博士論文 (要約)

論文題目: Geometry of cluster modular groups
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1. Cluster algebra and the Teichmüller-Thurston theory

The study of cluster algebra is initiated by Fomin–Zelevinsky [FZ02] and Fock–Goncharov [FG03] independently, and has been developed with fruitful connections with other areas of mathematics, such as discrete integrable systems [GK13, FM, FH14], Teichmüller theory and its higher analogue [FG03, FG07, GS19], and so on. The central objects of study are *seeds* and their *mutations*.

A seed consists of two bunches of commutative variables called the *A-variables* and the *X-variables* respectively, and a square matrix called the *exchange matrix*. A seed mutation (in a specified direction) produces a new seed from a given one, transforming the variables according to a rule determined by the exchange matrix, and changing the exchange matrix to another one at the same time. We call the transformation of A-variables (resp. X-variables) the *cluster $\mathcal{A}$-transformation* (resp. *cluster $\mathcal{X}$-transformation*), which are birational transformations. All the seeds obtained in this way are coherently assigned to vertices of a regular tree so that any two seeds connected by an edge is related by a seed mutation, and we call such an assignment a *(Fomin–Zelevinsky) seed pattern*. We are interested in algebraic, combinatorial or geometric objects that are determined by an isomorphism class of a seed pattern.

The transformation rule of the exchange matrices are purely combinatorial, and it unifies and generalizes several classical notions such as sink/source sequence of acyclic quivers (e.g. [Rin94]), flips of ideal triangulations of a marked surface (e.g. [Pen, FST08]), braid moves of wiring diagrams or Thurston diagrams (e.g. [FWZ17, FM]), spider moves of surface bipartite graphs (e.g. [GK13, FM]), and so on. The cluster transformations are *positive maps*, which means that they admit a rational expression without subtraction. It has turned out that this positivity nature also unifies some positivity notions emerging from several different contexts, including Lusztig's total positivity [Lus98] in the representation theory of quantum groups, positive representations of the fundamental group of a surface in the higher Teichmüller theory [FG03], and the positivity of wall-crossing automorphisms in the theory of wall-crossing structures [GHKK18]. The positivity of cluster transformations also allows one to *tropicalize* it, by considering the cluster variables as elements in a semifield $\mathbb{P}$. These features make the cluster algebra strongly related to broad topics in mathematics.

From a dynamical point of view, an interesting object is a *mutation loop*. A finite sequence of seed mutations and permutations of indices is called a *mutation sequence*, and it is a *mutation loop* if it preserves the exchange matrix. A mutation loop defines a pair of discrete dynamical system as the corresponding composition of cluster transformations and permutations of coordinates. The mutation loops form a group $\Gamma_{\boldsymbol{s}}$ called the *cluster modular group* [FG09]. The cluster modular group acts on two schemes $\mathcal{A}_{\boldsymbol{s}}$ and $\mathcal{X}_{\boldsymbol{s}}$ called the *cluster $\mathcal{A}$- and $\mathcal{X}$-varieties*, respectively. In this geometric context, the cluster A- (resp. X-)variables are regarded as coordinate functions on the $\mathcal{A}$- (resp. $\mathcal{X}$-)variety, and the bunch of coordinates that belong to a common seed form a rational chart on the respective cluster variety. The action of the cluster modular group is given by a composition of cluster transformations on such a rational chart. The tropicalization procedure with respect to a semifield $\mathbb{P}$ gives the sets of $\mathbb{P}$-valued points $\mathcal{A}_{\boldsymbol{s}}(\mathbb{P})$ and $\mathcal{X}_{\boldsymbol{s}}(\mathbb{P})$, which also admit actions of the cluster modular group. Thus the discrete dynamical systems induced by a mutation loop take place in these spaces.

It is known that many interesting discrete dynamical systems emerge in this way, and in some special cases a geometric construction ensures an integrability nature [GK13, FM]. As an outstanding feature, such a realization of a dynamical system by a mutation loop automatically provides a quantization (*i.e.*, a non-commutative deformation and its operator representations) of this system via the theory of *quantum cluster varieties* [FG08].

The combinatorial framework of cluster algebra especially fits into the Teichmüller–Thurston theory. An ideal triangulation Δ of a marked hyperbolic surface Σ determines a seed, and flips of ideal triangulations precisely induce mutations of the corresponding seeds. Thus we get a seed pattern $\boldsymbol{s}_\Sigma$, whose isomorphism class only depends on the surface Σ. The associated objects are related to the following geometric objects (see Table 1):

- The smooth manifold $\mathcal{A}_{\boldsymbol{s}_\Sigma}(\mathbb{R}_{>0})$ (resp. $\mathcal{X}_{\boldsymbol{s}_\Sigma}(\mathbb{R}_{>0})$) of $\mathbb{R}_{>0}$-valued points is naturally isomorphic to the decorated (resp. enhanced) Teichmüller space of Σ equipped with the Weil-Petersson presymplectic (resp. Poisson) structure [Pen, FG07, FST08]. Roughly speaking they parametrize certain hyperbolic structures on the surface Σ.
- The piecewise-linear manifold $\mathcal{A}_{\boldsymbol{s}_\Sigma}(\mathbb{R}^{\mathrm{trop}})$ (resp. $\mathcal{X}_{\boldsymbol{s}_\Sigma}(\mathbb{R}^{\mathrm{trop}})$) of $\mathbb{R}^{\mathrm{trop}}$-valued points is naturally isomorphic to the decorated (resp. enhanced) space of measured laminations on Σ equipped with the canonical piecewise-linear structure [FG07]. Here $\mathbb{R}^{\mathrm{trop}} = (\mathbb{R}^{\mathrm{trop}}, \max, +)$ is the *real tropical semifield* or the *max-plus algebra*. The space of measured laminations is particularly useful in the study of the *mapping class group* of a surface. See, for instance, [FLP, PH].
- A simplicial complex called the *cluster complex* is defined [FG09, GHKK18], and in this case it coincides with the tagged arc complex introduced by [FST08]. It describes a certain "combinatorial part" of the space of enhanced measured laminations.
- The cluster modular group includes the mapping class group $MC(\Sigma)$ of Σ as a subgroup of finite index [BS15]. They coincide with each other if the marked surface has no punctures. The actions of the cluster modular group on the spaces listed above restrict to the geometric actions of the mapping class group.
- The theory of quantum cluster varieties provides a deformation quantization of the enhanced Teichmüller space of Σ, and its operator representation on some Hilbert

Teichmüller–Thurston theory	**Cluster algebra**
Decorated Teichmüller space	Positive points of $\mathcal{A}$-variety
Enhanced Teichmüller space	Positive points of $\mathcal{X}$-variety
The space of decorated measured laminations	Tropical points of $\mathcal{A}$-variety
The space of enhanced measured laminations	Tropical points of $\mathcal{X}$-variety
Mapping class group	Cluster modular group
Arc complex	Cluster complex
Quantum Teichmüller theory	Representations of quantum $\mathcal{X}$-variety

TABLE 1. Correspondence between Teichmüller–Thurston theory and the cluster algebra.

space. The resulting quantum theory is explored by [CP07, BW11, GS19]. It should be related to the Liouville conformal field theory [Tes07], which is a model of $(1+1)$-dimensional theory of gravity.

2. Classification and dynamical properties of mutation loops

2.1. **Nielsen–Thurston classification theory.** From Table 1, it is natural to regard the cluster modular group as a generalization of the mapping class group of a marked surface. Then one can ask for a generalization of the theory of mapping class group to that of cluster modular group. From this viewpoint, the author proposed a kind of generalization of the *Nielsen–Thurston classification theory* for mapping class groups in [Ish19]. The original Nielsen–Thurston theory (see *e.g.* [FLP]) gives a classification of mapping classes into three types: periodic, reducible, and pseudo-Anosov. These types are characterized by fixed point properties of the action on the Thurston compactification of the Teichmüller space. In the same spirit, the author introduced a trichotomy for mutation loops: periodic, cluster-reducible, and cluster-pseudo-Anosov. By studying the action of the cluster modular group on the *tropical compactification* [FG16] of the (positive points of) cluster varieties, he gave a characterization of these types in terms of fixed point properties of this action.

2.2. **Cluster Dehn twists.** As an analogue of a *Dehn twist* on a surface, the author introduced a *cluster Dehn twist* as a mutation loop which admits certain simple expression as a sequence of mutations in [Ish19]. Dehn twists and half-twists in mapping class groups are cluster Dehn twists. It is proved that a cluster Dehn twist induces a parabolic dynamics on the tropical compactification of the cluster $\mathcal{A}$-variety, generalizing the corresponding fact on the action of a Dehn twist on the Thurston compactification. Later we discuss a problem whether a cluster modular group can be generated by cluster Dehn twists as in the case of mapping class groups.

2.3. **Sign stability.** One drawback of the Nielsen–Thurston classification for cluster modular groups discussed above is that there exists a slight discrepancy between pseudo-Anosov and cluster-pseudo-Anosov even for a mutation loop given by a mapping class: a pseudo-Anosov mapping class provides a cluster-pseudo-Anosov mutation loop, but the converse is not true for a general marked surface other than the once-punctured torus.

Our first aim in this paper is to introduce a property of mutation loops called the *sign stability*, which is more closely related to being pseudo-Anosov. An idea of the definition comes from the theory of *train tracks*, which is a combinatorial model of measured laminations and commonly used to study pseudo-Anosov mapping classes.

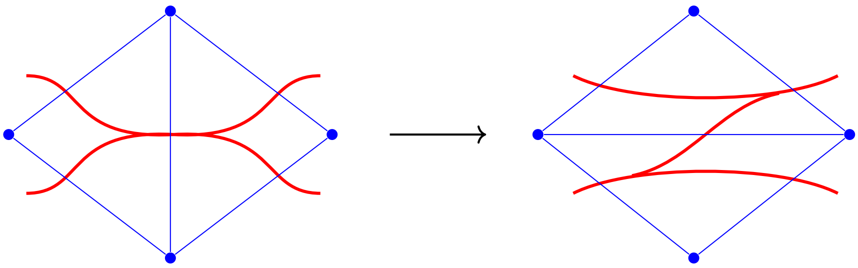

FIGURE 1. Train track splitting

Our key observation is that train track splittings can be translated into $\mathbb{R}^{\mathrm{trop}}$-valued cluster transformations. More precisely, some variants of train track splittings and their reverse operations can be unified to "signed" mutations introduced in [IN14], which is obtained by generalizing the usual seed mutation by putting an arbitrary sign in the formula.

An intuitive (but not exact) definition of the sign stability is a stabilization property of the presentation matrix of the piecewise-linear map obtained as the $\mathbb{R}^{\mathrm{trop}}$-valued cluster $\mathcal{X}$-tranformation. More precisely, given a mutation sequence and a point of $\mathcal{X}_{\boldsymbol{s}}(\mathbb{R}^{\mathrm{trop}})$, we define a sequence of signs indicating which presentation matrices (among three choices at each step of mutation) are applied to that point. A mutation loop is said to be sign-stable on a $\mathbb{R}_{>0}$-invariant set $\mathcal{L}$ if the sign of each orbit in $\mathcal{L}$ stabilizes to a common one. This stabilization property is also related to the *sign coherence property* of c-vectors [FZ07, NZ12] found in the algebraic theory of cluster algebra.

A sign-stable mutation loop is associated with a numerical invariant which we call the *cluster stretch factor*, which is a positive number greater or equal to 1. Now our main theorem is the following:

Theorem 1. *Let $\phi = [\gamma]_{\boldsymbol{s}}$ be a mutation loop with a representation path γ which is sign-stable on the set $\mathcal{L}_{(t_0,\ell_0)}$. Assuming that the specral radius conjecture holds true, we have*

$$\mathcal{E}^a_\phi = \mathcal{E}^x_\phi = \log \lambda_\phi,$$

where $\mathcal{E}^a_\phi$ (resp. $\mathcal{E}^x_\phi$) denotes the algebraic entropy of the cluster $\mathcal{A}$- (resp. $\mathcal{X}$-)transformation induced by ϕ, and $\lambda_\phi \geq 1$ is the cluster stretch factor.

This theorem gives a cluster algebraic analogue of the fact that the topological entropy of a pseudo-Anosov mapping class coincides with the logarithm of the stretch factor.

As testing examples we study the mutation loops of length one, which are classified by Fordy–Marsh [FM11]. We give a sufficient condition for such a mutation loop to be sign-stable, and compute its cluster stretch factor when the condition is satisfied. As a byproduct, we obtain a partial confirmation of [FH14, Conjecture 3.1] for these mutation loops.

The precise relation between the sign stability and the pseudo-Anosov property for mapping classes is investigated in [IK-a]. It is proved that the mutation loop obtained from a mapping class is sign-stable on $\mathbb{R}_{>0}\mathcal{X}_{\boldsymbol{s}}(\mathbb{Z}^{\mathrm{trop}})$ if and only if the mapping class is pseudo-Anosov, so we can honestly regard the sign-stable property as a cluster algebraic generalization of the pseudo-Anosov property.

3. Cluster realizations of groups

Let us turn our attention to examples of cluster modular groups. As mentioned above, the mapping class groups are particularly interesting examples.

In general, we call an embedding of a group G into some cluster modular group a *cluster realization* of G. Such a realization provides several actions of G on the spaces obtained from the cluster varieties, a quantum representation on a Hilbert space, and also allows us to apply the dynamical study on cluster modular groups to G as discussed above. A cluster realization of a group may not be unique if exists, and different cluster realizations can provide different actions and representations, and can have different dynamical property. For example, the *higher Teichmüller theory* [FG03, Le16b, GS19] allows us to construct different cluster realizations of the mapping class group of a marked surface, depending on the choice of the gauge group G (which is a semisimple algebraic group) of the theory. Other known examples of groups which can be realized in cluster modular groups are: Thomson's group T [FG09], Artin-Tits braid groups of finite type [FG03, GS19], Weyl groups of type A_n and $\widetilde{A}_n$ [ILP19], and so on. While the first two realization arise from the universal and higher Teichmüller theory, the third one arises from a completely different context in the discrete integrable systems. Our aim here is to add the Weyl groups associated with symmetrizable Kac-Moody Lie algebras to this list, generalizing the construction of [ILP19] for type A_n and $\widetilde{A}_n$.

3.1. Cluster realizations of Weyl groups.

For a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ and an integer $m \geq 2$, we define a weighted quiver $Q_m(\mathfrak{g})$ and a mutation loop $R(s) \in \Gamma_{Q_m(\mathfrak{g})}$ which corresponds to each Coxeter generator r_s of the Weyl group $W(\mathfrak{g})$. Now our main theorem is the following:

Theorem 2. (1) *We have an injective group homomorphism* $\phi_m : W(\mathfrak{g}) \to \Gamma_{Q_m(\mathfrak{g})}$ *which extends* $r_s \mapsto R(s)$*. Moreover, the mutation loop* $\phi_m(w)$ *for each* $w \in W(\mathfrak{g})$ *acts on the symplectic leaf* $\mathcal{U}_{Q_m(\mathfrak{g})} = p(\mathcal{A}_{Q_m(\mathfrak{g})})$ *trivially.*

(2) *We have a* $W(\mathfrak{g})$*-equivariant embedding of the root lattice* $L(\mathfrak{g})$ *to the group* $Z(\mathcal{X}_{|Q_m(\mathfrak{g})|})$ *of monomial Poisson Casimirs on* $\mathcal{X}_{Q_m(\mathfrak{g})}$.

Thus we get a one-parameter family of cluster realization of Weyl groups, depending on the integer m. These constructions give the first examples of infinite cluster modular groups which act on the cluster $\mathcal{U}$-variety trivially. In particular, the action on the cluster $\mathcal{X}$-variety is not properly discontinuous.

It turns out that these mutation loops $R(s)$ are of special type, known as *green sequences*. Moreover for $\mathfrak{g}$ of finite type, we can give a *cluster Donaldson-Thomas transformation*, whose existence is a highly non-trivial problem for a general cluster algebra and implies several good properties of the cluster ensemble [GHKK18].

Theorem 3. (1) *For each reduced expression* $w = r_{s_1} \dots r_{s_k} \in W(\mathfrak{g})$*, the mutation sequence* $R(w) = R(s_1) \dots R(s_k)$ *of* $Q_m(\mathfrak{g})$ *is a green sequence.*

(2) *Moreover if* $\mathfrak{g}$ *is of finite type, then the cluster Donaldson-Thomas transformation for the quiver* $Q_m(\mathfrak{g})$ *is given by* $\sigma \circ R(w_0)$*. Here* $w_0 = r_{s_1} \dots r_{s_l} \in W(\mathfrak{g})$ *is a fixed reduced expression of the longest element and* σ *is a certain explicit seed isomorphism.*

Our construction for type $\widetilde{A}_n$ includes the mutation loops used for the cluster realization of discrete Painlevé equations in [BGM17].

3.2. **Relation with the higher Teichmüller theory.** Although the cluster realization of Weyl groups given in Section 3.1 stems from the context of discrete integrable systems, it turns out that it is also related to the *higher Teichmüller theory*.

For a marked surface Σ and a simply-connected semisimple algebraic group G with the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, the moduli space of decorated twisted G-local systems on Σ has a special coordinate systems depending on ideal triangulations of Σ and other data, and birationally isomorphic to the cluster $\mathcal{A}$-variety associated with a suitable mutation class $\mathcal{C}_{\mathfrak{g},\Sigma}$ of weighted quivers [FG03, Le16b, GS19]. Explicit weighted quivers in $\mathcal{C}_{\mathfrak{g},\Sigma}$ are constructed by Fock–Goncharov [FG03] for type A_n, and by Le [Le16b] and Goncharov–Shen [GS19] for other types.

The basic idea behind these constructions, which is due to Fock–Goncharov [FG03], is that the additional data of decoration allows us to decompose the moduli space of local systems on a marked surface into several copies of the corresponding moduli space on a triangle, and the latter "local" moduli space has a canonical positive structure extending the one studied by Lusztig (see *e.g.* [Lus98]). This extension of moduli space by decorations in turn provides us a larger group of symmetry, the cluster modular group

$\Gamma_{\mathfrak{g},\Sigma} := \Gamma_{\mathcal{C}_{\mathfrak{g},\Sigma}}$, which is also called the *higher mapping class group* in this case. As we mentioned in Section 1, the cluster modular group $\Gamma_{\mathfrak{sl}_2,\Sigma}$ includes the mapping class group of Σ as a subgroup of finite index. It has been known that the higher mapping class group $\Gamma_{\mathfrak{g},\Sigma}$ for general $\mathfrak{g}$ contains the mapping class group $MC(\Sigma)$ of Σ [FG03, Le16b] and the outer automorphism group $\mathrm{Out}(G)$ of G [GS16, Le16b]. For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, it also contains the direct product of p copies of the Weyl group $W(\mathfrak{sl}_{n+1})$, where p denotes the number of punctures [GS16].

On the relation to our work, a crucial observation is that if we attach suitable frozen vertices and arrows to our quiver $Q_{kh}(\mathfrak{g})$ with h being the Coxeter number, then the resulting quiver $\widetilde{Q}_{kh}(\mathfrak{g})$ belongs to the mutation class $\mathcal{C}_{\mathfrak{g},\mathbb{D}_{2k}}$. Here $\mathbb{D}_n$ denotes the once-punctured disk with n marked points on its boundary. Hence our embedding $W(\mathfrak{g}) \subset \Gamma_{\mathfrak{g},\mathbb{D}_{2k}}$ induces an action of $W(\mathfrak{g})$ on the moduli space $\mathcal{A}_{G,\mathbb{D}_{2k}}$. By viewing $\mathbb{D}_{2k}$ as a local model around a puncture, we get an action of $W(\mathfrak{g})^p$ on the moduli space $\mathcal{A}_{G,\Sigma}$ for an *admissible* pair $(\Sigma, \mathfrak{g})$. We call this action the *cluster action*.

On the other hand, Goncharov-Shen [GS16] gave a natural action of $W(\mathfrak{g})^p$ on the moduli space $\mathcal{A}_{G,\Sigma}$ for an arbitrary marked surface Σ. This action only changes the decorations and keeps the underlying G-local systems intact. We call this action the *geometric action*. Therefore it is natural to ask whether the cluster action coincides with the geometric one. See [GS16, Conjectures 1.13 and 1.20] for related conjectures. Our goal is the following:

Theorem 4. *Assume $\mathfrak{g}$ is of classical finite type. For an admissible pair $(\Sigma, \mathfrak{g})$, the cluster action of $W(\mathfrak{g})^p$ on the moduli space $\mathcal{A}_{G,\Sigma}$ coincides with the geometric action.*

Theorem 4 gives us a geometric interpretation of our cluster realization of Weyl groups. We remark that another proof of Theorem 4 is recently given by Goncharov–Shen [GS19], as well as a similar statement for the $\mathcal{X}$-variety. Combining with the results by other authors [FG03, Le16b, GS16], we conclude that the higher mapping class group $\Gamma_{\mathfrak{g},\Sigma}$ for an admissible pair $(\Sigma, \mathfrak{g})$ contains the subgroup

$$\mathbb{G}_{\mathfrak{g},\Sigma} := (MC(\Sigma) \times \mathrm{Out}(G)) \ltimes W(\mathfrak{g})^p.$$

The following is still an open problem, except for the case $\mathfrak{g} = \mathfrak{sl}_2$:

Problem 5. *Is the index of the subgroup $\mathbb{G}_{\mathfrak{g},\Sigma} \subset \Gamma_{\mathfrak{g},\Sigma}$ finite or not ?*

4. Presentations of cluster modular groups and generation by cluster Dehn twists

Next we discuss a problem to find presentations of cluster modular groups. Problem 5 is one of our motivation of study. However such a problem is apparently too difficult for a general cluster modular group: we are going to present a partial progress on the cluster modular groups of *finite mutation type*.

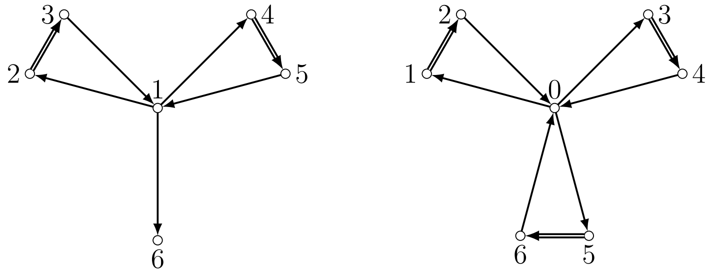

FIGURE 2. Quivers of type X_6 and X_7

4.1. **Presentations of saturated cluster modular groups.** Once trying to find a presentation of a given cluster modular group, one immediately encounters the difficulty which arises from the fact that a complete list of relations among the cluster transformations is not known in general. In simple cases they are exhaused by the *standard* $(h+2)$-*gon relations* [FG09] such as the involutivity and the *pentagon relation*, while there are "non-standard" relations in general, even for those associated with marked surfaces [FST08]. A survey on this problem is also found in [KY16]: not only an annoying thing is this, but also related to certain "dualities" between supersymmetric gauge theories.

In order to isolate such a problem, we consider the *saturated cluster modular group* $\widehat{\Gamma}_{\boldsymbol{s}}$ instead. It is defined by restricting the relations among cluster transformations to those generated by standard ones, so that the cluster modular group is obtained as a quotient of the saturated cluster modular group. We introduce a simplicial complex called the *saturated modular complex* on which the saturated cluster modular group acts, and show that it is simply-connected. Hence we can compute a presentation of the saturated cluster modular group from the data of this action.

When the seed pattern is of *finite mutation type*, namely the mutation class of the exchange matrix is finite, this method works well. In this case the "fundamental domain" of the saturated modular complex is finite, and we can obtain a finite presentation of the saturated cluster modular group. The mutation classes of finite mutation type has been completely classified in [FeST12a, FeST12b]. In the case of skew-symmetric exchange matrices, the list consists of the mutation classes associated with marked surfaces, several classes associated with generalized Dynkin diagrams, and two mysterious classes called X_6 and X_7. The initial quivers of type X_6 and X_7 are shown in Figure 2. Our main result here is a computation of finite presentations of the saturated cluster modular groups of type X_6 and X_7.

Theorem 6. *The saturated cluster modular group $\widehat{\Gamma}_{X_7}$ of type X_7 is generated by elements ψ_k, ϕ_k for $k = 1, 3, 5$ and the permutation group $\mathcal{S}_3$ of numbers $\{1, 3, 5\}$, and the complete* *set of relations among them is given as follows:*

$$\begin{aligned}
\sigma\psi_k\sigma^{-1} &= \psi_{\sigma(k)}, \\
\sigma\phi_k\sigma^{-1} &= \phi_{\sigma(k)} \quad \textit{for } \sigma \in \mathcal{S}_3,\ k = 1, 3, 5, \\
\phi_1\phi_3 &= \phi_3\phi_1, \\
\phi_1 &= \psi_1^2\sigma_{35}, \\
1 &= (\psi_3^{-1}\psi_1)^2, \\
1 &= \sigma_{153}\psi_5\psi_1\psi_3\psi_5\psi_1,
\end{aligned}$$

and the usual relations of permutations. Here σ_{ij} denotes the tranposition of i and j and σ_{135} denotes the cyclic permutation $1 \mapsto 3 \mapsto 5 \mapsto 1$.

Theorem 7. *The saturated cluster modular group $\widehat{\Gamma}_{X_6}$ of type X_6 is generated by five elements α_1, α_2, β_1, β_2 and σ, and the complete set of relations among them is given as follows:*

$$\begin{aligned}
&\sigma^2 = 1, \\
&\alpha_2 = \mathrm{Ad}_\sigma\alpha_1, \quad \beta_2 = \mathrm{Ad}_\sigma\beta_1, \\
&\alpha_1\alpha_2 = \alpha_2\alpha_1, \\
&\mathrm{Ad}_{\beta_1}^{-1}\alpha_2 = \alpha_1, \\
&(\beta_2\beta_1^{-1})^2 = 1, \\
&(\alpha_2\beta_2\beta_1^{-1}\alpha_1^{-1})^3 = 1, \\
&\beta_2(\beta_1\alpha_1)^{-1}\beta_2 = \mathrm{Ad}_{\alpha_2\beta_1}\sigma, \\
&\beta_1 = \alpha_1\mathrm{Ad}_{\beta_1}(\alpha_2)\alpha_2^{-1}\sigma^{-1}.
\end{aligned}$$

Here $\mathrm{Ad}_x y := xyx^{-1}$ denotes the conjugation.

First homology groups. As the first application, we compute the first homology groups of $\widehat{\Gamma}_{X_6}$ and $\widehat{\Gamma}_{X_7}$. Here the first homology group (=abelianization) of a group G is defined to be $H_1(G;\mathbb{Z}) = G/[G,G]$. Here we present the results with proofs based on Theorems 6 and 7.

Corollary 8. *We have $H_1(\widehat{\Gamma}_{X_7};\mathbb{Z}) \cong \mathbb{Z}/5 \times \mathbb{Z}/2$. The generators are the images of ψ_1 and σ_{13}.*

Proof. It is well-known that the signature function gives an isomorphism $H_1(\mathfrak{S}_n;\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/2$. From the first relation in Theorem 6 we get $\psi_1 = \psi_3 = \psi_5$ in the abelianization, and the last relation implies $\psi_1^5 = 1$. ∎

Corollary 9. *We have $H_1(\widehat{\Gamma}_{X_6};\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}/2$. The generators are the images of α_1 and β_1.*

Proof. From the relations in the second line of Theorem 7, we get $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ in the abelianization. The last two relations imply that $\sigma = \alpha_1^{-1}\beta_1$ and $\alpha_1^2 = \beta_1^2$. ■

4.2. **Generation by cluster Dehn twists.** It is a classical fact that the mapping class group of a marked surface is generated by Dehn twists and half-twists. Since they are also cluster Dehn twists, one can expect that a cluster modular group is generated by cluster Dehn twists. We confirm it is indeed true for certain cluster modular groups of finite mutation type:

Theorem 10. *The cluster modular groups of finite mutation type $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $G_2^{(*,*)}$ and X_7 are generated by finitely many cluster Dehn twists. The cluster modular group of type X_6 is virtually generated by four cluster Dehn twists.*

For the former three cases, the theorem follows from the computation of the cluster modular groups given by Assem–Schiffler–Shramchenko [ASS12] using the cluster categories. The saturated cluster modular group of type $G_2^{(*,*)}$ is computed by Fock–Goncharov [FG06]. For the last two cases we use Theorems 6 and 7.

We can also find cluster Dehn twists in the remaining cluster modular groups of finite mutation type, at least for skew-symmetric cases. Our general expectation is that any cluster modular group of finite mutation type is virtually generated by cluster Dehn twists. It will be especially interesting to study the cases $E_7^{(1,1)}$ and $E_8^{(1,1)}$, since they appear as the unfrozen parts of certain quivers in the mutation class $\mathcal{C}_{\mathfrak{g},\Sigma}$. See Table 2. For example, we have the equality $\Gamma_{E_8^{(1,1)}} = \Gamma^{\mathrm{uf}}_{\mathfrak{sl}_3,6\text{-gon}}$. The groups $\Gamma^{\mathrm{uf}}_{\mathfrak{sl}_3,6\text{-gon}}$ and $\Gamma_{\mathfrak{sl}_3,6\text{-gon}}$ are related by the *elimination homomorphism*. The mutation classes of $E_7^{(1,1)}$ and $E_8^{(1,1)}$ consist of 506 and 5739 quivers, respectively.

	3-gon	4-gon	5-gon	6-gon	7-gon
$\mathfrak{sl}_2$	$\emptyset$	A_1	A_2	A_3	A_4
$\mathfrak{sl}_3$	A_1	D_4	E_7	$E_8^{(1,1)}$	∞_{13}
$\mathfrak{sl}_4$	A_3	$E_7^{(1,1)}$	∞_{15}	∞_{21}	∞_{27}
$\mathfrak{sl}_5$	D_6	∞_{16}	∞_{26}	∞_{36}	∞_{46}
$\mathfrak{sl}_6$	$E_8^{(1,1)}$	∞_{25}	∞_{40}	∞_{55}	∞_{70}
$\mathfrak{sl}_7$	∞_{15}	∞_{36}	∞_{57}	∞_{78}	∞_{99}

TABLE 2. Type of the mutation class $\mathcal{C}_{\mathfrak{g},\Sigma}$ for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and Σ an unpunctured $(m+2)$-gon with $n = 1, \dots, 6$ and $m = 1, \dots, 5$. Here $\emptyset$ denotes the empty quiver, and ∞_N denotes a mutation class of a quiver of infinite mutation type with N vertices.

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