博士論文（要約）

論文題目

A higher rank Euler system for the multiplicative group over a totally real field

(総実体上の乗法群の高階オイラー系)

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Euler systems were introduced by Kolyvagin in order to study the structure of Selmer groups of elliptic curves. The theory of Euler systems has been playing a crucial role in studying the arithmetic of global Galois representations. However, there are very few known examples of Euler systems (see [11, 14, 15, 16, 21]).

In the paper [19], Perrin-Riou introduced the notion of higher rank Euler systems. However, at the present moment, we know an example of a higher rank Euler system only in the $\mathrm{GL}_2 \times \mathrm{GL}_2$-case (see [8, 9, 10]). In this paper, we construct a higher rank Euler system for the multiplicative group over a totally real field by using the classical Iwasawa main conjecture for totally real fields proved by Wiles in [23].

To explain the result we obtain in this article more precisely, we introduce some notations. Let k be a totally real field with degree $r := [k \colon \mathbb{Q}]$ and

$$\chi \colon G_k \longrightarrow \overline{\mathbb{Q}}^{\times}$$

a non-trivial finite order even character. Here, for a field k', we write $G_{k'}$ for the absolute Galois group of k'. Put $L := \overline{k}^{\ker(\chi)}$. Let p be an odd prime such that both the order of χ and the class number of k are coprime to p. Fix a finite abelian p-extension K/k satisfying $S_{\mathrm{ram}}(K/k) \cap (S_p(k) \cup S_{\mathrm{ram}}(L/k)) = \emptyset$. Here $S_p(k)$ denotes the set of all primes of k above p and, for an algebraic extension K'/k, we write $S_{\mathrm{ram}}(K'/k)$ for the set of all primes of k at which K'/k is ramified. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and put

$$\mathcal{O} := \mathbb{Z}_p[\mathrm{im}(\chi)] \quad \text{and} \quad T := \mathcal{O}(1) \otimes \chi^{-1}.$$

We write $M_{KL,\infty}$ for the maximal p-ramified pro-p abelian extension of $KL(\mu_{p^\infty})$. We note that $M_{KL,\infty}/k$ is a Galois extension. Since $M_{KL,\infty}/KL(\mu_{p^\infty})$ is an abelian extension, the Galois group $\mathrm{Gal}(KL(\mu_{p^\infty})/k)$ acts on the Galois group $\mathrm{Gal}(M_{KL,\infty}/KL(\mu_{p^\infty}))$. Furthermore, by using this Galois action, we see that the module

$$X_K := \mathcal{O} \otimes_{\mathbb{Z}_p} \mathrm{Gal}(M_{KL,\infty}/KL(\mu_{p^\infty}))$$

has an $\mathcal{O}[[\mathrm{Gal}(KL(\mu_{p^\infty})/k)]]$-module structure. Let k_∞/k denote the cyclotomic $\mathbb{Z}_p$-extension. Then we have the canonical isomorphism

$$\mathrm{Gal}(KL(\mu_{p^\infty})/k) \xrightarrow{\sim} \mathrm{Gal}(k_\infty K/k) \times \mathrm{Gal}(L(\mu_p)/k)$$

since $p \nmid [L \colon k]$. Set

$$e_\chi := \frac{1}{[L(\mu_p) \colon k]} \sum_{\sigma \in \mathrm{Gal}(L(\mu_p)/k)} \chi(\sigma)\sigma^{-1}.$$

By the above isomorphism, we see that the Iwasawa module

$$X_K^\chi := e_\chi X_K$$

is a $\Lambda_K := \mathcal{O}[[\mathrm{Gal}(k_\infty K/k)]]$-module.

Suppose that

- $H^0(G_{k_\mathfrak{p}}, T/\mathfrak{m}T) = H^2(G_{k_\mathfrak{p}}, T/\mathfrak{m}T) = 0$ for each prime $\mathfrak{p} \in S_p(k)$,

where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}$. Then one can show that, for any finite abelian extension K'/k and prime $\mathfrak{p} \in S_p(k)$, the $\mathcal{O}[\mathrm{Gal}(K'/k)]$-module

$$H^1(G_{k_\mathfrak{p}}, \mathrm{Ind}_{G_k}^{G_{K'}}(T))$$

is free of rank $[k_\mathfrak{p} \colon \mathbb{Q}_p]$. Fix an isomorphism

$$\varprojlim_{K'} \bigoplus_{\mathfrak{p} \mid p} H^1(G_{k_\mathfrak{p}}, \mathrm{Ind}_{G_k}^{G_{K'}}(T)) \xrightarrow{\sim} \varprojlim_{K'} \mathcal{O}[\mathrm{Gal}(K'/k)]^r,$$

where K' runs over all the finite abelian extensions of k.

For each integer $s \geq 0$, we denote by $\mathrm{ES}_s(T)$ the module of Euler systems of rank s for T. The above fixed isomorphism induces an injective 'rank reduction' homomorphism

$$\mathrm{ES}_r(T) \hookrightarrow \mathrm{ES}_0(T).$$

Remark 0.1. A rank reduction homomorphism $\mathrm{ES}_r(T) \longrightarrow \mathrm{ES}_s(T)$ (for $0 \leq s \leq r$) was considered firstly by Rubin in [20] and Perrin-Riou in [19]. After that, in the papers [4, 5, 6], Büyükboduk developed a machinery for higher rank Euler systems by using the rank reduction homomorphism.

An Euler system of rank 0 is a collection of elements in group rings satisfying certain norm-relations, and, philosophically, it should be related to p-adic L-functions. We actually see that the collection of p-adic L-functions for the multiplicative group over totally real fields is an Euler system of rank 0. Let $\mathcal{L}_p^\chi \in \mathrm{ES}_0(T)$ denote this Euler system.

Under this setting, the following is the main result of this paper.

Theorem 0.2. *Suppose that*

- $H^0(G_{k_\mathfrak{p}}, T/\mathfrak{m}T) = H^2(G_{k_\mathfrak{p}}, T/\mathfrak{m}T) = 0$ *for each prime* $\mathfrak{p}$ *of* k *above* p, *and*
- *the module* X_k^χ *is* p*-torsion-free.*

Then there is a unique Euler system of rank r *in* $\mathrm{ES}_r(T)$ *such that its image under the injection* $\mathrm{ES}_r(T) \hookrightarrow \mathrm{ES}_0(T)$ *is* $\mathcal{L}_p^\chi$.

A key ingredient of the proof of Theorem 0.2 is to generalize the notion of the characteristic ideal. In order to define the characteristic ideal of a finitely generated torsion module over a noetherian ring R, we need to impose that R is a normal ring since the definition relies on the structure theorem for finitely generated modules over a normal ring. In this paper, by using an exterior bi-dual instead of the structure theorem, we give a new definition of the characteristic ideal. This new definition is applicable to a finitely generated module over any noetherian ring, and we can compute the image of the injection $\mathrm{ES}_r(T) \hookrightarrow \mathrm{ES}_0(T)$ by using characteristic ideals of certain Iwasawa modules. Furthermore, by using the classical Iwasawa main conjecture for totally real fields proved by Wiles in [23], we show that the image contains the Euler system $\mathcal{L}_p^\chi$.

As an application of Theorem 0.2, by using the theory of higher rank Euler, Kolyvagin and Stark systems (which have been developed by Rubin in [21], Mazur–Rubin in [17, 18], Büyükboduk in [4, 5, 6, 7], the author in [22], Burns–Sano in [3], and Burns–Sano and the author in [1, 2]), we prove that all higher Fitting ideals of X_K^χ are described by analytic invariants canonically associated with Stickelberger elements. To be more precise, one can define an increasing sequence $\{\Theta_{K,\infty}^i\}_{i\geq 0}$ of ideals of the Iwasawa algebra $\Lambda_K = \mathcal{O}[[\mathrm{Gal}(k_\infty K/k)]]$. The ideals $\Theta_{K,\infty}^i$ are generated by some Kolyvagin derivative classes associated with Stickelberger elements.

Corollary 0.3. *Suppose that*

- *the same assumptions as in Theorem 0.2 hold, and*
- $H^2(k_\mathfrak{q}, T/\mathfrak{m}T)$ *vanishes for each prime* $\mathfrak{q} \in S_{\mathrm{ram}}(K/k)$.

Then, for any integer $i \geq 0$, *we have*

$$\Theta_{K,\infty}^i = \mathrm{Fitt}_{\Lambda_K}^i(X_K^\chi).$$

Remark 0.4. When $K = k$, Corollary 0.3 has already been proved by Kurihara in [12, 13]. Hence Corollary 0.3 can be viewed as an equivariant generalization of Kurihara's result.

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