博士論文

論文題目：**Magnitude homology and Vietoris-Rips homology of geodesic metric spaces**
（測地的距離空間のマグニチュードホモロジー及びヴィートリス・リップスホモロジー）

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Introduction

Magnitude is an invariant of enriched categories introduced by Leinster ([**12**]) as a generalization of the cardinality of sets, the rank of vector spaces, and the Euler number of topological spaces. In this thesis, we take a look on magnitude of metric spaces which can be seen as a $[0, \infty)$-enriched category. The definition of magnitude for a finite metric space is given as follows.

Definiton . We call a metric spaces which consists of finitely many points *a finite metric space*. For a finite metric space (X, d), let A_X be the symmetric matrix whose entries are the number $e^{-d(x,y)}$ for $x, y \in X$. If A_X is invertible, *magnitude of* X is defined by the summation of all entries of A_X^{-1}. Further, for $t \in \mathbb{R}_{\geq 0}$, we call magnitude of the metric space (X, td) *magnitude function* of X which is a function on t.

We note that one can define magnitude for almost all finite metric spaces, since the set of invertible matrices are dense in the space of all matrices. It is pointed out by Leinster that magnitude measures the number of efficient points of a metric space.

Magnitude homology, introduced by Hepworth-Willerton and Leinster-Shulman ([**10**], [**13**]), is a bigraded $\mathbb{Z}$-module MH_n^ℓ indexed by positive real numbers ℓ and positive integers n, and is defined for general metric spaces. As shown by them, for a finite metric space X, magnitude homology is a categorification of magnitude in a sense that magnitude function of X is equal to the "Euler charactersitic" $\sum_{\ell,n}(-1)^n \mathrm{rank} MH_n^\ell(X) e^{-t\ell}$. The definition of magnitude homology is given as follows. Let (X, d) denotes a metric space.

Definiton . *Magnitude chain complex*

$$\left(MC_*^\ell(X), \partial_* := \sum_{i=1}^{*} (-1)^i \partial_n^i \right)$$

is a graded free $\mathbb{Z}$-module generated by tuples $(x_0, \ldots, x_n) \in X^{n+1}$ satisfying $\sum_{i=0}^{n-1} d(x_i, x_{i+1}) = \ell$, with a differential defined by

$$\partial_n^i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, \hat{x}_i, \ldots, x_n) & d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1}) \\ 0 & \text{otherwise.} \end{cases}$$

The homology group of magnitude chain complex is called *magnitude homology group* of X, and is denoted by $MH_*^\ell(X)$.

Magnitude homology is defined as a homology of a pointed simplicial set, however, sometimes it behaves like usual homology theory and it looks something more than just a homology of a chain complex. For example, Hepworth-Willerton [**10**] shows a kind of Künneth theorem and Mayer-Vietoris theorem for magnitude homology by restricting it on graphs. Gu [**7**] computes magnitude homology for several classes of graphs by using an algebraic Morse method. Kaneta-Yoshinaga [**11**] describes magnitude chain complex from a view point of ordered complex of posets, and the author [**1**] computed magnitude homology of CAT(κ) spaces by using their frameworks. Gomi [**5**][**6**] constructed a spectral sequence for magnitude homology by using a filtration on numbers of "smooth points", and completely computed magnitude homology for a wide class of geodesic metric spaces. Their computations show that magnitude homology has some information about uniqueness of geodesics for a wide class of metric spaces including Riemannian manifolds. Further, Otter [**16**] studies "blurred" version of magnitude homology, and relates it to singular homology theory. She and Cho [**3**] studies magnitude homology from a view point of persistent theory which plays a central role in the area of Topological Data Analysis.

On the other hand, we can consider a simplicial complex which is associated to metric spaces called *Vietoris-Rips complex*. It was first introduced by Vietoris ([**18**]) as one of the first homology theory for compact metric spaces. Later Rips used this simplicial complex to construct a contractible space with hyperbolic group action([**4**]). The n-simplex of this complex consists of $(n + 1)$-points subsets $\{x_0, \ldots, x_n\}$ whose diameter is not greater than ε. Another chain complex whose n-simplex consists of $(n + 1)$-points subsets $\{x_0, \ldots, x_n\}$ with the diameter strictly smaller than ε is often referred to by the same name. We discuss both of the complexes in this thesis. The associated chain complex $VC_*^{\leq\varepsilon}$ or $VC_*^{<\varepsilon}$ of the above complex is called *Vietoris-Rips chain complex*, and its homology $VH_*^{\leq\varepsilon}$ or $VH_*^{<\varepsilon}$ is called *Vietoris-Rips homology*.

The idea and the definition of magnitude homology and Vietoris-Rips homology is quite similar, however, there are few studies comparing them each other. Further, the author considers that these ideas have been getting more and more important from a view point of algebraic topology for metric spaces which is effectively applicable to data science, in particular to Topological Data Analysis. For example, magnitude homology is a candidate for another criterion of approximative size of data sets instead of Vietoris-Rips homology which is now majorly used. Hence to reveal the relationship between each other is a pivotal problem. Further, the author has an eye on constructing a parametrized view point of geometry which connects discrete and continuous studies of geometry so far, and studying the above two homology theories are our first step to realize it.

In this thesis, we reveal some properties of magnitude homology and Vietoris-Rips homology restricted to the category of geodesic metric spaces. This thesis consists of four chapters, and the first one is devoted to some preliminaries on fundamental elements of topology and geometry. In the remained three chapters, we prove our main theorems. The following is a summary of this thesis.

In chapter 2, we prove an analogy to the fundamental fact that " the ordinary homology of a contractible space is trivial" for *blurred magnitude homology*. Blurred magnitude homology is a variant of magnitude homology coined by Otter ([**16**]), and it is defined as follows. Let (X, d) be a metric space.

Definiton . Let $\ell \in \mathbb{R}_{\geq 0}$. Let us consider a simplicial set whose n-simplices consist of tuples $(x_0, \dots, x_n)$ satisfying

$$\sum_{i=0}^{n-1} d(x_i, x_{i+1}) \leq \ell.$$

We denote its associated chain complex by $(MC_*^{\leq \ell}(X), \partial_*^{\leq \ell})$, and we denote its homology group by $MH_*^{\leq \ell}(X)$. We call $MH_*^{\leq \ell}(X)$ *blurred magnitude homology of* X. Similarly, we define a chain complex $(MC_*^{< \ell}(X), \partial_*^{< \ell})$ to be a chain complex which consists of tuples $(x_0, \dots, x_n)$ satisfying

$$\sum_{i=0}^{n-1} d(x_i, x_{i+1}) < \ell.$$

We denote its homology by $MH_*^{< \ell}(X)$. We also refer to it as blurred magnitude homology.

As remarked in the above, the definition of magnitude homology and blurred magnitude homology is quite simple, however, little property of them has been revealed so far. For example, the invariance of magnitude homology is one of the tantalizing problems. In this chapter, we discuss magnitude homology of *crushable spaces*.

Definiton . A metric space is called crushable if there exists a map $F : X \times [0, 1] \longrightarrow X$ satisfying the following conditions.

(1) $F(x, 1) = x$ for all $x \in X$,
(2) $F(x, 0) = a$ for all $x \in X$,
(3) $F(a, t) = a$ for all $t \in [0, 1]$,
(4) $d(F(x, s), F(y, s)) \leq d(F(x, t), F(y, t))$ for all $x, y \in X$ and $s \leq t \in [0, 1]$.

To study magnitude homology, the notion of blurred magnitude homology seems to make it easy to deal with algebraically. Our main result is the following which appears as Proposition 2.5.

THEOREM . If (X, d) is crushable, then

$$MH_n^{<\ell}(X) = 0$$

for all $\ell \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{>0}$.

We use some techniques from singular homology theory for the proof. As a corollary of these theorems, we can compute blurred magnitude homology of Euclidean spaces (Corollary 2.6).

In chapter 3, we study magnitude homology of geodesic metric spaces of curvature $\leq \kappa$, especially CAT(κ) spaces. We will show that magnitude homology $MH_n^\ell(X)$ of such a metric space X vanishes for small ℓ and all $n > 0$. Consequently, we can compute a total $\mathbb{Z}$-degree magnitude homology for small ℓ for the spheres $\mathbb{S}^n$, the Euclid spaces $\mathbb{E}^n$, the hyperbolic spaces $\mathbb{H}^n$, and real projective spaces $\mathbb{RP}^n$ with the standard metric. We also show that an existence of closed geodesic in a metric space guarantees the non-triviality of magnitude homology.

Some computations of magnitude homology is studied for graphs by Hepworth-Willerton([**10**]), for convex subsets in $\mathbb{R}^n$ by Leinster-Shulman ([**13**]), for the geodesic circle by Kaneta-Yoshinaga ([**11**]), and for geodesic spheres by Gomi ([**4**]).

Since the motivation and the formulation of magnitude homology are algebraic, several authors study it from a view point of algebra and category theory. On the other hand, its geometric meaning is gradually getting clarified in the study of Leinster-Shulman, Kaneta-Yoshinaga, and Gomi ([**13**], [**11**], [**4**]).

In this chapter, we study magnitude homology from a view point of metric geometry, and clarify that the curvature of metric spaces effects heavily on the triviality of magnitude homology. We also investigate its connection with closed geodesics.

Let (X, d) be a metric space. A quadruple $(x_0, x_1, x_2, x_3) \in X^4$ is called *4-cut* if (1) $x_i \neq x_{i+1}$ for $0 \leq i \leq 2$, (2) $d(x_i, x_{i+2}) = \sum_{j=i}^{i+1} d(x_j, x_{j+1})$ for $0 \leq i \leq 1$, and (3) $d(x_0, x_3) < \sum_{j=0}^{2} d(x_j, x_{j+1})$ are satisfied. Let m_X be the infimum of the length $\sum_{j=0}^{2} d(x_j, x_{j+1})$ of 4-cuts (x_0, x_1, x_2, x_3) in X. This invariant is first introduced in ([**11**]) and is very important for the study of magnitude homology. We clarify its metric geometrical interpretation by the following theorem stated as Theorem 3.5 in this artcle. The invariant l_X measures the infimum length of locally geodesic path which is not a geodesic. See Definition 3.3 for the detail. Let D_κ be $\pi/\sqrt{\kappa}$ for $\kappa > 0$ or $+\infty$ for $\kappa \leq 0$. The main result of this chapter is the following.

THEOREM . For a geodesic CAT(κ) space (X, d), we have

$$D_\kappa \leq m_X \leq l_X.$$

The following is stated as Corollary 3.6.

Corollary . Let (X, d) be a geodesic CAT(κ) space. Then for any $n > 0$ and $0 < \ell < D_\kappa$, magnitude homology $MH_n^\ell(X)$ vanishes.

The following is stated as Theorem 4.3. See Definition 4.1 for the definition of closed geodesics.

Theorem . Let X be a metric space. If there exists a closed geodesic of radius r in X, then we have $MH_2^{\pi r}(X) \neq 0$.

Most part of our results in this chapter overlaps with Gomi's works ([**5**], [**6**]). He computes magnitude homology for a wide class of geodesic spaces by using spectral sequence. Only he requires on a metric space is a non-branching condition on geodesics, hence every complete Riemannian manifold and every CAT(κ) space with non-positive κ can be dealt with in his frame work. However, the approach of ours is quite different from his, and the author does not know whether CAT(κ) spaces with positive κ can be covered by his work.

In chapter 4, we study Vietoris-Rips homology of geodesic spaces for small parameters ε, and see that it is similar to magnitude homology case. Vietoris-Rips homology theory is an origin of modern homology theory, and getting more and more interested in recent growth of Topological Data Analysis. From a view point of Topological Data Analysis, it is important to know about Vietoris-Rips homology of Riemannian manifolds as an "ideal" state of datasets. In this thesis, we discuss Vietoris-Rips homology with small and large parameters for Riemannian manifolds with some curvature requirements. We prove that Vietoris-Rips homology is isomorphic to singular homology when the parameter is small, and not isomorphic in large case. The techniques we use are analogy from the singular homology theory, and they are applicable to two slightly different definitions of Vietoris-Rips homology. We also consider a relationship with magnitude homology.

Although its origin is natural and fundamental, the Vietoris-Rips homology is not widely studied compared to other homology theories. In this chapter, we study the theory for geodesic metric spaces with some curvature requirements. Hausmann ([**9**]) proves the following in this context.

Theorem (Hausmann, Proposition 3.4[**9**]). The Vietoris-Rips homology with sufficiently small ε is isomorphic to the singular homology for Riemannian manifolds with some curvature requirement.

His proof depends on the strictness of the inequality in the definition of complex. We give another proof to the above statement, which can be applied to the both definition of Vietoris-Rips complex and is slightly a refinement of Hausmann's. Let $r(X)$ be the supremum of non-negative real numbers r satisfying the following:

(1) There uniquely exists a geodesic between every pair of points x, y with $d(x, y) \leq r$,

(2) For any three points x, y, z with diam $\{x, y, z\} \leq r$, and for every point w on the geodesic between y, z, they satisfy

$$d(x, w) \leq \max\{d(x, y), d(x, z)\}.$$

It is well known that Riemannian manifolds X with strictly positive injectivity radius satisfies (1) above. It is also easy to see such an X with an upper bound on its sectional curvature satisfies (2), hence we have $r(X) > 0$. In particular, we have $r(X) > 0$ for compact Riemannian manifolds. Hausmann considers similar number as the threshold but not greater than ours, hence our result is a refinement of the above theorem. The following is our main theorem, which appears as Theorem 2.3 in this chapter.

THEOREM . For any $0 < \varepsilon' \leq \varepsilon < r(X)$, the following inclusion of chain complexes is a chain homotopy equivalence :

$$VC_n^{\leq \varepsilon'}(X) \longrightarrow VC_n^{\leq \varepsilon}(X).$$

As a corollary of this theorem, we obtain the following, which appears as Corollary 2.23.

COROLLARY . For Riemannian manifolds X with $r(X) > 0$, we have

$$VH_*^{\leq \varepsilon}(X) \cong H_*(X; \mathbb{Z}),$$

for sufficiently small $0 < \varepsilon$.

Vietoris-Rips complex also plays a pivotal role in recent and fast growth of Topological Data Analysis, and is receiving more and more attention. One of the novel and remarkable viewpoints on it is coined by Otter ([**16**]) via magnitude homology. Cho ([**3**]) also relates them from a view point of quantales.

This thesis is organized as follows. In Chapter 1, we review some fundamental elements on metric geometry and algebraic topology which are used in the following chapters. In Chapter 2, we study blurred magnitude homology of crushable spaces. First we review on blurred magnitude homology defined by Otter [**16**] and make another description for magnitude homology from this view point by using general theory of simplicial abelian groups in section 1. In section 2, we prove the main theorem by formulating "prism operator" for blurred magnitude homology analogous to singular homology theory. We also indicate a computation of blurred magnitude homology of Euclidean space, hyperbolic space, and the semi-sphere.

In Chapter 3, we study magnitude homology of CAT(κ) spaces. In section 1, we briefly recall some notation and fundamental facts on metric geometry. In section 2, we briefly recall the definition of magnitude homology and some facts on them studied in ([**11**]). In section 3, we study magnitude homology of CAT(κ) spaces through the invariants m_X and l_X. We also compute magnitude homology of some simply connected complete Riemannian manifolds as examples. In section 4, we study how closed geodesic in a metric space effects on magnitude homology. This is motivated by the fact that magnitude homology is defined as a generalization of Hochschild homology in ([**13**]). In section 5, we study magnitude homology of non-CAT(κ) metric spaces whose curvature is bounded from above through the invariants ι_X and $\mathrm{Sys}(X)$. We also compute magnitude homology of real projective spaces $\mathbb{RP}^n$ with standard metric as an example.

In Chapter 4, we study the Vietoris-Rips homology of Riemannian manifolds for small parameters. We briefly review the construction of Vietoris-Rips homology in section 1. We also construct chain complexes used in this thesis which are not isomorphic to the associated chain complex of the Vietoris-Rips complex but are chain homotopy equivalent to it. We prove the main theorem in section 2. For that purpose, we discuss some analogies from singular homology theory.

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CHAPTER 1

Preliminary

In this chapter, we briefly review some fundamental elements which appear in this thesis.

1. Metric space and metric geometry

DEFINITION 1.1. A set X equipped with a map $d : X \times X \to \mathbb{R}_{\geq 0}$ is called *a metric space* if they satisfies the following:

$$d(x, y) = 0 \text{ if and only if } x = y,$$
$$d(x, y) = d(y, x) \text{ for any } x, y \in X,$$
$$d(x, z) \leq d(x, y) + d(y, z) \text{ for any } x, y, z \in X.$$

DEFINITION 1.2. Let (X, d) be a metric space, and c be a positive number. A path $\gamma\colon [0, c] \to X$ is a *linearly reparametrized geodesic connecting* $\gamma(0)$ *and* $\gamma(1)$ if it satisfies $d(\gamma(t), \gamma(s)) = \lambda|t - s|$ for any $0 \leq s \leq t \leq 1$ and for some $\lambda \geq 0$. When $\lambda = 1$ we call such a path *a geodesic*.

DEFINITION 1.3. A metric space X is *geodesic* if for each pair of points $a, b \in X$, there exists a geodesic connecting them. Furthermore, if such a geodesic is unique, then X is called *uniquely geodesic*.

We sometimes denote a geodesic from some interval connecting a and b in a metric space X, by $[a, b]$.

DEFINITION 1.4. Let a, b, c be points in a metric space X. The union of these geodesic images $\Delta_{abc} := [a, b] \cup [b, c] \cup [c, a]$ is called a *geodesic triangle*.

Let S_κ be a simply connected surface of constant sectional curvature κ for $\kappa \in \mathbb{R}$. Note that for every geodesic triangle Δ_{abc} in a metric space X, there exists a geodesic triangle $\tilde{\Delta}_{abc}$ in S_κ whose sides have the length precisely equal to the length of corresponding sides of Δ_{abc}. We also note that if the inequality

$$d(a, b) + d(b, c) + d(c, a) < 2D_\kappa$$

is satisfied for

$$D_\kappa := \begin{cases} \pi/\sqrt{\kappa} & \kappa > 0 \\ +\infty & \kappa \leq 0 \end{cases},$$

then such a triangle is uniquely determined up to congruence.

Definition 1.5. A geodesic triangle Δ_{abc} in a metric space X is *κ-small* if the inequality $d(a,b) + d(b,c) + d(c,a) < 2D_\kappa$ is satisfied. For a κ-small geodesic triangle Δ_{abc}, the corresponding triangle $\tilde{\Delta}_{abc}$ in S_κ is called the *comparison triangle in S_κ*.

Definition 1.6. Let $\Delta_{abc} = [a,b] \cup [b,c] \cup [c,a]$ be a geodesic triangle in a metric space (X,d), and let $\tilde{\Delta}_{abc} = [\tilde{a},\tilde{b}] \cup [\tilde{b},\tilde{c}] \cup [\tilde{c},\tilde{a}]$ be its comparison trinangle in S_κ. A point $\tilde{s} \in [\tilde{a},\tilde{b}]$ is *the comparison point* of $s \in \Delta_{abc}$ if $s \in [a,b]$ and $d(\tilde{a},\tilde{s}) = d(a,s)$ is satisfied. We similarly define the comparison points for points on $[\tilde{b},\tilde{c}]$ and $[\tilde{c},\tilde{a}]$.

The following notion of CAT(κ) space plays a fundamental role in this thesis.

Definition 1.7. A metric space (X,d) is CAT(κ) if for every κ-small geodesic triangle Δ_{abc} in X and for every pair of points $s,t \in \Delta_{abc}$, the CAT(κ) *inequality*

$$d(s,t) \leq d_{S_\kappa}(\tilde{s},\tilde{t})$$

holds for the comparison points $\tilde{s}$ and $\tilde{t}$.

Example 1.8. Apparently, the 2-sphere S_κ of constant sectional curvature κ, the Euclid plane S_0, and the hyperbolic plane $S_{-\kappa}$ of sectional curvature $-\kappa < 0$ are CAT(κ), CAT(0), and CAT($-\kappa$) respectively.

More generally, the following is well known.

Fact 1.9. A complete simply connected Riemannian manifold with sectional curvature $\leq \kappa$ is CAT(κ).

We have the following examples of CAT(κ) spaces. Let $\mathbb{S}^n_\kappa = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = 1/\kappa\}$ be the n-sphere of radius $1/\sqrt{\kappa}$ equipped with the geodesic metric.

Example 1.10. Let $n \geq 2$. The n-sphere $\mathbb{S}^n_\kappa$ of radius $1/\sqrt{\kappa}$, the n-dimensional Euclid space $\mathbb{E}^n$, and the n-dimensional hyperbolic space $\mathbb{H}^n$ of sectional curvature $-\kappa < 0$ are CAT(κ), CAT(0), and CAT($-\kappa$) respectively.

We denote the circle $\{(x_0,x_1) \in \mathbb{R}^2 \mid x_0^2 + x_1^2 = r^2\}$ of radius r equipped with the geodesic metric by C_r.

Example 1.11. The circle $C_{1/\sqrt{\kappa}}$ is CAT(κ).

Proof. Because the perimeter of the circle $C_{1/\sqrt{\kappa}}$ is $2D_\kappa$, every κ-small triangle in this space lies in a semi-circle, which implies the triangle is degenerated. Hence the CAT(κ) inequality holds. □

A connected undirected metric graph with no cycles is called *a tree*.

Example 1.12. Every tree is CAT(0).

Proof. Because there is no cycles in a tree, every triangle in a tree is degenerated. Hence the CAT(0) inequality holds. □

We introduce the following notion of the angle.

DEFINITION 1.13. Let c_1 and c_2 be geodesics in X with the same start point C. We define the *angle* between c_1 and c_2 at C by

$$\angle C := \limsup_{\epsilon,\epsilon'\to 0}\tilde{\angle} c_1(\epsilon)Cc_2(\epsilon'),$$

where $\tilde{\angle} c_1(\epsilon)Cc_2(\epsilon')$ is the angle of the comparison triangle $\tilde{\Delta}_{c_1(\epsilon)Cc_2(\epsilon')}$ at $\tilde{C}$ in the Euclid plane S_0.

2. Simplicial sets

The notion of simplicial sets is sometimes very useful, and it appears in this thesis. We briefly review its definition.

DEFINITION 2.1. The simplex category Δ is a category whose objects are ordered sets

$$[n] := \{0 \leq \cdots \leq n\}$$

for $n \in \mathbb{N}$, and morphisms are order preserving maps between them.

REMARK 2.2. Every morphism in Δ is generated by the following morphisms:

$$d_i : [n] \longrightarrow [n+1];\ j \mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases},$$

$$s_i : [n+1] \longrightarrow [n];\ j \mapsto \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases},$$

for $0 \leq i \leq n$ and all $n \in \mathbb{N}$.

DEFINITION 2.3. A *simplicial object* in a category C is a functor

$$\Delta^{\mathrm{op}} \longrightarrow C.$$

A simplicial object in the category of sets Set is called a *simplicial set*, and a simplicial object in the category of abelian groups Ab is called a *simplicial abelian group*.

DEFINITION 2.4. Let S be a simplicial abelian group. We define the *associated chain complex* C_*S of S by

$$C_nS := S[n],$$

$$\partial_n := \sum_{i=0}^{n}(-1)^i d_i : C_nS \longrightarrow C_{n-1}S.$$

DEFINITION 2.5. Let S be a simplicial abelian group. The images of degeneracy maps of S are called *degenerated simplices*. We denote the set of degenerated n-simplices by $N_n(S)$.

The following are fundamental facts on simplicial abelian group theory.

FACT 2.6. The family of set $N = \{N_*(S)\}$ is a sub-simplicial abelian group of S.

Fact 2.7. The quotient map $S \longrightarrow S/N$ induces chain homotopy equivalence

$$C_*S \longrightarrow C_*S/N$$

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3. Magnitude homology

One of the main subjects of this thesis is the notion of magnitude homology. Through this section, (X, d) denotes a metric space unless otherwise mentioned.

Definition 3.1. An $(n+1)$-tuple $(x_0, \ldots, x_n) \in X^{n+1}$ is called an *n-chain* of X. An n-chain is *proper* if $x_i \neq x_{i+1}$ for all $0 \leq i \leq n$. The *length* $|x|$ of n-chain x is defined by

$$|x| := \sum_{i=0}^{n-1} d(x_i, x_{i+1}).$$

We denote the set of all proper n-chains of length ℓ by $P_n^\ell(X)$, and $P_n(X)$ denotes the union of them running through all $\ell \geq 0$. Let $MC_n^\ell(X)$ be the abelian group freely generated by $P_n^\ell(X)$.

Definition 3.2. Let a and b be points in X. A point $c \in X$ is *smooth between a and b* if the equality $d(a, b) = d(a, c) + d(c, b)$ holds. We denote $a \prec c \prec b$ if c is a smooth point between a and b with $a \neq c$ and $b \neq c$.

Definition 3.3. *Magnitude chain complex*

$$(MC_*^\ell(X), \partial_* := \sum_{i=1}^{*} (-1)^i \partial_n^i)$$

is defined by

$$\partial_n^i(x_0, \ldots, x_n) = \begin{cases} (x_0, \ldots, \hat{x}_i, \ldots, x_n) & (\text{if } x_{i-1} \prec x_i \prec x_{i+1}) \\ 0 & (\text{otherwise}). \end{cases}$$

The homology group of magnitude chain complex is called the *magnitude homology group* of X, and is denoted by $MH_*^\ell(X)$.

Remark 3.4. It is easily seen that the first magnitude homology measures *Menger convexity*, where a metric space is Menger convex if for all $x, y \in X$ there exists $z \neq x, y \in X$ such that $d(x, z) + d(z, y) = d(x, y)$.

CHAPTER 2

Magnitude homology of crushable space is trivial

1. Blurred magnitude homology

Let (X, d) be a metric space.

Definition 1.1. Let $\ell \in \mathbb{R}_{\geq 0}$. Let us consider a simplicial set whose n-cells consist of tuples $x = (x_0, \ldots, x_n)$ satisfying

$$|x| := \sum_{i=0}^{n-1} d(x_i, x_{i+1}) \leq \ell.$$

We denote its associated chain complex by $(MC_*^{\leq \ell}(X), \partial_*^{\leq \ell})$, and we denote its homology group by $MH_*^{\leq \ell}(X)$. We call $MH_*^{\leq \ell}(X)$ *blurred magnitude homology of* X. Similarly, we define chain complex $(MC_*^{<\ell}(X), \partial_*^{<\ell})$ to be a chain complex which consists of tuples $x = (x_0, \ldots, x_n)$ satisfying

$$|x| := \sum_{i=0}^{n-1} d(x_i, x_{i+1}) < \ell.$$

We denote its homology by $MH_*^{<\ell}(X)$. We also refer to it as blurred magnitude homology.

Definition 1.2. Let $\ell \leq \ell'$. We denote by $\iota_\sharp^{\ell \leq \ell'}$ either the chain maps

$$MC_*^{\leq \ell}(X) \longrightarrow MC_*^{\leq \ell'}(X)$$

or

$$MC_*^{<\ell}(X) \longrightarrow MC_*^{<\ell'}(X)$$

both induced from the natural inclusion. The homomorphism induced on the homology is denoted by $\iota_*^{\ell \leq \ell'}$.

Definition 1.3. *Magnitude chain complex* $MC_*^{\ell}(X)$ is the quotient complex of $MC_*^{\leq \ell}(X)$ by a subcomplex $MC_*^{<\ell}(X)$. Its homology denoted by $MH_*^{\ell}(X)$ is called *magnitude homology* of X.

Remark 1.4. Although this definition of magnitude chain complex is different from the original definition due to Hepworth-Willerton [**10**] and Leinster- Shulman [**13**], we can easily see that its homology is isomorphic to the original magnitude homology by the standard argument of simplicial abelian group as follows. This is also remarked in [**10**] Remark 45.

Definition 1.5. Let S be a simplicial abelian group. The image of degeneracy maps of S are called *degenerated simplices*. We denote the set of degenerated n-simplices by $N_n(S)$.

Lemma 1.6 ([**15**]). The family of set $N = \{N_*(S)\}$ is a sub-simplicial abelian group of S.

We denote the associated chain complex of S by C_*S.

Theorem 1.7 ([**15**] Corollary 22.3). The quotient map $S \longrightarrow S/N$ induces chain homotopy equivalence

$$C_*S \longrightarrow C_*S/N.$$

If we take C_*S as our definition of magnitude homology, C_*S/N is the original definition.

The following is clear from the standard argument of homological algebra.

Proposition 1.8. There is a long exact sequence

$$\cdots \longrightarrow MH^{\leq \ell}_{n+1}(X) \longrightarrow MH^{<\ell}_n(X) \longrightarrow MH^{\leq \ell}_n(X) \longrightarrow MH^{\ell}_n(X) \longrightarrow \dots.$$

Proof. It is a fundamental fact of homological algebra that if we have a short exact sequence of chain complexes $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, then we can derive a long exact sequence

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow \dots.$$

We substitute $MH^{<\ell}_*(X)$, $MH^{\leq\ell}_*(X)$ and $MH^{\ell}_*(X)$ for A, B and C respectively, and this completes a proof. □

2. Results

Let (X, d) be a metric space, and $a \in X$.

Definition 2.1. (X, d) is *crushable* if there exists a map $F : X \times [0, 1] \longrightarrow X$ satisfying the following conditions.

(1) $F(x, 1) = x$ for all $x \in X$,
(2) $F(x, 0) = a$ for all $x \in X$,
(3) $F(a, t) = a$ for all $t \in [0, 1]$,
(4) $d(F(x, s), F(y, s)) \leq d(F(x, t), F(y, t))$ for all $x, y \in X$ and $s \leq t \in [0, 1]$.

Remark 2.2. This definition of crushable space is coined by Hausmann [**9**] in his study of Vietoris-Rips complex of metric space.

Definition 2.3. For a chain $x \in MC^{<\ell}_n(X)$ expressed as

$$x = \sum_{i=0}^{k} \alpha_i x^i \text{ with } \alpha_i \in \mathbb{Z},$$

we set $L(x) := \max_i |x^i|$ and $\delta(x) := \ell - L(x)$.

Lemma 2.4. Let $f, g : X \longrightarrow X$ be maps satisfying

$$d(g(a), g(b)) \leq d(f(a), f(b)) \text{ for all } a, b \in X.$$

Let $x = (x_0, \dots, x_n)$ be a chain in $MC_n^{<\ell}(X)$. We denote

$$fx = (f(x_0), \dots, f(x_n)),\ \ gx = (g(x_0), \dots, g(x_n)),$$

and assume that they are in $MC_n^{<\ell}(X)$. Suppose that they satisfy

$$d(f(x_i), g(x_i)) < \delta(fx)$$

for all $0 \le i \le n$. Then an $n+1$-chain

$$P_{f,g}x := \sum_{i=0}^{n}(-1)^i(f(x_0), \dots, f(x_i), g(x_i), \dots, g(x_n)) \in MC_{n+1}^{<\ell}(X)$$

can be defined, and it satisfies

$$\partial^{<\ell}P_{f,g}x + P_{f,g}\partial^{<\ell}x = gx - fx.$$

Proof. The former part is clear. The latter part can be checked by calculating $\partial^{<\ell}P_{f,g}x + P_{f,g}\partial^{<\ell}x$ as follows. We calculate $\partial^{<\ell}P_{f,g}x$ first by setting $x = (x_0, \dots, x_n)$ as follows.

$$\begin{aligned}
\partial^{<\ell}P_{f,g}x &= \partial^{<\ell}\sum_{i=0}^{n}(-1)^i(f(x_0), \dots, f(x_i), g(x_i), \dots, g(x_n)) \\
&= \sum_{i=0}^{n}(-1)^i\Big(\sum_{j=0}^{i}(-1)^j(f(x_0), \dots, f(\check{x}_j), \dots, f(x_i), g(x_i), \dots, g(x_n)) \\
&\quad + \sum_{j=i}^{n}(-1)^{j+1}(f(x_0), \dots, f(x_i), g(x_i), \dots, g(\check{x}_j), \dots, g(x_n))\Big) \\
&= \sum_{0\le j\le i\le n}(-1)^{i+j}(f(x_0), \dots, f(\check{x}_j), \dots, f(x_i), g(x_i), \dots, g(x_n)) \\
&\quad - \sum_{0\le i\le j\le n}(-1)^{i+j}(f(x_0), \dots, f(x_i), g(x_i), \dots, g(\check{x}_j), \dots, g(x_n))
\end{aligned}$$

The $i = j$ part in the above is further calculated as follows.

$$\begin{aligned}
&\sum_{i=0}^{n}\Big(f(x_0), \dots, f(x_{i-1}), g(x_i), \dots, g(x_n)\Big) \\
&- \sum_{i=0}^{n}\Big((f(x_0), \dots, f(x_i), g(x_{i+1}), \dots, g(x_n))\Big) \\
&= (g(x_0), \dots, g(x_n)) - (f(x_0), \dots, f(x_n))
\end{aligned}$$

On the other hand, we can see that the $i \ne j$ part is equal to $-P_{f,g}\partial^{<\ell}x$. This completes a proof. □

Proposition 2.5. If (X, d) is crushable, then

$$MH_n^{<\ell}(X) = 0,$$

for all $\ell \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{>0}$.

Proof. Assume that there exists a non-zero homology class

$$[x] \in MH_n^{<\ell}(X),$$

for some $\ell > 0$ and $n > 0$. We fix such a chain $x = \sum_{i=0}^{k} \alpha_i x^i$ with $x^i = (x_0^i, \dots, x_n^i)$. Take a sequence $t_0 = 0 < t_1 < \cdots < t_N = 1$ satisfying

$$d(F(x_j^i, t_m), F(x_j^i, t_{m+1})) \leq \delta(x),$$

for all $0 \leq i \leq k$, $0 \leq j \leq n$ and $0 \leq m \leq N-1$. We set

$$F(x, t_m) := \sum_i \alpha_i F(x^i, t_m) \in MC_n^{<\ell}(X).$$

By Lemma 2.4, we know inductively that two chains $F(x, t_m)$ and $F(x, t_{m+1})$ are homologous in $MC_n^{<\ell}(X)$. Hence two chains x and

$$F(x, t_0) = \sum_i \alpha_i (a, \dots, a)$$

are homologous, where the latter is null-homologous. Therefore we obtain $[x] = 0$, which is a contradiction. □

Corollary 2.6. Blurred magnitude homology of Euclidean space $\mathbb{R}^N$ and hyperbolic spaces $\mathbb{H}^N$ are trivial, namely

$$MH_n^{<\ell}(\mathbb{R}^N) = MH_n^{<\ell}(\mathbb{H}^N) = 0,$$

for any $\ell > 0$ and any $n > 0$.

Proof. We have a distance decreasing contraction $x \mapsto tx$ for $x \in \mathbb{R}^N$ and $0 \leq t \leq 1$. For hyperbolic spaces, we can apply the same contraction for the ball model. □

Example 2.7. Let X be the semi-sphere of the standard sphere $\mathbb{S}^N$ with the geodesic metric. This space is crushable, hence we have $MH_*^{<\ell}(X) = 0$. On the other hand, by Gomi's work [**6**] and the authors work in chapter 2, we know that magnitude homology of X is non-trivial, because there are infinitely many shortest geodesics connecting two distinct points. Therefore, by considering the long exact sequence for magnitude homologies, we obtain that blurred magnitude homology $MH_*^{\leq \ell}(X)$ is non-trivial.

CHAPTER 3

Magnitude homology of $\mathrm{CAT}(\kappa)$ spaces

1. Preliminary on metric geometry

In this section, we briefly recall some notations on metric geometry from ([**2**]).

Definition 1.1. Let (X, d) be a metric space, and c be a positive number. A path $\gamma\colon [0, c] \to X$ is a *linearly reparametrized geodesic connecting* $\gamma(0)$ *and* $\gamma(1)$ if it satisfies $d(\gamma(t), \gamma(s)) = \lambda|t - s|$ for any $0 \le s \le t \le 1$ and for some $\lambda \ge 0$. When $\lambda = 1$ we call such a path *geodesic*.

Definition 1.2. A metric space X is *geodesic* if for each pair of points $a, b \in X$, there exists a geodesic connecting them. Furthermore, if such a geodesic is unique, then X is called *uniquely geodesic*.

We sometimes denote a geodesic from some interval connecting a and b in a metric space X, by $[a, b]$.

Definition 1.3. Let a, b, c be points in a metric space X. The union of these geodesic images $\Delta_{abc} := [a, b] \cup [b, c] \cup [c, a]$ is called a *geodesic triangle*.

Let S_κ be a simply connected surface of constant sectional curvature κ for $\kappa \in \mathbb{R}$. Note that for every geodesic triangle Δ_{abc} in a metric space X, there exists a geodesic triangle $\tilde{\Delta}_{abc}$ in S_κ whose sides have the length precisely equal to the length of corresponding sides of Δ_{abc}. We also note that if the inequality

$$d(a, b) + d(b, c) + d(c, a) < 2D_\kappa$$

is satisfied for

$$D_\kappa := \begin{cases} \pi/\sqrt{\kappa} & \kappa > 0 \\ +\infty & \kappa \le 0 \end{cases},$$

then such a triangle is uniquely determined up to congruence. The following is a fundamental fact on elementary geometry known as Alexandrov's lemma.

Proposition 1.4 ([**2**] Lemma I.2.16). Let A, B, B', C be distinct points in S_κ. If $\kappa > 0$, we assume that $d(A, B) + d(B, C) + d(C, B') + d(B', A) < 2D_\kappa$ and $d(B, C) + d(C, B') < D_\kappa$. We suppose that B and B' lie on opposite sides of the line through A and C. Let α, β, γ(*respectively* α', β', γ') be the angles of a triangle Δ(*resp.* Δ') with vertices A, B, C(*resp.* A, B', C). We assume that $\gamma + \gamma' \geq \pi$. Let $\overline{\Delta}$ be a triangle on S_κ with vertices $\overline{A}, \overline{B}, \overline{B'}$ such that $d(\overline{A}, \overline{B}) = d(A, B), d(\overline{A}, \overline{B'}) = d(A, B')$ and $d(\overline{B}, \overline{B'}) = d(B, C) + d(C, B')$. Let $\overline{\alpha}, \overline{\beta}, \overline{\beta'}$ be the angles of $\overline{\Delta}$ at $\overline{A}, \overline{B}, \overline{B'}$. Then we have

$$\overline{\beta} \geq \beta,\ \overline{\beta'} \geq \beta'.$$

Definition 1.5. A geodesic triangle Δ_{abc} in a metric space X is *κ-small* if the inequality $d(a, b) + d(b, c) + d(c, a) < 2D_\kappa$ is satisfied. For a κ-small geodesic triangle Δ_{abc}, the corresponding triangle $\tilde{\Delta}_{abc}$ in S_κ is called the *comparison triangle in S_κ*.

Definition 1.6. Let $\Delta_{abc} = [a, b] \cup [b, c] \cup [c, a]$ be a geodesic triangle in a metric space (X, d), and let $\tilde{\Delta}_{abc} = [\tilde{a}, \tilde{b}] \cup [\tilde{b}, \tilde{c}] \cup [\tilde{c}, \tilde{a}]$ be its comparison trinangle in S_κ. A point $\tilde{s} \in [\tilde{a}, \tilde{b}]$ is *the comparison point* of $s \in \Delta_{abc}$ if $s \in [a, b]$ and $d(\tilde{a}, \tilde{s}) = d(a, s)$ is satisfied. We similarly define the comparison points for points on $[\tilde{b}, \tilde{c}]$ and $[\tilde{c}, \tilde{a}]$.

The following notion of CAT(κ) space plays a fundamental role in this thesis.

Definition 1.7. A metric space (X, d) is CAT(κ) if for every κ-small geodesic triangle Δ_{abc} in X and for every pair of points $s, t \in \Delta_{abc}$, the CAT(κ) *inequality*

$$d(s, t) \leq d_{S_\kappa}(\tilde{s}, \tilde{t})$$

holds for the comparison points $\tilde{s}$ and $\tilde{t}$.

Example 1.8. Apparently, the 2-sphere S_κ of constant sectional curvature κ, the Euclid plane S_0, and the hyperbolic plane $S_{-\kappa}$ of sectional curvature $-\kappa < 0$ are CAT(κ), CAT(0), and CAT($-\kappa$) respectively.

More generally, we have the following.

Proposition 1.9. A complete simply connected Riemannian manifold with sectional curvature $\leq \kappa$ is CAT(κ).

We have the following examples of CAT(κ) spaces. Let

$$\mathbb{S}^n_\kappa = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1/\kappa\}$$

be the n-sphere of radius $1/\sqrt{\kappa}$ equipped with the geodesic metric.

Example 1.10. Let $n \geq 2$. The n-sphere $\mathbb{S}^n_\kappa$ of radius $1/\sqrt{\kappa}$, the n-dimensional Euclid space $\mathbb{E}^n$, and the n-dimensional hyperbolic space $\mathbb{H}^n$ of sectional curvature $-\kappa < 0$ are CAT(κ), CAT(0), and CAT($-\kappa$) respectively.

We denote the circle $\{(x_0, x_1) \in \mathbb{R}^2 \mid x_0^2 + x_1^2 = r^2\}$ of radius r equipped with the geodesic metric by C_r. The following is immediate.

Example 1.11. The circle $C_{1/\sqrt{\kappa}}$ is CAT(κ).

Proof. Because the perimeter of the circle $C_{1/\sqrt{\kappa}}$ is $2D_\kappa$, every κ-small triangle in this space lies in a semi-circle, which implies the triangle is degenerated. Hence the CAT(κ) inequality holds. □

A connected undirected metric graph with no cycles is called *a tree*.

Example 1.12. Every tree is CAT(0).

Proof. Because there is no cycles in a tree, every triangle in a tree is degenerated. Hence the CAT(0) inequality holds. □

We introduce the following notion of the angle.

Definition 1.13. Let c_1 and c_2 be geodesics in X with the same start point C. We define the *angle* between c_1 and c_2 at C by

$$\angle C := \limsup_{\epsilon,\epsilon'\to 0} \tilde{\angle} c_1(\epsilon) C c_2(\epsilon'),$$

where $\tilde{\angle} c_1(\epsilon) C c_2(\epsilon')$ is the angle of the comparison triangle $\tilde{\Delta}_{c_1(\epsilon)Cc_2(\epsilon')}$ at $\tilde{C}$ in the Euclid plane S_0.

The following are fundamental. See for example [**2**] for the proof.

Proposition 1.14 ([**2**] Proposition I.1.14). Let X be a metric space and let c, c' and c'' be geodesics in X starting from the same point p. Then we have

$$\angle(c', c'') \leq \angle(c, c') + \angle(c, c'').$$

Proposition 1.15 ([**2**] Proposition II.1.7). Let $\kappa \in \mathbb{R}$. For a geodesic metric space (X, d), the following are equivalent.

(i) (X, d) is CAT(κ).
(ii) For every κ-small geodesic triangle Δ_{abc}, the angles at a, b and c are not greater than the corresponding angles of the comparison triangle $\tilde{\Delta}_{abc}$ in S_κ.

The following proposition is technical but significant in this chapter.

Proposition 1.16 ([**2**] Lemma II.4.11). Let κ be a real number, and X be a metric space of curvature $\leq \kappa$. Let $q\colon [0, 1] \to X$ be a linearly reparametrized geodesic connecting two distinct points $q(0)$ and $q(1)$, and let p be a point in X which is not on the image of q. Assume that for each $s \in [0, 1]$, there is a linearly reparametrized geodesic $c_s\colon [0, 1] \to X$ connecting p and $q(s)$, varying continuously with s. We further assume that the geodesic triangle $\Delta_{q(0)pq(1)}$ is κ-small. Then the angles of $\Delta_{q(0)pq(1)}$ at $q(0), p$ and $q(1)$ are not greater than the corresponding angles of any comparison triangle $\tilde{\Delta}_{q(0)pq(1)}$ in S_κ.

We study not only CAT(κ) spaces, but also locally CAT(κ) spaces. We recall some fundamental notions on them.

Definition 1.17. A metric space X is of *curvature* $\leq \kappa$ if for each point $x \in X$ there exists $r_x > 0$ such that the ball $B(x, r_x)$ with the induced metric is CAT(κ).

The following well-known fact due to Alexandrov supports the significance for studying metric spaces of curvature $\leq \kappa$.

Proposition 1.18 ([**2**] Theorem I.1A.6). A smooth Riemannian manifold is of curvature $\leq \kappa$ if and only if its sectional curvature is $\leq \kappa$.

Example 1.19. The standard projective space $\mathbb{RP}^n$ for $n \geq 2$ is of curvature ≥ 1. Furthermore, it is **not** CAT(1) since a closed geodesic which lifts to the geodesic semi-circle on $\mathbb{S}^n$ does not satisfy the CAT(1) angle condition.

Definition 1.20. For a metric space X, the *injectivity radius* ι_X is the supremum of $r \geq 0$ such that any two point of distance $< r$ is connected by the unique geodesic. The *systole* $\mathrm{Sys}(X)$ is the infimum of the length of closed geodesic in X if there exists some, or 0 otherwise.

The following proposition shows the significance of the notions of the injectivity radius and the systole. See for example [**2**] for the proof. A metric space X is called *cocompact* if there exists a compact subset $K \subset X$ such that $X = \bigcup_{f \in \mathrm{Isom}(X)} fK$ holds.

Proposition 1.21 ([**2**] Proposition II.4.16). Let X be a cocompact proper geodesic metric space of curvature $\leq \kappa$. Then X fails to be CAT(κ) if and only if there exists a closed geodesic of length $< 2D_\kappa$. Moreover, if there exists such a closed geodesic, then there exists a closed geodesic of length $\mathrm{Sys}(X) = 2\iota_X$.

2. Preliminary on magnitude homology

In this section, we briefly recall some notations on magnitude homology from [**11**]. Through this section, (X, d) denotes a metric space unless otherwise mentioned.

Definition 2.1. An $(n+1)$-tuple $(x_0, \ldots, x_n) \in X^{n+1}$ is called an *n-chain* of X. An n-chain is *proper* if $x_i \neq x_{i+1}$ for all $0 \leq i \leq n$. The *length* $|x|$ of n-chain x is defined by

$$|x| := \sum_{i=0}^{n-1} d(x_i, x_{i+1}).$$

We denote the set of all proper n-chains of length ℓ by $P_n^\ell(X)$, and $P_n(X)$ denotes the union of them running through all $\ell \geq 0$. Let $MC_n^\ell(X)$ be the abelian group freely generated by $P_n^\ell(X)$.

Definition 2.2. Let a and b be points in X. A point $c \in X$ is *smooth between a and b* if the equality $d(a, b) = d(a, c) + d(c, b)$ holds. We denote $a \prec c \prec b$ if c is a smooth point between a and b with $a \neq c$ and $b \neq c$.

Definition 2.3. The *magnitude chain complex*

$$(MC_*^\ell(X), \partial_* := \sum_{i=1}^{*} (-1)^i \partial_n^i)$$

is defined by

$$\partial_n^i(x_0, \dots, x_n) = \begin{cases} (x_0, \dots, \hat{x}_i, \dots, x_n) & (\text{if } x_{i-1} \prec x_i \prec x_{i+1}) \\ 0 & (\text{otherwise}). \end{cases}$$

The homology group of magnitude chain complex is called the *magnitude homology group* of X, and denoted by $MH_*^\ell(X)$.

DEFINITION 2.4. If the point x_i of $x = (x_0, \dots, x_n) \in P_n(X)$ is **not** a smooth point between x_{i-1} and x_{i+1}, then we call it a *singular point* of x. We set the endpoints x_0 and x_n singular points. Let $\varphi(x) = (x_{s_0} = x_0, x_{s_1}, \dots, x_{s_k} = x_n) \in P_k(X)$ be the tuple of all singular points of x. We call $\varphi(x)$ the *frame* of x. A chain $x \in P_n^\ell(X)$ is *geodesically simple* if $|\varphi(x)| = |x|$ holds.

Let $P_n^F(X)$ be the set of all geodesically simple n-chains whose frame is $F \in P_{\leq n}^{|F|}(X) := \bigcup_{k \leq n} P_k^{|F|}(X)$. We denote the abelian group freely generated by $P_n^F(X)$ by $MC_n^F(X)$. We set

$$MC_n^{\mathrm{simp},\ell}(X) := \bigoplus_{F \in P_{\leq n}^\ell(X)} MC_n^F(X).$$

Note that both $MC_*^F(X)$ and $MC_n^{\mathrm{simp},\ell}(X)$ are subcomplexes of $MC_*^*(X)$, and we denote their homology by $MH_*^F(X)$ and $MH_n^{\mathrm{simp},\ell}(X)$ respectively.

DEFINITION 2.5. A proper 3-chain $x = (x_0, x_1, x_2, x_3)$ is a 4-*cut* if $\varphi(x) = (x_0, x_3)$ and $d(x_0, x_3) < |x|$ holds.

DEFINITION 2.6. We define m_X to be the infimum of the length of 4-cuts in X.

The following theorem is shown in [**11**].

THEOREM 2.7 ([**11**] Theorem 3.12, Theorem 5.11). (1)

$$MH_n^{\mathrm{simp},\ell}(X) \cong \bigoplus_{F \in P_{\leq n}^\ell} MH_n^F(X).$$

(2) For $n > 0$ and $0 < \ell < m_X$,

$$MH_n^{\mathrm{simp},\ell}(X) \cong MH_n^\ell(X).$$

(3) For a proper 1-chain $F = (x_0, x_1)$, the natural map

$$MH_n^F(X) \to MH_n^{|F|}(X)$$

is injective.

DEFINITION 2.8. Let a and b be points in X. The *interval poset* $I(a, b)$ is a poset which consists of smooth points between a and b, and the partial order $\leq$ among them is defined by

$$x \leq y \Leftrightarrow a \prec x \leq y.$$

Note that this definition is equivalent to

$$x \leq y \Leftrightarrow x \leq y \prec b.$$

We recall the definition of the order complex and its reduced chain complex of a poset.

DEFINITION 2.9. Let P be a poset. The *order complex of P* denoted by $\Delta(P)$ is the abstract simplicial complex whose n-simplices are the subsets $\{x_0, \ldots, x_n\}$ of P such that $x_0 \prec \cdots \prec x_n$. Its *reduced chain complex* denoted by $(C_*(\Delta(P)), \partial_*)$ is defined by

$$C_n(\Delta(P)) = \begin{cases} \bigoplus_{x_0 \prec \cdots \prec x_n} \mathbb{Z}\langle x_0, \ldots, x_n \rangle & n \geq 0, \\ \mathbb{Z} & n = -1, \\ 0 & n < -1, \end{cases}$$

and $\partial_n = \sum_{i=0}^{n} (-1)^i \partial_n^i$ with

$$\partial_n^i(\langle x_0, \ldots, x_n \rangle) = \begin{cases} \langle x_0, \ldots, \hat{x}_i, \ldots, x_n \rangle & n \geq 0, \\ 0 & n < 0. \end{cases}$$

The following theorem is also shown in [**?**].

THEOREM 2.10 ([**11**] Corollary 4.5). For a proper chain $F = (x_0, \ldots, x_m)$, we have

$$MH_n^F(X) \cong H_{n-2m}(C_*(\Delta_{0,1}) \otimes C_*(\Delta_{1,2}) \otimes \cdots \otimes C_*(\Delta_{m-1,m})),$$

where $C_*(\Delta_{i,i+1})$ is the reduced chain complex of the complex $\Delta(I(x_i, x_{i+1}))$.

3. Magnitude homology of CAT(κ) spaces

In this section, we study magnitude homology of a CAT(κ) space X. We will show that magnitude homology $MH_n^\ell(X)$ vanishes for $0 < \ell < D_\kappa$ and $n > 0$. Consequently, we can compute a total $\mathbb{Z}$-degree magnitude homology for some length ℓ for the spaces in Examples 1.10, 1.11, and 1.12. For the purpose, we introduce a quantity l_X for a metric space X.

DEFINITION 3.1. Let X be a metric space. A continuous map $\gamma\colon [0, c] \to X$ is *locally geodesic* if for every $t \in [0, c]$, there exists a neighborhood U of t such that $\gamma|_U$ is a geodesic.

DEFINITION 3.2. Let (X, d) be a metric space, and let $\gamma\colon [0, c] \to X$ be a continuous map. We define *the length of* γ by

$$|\gamma| := \sup_{0 = t_0 \leq t_1 \leq \cdots \leq t_n = c} \sum_{i=0}^{n} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all partitions of $[0, c]$.

DEFINITION 3.3. For a metric space X, we define l_X to be the infimum of length of locally geodesic paths which is not geodesics.

LEMMA 3.4. For a geodesic CAT(κ) space (X, d), we have

$$D_\kappa \leq l_X.$$

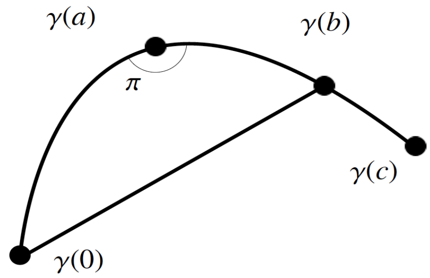

FIGURE 1

PROOF. Assume $l_X < D_\kappa$. Then there exists a map $\gamma\colon [0,c] \to X$ which is locally geodesic but not a geodesic, satisfying $l_X \leq |\gamma| < D_\kappa$. Let a be the supremum of the numbers $0 < t < c$ such that $\gamma|_{[0,t]}$ is a geodesic. Then $\gamma|_{[0,a]}$ is a geodesic by the continuity of d. Note that a is positive since γ is locally geodesic. Let b be the supremum of numbers $a < t \leq c$ such that $\gamma|_{[a,t]}$ is a geodesic. Then $\gamma|_{[a,b]}$ is also a geodesic, and b is positive. Let δ be a geodesic between $\gamma(0)$ and $\gamma(b)$. See Figure 1. By the assumption $|\gamma| < D_\kappa$, the geodesic triangle $\Delta_{\gamma(0),\gamma(a),\gamma(b)}$ satisfies $d(\gamma(0),\gamma(a)) + d(\gamma(a),\gamma(b)) + d(\gamma(b),\gamma(0)) < 2D_\kappa$. Hence $\Delta_{\gamma(0),\gamma(a),\gamma(b)}$ is κ-small. We note that its angle at $\gamma(a)$ is π because γ is locally geodesic. Therefore by the CAT(κ) condition for angles in Proposition 1.15, the angles of the comparison triangle of $\Delta_{\gamma(0),\gamma(a),\gamma(b)}$ at $\tilde{\gamma}(0)$ and $\tilde{\gamma}(b)$ are both 0. By the CAT(κ) condition for length, there is a 1-1 correspondence between $\gamma|_{[0,b]}$ and δ, which implies $\gamma|_{[0,b]}$ is a geodesic. This contradicts the definition of a. □

We recall that the quantity m_X is the infimum of the length of 4-cuts in X. (Definition 2.6.)

THEOREM 3.5. For a geodesic CAT(κ) space (X,d), we have

$$D_\kappa \leq m_X \leq l_X.$$

PROOF. We first show the former inequality. Assume $m_X < D_\kappa$. Then there exists a 4-cut $x = (x_0, x_1, x_2, x_3)$ in X with $m_X \leq |x| < D_\kappa$. Let $\{\gamma_{ij}\colon [0, d(x_i, x_j)] \to X \mid 0 \leq i < j \leq 3, (i, j) \neq (0, 3)\}$ be a family of geodesics between these points, and $\Delta_{x_0x_1x_2}$ and $\Delta_{x_1x_2x_3}$ be its geodesic triangles. See Figure 2. Because $|x|$ is smaller than D_κ, the geodesic triangles $\Delta_{x_0x_1x_2}$ and $\Delta_{x_1x_2x_3}$ are both κ-small. Note that the comparison triangles $\tilde{\Delta}_{x_0x_1x_2}$ and $\tilde{\Delta}_{x_1x_2x_3}$ are both degenerated because they are on some semispheres of S_κ. Hence by the CAT(κ) inequality, the unions of geodesics $\gamma_{01} \cup \gamma_{12}$ and $\gamma_{12} \cup \gamma_{23}$ coincide with γ_{02} and γ_{13} respectively. Therefore the map $\gamma_{02} \cup \gamma_{23}$ can be defined and is locally geodesic, which is actually a geodesic because of the inequality $|x| < D_\kappa \leq l_X$ as shown in Lemma 3.4. Then we have $d(x_0, x_3) = d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3)$, which contradicts that the chain x is a 4-cut. Thus we obtain $m_X \geq D_\kappa$.

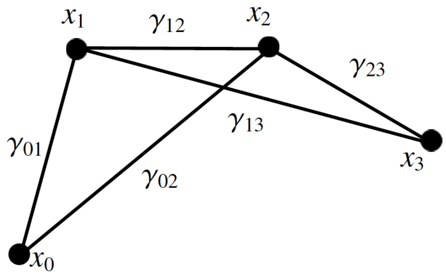

FIGURE 2

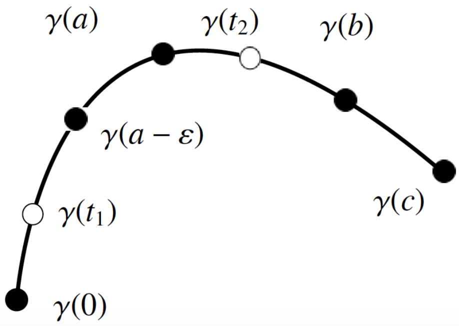

FIGURE 3

Next we show the latter inequality. Assume $l_X < m_X$. Then there exists a locally geodesic map $\gamma\colon [0,c] \to X$ which is not a geodesic and is satisfying $l_X \leq |\gamma| < m_X$. Take $a > 0$ as the supremum of the number t such that $\gamma|_{[0,t]}$ is a geodesic. Let ϵ be a sufficiently small positive number. Let $b > a$ be the supremum of the number t such that $\gamma|_{[a-\epsilon,t]}$ is a geodesic. See Figure 3. If $\left|\gamma|_{[0,b]}\right|$ is smaller than l_X, then $\gamma|_{[0,b]}$ is a geodesic, which contradicts the assumption $a < c$. Hence we have $\left|\gamma|_{[0,b]}\right| \geq l_X$. Note that the proper chain $(\gamma(0), \gamma(a-\epsilon), \gamma(a), \gamma(b))$ has no singular points other than end points, but is not a 4-cut because we have $\left|\gamma|_{[0,b]}\right| \leq |\gamma| < m_X$. Similarly, neither is $(\gamma(t_1), \gamma(a-\epsilon), \gamma(a), \gamma(t_2))$ for $0 \leq t_1 \leq a-\epsilon$ and $a \leq t_2 \leq b$. Hence we have $d(\gamma(t_2), \gamma(t_1)) = (a-\epsilon-t_1) + (a-(a-\epsilon)) + (t_2-a) = t_2 - t_1$, which implies that $\gamma|_{[0,b]}$ is a geodesic. Thus we obtain a contradiction and conclude $l_X \geq m_X$. □

COROLLARY 3.6. Let (X,d) be a geodesic CAT(κ) space. Then for any $n > 0$ and $0 < \ell < D_\kappa$, magnitude homology $MH_n^\ell(X)$ vanishes.

PROOF. By Theorem 2.7 (3) and Theorem 3.5, we have

$$MH_n^\ell(X) \cong \bigoplus_{|F|=\ell} MH_n^F(X)$$

for $0 < \ell < D_\kappa$. We show that every interval poset $I(x,y)$ is totally ordered for $d(x,y) < D_\kappa$, which implies that $MH_n^F(X) = 0$ for $|F| < D_\kappa$ by Theorem 2.10. Let a be a smooth point between x and y. Let $\overline{xy}, \overline{xa}$ and $\overline{ay}$ be geodesics connecting each pair of points. Then the geodesic triangle Δ_{xay} is κ-small since $d(x, y) < D_\kappa$, hence the point a is on $\overline{xy}$. Thus we conclude that $\overline{xy}$ is the unique geodesic connecting x and y, and the interval poset $I(x, y)$ is precisely equal to $\overline{xy}$ which is totally ordered. □

Example 3.7. For the circle $C_{1/\sqrt{\kappa}}$ of radius $1/\sqrt{\kappa}$ and the n-sphere $\mathbb{S}^n_\kappa$ of radius $1/\sqrt{\kappa}$, we have

$$MH^\ell_n(C_{1/\sqrt{\kappa}}) = MH^\ell_n(\mathbb{S}^n) = 0,$$

for $0 < \ell < \pi/\sqrt{\kappa}$ and $n > 0$.

Example 3.8. For the Euclidean space $\mathbb{E}^n$, the hyperbolic space $\mathbb{H}^n$, and every tree T, magnitude homology $MH^\ell_n(\mathbb{E}^n)$, $MH^\ell_n(\mathbb{H}^n)$, and $MH^\ell_n(T)$ vanishes for all $\ell > 0$ and $n > 0$.

4. Closed geodesics represent non-trivial Magnitude homology classes

In this section, we show that an existence of closed geodesic in a metric space X guarantees the non-triviality of the second magnitude homology $MH^*_2(X)$. As a corollary, we give a criterion of being CAT(κ) for a cocompact proper geodesic metric space X of curvature $\leq \kappa$ from a viewpoint of the second magnitude homology. We begin by clarifying the definition of a closed geodesic.

Definition 4.1. Let X be a metric space, and let C_r be the circle of radius r. An isometry $\rho\colon C_r \to X$ is called a *closed geodesic of radius r* (or *of length $2\pi r$*) in X.

Proposition 4.2. Let (X, d) be a metric space, and let $\rho\colon C_r \to X$ be a closed geodesic. Let $0, 1 \in C_r$ be a pair of antipodal points. Then the interval poset $I(\rho(0), \rho(1))$ has at least two connected components.

Proof. Let U, V be semicircles in C_r with $C_r = U \cup V$ and $U \cap V = \{0, 1\}$. Take points $x \in U - \{0, 1\}$ and $y \in V - \{0, 1\}$, and assume $\rho(x) \leq \rho(y)$ in the poset $I(\rho(0), \rho(1))$. Namely, we have

$$d(\rho(0), \rho(y)) = d(\rho(0), \rho(x)) + d(\rho(x), \rho(y)).$$

Since we have $\rho(0) \prec \rho(x) \prec \rho(1)$ and $\rho(0) \prec \rho(y) \prec \rho(1)$, we obtain

$$\begin{aligned} d(\rho(x), \rho(1)) &= d(\rho(0), \rho(1)) - d(\rho(0), \rho(x)) \\ &= d(\rho(0), \rho(1)) - d(\rho(0), \rho(y)) + d(\rho(x), \rho(y)) \\ &= d(\rho(y), \rho(1)) + d(\rho(x), \rho(y)) \end{aligned}$$

We also have either

$$d(\rho(x), \rho(y)) = d(\rho(x), \rho(1)) + d(\rho(1), \rho(y))$$

or

$$d(\rho(x), \rho(y)) = d(\rho(x), \rho(0)) + d(\rho(0), \rho(y)).$$

Hence we obtain

$$d(\rho(y), \rho(1)) = 0,$$

or

$$d(\rho(x), \rho(1)) - d(\rho(x), \rho(0)) = \pi r,$$

respectively. Each case implies $y = 1$ or $x = 0$, which is a contradiction. Hence any pair of points in $\rho(U - \{0, 1\})$ and $\rho(V - \{0, 1\})$ are not comparable, which implies the order complex $\Delta(I(\rho(0), \rho(1)))$ is not connected. □

Theorem 4.3. Let X be a metric space. If there exists a closed geodesic of radius r in X, then we have $MH_2^{\pi r}(X) \neq 0$.

Proof. Let ρ be a closed geodesic of radius r, and let $\rho(0)$ and $\rho(1)$ be antipodal points of it. Then for the proper 1-chain $F = (\rho(0), \rho(1))$, there exists an injection $MH_2^F(X) \to MH_2^{|F|}(X)$ by Theorem 2.7 (3). Furthermore, we have $MH_2^F(X) \cong H_0(C_*(\Delta(I(\rho(0), \rho(1)))))$ by Theorem 2.10, and Proposition 4.2 implies this is non-zero. Hence the statement follows. □

Corollary 4.4. Let X be a cocompact proper geodesic metric space of curvature $\leq \kappa$. Then the following are equivalent.

(i) X fails to be CAT(κ).
(ii) there exists a closed geodesic of length $< 2D_\kappa$.
(iii) $MH_2^{<D_\kappa}(X) \neq 0$.

Proof. By Proposition 1.21, (i) implies (ii). By Theorem 4.3, (ii) implies (iii). By Corollary 3.6, (iii) implies (i). □

In particular, the equivalence of (i) and (iii) in Corollary 4.4 gives a criterion of being CAT(κ) for a cocompact proper geodesic metric space (especially for compact or homogeneous Riemannian manifold) whose curvature is bounded from above.

5. Magnitude homology of non-CAT(κ) metric spaces of curvature $\leq \kappa$

In this section, we study magnitude homology of proper geodesic metric spaces by using the injectivity radi and the systoles. As a corollary, we obtain a vanishing theorem for magnitude homology of cocompact proper geodesic metric spaces, and give a partial computation of magnitude homology of the projective spaces $\mathbb{RP}^n$ with the standard metric.

Proposition 5.1. For a metric space X, we have

$$2\iota_X \leq \mathrm{Sys}(X),$$

and

$$2l_X \leq \mathrm{Sys}(X).$$

Proof. The former inequality is immediate. For the latter, suppose

$$\mathrm{Sys}(X) < 2l_X.$$

Then there exists a closed geodesic $c\colon C_r \to X$ of length $2\pi r < 2l_X - 2\delta$ for small $\delta > 0$. Then the restriction of c on the interval $[0, \pi r + \delta]$ is locally

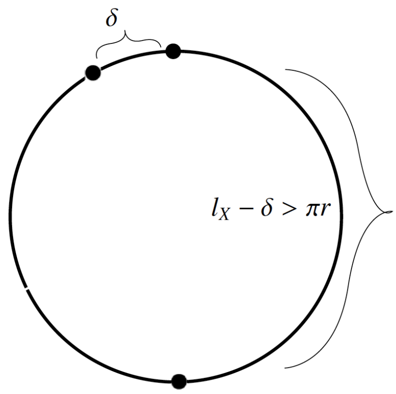

FIGURE 4

geodesic but not a geodesic, with length $\pi r + \delta < l_X$. Hence we obtain a contradiction. □

PROPOSITION 5.2. Let (X, d) be a proper geodesic metric space of curvature $\leq \kappa$. If the injectivity radius ι_X is not greater than D_κ, then we have

$$\iota_X \leq l_X.$$

For the proof of Proposition 5.2, we use the Arzelà-Ascoli theorem of the following form. Recall that a sequence of maps $\{f_n : Y \to X\}$ between metric spaces is called *equicontinuous* if for any positive number ϵ there exists a positive number δ such that $d(f_n(a), f_n(b)) < \epsilon$ holds for any n and $a, b \in Y$ with $d(a, b) < \delta$.

LEMMA 5.3. Let X be a compact metric space, and $\{\gamma_n : [0, 1] \to X\}$ be an equicontinuous sequence of maps. Then there exists a subsequence $\{\gamma_{n_i}\}$ which uniformly converges to a continuous map $\gamma : [0, 1] \to X$.

PROOF. See for example [**2**] Chapter I Lemma 3.10. □

PROOF OF PROPOSITION 5.2. Suppose $l_X < \iota_X$. Then there exists a locally geodesic path $\gamma : [0, L] \to X$ which is not a geodesic, satisfying $l_X \leq |\gamma| < \iota_X$. Let $0 < a < 1$ be the supremum of the number $0 \leq t < 1$ such that $\gamma|_{[0,t]}$ is a geodesic. Then $\gamma|_{[0,a]}$ is a geodesic by the continuity of d. Let α be a positive number such that $\gamma|_{[a,a+\alpha]}$ is a geodesic. We have $\alpha > 0$ by the assumption. We linearly reparametrize $\gamma|_{[0,a]}$ and $\gamma|_{[a,a+\alpha]}$ so that the domain of each map is $[0, 1]$. Let $\delta : [0, 1] \to X$ be the linearly reparametrized geodesic connecting $\gamma(0)$ and $\gamma(a + \alpha)$. Let $\gamma_s : [0, 1] \to X$ be the linearly reparametrized geodesic connecting $\gamma(a)$ and $\delta(s)$ for $0 \leq s \leq 1$. Note that $\gamma|_{[0,a]} = \gamma_0$ and $\gamma|_{[a,a+\alpha]} = \gamma_1$. We show that the maps $\{\gamma_s\}$ vary continuously with s. Let $\{s_n\}$ be a sequence of points of the interval $[0, 1]$ converging to a point $s' \in [0, 1]$. If γ_{s_n} does not converge, there exists a number $t_0 \in [0, 1]$, a number $\delta > 0$ and a subsequence $\{s_{n_i}\}$ such that $d(\gamma_{s_{n_i}}(t_0), \gamma_{s'}(t)) > \delta$. Then by Lemma 5.3, there exists a geodesic connecting $\gamma(a)$ and $\gamma(s')$ which is different from $\gamma_{s'}$. It contradicts the uniqueness of geodesic. Therefore the sequence $\{\gamma_{s_n}\}$ uniformly converges to $\gamma_{s'}$, which implies that geodesics vary continuously. Then by Proposition 1.16, the angles of $\Delta_{\gamma(0)\gamma(a)\gamma(a+\alpha)}$ are not greater than the corresponding angles of comparison triangle in S_κ. Since γ is locally geodesic, the angle at $\gamma(a)$ is π, hence the comparison triangle is degenerated. Thus the angles at $\gamma(0)$ and $\gamma(a+\alpha)$ are both 0. Let p be a point on δ apart from $\gamma(a+\alpha)$ by α. Then by Proposition 1.4 and Proposition 1.16, the angle at $\gamma(a+\alpha)$ of the comparison triangle $\tilde{\Delta}_{\gamma(a)p\gamma(a+\alpha)}$ in S_κ is 0. Hence $\tilde{\Delta}_{\gamma(a)p\gamma(a+\alpha)}$ is degenerated, and we obtain $\gamma(a) = p$. Then we have $\gamma|_{[0,a+\alpha]} = \delta$ by $|\gamma| < \iota_X$, which contradicts to the definition of a. Therefore we obtain $l_X \geq \iota_X$. □

Proposition 5.4. Let (X, d) be a proper geodesic metric space of curvature $\leq \kappa$. If the injectivity radius ι_X is not greater than D_κ, then we have

$$\iota_X \leq m_X.$$

Proof. Suppose $m_X < \iota_X$. Then there exists a 4-cut $x = (x_0, x_1, x_2, x_3)$ with $m_X \leq |x| < \iota_X$. Let γ_{01}, γ_{12} and γ_{02} be the linearly reparametrized geodesics connecting each pair of points x_0, x_1 and x_2. By a similar argument as in the proof of Proposition 5.2, the geodesics connecting x_1 and points on γ_{02} varies continuously. Then by Proposition 1.16, the angles of $\Delta_{x_0x_1x_2}$ are not greater than the corresponding angles of comparison triangle in S_κ. Note that the comparison triangle $\tilde{\Delta}_{x_0x_1x_2}$ is degenerated because it is κ-small and x_1 is smooth between x_0 and x_2, hence x_1 is on the image of γ_{02} by the similar argument in the proof of Proposition 5.2. Similarly, we obtain a geodesic γ_{13} connecting x_1 and x_3, and going through x_2. Then geodesics γ_{02} and γ_{13} coincide between x_1 and x_2 by the assumption $m_X < \iota_X$, hence we obtain a locally geodesic path by glueing them. The obtained path turns out to be a geodesic since we have $|x| < \iota_X \leq l_X$ by Proposition 5.2, which contradicts that $d(x_0, x_3) < |x|$. □

Corollary 5.5. Let X be a cocompact proper geodesic metric space of curvature $\leq \kappa$ which is not CAT(κ), or the standard sphere. Then for any $n > 0$ and any $0 < \ell < \iota_X = \mathrm{Sys}(X)/2$, magnitude homology $MH_n^\ell(X)$ vanishes.

Proof. For the standard sphere, it follows from Corollary 3.6. For X being a proper geodesic metric space of curvature $\leq \kappa$ which is not CAT(κ), we have $\iota_X = \mathrm{Sys}(X)/2 < D_\kappa$ by Proposition 1.21. Furthermore, we have $\iota_X \leq m_X$ by Proposition 5.4. Hence the statement follows from the similar argument in the proof of Corollary 3.6 and Proposition 5.4. □

Example 5.6. As mentioned in Example 1.19, the standard projective space $\mathbb{RP}^n$ for $n \geq 2$ is of curvature ≥ 1 and is not CAT(1). Furthermore, we can immediately check that $\mathrm{Sys}(\mathbb{RP}^n) = \pi$. Hence by Corollary 5.5, we obtain

$$MH_*^\ell(\mathbb{RP}^n) = 0,$$

for $0 < \ell < \pi/2$ and $* > 0$.

CHAPTER 4

Vietoris-Rips and singular homology of Riemannian manifolds

1. Vietoris-Rips homology

Let (X, d) be a metric space. Let ε be a positive number or ∞. In this chapter, we consider two kinds of simplicial complexes which are both referred to as Vietoris-Rips complex.

Definition 1.1. The *Vietoris-Rips complex with parameter* ε is a simplicial complex whose n-simplices are $(n+1)$ points subsets $\sigma = \{x_0, \dots, x_n\} \subset X$ with diam $\sigma \leq \varepsilon$.

Remark 1.2. Consider a simplicial set whose n-simplices consist of $(n+1)$-tuples $(x_0, \dots x_n)$ with diam $\sigma \leq \varepsilon$ equipped with the standard face and degeneracy maps. It is well known that the associated chain complex is naturally homotopy equivalent to that of Vietoris-Rips complex. See [**14**] Theorem 13.6 for example. We consider this chain complex in this chapter.

Definition 1.3. We denote $VC_*^{\leq\varepsilon}(X)$ the chain complex associated to the simplicial set constructed in Remark 1.2, and we call it *Vietoris-Rips chain complex*. We denote its homology by $VH_*^{\leq\varepsilon}(X)$, and call it *Vietoris-Rips homology*. We also define another chain complex $VC_*^{<\varepsilon}(X)$ and its homology $VH_*^{<\varepsilon}(X)$ by considering strict inequality instead. We also refer to them as Vietoris-Rips chain complex and homology.

Remark 1.4. Although we only discuss chain complex with $\leq$ in this chapter, the same argument is applicable for strict inequality case.

2. Results

Let (X, d) be a metric space.

Definition 2.1. Let $r(X)$ be the supremum of non-negative real numbers r satisfying the following:

(1) There uniquely exists a geodesic between every pair of points x, y with $d(x, y) < r$,
(2) For any three points x, y, z with diam $\{x, y, z\} < r$, and for every point w on the geodesic between y, z, they satisfy

$$d(x, w) \leq \max\{d(x, y), d(x, z)\}.$$

Remark 2.2. It is well known that $r(X) > 0$ for Riemannian manifolds X with strictly positive injectivity radius and an upper bound on its sectional curvature. In particular, $r(X) > 0$ for compact Riemannian manifolds.

The purpose of this section is to prove the following.

Theorem 2.3. The following inclusions of chain complexes are chain homotopy equivalences :

$$VC_n^{\le\varepsilon'}(X) \longrightarrow VC_n^{\le\varepsilon}(X),$$
$$VC_n^{<\varepsilon'}(X) \longrightarrow VC_n^{\le\varepsilon}(X),$$
$$VC_n^{<\varepsilon'}(X) \longrightarrow VC_n^{<\varepsilon}(X),$$

for any $0 < \varepsilon' \le \varepsilon < r(X)$, and

$$VC_n^{\le\varepsilon'}(X) \longrightarrow VC_n^{<\varepsilon}(X),$$

for any $0 < \varepsilon' < \varepsilon \le r(X)$.

To prove Theorem 2.3, we transfer some standard arguments in singular homology theory to Vietoris-Rips homology theory. For a reference for original singular homology theory, see [**8**] section 2.1 for example. After preparing a series of definitions and propositions, we prove Proposition 2.22, which completes a proof of Theorem 2.3.

Definition 2.4. Let b be a point on X. We define a homomorphism

$$b_n : VC_n^{\le\varepsilon}(X) \longrightarrow VC_{n+1}^{\le\infty}(X)$$

by

$$b_n(x_0, \dots, x_n) = (b, x_0, \dots, x_n).$$

Proposition 2.5. If $\mathrm{Im} b_* \subset VC_*^{\le\varepsilon}(X)$, then b_* satisfies

$$b_{n-1}\partial_n + \partial_{n+1}b_n = id_n.$$

Proof. It is immediate from the following calculation:

$$\begin{aligned}\partial_{n+1}b_n(x_0, \dots, x_n) &= \partial_{n+1}(b, x_0, \dots, x_n)\\ &= (x_0, \dots, x_n) + \sum_{i=0}^{n}(-1)^{i+1}(b, x_0, \dots, \check{x}_i, \dots, x_n)\\ &= (x_0, \dots, x_n) - b_{n-1}\partial_n(x_0, \dots, x_n).\end{aligned}$$

□

Definition 2.6. Let (X, d) be a metric space. Suppose that

$$\sigma = (x_0, \dots, x_n) \in X^{n+1}$$

satisfies $\operatorname{diam}\{x_0, \dots, x_n\} < r(X)$. We define the *barycenter* $b_\sigma \in X$ of σ inductively on n as follows:

- $n = 0$ **:** $b_{(x)} = x$

- $n = k$ **:** Let $\sigma_i = (x_0, \ldots, x_i)$. b_{σ_k} is defined to be the point on the geodesic between $b_{\sigma_{k-1}}$ and x_k satisfying $2^k d(b_{\sigma_k}, b_{\sigma_{k-1}}) = d(x_k, b_{\sigma_{k-1}})$.

Proposition 2.7. Let $\sigma = (x_0, \ldots, x_n)$ be a tuple with

$$\operatorname{diam}\{x_0, \ldots, x_n\} < r(X).$$

Then it satisfies

$$d(b_\sigma, x_i) \leq (1 - \frac{1}{2^n})\operatorname{diam}\sigma,$$

for every $0 \leq i \leq n$.

Proof. We prove by the induction on n:

- $n = 0$ **:** It is trivial.
- $n = k$ **:** By definition, we have

$$d(b_\sigma, x_k) = (1 - \frac{1}{2^k})d(x_k, b\sigma_{k-1}) \leq (1 - \frac{1}{2^k})\operatorname{diam}\sigma.$$

For the other x_i's, we have

$$\begin{aligned} d(b_\sigma, x_i) &\leq d(b_\sigma, b_{\sigma_{k-1}}) + d(b_{\sigma_{k-1}}, x_i) \\ &\leq \frac{1}{2^k} d(x_k, b\sigma_{k-1}) + (1 - \frac{1}{2^{k-1}})\operatorname{diam}\sigma \\ &\leq (1 - \frac{1}{2^k})\operatorname{diam}\sigma. \end{aligned}$$

□

Proposition 2.8. Let $\sigma = (x_0, \ldots, x_n)$ be a tuple with

$$\operatorname{diam}\{x_0, \ldots, x_n\} < r(X).$$

Let λ be a subtuple of σ. Then it satisfies

$$d(b_\sigma, b_\lambda) \leq \operatorname{diam}\sigma.$$

Proof. We prove inductively on the size of λ. We denote $|\lambda| = m$ if λ consists of m coordinates.

- $|\lambda| = 0$ **:** It reduces to Proposition 2.7.
- $|\lambda| = k \leq n$ **:** By definition of $r(X)$, we have

$$d(\sigma, \lambda) \leq \max\{d(b_\sigma, b_{\lambda'}), d(b_\sigma, x_i)\},$$

where λ' is the subtuple of λ obtained by eliminating the last coordinate, and b_λ locates between $b_{\lambda'}$ and x_i. Then it reduces to lower size case.

□

Definition 2.9. We inductively define a homomorphism

$$S_n : VC_n^{\leq\varepsilon}(X) \longrightarrow VC_n^{\leq\infty}(X),$$

for any $n \geq 0$ and $\varepsilon < r(X)$ by

$$S_n\sigma = b_\sigma(S_{n-1}\partial_n\sigma),$$

for any $(n+1)$-tuples $\sigma \in VC_n^{\le\varepsilon}(X)$.

PROPOSITION 2.10. The image of S_n is contained in $VC_n^{\le\varepsilon}(X)$ for every n.

PROOF. It follows from Proposition 2.8 and by induction on n. The $n = 0$ case is clear. Assume that the statement is true for $n \le k$. Then the chain $S_{k+1}\sigma$ consists of simplices of the form $b_\sigma *$, whose diameter is less than $\varepsilon < r(X)$ by the assumption and Proposition 2.8. This completes a proof. □

PROPOSITION 2.11. The family of homomorphisms S_* is a chain map.

PROOF. We extend the grading of the chain complex from $\mathbb{Z}_{\ge 0}$ to $\mathbb{Z}$ by putting 0-modules on the negative part. We prove the statement by the induction on n:

- $S_{-1}\partial_0 = \partial_0 S_0$ **:** It is trivial.
- $S_k\partial_{k+1} = \partial_{k+1}S_{k+1}$ **:** It follows from the following calculation:

$$\begin{aligned}\partial_{k+1}S_{k+1}\sigma &= \partial_{k+1}b_\sigma(S_k\partial_{k+1}\sigma)\\ &= (-b_\sigma\partial_k + id_k)(S_k\partial_{k+1}\sigma)\\ &= -b_\sigma\partial_k S_k\partial_{k+1}\sigma + S_k\partial_{k+1}\sigma\\ &= -b_\sigma S_{k-1}\partial_k\partial_{k+1}\sigma + S_k\partial_{k+1}\sigma\\ &= S_k\partial_{k+1}\sigma,\end{aligned}$$

  where the second equality is by Proposition 2.5 and the 4th equality is by the assumption.

□

DEFINITION 2.12. We inductively define a homomorphism

$$T_n : VC_n^{\le\varepsilon}(X) \longrightarrow VC_n^{\le\infty}(X),$$

for any $n \ge 0$ and $\varepsilon < r(X)$ by

$$T_n\sigma = b_\sigma(\sigma - T_{n-1}\partial_n\sigma),$$

for any $(n+1)$-tuples $\sigma \in VC_n^{\le\varepsilon}(X)$.

PROPOSITION 2.13. The image of S_n is contained in $VC_n^{\le\varepsilon}(X)$ for every n.

PROOF. It follows from Proposition 2.8 and by induction on n. The $n = 0$ case is clear. Assume that the statement is true for $n \le k$. Then the chain $T_{k+1}\sigma$ consists of simplices of the form $b_\sigma *$, whose diameter is less than $\varepsilon < r(X)$ by the assumption and Proposition 2.8. This completes a proof.

□

PROPOSITION 2.14. The family of homomorphisms T_* is a chain homotopy between id_* and S_*, namely,

$$\partial_{n+1}T_n + T_{n-1}\partial_n = id_n - S_n,$$

for every n.

Proof. We prove inductively on n. It is trivial for $n = 0$. For the general case, it follows from the following calculation:

$$\begin{aligned}\partial_{k+1}T_k\sigma &= \partial_{k+1}b_\sigma(\sigma - T_{k-1}\partial_k\sigma)\\ &= (-b_\sigma\partial_k + id_k)(\sigma - T_{k-1}\partial_k\sigma)\\ &= -b_\sigma\partial_k\sigma + b_\sigma\partial_k T_{k-1}\partial_k\sigma + \sigma - T_{k-1}\partial_k\sigma\\ &= -b_\sigma\partial_k\sigma + b_\sigma(-T_{k-2}\partial_{k-1} + id_{k-1} - S_{k-1})\partial_k\sigma + \sigma - T_{k-1}\partial_k\sigma\\ &= -b_\sigma S_{k-1}\partial_k\sigma + \sigma - T_{k-1}\partial_k\sigma\\ &= -S_k\sigma + \sigma - T_{k-1}\partial_k\sigma,\end{aligned}$$

where the second equality is by Proposition 2.5 and the 4th equality is by the assumption. □

Definition 2.15. We define a homomorphism

$$D_n^m : VC_n^{\le\varepsilon}(X) \longrightarrow VC_n^{\le\varepsilon}(X)$$

by

$$D_n^m := \sum_{i=0}^{m-1} T_n S_n^i,$$

for any m and n.

Proposition 2.16. For any m and n, we have

$$\partial_{n+1}D_n^m + D_{n-1}^m\partial_n = id_n - S_n^m.$$

Proof. It follows from the following calculation:

$$\begin{aligned}\partial_{n+1}D_n^m + D_{n-1}^m\partial_n &= \sum_{i=0}^{m-1} \partial_{n+1}T_nS_n^i + T_{n-1}S_{n-1}^i\partial_n\\ &= \sum_{i=0}^{m-1} \partial_{n+1}T_nS_n^i + T_{n-1}\partial_nS_n^i\\ &= \sum_{i=0}^{m-1}(id_n - S_n)S_n^i\\ &= id_n - S_n^m.\end{aligned}$$

□

Definition 2.17. Let $\sigma \in X^{n+1}$ and $\varepsilon > 0$. We define $m^\varepsilon(\sigma)$ to be the minimum number m satisfying $S^m\sigma \in VC_n^{\le\varepsilon}(X)$.

Definition 2.18. Let $0 < \varepsilon' \le \varepsilon$. We define a homomorphism

$$D_n^{\varepsilon'} : VC_n^{\le\varepsilon}(X) \longrightarrow VC_{n+1}^{\le\varepsilon}(X)$$

by

$$D_n^{\varepsilon'}\sigma := D_n^{m^{\varepsilon'}(\sigma)}\sigma,$$

for every n. We further define a homomorphism

$$\rho_n^{\varepsilon'} : VC_n^{\le\varepsilon}(X) \longrightarrow VC_n^{\le\varepsilon}(X)$$

by

$$\rho_n^{\varepsilon'} := id_n - \partial_{n+1} D_n^{\varepsilon'} - D_{n-1}^{\varepsilon'} \partial_n,$$

for every n.

PROPOSITION 2.19. The family of homomorphisms $\rho_*^{\varepsilon'}$ is a chain map.

PROOF. It follows from the following calculation:

$$\begin{aligned} \partial_n \rho_n^{\varepsilon'} &= \partial_n (id_n - \partial_{n+1} D_n^{\varepsilon'} - D_{n-1}^{\varepsilon'} \partial_n) = \partial_n - \partial_n D_{n-1}^{\varepsilon'} \partial_n \\ \rho_{n-1}^{\varepsilon'} \partial_n &= (id_{n-1} - \partial_n D_{n-1}^{\varepsilon'} - D_{n-2}^{\varepsilon'} \partial_{n-1}) \partial_n = \partial_n - \partial_n D_{n-1}^{\varepsilon'} \partial_n. \end{aligned}$$

□

PROPOSITION 2.20. The image of $\rho_*^{\varepsilon'}$ is contained in the chain complex $VC_*^{\leq \varepsilon'}(X)$.

PROOF. Let $\sigma \in X^{n+1}$. By Proposition 2.16, we have

$$\begin{aligned} \rho_n^{\varepsilon'} \sigma &= \sigma - D_{n-1}^{\varepsilon'} \partial_n \sigma - \partial_{n+1} D_n^{\varepsilon'} \sigma \\ &= \sigma - D_{n-1}^{\varepsilon'} \partial_n \sigma - (id_n - S_n^{m^{\varepsilon'}(\sigma)} - D_{n-1}^{m^{\varepsilon'}(\sigma)} \partial_n) \sigma \\ &= S_n^{m^{\varepsilon'}(\sigma)} \sigma - D_{n-1}^{\varepsilon'} \partial_n \sigma + D_{n-1}^{m^{\varepsilon'}(\sigma)} \partial_n \sigma. \end{aligned}$$

Here, $S_n^{m^{\varepsilon'}(\sigma)} \sigma \in VC_n^{\leq \varepsilon'}(X)$ by definition. For each component σ' of $\partial_n \sigma$, it is clear that $m^{\varepsilon'}(\sigma') \leq m^{\varepsilon'}(\sigma)$, and we have

$$D_n^{m^{\varepsilon'}(\sigma)} - D_n^{m^{\varepsilon'}(\sigma')} = \sum_{i=m^{\varepsilon'}(\sigma')}^{m^{\varepsilon'}(\sigma)-1} T_n S_n^i,$$

whose image is contained in $VC_*^{\leq \varepsilon'}(X)$. Hence the statement follows. □

REMARK 2.21. By Proposition 2.20, the chain map $\rho_*^{\varepsilon'} : VC_*^{\leq \varepsilon}(X) \longrightarrow VC_*^{\leq \varepsilon}(X)$ factors through $VC_*^{\leq \varepsilon'}(X)$. We also denote the obtained chain map $VC_*^{\leq \varepsilon}(X) \longrightarrow VC_*^{\leq \varepsilon'}(X)$ by $\rho_*^{\varepsilon'}$.

PROPOSITION 2.22. The chain map $\rho_*^{\varepsilon'} : VC_*^{\leq \varepsilon}(X) \longrightarrow VC_*^{\leq \varepsilon'}(X)$ is the chain homotopy inverse to the inclusion $\iota_*^{\varepsilon' \leq \varepsilon} : VC_n^{\leq \varepsilon'}(X) \longrightarrow VC_n^{\leq \varepsilon}(X)$.

PROOF. The composition $\rho_*^{\varepsilon'} \iota_*^{\varepsilon' \leq \varepsilon}$ is an indentity on $VC_n^{\leq \varepsilon'}(X)$ by definition of $m^{\varepsilon'}$. On the other hand, by definition of $\rho_*^{\varepsilon'}$, we have

$$\partial_{n+1} D_n^{\varepsilon'} - D_{n-1}^{\varepsilon'} \partial_n = id_n - \rho_n^{\varepsilon'},$$

for every n. It implies $\rho_*^{\varepsilon'}$ is chain homotopic to id_*. This completes a proof. □

PROOF OF THEOREM 2.3. We can also consider a chain map

$$\rho_*^{\varepsilon'} : VC_*^{\leq \varepsilon}(X) \longrightarrow VC_*^{< \varepsilon'}(X),$$

and we can similarly show that the other inclusions for example

$$VC_n^{< \varepsilon'}(X) \longrightarrow VC_n^{\leq \varepsilon}(X)$$

are chain homotopy equivalences. This completes a proof of Theorem 2.3. □

Corollary 2.23. For a Riemannian manifold X with $r(X) > 0$, we have

$$VH_*^{\leq\varepsilon}(X) \cong H_*(X;\mathbb{Z}),$$

and

$$VH_*^{<\varepsilon}(X) \cong H_*(X;\mathbb{Z}),$$

for sufficiently small $0 < \varepsilon$.

Proof. It follows from Theorem 2.3 and Hasumann's Proposition 3.4 ([**9**]). □

Bibliography

[1] Y. Asao, Magnitude homology of geodesic metric spaces with an upper curvature bound, preprint, 2019, arXiv:1903.11794.

[2] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999.

[3] S. Cho, *Quantales, persistence, and magnitude homology*, preprint, 2019, arXiv:1910.02905.

[4] M. Gromov, *Hyperbolic groups, in Essays in group theory*, S.M. Gersten(ed.), MSRI Publ. 8, 1987, 75-263.

[5] K. Gomi, *Smoothness filtration of magnitude complex*, preprint, 2018, arXiv:1809.06593.

[6] K. Gomi, *Magnitude homology of geodesic space*, preprint, 2019, arXiv:1902.07044.

[7] Y. Gu, *Graph magnitude homology via algebraic Morse theory* preprint, 2018, arXiv:1809.07240.

[8] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[9] J.-C. Hausmann, *On the Vietoris - Rips complexes and a cohomology theory for metric spaces*, pages 175 - 188, Princeton U. Press, Princeton, 1995.

[10] R. Hepworth and S. Willerton, *Categorifying magnitude of a graph*. Homology Homotopy Appl. 19 (2017), no. 2, 31 - 60.

[11] R. Kaneta and M. Yoshinaga, *Magnitude homology of metric spaces and order complexes*. preprint, arXiv:1803.04247, 2018.

[12] T. Leinster, *magnitude of metric spaces*. Doc. Math. 18 (2013), 857 - 905.

[13] T. Leinster and M. Shulman, *Magnitude homology of enriched categories and metric spaces*. preprint, arXiv:1711.00802.

[14] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[15] P. May, *Simplicial objects in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1992.

[16] N. Otter, *Magnitude meets persistence. Homology theories for filtered simplicial sets*, preprint, 2018, arXiv:1807.01540.

[17] R. E. Reed, *Foundations of Vietoris homology with applications to non-compact spaces*, 1980. PhD thesis, Polska Akademia Nauk, Instytut Matematyczny.

[18] L. Vietoris, *Uber den höheren Zusammenhang kompakter Raüme und eine Klasse von zusammenhangstreuen Abbildungen*, Mathematische Annalen, 97:454 - 472, 1927.