}(K_{ij}) \subset T^*X_{ij}$ $(ij = 12, 23)$. Assume*

(1) *q_{13} is proper on $q_{12}^{-1}\operatorname{Supp}(K_{12}) \cap q_{23}^{-1}\operatorname{Supp}(K_{23})$,*

(2) *$p_{12^a}^{-1}\Lambda_{12} \cap p_{23}^{-1}\Lambda_{23} \cap (T^*_{X_1}X_1 \times T^*X_2 \times T^*_{X_3}X_3) \subset T^*_{X_{123}}X_{123}$.*

Then

$$\mathrm{SS}(K_{12} \underset{X_2}{\circ} K_{23}) \subset \Lambda_{12} \circ \Lambda_{23}. \tag{2.9}$$

2.5 *μhom* functor

See [KS90, §4.4] for details concerning this subsection. Let $q_1, q_2 \colon X \times X \to X$ be the projections. We identify $T^*_{\Delta_X}(X \times X)$ with T^*X through the first projection $(x, x; \xi, -\xi) \mapsto (x; \xi)$.

Definition 2.9. For $F, G \in \mathbf{D}^{\mathrm{b}}(X)$, one defines

$$\mu hom(F, G) := \mu_{\Delta_X}\, R\mathcal{H}om(q_2^{-1}F, q_1^!G) \in \mathbf{D}^{\mathrm{b}}(T^*X). \tag{2.10}$$

For a closed submanifold M of X and $F \in \mathbf{D}^{\mathrm{b}}(X)$, we have an isomorphism

$$\mu hom(\mathbf{k}_M, F) \simeq i_{M*}\mu_M(F), \tag{2.11}$$

where $i_M \colon T^*_MX \to T^*X$ is the embedding.

Proposition 2.10 ([KS90, Cor. 5.4.10 and Cor. 6.4.3])**.** *Let $F, G \in \mathbf{D}^{\mathrm{b}}(X)$. Then*

$$\operatorname{Supp}(\mu hom(F, G)) \subset \mathrm{SS}(F) \cap \mathrm{SS}(G), \tag{2.12}$$

$$\mathrm{SS}(\mu hom(F, G)) \subset -\mathbf{h}^{-1}(C(\mathrm{SS}(G), \mathrm{SS}(F))), \tag{2.13}$$

where $C(S_1, S_2)$ is the normal cone and $\mathbf{h} \colon T^*T^*X \xrightarrow{\sim} TT^*X$ is the Hamiltonian isomorphism (see Subsection 2.1).

The functor *μhom* gives $\mathbf{D}^{\mathrm{b}}(X)$ an enrichment in $\mathbf{D}^{\mathrm{b}}(T^*X)$. For each $F \in \mathbf{D}^{\mathrm{b}}(X)$, a morphism

$$\mathrm{id}^{\mu}_F \colon \mathbf{k}_{T^*X} \to \mu hom(F, F) \tag{2.14}$$

is given. For each $F, G, H \in \mathbf{D}^{\mathrm{b}}(X)$, a composition morphism

$$\circ^{\mu}_{F,G,H} \colon \mu hom(G, H) \otimes \mu hom(F, G) \to \mu hom(F, H) \tag{2.15}$$

is defined. This composition is unital and associative.

For an open subset $\Omega \subset T^*X$, the restriction of *μhom* to Ω also gives $\mathbf{D}^{\mathrm{b}}(X; \Omega)$ an enrichment in $\mathbf{D}^{\mathrm{b}}(\Omega)$. The 0-th cohomology of *μhom* gives a new category whose objects are those of $\mathbf{D}^{\mathrm{b}}(X; \Omega)$ and Hom-set is defined by $\mathrm{Hom}^{\mu}_{\Omega}(-, -) := H^0(\Omega; \mu hom(-, -)|_{\Omega})$. Moreover for $F, G \in \mathbf{D}^{\mathrm{b}}(X; \Omega)$ there exist a natural map $m_{F,G} \colon \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X;\Omega)}(F, G) \to H^0(\Omega; \mu hom(F, G)|_{\Omega})$ and these maps give a functor from $\mathbf{D}^{\mathrm{b}}(X; \Omega)$ to the new category. For $v \in \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X;\Omega)}(F, G)$, we denote v^{μ} by the corresponding morphism $\mathbf{k}_{\Omega} \to \mu hom(F, G)|_{\Omega}$. This notation is compatible with (2.14).

2.6 Simple sheaves

Let $\Lambda \subset \mathring{T}^*X$ be a locally closed conic Lagrangian submanifold and $p \in \Lambda$. Simple sheaves along Λ at p are defined in [KS90, Def. 7.5.4]. In this subsection, we recall them.

For a C^∞-function $\varphi\colon X \to \mathbb{R}$ such that $\varphi(\pi(p)) = 0$ and $\Gamma_{d\varphi}$ intersects Λ transversally at p, one can define $\tau_\varphi = \tau_{p,\varphi} \in \mathbb{Z}$ (see [KS90, §7.5 and A.3]).

Proposition 2.11 ([KS90, Prop. 7.5.3]). *For $i = 1, 2$, let $\varphi_i\colon X \to \mathbb{R}$ be a C^∞-function such that $\varphi_i(\pi(p)) = 0$ and $\Gamma_{d\varphi_i}$ intersects Λ transversally at p. Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and assume that $\mathrm{SS}(F) \subset \Lambda$ in a neighborhood of p. Then*

$$R\Gamma_{\{\varphi_1 \geq 0\}}(F)_{\pi(p)} \simeq R\Gamma_{\{\varphi_2 \geq 0\}}(F)_{\pi(p)} \left[\tfrac{1}{2}(\tau_{\varphi_2} - \tau_{\varphi_1})\right]. \tag{2.16}$$

Definition 2.12 ([KS90, Def. 7.5.4]). In the situation of Proposition 2.11, F is said to have microlocal type $L \in \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbf{k}))$ with shift $d \in \frac{1}{2}\mathbb{Z}$ at p if

$$R\Gamma_{\{\varphi \geq 0\}}(F)_{\pi(p)} \simeq L\left[d - \tfrac{1}{2}\dim X - \tfrac{1}{2}\tau_\varphi\right] \tag{2.17}$$

for some (hence for any) C^∞-function φ such that $\varphi(\pi(p)) = 0$ and $\Gamma_{d\varphi}$ intersects Λ transversally at p. If moreover $L \simeq \mathbf{k}$, F is said to be *simple* along Λ at p. If F is simple at all points of Λ, one says that F is simple along Λ.

One can prove that if $F \in \mathbf{D}^{\mathrm{b}}(X)$ is simple along Λ, then $\mathrm{id}^\mu_F|_\Lambda\colon \mathbf{k}_\Lambda \to \mu hom(F,F)|_\Lambda$ is isomorphic. When Λ is a conormal bundle to a closed submanifold M of X in a neighborhood of p, that is, $\pi|_\Lambda\colon \Lambda \to X$ has constant rank, then $F \in \mathbf{D}^{\mathrm{b}}(X)$ is simple along Λ at p if $F \simeq \mathbf{k}_M[d]$ in $\mathbf{D}^{\mathrm{b}}(X;p)$ for some $d \in \mathbb{Z}$.

Lemma 2.13 ([Gui12, Lem. 6.14]). *If two conic Lagrangian submanifolds $\Lambda_1, \Lambda_2 \subset T^*X$ intersect cleanly along $N = \Lambda_1 \cap \Lambda_2$. Then $-\mathbf{h}^{-1}C(\Lambda_1, \Lambda_2) = T^*_N X$. Moreover, let $F \in \mathbf{D}^{\mathrm{b}}_{(\Lambda_1)}(X), G \in \mathbf{D}^{\mathrm{b}}_{(\Lambda_2)}(X)$ be simple along Λ_1 and Λ_2 respectively. Then there exist $d \in \mathbb{Z}$ and a rank 1 local system L on N such that $\mu hom(F,G)|_N \simeq L[d]$.*

2.7 Triangulated orbit category

For a C^∞-manifold X, Guillermou [Gui12] introduced the triangulated orbit category $\mathbf{D}^{\mathrm{b}}_{/[1]}(X)$ [1], in which any object F is isomorphic to its shift $F[n]$ for any $n \in \mathbb{Z}$. Guillermou also defined the microsupport $\mathrm{SS}(F) \subset T^*X$ of an object $F \in \mathbf{D}^{\mathrm{b}}_{/[1]}(X)$. See [Gui12, Gui19] for details.

Let $\mathbf{K}$ be the $\mathbf{k}$-algebra $\mathbf{k}[\varepsilon]/(\varepsilon^2)$ and $\mathrm{perf}(\mathbf{K}_X)$ be the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathbf{K}_X)$ generated by the image of the functor $\mathfrak{e}\colon \mathbf{D}^{\mathrm{b}}(\mathbf{k}_X) \to \mathbf{D}^{\mathrm{b}}(\mathbf{K}_X), F \mapsto \mathbf{K} \otimes_{\mathbf{k}} F$. We denote by $\mathbf{D}^{\mathrm{b}}_{/[1]}(X)$ or $\mathbf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_X)$ the quotient category $\mathbf{D}^{\mathrm{b}}(\mathbf{K}_X)/\mathrm{perf}(\mathbf{K}_X)$. We also denote by $\mathfrak{i}$ the composite functor $\mathfrak{i}\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(\mathbf{K}_X) \to \mathbf{D}^{\mathrm{b}}_{/[1]}(X)$ where the former is the functor induced by the $\mathbf{k}$ as the $\mathbf{K}$-algebra with the trivial ε-action and the latter is the quotient functor.

Define $\mathrm{SS}(F) := \bigcap_{F' \simeq F} \mathrm{SS}_{\mathbf{K}}(F')$ where F' runs objects of $\mathbf{D}^{\mathrm{b}}(\mathbf{K}_X)$ which are isomorphic to F in $\mathbf{D}^{\mathrm{b}}_{/[1]}(X)$ and $\mathrm{SS}_{\mathbf{K}}(F')$ is the usual microsupport of F' as an object of $\mathbf{D}^{\mathrm{b}}(\mathbf{K}_X)$.

The Grothendieck's six operations are defined on the orbit categories. Similar properties for the Grothendieck's six operations and microsupports also hold. The same results as Proposition 2.4 (i) (ii) and Proposition 2.5 are proved in [Gui19].

[1] The original idea is due to Keller[Kel05].

It is also proved in [Gui19] that for $F \in \mathbf{D}^{\mathrm{b}}_{/[1]}(\mathbb{R}^n)$ with $\mathrm{SS}(F) \subset T^*_{\mathbb{R}^n}\mathbb{R}^n$ there exists an $L \in \mathrm{Mod}(\mathbf{k})$ such that $F \simeq L_{\mathbb{R}^n}$. The orbit categorical version of Proposition 2.2 can be seen as a corollary of this result.

The categories $\mathbf{D}^{\mathrm{b}}_{/[1]}(X;\Omega), \mathbf{D}^{\mathrm{b}}_{/[1],A}(X)$ and $\mathbf{D}^{\mathrm{b}}_{/[1],A}(X;\Omega)$ are similarly defined.

A cohomological functor $H^* \colon \mathbf{D}^{\mathrm{b}}_{/[1]}(X) \to \mathrm{Mod}(\mathbf{k}_X)$ is defined so that $H^*(F)$ is the sheafification of the presheaf $(U \mapsto \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}_{/[1]}(U)}(\mathbf{k}_U, F|_U))$ on X. This functor satisfies $H^*(\mathfrak{i}(F)) = \bigoplus_{n\in\mathbb{Z}} H^n(F)$ for the image of an object F of $\mathbf{D}^{\mathrm{b}}(\mathbf{k}_X)$. The functor H^* for $X = \mathrm{pt}$ gives an equivalence between $\mathbf{D}^{\mathrm{b}}_{/[1]}(\mathrm{pt})$ and $\mathrm{Mod}(\mathbf{k})$.

We define $\mathrm{Hom}^{\mu}_{\Omega}(F,G) := H^*R\Gamma(\Omega; \mu hom(F,G)|_U)$ and $\mathrm{End}^{\mu}_{\Omega}(F) := \mathrm{Hom}^{\mu}_{\Omega}(F,F)$ for $F, G \in \mathbf{D}^{\mathrm{b}}_{/[1]}(X;\Omega)$.

Definition 2.14. Let $\mathcal{U} = \{U_\alpha\}_\alpha$ be an open covering of X and $\mathcal{V} = \{V_\alpha\}_\alpha$ be an open covering of an open subset $\Omega \subset T^*X$.

(i) An object $F \in \mathbf{D}^{\mathrm{b}}_{/[1]}(X;\Omega)$ is said to be *locally tame with respect to* $\mathcal{U}$ if for each $U_\alpha \in \mathcal{U}$ $F|_{U_\alpha}$ is isomorphic to some $\mathfrak{i}(G)$ as an object of $\mathbf{D}^{\mathrm{b}}_{/[1]}(U; \Omega \cap T^*U)$.

(ii) An object $F \in \mathbf{D}^{\mathrm{b}}_{/[1]}(X;\Omega)$ is said to be *locally tame* if F is locally tame with respect to some open covering of X.

(iii) An object $F \in \mathbf{D}^{\mathrm{b}}_{/[1]}(X;\Omega)$ is said to be *microlocally tame with respect to* $\mathcal{V}$ if for each $V_\alpha \in \mathcal{V}$ F is isomorphic to some $\mathfrak{i}(G)$ as an object of $\mathbf{D}^{\mathrm{b}}_{/[1]}(X;V)$.

(iv) An object $F \in \mathbf{D}^{\mathrm{b}}_{/[1]}(X;\Omega)$ is said to be *microlocally tame* if F is microlocally tame with respect to some open covring of Ω.

If $F, G \in \mathbf{D}^{\mathrm{b}}_{/[1]}(X;\Omega)$ are microlocally tame, then $\mu hom(F,G)|_\Omega \in \mathbf{D}^{\mathrm{b}}_{/[1]}(\Omega)$ is locally tame.

Definition 2.15. Let $\Lambda \subset \mathring{T}^*X$ be a locally closed conic Lagrangian submanifold. $F \in \mathbf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_X)$ is simple along Λ if for each $p \in \Lambda$ there is an open neighborhood U of p in $\mathring{T}^*X$ such that there exist an object $G \in \mathbf{D}^{\mathrm{b}}(X;U)$ which is simple along Λ and an isomorphism $\mathfrak{i}(G) \simeq F$ in $\mathbf{D}^{\mathrm{b}}_{/[1]}(X;U)$.

Remark 2.16. We have not prepared all the counterparts of the statements in the previous subsections for the triangulated orbit categories. Some of such statements for locally tame or microlocally tame objects can be verified from the original statements easily. For example, Proposition 2.4(iii) for locally tame objects is deduced from the original statement directly.

The statements for the triangulated orbit categories which we will use are stated above in this subsection (whose proofs are given in [Gui19]) or direct corollaries of the corresponding statements for the usual derived categories under locally tame or microlocally tame conditions.

3 Parametrized and circled Tamarkin category and sheaf-theoretic energy estimate

From now on, until the end of this paper, let M be a non-empty connected manifold without boundary and $\mathbf{k}$ be the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

In this section, we give generalizations of the Tamarkin categories defined in [Tam18] and [GS14]. Our generalization consists of three aspects. The first one is generalizing the definition to the setting parametrized by another manifold. This modification is crucial to give better estimate of displacement energy. The second one is replacing the additive variable space $\mathbb{R}$ with S^1. The last one is to use the orbit category (see Subsection 2.7). We omit the details and refer to Appendix A.

Let $\theta \in \mathbb{R}_{\geq 0}$ and put $S^1_\theta := \mathbb{R}/\theta\mathbb{Z}$. Note that if $\theta = 0$ then $S^1_\theta = \mathbb{R}$. We denote the image of $t \in \mathbb{R}$ under the quotient map $\mathbb{R} \to S^1_\theta$ by $[t]$ or t simply. Moreover, let P be a manifold. Denote by $(x;\xi)$ a local homogeneous coordinate system on T^*M, by $(y;\eta)$ that on T^*P, and by $(t;\tau)$ the homogeneous coordinate system on $T^*S^1_\theta$ and $T^*\mathbb{R}$. We define maps $\tilde{q}_1, \tilde{q}_2, s_{S^1_\theta} \colon M \times P \times S^1_\theta \times S^1_\theta \longrightarrow M \times P \times S^1_\theta$ by

$$\begin{aligned} \tilde{q}_1(x,y,t_1,t_2) &= (x,y,t_1), \\ \tilde{q}_2(x,y,t_1,t_2) &= (x,y,t_2), \\ s_{S^1_\theta}(x,y,t_1,t_2) &= (x,y,t_1+t_2). \end{aligned} \tag{3.1}$$

If there is no risk of confusion, we simply write s for $s_{S^1_\theta}$. We also set

$$\begin{aligned} i &\colon M \times P \times S^1_\theta \to M \times P \times S^1_\theta,\ (x,y,t) \longmapsto (x,y,-t), \\ \ell &\colon M \times P \times \mathbb{R} \to M \times P \times S^1_\theta,\ (x,y,t) \longmapsto (x,y,[t]). \end{aligned} \tag{3.2}$$

Note also that if $\theta = 0$ then ℓ is the identity map.

Definition 3.1. For $F, G \in \mathbf{D}^{\mathrm{b}}_{/[1]}(M \times P \times S^1_\theta)$, one sets

$$F \star G := Rs_!(\tilde{q}_1^{-1}F \otimes \tilde{q}_2^{-1}G), \tag{3.3}$$

$$\begin{aligned} \mathcal{H}om^\star(F,G) &:= R\tilde{q}_{1*}\, R\mathcal{H}om(\tilde{q}_2^{-1}F, s^!G) \\ &\simeq Rs_*\, R\mathcal{H}om(\tilde{q}_2^{-1}i^{-1}F, \tilde{q}_1^!G). \end{aligned} \tag{3.4}$$

Note that the functor $\star$ is a left adjoint to $\mathcal{H}om^\star$.

We set $\Omega_+(N)_\theta := \{(x,t;\xi,\tau) \mid \tau > 0\} \subset T^*(N \times S^1_\theta)$ for any manifold N and write $\Omega_+ = \Omega_+(M \times P)_\theta$ for short.

One can show the equivalence of categories

$$\begin{aligned} P_l := R\ell_!\mathbf{k}_{M\times P\times[0,+\infty)} \star (*) &\colon \mathbf{D}^{\mathrm{b}}_{/[1]}(M \times P \times S^1_\theta;\Omega_+) \xrightarrow{\sim} {}^\perp\mathbf{D}^{\mathrm{b}}_{/[1],\{\tau\leq 0\}}(M \times P \times S^1_\theta), \\ P_r := \mathcal{H}om^\star(R\ell_!\mathbf{k}_{M\times P\times[0,+\infty)}, *) &\colon \mathbf{D}^{\mathrm{b}}_{/[1]}(M \times P \times S^1_\theta;\Omega_+) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{/[1],\{\tau\leq 0\}}(M \times P \times S^1_\theta)^\perp. \end{aligned}$$

We define the map

$$\begin{array}{rcl} \rho\colon \Omega_+ & \longrightarrow & T^*M \\ \cup & & \cup \\ (x,y,t;\xi,\eta,\tau) & \longmapsto & (x;\xi/\tau). \end{array} \tag{3.5}$$

Definition 3.2. We define a category $\mathcal{D}^P(M)_\theta$ by

$$\mathcal{D}^P(M)_\theta := \mathbf{D}^{\mathrm{b}}_{/[1]}(M \times P \times S^1_\theta;\Omega_+) \tag{3.6}$$

and identify it with the left orthogonal ${}^\perp\mathbf{D}^{\mathrm{b}}_{/[1],\{\tau\leq 0\}}(M \times P \times S^1_\theta)$ or the right orthogonal $\mathbf{D}^{\mathrm{b}}_{/[1],\{\tau\leq 0\}}(M \times P \times S^1_\theta)^\perp$. Set $\mathrm{SS}_+(F) := \mathrm{SS}(F) \cap \Omega_+$ for $F \in \mathcal{D}^P(M)_\theta$.

For a closed subset A of T^*M, we define a full subcategory $\mathcal{D}^P_A(M)_\theta$ by

$$\mathcal{D}^P_A(M)_\theta := \mathbf{D}^{\mathrm{b}}_{/[1],\rho^{-1}(A)}(M \times P \times S^1_\theta; \Omega_+). \tag{3.7}$$

For an open subset U of T^*M and a closed subset A of U, we define categories

$$\mathcal{D}^P(M;U)_\theta := \mathbf{D}^{\mathrm{b}}_{/[1]}(M \times P \times S^1_\theta; \rho^{-1}(U)), \tag{3.8}$$

$$\mathcal{D}^P_A(M;U)_\theta := \mathbf{D}^{\mathrm{b}}_{/[1],\rho^{-1}(A)}(M \times P \times S^1_\theta; \rho^{-1}(U)). \tag{3.9}$$

We omit P from above notation if $P = \mathrm{pt}$.

Next we consider Hamiltonian deformations of sheaves. Let I be an open interval containing the closed interval $[0,1]$. Let $H\colon T^*M \times I \to \mathbb{R}$ be a compactly supported Hamiltonian function and denote by $\phi^H\colon T^*M \times I \to T^*M$ the Hamiltonian isotopy generated by H. Following Hofer [Hof90], we set

$$\|H\| := \int_0^1 \left(\max_p H_s(p) - \min_p H_s(p) \right) ds. \tag{3.10}$$

Let $K^H \in \mathbf{D}^{\mathrm{b}}(M \times S^1_\theta \times M \times S^1_\theta \times I)$ be the sheaf quantization associated with ϕ^H, whose existence was proved by Guillermou–Kashiwara–Schapira [GKS12]. For $s \in I$, we set $K^H_s := K^H|_{M \times S^1_\theta \times M \times S^1_\theta \times \{s\}}$. Then the composition with $K^H_s \boxtimes \mathbf{k}_{\Delta_P}$ induces a functor

$$\Phi^H_s := (K^H_s \boxtimes \mathbf{k}_{\Delta_P}) \circ (*)\colon \mathcal{D}^P(M)_\theta \longrightarrow \mathcal{D}^P(M)_\theta, \tag{3.11}$$

which restricts to $\mathcal{D}^P_A(M) \to \mathcal{D}^P_{\phi^H_s(A)}(M)$ for any compact subset A of T^*M.

Now we measure the difference between $G \in \mathcal{D}^P(M)_\theta$ and its deformation $\Phi^H_s(G)$, by introducing a distance on $\mathcal{D}^P(M)_\theta$. For $c \in \mathbb{R}$ or S^1_θ, we define the translation map

$$T_c\colon M \times P \times S^1_\theta \longrightarrow M \times P \times S^1_\theta, (x, y, t) \longmapsto (x, y, t + c). \tag{3.12}$$

For $F \in \mathcal{D}^P(M)_\theta$ and $d \in \mathbb{R}_{\geq 0}$, there is the canonical morphism

$$\tau_{c,d}(F)\colon T_{c*}F \longrightarrow T_{c+d*}F. \tag{3.13}$$

Using the morphism, we define the translation distance $d_{\mathcal{D}^P(M)_\theta}$ as in [AI17].

Definition 3.3. Let $F, G \in \mathcal{D}^P(M)_\theta$.

(i) Let $a, b \in \mathbb{R}_{\geq 0}$. Then the pair (F, G) is said to be (a,b)-*interleaved* if there exist morphisms $\alpha, \delta\colon F \to T_{a*}G$ and $\beta, \gamma\colon G \to T_{b*}F$ satisfying the following conditions:

(1) $F \xrightarrow{\alpha} T_{a*}G \xrightarrow{T_{a*}\beta} T_{a+b*}F$ is equal to $\tau_{0,a+b}(F)\colon F \to T_{a+b*}F$ and

(2) $G \xrightarrow{\gamma} T_{b*}F \xrightarrow{T_{b*}\delta} T_{a+b*}G$ is equal to $\tau_{0,a+b}(G)\colon G \to T_{a+b*}G$.

(ii) One defines

$$d_{\mathcal{D}^P(M)_\theta}(F, G) := \inf\{a + b \in \mathbb{R}_{\geq 0} \mid a, b \in \mathbb{R}_{\geq 0}, (F, G) \text{ is } (a,b)\text{-interleaved}\}, \tag{3.14}$$

and calls $d_{\mathcal{D}^P(M)_\theta}$ the *translation distance*.

(iii) One defines

$$\begin{aligned} e_{\mathcal{D}^P(M)_\theta}(F, G) :=& d_{\mathcal{D}(\mathrm{pt})_\theta}\left(Rq_{S^1_\theta *} \mathcal{H}om^\star(F, G), 0 \right) \\ =& \inf\left\{ c \in \mathbb{R}_{\geq 0} \,\middle|\, \tau_{0,c}(Rq_{S^1_\theta *} \mathcal{H}om^\star(F, G)) = 0 \right\}. \end{aligned} \tag{3.15}$$

Note that by Proposition A.3, we have

$$e_{\mathcal{D}^P(M)_\theta}(F,G) \geq \inf \left\{ c \in \mathbb{R}_{\geq 0} \;\middle|\; \mathrm{Hom}_{\mathcal{D}^P(M)_\theta}(F,G) \to \mathrm{Hom}_{\mathcal{D}^P(M)_\theta}(F, T_{c*}G) \text{ is zero} \right\}. \tag{3.16}$$

Proposition 3.4. *Let $\phi^H : T^*M \times I \to T^*M$ be the Hamiltonian isotopy generated by a compactly supported Hamiltonian function $H\colon T^*M \times I \to \mathbb{R}$. Denote by $\Phi_1^H : \mathcal{D}^P(M)_\theta \to \mathcal{D}^P(M)_\theta$ the functor associated with ϕ_1^H. Let $G \in \mathcal{D}^P(M)_\theta$. Then $d_{\mathcal{D}^P(M)_\theta}(G, \Phi_1^H(G)) \leq \|H\|$.*

Combining Proposition A.4 with Proposition 3.4, we obtain the following generalization of the main theorem of [AI17]. We may use this to estimate the the displacement energy of Lagrangian immersions. However, we need more precise arguments for the intersection number estimates and will not use this proposition in this paper.

Proposition 3.5. *Let A and B be compact subsets of T^*M. Then for any $F \in \mathcal{D}^P_A(M)_\theta$ and $G \in \mathcal{D}^P_B(M)_\theta$ such that $q_{S^1_\theta}$ is proper on $\mathrm{Supp}(F) \cup \mathrm{Supp}(G)$, one has*

$$e(A,B) \geq e_{\mathcal{D}^P(M)_\theta}(F,G). \tag{3.17}$$

In particular, for any $F \in \mathcal{D}^P_A(M)_\theta$ and $G \in \mathcal{D}^P_B(M)_\theta$,

$$e(A,B) \geq \inf\{c \in \mathbb{R}_{\geq 0} \mid \mathrm{Hom}_{\mathcal{D}^P(M)_\theta}(F,G) \to \mathrm{Hom}_{\mathcal{D}^P(M)_\theta}(F, T_{c*}G) \textit{ is zero}\}. \tag{3.18}$$

4 Sheaf quantization of rational Lagrangian immersions

In this section, we prove the existence of sheaf quantizations of certain class of Lagrangian immersions, by following ideas of Guillermou [Gui12, Gui19].

4.1 Definitions

First we introduce some notions for Lagrangian immersions. We assume that L is compact and connected.

Definition 4.1. (i) A Lagrangian immersion $\iota\colon L \to T^*M$ is said to be *strongly rational* if there exists a non-negative number $\theta_\iota \in \mathbb{R}_{\geq 0}$ such that the image of the pairing map with $\iota^*\alpha; H_1(L;\mathbb{Z}) \to \mathbb{R}, \gamma \mapsto \int_\gamma \alpha$ is $\theta_\iota\mathbb{Z}$. We call θ_ι the *period* of ι.

(ii) For a strongly rational Lagrangian immersion $\iota\colon L \to T^*M$, one defines

$$r_\iota := \inf \left(\left(\left\{ \int_l \iota^*\alpha \;\middle|\; \begin{array}{l} l\colon [0,1] \to L, l(0) \neq l(1), \\ \iota \circ l(0) = \iota \circ l(1) \end{array} \right\} \cup \{\theta_\iota\} \right) \cap \mathbb{R}_{>0} \right). \tag{4.1}$$

The infimum of the empty set is defined to be $+\infty$.

Notation 4.2. Let $\iota\colon L \to T^*M$ be a compact strongly rational Lagrangian immersion with period θ and $f\colon L \to S^1_\theta$ be a function satisfying $\iota^*\alpha = df$. We define a conic Lagrangian immersion

$$\widehat{\iota} := \widehat{\iota}_f \colon L \times \mathbb{R}_{>0} \to T^*(M \times S^1_\theta), (y,\tau) \mapsto (\tau\iota(y), (-f(y);\tau)) \tag{4.2}$$

and put its image

$$\Lambda := \Lambda_{\iota,f} := \{(x,t;\xi,\tau) \in T^*(M \times S^1_\theta) \mid \tau > 0, \exists\, y \in L, (x;\xi/\tau) = \iota(y), t = -f(y)\}. \quad (4.3)$$

We also set

$$\Lambda_q := \Lambda_{\iota,f,q} := \{(x,u,t;\xi,0,\tau) \in T^*(M \times (0,r_\iota) \times S^1_\theta) \mid (x,t;\xi,\tau) \in \Lambda_{\iota,f}\},$$
$$\Lambda_r := \Lambda_{\iota,f,r} := \{(x,u,t;\xi,-\tau,\tau) \in T^*(M \times (0,r_\iota) \times S^1_\theta) \mid (x,t-u;\xi,\tau) \in \Lambda_{\iota,f}\}.$$

Assumption 4.3. *There exists no curve $l\colon [0,1] \to L$ with $l(0) \neq l(1)$, $\iota \circ l(0) = \iota \circ l(1)$ and $\int_l \iota^*\alpha = 0$.*

Note that if Assumption 4.3 is satisfied, $\widehat{\iota}$ is an embedding.

Theorem 4.4. *Let $\iota\colon L \to T^*M$ be a compact strongly rational Lagrangian immersion satisfying Assumption 4.3. Then for each $a \in (0, r_\iota)$ there exists an object $G_{(0,a)} \in \mathcal{D}^{(0,a)}_L(M)_\theta$ satisfying the following conditions.*

1. *$\mathring{\mathrm{SS}}(G_{(0,a)}) \subset (\Lambda_{\iota,f,q} \cup \Lambda_{\iota,f,r}) \cap T^*(M \times (0,a) \times S^1_\theta)$,*
2. *$G_{(0,a)}$ is simple along $\Lambda_{\iota,f,q} \cap T^*(M \times (0,a) \times S^1_\theta)$,*
3. *$F_0 := (Rj_*G_{(0,a)})|_{M \times \{0\} \times S^1_\theta}$ is isomorphic to 0, where j is the inclusion $M \times (0,a) \times S^1_\theta \to M \times \mathbb{R} \times S^1_\theta$,*
4. *there is an open covering $\{V_\alpha\}_\alpha$ of $\Omega_+(M)_\theta$ such that $G_{(0,a)}$ is microlocally tame with respect to $\{V_\alpha \times T^*(0,a)\}_\alpha$.*

Moreover, this $G_{(0,a)}$ automatically satisfies $d(G_{(0,a)}, 0) \leq a$.

4.2 Construction

Theorem 4.4 is essentially proved by Guillermou [Gui19, Thm. 12.2.2 and Sec. 12.3]. We prepare some notions introduced in [Gui19].

4.2.1 Kashiwara-Schapira stacks

Notation 4.5. Let Λ be a locally closed conic Lagrangian submanifold of T^*X.

(i) For $V \subset \Lambda$, $\mu\mathrm{Sh}^{mt,0}_{/[1],\Lambda}(V)$ is a category with the same objects to $\mathbf{D}^{\mathrm{b}}_{(V)}(\mathbf{k}_X)$. For $F, G \in \mu\mathrm{Sh}^{mt,0}_{/[1],\Lambda}(V)$, $\mathrm{Hom}_{\mu\mathrm{Sh}^{mt,0}_{/[1],\Lambda}(V)}(F,G) := \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_X;V)}(\mathfrak{i}(F), \mathfrak{i}(G))$. This $\mu\mathrm{Sh}^{mt,0}_{/[1],\Lambda}$ forms a prestack on Λ.

(ii) We define the Kashiwara–Schapira stack $\mu\mathrm{Sh}^{mt}_{/[1],\Lambda}$ on Λ as the associated stack to $\mu\mathrm{Sh}^{mt,0}_{/[1],\Lambda}$.

(iii) The quotient functor gives a functor $\mathfrak{m}_\Lambda\colon \mathbf{D}^{\mathrm{b},mt}_{/[1],(\Lambda)}(\mathbf{k}_X) \to \mu\mathrm{Sh}^{mt}_{/[1],\Lambda}(\Lambda)$ where $\mathbf{D}^{\mathrm{b},mt}_{/[1],(\Lambda)}(\mathbf{k}_X)$ is the full subcategory of $\mathbf{D}^{\mathrm{b}}_{/[1]}(X)$ consisting of F for which there exists a neighborhood U of Λ such that $\mathring{\mathrm{SS}}(F) \cap U \subset \Lambda$ and $F \in \mathbf{D}^{\mathrm{b}}_{/[1]}(X;U)$ is microlocally tame.

(iv) $\mathcal{F} \in \mu\mathrm{Sh}_{/[1],\Lambda}(V)$ is said to be simple if $\mathcal{F}$ is obtained by gluing simple objects.

Remark 4.6. The stack $\mu\mathrm{Sh}^{mt}_{/[1],\Lambda}$ is smaller than or equal to the Kashiwara–Schapira stack $\mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda)$ in [Gui19]. We put the microlocally tameness condition since they are easier to treat. They would become equivalent to each other after additional arguments.

The next lemma is proved in [Gui19, §10.4].

Proposition 4.7. *Let Λ is a locally closed conic Lagrangian submanifold of T^*X. The category $\mu\mathrm{Sh}^{mt}_{/[1],\Lambda}(\Lambda)$ has a unique simple object.*

4.2.2 doubling functor Ψ

We introduce a variant of the convolution functor $\star$ in Section 3. Set $\gamma := \{(u,t) \mid 0 \leq t < u\} \subset \mathbb{R}_{>0} \times \mathbb{R}$.

For an open subset $U \subset M \times S^1_\theta$, define $U_\gamma := \{(x,u,t) \in M \times \mathbb{R}_{>0} \times S^1_\theta \mid (x, t-[a]) \in U (\forall a \in [0,u])\}$.

The functor $\Psi_U \colon \mathbf{D}^{\mathrm{b}}_{/[1]}(U) \to \mathbf{D}^{\mathrm{b}}_{/[1]}(U_\gamma)$ is defined by $\Psi_U(F) := Rs_{U!}(F \boxtimes \mathbf{k}_\gamma)|_{U_\gamma}$ where $s_U \colon U \times \mathbb{R}_{>0} \times \mathbb{R} \to M \times \mathbb{R}_{>0} \times S^1_\theta$ is $(x, t_1, u, t_2) \mapsto (x, u, t_1 + [t_2])$.

We will construct $G_{(0,\varepsilon)}$ for sufficiently small $\varepsilon > 0$ so that $G_{(0,\varepsilon)}$ is locally isomorphic to an image of Ψ_U.

The next lemma follows from [Gui19, Thm. 11.1.7].

Lemma 4.8. *Let s_{23} be the swapping map $M \times S^1_\theta \times \mathbb{R} \to M \times \mathbb{R} \times S^1_\theta, (x,t,u) \mapsto (x,u,t)$ and j_U be the open embedding $U_\gamma \to U_\gamma \cup s_{23}(U \times \mathbb{R}_{\leq 0})$. Then $Rj_{U*}\Psi_U(F)|_{s_{23}(U \times \{0\})} \simeq 0$.*

4.2.3 doubled sheaves

Definition 4.9. Let $\Lambda \subset \Omega_+(M)_\theta$ be a conic Lagrangian submanifold such that $\Lambda/\mathbb{R}_{>0}$ is compact and $\Lambda/\mathbb{R}_{>0} \to M$ is finite. A finite family $\mathcal{V} = \{V_a \mid a \in A\}$ of open subsets of $M \times S^1_\theta$ is said adapted to $\widehat{\iota}$ if the following conditions hold:

(i) $\pi_{M \times S^1_\theta}(\Lambda) \subset \bigcup_a V_a$,

(ii) for each $a \in A$, $V_a = W_a \times I_a$ where W_a is an open subset of M and I_a is a contractible open subset of S^1_θ, and $\pi_{M \times S^1_\theta}(\Lambda) \cap V_a \subset W_a \times K$ for some compact subset K of I_a .

(iii) for all $A_1 \subset A$ $R\mathcal{H}om(\mathbf{k}_{V^{A_1}}, \mathbf{k}_{M \times S^1_\theta}) \simeq \mathbf{k}_{\overline{V^{A_1}}}$ where $V^{A_1} := \bigcup_{a \in A_1} V_a$.

(iv) put $\Lambda_+ := \Lambda \cup 0_{M \times S^1_\theta}$ then

$$(\mathrm{SS}(\mathbf{k}_{V^{A_1}}) \widehat{+} \mathrm{SS}(\mathbf{k}_{V^{A_2}})^a) \cap (\Lambda_+ \widehat{+} (\Lambda_+)^a) \subset 0_{M \times S^1_\theta} \tag{4.4}$$

for each $A_1, A_2 \subset A$.

See [KS90] or [Gui19] for the definition of $\widehat{+}$ in (4.4).

Lemma 4.10. *Let $\Lambda \subset \Omega_+(M)_\theta$ be a conic Lagrangian submanifold such that $\Lambda/\mathbb{R}_{>0}$ is compact and let $\{\Lambda_j\}_{j \in J}$ be a finite open covering of Λ_i by conic subsets. Then there exists a homogeneous Hamiltonian isotopy $\widehat{\phi}$ on $\Omega_+(M)_\theta$, as closed to* id *as desired, and a finite family $\{V_a\}_{a \in A}$ of open subsets of $M \times S^1_\theta$ which is adapted to $\widehat{\phi}(\Lambda)$ such that each connected component of $\widehat{\phi}(\Lambda) \cap T^*V_a$, for each $a \in A$, is contained in $\widehat{\phi}(\Lambda_j)$, for some $j \in J$.*

Definition 4.11. Let Λ and $\mathcal{V} = \{V_a\}_{a \in A}$ be as in Definition 4.9. Let $V \subset M \times S^1_\theta$ be an open subset. We denote by $\mathbf{D}^{dbl}_{/[1],\Lambda,\mathcal{V}}(\mathbf{k}_V)$ the subcategory of $\mathbf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_{V \times \mathbb{R}_{>0}})$ formed by G such that, for sufficiently small $\varepsilon > 0$,

(i) $\mathrm{Supp}(G) \cap (V \times (0,\varepsilon)) \subset \{(x, t + [a], u) \in V \times (0,\varepsilon) \mid (x,t) \in \pi(\Lambda), a \in [0,u]\}$

(ii) every point of V has a neighborhood W such that $\pi_0(\Lambda \cap T^*W) = \{\Lambda_i\}$ is finite, and for each Λ_i there exist $A_i \subset A$ and $F_i \in \mathbf{D}^{\mathrm{b}}_{\Lambda_i}(\mathbf{k}_W)$ with $\mathring{\mathrm{SS}}(F_i) = \Lambda_i$ such that

$$G|_{W^\varepsilon} \simeq \bigoplus_i \Psi_W(R\Gamma_{V^{A_i}}(\mathfrak{i}(F_i)))|_{W^\varepsilon}. \tag{4.5}$$

As well as the alteration remarked in Remark 4.6, our definition of doubled sheaves is slightly different from Guillermou's definition. These alternations do not revoke the arguments in [Gui19].

For $G \in \mathbf{D}^{dbl}_{/[1],\Lambda,\mathcal{V}}(\mathbf{k}_V)$, there exists a well-defined open subset $\mathrm{SS}^{dbl}(G)$ of $\Lambda \cap T^*V$ locally defined by $\mathrm{SS}^{dbl}(G) \cap T^*W := \bigcup_i (\Lambda_i \cap T^*V^{A_i})$.

$\mathfrak{m}^{dbl}_\Lambda(G) \in \mu\mathrm{Sh}^{mt}_{/[1],\Lambda}(\mathrm{SS}^{dbl}(G))$ is also defined so that $\mathfrak{m}^{dbl}_\Lambda(G)|_{\Lambda_i \cap T^*V^{A_i}} \simeq \mathfrak{m}_{\Lambda_i}(F_i)|_{\Lambda_i \cap T^*V^{A_i}}$.

Replacing some $\mathbb{R}$'s in [Gui19] to S^1_θ's, one can check the following theorem.

Theorem 4.12 ([Gui19, Thm. 2.2.2])**.** *Let Λ and $\mathcal{V} = \{V_a\}_{a\in A}$ be as in Definition 4.9. For any object $\mathcal{F} \in \mu\mathrm{Sh}^{mt}_{/[1],\Lambda}(\Lambda)$ there exists an object $F \in \mathbf{D}^{dbl}_{/[1],\Lambda,\mathcal{V}}(M \times S^1_\theta)$ with $\mathrm{SS}^{dbl}(F) = \Lambda$ and $\mathfrak{m}^{dbl}_\Lambda(F) \simeq \mathcal{F}$.*

Proof of Theorem 4.4. Since the conditions in Theorem 4.4 are preserved by the actions of the homogeneous Hamiltonian isotopies on $\Omega_+(M)_\theta$, we may assume that there exists a family $\mathcal{V}$ adapted to Λ by Lemma 4.10. By Proposition 4.7 and Theorem 4.12, there exists $F \in \mathbf{D}^{dbl}_{/[1],\Lambda,\mathcal{V}}(M \times S^1_\theta)$ with $\mathrm{SS}^{dbl}(F) = \Lambda$ and $\mathfrak{m}^{dbl}_\Lambda(F)$ is simple. The additional condition is equivalent to

$$F|_{W^\varepsilon} \simeq \bigoplus_i \Psi_W(\mathfrak{i}(F_i))|_{W^\varepsilon} \tag{4.6}$$

where each F_i is simple. Then, for sufficiently small $0 < a$, $F|_{M\times(0,a)\times S^1_\theta} \in \mathbf{D}^{\mathrm{b}}(M \times (0,a) \times S^1_\theta)$ satisfies (1)–(4) in Theorem 4.4, where (3) follows from Lemma 4.8.

The construction for larger $a \in (0, r_\iota)$ is parallel to [Gui19, Subsection 12.3.]. Although there may exist some Reeb chords in our situation, the length of any Reeb chord is longer than a. Hence a parallel argument will go on $M \times (0,a) \times S^1_\theta$. (Then (1)–(4) are obvious. The boundedness is maintained since we don't touch $\{u = r_\iota\}$-part.) [2]

Take $a' \in (a, r)$ and $G_{(0,a')} \in \mathcal{D}^{(0,a')}_L(M)_\theta$ satisfying the conditions in Theorem 4.4 so that $G_{(0,a')}|_{M\times(0,a)\times S^1_\theta}$ is isomorphic to $G_{(0,a)}$. Define $D_{a'} := \{(u,s) \in \mathbb{R}^2 \mid 0 < u < s < a'\}$ and $p\colon M \times D_{a'} \times S^1_\theta, (x,u,s,t) \to (x,u,t)$ and $\mathcal{G} := p^{-1}G_{(0,a)} \in \mathcal{D}^{D_{a'}(0,a')}_L(M)_\theta$. Consider $g\colon M \times D_{a'} \times S^1_\theta \to M \times (-\infty,a) \times (-\infty,a') \times S^1_\theta, (x,u,s,t) \to (x, u-s+a, s, t)$ and $Rg_!\mathcal{G} \simeq Rg_*\mathcal{G} \in \mathcal{D}^{(-\infty,a)\times(-\infty,a')}_L(M)_\theta$. Then $\mathcal{G}' := Rg_!\mathcal{G}|_{M\times(0,a)\times(-\infty,a')\times S^1_\theta}$ satisfies $\mathcal{G}'|_{\{s=0\}} \simeq 0$, $\mathcal{G}'|_{\{s=a\}} \simeq G_{(0,a)}$ and $\mathrm{SS}(\mathcal{G}') \subset \{0 \leq \sigma \leq \tau\}$. By Lemma A.6, $d(G_{(0,a)}, 0) \leq a$ holds. □

5 Intersection and displacement energy of rational Lagrangian immersions

In this section, we give an estimate of their displacement energy and the number of intersections, which overlaps with theorems of Chekanov [Che98] and Akaho [Aka15] (see Theorem 5.4), and that of Liu [Liu05] (see Theorem 5.5).

[2] If the boundedness is presereved near $\{u = r\}$, we can prove Theorem 4.4 for $a = r$ and a better estimate than that in Theorem 5.5.

5.1 Statements of main thoerems

Definition 5.1. (i) A Lagrangian immersion $\iota\colon L \to T^*M$ is said to be *rational* if there exists $\sigma_\iota \in \mathbb{R}_{\geq 0}$ such that

$$\left\{ \int_{D^2} v^*\omega \;\middle|\; (v,\bar{v}) \in \Sigma_\iota \right\} = \sigma_\iota \mathbb{Z}, \tag{5.1}$$

where

$$\Sigma_\iota := \left\{ (v,\bar{v}) \;\middle|\; \begin{array}{l} v\colon D^2 \to T^*M, \bar{v}\colon \partial D^2 \to L, \\ v|_{\partial D^2} = \iota \circ \bar{v} \end{array} \right\}. \tag{5.2}$$

We call σ_ι the *rationality constant* of ι.

(ii) For a rational Lagrangian immersion $\iota\colon L \to T^*M$, one define

$$e_\iota := \inf \left(\left(\left\{ \int_{D^2} v^*\omega \;\middle|\; (v,\bar{v}) \in K_\iota \right\} \cup \{\sigma_\iota\} \right) \cap \mathbb{R}_{>0} \right), \tag{5.3}$$

where

$$K_\iota := \left\{ (v,\bar{v}) \;\middle|\; \begin{array}{l} v\colon D^2 \to T^*M, \bar{v}\colon [0,1] \to L, \\ \bar{v}(0) \neq \bar{v}(1), \iota \circ \bar{v}(0) = \iota \circ \bar{v}(1), \\ v|_{\partial D^2} = \iota \circ \bar{v} \end{array} \right\}. \tag{5.4}$$

Remark 5.2. A strongly rational Lagrangian immersion is rational. Indeed, if such θ_ι exists, then ι is rational and its rationality constant is $n\theta_\iota$ for some $n \in \mathbb{Z}_{\geq 0}$. However the converse is not true. For example, the graph of any closed 1-form β on a closed manifold M is rational with rationality constant 0, though this embedding has a period θ_ι if and only if there exist a primitive element $b \in H^1(M;\mathbb{Z})$ such that $[\beta] = \theta_\iota b \in H^1(M;\mathbb{R})$.

Assumption 5.3. *There exists no $(v,\bar{v}) \in K_\iota$ with $\int_{D^2} v^*\omega = 0$.*

The relation between Assumption 4.3 and Assumption 5.3 will be explained in Subsection 5.2 below.

Theorem 5.4 (cf. Chekanov [Che98] and Akaho [Aka15])**.** *Let $\iota\colon L \to T^*M$ be a compact rational Lagrangian immersion satisfying Assumption 5.3. If $\|H\| < e_\iota$ and $\iota\colon L \to T^*M$ intersects $\phi_1^H \circ \iota\colon L \to T^*M$ transversally, then*

$$\#\left\{ (y,y') \in L \times L \;\middle|\; \iota(y) = \phi_1^H \circ \iota(y') \right\} \geq \sum_{i=0}^{\dim L} b_i(L). \tag{5.5}$$

Theorem 5.5 (cf. Liu [Liu05])**.** *Let $\iota\colon L \to T^*M$ be a compact rational Lagrangian satisfying Assumption 5.3. If $\|H\| < \min\left(\{e_\iota\} \cup (\{\sigma_\iota/2\} \cap \mathbb{R}_{>0})\right)$, then*

$$\#\left\{ (y,y') \in L \times L \;\middle|\; \iota(y) = \phi_1^H \circ \iota(y') \right\} \geq \mathrm{cl}(L) + 1, \tag{5.6}$$

where $\mathrm{cl}(L)$ is the cup-length of L over $\mathbb{F}_2$.

Remark 5.6. There seem to be proofs via Floer theory of slightly stronger statements than our main results.

(i) Floer theoretic proofs will not need Assumption 5.3.

(ii) In [Liu05], $\min\left(\{e_\iota\} \cup (\{\sigma_\iota/2\} \cap \mathbb{R}_{>0})\right)$ in Theorem 5.5 is replaced by e_ι. Note that $\min\left(\{e_\iota\} \cup (\{\sigma_\iota/2\} \cap \mathbb{R}_{>0})\right) = e_\iota$ if $\sigma_\iota \neq e_\iota$.

(iii) It is possible to study the cases $\|H\| \geq e_\iota$ if bounding cochains exists [FOOO09, FOOO13].

5.2 Proof of main theorems

This subsection is devoted to the proofs of Theorems 5.4 and 5.5.

First we reduce the problems to the strongly rational cases.

Lemma 5.7. *Let $\iota\colon L \to T^*M$ be a compact (connected) rational Lagrangian immersion with rationality constant σ_ι. Assume that $\pi_1(\pi \circ \iota)\colon \pi_1(L) \to \pi_1(M)$ is surjective. Then, there exists a closed 1-form β on M such that the immersion $\iota + \beta\colon L \to T^*M, y \mapsto \iota(y) + \beta(\pi \circ \iota(y))$ is strongly rational with period σ_ι. Moreover, $r_{\iota+\beta} = e_{\iota+\beta} = e_\iota$.*

Proof. If $\pi_1(L) \to \pi_1(M)$ is surjective, then the induced homomorphism $[\pi_1(L), \pi_1(L)] \to [\pi_1(M), \pi_1(M)]$ is also surjective. By this surjectivity and the nine lemma for groups, one can check $\mathrm{Ker}(\pi_1(\pi \circ \iota)) \to \mathrm{Ker}(H_1(\pi \circ \iota; \mathbb{Z}))$ is surjective.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & [\pi_1(L), \pi_1(L)] & \twoheadrightarrow & [\pi_1(M), \pi_1(M)] & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Ker}(\pi_1(\pi \circ \iota)) & \hookrightarrow & \pi_1(L) & \twoheadrightarrow & \pi_1(M) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(H_1(\pi \circ \iota; \mathbb{Z})) & \hookrightarrow & H_1(L; \mathbb{Z}) & \twoheadrightarrow & H_1(M; \mathbb{Z}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{5.7}
$$

Choose a retraction $r\colon H^1(L; \mathbb{R}) \to H^1(M; \mathbb{R})$ of $\iota^*\colon H^1(M; \mathbb{R}) \to H^1(L; \mathbb{R})$ and take an 1-form β on M so that $[\beta] = -r([\iota^*\alpha]) \in H^1(M; \mathbb{R})$. Then $r[(\iota + \beta)^*\alpha] = 0 \in H^1(M; \mathbb{R})$ and

$$
\begin{aligned}
\left\{ \int_\gamma (\iota + \beta)^*\alpha \;\middle|\; \gamma \in H_1(L; \mathbb{Z}) \right\} &= \left\{ \int_\gamma (\iota + \beta)^*\alpha \;\middle|\; \gamma \in \mathrm{Ker}(H_1(\pi \circ \iota; \mathbb{Z})) \right\} \\
&= \left\{ \int_\gamma \iota^*\alpha \;\middle|\; \gamma \in \mathrm{Ker}(H_1(\pi \circ \iota; \mathbb{Z})) \right\} \\
&= \left\{ \int_\gamma \iota^*\alpha \;\middle|\; \gamma \in \mathrm{Ker}(\pi_1(\pi \circ \iota)) \right\} = \sigma_\iota \mathbb{Z}
\end{aligned}
\tag{5.8}
$$

For the third equality, we used the surjectivity of $\mathrm{Ker}(\pi_1(\pi \circ \iota)) \to \mathrm{Ker}(H_1(\pi \circ \iota; \mathbb{Z}))$.

Let $E_{\iota'}$ be the set $\{(y, y') \in L \times L \mid y \neq y', \iota'(y) = \iota'(y')\}$ of the non-injective points for a Lagrangian immersion $\iota'\colon L \to T^*M$. By the surjectivity of $\pi_1(L) \to \pi_1(M)$, for any $(y_0, y_1) \in E_{\iota+\beta} = E_\iota$ there is an element $(v, \bar{v}) \in K_\iota$ such that $\bar{v}(i) = y_i$ $(i = 0, 1)$. Hence $r_{\iota+\beta} = e_{\iota+\beta} = e_\iota$. (Take a path conncting y_0 and y_1 in L and composing $\pi \circ \iota$ to this path gives an element of $\pi_1(M, \pi \circ \iota(y_0))$. Choose a preimage of this element in $\pi_1(L, y_0)$. Concatenating a representative path of the preimage to the original path on L, we obtain a path connecting y_0 and y_1 which bounds a disk in T^*M.) □

Lemma 5.8. *Assume that Theorems 5.4 and 5.5 are true for any strongly rational Lagrangian immersion $\iota\colon L \to T^*M$ satisfying Assumption 4.3, $\theta_\iota = \sigma_\iota$ and $r_\iota = e_\iota$. Then Theorems 5.4 and 5.5 are true for any rational Lagrangian immersion.*

Proof. Take the covering $p\colon \widetilde{M} \to M$ corresponding to $\iota_*(\pi_1(L)) \subset \pi_1(M)$. Note that $\sigma_{\widetilde{\iota}} = \sigma_\iota$. A lift $\widetilde{\iota}\colon L \to T^*\widetilde{M}$ of ι induces a surjection on the fundamental groups. By the construction of p, a non-injective point $(y, y') \in E_\iota$ of ι, whose definition is given in the proof of Lemma 5.7, is a non-injective point of $\widetilde{\iota}$ if and only if there exists $(v, \bar{v}) \in K_\iota$ with $(y, y') = (\bar{v}(0), \bar{v}(1))$. Hence $e_{\widetilde{\iota}} = e_\iota$ and Assumption 5.3 for ι is equivalent to Assumption 5.3 for $\widetilde{\iota}$.

Take an 1-form β on $\widetilde{M}$ satisfying the conclusion of Lemma 5.7 for $\widetilde{\iota}$. By the surjectivity of $\pi_1(\widetilde{\iota})$, Assumption 4.3 for $\widetilde{\iota} + \beta$ is equivalent to Assumption 5.3 for $\widetilde{\iota} + \beta$, which is also equivalent to Assumption 5.3 for $\widetilde{\iota}$ and ι.

For a Hamiltonian function H on T^*M, let $\widetilde{H}$ be the Hamiltonian function on $T^*\widetilde{M}$ obtained by pulling H back. Note that $\|\widetilde{H}\| = \|H\|$. By the inclusions between intersection points

$$\begin{aligned}\{(y, y') \in L \times L \mid \iota(y) = \phi_1^H \circ \iota(y')\} &\supset \left\{(y, y') \in L \times L \,\middle|\, \widetilde{\iota}(y) = \phi_1^{\widetilde{H}} \circ \widetilde{\iota}(y')\right\} \\ &= \left\{(y, y') \in L \times L \,\middle|\, (\widetilde{\iota} + \beta)(y) = \phi_1^{\widetilde{H}} \circ (\widetilde{\iota} + \beta)(y')\right\},\end{aligned} \tag{5.9}$$

Theorems 5.4 and 5.5 for ι is reduced to those for $\widetilde{\iota} + \beta$. □

Hereafter we assume the following.

Assumption 5.9. *An immersion $\iota\colon L \to T^*M$ is a strongly rational Lagrangian immersion satisfying Assumption 4.3, $\theta_\iota = \sigma_\iota$ and $r_\iota = e_\iota$.*

Notation 5.10. We sometimes write $\{P(x)\}_X$ as an abbreviation for $\{x \in X \mid P(x)\}$ if there is no fear of confusion.

Notation 5.11. For cones $C_1 \subset \Omega_+(M)_\theta$ and $C_2 \subset \Omega_+(P)_\theta$, define

$$C_1 \boxplus C_2 := \{(x, y, t_1 + t_2; \xi, \eta, \tau) \mid (x, t_1; \xi, \tau) \in C_1, (y, t_2; \eta, \tau) \in C_2\} \subset \Omega_+(M \times P)_\theta. \tag{5.10}$$

Notation 5.12. Let $a \in (0, r_\iota)$ and denote by j_a the open embedding $M \times (0, a) \times S^1_\theta \to M \times \mathbb{R} \times S^1_\theta$

Define

$$F_{(0,a)} := Rj_{a!} G_{(0,a)}, \tag{5.11}$$

$$F_{[0,a]} := Rj_{a*} G_{(0,a)}. \tag{5.12}$$

$\mathrm{Hom}(F_{(0,a)}, F_{[0,a]})$ is naturally isomorphic to $\mathrm{End}(G_{(0,a)})$ by the adjunction $Rj_{a!} \dashv {j_a}^!$ (or ${j_a}^{-1} \dashv Rj_{a*}$).

Moreover let $\phi^H\colon T^*M \times I \to T^*M$ be the Hamiltonian isotopy generated by a compactly supported Hamiltonian function $H\colon T^*M \times I \to \mathbb{R}$ and denote by $\Phi_1^H\colon \mathcal{D}^P(M)_\theta \to \mathcal{D}^P(M)_\theta$ the functor associated with ϕ_1^H. One sets

$$F^H_{[0,a]} := \Phi_1^H(F_{[0,a]}). \tag{5.13}$$

Notation 5.13. We define subsets of $T^*\mathbb{R}$, for $a > 0$,

$$\begin{aligned}\mathbf{c}(a) &:= \{(0; v) \mid -1 \le v \le 0\} \cup \{(u; v) \mid 0 \le u \le a, v = 0, -1\}, \\ \mathbf{d}(a) &:= \mathbf{c}(a) \cup \{(a; v) \mid v \ge -1\}, \\ \mathbf{q}(a) &:= \mathbf{c}(a) \cup \{(a; v) \mid v \le 0\}, \\ \mathbf{l}(a) &:= \{(a; v) \mid -1 < v < 0\}.\end{aligned}$$

We also define their "conifications", which are subsets of $\Omega_+(\mathbb{R})_\theta$, by

$$\begin{aligned}
\widehat{\mathbf{c}}(a) &:= \{(u, -uv; \tau v, \tau) \mid (u; v) \in \mathbf{c}(a)\},\\
\widehat{\mathbf{d}}(a) &:= \widehat{\mathbf{c}}(a) \cup \{(a, 0; v, \tau) \mid v > 0\} \cup \{(a, [a]; v, \tau) \mid v > -\tau\},\\
\widehat{\mathbf{q}}(a) &:= \widehat{\mathbf{c}}(a) \cup \{(a, 0; v, \tau) \mid v < 0\} \cup \{(a, [a]; v, \tau) \mid v < -\tau\},\\
\widehat{\mathbf{l}}(a) &:= \{(a, [a]; v, \tau) \mid -\tau < v < 0\}.
\end{aligned}$$

Note that there uniquely exist continuous maps $\Lambda \boxplus \widehat{\mathbf{q}}(a) \to \Lambda$ and $\Lambda \boxplus \widehat{\mathbf{d}}(a) \to \Lambda$ satisfying $(x, u, t; \xi, v, \tau) \mapsto (x, t'; \xi, \tau)$ for some t'.

Lemma 5.14. *The microsupports of $F_{(0,a)}$ and $F_{[0,a]}$ satisfy the following.*

(i) $\mathrm{SS}_+(F_{[0,a]}) \subset \Lambda \boxplus \widehat{\mathbf{q}}(a)$ *and* $\mathrm{SS}_+(F_{(0,a)}) \subset \Lambda \boxplus \widehat{\mathbf{d}}(a)$.

(ii) $\mathring{\mathrm{SS}}(F_{(0,a)}) \cap \{\tau = 0\}_{T^*(M\times\mathbb{R}\times S^1_\theta)} \subset \{u = a, \xi = 0, v \geq 0\}_{T^*(M\times\mathbb{R}\times S^1_\theta)}$, $\mathring{\mathrm{SS}}(F_{[0,a]}) \cap \{\tau = 0\}_{T^*(M\times\mathbb{R}\times S^1_\theta)} \subset \{u = a, \xi = 0, v \leq 0\}_{T^*(M\times\mathbb{R}\times S^1_\theta)}$. *Here $F_{(0,a)}$ and $F_{[0,a]}$ are regarded as objects of $\mathbf{D}^{\mathrm{b}}_{/[1]}(M \times \mathbb{R} \times S^1_\theta)$ by P_l.*

Proof. (By Proposition 2.6 and Theorem 4.4(1)(3).) (We regard $G_{(0,r)}$ as an object of $\mathbf{D}^{\mathrm{b}}$ by P_l.) ($\{P\}$ are abbreviations of $\{(x, u, t; \xi, v, \tau) \in T^*(M \times \mathbb{R} \times S^1_\theta) \mid P\}$)

$\Lambda_q \cup \Lambda_r \subset \{v = 0, -\tau\}, N^*(M \times (0, a) \times S^1_\theta) \subset \{\tau = 0\}$ and hence $\overline{\mathrm{SS}(G_{(0,a)})} \cap N^*(M \times (0, a) \times S^1_\theta)$ and $\overline{\mathrm{SS}(G_{(0,a)})} \cap N^*(M \times (0, a) \times S^1_\theta)^a$ are contained in the zero-section. By Proposition 2.6, $\mathrm{SS}(F_{(0,a)}) \subset \overline{\mathrm{SS}(G_{(0,a)})} + N^*(M \times (0, a) \times S^1_\theta)^a$ and $\mathrm{SS}(F_{[0,a]}) \subset \overline{\mathrm{SS}(G_{(0,a)})} + N^*(M \times (0, a) \times S^1_\theta)$ are satisfied. Fiberwise comuptations show

$$\begin{aligned}
\mathrm{SS}(F_{(0,a)}) \cap \{u = 0\} \subset &\{(x, t; \xi, \tau) \in \Lambda, v \leq 0\} \cup \{\tau = 0, \xi = 0, v \leq 0\},\\
\mathrm{SS}(F_{[0,a]}) \cap \{u = 0\} \subset &\{(x, t; \xi, \tau) \in \Lambda, v \geq -\tau\} \cup \{\tau = 0, \xi = 0, v \geq 0\},\\
\mathrm{SS}(F_{(0,a)}) \cap \{u = a\} \subset &\{(x, t; \xi, \tau) \in \Lambda, v \geq 0\} \cup \{(x, t - a; \xi, \tau) \in \Lambda, v \geq -\tau\}\\
&\cup \{\tau = 0, \xi = 0, v \geq 0\},\\
\mathrm{SS}(F_{[0,a]}) \cap \{u = a\} \subset &\{(x, t; \xi, \tau) \in \Lambda, v \leq -\tau\} \cup \{(x, t - a; \xi, \tau) \in \Lambda, v \leq -\tau\}\\
&\cup \{\tau = 0, \xi = 0, v \leq 0\}.
\end{aligned}$$

The cone of the natural map $F_{(0,a)} \to F_{[0,a]}$ is supported on $\{u = a\}$ since $F_{[0,a]}|_{\{u=0\}}$ is isomorphic to 0 by Theorem 4.4(3). Hence $\mathrm{SS}(F_{(0,a)}) \cap \{u = 0\} = \mathrm{SS}(F_{[0,a]}) \cap \{u = 0\}$ by the triangle inequality and

$$\mathring{\mathrm{SS}}(F_{(0,a)}) \cap \{u = 0\} = \mathring{\mathrm{SS}}(F_{[0,a]}) \cap \{u = 0\} \subset \{(x, t; \xi, \tau) \in \Lambda, -\tau \leq v \leq 0\} \tag{5.14}$$

by the above inclusions. These inclusions show $\mathrm{SS}(F_{[0,a]}) \cap \Omega_+ \subset \Lambda \boxplus \widehat{\mathbf{q}}(a)$, $\mathrm{SS}(F_{(0,a)}) \cap \Omega_+ \subset \Lambda \boxplus \widehat{\mathbf{d}}(a)$, $\mathring{\mathrm{SS}}(F_{(0,a)}) \cap \{\tau = 0\} \subset \{u = a, \xi = 0, v \geq 0\}$, $\mathring{\mathrm{SS}}(F_{[0,a]}) \cap \{\tau = 0\} \subset \{u = a, \xi = 0, v \leq 0\}$. □

From now on, we omit $_*$ from T_{c*}.

Proposition 5.15. *Let H be a Hamiltonian function.*

(i) *One has $d_{\mathcal{D}(M)_\theta}(F_{[0,a]}, F^H_{[0,a]}) \leq \|H\|$. In particular, for any $b > \|H\|$, there exist $c \in [0, b]$ and morphisms $\alpha\colon F_{[0,a]} \to T_c F^H_{[0,a]}, \beta\colon F^H_{[0,a]} \to T_{b-c} F_{[0,a]}$ such that $\tau_{0,b}\colon F_{[0,a]} \to T_b F_{[0,a]}$ is equal to $T_c\beta \circ \alpha$.*

(ii) *One has*

$$\pi(\mathring{\mathrm{SS}}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]}))) \subset \left\{ -c' \in \mathbb{R} \,\middle|\, \begin{array}{l} \exists (y,y') \in C(\iota, H), \\ c(y,y') \equiv c' \mod \theta \text{ or} \\ c(y,y') \equiv c' - a \mod \theta \end{array} \right\} \tag{5.15}$$

where ℓ is the quotient map $\mathbb{R} \to S^1_\theta$.

(iii) *If $b < a$, then $\tau_{0,b}\colon \mathrm{Hom}(F_{(0,a)}, F_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_b F_{[0,a]})$ is an isomorphism.*

Proof. (i) By Proposition 3.4 and Definition 3.3.

(ii) Let $T'_{c'}$ be the translation map $\Omega_+ \to \Omega_+$ or $\Omega_+(\mathbb{R})_\theta \to \Omega_+(\mathbb{R})_\theta$ which is the lift of $T_{c'}$. For $F_1, F_2 \in \mathcal{D}^{\mathbb{R}}(M)_\theta$ with $\mathrm{SS}(i^{-1}F_1) \cap \mathrm{SS}(F_2) \subset T^*_{M\times\mathbb{R}\times S^1_\theta}(M \times \mathbb{R} \times S^1_\theta)$,

$$\pi(\mathring{\mathrm{SS}}(\ell^! Rq_* \mathcal{H}om^\star(F_1, F_2)) \subset \{-c' \mid \mathrm{SS}_+(F_1) \cap T'_{c'}(\mathrm{SS}_+(F_2)) \neq \emptyset\} \tag{5.16}$$

by Proposition 2.5(ii) and Proposition 2.4. By Lemma 5.14 and a property of the composition, $\mathring{\mathrm{SS}}(i^{-1}F_{(0,a)}) \cap \mathring{\mathrm{SS}}(F^H_{[0,a]}) = \emptyset$. Hence again by Lemma 5.14,

$$\begin{aligned} \pi(\mathring{\mathrm{SS}}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]})) &\subset \{-c' \mid \mathrm{SS}_+(F_{(0,a)}) \cap T'_{c'}(\mathrm{SS}_+(F^H_{[0,a]})) \neq \emptyset\} && (5.17) \\ &\subset \{-c' \mid \Lambda \boxplus \widehat{\mathbf{d}}(a) \cap T'_{c'}(\Lambda' \boxplus \widehat{\mathbf{q}}(a)) \neq \emptyset\}. && (5.18) \end{aligned}$$

If $(x,u,t;\xi,\upsilon,\tau) \in \Lambda \boxplus \widehat{\mathbf{d}}(a) \cap T'_{c'}(\Lambda' \boxplus \widehat{\mathbf{q}}(a))$, then there exist $t_1, t_2, t_3, t_4 \in S^1_\theta$ with $t = t_1 + t_2 = t_3 + t_4 + c'$ such that $(x,t_1;\xi,\tau) \in \Lambda, (u,t_2;\upsilon,\tau) \in \widehat{\mathbf{d}}(a), (x,t_3;\xi,\tau) \in \Lambda', (u,t_4;\upsilon,\tau) \in \widehat{\mathbf{q}}(a)$. Then $(u,t_2;\upsilon,\tau) \in \widehat{\mathbf{d}}(a) \cap T'_{t_2-t_4}(\widehat{\mathbf{q}}(a))$. Since

$$\widehat{\mathbf{d}}(a) \cap T'_{c'}(\widehat{\mathbf{q}}(a)) = \begin{cases} \widehat{\mathbf{c}}(a) & (c' = 0) \\ \overline{\widehat{\mathbf{l}}(a)} & (c' = [a]) \\ \emptyset & (\text{otherwise}), \end{cases} \tag{5.19}$$

$t_2 - t_4 = 0, [a]$. There exist $y, y' \in L$ such that $\iota(y) = \iota'(y') = (x, \xi/\tau), t_1 = -f(y)$ and $t_3 = -f'(y')$ and then $c(y,y') = -t_3 + t_1 = c' - (t_2 - t_4)$.

(iii) By Proposition A.3,

$$\mathrm{Hom}(F_{(0,a)}, T_b F_{[0,a]}) \simeq H^* R\Gamma_{[-b,+\infty)}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, T_b F_{[0,a]})) \tag{5.20}$$

. By (ii),

$$\pi(\mathrm{SS}_+(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, T_b F_{[0,a]}))) \subset \left\{ -c' \in \mathbb{R} \,\middle|\, \begin{array}{l} \exists (y,y') \in C(\iota, 0), \\ c(y,y') \equiv c' \mod \theta \text{ or} \\ c(y,y') \equiv c' - a \mod \theta \end{array} \right\}. \tag{5.21}$$

Hence $[-b, 0) \cap \pi(\mathrm{SS}_+(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, T_b F_{[0,a]}))) = \emptyset$. By the microlocal Morse lemma (Proposition 2.2) for $\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, T_b F_{[0,a]})$ and the five lemma,

$$R\Gamma_{[0,+\infty)}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, T_b F_{[0,a]})) \to R\Gamma_{[-b,+\infty)}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, T_b F_{[0,a]})) \tag{5.22}$$

is an isomorphism. □

Now fix $a, b \in \mathbb{R}_{>0}$ such that $\|H\| < b < a$. Then by the above proposition, the isomorphism $\tau_{0,b}$ factors

$$\mathrm{Hom}(F_{(0,a)}, F_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_b F_{[0,a]}) \tag{5.23}$$

for some $c \in [0, b]$. We also fix this c in what follows.

Proposition 5.16. *Let H be a Hamiltonian function with $\|H\| < a$.*

(i) *Assume that $c' \in \mathbb{R}$ is not an accumulation point of $\pi(\mathrm{SS}_+(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]}))$. Let $d, d' \in \mathbb{R}$ satisfying*

  (1) $d < -c' \leq d'$ *and*

  (2) $[-d', -d] \cap \pi(\mathrm{SS}_+(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]})) \subset \{c'\}$.

  For a sufficiently small ε, define

$$W_{c'} := H^* R\Gamma_{[c', c'+\varepsilon)}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]}))_{c'} \tag{5.24}$$

  and

$$A_{c'} := \mathrm{Coker}(\mathrm{Hom}(F_{(0,a)}, T_d F^H_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_{d'} F^H_{[0,a]})), \tag{5.25}$$
$$B_{c'} := \mathrm{Ker}(\mathrm{Hom}(F_{(0,a)}, T_d F^H_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_{d'} F^H_{[0,a]})). \tag{5.26}$$

  Then the modules $W_{c'}, A_{c'}$ and $B_{c'}$ are independent of the choice of d, d', ε and canonically isomorphic to each other. Moreover, there is a short exact sequence of right $\mathrm{End}(G_{(0,a)})$-modules

$$0 \longrightarrow A_{c'} \longrightarrow W_{c'} \longrightarrow B_{c'} \longrightarrow 0. \tag{5.27}$$

(ii) *Assume that $\pi(\mathrm{SS}_+(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]})) \cap [-c-a, -c)$ is a finite set and let*

$$\{c_1, \dots, c_n\} = \pi(\mathrm{SS}_+(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]})) \cap [-c-a, -c) \tag{5.28}$$

  with $c_1 < \cdots < c_n$. Take $d_1, \dots, d_{n-1} \in \mathbb{R}$ satisfying

$$c_1 + c < -d_1 < c_2 + c < \cdots < -d_{n-1} < c_n + c \tag{5.29}$$

  and set $d_0 = a, d_n = 0$. Define

$$V_d := \mathrm{Im}(\mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_{c+d} F^H_{[0,a]})) \tag{5.30}$$

  for $d \in [0, a]$. Then for any $i = 0, \dots, n$ there exists a submodule $\widetilde{B}_{c_i}$ of B_{c_i} and an exact sequence of right $\mathrm{End}(G_{(0,a)})$-modules

$$0 \longrightarrow \widetilde{B}_{c_i} \longrightarrow V_{d_i} \longrightarrow V_{d_{i-1}} \longrightarrow 0. \tag{5.31}$$

(iii) *Assume that $\pi(\mathring{\mathrm{SS}}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]})) \cap [-c, -c+a)$ is a finite set and let*

$$\{c_{n+1}, \dots, c_{n+m}\} = \pi(\mathring{\mathrm{SS}}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]})) \cap [-c, -c+a) \tag{5.32}$$

  with $c_{n+1} < \cdots < c_{n+m}$. Take $d_1, \dots, d_{n-1} \in \mathbb{R}$ satisfying

$$c_{n+1} + c < -d_{n+1} < c_{n+2} < \cdots < -d_{n-1} < c_{n+m} + c \tag{5.33}$$

and set $d_n = 0, d_{n+m} = -a$. *Define*

$$V_d := \mathrm{Im}(\mathrm{Hom}(F_{(0,a)}, T_{c+d}F^H_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]})) \tag{5.34}$$

for $d \in [-a, 0]$. *Then for any* $i = n+1, \dots, n+m$ *there exists a quotient module of* A_{c_i} *and an exact sequence of right* $\mathrm{End}(G_{(0,a)})$*-modules*

$$0 \longrightarrow V_{d_i} \longrightarrow V_{d_{i-1}} \longrightarrow \widetilde{A}_{c_i} \longrightarrow 0. \tag{5.35}$$

(iv) $V_{d_0} = 0, V_{d_{n+m}} = 0$.

Proof. (i) Set $\mathcal{H} = Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]}) \in \mathcal{D}(\mathrm{pt})_\theta$. Then there is a distingushed triangle

$$R\Gamma_{[-d_i-c,+\infty)}(\mathbb{R}; \ell^! \mathcal{H}) \longrightarrow R\Gamma_{[-d_{i-1}-c,+\infty)}(\mathbb{R}; \ell^! \mathcal{H}) \longrightarrow R\Gamma_{[c_i,-d_i-c)}(\ell^! \mathcal{H})_c \xrightarrow{+1} \tag{5.36}$$

in $\mathbf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k})$, whenever $-d_{i-1} - c$ is not an accumulated point of $\pi(\mathring{\mathrm{SS}}(\ell^! \mathcal{H}))$. The cohomology of the distinguished triangle is isomorphic to

$$\mathrm{Hom}(F_{(0,a)}, T_{c+d_i}F^H_{[0,a]}) \longrightarrow \mathrm{Hom}(F_{(0,a)}, T_{c+d_{i-1}}F^H_{[0,a]}) \longrightarrow W_{c_i} \xrightarrow{+1}. \tag{5.37}$$

There exists a short exact sequence

$$0 \longrightarrow A_{c_i} \longrightarrow W_{c_i} \longrightarrow B_{c_i} \longrightarrow 0 \tag{5.38}$$

where $A_{c_i} = \mathrm{Coker}(\mathrm{Hom}(F_{(0,a)}, T_{c+d_i}F^H_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_{c+d_{i-1}}F^H_{[0,a]}))$ and $B_{c_i} = \mathrm{Ker}(\mathrm{Hom}(F_{(0,a)}, T_{c+d_i}F^H_{[0,a]}) \to \mathrm{Hom}(F_{(0,a)}, T_{c+d_{i-1}}F^H_{[0,a]}))$. Defining $\widetilde{B}_{c_i} := B_{c_i} \cap V_{d_i}$ gives (5.31).

The induced morphism $A_{c_i} \to \mathrm{Coker}(V_{d_i} \to V_{d_{i-1}})$ is surjective (and $\mathrm{Coker}(V_{d_i} \to V_{d_{i-1}})$ is isomorphic to a quotient module $\widetilde{A}_{c_i}$ of A_{c_i}).

This construction is natural with respect to $\mathcal{H}$ and hence we obtain left actions of $\mathrm{End}(\mathcal{H})$ and maps become $\mathrm{End}(\mathcal{H})$-equivariant. Through the natural ring homomorphism

$$\mathrm{End}(G_{(0,a)})^{\mathrm{op}} \to \mathrm{End}(F_{(0,a)})^{\mathrm{op}} \to \mathrm{End}(\mathcal{H}), \tag{5.39}$$

we obtain an exact sequence of right $\mathrm{End}(G_{(0,a)})$-modules.

(iv) We only give a proof for $V_{d_0} \simeq 0$ since the other is parallel. It is enough to show $\tau_{0,a}(G_{(0,a)}) = 0$ since $V_{d_0} \simeq \mathrm{Im}(\mathrm{Hom}(G_{(0,a)}, T_c G^H_{(0,a)}) \to \mathrm{Hom}(G_{(0,a)}, T_{c+a} G^H_{(0,a)}))$.

$\tau_{0,a}(G_{(0,a)}) = 0$ is equivalent to $\mathrm{Im}(\mathrm{Hom}(G_{(0,a)}, G_{(0,a)}) \to \mathrm{Hom}(G_{(0,a)}, T_a G_{(0,a)})) \simeq 0$.

$\mathrm{Im}(\mathrm{Hom}(G_{(0,a)}, G_{(0,a)}) \to \mathrm{Hom}(G_{(0,a)}, T_a G_{(0,a)}))$ is isomorphic to $\mathrm{Im}(\mathrm{Hom}(G_{(0,a)}, G_{(0,a)}) \to \mathrm{Hom}(G_{(0,a)}, T_{a+\varepsilon} G_{(0,a)}))$ for sufficiently small $\varepsilon > 0$ as in (ii). Hence it is deduced from $d(G_{(0,a)}, 0) \leq a$ stated in Theorem 4.4.

□

Let $\iota' := \phi^H_1 \circ \iota$ and $f' := f - h \colon L \to S^1_\theta$, where $h(y) := \int_0^1 (H_s - \alpha(X_s))(\phi^H_s(\iota(y)))ds$. One sets $C(\iota, H) := \{(y, y') \in L \times L \mid \iota(y) = \iota'(y')\}$ and $c(y, y') := f'(y') - f(y)$. For $y \in L$, $l(y) := \{(x, -f(y); \xi, \tau) \in \Lambda \mid (x; \xi/\tau) = \iota(y)\} \subset T^*(M \times S^1_\theta)$.

Proposition 5.17. *Let* H *be a Hamiltonian function and* $c' \in S^1_\theta$.

(i) *There is an isomorphism*

$$H^* R\Gamma_{[-c', -c'+\varepsilon)}(Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]}))_{-c'} \simeq H^* R\Gamma(\Omega_+; \mu hom(F_{(0,a)}, T_{c'} F^H_{[0,a]})). \tag{5.40}$$

(ii) *One has*

$$\operatorname{Supp}(\mu hom(F_{(0,a)}, T_{c'} F^H_{[0,a]})|_{\Omega_+}) \subset C_1(a,c') \cup \overline{C_2(a,c')}, \tag{5.41}$$

where $C_1(a,c') := \bigcup_{c(y,y')=c'} l(y) \boxplus \widehat{\mathbf{c}}(a)$ *and* $C_2(a,c') := \bigcup_{c(y,y')+a=c'} l(y) \boxplus \widehat{\mathbf{l}}(a)$.

Proof. (i) Similar to [Ike17, §4.3].

(ii) $\operatorname{Supp}(\mu hom(F_{(0,a)}, T_{c'} F^H_{[0,a]})|_{\Omega_+}) \subset \mathrm{SS}_+(F_{(0,a)}) \cap T'_{c'}(\mathrm{SS}_+(F^H_{[0,a]})) \subset \Lambda \boxplus \widehat{\mathbf{d}}(a) \cap T'_{c'}(\Lambda' \boxplus \widehat{\mathbf{q}}(a))$ by Proposition 2.10 and Lemma 5.14. $\Lambda \boxplus \widehat{\mathbf{d}}(a) \cap T'_{c'}(\Lambda' \boxplus \widehat{\mathbf{q}}(a)) = C_1(a,c') \cap \overline{C_2(a,c')}$ is checked in the proof of Proposition 5.15(ii).

□

Proposition 5.18. (i) $F_{(0,a)}$ *is simple along* $\Lambda \boxplus \widehat{\mathbf{d}}(a) \setminus \widehat{\mathbf{c}}(a)$ *and* $F_{[0,a]}$ *is simple along* $\Lambda \boxplus \widehat{\mathbf{q}}(a) \setminus \widehat{\mathbf{c}}(a)$.

(ii) *There exists an isomorphism* $\mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+} \simeq \mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{k}_{\Omega_+} & \longrightarrow & \mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)} \\ & \searrow^{\mathrm{id}^\mu_{F_{(0,a)}}} & \downarrow \\ & & \mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+} \end{array} \tag{5.42}$$

commutes.

In particular, $\operatorname{End}^\mu_{\Omega_+}(F_{(0,a)}) \simeq H^*(L)$ *(as* $\mathbf{k}$*-vector spaces).*

Moreover, $\circ_{F_{(0,a)},F_{(0,a)},F_{(0,a)}}$ *induces the cup product on* $H^*(L)$ *through this isomorphism.*

(iii) *There exists an isomorphism* $\mu hom(F_{(0,a)}, F_{[0,a]})|_{\Omega_+} \simeq \mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{c}}(a)}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)} & \longrightarrow & \mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{c}}(a)} \\ \downarrow & & \downarrow \\ \mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+} & \xrightarrow{\mu hom(F_{(0,a)},\eta)|_{\Omega_+}} & \mu hom(F_{(0,a)}, F_{[0,a]})|_{\Omega_+} \end{array} \tag{5.43}$$

commutes where $\eta\colon F_{(0,a)} \to F_{[0,a]}$ *is the unit morphism of the adjunction* $j_a^{-1} \dashv Rj_{a*}$ *and the morphism in the left column is the isomorphism given in (ii).*

In particular, $\operatorname{Hom}^\mu_{\Omega_+}(F_{(0,a)}, F_{[0,a]}) \simeq H^*(L)$ *(as* $\mathbf{k}$*-vector spaces).*

Moreover, $\circ_{F_{(0,a)},F_{(0,a)},F_{[0,a]}}$ *induces the usual right* $H^*(L)$*-module structure on* $H^*(L)$ *through this isomorphism and the isomorphism in (ii).*

Proof. (ii) By Proposition 2.10 and Lemma 5.14, $\mathrm{SS}(\mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+}) \subset -\mathbf{h}^{-1}(C(\Lambda \boxplus \widehat{\mathbf{d}}(a), \Lambda \boxplus \widehat{\mathbf{d}}(a)))$. On the other hand, $\mathrm{SS}(\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)}) \subset -\mathbf{h}^{-1}(C(\Lambda \boxplus \widehat{\mathbf{d}}(a), \Lambda \boxplus \widehat{\mathbf{d}}(a)))$.

Since $\operatorname{Supp}(\mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+}) \subset \Lambda \boxplus \widehat{\mathbf{d}}(a)$, $\mathrm{id}^\mu_{F_{(0,a)}} \colon \mathbf{k}_{\Omega_+} \to \mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+}$ factors through $\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)}$. Let $\mathcal{E}$ be the cone of $\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)} \to \mu hom(F_{(0,a)}, F_{(0,a)})$. By the triangle inequality, $\mathrm{SS}(\mathcal{E}) \subset -\mathbf{h}^{-1}(C(\Lambda \boxplus \widehat{\mathbf{d}}(a), \Lambda \boxplus \widehat{\mathbf{d}}(a))) \subset -\mathbf{h}^{-1}((d\rho)^{-1} C(\mathbf{d}(a), \mathbf{d}(a)))$.

Decompose $\widehat{\mathbf{d}}(a)$ into nine parts

$$\begin{aligned} &D_1 := \{(a,0;\upsilon,\tau) \mid \upsilon > 0\}, && D_2 := \{(a,0;0,\tau)\}, \\ &D_3 := \{(u,0;0,\tau) \mid 0 < u < a\}, && D_4 := \{(0,0;0,\tau)\}, \\ &D_5 := \{(0,0;\upsilon,\tau) \mid -\tau < \upsilon < 0\}, && D_6 := \{(0,0;-\tau,\tau)\}, \\ &D_7 := \{(u,[u];-\tau,\tau) \mid 0 < u < a\}, && D_8 := \{(a,[a];-\tau,\tau)\}, \\ &D_9 := \{(a,[a];\upsilon,\tau) \mid \upsilon > -\tau\}. \end{aligned}$$

Let $p\colon \Lambda \boxplus \widehat{\mathbf{d}}(a) \to \Lambda$ be the unique continuous map satisfying $p(x,u,t;\xi,\upsilon,\tau) = (x,t';\xi,\tau)$ for any $(x,u,t;\xi,\upsilon,\tau) \in \Lambda \boxplus \widehat{\mathbf{d}}(a)$ and some t'. Define $\Lambda_i := \Lambda \boxplus D_i$ and $\Omega_i :=$for $i = 1,\dots,9$. Let p_i be the projection $\Lambda_i \to \Lambda$ (restriction of p). (For even i, p_i is bijective.) For odd i, consider extensions of p_i's,

$$\begin{aligned} &q_3\colon \Omega_3 = \{0 < u < a, \upsilon = 0\}_{\Omega_+} \to \Omega_+(M)_\theta, (x,u,t;\xi,0,\tau) \mapsto (x,t;\xi,\tau), \\ &q_5\colon \Omega_5 = \{u = 0, -\tau < \upsilon < 0\}_{\Omega_+} \to \Omega_+(M)_\theta, (x,0,t;\xi,\upsilon,\tau) \mapsto (x,t;\xi,\tau) \\ &q_7\colon \Omega_7 = \{0 < u < a, \upsilon = -\tau\}_{\Omega_+} \to \Omega_+(M)_\theta, (x,u,t;\xi,-\tau,\tau) \mapsto (x,t-u;\xi,\tau), \\ &q_9\colon \Omega_9 = \{u = a, -\tau < \upsilon\}_{\Omega_+} \to \Omega_+(M)_\theta, (x,a,t;\xi,\upsilon,\tau) \mapsto (x,t-a;\xi,\tau). \end{aligned}$$

The image of $(q_i)_d$ contains $\mathrm{SS}(\mathcal{E}|_{\Omega_i})$ for each even i. We treat $i = 7$ case here for example. We write the conormal coordinates of $T^*\Omega_+$ by putting tildes on the coordinate functions of the base manifold Ω_+ and hence $(x,u,t,\xi,\upsilon,\tau;\tilde{x},\tilde{u},\tilde{t},\tilde{\xi},\tilde{\upsilon},\tilde{\tau})$ denotes a point of $T^*\Omega_+$. Let $i_7\colon \Omega_7 \to \Omega_+$ be the inclusion. It is enough to check $\mathrm{SS}(\mathcal{E}) \cap T^*\Omega_+|_{\Omega_7}$ is contained in $((i_7)_d)^{-1}(\mathrm{Im}(q_7)_d)$. A direct computation shows $((i_7)_d)^{-1}(\mathrm{Im}(q_7)_d) = \{0 < u < a, \upsilon = -\tau, \tilde{u} = -\tilde{t}\}_{T^*\Omega_+}$. On the other hand, $-\mathbf{h}^{-1}(C(\Lambda \boxplus \widehat{\mathbf{d}}(a), \Lambda \boxplus \widehat{\mathbf{d}}(a))) \cap T^*\Omega_+|_{\Omega_7}$ is the conormal bundle of Λ_7 and hence contained in $\{\tilde{u} = -\tilde{t}\}_{T^*\Omega_+}$. Hence the image of $(q_7)_d$ contains $\mathrm{SS}(\mathcal{E}|_{\Omega_7})$.

By Proposition 2.4(iii) and Theorem 4.4(4), there exists an $E_i \in \mathbf{D}^{\mathrm{b}}_{/[1]}(\Omega_+(M)_\theta)$ with $\mathrm{Supp}(E_i) \subset \Lambda$ satisfying $\mathcal{E}|_{\Omega_i} \simeq q_i^{-1}E_i$. We also define $E_i := \mathcal{E}|_{\Omega_+(M)_\theta \boxplus D_i}$ for even i. By Theorem 4.4(2), $\mathcal{E}|_{\Omega_+(M)_\theta \boxplus D_3}$ is 0.

On a neighborhood of Λ_2, $-\mathbf{h}^{-1}C(\Lambda \boxplus \widehat{\mathbf{d}}(a), \Lambda \boxplus \widehat{\mathbf{d}}(a))$ doesn't intersect to $\{\tilde{u}\tilde{\upsilon} < 0\}_{T^*\Omega_+}$. Using Lemma 2.7(ii) for $\phi = \pm(u + \frac{\upsilon}{\tau} - a)$, we obtain $E_1 \simeq E_2 \simeq E_3$ and moreover that $\widetilde{E}$ is of the form obtained by pulling back E_1 on this neighborhood. By similar arguments for Λ_4, Λ_6 and Λ_8, we get $\mathcal{E}|_{\Lambda \boxplus \widehat{\mathbf{d}}(a)} \simeq p^{-1}(E_3|_\Lambda) \simeq 0$.

Via the isomorophism $\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)} \simeq \mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+}$, $\mathrm{id}^\mu_{F_{(0,a)}}$ corresponds to $1 \in H^*(L) \simeq H^*R\Gamma(\Omega_+; \mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)})$. The composition morphism $\mu hom(F_{(0,a)}, F_{(0,a)})|^{\otimes 2}_{\Omega_+} \to \mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+}$ is determined as follows. A morphism $v \in \mathrm{Hom}((\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)})^{\otimes 2}, \mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{d}}(a)})$ is determined by the image $H^*(v)(1 \otimes 1) \in H^*(L)$ of $1 \otimes 1 \in H^*(L)^{\otimes 2}$. The composition morphism has to correspond to $1 \in H^*(L)$ by the unitality. This is the unique morphism which comes from a morphism of $\mathbf{D}^{\mathrm{b}}(\Omega_+)$.

Note alos that $\mu hom(F_{[0,a]}, F_{[0,a]})|_{\Omega_+} \simeq \mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{q}}(a)}$ is proved parallelly.

(i) The statement is a corollary of (ii) (and the parallel statement for $F_{[0,a]}$) and Theorem 4.4 (4).

(iii) $\mu hom(F_{(0,a)}, \eta)|_{\Omega_+}$ factors through $\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{c}}(a)}$. Since $\eta|_{\{u<a\}_{\Omega_+}}$ is an isomorphism, $\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{c}}(a)} \to \mu hom(F_{(0,a)}, F_{[0,a]})$ is also isomorphic on $\{u < a\}_{\Omega_+}$. The cone $\mathcal{E}'$ of $\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{c}}(a)} \to \mu hom(F_{(0,a)}, F_{[0,a]})|_{\Omega_+}$ is supported on $\{u = a\}_{\Omega_+}$. Since the microsupports of both $\mathbf{k}_{\Lambda \boxplus \widehat{\mathbf{c}}(a)}$ and $\mu hom(F_{(0,a)}, F_{[0,a]})|_{\Omega_+}$ are contained in $-\mathbf{h}^{-1}C(\Lambda \boxplus \widehat{\mathbf{q}}(a), \Lambda \boxplus \widehat{\mathbf{d}}(a))$, $\mathrm{SS}(\mathcal{E}')$ does not intersect to $\{\tilde{u} > 0\}_{T^*\Omega_+}$. These two properties requires $\mathcal{E}' \simeq 0$.

The composition morphism

$$\mu hom(F_{(0,a)}, F_{[0,a]})|_{\Omega_+} \otimes \mu hom(F_{(0,a)}, F_{(0,a)})|_{\Omega_+} \to \mu hom(F_{(0,a)}, F_{[0,a]})|_{\Omega_+}$$

is also determined by the unitality as in (ii).

□

Proposition 5.19. $\operatorname{End}(G_{(0,a)})$ *is isomorphic to* $H^*(L)$ *as a ring.*

Proof. By the functoriality of $m_{-,-}$ and Proposition 5.18, the ring homomorphism

$$\operatorname{End}(G_{(0,a)}) \xrightarrow{Rj_{a!}} \operatorname{End}(F_{(0,a)}) \xrightarrow{m_{F_{(0,a)},F_{(0,a)}}} \operatorname{End}^{\mu}_{\Omega_+}(F_{(0,a)}) \simeq H^*(L) \tag{5.44}$$

is obtained. We check this is a bijection.

Take $0 < \varepsilon < r - a$. There is an exact triangle of $\operatorname{End}(G_{(0,a)})$ modules

$$\operatorname{Hom}(F_{(0,a)}, T_{-\varepsilon}F_{[0,a]}) \to \operatorname{Hom}(F_{(0,a)}, F_{[0,a]}) \to H^*(R\Gamma_{[0,\varepsilon')}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F_{[0,a]}))_0) \to . \tag{5.45}$$

The second module is isomorphic to $\operatorname{End}(G_{(0,a)})$ and the third module is isomorphic to $\operatorname{Hom}^{\mu}(F_{(0,a)}, F_{[0,a]})$ by Proposition 5.17(i), which is isomorphic to $H^*(L)$ by Proposition 5.18(iii).

By the commutativity of the following diagram, it is enough to prove the first module is 0.

$$\begin{array}{ccc}
\operatorname{End}(G_{(0,a)}) & & \\
\downarrow{\scriptstyle R(j_a)_!} & \searrow & \\
\operatorname{End}(F_{(0,a)}) & \xrightarrow{\eta\circ -} & \operatorname{Hom}(F_{(0,a)}, F_{[0,a]}) \\
\downarrow{\scriptstyle m_{F_{(0,a)},F_{(0,a)}}} & & \downarrow{\scriptstyle m_{F_{(0,a)},F_{[0,a]}}} \\
\operatorname{End}^{\mu}_{\Omega_+}(F_{(0,a)}) & \longrightarrow & \operatorname{Hom}^{\mu}_{\Omega_+}(F_{(0,a)}, F_{[0,a]}) \\
\downarrow & & \downarrow \\
H^*(L) & \longrightarrow & H^*(L)
\end{array} \tag{5.46}$$

All the morphism in this diagram are right $\operatorname{End}(G_{(0,a)})$-module morphism and morphisms in the left column are ring homomorphism. Note that unlabeled arrows above are isomorphisms.

If $a < r/2$, we can choose $0 < \varepsilon_1, \varepsilon_2 < r - a$ so that $\varepsilon_2 - \varepsilon_1 > a$. The isomorphism $\operatorname{Hom}(F_{(0,a)}, T_{-\varepsilon_2}F_{[0,a]}) \to \operatorname{Hom}(F_{(0,a)}, T_{-\varepsilon_1}F_{[0,a]})$ is induced by $\tau_{-\varepsilon_2,\varepsilon_2-\varepsilon_1}(G)$ and this is 0 since $d(G_{(0,a)}, 0) \leq a$ by Theorem 4.4.

For the remained case, consider an object $\mathcal{H} \in \mathcal{D}^{(0,a')}(\mathrm{pt})_\theta$ on such that $\mathcal{H}|_{S^1_\theta \times \{a\}} \simeq Rq_{S^1_\theta *} \mathcal{H}om^\star(F_{(0,a)}, F_{[0,a]})$ for any $a \in (0, a')$. This $\mathcal{H}$ is constructed as follows...

Let $j\colon M \times D_{a'} \times S^1_\theta \to M \times \mathbb{R} \times (0, a') \times S^1_\theta$ be the inclusion. Set $\mathcal{F}_1 := Rj_! \mathcal{G}, \mathcal{F}_2 := Rj_* \mathcal{G}$ and $\mathcal{H} := q_{(0,a') \times S^1_\theta} \mathcal{H}om^\star(\mathcal{F}_1, \mathcal{F}_2)$ where $q_{(0,a') \times S^1_\theta}$ is the projection to $(0, a') \times S^1_\theta$.

$\mathcal{H}|_{\{-\varepsilon\} \times S^1_\theta}$ is locally constant for $0 < \varepsilon < r - a'$. This shows $(\operatorname{Hom}(F_{(0,a)}, T_{-\varepsilon}F_{[0,a]}))_{a \in (0,a')}$ are isomorphic to each other and hence they are 0.

□

5.2.1 Betti number estimate: Proof of Theorem 5.4

In this subsection, we assume the following.

Assumption 5.20. *The strongly rational Lagrangian immersion $\iota\colon L \to T^*M$ and the Hamiltonian function $H\colon T^*M \times I \to \mathbb{R}$ satisfy*

(1) $\|H\| < r_\iota$,

(2) ι *and* $\phi_1^H \circ \iota$ *intersect transversally.*

Note that under the assumption, $\pi(\mathrm{SS}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]}) \cap \Gamma_{dt}) \cap [-c-a, -c+a)$ is a finite set.

Lemma 5.21. *Let*

$$\{c_1, \dots, c_{n+m}\} = \pi(\mathrm{SS}(\ell^! Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]}) \cap \Gamma_{dt}) \cap [-c-a, -c+a), \tag{5.47}$$

with $c_1 < \dots < c_n < -c \le c_{n+1} < \dots c_{n+m}$. For $i = 1, \dots, n+m$, define W_{c_i}, A_{c_i}, and B_{c_i} as in Proposition 5.16(i). For $t \in S^1_\theta$, take any $\tilde{t} \in \ell^{-1}(t)$ and define $W_t := W_{\tilde{t}}, A_t := A_{\tilde{t}}$, and $B_t := B_{\tilde{t}}$. Then

$$\begin{aligned} \sum_{t \in S^1_\theta} \dim B_t &\ge \sum_{i=1}^{n} \dim B_{c_i} \ge \dim H^*(L), \\ \sum_{t \in S^1_\theta} \dim A_t &\ge \sum_{i=n+1}^{n+m} \dim A_{c_i} \ge \dim H^*(L). \end{aligned} \tag{5.48}$$

In particular,

$$\sum_{t \in S^1_\theta} \dim W_t \ge 2 \dim H^*(L). \tag{5.49}$$

Proof. Since the composite (5.23) is an isomorphism and $\mathrm{Hom}(F_{(0,a)}, F_{[0,a]}) \simeq \mathrm{End}(G_{(0,r_\iota)})$ contains $H^*(L)$ as a submodule, we have

$$\dim \mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]}) \ge \dim H^*(L). \tag{5.50}$$

By Proposition 5.16(iii), noticing that $V_{d_0} \simeq 0$ and $V_{d_n} \simeq \mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]})$, we get

$$\sum_i^n \dim B_{c_i} \ge \sum_i^n \dim \widetilde{B}_{c_i} = \dim \mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]}). \tag{5.51}$$

By Proposition 5.16(iv), noticing that $V_{d_{n+m}} \simeq 0$ and $V_{d_n} \simeq \mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]})$, we get

$$\sum_{i=n+1}^{n+m} \dim A_{c_i} \ge \dim \mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]}). \tag{5.52}$$

By Proposition 5.16(ii), $\dim B_t + \dim A_t = \dim W_t$ and hence

$$\sum_{t \in S^1_\theta} \dim B_t + \sum_{t \in S^1_\theta} \dim A_t = \sum_{t \in S^1_\theta} \dim W_t. \tag{5.53}$$

Combining these inequalities, we obtain the result. □

Proposition 5.22. *One has*

$$\mu hom(F_{(0,a)}, T_{c*}F^H_{[0,a]})|_{\Omega_+} \simeq \bigoplus_{c(y,y')=c} \mathbf{k}_{l(y)\boxplus\widehat{\mathbf{c}}(a)} \oplus \bigoplus_{c(y,y')+a=c} \mathbf{k}_{l(y)\boxplus\widehat{\mathbf{l}}(a)}. \tag{5.54}$$

Proof. We use the same notation to Proposition 5.17 and the proof of Proposition 5.18. We additionally put $D_{11} := \widehat{\mathbf{l}}(a), D_{10} := \{(a,[a];0,\tau)\}$, $q_{11}\colon \Omega_{11} = \{u=a, -\tau < \upsilon < 0\} \to \Omega_+(M)_\theta, (x,a,t;\xi,\upsilon,\tau) \mapsto (x,t-a;\xi,\tau)$ and $\mathcal{F} := \mu hom(F_{(0,a)}, F^H_{[0,a]})|_{\Omega_+}$.

By Proposition 5.17(ii), $\mathrm{Supp}(\mathcal{F}) \subset C_1(a,c) \cup \overline{C_2(a,c)}$. Since $C_1(a,c)$ and $\overline{C_2(a,c)}$ are disjoint, $\mathcal{F}$ admits a direct sum decomposition $\mathcal{F} \simeq \mathcal{F}' \oplus \mathcal{F}''$ with $\mathrm{Supp}(\mathcal{F}') \subset C_1(a,c)$ and $\mathcal{F}'' \subset \overline{C_2(a,c)}$.

By Proposition 2.10 and Lemma 5.14, $\mathrm{SS}(\mathcal{F}) \subset -\mathbf{h}^{-1}(C(\Lambda \boxplus \widehat{\mathbf{d}}(a), \Lambda' \boxplus \widehat{\mathbf{q}}(a))) \subset -\mathbf{h}^{-1}((d\rho)^{-1}C(\mathbf{d}(a),\mathbf{q}(a)))$.

The image of $(q_i)_d$ contains $\mathrm{SS}(\mathcal{F}'|_{\Omega_i})$for each odd i. By Proposition 2.4(iii), there exists a locally tame $F'_i \in \mathbf{D}^{\mathrm{b}}_{/[1]}(\Omega_+(M)_\theta)$ with $\mathrm{Supp}(F'_i) \subset \Lambda$ satisfying $\mathcal{F}'|_{\Omega_i} \simeq q_i^{-1}F'_i$. We also define $F'_i := \mathcal{F}'|_{\Omega_+(M)_\theta \boxplus D_i}$ for odd i.

By Theorem 4.4 and Lemma 2.13, $F'_3 \simeq \bigoplus_{c(y,y')=c} \mathbf{k}_{l(y)\boxplus D_3}$.

On a neighborhood of Λ_2, $-\mathbf{h}^{-1}C(\Lambda \boxplus \widehat{\mathbf{d}}(a), \Lambda \boxplus \widehat{\mathbf{q}}(a))$ doesn't intersect to $\{\tilde{u}\tilde{\upsilon} < 0\}$. Using Lemma 2.7(ii) for $\phi = u - \frac{\upsilon}{\tau} - a$, we obtain $F'_2 \simeq F'_3$ and moreover that $\mathcal{F}'|_{\Lambda\boxplus\widehat{\mathbf{c}}(a)}$ is of the form obtained by pulling back F'_2 on this neighborhood. By similar arguments for Λ_4, Λ_6 and Λ_8, we get $\mathcal{F}' \simeq \bigoplus_{c(y,y')=c} \mathbf{k}_{l(y)\boxplus\widehat{\mathbf{c}}(a)}$.

By Proposition 5.18(i) and Lemma 2.13, $\mathcal{F}''|_{\Omega_{11}} \simeq \bigoplus_{c(y,y')+a=c} \mathbf{k}_{l(y)\boxplus\widehat{\mathbf{l}}(a)}|_{\Omega_{11}}$. By Lemma 2.7(i) for $\phi = u - \frac{\upsilon}{\tau} - a, -(u + \frac{\upsilon}{\tau} - a) - 1$, $\mathcal{F}'' \simeq \bigoplus_{c(y,y')+a=c} \mathbf{k}_{l(y)\boxplus\widehat{\mathbf{l}}(a)}$. Hence we obtain the assertion.

□

Proof of Theorem 5.4. By Proposition 5.17(i) and Proposition 5.22, we have

$$\begin{aligned} \dim W_t = {} & \dim H^* R\Gamma(\Omega_+; \mu hom(F_{(0,a)}, T_{-c*}F^H_{[0,a]})) \\ & = \#\{(y,y') \in C(\iota,H) \mid c(y,y') = -t\} \\ & \quad + \#\{(y,y') \in C(\iota,H) \mid c(y,y') = -t-a\}. \end{aligned} \tag{5.55}$$

Hence, we get $\sum_{t\in S^1_\theta} \dim W_t = 2\#C(\iota,H)$. Combining this with Lemma 5.21, we obtain the result. □

5.2.2 Cup-length estimate: Proof of Theorem 5.5

First we introduce the algebraic counterpart of cup-length and study some properties.

Definition 5.23. Let R be an associative algebra[3] over $\mathbf{k}$. For a right R-module A, define

$$\mathrm{cl}_R(A) := \inf\left\{ k-1 \,\middle|\, \begin{array}{l} k \in \mathbb{Z}_{\geq 0}, \forall (r_i)_i \in R^k, \forall a_0 \in A, \\ a_0 \cdot r_1 \cdots r_k = 0 \end{array} \right\} \in \mathbf{Z}_{\geq -1} \cup \{\infty\}. \tag{5.56}$$

Note that

(i) $\mathrm{cl}_R(A) = -1$ if and only if $A = 0$.

(ii) $\mathrm{cl}_R(A) = 0$ if and only if $A \neq 0$ and $ar = 0$ for any $a \in A$ and any $r \in R$.

[3] not necessarily commutative nor unital

If there is no risk of confusion, we simply write $\mathrm{cl}(A)$ for $\mathrm{cl}_R(A)$.

Lemma 5.24. *For an exact sequence $A \to B \to C$ of right R-modules, one has* $\mathrm{cl}(B) \le \mathrm{cl}(A) + \mathrm{cl}(C) + 1$.

Lemma 5.25. *Let R' be a non-zero unital ring and $R \to R'$ is a ring homomorphism. Assume that the action of R on A is factor as $R \to R' \to \mathrm{End}(A)^{op}$. Then $\mathrm{cl}_R(A) \le \mathrm{cl}_R(R')$.*

Let X be a topological space. We define the ring $R_X := \bigoplus_{i \ge 1} H^i(X; \mathbf{k})$ equipped with the cup product and $\mathrm{cl}(X) := \mathrm{cl}_{R_X}(H^*(X; \mathbf{k}))$. The number $\mathrm{cl}(X) \in \mathbf{Z}_{\ge -1} \cup \{\infty\}$ is called the *cup-length* of X.

Now we start the proof of Theorem 5.5. We assume the following.

Assumption 5.26. *The Hamiltonian function H satisfies $\|H\| < \min(r_\iota, \theta_\iota/2)$. Let $a \in \mathbb{R}$ satisfying $\|H\| < a < \min(r_\iota, \theta_\iota/2)$.*

From now on, until the end of this subsection, set $R := \bigoplus_{i=1}^{\dim L} H^i(L)$. Note that $H^*(L)$ is isomorphic to the unitization of R, since L is connected.

Proposition 5.27. *Assume that $\pi(\mathring{\mathrm{SS}}(Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]})) \cap [-a-c, -c)$ is a finite set and let*

$$\{c_1, \dots, c_n\} = \pi(\mathring{\mathrm{SS}}(Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]})) \cap [-a-c, -c) \tag{5.57}$$

with $c_1 < \cdots < c_n$. For $i = 1, \dots, n$, set

$$W_{c_i} := H^* R\Gamma_{[c_i, c_i+\varepsilon)}(Rq_* \mathcal{H}om^\star(F_{(0,a)}, F^H_{[0,a]}))_{c_i}. \tag{5.58}$$

Then

$$\sum_{i=1}^{n} \mathrm{cl}(W_{c_i}) + n \ge \mathrm{cl}(H^*(L)) + 1. \tag{5.59}$$

Proof. Since the composite (5.23) is an isomorphism and $\mathrm{Hom}(F_{(0,a)}, F_{[0,a]})$ is isomorphic to $H^*(L)$ as an R-module by Proposition 5.19, we have

$$\mathrm{cl}(\mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]})) \ge \mathrm{cl}(H^*(L)). \tag{5.60}$$

Note also the exact sequence (5.31) is equipped with R-module structure. Applying Lemma 5.24 to the exact sequence (5.31), we have

$$\mathrm{cl}(V_{d_i}) \le \mathrm{cl}(W_{c_i}) + \mathrm{cl}(V_{d_{i-1}}) + 1. \tag{5.61}$$

Noticing that $V_{d_0} \simeq 0$ and $V_{d_n} \simeq \mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]})$, by induction we obtain

$$\sum_{i}^{n} \mathrm{cl}(W_{c_i}) + n \ge \mathrm{cl}(\mathrm{Hom}(F_{(0,a)}, T_c F^H_{[0,a]})) + 1, \tag{5.62}$$

which proves the result. □

It remains to see the action of R on each W_{c_i}, which is isomorphic to the section of μhom (Proposition 5.17).

Proposition 5.28. *Let U be an open subset of T^*M. Then*

$$\mathrm{cl}_R\left(H^*R\Gamma(\rho^{-1}(U);\mu hom(F_{(0,a)},F^H_{[0,a]})|_{\rho^{-1}(U)})\right)\le \mathrm{cl}_R(H^*(\iota^{-1}(U))). \tag{5.63}$$

Proof. The action of R on $H^*R\Gamma(\rho^{-1}(U);\mu hom(F_{(0,a)},F^H_{[0,a]})|_{\rho^{-1}(U)})$ factors an action of $H^*R\Gamma(\rho^{-1}(U);\mu hom(F_{(0,a)},F_{(0,a)})|_{\rho^{-1}(U)})$. $H^*R\Gamma(\rho^{-1}(U);\mu hom(F_{(0,a)},F_{(0,a)})|_{\rho^{-1}(U)})$ is isomorphic to $H^*R\Gamma(\rho^{-1}(U);\mathbf{k}_{\Lambda\boxplus d(a)\cap\rho^{-1}(U)})\simeq H^*(\iota^{-1}(U))$ by . Hence the assertion is obtained by Lemma 5.25.

□

Proof of Theorem 5.5. We may assume that $C(\iota,H)$ is discrete and let $c_1,\dots,c_n$ be as in Proposition 5.27. Since $a<\theta/2$, for any $(y,y')\in C(\iota,H)$, the set

$$\left\{c'\in\mathbb{R}\;\middle|\;\begin{array}{l}c(y,y')\equiv -c'\mod\theta\text{ or}\\ c(y,y')\equiv -c'-a\mod\theta\end{array}\right\}\cap[-a-c,-c) \tag{5.64}$$

is a singleton or empty. Hence, we have $\#C(\iota,H)\ge n$.

Let c' be any of $c_1,\dots,c_n$. Let $(y_1,y'_1),\dots,(y_k,y'_k)\in C(\iota,H)$ satisfying $c(y,y')\equiv -c'$ mod θ or $c(y,y')\equiv -c'-a \mod \theta$. Set $p_j:=\iota(y_j)=\iota'(y'_j)$ and take a sufficiently small contractible open neighborhood U_j of p_j in T^*M and set $U:=\bigcup_{j=1}^k U_j$ Then, by Proposition 5.17(i), we obtain

$$\begin{aligned}W_{c'}&\simeq H^*R\Gamma(\Omega_+;\mu hom(F_{(0,a)},T_{-c'*}F^H_{[0,a]}))\\ &\simeq H^*R\Gamma(\rho^{-1}(U);\mu hom(F_{(0,a)},T_{-c'*}F^H_{[0,a]})|_{\rho^{-1}(U)}).\end{aligned} \tag{5.65}$$

Therefore, by Proposition 5.28, we have $\mathrm{cl}(W_{c'})\le\mathrm{cl}(H^*(\iota^{-1}(U)))=0$. Thus, by Proposition 5.27, we obtain the result. □

A Parametrized and circled Tamarkin category

In this section, we give more detailed definitions and proofs in Section 3. We continue to use the notation given in Section 3.

A.1 Separation theorem

First, noticing that $\ell\colon M\times P\times\mathbb{R}\to M\times P\times S^1_\theta$ is a covering map, we obtain the following.

Lemma A.1. (i) *Let $G\in\mathbf{D}^{\mathrm{b}}_{/[1]}(M\times P\times S^1_\theta)$. If $\ell^!G\simeq 0$ then $G\simeq 0$.*

(ii) *The functors $\ell^!\colon\mathbf{D}^{\mathrm{b}}_{/[1]}(M\times P\times S^1_\theta)\to\mathbf{D}^{\mathrm{b}}_{/[1]}(M\times P\times\mathbb{R})$ is conservative. That is, for any morphism f in $\mathbf{D}^{\mathrm{b}}_{/[1]}(M\times P\times S^1_\theta)$, $\ell^!f$ is an isomorphism if and only if f is an isomorphism.*

The proper base change and the projection formula prove the following.

Lemma A.2. *For $F,G\in\mathbf{D}^{\mathrm{b}}_{/[1]}(M\times P\times\mathbb{R})$, there is an natural isomorphism*

$$R\ell_!F\star R\ell_!G\simeq R\ell_!(F\star G) \tag{A.1}$$

where $\star$ in the right hand side is the star product $\star$ for the case $\theta=0$.

As in [GS14], using Lemmas A.1 and A.2 additionally, one can show the equivalence of categories (recall that we have set $\Omega_+ := \{\tau > 0\} \subset T^*(M \times S^1_\theta)$)

$$P_l := R\ell_! \mathbf{k}_{M\times P\times[0,+\infty)} \star (*) \colon \mathbf{D}^{\mathrm{b}}_{/[1]}(M \times P \times S^1_\theta; \Omega_+) \xrightarrow{\sim} {}^\perp\mathbf{D}^{\mathrm{b}}_{/[1],\{\tau\le 0\}}(M \times P \times S^1_\theta),$$
$$P_r := \mathcal{H}om^\star(R\ell_! \mathbf{k}_{M\times P\times[0,+\infty)}, *) \colon \mathbf{D}^{\mathrm{b}}_{/[1]}(M \times P \times S^1_\theta; \Omega_+) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}_{/[1],\{\tau\le 0\}}(M \times P \times S^1_\theta)^\perp.$$

For $F \in \mathcal{D}^P(M)$, we take the canonical representative $P_l(F) \in {}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M \times P \times \mathbb{R})$ unless otherwise specified. For a compact subset A of T^*M and $F \in \mathcal{D}^P_A(M)$, the canonical representative $P_l(F) \in {}^\perp\mathbf{D}^{\mathrm{b}}_{\{\tau\le 0\}}(M \times P \times \mathbb{R})$ satisfies $\mathrm{SS}(P_l(F)) \subset \overline{\rho^{-1}(A)}$. The support of an object $F \in \mathcal{D}^P(M)$ is that of $P_l(F)$. One can show that the functor $\mathcal{H}om^\star$ induces an internal Hom functor $\mathcal{H}om^\star \colon \mathcal{D}^P(M)^{\mathrm{op}} \times \mathcal{D}^P(M) \to \mathcal{D}^P(M)$ (see [AI17] for the details).

Proposition A.3 (cf. [GS14, Lem. 4.18])**.** *For $F, G \in \mathcal{D}^P(M)_\theta$, there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}^P(M)_\theta}(F, G) \simeq H^* R\Gamma_{M\times[0,+\infty)}(M \times \mathbb{R}; \mathcal{H}om^\star(F, G)). \tag{A.2}$$

The following is a slight generalization of Tamarkin's separation theorem. Let $q_{S^1_\theta}$ denote the projection $M \times P \times S^1_\theta \to S^1_\theta$.

Proposition A.4. *Let A and B be compact subsets of T^*M. Let moreover $F \in \mathcal{D}^P_A(M)_\theta$ and $G \in \mathcal{D}^P_B(M)_\theta$. Assume*

(1) $A \cap B = \emptyset$,

(2) $q_{S^1_\theta}$ *is proper on* $\mathrm{Supp}(F) \cup \mathrm{Supp}(G)$.

Then one has $Rq_{S^1_\theta *} \mathcal{H}om^\star(F, G) \simeq 0$.

A.2 Sheaf quantization of Hamiltonian isotopies

In this subsection, we briefly recall the existence theorem of sheaf quantizations of Hamiltonian isotopies due to Guillermou–Kashiwara–Schapira [GKS12], with a slight modification so that it fits our setting. See [GKS12, Subsection A.3] for more details.

Let I be an open interval containing the closed interval $[0, 1]$. Let $H \colon T^*M \times I \to \mathbb{R}$ be a compactly supported Hamiltonian function and denote by $\phi^H : T^*M \times I \to T^*M$ the Hamiltonian isotopy generated by H. We consider conification of ϕ^H. Define $\widehat{H} \colon T^*M \times \mathring{T}^*S^1_\theta \times I \to \mathbb{R}$ by $\widehat{H}_s(x, t; \xi, \tau) := \tau \cdot H_s(x; \xi/\tau)$. Note that $\widehat{H}$ is homogeneous of degree 1, that is, $\widehat{H}_s(x, t; c\xi, c\tau) = c \cdot \widehat{H}_s(x, t; \xi, \tau)$ for any $c \in \mathbb{R}_{>0}$. The Hamiltonian isotopy $\widehat{\phi} \colon T^*M \times \mathring{T}^*S^1_\theta \times I \to T^*M \times \mathring{T}^*S^1_\theta$ associated with $\widehat{H}$ makes the following diagram commute (recall that we have set $\rho \colon \Omega_+ \to T^*M, (x, t; \xi, \tau) \mapsto (x; \xi/\tau)$):

$$\begin{array}{ccc} \Omega_+ \times I & \xrightarrow{\widehat{\phi}} & \Omega_+ \\ {\scriptstyle \rho\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \rho} \\ T^*M \times I & \xrightarrow[\phi^H]{} & T^*M. \end{array} \tag{A.3}$$

Moreover there exists a C^∞-function $u \colon T^*M \times I \to \mathbb{R}$ such that

$$\widehat{\phi}_s(x, t; \xi, \tau) = (x', t + u_s(x; \xi/\tau); \xi', \tau), \tag{A.4}$$

where $(x';\xi'/\tau) = \phi_s^H(x;\xi/\tau)$. By construction, $\widehat{\phi}$ is a homogeneous Hamiltonian isotopy: $\widehat{\phi}_s(x,t;c\xi,c\tau) = c\cdot\widehat{\phi}_s(x,t;\xi,\tau)$ for any $c \in \mathbb{R}_{>0}$. We define a conic Lagrangian submanifold $\Lambda_{\widehat{\phi}} \subset T^*M \times \mathring{T}^*S^1_\theta \times T^*M \times \mathring{T}^*S^1_\theta \times T^*I$ by

$$\Lambda_{\widehat{\phi}} := \left\{ \left(\widehat{\phi}_s(x,t;\xi,\tau),(x,t;-\xi,-\tau),(s;-\widehat{H}_s \circ \widehat{\phi}_s(x,t;\xi,\tau))\right) \,\middle|\, \begin{array}{c} (x;\xi) \in T^*M, \\ (t;\tau) \in \mathring{T}^*S^1_\theta, \\ s \in I \end{array} \right\}. \tag{A.5}$$

By construction, we have

$$\widehat{H_s} \circ \widehat{\phi}_s(x,t;\xi,\tau) = \tau \cdot (H_s \circ \phi_s^H(x;\xi/\tau)). \tag{A.6}$$

Note also that

$$\begin{aligned} \Lambda_{\widehat{\phi}} \circ T^*_sI = &\left\{ \left(\widehat{\phi}_s(x,t;\xi,\tau),(x,t;-\xi,-\tau)\right) \,\middle|\, (x,t;\xi,\tau) \in T^*M \times \mathring{T}^*S^1_\theta \right\} \\ &\subset T^*M \times \mathring{T}^*S^1_\theta \times T^*M \times \mathring{T}^*S^1_\theta \end{aligned} \tag{A.7}$$

for any $s \in I$ (see (2.7) for the definition of $A \circ B$). The following was essentially proved by Guillermou–Kashiwara–Schapira [GKS12].

Theorem A.5 (cf. [GKS12, Thm. 4.3])**.** *In the preceding situation, there exists a unique object $K^H \in \mathbf{D}^{\mathrm{b}}(M \times S^1_\theta \times M \times S^1_\theta \times I)$ satisfying the following conditions:*

(1) $\mathring{\mathrm{SS}}(K^H) \subset \Lambda_{\widehat{\phi}}$,

(2) $K^H|_{M\times S^1_\theta \times M \times S^1_\theta \times\{0\}} \simeq \mathbf{k}_{\Delta_{M\times S^1_\theta}}$, *where $\Delta_{M\times S^1_\theta}$ is the diagonal of $M \times S^1_\theta \times M \times S^1_\theta$.*

Moreover both projections $\mathrm{Supp}(K^H) \to M \times S^1_\theta \times I$ *are proper.*

The object K^H is called the sheaf quantization of $\widehat{\phi}$ or associated with ϕ^H.

A.3 Hamiltonian deformation of sheaves and translation distance

In this subsection, we recall the detail of the translation distance and give the outline of the proof of Proposition 3.4.

Let $F \in \mathcal{D}^P(M)_\theta$. Then the canonical morphism $R\ell_!\mathbf{k}_{M\times P\times[0,+\infty)} \star F \to F$ is an isomorphism. Moreover, for any $c \in \mathbb{R}$, we find that $T_{c*}(R\ell_!\mathbf{k}_{M\times P\times[0,+\infty)} \star F) \simeq R\ell_!\mathbf{k}_{M\times P\times[c,+\infty)} \star F$. Hence, for any $c,d \in \mathbb{R}$ with $c \leq d$, the canonical morphism $\mathbf{k}_{M\times P\times[c,+\infty)} \to \mathbf{k}_{M\times P\times[d,+\infty)}$ induces a morphism in $\mathcal{D}^P(M)_\theta$:

$$\tau_{c,d}(F)\colon T_{c*}F \longrightarrow T_{d*}F. \tag{A.8}$$

Using the morphism, we define the translation distance as in Definition 3.3.

Lemma A.6 (cf. [GS14, Prop. 6.9])**.** *Let $\mathcal{H} \in \mathbf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(M \times P \times S^1_\theta \times I)$ and $s_1 < s_2$ be in I. Denote by $q\colon M \times P \times S^1_\theta \times I \to M \times P \times S^1_\theta$ the projection. Assume that there exist $a,b,r \in \mathbb{R}_{>0}$ satisfying*

$$\mathrm{SS}(\mathcal{H}) \cap \pi^{-1}(M \times P \times S^1_\theta \times (s_1 - r, s_2 + r)) \subset T^*(M \times P) \times (S^1_\theta \times I) \times \gamma_{a,b}, \tag{A.9}$$

where $\gamma_{a,b} := \{(\tau,\sigma) \in \mathbb{R}^2 \mid -a\tau \leq \sigma \leq b\tau\} \subset \mathbb{R}^2$. Then

(i) $d_{\mathcal{D}^P(M)_\theta}(Rq_*(\mathcal{H}_{M\times P\times S^1_\theta\times[s_1,s_2)}),0) \leq a(s_2 - s_1)$,

(ii) $d_{\mathcal{D}^P(M)_\theta}(Rq_*(\mathcal{H}_{M\times P\times S^1_\theta\times(s_1,s_2]})),0) \le b(s_2-s_1)$,

(iii) $d_{\mathcal{D}^P(M)_\theta}(\mathcal{H}|_{M\times P\times S^1_\theta\times\{s_1\}},\mathcal{H}|_{M\times P\times S^1_\theta\times\{s_2\}}) \le (a+b)(s_2-s_1)$.

Outline of the proof. One can prove (i) and (ii) similarly to that of [AI17, Prop. 4.3], using Lemma A.7 below instead of the usual microlocal cut-off lemma. Similarly to [AI17, Lem. 4.14], we can show that if $F \longrightarrow G \longrightarrow H \xrightarrow{+1}$ is a distinguished triangle in $\mathbf{D}^{\mathrm{b}}_{\{\tau\ge 0\}}(M\times P\times S^1_\theta\times I)$ and $d_{\mathcal{D}^P(M)_\theta}(F,0)\le c$ with $c\in\mathbb{R}_{\ge 0}$, then $d_{\mathcal{D}^P(M)_\theta}(G,H)\le c$. Hence, applying it to the distinguished triangles

$$\begin{aligned} &Rq_*(\mathcal{H}_{M\times P\times S^1_\theta\times(s_1,s_2]}) \longrightarrow Rq_*(\mathcal{H}_{MP\times S^1_\theta\times[s_1,s_2]}) \longrightarrow \mathcal{H}|_{M\times P\times S^1_\theta\times\{s_1\}} \xrightarrow{+1},\\ &Rq_*(\mathcal{H}_{M\times P\times S^1_\theta\times[s_1,s_2)}) \longrightarrow Rq_*(\mathcal{H}_{M\times P\times S^1_\theta\times[s_1,s_2]}) \longrightarrow \mathcal{H}|_{M\times P\times S^1_\theta\times\{s_2\}} \xrightarrow{+1}, \end{aligned} \tag{A.10}$$

we obtain (iii) by the triangle inequality for $d_{\mathcal{D}^P(M)_\theta}$. □

Lemma A.7. *Define*

$$\begin{aligned} \bar{s}\colon M\times P\times S^1_\theta\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} &\to M\times P\times S^1_\theta\times\mathbb{R},\\ (x,y,t_1,s_1,t_2,s_2) &\mapsto (x,y,t_1+[t_2],s_1+s_2), \end{aligned}$$

and let $\bar{q}_1\colon M\times P\times S^1_\theta\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to M\times P\times S^1_\theta\times\mathbb{R}, \bar{q}_2\colon M\times P\times S^1_\theta\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to M\times P\times\mathbb{R}\times\mathbb{R}$ *be the first and second projections. Let* γ *be a closed convex cone in* $\mathbb{R}^2$ *with* $0\in\gamma$ *and let* $F\in\mathbf{D}^{\mathrm{b}}(M\times P\times S^1_\theta\times\mathbb{R})$. *Then* $\mathrm{SS}(F)\subset T^*(M\times P)\times(S^1_\theta\times\mathbb{R})\times\gamma^\circ$ *if and only if the canonical morphism* $R\bar{s}_*(\bar{q}_1^{-1}F\otimes\bar{q}_2^{-1}\mathbf{k}_{M\times P\times\gamma})\to F$ *is an isomorphism.*

Outline of the proof of Proposition 3.4. Let K^H be the sheaf quantization associated with ϕ^H. Define $\mathcal{H} := G\circ K^H\mathbf{D}^{\mathrm{b}}(M\times P\times S^1_\theta\times I)$. Note that $\mathcal{H}|_{M\times P\times S^1_\theta\times\{0\}}\simeq G$ and $\mathcal{H}|_{M\times P\times S^1_\theta\times\{1\}}\simeq\Phi^H_1(G)$. By Proposition 2.8 and (A.5), we get

$$\mathrm{SS}(\mathcal{H})\subset T^*(M\times P)\times\left\{(t,s;\tau,\sigma)\;\middle|\; -\max_p H_s(p)\cdot\tau\le\sigma\le-\min_p H_s(p)\cdot\tau\right\}. \tag{A.11}$$

Using Lemma A.6(iii) and arguing similar to [AI17, Prop. 4.15], we obtain

$$d_{\mathcal{D}^P(M)_\theta}(G,\Phi^H_1(G))\le\sum_{k=0}^{n-1}\frac{1}{n}\cdot\left(\max_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]}f(s)+\max_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]}g(s)\right), \tag{A.12}$$

where $f(s)=\max_p H_s(p)$ and $g(s)=-\min_p H_s(p)$. For any $\varepsilon\in\mathbb{R}_{>0}$, there exists $n\in\mathbb{Z}_{\ge 0}$ such that the right-hand side of (A.12) is less than $\|H\|+\varepsilon$, which proves the result. □

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