# 博士論文

論文題目　Existence problems of fibered links
(ファイバー絡み目の存在問題)

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# Existence problems of fibered links

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A fibered link in a 3-manifold induces a fiber structure on the complement of the link in the manifold. The fiber structure reduces the structure of the manifold into the property of the mapping class of the monodromy. Conversely, for a mapping class of a surface with a boundary we can construct a 3-manifold and a fibered link whose monodromy is the mapping class. Especially fibered knots are important since the mapping class group of a surface with a connected boundary is widely studied. Myers [16] showed that every connected orientable closed 3-manifold has a fibered knot. However, it is difficult in general to find fibered links with a fibered surface of a fixed homeomorphism type in a given 3-manifold. There is an important topological invariant, called the *openbook genus* concerning with this problem. This is defined as the minimum number of genus among fibered knots which a manifold has. To attack this problem, there are several approaches. In this thesis, we argue by following some of them in two parts, one concerns with Heegaard splittings and genus one fibered knots and the other concerns with *homologically fibered links*.

We state the background of Part I briefly. By thickening a fiber surface of a fibered link, we get a Heegaard splitting. Morimoto [15] started a study of genus one fibered knots by such an approach. After that Baker [2] completely determined the number of genus one fibered knots in a given lens space. At Section 1, we give a criterion for a simple closed curve on a genus $2g$ Heegaard surface being a genus $g$ fibered knot in terms of its Heegaard diagram. This is the main purpose of Part I. We use this criterion especially when $g = 1$ in this thesis. Genus one fibered knots can be handled because of their low genus. Applying the criterion, we research the openbook genera for some Seifert manifolds at Sections 2, 3. In general, the technique for genus one cases is not similarly used for higher genus cases. However using genus one fibered knots as building blocks, we get some information for higher genus fibered links. At Section 4, we make examples of higher genus fibered knots in lens spaces which are analogues of genus one fibered knots in lens spaces. There is a generalization for Heegaard splittings, called *relative trisections*. They consist of multi curves on a surface and represent 4-manifolds with boundaries with fibered links on their boundaries. Trisections are a way to study fibered links since considering a 4-manifold bounded by a given 3-manifold is sometimes useful. At Section 5, we give an example of a relative trisection for a 4-manifold whose boundary is a lens space and a fibered link on it, which are obtained by a standard way.

We state the background of Part II. There is a generalization of fibered links, called homologically fibered links. This loosens the condition for fibered links, inducing product structures into the condition, inducing homologically product structures. Thus the existence of homologically fibered links is a necessary condition for that of fibered links (with fiber surfaces are homeomorphic to given surfaces). We will give an algebraic condition of existing of some homologically fibered links in some manifolds at Section 6. This is the purpose of Part II. Throughout this thesis, $\Sigma_{g,b}$ denotes the connected orientable surface of genus $g$ with $b$ boundary components.

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# Part I

## 1 Genus $g$ fibered knots and simple closed curves on genus $2g$ Heegaard surfaces

### 1.1 Introduction

Let $M$ be a connected, closed orientable 3-manifold. A link $L$ in $M$ is called a *fibered link* if there is a Seifert surface $F$ of $L$ such that there is a homeomorphism between the complement of $L$ in $M$ and a surface bundle over $S^1$ mapping $F$ to (a surface)$\times\{*\}$. The surface $F$ in the definition is called the *fiber surface* of $L$. For the case of knots, we say a fibered knot is a *genus g fibered knot* if its fiber surface is of genus $g$. By thickening a fiber surface $F$ of a genus $g$ fibered knot $K$ in $M$, we get a genus $2g$ Heegaard splitting of $M = V \cup_S W$. In this construction, we can put $K$ on $S$ being a *binding* of both of $V$ and $W$, where a *binding* of a genus $2g$ handlebody means a simple closed curve $l$ on the boundary of a genus $2g$ handlebody such that there exists a homeomorphism between $(V, l)$ and $(\Sigma_{g,1} \times [0,1], \partial\Sigma_{g,1} \times \{\frac{1}{2}\})$. It is like a *connected binding* in [11]. Conversely, suppose that we are given a genus $2g$ Heegaard splitting $M = V \cup_S W$ and a simple closed curve $l$ on $S$ such that it is a binding of both of $V$ and $W$. Then the corresponding fiber structures of handlebodies induce the global fiber structure of $M \setminus L$ and we know that $l$ is a genus $g$ fibered knot. Hence there is a one-to-one correspondence between the set of genus $g$ fibered knots with their fiber surfaces and the set of simple closed curves on genus $2g$ Heegaard surfaces such that they are binding of both of handlebodies. In this section, we give a criterion for whether a given simple closed curve on the boundary of a genus $2g$ handlebody is a binding or not in terms of a cut system of the handlebody. Applying this criterion, we know whether a simple closed curve on a genus $2g$ Heegaard surface is a genus $g$ fibered knot inducing this Heegaard splitting using the Heegaard diagram.
In this section, curves and disks are considered up to isotopy, and always assumed to intersect minimally and transversely. In particular, the intersection of any disks in a handlebody contains no circle components because of the irreducibility of the handlebody.

**The setting** Let $V$ be a handlebody of genus $2g$, and $l$ be an oriented simple closed curve on $\partial V$. We fix a *cut system* $\{D_1, \dots, D_{2g}\}$ of $V$. A *cut system* of $V$ means a set of pairwise disjoint properly embedded $2g$ disks in $V$ such that cutting $V$ along these disks produces a 3-ball. Make $l$ intersect with $\{\partial D_1, \dots, \partial D_{2g}\}$ minimally and transversely. By giving $\{\partial D_1, \dots, \partial D_{2g}\}$ orientations and letters $\{x_1, \dots, x_{2g}\}$, we assign a cyclic sequence of letters to $l$ by following $l$ and counting the intersections with the disks. This is called the *sequence of letters represented by* $l$ and we call the cyclically reduced word obtained by reducing the sequence of letters represented by $l$ the *word represented by* $l$. We define the sign of an intersection of $l$ with $\partial D_i$ such that it is positive $(x_i)$ if $l$ hits $\partial D_i$ from the left side of $\partial D_i$, negative $(x_i^{-1})$ if $l$ hits $\partial D_i$ from the right side of $\partial D_i$, where we see $\partial D$ runs from bottom to top.
Let $F_{2g}$ be the free group generated by the alphabets $\{x_1, \dots, x_{2g}\}$. The automorphism group $Aut(F_{2g})$ of $F_{2g}$, and the outer automorphism group, $Out(F_{2g})$ naturally act on the set of cyclic words of alphabets $\{x_1, \dots, x_{2g}\}$.

For $g = 1$, we give one definition, and there are useful facts below due to Cho and Koda (we set $x_1 = x, x_2 = y$):

**Definition 1.1.** (corresponding arc) Let $B$ be the ball $V \setminus (D_1 \cup D_2)$. There are four cut ends on $\partial B$, $D_1^+$ and $D_1^-$ coming from $D_1$, $D_2^+$ and $D_2^-$ coming from $D_2$. Let $l$ be a simple closed curve on $\partial V$. On $\partial B$, $l$ is cut into arcs with their endpoints on the cut ends. Let $c$ be a subarc of $l$ on $\partial B$ representing a trivial word, say $xx^{-1}$. It is on $\partial B$, putting its endpoints on $\partial D_1^+$ and being disjoint from cut ends in its interior. $c$ separates the other three cut ends into one and two. Let $A$ denote the cut end in the part separated by $c$ which contains just one cut end. Hence the interior of $c$ is realized as a boundary of a neighborhood of $A \cup \alpha$ in $\partial B \setminus (D_1^\pm \cup D_2^\pm)$ using an arc $\alpha$ starting from $\partial D_1^+$ and ending on $A$. Note that $A$ cannot be $D_1^-$. If this is the case, every subarc of $l$ on $B$ with one of whose endpoints on $\partial D_1^-$ must have the other endpoint on $\partial D_1^+$. It leads a contradiction since the number of the intersections of $l$ with $\partial D_1^+$, and with $\partial D_1^-$ must be same. We call the arc $\alpha$ the *corresponding arc* for $c$.

**Fact 1.1.** [18] *If the sequence of letters represented by $l$ contains $xx^{-1}$, then it contains also $x^{-1}x$.*

**Fact 1.2.** [4] [5] *If the sequence of letters represented by $l$ contains $xy^n x^{-1}$ for non zero integer $n$, then this is cyclically reduced.*

**Fact 1.3.** [4] [5] *If the sequence of letters represented by $l$ contains both $x^2$ (or $x^{-2}$) and $y^2$ (or $y^{-2}$), then this is cyclically reduced.*

These Definition and Facts are heavily used in Sections 2, 3.

Under this setting, the author previously proved the following (here, "commutator" means $xyx^{-1}y^{-1}$ or $yxy^{-1}x^{-1}$):

**Theorem 1.1.** *(criterion for $g = 1$)* [18] *For a genus 2 handlebody $V$ and a simple closed curve $l$ on $\partial V$, the following are equivalent:*

*(1) $l$ is a binding of $V$.*

*(2) For some cut system $\{D, E\}$, the word represented by $l$ is a commutator.*

*(3) For every cut system $\{D, E\}$, the word represented by $l$ is a commutator.*

In this section, we generalize this theorem:

**Theorem 1.2.** *(criterion for general $g$) For a genus $2g$ handlebody $V$ and a simple closed curve $l$ on $\partial V$, the following are equivalent:*

*(1) $l$ is a binding of $V$.*

*(2) For some cut system $\{D_1, \dots, D_{2g}\}$, the word represented by $l$ is in $Out(F_{2g})\cdot[x_1, x_2]\cdots[x_{2g-1}, x_{2g}]$.*

*(3) For every cut system $\{D_1, \dots, D_{2g}\}$, the word represented by $l$ is in $Out(F_{2g})\cdot[x_1, x_2]\cdots[x_{2g-1}, x_{2g}]$.*

In the above, $[x_i, x_j]$ means $x_i x_j x_i^{-1} x_j^{-1}$ and $Out(F_{2g})\cdot$ means the orbit of the action. Note that when $g = 1$, the orbit of $[x, y]$ is the set of commutators.

**Operations** Here we give some operations needed for arguments. Let $V$ be a genus $2g$ handlebody and $D$, $E$ properly embedded essential non-separating disks in $V$.

**Definition 1.2.** (band-sum) Suppose $D$ and $E$ are disjoint, and $V \setminus (D \cup E)$ is connected.
Let $\alpha$ be an oriented arc on $\partial V$ starting from $\partial D$, ending on $\partial E$ and disjoint from $\partial D$ and $\partial E$ in its interior. Then the boundary component of the regular neighborhood of $\partial D \cup \alpha \cup \partial E$ in $\partial V$ which is neither $\partial D$ nor $\partial E$ bounds an essential non-separating disk $D'$. We call $D'$ the disk obtained by the *band-sum* from $D$ by using $\alpha$. We give $D'$ an orientation coming from $D$: A part of $\partial D'$, which is near to $\partial D$ has the parallel orientation to $\partial D$.

**Definition 1.3.** (band-sum for cut system) Let $\mathbb{D} = \{D_1, \dots, D_{2g}\}$ be a cut system of $V$. Take an oriented arc $\alpha$ starting from $\partial D_i$, ending on $\partial D_j$ ($i \neq j$) and disjoint from disks in $\mathbb{D}$ in its interior. Then let $D_i'$ be the disk obtained by the band-sum from $D_i$ by using $\alpha$. Set $\mathbb{D}'$ be $(\mathbb{D} \setminus \{D_i\}) \cup \{D_i'\}$, which gives another cut system of $V$. We call $\mathbb{D}'$ the cut system obtained by the *band-sum for cut system* from $\mathbb{D}$ by using $D_i$ and $\alpha$.

**Definition 1.4.** (disk surgery) Suppose $D$ and $E$ intersect. Take an outermost disk $\Delta$ in $E$ cut-offed by $D$. The arc $\Delta \cap D$ cut $D$ into two disks $D'$ and $D''$. Then $\Delta \cup D'$ and $\Delta \cup D''$ are essential disks in $V$. Note that at least one of these disks is non-separating. We say that such a non-separating disk (if there are two, choose one), say $D'$, is obtained by a *disk surgery* from $D$ and $E$ using $\Delta$. We give $D'$ an orientation coming from $\Delta$. Note that $D'$ is disjoint from $D$ and the number of components of $D' \cap E$ is less than that of $D \cap E$.

Concerning with cut-systems, we claim the following:

**Claim 1.1.** *Let $\mathbb{D} = \{D_1, \ldots, D_{2g}\}$ and $\mathbb{E} = \{E_1, \ldots, E_{2g}\}$ be cut systems of $V$. Then $\mathbb{E}$ is obtained from $\mathbb{D}$ by applying finitely many times band-sums for cut system by using appropriate arcs(, and reversing the orientations of the disks in the cut systems).*

*Proof.* Let $B$ be $V \setminus (\cup_{D \in \mathbb{D}} D)$, a ball having $4g$ cut ends of disks on the boundary.
At first, suppose there is a disk in $\mathbb{E}$, say $E_1$, intersecting some disks of $\mathbb{D}$. Take an outermost disk $\Delta$ in $E_1$ cut-offed by disks of $\mathbb{D}$. Suppose the disk $D_i$ cut off $\Delta$ from $E_1$. Then $\Delta$ cuts $B$ into two balls $B_1$ and $B_2$. Assume $B_2$ has the other cut end coming from $D_i$ (called $D_i^-$), which is not the one $\Delta$ stemming from (called $D_i^+$). Note that $B_2$ has a cut end which is not coming from $D_i$ since otherwise the number of the intersections with the boundaries of the disks in $\mathbb{E}$ of $\partial D^+$ and $\partial D_i^-$ would differ. Take an oriented arc $\alpha$ starting from $\partial D_i$, ending on the boundary of a cut end on $B_1$ and it is on the boundary of $B_1$. See Figure 1. Let $D_i'$ be the disk obtained by band-sum from $D_i$ using $\alpha$. Note that $\mathbb{D}' = (\mathbb{D} \setminus \{D_i\}) \cup \{D_i'\}$ is another cut system and cut the unique outermost disk $\Delta'$ from $\Delta$. $\mathbb{D}'$ is obtained by a band-sum for cut system from $\mathbb{D}$. Let $B'$ be $V \setminus (\cup_{D \in \mathbb{D}'} D)$. Then $\Delta'$ cuts $B'$ into two balls $B_1'$ and $B_2'$. Assume $B_2'$ has the other cut end coming from $D_i'$, which is not the one $\Delta'$ intersecting. Note that the number of cut ends on $B_1'$ is less than that of $B_1$. By repeating this operation, we get a cut system $\tilde{\mathbb{D}} = (\mathbb{D} \setminus \{D_i\}) \cup \{\tilde{D_i}\}$. $\tilde{D_i}$ is a disk obtained by disk surgery from $D_i$ and $E_1$ using $\Delta$. Then the number of components of the intersection of disks of $\tilde{\mathbb{D}}$ and disks of $\mathbb{E}$ is less than that of $\mathbb{D}$ and $\mathbb{E}$.

By the above operation, we assume the disks of $\mathbb{D}$ and $\mathbb{E}$ are disjoint. If every disk of $\mathbb{E}$ is a disk of $\mathbb{D}$, then $\mathbb{D} = \mathbb{E}$. Suppose there exists $E_k$ such that it is not a disk of $\mathbb{D}$. $E_k$ cut $B$ into two balls $B_1$ and $B_2$. Since $\mathbb{D}$ and $\mathbb{E}$ are cut systems, there exists a disk $D_i$ such that the cut ends coming from $D_i$ is separated by $E_k$ in $B$ and $D_i$ is not a disk of $\mathbb{E}$. Take an oriented arc $\alpha$ starting from $\partial D_i$, ending on a cut end on $B_1$ and being on the boundary of $B_1$. Let $D_i'$ be the disk obtained from a band-sum from $D_i$ using $\alpha$. Note that $\mathbb{D}' = (\mathbb{D} \setminus \{D_i\}) \cup \{D_i'\}$ is another cut system since $D_i$ is not a disk in $\mathbb{E}$, and $\mathbb{D}'$ is obtained from $\mathbb{D}$ by the band-sum for cut system operation. Let $B'$ be $V \setminus (\cup_{D \in \mathbb{D}'} D)$. $E_k$ cuts $B'$ into two balls $B_1'$ and $B_2'$. The cut ends coming from $D_i'$ are separated by $E_k$, and the one of $B_1'$ and $B_2'$ has a smaller number of cut ends than $B_1$ has. See Figure 2. By repeating this operation, we get a cut system $\tilde{\mathbb{D}} = (\mathbb{D} \setminus \{D_i\}) \cup \{\tilde{D_i}\}$ such that $\tilde{D_i}$ is $E_k$. This increases the number of common disks in two cut systems. Hence by band-sums for cut system, we can change $\mathbb{D}$ into $\mathbb{E}$.

$\square$

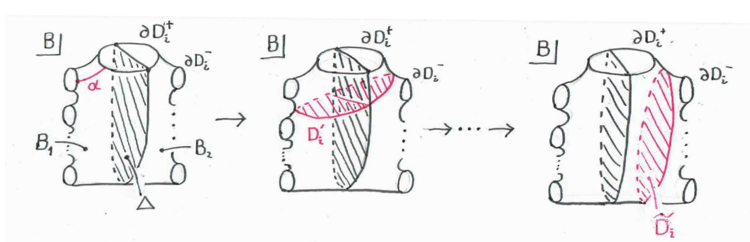


Figure 1: change cut system

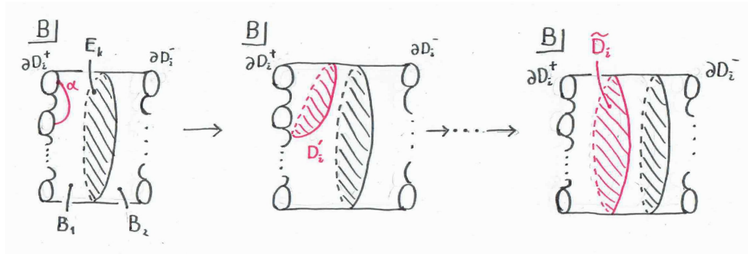


Figure 2: change cut system

**Definition 1.5.** (reducing the word) Let $l$ be a simple closed curve on $\partial V$, $\mathbb{D} = \{D_1, \dots, D_{2g}\}$ a cut system of $V$ and $B$ a ball $V \setminus \cup_{D \in \mathbb{D}} D$. Suppose the sequence of letters represented by $l$ is non-reduced i.e. there exists a subarc $c$ of $l$ representing $x_i x_i^{-1}$, for example. We call two cut ends on $\partial B$ coming from $D_i$ $D_i^+$ and $D_i^-$. The endpoints of $c$ are on the same cut end coming from $D_i$, say $\partial D_i^+$. Let $\delta$ be a properly embedded arc on $D_i^+$ connecting the endpoints of $c$. Then $c \cup \delta$ bounds a properly embedded disk $\Delta$ in $B$. Performing a disk surgery from $D_i$ using $\Delta$, we get two disks. Exactly one of the two disks separates $D_i^+$ and $D_i^-$ in $B$. Take such a disk $D_i'$. Then $\mathbb{D}' = (\mathbb{D} \setminus \{D_i\}) \cup \{D_i'\}$ is another cut system. We call this operation of changing cut systems *reducing the word.*

Note that after reducing the word, the length of the sequence of letters represented by $l$ with respect to $\mathbb{D}'$ is less than that with respect to $\mathbb{D}$. Moreover the length as not only the sequence of letters but also the word does not increase as in claim below:

**Claim 1.2.** *After reducing the word, the word represented by $l$ with respect to $\mathbb{D}'$ is obtained by deleting some letters (, the deleted letters may be empty) from the word represented by $l$ with respect to $\mathbb{D}$ and then reducing as a cyclic word.*

*Proof.* In the definition, $\delta$ separates $D_i^+$. It is enough to show that there is no subarc of $l$ on $\partial V$ starting from and ending on $\partial D_i^+$, each side separated by $\delta$ and representing a trivial word with respect to $\mathbb{D}$. If we show this, then all subarcs of $l$ representing trivial words with respect to $\mathbb{D}$ also represent trivial word with respect to $\mathbb{D}'$. Suppose there exists a subarc $\tilde{c}$ of $l$ on $\partial V$ starting from and ending on $\partial D_i^+$, each side separated by $\delta$ and representing a trivial word with respect to $\mathbb{D}$. Taking the subarc of $\tilde{c}$ if necessary, we assume $\tilde{c}$ is disjoint from $\partial D_i$ in its interior. By reducing the word operations using subarcs of $\tilde{c}$ representing reducible sequences, like $x_j x_j^{-1}$ if necessary, we assume the interior of $\tilde{c}$ is disjoint from the disks of $\mathbb{D}$. Such $\tilde{c}$ must intersect $c$. It contradicts for $l$ being a simple closed curve. □

## 1.2 The proof of Theorem 1.2

(1)⇒(2)
Fix a homeomorphism between $(V, l)$ and $(\Sigma_{g,1} \times [0,1], \partial\Sigma_{g,1} \times \{\frac{1}{2}\})$. Take arcs $\{a_1, \dots, a_{2g}\}$ on $\Sigma_{g,1}$ so that they cut $\Sigma_{g,1}$ into a disk as in Figure 3. Let $D_i$ be $a_i \times [0,1]$ (with appropriate orientations). Then $\mathbb{D} = \{D_1, \dots, D_{2g}\}$ is a cut system of $V$. About this cut system, the sequence of letters represented by $l$ is exactly $[x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$.

(2)⇒(3)
First, note that under a band-sum for cut system $\{D_1, \dots, D_{2g}\}$ using $D_j$ and an arc $\alpha$ starting $\partial D_j$ and ending on $\partial D_i$, the word represented by $l$ changes like $(x_i \mapsto x_i x_j^\epsilon)$ or $(x_i \mapsto x_j^\epsilon x_i)$ in $Out(F_{2g})$ action for $\epsilon \in \{\pm 1\}$: Realize $\partial D_j'$, the result of a band-sum, as the boundary component of a small neighborhood of $\partial D_j \cup \alpha \cup \partial D_i$. Give $D_j'$ the letter $x_j$, same as the letter assigned to $D_i$. The intersections of $l$ with $\partial D_j'$ near $\alpha$ make no effect on the word (representing $x_j x_j^{-1}$ for example). Hence it is enough to focus on the intersections of $l$ with disks of $\mathbb{D}$. For a disk $D_k$ of $\mathbb{D}$ which is

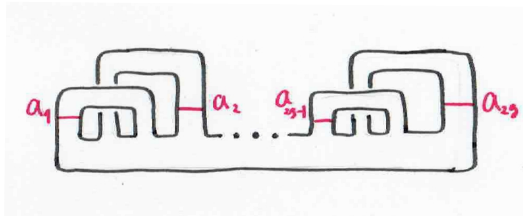


Figure 3: cut system of surface

neither $D_i$ nor $D_j$, the intersections do not change ($x_k \mapsto x_k$). Moreover, in our definition of the orientation of $D'_j$, the intersections near $\partial D_j$ does not change ($x_j \mapsto x_j$). Near $\partial D_i$, the intersections changes as ($x_i \mapsto x_i x_j^{\epsilon}$) or ($x_i \mapsto x_j^{\epsilon} x_i$). The position of $x_i$ and $x_j$ and $\epsilon$ depend on which side of $\partial D_j$ and $\partial D_i$ an arc $\alpha$ starting and ending.
Next, we assume that the word represented by $l$ is in $Out(F_{2g}) \cdot [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$ for a cut system $\mathbb{D}$. Take another cut system $\mathbb{E}$. By Claim 1.1, we can change $\mathbb{D}$ into $\mathbb{E}$ by band-sums for cut system. Note that if we change the orientation of $D_k$, a member of a cut system assigned the letter $x_k$, then the word represented by $l$ changes as ($x_k \mapsto x_k^{-1}$) and that if we exchange the letters assigned to $D_i$ and $D_j$, the word represented by $l$ changes as ($x_i \leftrightarrow x_j$). By the above observation, we see that the word represented by $l$ for $\mathbb{E}$ is also in $Out(F_{2g}) \cdot [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$.

(3)⇒(1)

Suppose $l$ represents a cyclic word in $Out(F_{2g}) \cdot [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$ with respect to $\mathbb{D} = \{D_1, \ldots, D_{2g}\}$. As the above, band-sums for cut system using an arc starting from $\partial D_j$ and ending on $\partial D_i$ changes the alphabets ($x_j \mapsto x_i^{\epsilon} x_j$) or ($x_j \mapsto x_j x_i^{\epsilon}$), and reversing the orientation of $\partial D_i$ changes alphabets ($x_i \mapsto x_i^{-1}$). Since theses type of changes form Nielsen's generators for $Aut(F_{2g})$, the action of every element of $Out(F_{2g})$ is realized as band-sums for cut system and changing the orientations of disks. Thus we assume that the word represented by $l$ with respect to $\mathbb{D}$ is exactly $[x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$. If the sequence of letters represented by $l$ with respect to $\mathbb{D}$ is reducible, then we perform reducing the word operations until the sequence of letters becomes reduced. We get a cut system $\mathbb{D}' = \{D'_1, \ldots, D'_{2g}\}$ such that the sequence of letters represented by $l$ is reduced and length of equal to or less than $4g$ by Claim 1.2.

**Claim 1.3.** *Let $w$ be a cyclically reduced word in $Out(F_{2g}) \cdot [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$. Then for all $i$,*

- *the sum of $x_i$ in $w$ is zero.*
- *there exists $x_i$ in $w$.*

*Proof.* Assume $w$ be a cyclically reduced word in $Out(F_{2g}) \cdot [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$. As seen before, there are a binding $K$ of $V$ and cut system $\mathbb{D}$ such that $K$ represents the word $[x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$, and the action of $Out(F_{2g})$ is realized by band-sums for cut system and orientations changes. Thus there exist a binding $K$ of $V$ and cut system $\tilde{\mathbb{D}}$ such that $K$ represents the word $w$. Since $K$ is separating on $\partial V$, the algebraic intersection number of $K$ and every essential disk in $V$ is zero. This is the first part of the statement.
Suppose $w$ has no $x_i$'s. If the sequence of letters represented by $K$ is reducible, perform reducing the word operations until the sequence of letters becomes reduced. Note that the obtained reduced sequence of letters also contains no $x_i$ by Claim 1.2. This means a binding $K$ is disjoint from an essential disk in $V$. This is a contradiction. Thus the second part of the statement is proved. □

By this claim, the sequence of letters represented by $l$ with respect to $\mathbb{D}'$ is reduced and has length exactly $4g$ and $l$ intersects each disk of $\mathbb{D}'$ exactly two times in the opposite directions. Then on $B' = V \setminus \cup_{D \in \mathbb{D}'} D$, a simple closed curve consisting of the image of $l$ and $2g$-properly embedded arcs, each of which is on one cut end, becomes a binding of $B'$, a handlebody of genus 0. This fiber structure extends to $V$ by pasting the cut ends coming from $D'_i$ for all $i$. Thus we know that $l$ is a binding of $V$. This completes the proof. □

## 1.3 Correspondence

Let $M$ be a connected, closed and orientable 3-manifold. For a genus $2g$ Heegaard splitting $M = V \cup_S W$, take a Heegaard diagram $(S; \partial\mathbb{D}, \partial\mathbb{E})$, where $\mathbb{D}$, $\mathbb{E}$ are cut systems of $V$, $W$ respectively and $\partial\mathbb{D}$, $\partial\mathbb{E}$ represent the sets of curves on $S$ which are the boundary of disks of $\mathbb{D}$, $\mathbb{E}$ respectively. Give orientations and letters $\{x_1, \ldots, x_{2g}\}$, $\{y_1, \ldots, y_{2g}\}$ to disks of $\mathbb{D}$ and $\mathbb{E}$ respectively. By the argument in Section 1 and theorems, we have the following:

**Corollary 1.1.** *There exists a one-to-one correspondence between the set of genus $g$ fibered knots with their fiber surfaces and the set of simple closed curves on some genus $2g$ Heegaard surfaces representing the words in $Out(F_{2g}) \cdot [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$, $Out(F_{2g}) \cdot [y_1, y_2] \cdots [y_{2g-1}, y_{2g}]$ with respect to $\mathbb{D}$, $\mathbb{E}$ respectively.*

We will give applications of this corollary where $g = 1$ in Sections 2, 3.

## 2 Non-existence of genus one fibered knots in a Seifert manifold whose base orbifold is a sphere with three exceptional points of sufficiently complex coefficients

Every Seifert manifold $M$ whose base orbifold is sphere with three exceptional points has a surgery presentation as in Figure 4, $(a,b),(c,d)$ and $(e,f)$ are coprime pairs. We will show that there are no GOF-knots in $M$ if $b \not\equiv \pm 1 \pmod{a}$, $d \not\equiv \pm 1 \pmod{c}$, and $f \not\equiv \pm 1 \pmod{e}$. We assume $b \not\equiv \pm 1 \pmod{a}$, $d \not\equiv \pm 1 \pmod{c}$, and $f \not\equiv \pm 1 \pmod{e}$. In particular, $|a|, |c|, |e| \geq 5$, and $|b|, |d|, |f| \geq 2$.

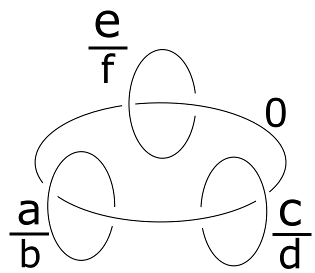


Figure 4: surgery presentation of $M$

### 2.1 Heegaard splittings of genus two of $M$

By Corollary 1.1, for non-existence of GOF-knots, it is enough to check that we cannot put simple closed curves on any Heegaard surface of genus two of $M$ satisfying the condition in Corollary 1.1 for $g = 1$. Note that the elements in $Out(F_2) \cdot [x,y]$ are $[x,y]$ and $[x,y^{-1}]$ as cyclic words.

**Facts 2.1.**

- *(See* [12] *for example) $M$ is irreducible.*
- [14] *Every irreducible Heegaard splitting of Seifert manifolds is either horizontal or vertical.*

It is known that every Heegaard splitting of genus grater than zero of every irreducible 3-manifold is irreducible. Therefore, every Heegaard splitting of genus two of $M$ is horizontal or vertical.

### 2.2 Horizontal Heegaard splittings

A Seifert manifold has a horizontal Heegaard splitting if there is a Seifert fiber which is a fibered knot. By thickening the fiber surface, we get the horizontal Heegaard splitting. It is known that the monodromy of this fibered knot is periodic as a boundary-free map. Hence for a genus two horizontal Heegaard splitting, there is a Seifert fiber which is a genus one fibered knot with periodic monodromy. It is known that the order of a periodic self-homeomorphism of $\Sigma_{1,1}$ is $1, 2, 3, 4$ or $6$. This means that this manifold admits a Seifert structure which has an exceptional fiber of coefficient $\frac{p}{q}$, for $p \in \{1, 2, 3, 4, 6\}$. Note that $M$ is not an exceptional Seifert manifold i.e. admits the unique Seifert structure up to homeomorphism (see [12] for example). Thus $|p| = |a|$ for example. This contradicts to $b \not\equiv \pm 1 \pmod{a}$. Hence $M$ has no horizontal Heegaard splittings of genus two.

Therefore, we consider only vertical Heegaard splittings of genus two of $M$.

### 2.3 Vertical Heegaard splittings

Every vertical Heegaard splitting of genus two $M = V \cup_S W$ is obtained by the following way: In Figure 4, the $0-$framed unknot bounds a disk. Take an arc $\alpha$ on this disk that connects two exceptional knots $K_1$ and $K_2$ and is disjoint from exceptional knots in its interior. Let $W$ be a regular neighborhood of

$K_1 \cup \alpha \cup K_2$, and $V$ be $M \setminus W$.

By the above construction, we get standard Heegaard diagrams as in Figure 5 and Figure 6. In figures, boxes represent curves as in Figure 7. $\{D_L, E\}$ (and $\{D_R, E\}$) are cut systems of the inner handlebodies, and $\{D', E'\}$ is a cut system of the outer handlebody. Note that $D_R$ is obtained from $D_L$ by $|f|$ times band-sums using $E$, hence these Heegaard splittings are equivalent. By applying homeomorphisms on this Heegaard surface and band-sums, we assume $a, b, c, d > 0$ and $a-b, c-d > 0$. Under these homeomorphisms, the condition $f \not\equiv \pm 1 \pmod{e}$ is preserved. Note also that in some cases, the diagrams in Figure 5 and Figure 6 may not be minimally and transversely. See Figure 8. In our assumption, $a-b, c-d \geq 2$, and $||e| - |f|| \geq 2$.

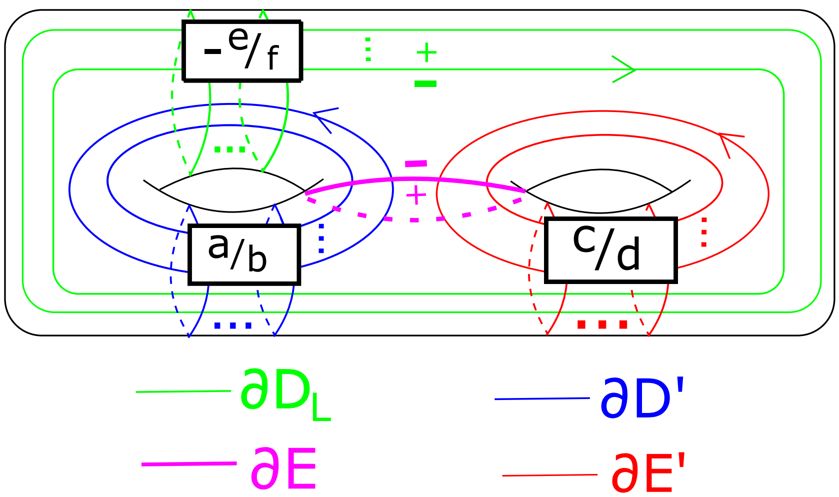


Figure 5: left-standard

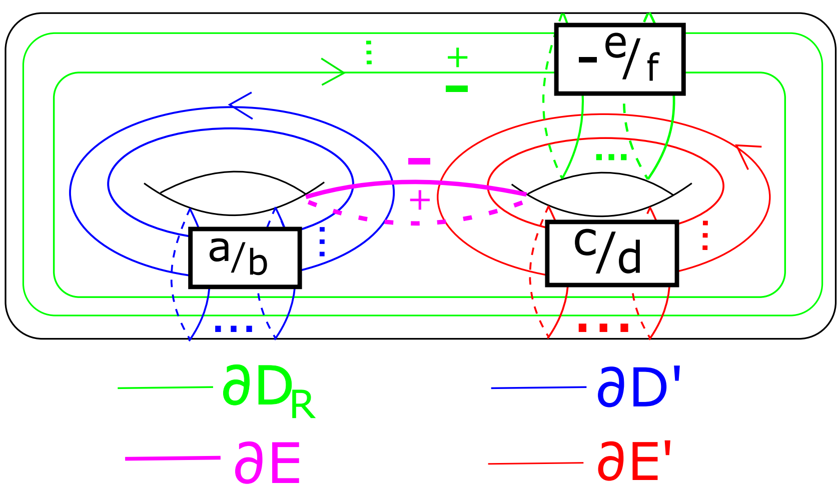


Figure 6: right-standard

Give $D_L$, $D_R$, $E$, $D'$, $E'$ letters $x_L$, $x_R$, $y$, $x'$, $y'$ respectively. For a contradiction, we suppose there exists a simple closed curve $l$ on the left-standard Heegaard diagram satisfying the condition in Corollary 1.1. Since the left-standard diagram and the right-standard diagram are equivalent, $l$

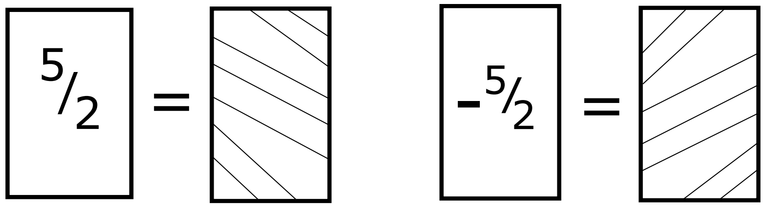


Figure 7: sign rule

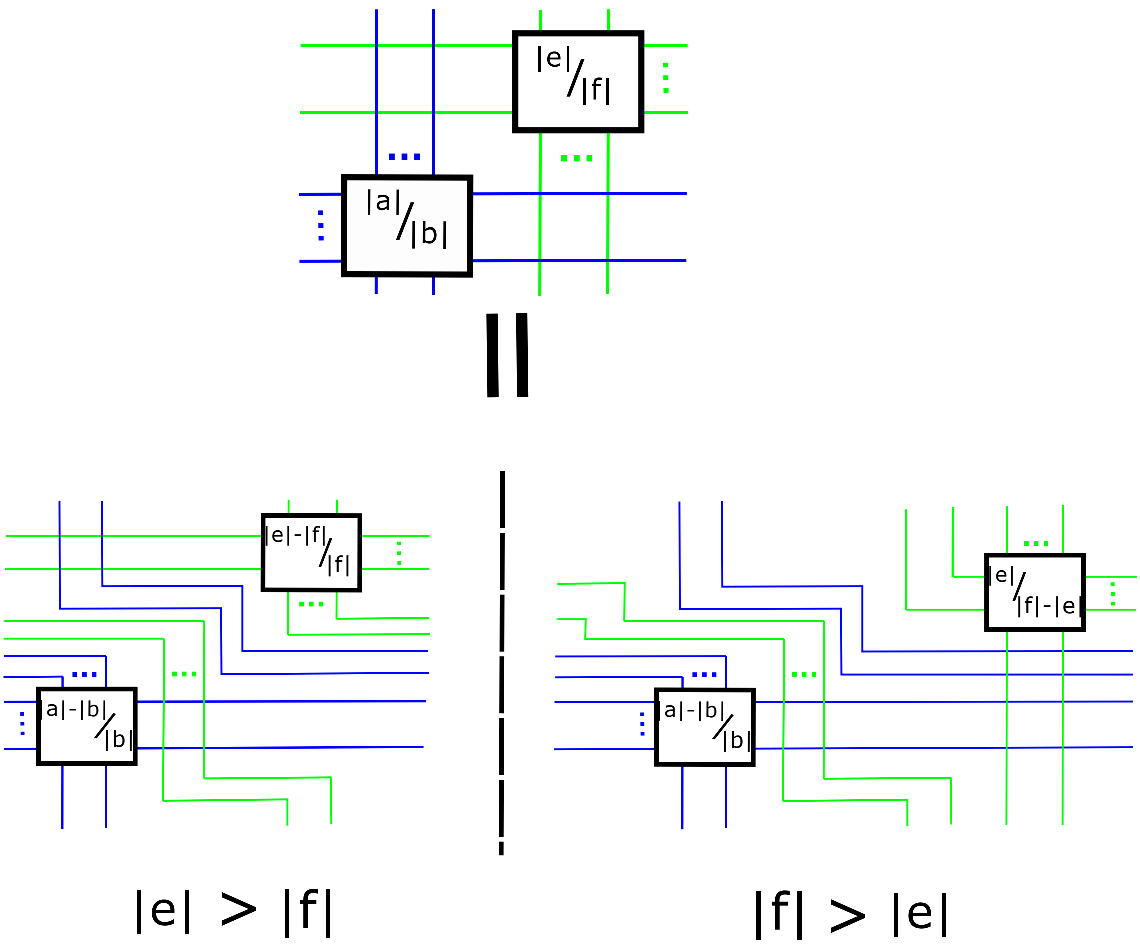


Figure 8: reduction

satisfies the condition also in the right-standard diagram.
First, we will show that $l$ must represent cyclically reduced sequences of letters in both $\{x_L, y\}$ and $\{x', y'\}$. Next, we will show that such $l$ cannot exist.

### 2.3.1 $l$ must be cyclically reduced

**Lemma 2.1.** *If the sequence of letters represented by $l$ in $\{x', y'\}$ is cyclically reduced, then that in $\{x_L, y\}$ is also cyclically reduced.*

*Proof.* Since $c > d$, there exist consecutive $(c-d)$ arcs with the same orientations connecting $\partial E^+$ and $\partial E^-$ in the left-standard diagram, which are subarcs of $\partial E'$. Hence if there exists a $x_L x_L^{-1}$ or $x_L^{-1} x_L$ in the sequence of letters represented by $l$ in $\{x_L, y\}$, then that in $\{x', y'\}$ has a subsequence $y'^{\pm(c-d)}$. However since the sequence of letters represented by $l$ in $\{x', y'\}$ is cyclically reduced and is a commutator, $l$ must intersect $\partial E'$ exactly two times in the opposite directions. It is a contradiction since $c - d \geq 2$.
Note that in any choice of signs, even after making the diagrams minimally and transversely, there are *sharp shaped* subdiagrams in the left-standard and the right-standard diagrams as in Figure 9. Hence if there exists a $yy^{-1}$ or $y^{-1}y$ in the sequence of letters represented by $l$ in $\{x_L, y\}$, that in $\{x', y'\}$ has a subsequence $x'^{\pm 2}$ (and $y'^{\pm 2}$). However since the sequence of letters represented by $l$ in $\{x', y'\}$ is cyclically reduced and is a commutator, $l$ must intersect $\partial D'$ exactly two times in the opposite directions. It is a contradiction.

□

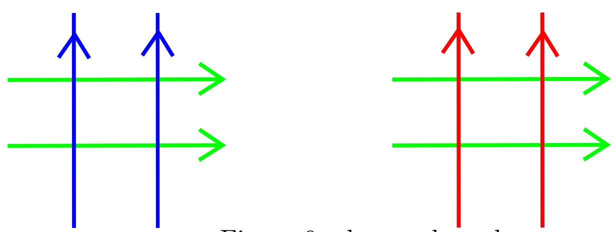

Figure 9: sharpe shaped

Similarly to the proof of Lemma 2.1, we can show that if the sequence of letters represented by $l$ is cyclically reduced in $\{x_L, y\}$ or $\{x_R, y\}$, then that in $\{x', y'\}$ is also cyclically reduced.

**Lemma 2.2.** *The sequences of letters represented by $l$ in $\{x_L, y\}$ and $\{x', y'\}$ are cyclically reduced.*

*Proof.* If the sequence of letters represented by $l$ in $\{x_L, y\}$ contains $yy^{-1}$, then it has $x'^{\pm 2}$ and $y'^{\pm 2}$ because of sharp shaped subdiagrams as in Figure 9. Hence by Fact 1.3, $l$ represents a cyclically reduced sequence of letters in $\{x', y'\}$ and it cannot represent a commutator. Similarly, we see that the sequence of letters represented by $l$ in $\{x_R, y\}$ does not contain $yy^{-1}$.
If the sequence of letters represented by $l$ in $\{x_L, y\}$ contains $x_L x_L^{-1}$, then it has $y'^{\pm(c-d)}$. And If the sequence of letters represented by $l$ in $\{x_R, y\}$ contains $x_R x_R^{-1}$, then it has $x'^{\pm(a-b)}$. Therefore the sequence of letters represented by $l$ at least one of in $\{x_L, y\}$ and in $\{x_R, y\}$ is cyclicaly reduced, since otherwise the sequence of letters represented by $l$ in $\{x', y'\}$ has $x'^{\pm 2}$ and $y'^{\pm 2}$. In both case, the sequence of letters represented by $l$ in $\{x', y'\}$ is cyclically reduced and hence that in $\{x_L, y\}$ must be cyclically reduced by Lemma 2.1. □

### 2.3.2 $l$ cannot exist

By the above, $l$ represents cyclically reduced sequences of letters which are commutators in both of $\{x_L, y\}$ and $\{x', y'\}$. In the left-standard diagram, there is a subarc of $l$ starting from $\partial E^+$ and ending on $\partial D_L^+$ by reversing the orientation of $l$ if necessary. Next to this subarc, $l$ starts from $\partial D_L^-$ and must end on $\partial E^+$ since it represents a commutator in $\{x_L, y\}$. However these two subarcs prevent $l$ from being a commutator in $\{x', y'\}$. See Figures 10,11,12,13 below.

Here we give figures of left-standard diagrams for four cases, where $\frac{e}{f} > 0$ with $e > f$, $\frac{e}{f} > 0$ with $e < f$, $\frac{e}{f} < 0$ with $|e| > |f|$, and $\frac{e}{f} < 0$ with $|e| < |f|$. In the figures, the disks with three-holes are obtained by cutting the Heegaard surfaces along $\partial D_L$ and $\partial E$. We can reconstruct the Heegaard surfaces by pasting $\partial D_L^+$ and $\partial D_L^-$ so that the spade marks ♠ match, and pasting $\partial E^+$ and $\partial E^-$ so that the circle marks • match. Boxes with numbers represent parallel copies of arcs. In the cases where $|e| > |f|$, $s_n$ for an integer $n$ denotes a number of multiples of $|f|$ in $((n-1)|e|, n|e|)$, and $i$ denotes the least positive number such that $(1 + \Sigma_{n=1}^{i} s_n)|f| \equiv 1 \mod |e|$, this number is equal to $1 + i|e|$ by definition. In the cases where $|e| < |f|$, $t_n$ for an integer $n$ denotes a number of multiples of $|e|$ in $((n-1)|f|, n|f|)$, and $j$ denotes the least positive number such that $(1 + \Sigma_{n=1}^{j} t_n)|e| \equiv 1 \mod |f|$. This number is equal to $1 + j|f|$ by definition. Note that $s_n, t_n > 0$ for any $n$.

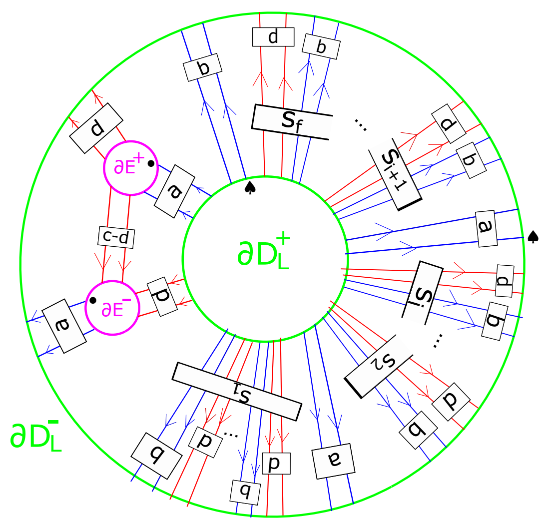


Figure 10: $\frac{e}{f} > 0$ with $e > f$

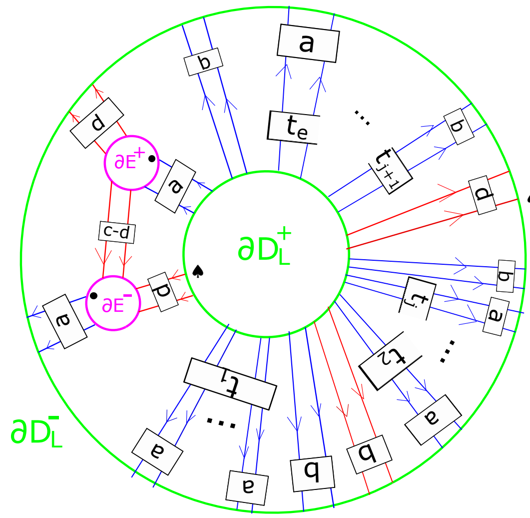


Figure 11: $\frac{e}{f} > 0$ with $e < f$

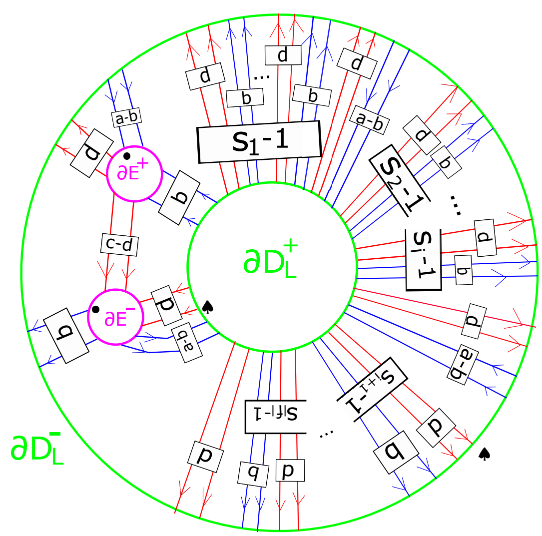


Figure 12: $\frac{e}{f} < 0$ with $|e| > |f|$

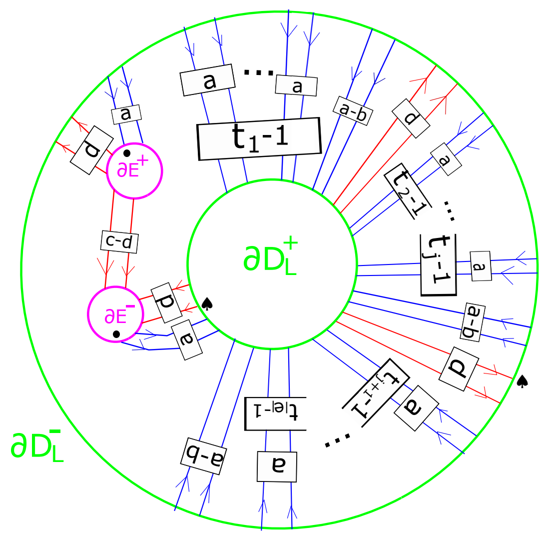


Figure 13: $\frac{e}{f} < 0$ with $e < f$

# 3 Openbook genus of a Seifert manifold whose base orbifold is sphere with three exceptional points of integral coefficients

For a connected, orientable closed 3-manifold $X$, its *openbook genus* $\mathrm{op}(X)$ is defined by
$\mathrm{op}(X) = \min\{g \geq 0 \mid X$ has a genus g fibered knot$\}$.
We will compute the openbook genus for a Seifert manifold whose base orbifold is a sphere with three exceptional points of integer coefficients. Every such manifold has a surgery presentation as in the left-hand side of Figure 14 by introducing integer parameters $a$,$b$ and $c$. It is denoted by $M\{a, b, c\}$. It can be presented as a surgered manifold along the three component link as in the right-hand side of Figure 14. Note that the homeomorphism type of $M\{a, b, c\}$ is independent of the order of $\{a, b, c\}$, and that $M\{a, b, c\}$ is homeomorphic to $M\{-a, -b, -c\}$.

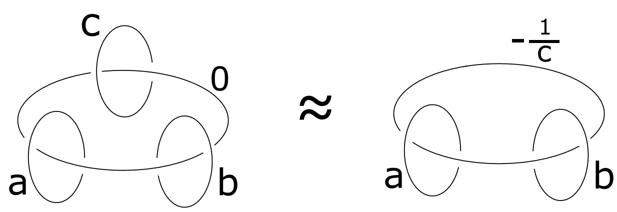


Figure 14: surgery presentation of $M\{a, b, c\}$

In this section, we will prove the following:

**Theorem 3.1.** $\mathrm{op}(M\{a, b, c\})$ *is computed as follows, where $n$ is an integer:*

- $\mathrm{op}(M\{a,b,c\}) = 0$ *if and only if* $\{a, b, c\} = \{0, \pm1, \pm1\}, \{1, -1, n\}$, $\{\pm1, \mp2, \mp3\}$.
- $\mathrm{op}(M\{a,b,c\}) \leq 1$ *if and only if at least one of* $\{a, b, c\}$ *is* $0$, $2$ *or* $-2$, *or* $\{a, b, c\} = \{1, -1, n\}$, $\{\pm1, \mp4, n\}$, $\{\pm1, \pm1, \pm1\}$, $\{\pm1, \pm1, \pm3\}$, $\{\pm1, \pm1, \mp3\}$, $\{\pm1, \mp3, \mp3\}$, $\{\pm1, \mp3, \mp5\}$.
- *Otherwise* $\mathrm{op}(M\{a,b,c\}) = 2$.

In Subsection 3.1, we prove that $\mathrm{op}(M\{a, b, c\}) \leq 2$ for every $\{a, b, c\}$. In Subsection 3.2, we give the necessary and sufficient condition for $\mathrm{op}(M\{a, b, c\}) = 0$. In Subsection 3.3, we give the necessary and sufficient condition for $\mathrm{op}(M\{a, b, c\}) \leq 1$ admitting some proposition. In Subsection 3.4, we prove the proposition.

## 3.1 An upper bound

Regard $M\{a, b, c\}$ as a surgered manifold as in the right-hand side of Figure 14. Let $K'_c$ denote the $(-\frac{1}{c})$-framed unknot. In general, $L(n, 1)$, the lens space of type $(n, 1)$, has a fibered link whose fiber surface is an annulus. Hence by the plumbing of fibered annuli, we get a fibered link whose fiber surface is homeomorphic to $\Sigma_{0,3}$ in $L(-a, 1)\#L(-b, 1)$, see Figure 15. Note that in the surgery presentation of Figure 15, $K'_c$ can be put on the fiber of $L(-a, 1)\#L(-b, 1)$.

**Fact 3.1.** *(See a part of "twisting" in* [10]*) Let $X$ be a 3-manifold with a fibered link and $F$ its fiber surface, and $K$ a knot on $F$. Give $K$ the framing coming from $F$, surface framing. Let $X(\frac{1}{n})$ be the manifold obtained from $X$ by $\frac{1}{n}$-surgery (according to the surface framing) along $K$ for an integer $n$. Then $X(\frac{1}{n})$ has a fibered link whose fiber surface is homeomorphic to $F$.*

Since the canonical framing and the surface framing of $K'_c$ are identical, $M\{a, b, c\}$ has a fibered link whose fiber surface is homeomorphic to $\Sigma_{0,3}$ by Fact 3.1. By the plumbings of Hopf annuli in $S^3$ in order to make the boundary connected, we get a genus two fibered knot in $M\{a, b, c\}$. Therefore $\mathrm{op}(M\{a, b, c\}) \leq 2$ for every $\{a, b, c\}$.

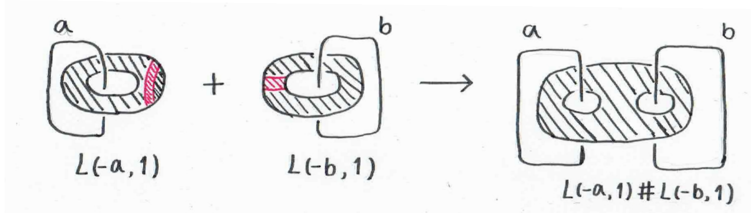


Figure 15: one way of the plumbing fibered annuli

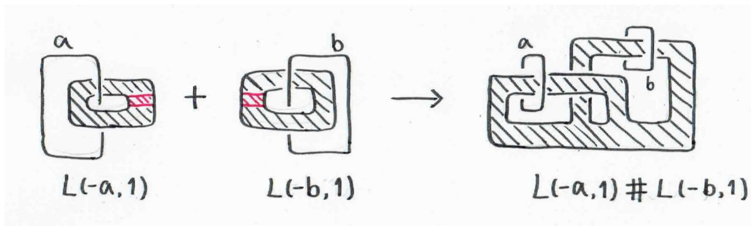


Figure 16: another way of the plumbing fibered annuli

## 3.2 The condition for $\mathrm{op}(M\{a,b,c\}) = 0$

Suppose that $\mathrm{op}(M\{a,b,c\}) = 0$. For a 3-manifold $X$, it is known that $\mathrm{op}(X) = 0$ if and only if $X = S^3$. Hence $\pi_1(M\{a,b,c\})$ must be trivial. We assume that $|a| \geq |b| \geq |c|$. From the surgery presentation, we get a presentation $\pi_1(M\{a,b,c\}) = \langle x, y, h |\ x^a h = 1,\ y^b h = 1, (xy)^{-c} h = 1, [x,h] = [y,h] = 1 \rangle$. Taking the quotient by the normal subgroup $\langle h \rangle$, we get a group $G = \langle x, y |\ x^a = 1,\ y^b = 1,\ (xy)^{-c} = 1 \rangle$, which must be also trivial. Since it is known that $G$ is finite if and only if $\frac{1}{|a|} + \frac{1}{|b|} + \frac{1}{|c|} > 1$, we get $|c| \leq 2$. Moreover, $|c| \leq 1$ since $|G| = |a||b| \geq |c|^2 \neq 1$ under $|c| = 2$. If $c = 0$, $|a|$ and $|b|$ must be one since $M\{a,b,c\}$ represents $L(-a,1)\#L(-b,1)$. We assume $c = 1$. Then $\pi_1(M\{a,b,c\}) = \langle x |\ x^{ab+a+b} \rangle$. Therefore $\{a,b\} = \{-1,n\}, \{0,1\}, \{-2,-3\}$ for every integer $n$.
By the above, we see that $\{a,b,c\} = \{0,\pm1,\pm1\}$, $\{1,-1,n\}$, $\{\pm1,\mp2,\mp3\}$ must hold for $\mathrm{op}(M\{a,b,c\}) = 0$. We see that this is also the sufficient condition for $\mathrm{op}(M\{a,b,c\}) = 0$ by using the Kirby calculus.

## 3.3 The condition for $\mathrm{op}(M\{a,b,c\}) \leq 1$

### 3.3.1 The case where at least one of $\{a,b,c\}$ is zero

Assume one of $\{a,b,c\}$, say $c$, is zero. Then $M\{a,b,0\} = L(-a,1)\#L(-b,1)$. This has a genus one fibered knot, which is obtained by a plumbing of fibered annuli in $L(-a,1)$ and $L(-b,1)$, see Figure 16. Hence $\mathrm{op}(M\{a,b,0\}) \leq 1$.

### 3.3.2 The case where at least one of $\{a,b,c\}$ is $\pm1$

Assume one of $\{a,b,c\}$, say $c$, is one. Then $M\{a,b,1\} = L(ab+a+b, -(b+1))$ by the Kirby calculus and it is homeomorphic to $L(ab+a+b, b+1)$. Whether a given lens space has a genus one fibered knot is completely determined as follows:

**Fact 3.2.** [2] *A lens space has a genus one fibered knot if and only if there exist two non-negative integers $x$, $y$ and $\epsilon$, which is $0$ or $1$ such that it is homeomorphic to $L(2xy+x+y+\epsilon, 2y+1)$.*

By noting that every lens space has at most four representations of the form $L(p,q)$ with $p \geq q \geq 0$, we can determine whether a solution $x, y, \epsilon$ of Fact 3.2 exists for each $\{a, b\}$ as follows:

$L(ab+a+b, b+1)$ has a genus one fibered knot if and only if at least one of $\{a, b\}$ is $0, -1, \pm 2, -4$ or $\{a, b\} = \{1, 1\}, \{1, \pm 3\}, \{-3, -3\}, \{-3, -5\}$.

These are conditions for $\mathrm{op}(M\{a, b, c\}) \leq 1$ under $c = 1$. For under $c = -1$, $a$ and $b$ should be replaced with $-a$ and $-b$.

#### 3.3.3 The case where at least one of $\{a, b, c\}$ is $\pm 2$

Assume one of $\{a, b, c\}$, say $c$, is $-2$. We can put a knot $K$ on the fiber surface of genus one of $L(-a, 1)\#L(-b, 1)$ as in Figure 17 so that the surface framing of $K$ is $(+1)$-framing according to the canonical framing. Note that $(\frac{1}{2})$-framing of $K$ according to the canonical framing corresponds to the $(-\frac{1}{2})$-framing according to the surface framing. Hence by Fact 3.1, $M\{a, b, -2\}$ has a fibered knot whose fiber surface is homeomorphic to $\Sigma_{1,1}$ i.e. $\mathrm{op}(M\{a, b, -2\}) \leq 1$ for any $a$,$b$. For $c = 2$, the similar argument holds.

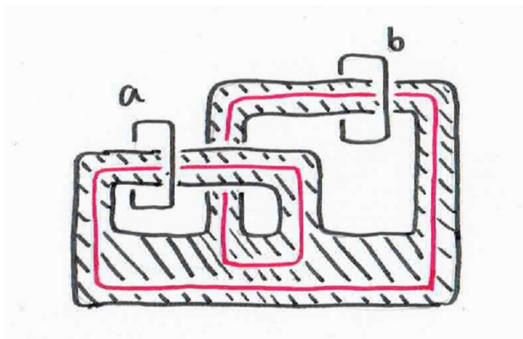


Figure 17: $K$ on the fiber surface

#### 3.3.4 The case where $|a|, |b|, |c| \geq 3$

Assume $|a|, |b|, |c| \geq 3$.
We postpone the proof of the proposition below to the next subsection:

**Proposition 3.1.** *$M\{a, b, c\}$ has no genus one fibered knots.*

By the above, we see that Theorem 3.1 holds.

### 3.4 Proof of Proposition 3.1

Assume $|a|, |b|, |c| \geq 3$. We use the setting of Section 1. Hence, for the non-existence of GOF-knots, it is enough to check that we cannot put simple closed curves on any Heegaard surfaces of genus two of $M$ satisfying the condition in Corollary 1.1. It is known that $M$ is irreducible (see [12] for example). By Facts 2.1, it is enough for Proposition 3.1 to prove the lemmas below:

**Lemma 3.1.** *$M\{a, b, c\}$ has no horizontal Heegaard splittings of genus two.*

**Lemma 3.2.** *There are no simple closed curves satisfying the condition in Corollary 1.1 on any vertical Heegaard surfaces of genus two of $M\{a, b, c\}$.*

### 3.4.1 Proof of Lemma 3.1

For a Seifert manifold $X$, a horizontal Heegaard splitting exists if there is a Seifert fiber which is a fibered knot. By thickening the fiber surface, we get the horizontal Heegaard splitting. It is known that the monodromies of the fibered knots of horizontal Heegaard splitting are periodic as boundary-free maps.
By the above, if $X$ has a horizontal genus two Heegaard splitting, there is a fibered knot which is a fiber of the Seifert structure whose fiber surface is homeomorphic to $\Sigma_{1,1}$, and the monodromy is periodic. It is known that the order of a periodic map of $\Sigma_{1,1}$ is $1, 2, 3, 4$ or $6$ and we have concrete homeomorphisms of such orders. For the case where the order is $1, 2, 4$ or $6$, there exists an exceptional fiber of coefficient $\frac{p}{q}$ with $|p| < 3$. For the case where the order is 3, we see that $X$ is homeomorphic to a Seifert manifold whose base orbifold is a sphere with three exceptional points with coefficients $\frac{3}{1}, \frac{3}{1}$ and $-\frac{2+3k}{1+2k}$ for an integer $k$. Note that $M\{a, b, c\}$ for $|a|, |b|, |c| \geq 3$ is not an exceptional Seifert space i.e. admits the unique Seifert fiber structure up to homeomorphism (see [12] for example). This implies $M\{a, b, c\}$ has no horizontal Heegaard splittings.

### 3.4.2 Proof of Lemma 3.2

Every vertical Heegaard splitting of $M\{a, b, c\}$ is obtained as follows: Choose an arc $\alpha$ connecting two of three exceptional fibers, say $A$ and $B$. Then $N(A \cup \alpha \cup B) \cup (M\{a, b, c\} \setminus N(A \cup \alpha \cup B))$ is a vertical Heegaard splitting. A Heegaard diagram obtained by this construction is as in Figure18. We call the top diagram left-standard and the bottom right-standard. $\{D_L, E\}$ and $\{D_R, E\}$ are cut systems of the inner handlebody, and $\{D', E'\}$ is a cut system of the outer handlebody. Give letters $x_L, x_R, y, x', y'$ to $D_L, D_R, E, D', E'$ respectively. Note that the positions of $a$, $b$, $c$ may interchange and when the sign of $a$ is different from that of $c$, the diagram is not minimally intersecting.

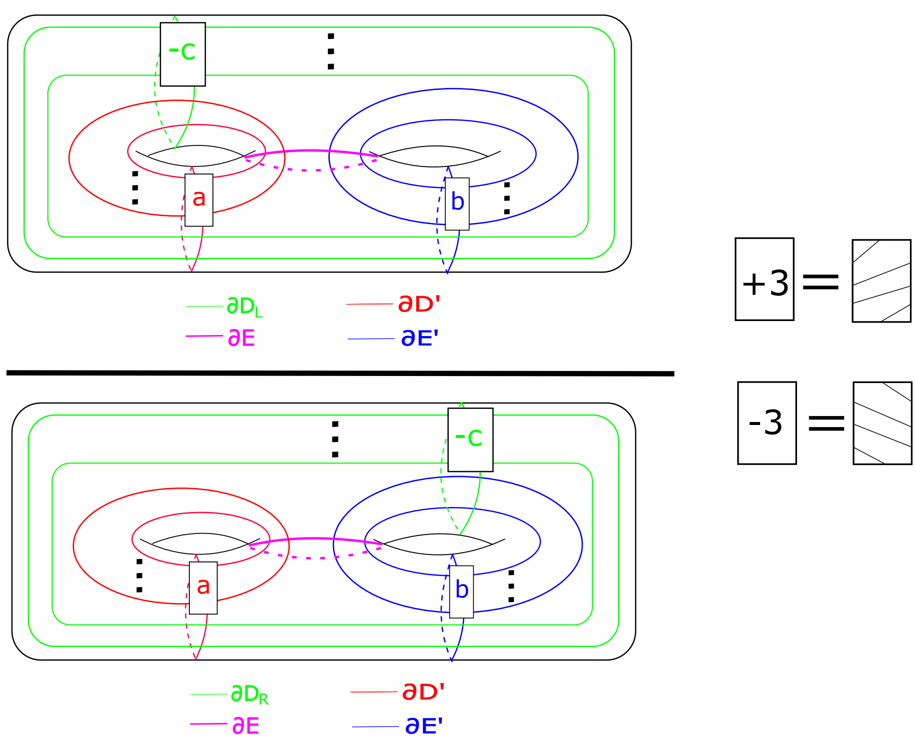


Figure 18: vertical diagram

Suppose there exists a simple closed curve $l$ on the Heegaard surface satisfying the condition in Corollary 1.1. First we show $l$ must represent reduced sequences of letters with respect to the inner and outer cut systems of some left (or right)-standard diagram. Next, we show that such $l$ cannot exist.
We consider three cases: (1)$a > 0, b > 0, c > 0$, (2)$a > 0, b > 0, c < 0$, (3)$a < 0, b > 0, c > 0$. Their left-standard diagrams are in Figures 19, 25, 31.
Note that in any case, if $l$ represents a reduced sequence of letters with respect to $\{D', E'\}$, then its representing sequence of letters with respect to $\{D_L, E\}$ (and $\{D_R, E\}$) are also reduced. This is because there are $(|b| - 1)$ arcs with the same orientations coming from $\partial E'$ connecting $\partial E^+$ and $\partial E^-$, $(|c| - 1)$ arcs with the same orientations coming from $\partial E'$ connecting $\partial D_L^+$ and $\partial D_L^-$ in the left-standard diagram. Therefore, if $l$ has a subarc representing $x_L x_L^{-1}$ must intersect with $\partial E'$ in the same orientation more than once. Since $l$ represents a reduced sequence of letters with respect to $\{x', y'\}$ and a commutator, this is a contradiction.
Similarly we can see that if $l$ represents a reduced sequence of letters with respect to $\{D_L, E\}$ (or $\{D_R, E\}$), then its representing sequence of letters with respect to $\{D', E'\}$ is also reduced.
If there exists a subarc of $l$ representing $x_L x_L^{-1}$ (or $x_R x_R^{-1}$) with respect to $\{D_L, E\}$ (or $\{D_R, E\}$), a part of this subarc represents $y'^{\pm(|b|-1)}$ (or $x'^{\pm(|a|-1)}$) with respect to $\{D', E'\}$. Hence with respect to one of $\{D_L, E\}$ and $\{D_R, E\}$, $xx^{-1}$-type subarcs of $l$ cannot exist. Otherwise, $l$ represents a reduced sequence of word by Fact 1.3. We assume that $l$ does not have a subarc representing $x_L x_L^{-1}$ with respect to $\{D_L, E\}$ for cases (1) and (2).

**(1) the case where** $a > 0, b > 0, c > 0$ Suppose that $l$ has a subarc $\tilde{c}$ representing $yy^{-1}$ with respect to $\{D_L, E\}$. We suppose that its corresponding arc $\alpha$ starts from $\partial E^-$. We may assume that $\alpha$ is put so that it is disjoint from $D'$ and $E'$ by Fact 1.2.
If $\alpha$ ends on $\partial D_L^+$, we change an (inner) cut system as in Figure 20. In this diagram we see that $l$ is reduced with respect to this new cut system $\{\bar{D}_L, E\}$, or that $l$ is reduced with respect to $\{D', E'\}$, or $l$ is reduced with respect to an (outer) cut system $\{\bar{D}', E'\}$ as in Figure 21. For suppose $l$ is not reduced with respect to this new cut system $\{\bar{D}_L, E\}$. There exists a reducible subarc of $l$. Let $\alpha$ denote the corresponding arc. If $\alpha$ intersects $\partial D' \cap \partial E'$, $l$ represents a reduced sequence of letters with respect to $\{D', E'\}$ by Fact 1.2. Suppose $\alpha$ is disjoint from $\partial D' \cap \partial E'$. If $\alpha$ starts at $\partial \bar{D}_L$ and ends on $\partial E^+$ or $\partial E^-$, $l$ represents a reduced sequence of letters with respect to $\{D', E'\}$ by Fact 1.3. If $\alpha$ started at $\partial E^-$ and ends on $\partial \bar{D}_L^-$, it would intersect $\tilde{c}$. If $\alpha$ starts at $\partial E^-$ and ends on $\partial \bar{D}_L^+$, $l$ represents a reduced sequence of letters with respect to $\{\bar{D}', \bar{E}'\}$ by Fact 1.3. First two cases imply that $l$ is reduced with respect to $\{D_L, E\}$. In the last case, the new outer cut system is equal to the outer cut system of right of Figure 22. The left of Figure 23 is a diagram obtained by cutting open the diagram in the right of Figure 22 along the inner cut system. The right of Figure 23 is another description of this diagram. Note that this diagram looks like the standard diagram. Hence $l$ can be put on some standard diagram so that it is reduced with respect to inner and outer cut systems.
If $\alpha$ ends on $\partial D_L^-$, we change an inner cut system as in the left of Figure 22. Note that the number of intersections of $l$ with the inner cut system decreases. Then we change also an outer cut system as in the right of Figure 22. Again, this diagram is equal to the standard diagram in the right of Figure 23. Return to the start of (1), and repeat the argument. After finitely many operations, $l$ can be put on some standard diagram so that it is reduced with respect to inner and outer cut systems.

Next we search such $l$ on the left-standard diagram. By reversing the orientation if necessary, we have a subarc $l'$ of $l$ starting from $\partial E^-$ and ending on $\partial D_L^-$. After $l'$, $l$ must start from $\partial D_L^+$ and end on $\partial E^-$. Before $l'$, $l$ must end on $\partial E^+$ and start from $\partial D_L^-$. They prevent $l$ from representing a reduced sequence of letters and a commutator. See Figure 24

**(2) the case where** $a > 0, b > 0, c < 0$ Suppose that $l$ has a subarc representing $yy^{-1}$ with respect to $\{D_L, E\}$. We suppose that its corresponding arc $\alpha$ starts from $\partial E^-$. We can assume that $\alpha$ is put

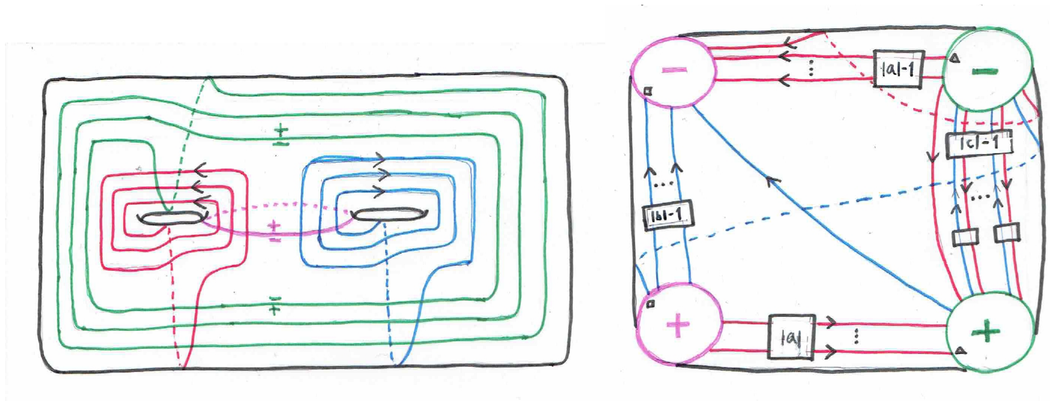


Figure 19: left-standard diagram for (1)

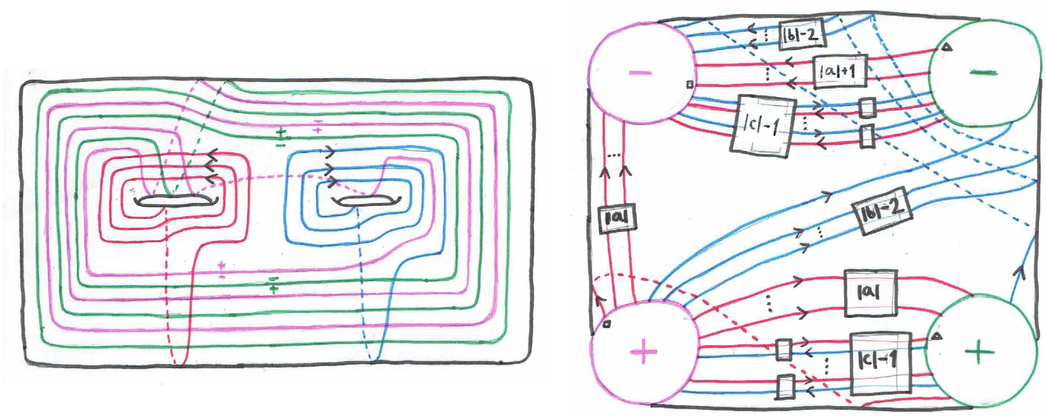


Figure 20: diagram for (1)

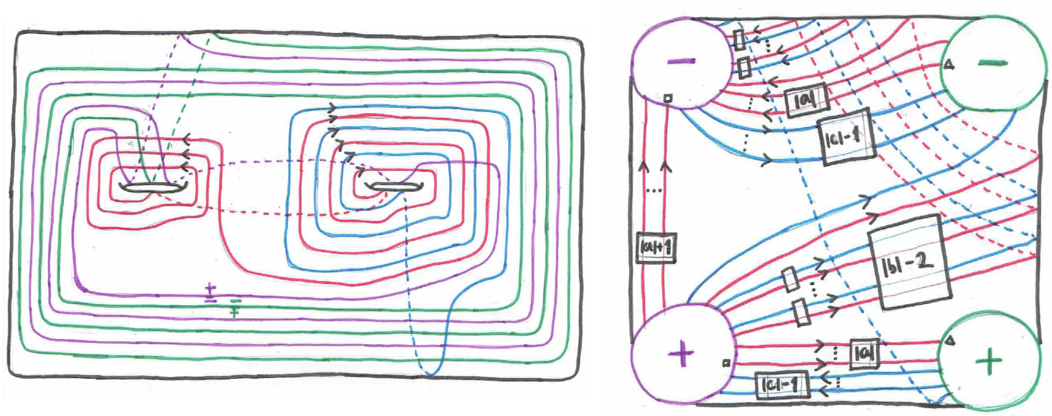


Figure 21: diagram for (1)

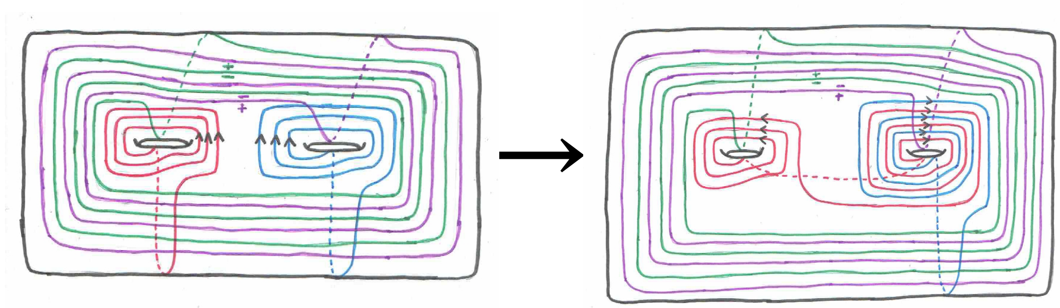

Figure 22: diagram for (1)

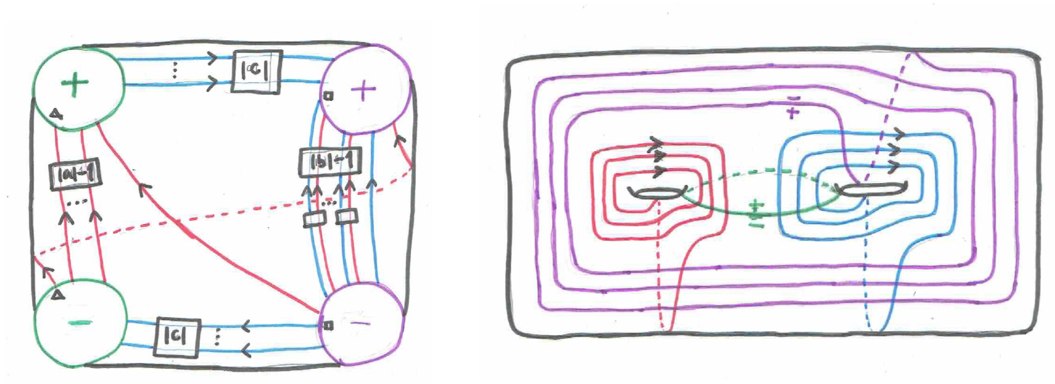


Figure 23: diagram for (1)

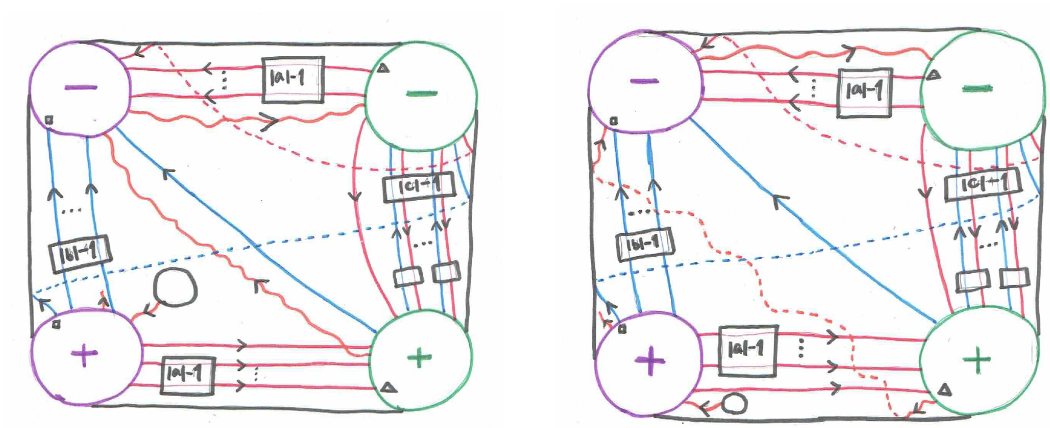


Figure 24: parts of $l$ for (1)

so that it is disjoint from $D'$ and $E'$ by Fact 1.2.
If $\alpha$ ends on $\partial D_L^-$, we change an (inner) cut system as in Figure 26. In this diagram we see that $l$ is reduced with respect to this new cut system, or that $l$ is reduced with respect to $\{D', E'\}$, or $l$ is reduced with respect to an (outer) cut system as in the right of Figure 27. First two cases imply that $l$ is reduced with respect to $\{D_L, E\}$. In the last case, note that the new outer cut system is equal to the outer cut system of right of Figure 28. The left of Figure 28 is a diagram obtained by cutting open the diagram in the right of Figure 28 along the inner cut system. Note that this diagram looks like the standard diagram of $a' = -b < 0, b' = a > 0, c' = -c > 0$. Hence $l$ can be put on some standard diagram in the case (3) so that it is reduced with respect to the inner and outer cut systems.
If $\alpha$ ends on $\partial D_L^-$, we change an inner cut system as in the left of Figure 27. Note that the number of intersections of $l$ with inner cut systems decreases. Then we change also an outer cut system as in the right of Figure 27. Again, this diagram is equal to the standard diagram in the right of Figure 28. Go to (3). After finitely many operations, $l$ can be put on some standard diagram so that it is reduced with respect to the inner and outer cut systems.

Next we search such $l$ on the left-standard diagram. By reversing the orientation if necessary, there is a subarc $l'$ of $l$ starting from $\partial E^+$ and ending on $\partial D_L^+$. After $l'$, $l$ must start from $\partial D_L^-$ and end on $\partial E^+$. Before $l'$, $l$ must end on $\partial E^-$ and start from $\partial D_L^+$. They prevent $l$ from representing a reduced sequence of letters and a commutator. See Figures 29, 30

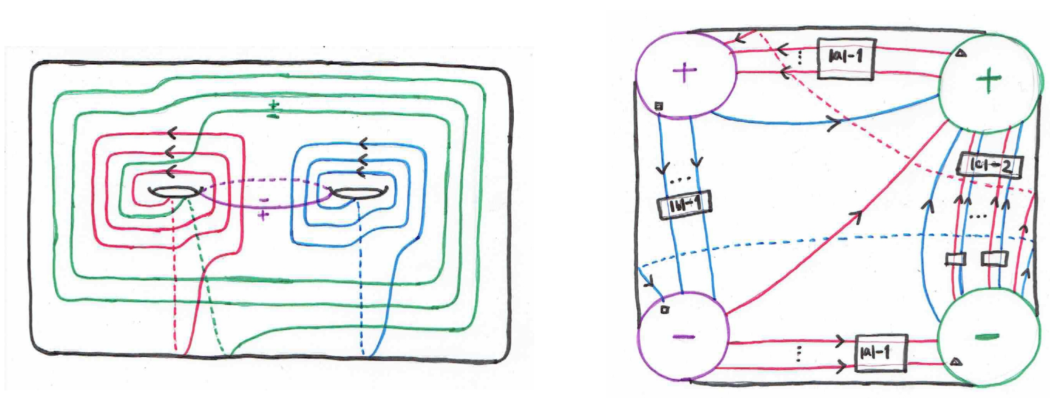


Figure 25: left-standard diagram for (2)

**(3) the case where** $a < 0, b > 0, c > 0$ The left (or right)-standard diagram is given in Figure 31 (or 32), respectively.
As mentioned before, not both of $x_R x_R^{-1}$ with respect to $\{x_R, y\}$ and $x_L x_L^{-1}$ with respect to $\{x_L, y\}$ in the sequence of letters represented by $l$ exist. Assume $l$ does not have a subarc representing $x_R x_R^{-1}$ with respect to $\{x_R, y\}$. If $l$ does not have a subarc representing $yy^{-1}$ with respect to $\{x_R, y\}$, $l$ is reduced with respect to $\{x_R, y\}$, moreover $\{x', y'\}$, $\{x_L, y\}$. If $l$ has a subarc representing $yy^{-1}$ with respect to $\{x_R, y\}$, its corresponding arc $\alpha$ is disjoint from $D'$ and $E'$ by Fact 1.2. We assume $\alpha$ starts from $\partial E^+$. If $\alpha$ ends on $\partial D_R^+$, we change the inner cut system as in Figure 34. Additionally, we change also the outer cut system as in Figure 35. Note that this diagram looks like a right-standard diagram (compare the rights of Figures 32 and 35) and note that the number of the intersections of $l$ with the inner cut systems decreases. If $\alpha$ ends on $\partial D_R^-$, we change the outer cut system as in Figure 33. Then $l$ is reduced with respect to this outer cut system by Fact 1.3. This leads that $l$ is reduced with respect to also the inner cut system, $\{D_R, E\}$. As before, $l$ is reduced with respect to $\{x_L, y\}$.
Assume $l$ does not have a subarc representing $x_L x_L^{-1}$ with respect to $\{x_L, y\}$. If $l$ does not have a subarc representing $yy^{-1}$ with respect to $\{x_L, y\}$, $l$ is reduced with respect to $\{x_L, y\}$. If $l$ has a subarc

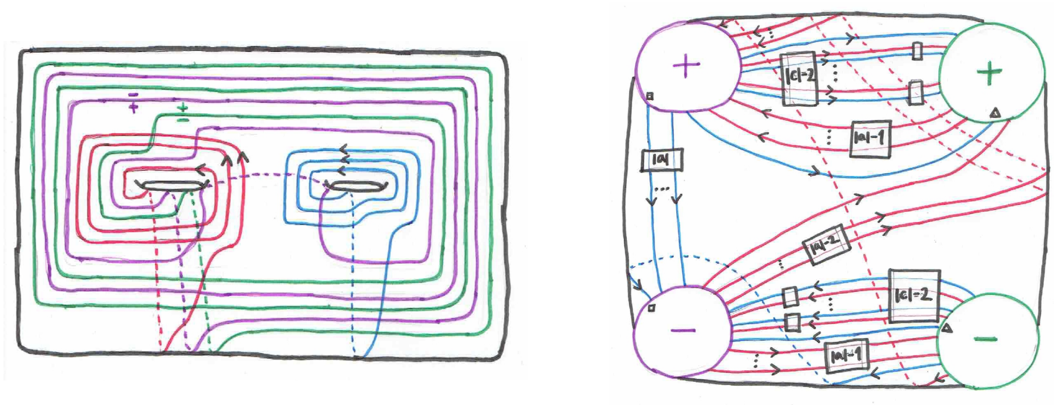


Figure 26: diagram for (2)

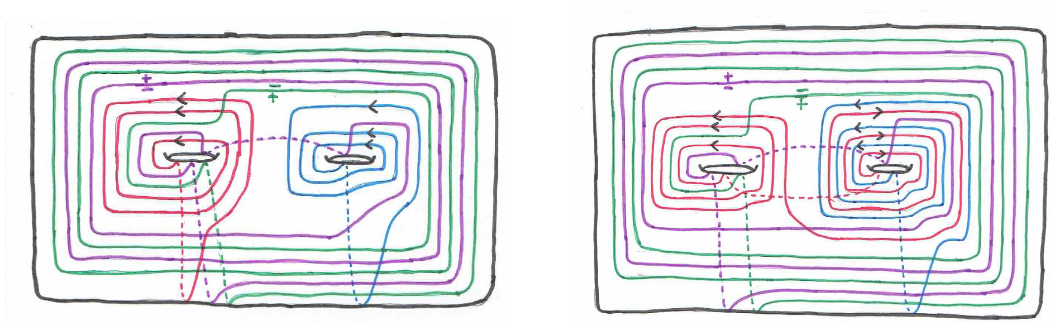

Figure 27: diagram for (2)

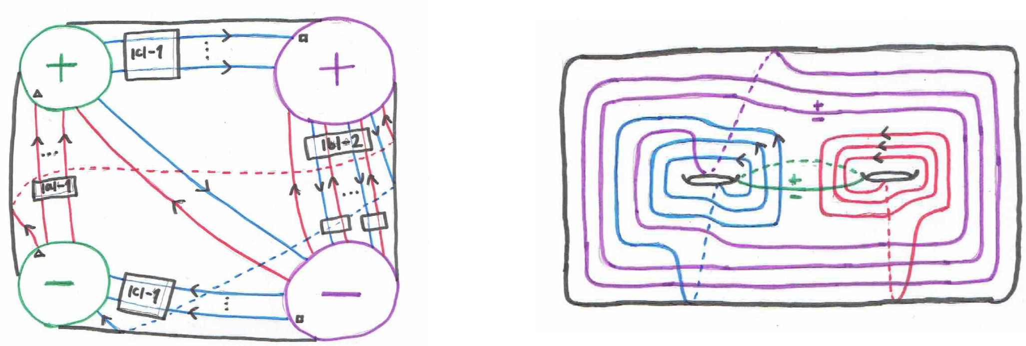


Figure 28: diagram for (2)

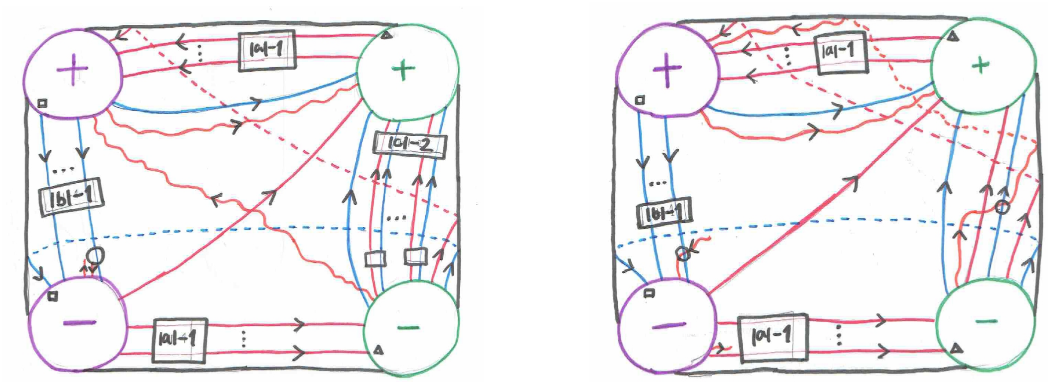


Figure 29: parts of $l$ for (2)

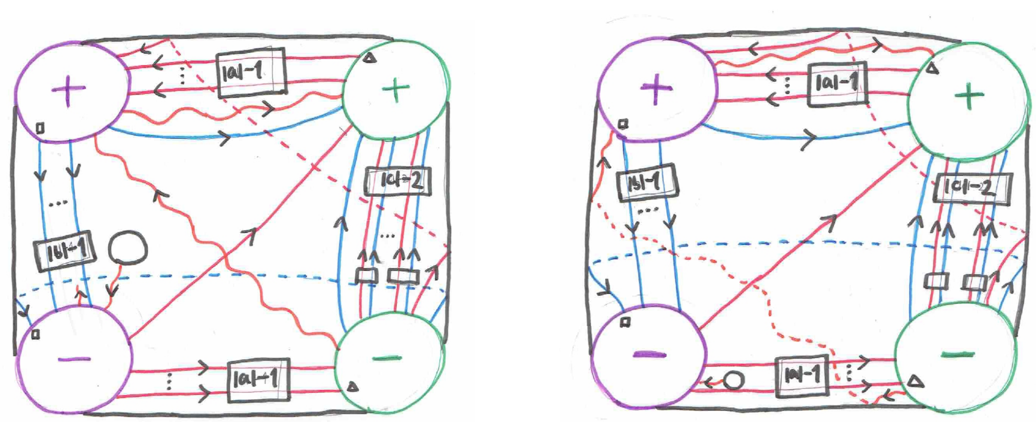


Figure 30: parts of $l$ for (2)

representing $yy^{-1}$ with respect to $\{x_L, y\}$, its corresponding arc $\alpha$ is disjoint from $D'$ and $E'$ by Fact 1.2. We assume that $\alpha$ starts from $\partial E^+$. If $\alpha$ ends on $\partial D_L^-$, $l$ is reduced with respect to $\{x', y'\}$ by Fact 1.2, and then $\{x_L, y\}$. If $\alpha$ ends on $\partial D_L^+$, we change the inner cut system as in Figure 34. In this diagram we can see that $l$ is reduced with respect to this new cut system, or that $l$ is reduced with respect to $\{D', E'\}$, or we have a right standard diagram whose inner cut system intersects $l$ less times than that of the left standard diagram where we start, or $l$ is reduced with respect to an (outer) cut system as in Figure 36. First two cases imply that $l$ is reduced with respect to $\{D_L, E\}$. Third case takes us to the paragraph above. In the last case, $l$ is reduced with respect to this outer cut system. Note that this diagram looks like the left-standard diagram of (2). Then in finitely many times, we can put $l$ on some standard diagram. As mentioned before, if $l$ is reduced with respect to $\{x_R, y\}$, then it is also reduced with respect to $\{x_L, y\}$.
Next we search such $l$ on the left-standard diagram. By reversing the orientation if necessary, there is a subarc $l'$ of $l$ starting from $\partial E^+$ and ending on $\partial D_L^+$. After $l'$, $l$ must start from $\partial D_L^-$ and end on $\partial E^+$. Before $l'$, $l$ must end on $\partial E^-$ and start from $\partial D_L^+$. They prevent $l$ from representing a reduced sequence of letters and a commutator. See Figure 37.

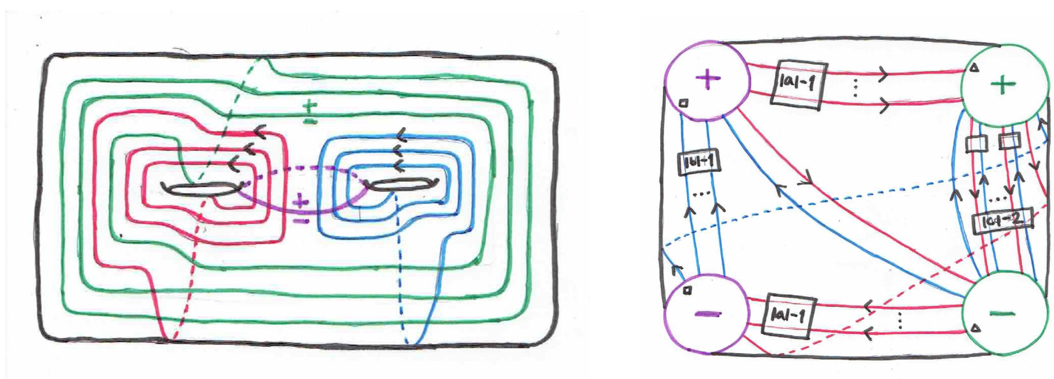


Figure 31: left-standard diagram for (3)

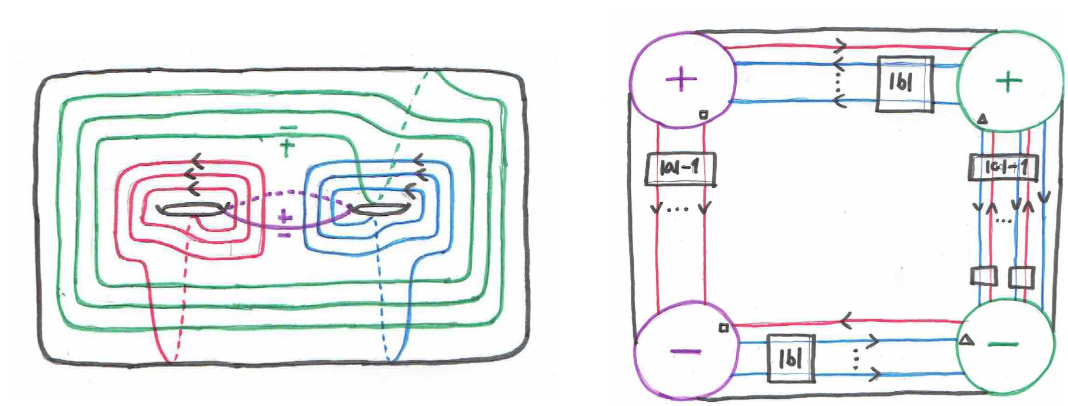


Figure 32: diagram for (3)

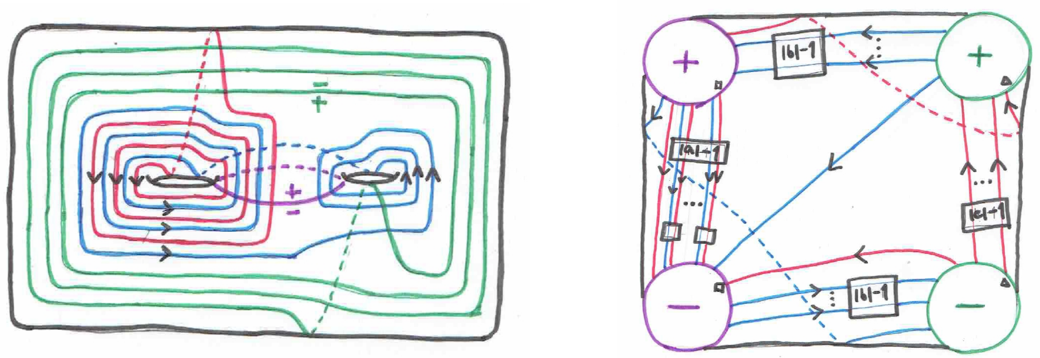


Figure 33: diagram for (3)

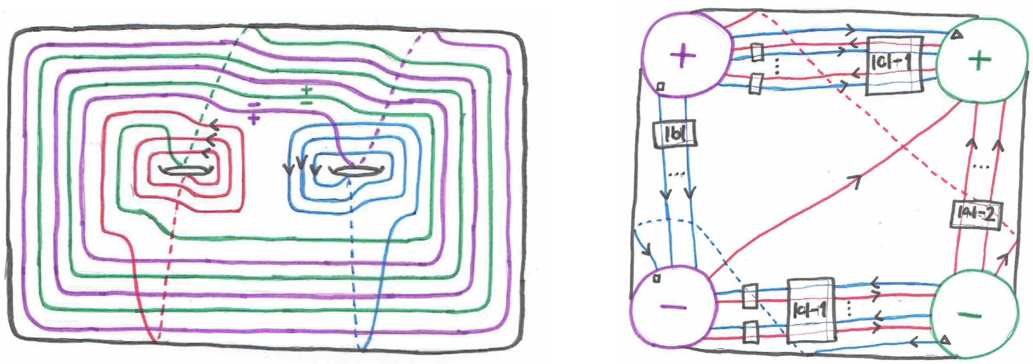


Figure 34: diagram for (3)

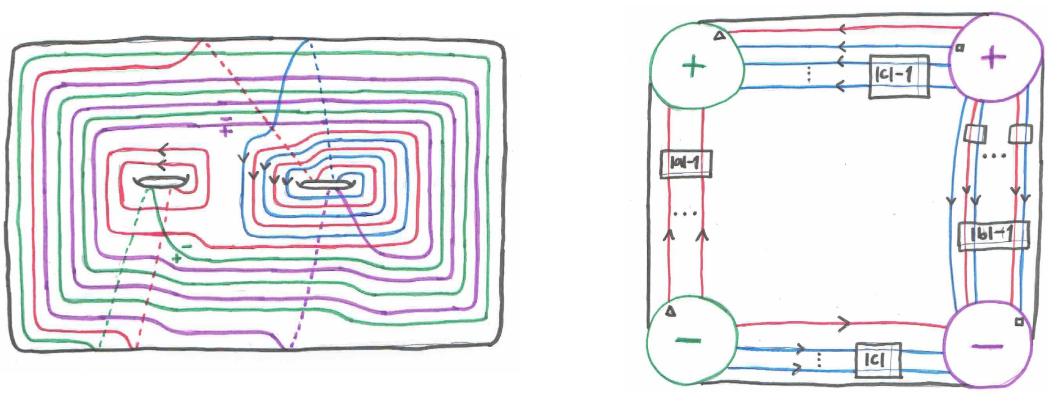


Figure 35: diagram for (3)

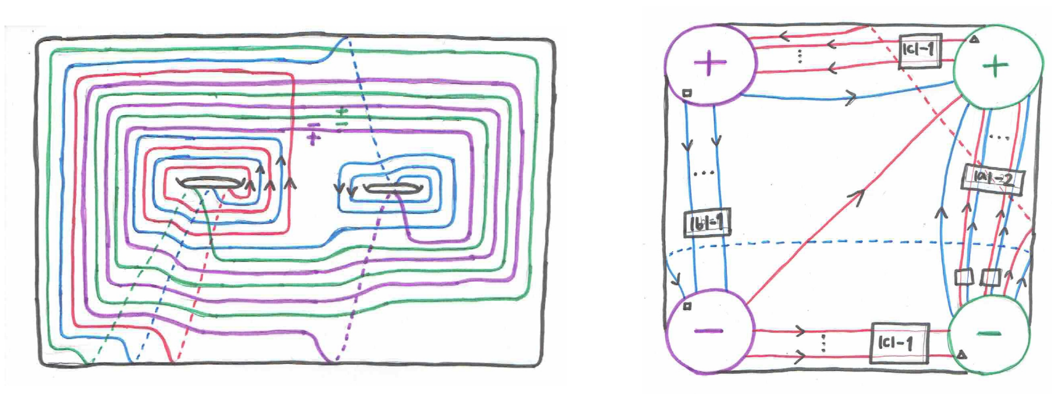


Figure 36: diagram for (3)

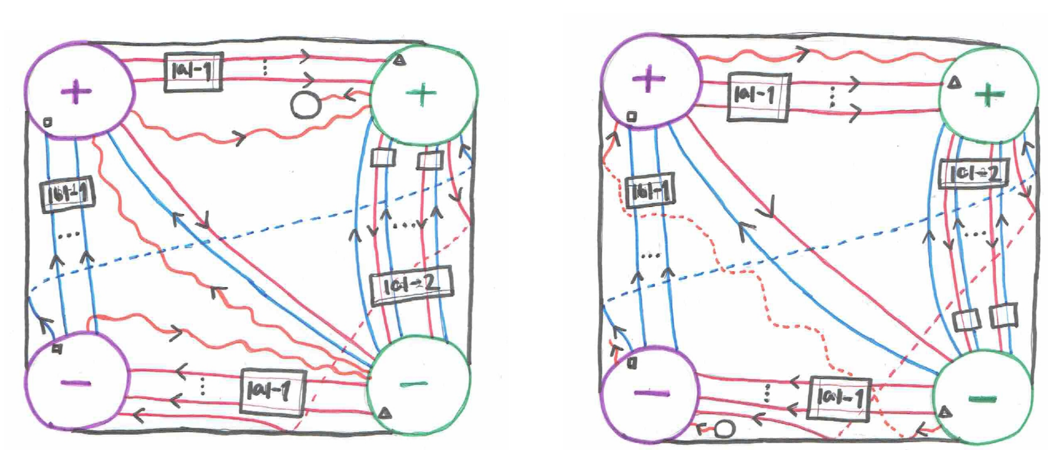


Figure 37: parts of $l$ for (3)

# 4 Examples of fibered knots in lens spaces

We use the notation of the continue fraction expansion as follows:

**Definition 4.1.** For integers $x_1, x_2, \ldots, x_n$ with $x_n \neq 0$, we define $[x_1, x_2, \ldots, x_n] = x_1 - \frac{1}{x_2 - \frac{1}{\cdots - \frac{1}{x_n}}}$.

It is known that the lens space $L(p, q)$ of type $(p, q)$ has a surgery presentation of an $n$-components chain link as in Figure 38 if $\frac{p}{q} = [x_1, x_2, \ldots, x_n]$ for some integers $x_1, x_2, \ldots, x_n$. Moreover by this presentation, we see that $L(p, q)$ has a fibered link whose fiber surface is homeomorphic to $\Sigma_{0,n+1}$. The surface is depicted in the top of Figure 41. Thus by the plumbings of Hopf annuli, we have $\mathrm{op}(L(p, q)) \leq n$ for such $L(p, q)$. However the existence of the universal upper bound for the openbook genus of every lens space is unknown.
In this section, we give some examples for fibered knots in lens spaces by using analogues of genus one fibered knots in lens spaces.

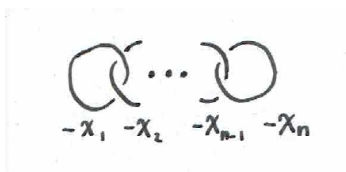


Figure 38: a surgery presentation of $L(p, q)$

## 4.1 Genus one fibered knots in lens spaces

As mentioned before, whether a given lens space has a genus one fibered knot is completely determined (Fact 3.2). By changing variables, we have the below, equivalent to Fact 3.2:

**Fact 4.1.** *A lens space has a genus one fibered knot if and only if it is homeomorphic to $L(2xy + x + y, 2y + 1)$ or $L(2xy + x + y + 1, 2y + 1)$ for some integers $x, y$.*

Since $\frac{2xy+x+y}{2y+1} = [x, -2, y]$ and $\frac{2xy+x+y+1}{2y+1} = [x, -2, -(y+1)]$, we can construct a genus one fibered knot in $L(2xy + x + y, 2y + 1)$ and $L(2xy + x + y + 1, 2y + 1)$ as in Figure 39. Note that a lens space corresponding to $[x, +2, y]$ is homeomorphic to a lens space corresponding to $[-x, -2, -y]$, and a lens space corresponding to $[x, 0, y]$ is homeomorphic to a lens space corresponding to $[0, -2, x + y]$. We give surgery presentations of these in Figure 39. We use them as building blocks for our examples.

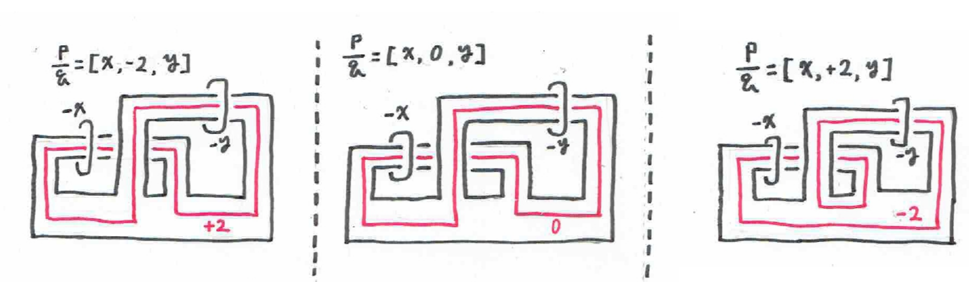


Figure 39: genus one fibered knots of $[x, -2, y]$, $[x, +2, y]$, $[x, 0, y]$.

## 4.2 Examples

**Proposition 4.1.** *Suppose a pair $(p, q)$ has a continue fraction expansion $\frac{p}{q} = [x_1, 2\epsilon_1, y_1, x_2, 2\epsilon_2, y_2, \ldots, x_n, 2\epsilon_n, y_n]$ for integers $x_i, y_i$, and $\epsilon_i \in \{0, \pm 1\}$ $(1 \leq i \leq n)$. Then $L(p, q)$ has a genus $n$ fibered knot.*

*Proof.* Note that
$[x_1, 2\epsilon_1, y_1, x_2, 2\epsilon_2, y_2, \ldots, x_n, 2\epsilon_n, y_n]$

$$= [x_1, 2\epsilon_1, y_1 - 1, -1, x_2 - 1, 2\epsilon_2, y_2 - 1, -1, \ldots, -1, x_{n-1} - 1, 2\epsilon_{n-1}, y_{n-1} - 1, -1, x_n - 1, 2\epsilon_n, y_n].$$

Put $\frac{p_1}{q_1} = [x_1, 2\epsilon_1, y_1 - 1]$, $\frac{p_k}{q_k} = [x_k - 1, 2\epsilon_k, y_k - 1]$ for $1 < k < n$, $\frac{p_n}{q_n} = [x_n - 1, 2\epsilon_n, y_n]$. By the observation in Section 4.1, $L(p_i, q_i)$ has a genus one fibered knot for all $1 \leq i \leq n$, each of which is as in Figure 39. By the plumbings (using inessential arcs), we construct a genus $n$ fibered knot in $L = L(p_1, q_1) \# L(p_2, q_2) \# L(p_n, q_n)$ as in the top of Figure 40. Moreover we put $(-1)$-framed knots $l_1, \ldots, l_{n-1}$ on the fiber surface as in the bottom of Figure 40. Note that since the canonical framings of $l_i$'s are identical with the surface framings, the manifold $L'$ obtained by the surgery of $L$ along $l_i's$ also has a genus $n$ fibered knot by Fact 3.1. Note also that since the surgery presentation of $L'$ on $S^3$ is a chain link with framing $-x_1, -2\epsilon_1, -(y_1 - 1), -(-1), \ldots, -(x_n - 1), -2\epsilon_n, -y_n$, $L'$ is homeomorphic to $L(p, q)$. This completes the proof. $\square$

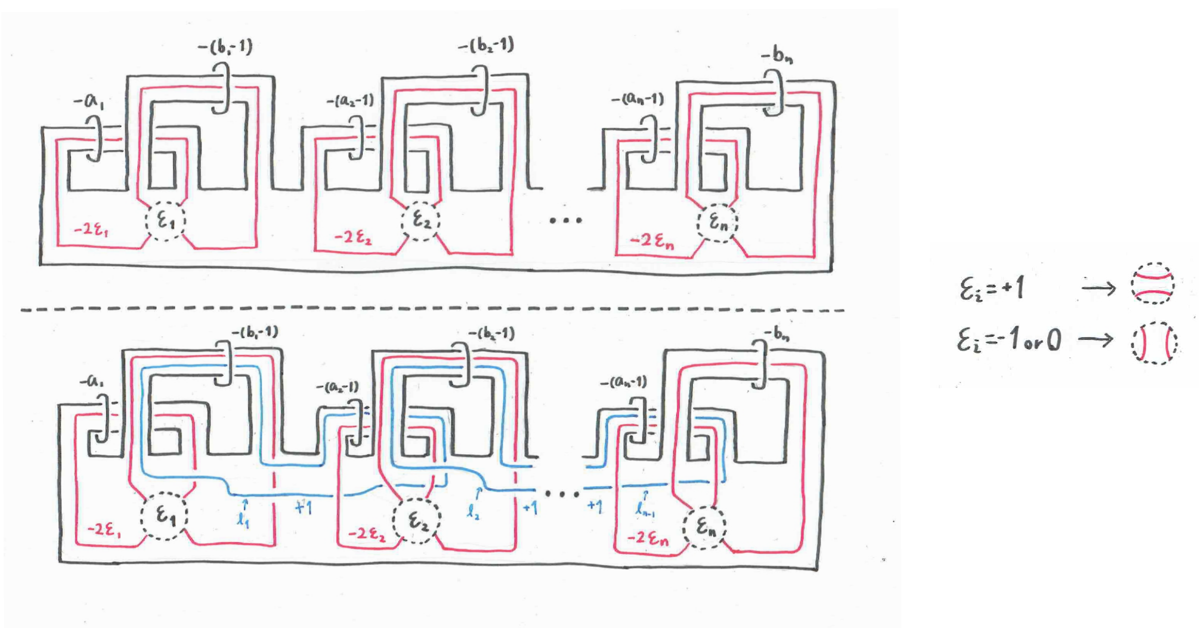


Figure 40: top: a surgery presentation of $L$, bottom: a surgery presentation of $L(p, q)$

# 5 A minimal genus relative trisection corresponding to a chain link

For a smooth, connected, orientable and compact 4-manifold $X$ and an openbook decomposition of its boundary $\partial X$, which is equivalent to a fibered link of $\partial X$ with its fiber surface, we can construct a *relative trisection* corresponding to these. For a chain link in $S^3$, we can associate a compact 4-manifold and an openbook decomposition of its boundary. In this section, we give a minimal genus relative trisection corresponding to these.

## 5.1 Relative trisection

We review the definition of relative trisections [8] (we consider only the case of 4-manifolds with connected boundaries).

**Definition 5.1.** A $(g, k_1, k_2, k_3; p, b)$-*relative trisection diagram* for $b \geq 1$, $p \leq g$, $2p + b - 1 \leq k_1, k_2, k_3 \leq g+p+b-1$ consists of $(S, \alpha, \beta, \gamma)$, where $S$ is homeomorphic to $\Sigma_{g,b}$ and $\alpha, \beta, \gamma$ are $(g-p)$ simple closed curves on $S$ such that:

- $(S, \alpha, \beta)$ is a sutured decomposition of genus $g$ of $(\Sigma_{p,b} \times [0,1])\#(\#^{k_1-(2p+b-1)}S^2 \times S^1)$ ( defined below).
- $(S, \beta, \gamma)$ is a sutured decomposition of genus $g$ of $(\Sigma_{p,b} \times [0,1])\#(\#^{k_2-(2p+b-1)}S^2 \times S^1)$.
- $(S, \gamma, \alpha)$ is a sutured decomposition of genus $g$ of $(\Sigma_{p,b} \times [0,1])\#(\#^{k_3-(2p+b-1)}S^2 \times S^1)$.

We call $g$ the genus of the relative trisection.

A *sutured decomposition of* $(\Sigma_{p,b} \times [0,1])\#(\#^{k-(2p+b-1)}S^2 \times S^1)$ is obtained from the product decomposition $(\Sigma_{p,b} \times [0, \frac{1}{2}]) \cup_{\Sigma_{p,b}\times\{\frac{1}{2}\}} (\Sigma_{p,b} \times [\frac{1}{2}, 1])$ of $\Sigma_{p,b} \times [0,1]$ by the (inner) connected sums of $S^2 \times S^1$'s and stabilizations. Note that for such decompositions a product cut system for $\Sigma_{p,b} \times [0,1]$ survives in this decomposition, and it cuts $(\Sigma_{p,b} \times [0,1])\#(\#^{k-(2p+b-1)}S^2 \times S^1)$ into a once punctured $(\#^{k-(2p+b-1)}S^2 \times S^1)$. By a given relative trisection diagram, we can construct a 4-manifold with boundary and an openbook decomposition of its boundary: Let $H_\alpha$,$H_\beta$,$H_\gamma$ be 3-dimensional handlebodies obtained by attaching 3-dimensional 2-handles along $\alpha, \beta, \gamma$ to $S$ respectively. Let $Y_1'$,$Y_2'$,$Y_3'$ be $H_\alpha \cup_S H_\beta$, $H_\beta \cup_S H_\gamma$,$H_\gamma \cup_S H_\alpha$ respectively. $Y_i'$ is $(\Sigma_{p,b} \times [0,1])\#(\#^{k_i-(2p+b-1)}S^2 \times S^1)$. Let $Y_1''$,$Y_2''$,$Y_3''$ be copies of $\Sigma_{p,b} \times [0,1]$ with a binding $\partial\Sigma_{p,b} \times \{\frac{1}{2}\}$. Let $Y_i$ be a result of pasting $Y_i'$ and $Y_i''$ along boundaries so that the upper (lower) boundary of $Y_i'$ is pasted to the upper (lower) boundary of $Y_i''$ by the "identical" way, which means the way so that the boundary of the product cut system of $Y_i'$ as mentioned before also bounds a product cut system of $Y_i''$. Then $Y$ is homeomorphic to $\#^{k_i}S^2 \times S^1$. It is known that smoothly there is the unique way for $\#^{k_i}S^2 \times S^1$ bounding a 4-dimensional 1-handlebody of genus $k_i$. Let $Z_i$ be such a handlebody. We call $Y_i'$, $Y_i''$ are the inner, the outer boundaries of $Z_i$. Let $X$ be a compact 4-manifold which is the result of identifying $H_\alpha, H_\beta, H_\gamma$ from $Z_1, Z_2, Z_3$. The boundary $\partial X$ consists of $Y_i''$'s, which are the product of the surface. Hence we have an openbook decomposition of $\partial X$ whose page is homeomorphic to $\Sigma_{p,b}$.
Note that for a relative trisection diagram representing a given a 4-manifold $X$ (and an openbook decomposition of $\partial X$), a new relative trisection diagram obtained from the old one by changing $\alpha$(,$\beta$,$\gamma$)-curves by the band-sums according to $\alpha$(,$\beta$,$\gamma$)-curves on the old one also represents $X$ and the openbook decomposition of $\partial X$.

There is a way to compute some of homology groups (we consider the case of integer coefficients) of the 4-manifold represented by a relative trisection diagram using the computation for non-relative trisections [6] and the above construction:

**Proposition 5.1.** [6] *Let $(S, \alpha, \beta, \gamma)$ be a relative trisection diagram for $X$. Let $L_\alpha$,$L_\beta$,$L_\gamma$ be submodules of $H_1(S)$ generated by the homology classes of curves in $\alpha$,$\beta$,$\gamma$-curves respectively. Then*

- $H_1(X) = H_1(S)/(L_\alpha + L_\beta + L_\gamma)$
- $H_2(X) = \{L_\gamma \cap (L_\beta + L_\alpha)\}/\{(L_\gamma \cap L_\beta) + (L_\gamma \cap L_\alpha)\}$

## 5.2 Relative trisections and Kirby diagrams

For a Kirby diagram, we can construct a compact 4-manifold $X$ by attaching 4-dimensional 2-handles along links in the diagram to the boundary of the 4-dimensional ball $B$. Additionally, for an openbook decomposition of $\partial X$, we can put its page $F$ on the $\partial B$ so that it is disjoint from surgery links. We put this page in the diagram. There is a method to obtain a relative trisection for $X$ and the openbook decomposition of $\partial X$ using the diagram due to [3]. We briefly review this in this section:

Regard $F$ as a union of a disk and 2-dimensional 1-handles. Introduce 4-dimensional $1,2$-canceling pairs and isotope $F$ so that every 1-handle of $F$ passes through some 4-dimensional 1-handle. Project surgery links onto one side of $F$ with finitely many double points with over/under information. Stabilize $F$ on the side ( and attach 3-dimensional 1-handles to $F \times [0,1]$ on $F \times \{1\}$) to get a surface $F'$ so that the surgery link is put on $F'$ with no double points and the framing number for surgery of every component is the same as the surface framing coming from $F'$. $F \times [0, \frac{1}{2}]$ is the outer boundary of $Z_1$, $(F \times [\frac{1}{2}, 1]) \cup$(1-handles) is $H_\beta$, the complement of these in the connected sums of $S^2 \times S^1$'s, which is the diagram with 4-dimensional 1-handles is $H_\alpha$. $F' \times [0,1] \cup$(3-dimensional 2-handles along surgery link) is $H_\gamma$. For these we can find $\alpha$,$\beta$,$\gamma$-curves.

## 5.3 Construction of a relative trisection diagram

Assume that a coprime pair $(p, q)$ satisfies $\frac{p}{q} = [x_1, \dots, x_n]$ for integers $x_1, \dots, x_n$. Then $L(p, q)$ has a surgery presentation as in Figure 38. Let $W$ be a compact 4-manifold obtained by attaching 4-dimensional 2-handles along the surgery link in Figure 38 to the boundary of the 4-dimensional ball $B$. Note that $\partial W$ is $L(p, q)$ and it has a fibered link whose fiber is homeomorphic to $\Sigma_{0,n+1}$, which is depicted in the top of Figure 41. We construct a $(2n, n, n, n; 0, n+1)$-relative trisection for $W$ and the openbook decomposition of $\partial W$ by following the method of Section 5.2. In Figure 41, the top is a page with the surgery link, the middle is the result of introducing 1,2-canceling pairs, and the bottom is the result of isotoping the link on the front side of the page. Next we do handle slides of the $(-x_i)$-framed knot along the 0-framed knot which pass through the $i$-th 1-handle $h_i$ for every $1 \leq i \leq n$. Stabilize the page so that the link is on the surface and has the surface framing which is the same as the surgery framing. Then we have a relative trisection diagram. Figure 42 is for $n = 1$, and Figure 43 is for $n \geq 2$. This is a $(2n, n, n, n; 0, n+1)$-relative trisection diagram.

## 5.4 Minimal relative trisection

In this section, we will show that every $(g, k_1, k_2, k_3; p, b)$-relative trisection $(S, \alpha, \beta, \gamma)$ for $W$ and an openbook decomposition of $\partial W$ mentioned in Section 5.2 satisfies $g \geq 2n$. This implies that a relative trisection constructed in Section 5.2 is of the minimal genus.

At first, $p = 0$ and $b = n+1$ hold since the page is homeomorphic to $\Sigma_{0,n+1}$. Note that $\alpha$,$\beta$,$\gamma$-curves are $g$-components. We set $\alpha = \{a_1, \dots, a_g\}$, $\beta = \{b_1, \dots, b_g\}$, $\gamma = \{c_1, \dots, c_g\}$. We may make $\alpha$,$\beta$-curves standard as in the left of Figure 44 by the band-sums since $(S, \alpha, \beta)$ is a sutured decomposition of genus $g$ of $(\Sigma_{p,b} \times [0,1]) \#(\#^{k_1-(2p+b-1)} S^2 \times S^1)$. Also we take pairwise disjoint simple closed curves $\{x_1, \dots, x_g, y_1, \dots, y_g, z_1, \dots, z_n\}$, which form a basis of $H_1(S)$. Every element in $H_1(S)$ is represented as the linear combination of the basis in the unique way. We call the coefficient of $x_i$ for an element the $x_i$-component for the element, and so on.

We compare the homology groups computed by the Kirby diagram and by the relative trisection diagram. By the Kirby diagram, we see $H_1(W) = \{0\}$ and $H_2(W) = \mathbb{Z}^n$. Since $\{0\} = H_1(W) = H_1(S)/(L_\alpha + L_\beta + L_\gamma)$, and $a_i$ and $b_i$ have no $z_j$-component for all $i, j$, there exists a $\tilde{c}_k$, which is

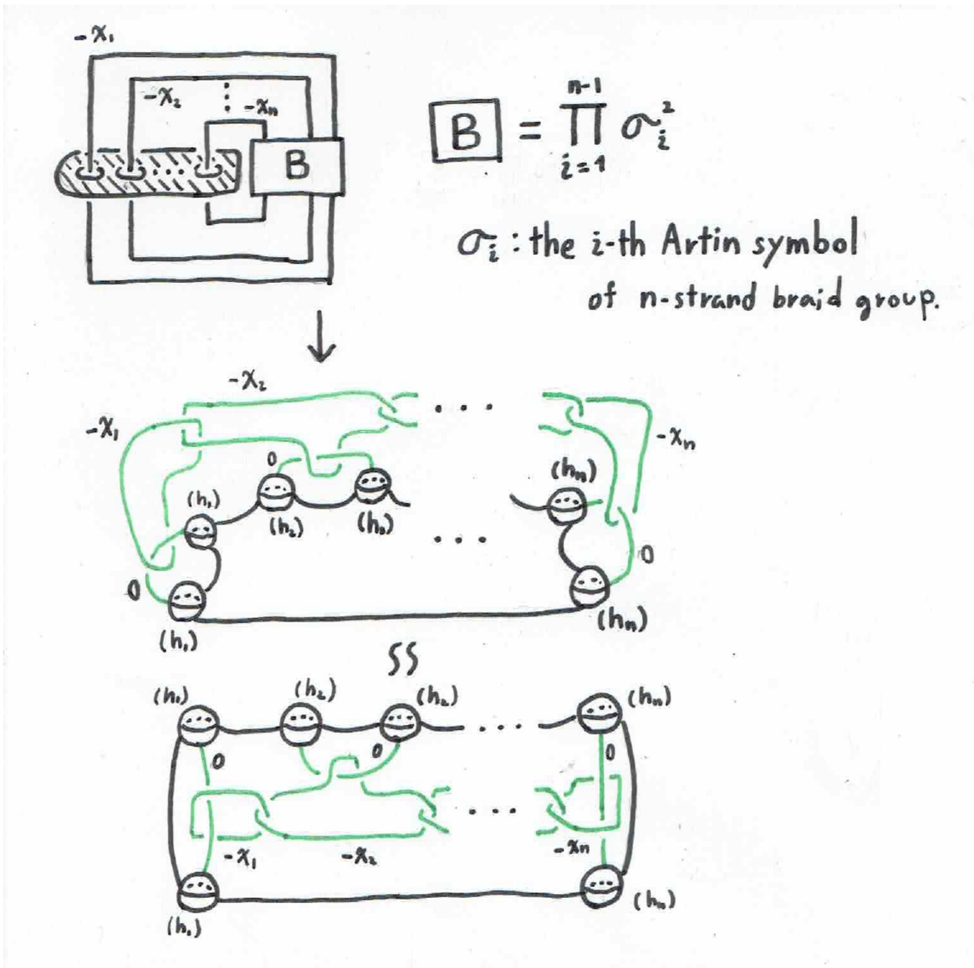


Figure 41: introduce canceling 1,2-canceling pairs

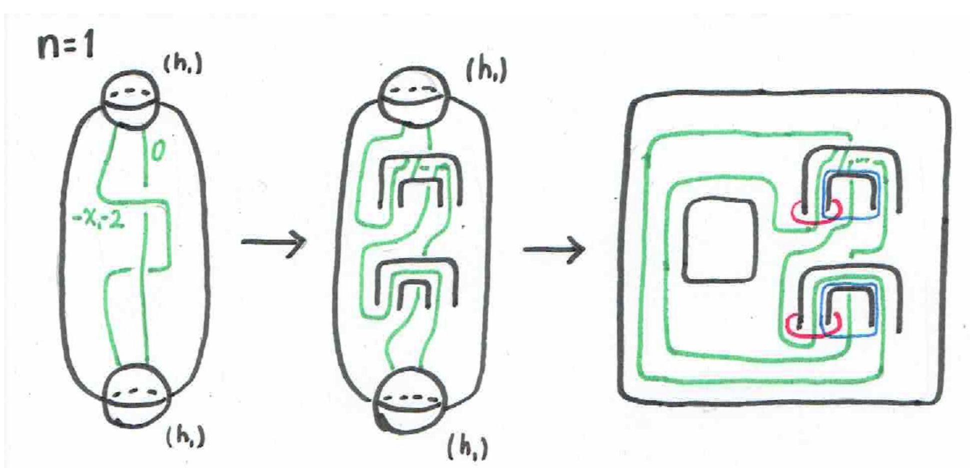


Figure 42: a relative trisection diagram for $n = 1$

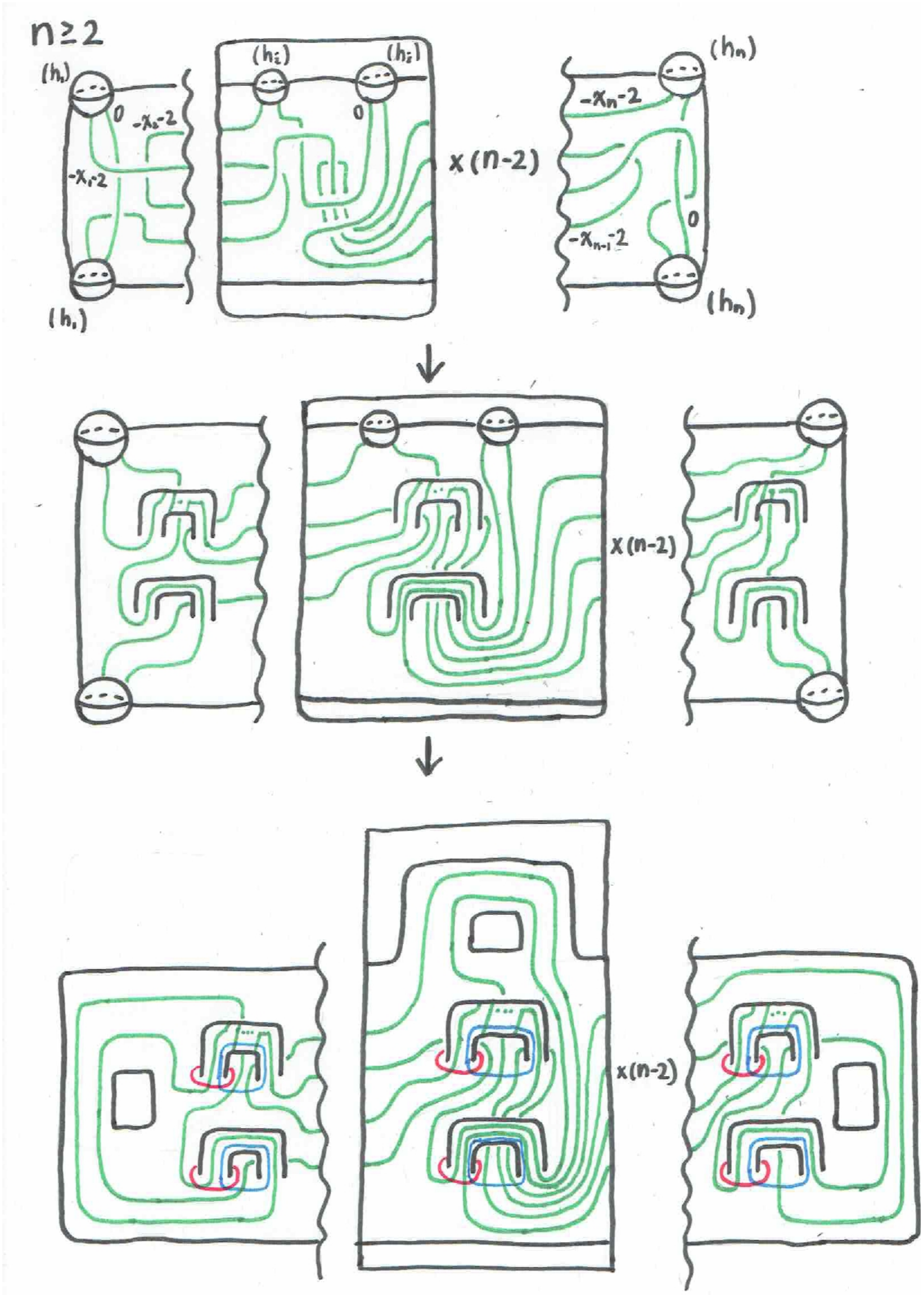


Figure 43: a relative trisection diagram for $n \geq 2$

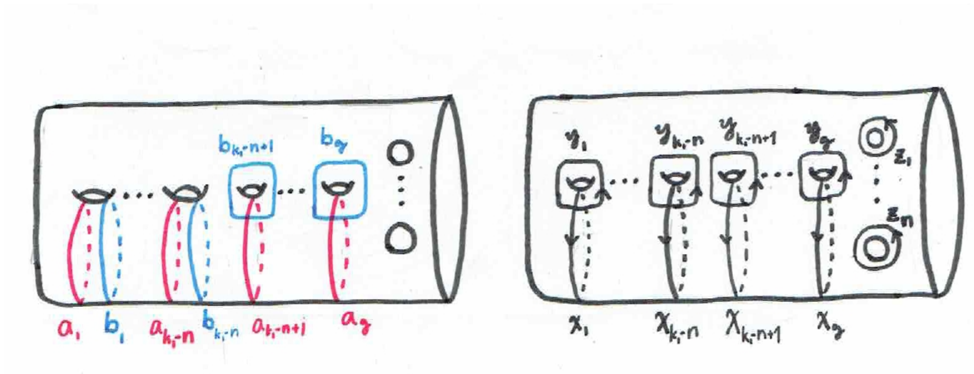


Figure 44: left: standard $\alpha$,$\beta$-curves   right: a basis of $H_1(S)$

a linear combination of $\gamma$-curves such that its $z_t$-component is zero if $k \neq t$ and one if $k = t$ for all $1 \leq k, t \leq n$. Note that $\{\tilde{c}_1, \ldots, \tilde{c}_n\}$ can be a subset of some basis of $L_\gamma$: Let $\tilde{c}_1 = \Sigma_i l_i c_i$. Since the $z_1$-component of $\tilde{c}_1$ is one, $\gcd(l_1, \ldots, l_g) = 1$. Thus $\tilde{c}_1$ is a member of some basis of $L_\gamma$. In this new basis, we can assume that all elements but $\tilde{c}_1$ has zero $z_1$-components by adding $\tilde{c}_1$'s if necessary. Then $\tilde{c}_2$ is represented as a linear combination of this basis without $\tilde{c}_1$'s, and so on. This implies $g \geq n$. This new basis is denoted by $\{\tilde{c}_1, \ldots, \tilde{c}_n, \tilde{c}_{n+1}, \ldots, \tilde{c}_g\}$. We choose $\tilde{c}_i$ for $n+1 \leq i \leq g$ so that it has zero $z_k$-component for all $1 \leq k \leq n$ by adding $\tilde{c}_k$'s. Then $L_\gamma \cap (L_\beta + L_\alpha)$ is a submodule of $\langle \tilde{c}_{n+1}, \ldots, \tilde{c}_g \rangle$. Since $\mathbb{Z}^n = H_2(W) = \{L_\gamma \cap (L_\beta + L_\alpha)\}/\{(L_\gamma \cap L_\beta) + (L_\gamma \cap L_\alpha)\}$, the rank of $\langle \tilde{c}_{n+1}, \ldots, \tilde{c}_g \rangle$ must be equal to or bigger than $n$. This implies $g \geq 2n$.

# Part II

## 6 A connected sums of lens spaces represented as a closure of a homology cobordism over $\Sigma_{0,n+1}$ or $\Sigma_{g,1}$

A fibered link $L$ with fiber surface $F$ in a 3-manifold $M$ requests that the sutured manifold $M \setminus F$ is a product sutured manifold. There is one generalization of fibered links, *homologically fibered links*. A homologically fibered link requests that the sutured manifold obtained by cutting the ambient manifold along a "fiber surface" is a *homology cobordism*.

**Definition 6.1.** (Section 2.4 of [7]) A *homology cobordism over $\Sigma_{g,n}$* ($n \geq 1$) is a triad $(X, \partial_+ X, \partial_- X)$, where $X$ is an oriented compact connected 3-manifold and $\partial_+ X \cup \partial_- X$ is a partition of $\partial X$, and $\partial_\pm X$ are homeomorphic to $\Sigma_{g,n}$ satisfying:

- $\partial_+ X \cup \partial_- X = \partial X$.
- $\partial_+ X \cap \partial_- X = \partial(\partial_+ X)$.
- The induced maps $(i_\pm)_* : H_*(\partial_\pm X) \to H_*(X)$ are isomorphisms, where $i_\pm : \partial_\pm X \to X$ are the inclusions.

Note that the third condition is equivalent to the condition that $X$ is connected and $i_\pm$ induce isomorphisms on $H_1(\cdot)$. Moreover by using the Poincaré duality, we see that inducing isomorphism of one of $(i_+)_*$ and $(i_-)_*$ is sufficient.

By pasting $\partial_+ X$ and $\partial_- X$ using any boundary-fixing homeomorphism, we get a closed 3-manifold $M$ and a surface $F$ in $M$, which is the image of $\partial_+ X$. In this situation, we say $M$ is representable as a closure of a homology cobordism over $\Sigma_{g,n}$.

**Definition 6.2.** [9] A link $L$ with Seifert surface $F$ in a 3-manifold $M$ is called a *homologically fibered link* if the sutured manifold $M \setminus F$ is a homology cobordism over (a surface which is homeomorphic to) $F$. The surface $F$ is called a *homologically fibered surface* of $L$.

By the above observation, for a link $L$ with its Seifert surface $F$ in a 3-manifold $M$ being homologically fibered link, it is enough to check that $H_1(M \setminus F) \cong H_1(F)$ and the push-up's (or push-down's) of simple closed curves on $F$ which represent a basis of $H_1(F)$ form a basis of $H_1(M \setminus F)$.

In this section, we give the conditions for a connected sums of lens spaces of having a homologically fibered link whose fiber is homeomorphic to $\Sigma_{0,n+1}$ or $\Sigma_{g,1}$ in terms of some algebraic equations.

Let $m$ be a positive integer. Let $(p_i, q_i)$ be a coprime integer pair for $1 \leq i \leq m$. Consider $M = \#_{i=1}^m L(p_i, q_i)$. Note that $M$ has a surgery presentation as in Figure 45. We call the $(-\frac{p_i}{q_i})$-framed unknot $U_i$.

We will show the below in Sections 6.2, 6.3. In Theorems 6.1, 6.2, $P$ denotes the $(m \times m)$- diagonal matrix whose $(i,i)$-entry is $p_i$ and $Q$ denotes the $(m \times m)$-diagonal matrix whose $(i,i)$-entry is $q_i$:

**Theorem 6.1.** *$M$ has a homologically fibered link whose homologically fibered surface is homeomorphic to $\Sigma_{0,n+1}$ if and only if there exist $(m \times n)$-matrix of integer coefficients $X$ and symmetric $(n \times n)$-matrix of integer coefficients $Y$ satisfying the following equation:*

$$\begin{vmatrix} P & -QX \\ {}^tX & Y \end{vmatrix} = \pm 1 \tag{1}$$

**Theorem 6.2.** *$M$ has a homologically fibered link whose homologically fibered surface is homeomorphic to $\Sigma_{g,1}$ if and only if there exist $(m \times 2g)$-matrix of integer coefficients $X$ and $(2g \times 2g)$-matrices of integer coefficients $Y$ which is symmetric, and $Z$ whose $(2i-1, 2i)$-entry is $1$ or $-1$ for $i = 1, \ldots, 2g$ and the other entries are $0$ satisfying the following equation:*

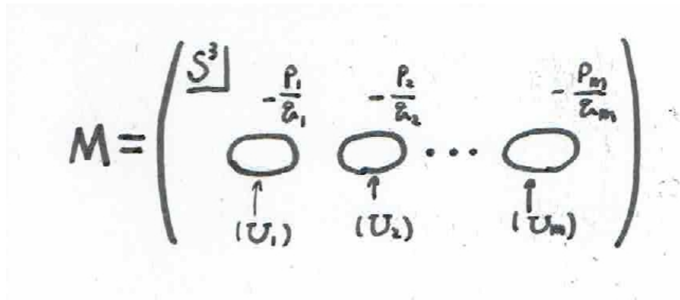


Figure 45: A surgery presentation of $M$

$$\begin{vmatrix} P & -QX \\ {}^tX & Y-Z \end{vmatrix} = \pm 1 \tag{2}$$

**Remark 6.1.** In [17], Nozaki showed that every lens space has a homologically fibere knot whose homologically fibered surface has genus one by solving the equation (2) for $m=1, g=1$. After that, in [19], the author showed that every lens space has a homologically fibered link whose homologically fiber surface is homeomorpic to $\Sigma_{0,3}$ by solving the equation (1) for $m=1, n=2$ in a way similar to Nozaki [17]. In Section 6.1, we will show this result admitting Theorem 6.1.

## 6.1 A solution for planar cases with $m=1, n=2$

We fix a lens space $L(p,q)$ with $p,q>0$ (every lens space has such a representation). We fix a positive integers $s$ and $r$ such that $ps-qr=1$.

By Theorem 6.1, that $L(p,q)$ is realizable as a closure of a homology cobordism over $\Sigma_{0,3}$ is equivalent to that there exist integers $a_1, a_2, l_{1,2}, t_1$, and $t_2$ satisfying $\begin{vmatrix} p & -qa_1 & -qa_2 \\ a_1 & t_1 & l_{1,2} \\ a_2 & l_{1,2} & t_2 \end{vmatrix} = \pm 1$. This equation is equivalent to $p(t_1t_2 - l_{1,2}^2) - q(2l_{1,2}a_1a_2 - t_2a_1^2 - t_1a_2^2) = \pm 1$, and also equivalent to that there exist integers $a_1, a_2, l_{1,2}, t_1, t_2, k$ and $\epsilon$, which is 1 or $-1$ satisfying

$$\begin{cases} t_2a_1^2 - 2l_{1,2}a_1a_2 + t_1a_2^2 = -\epsilon(r+kp) \\ \begin{vmatrix} t_2 & -l_{1,2} \\ -l_{1,2} & t_1 \end{vmatrix} = \epsilon(s+kq) \end{cases} \tag{3}$$

We use the following lemma (see Section 5.3 of [1] for example). Here the determinant of $ax^2 + 2bxy + cy^2$ is $ac - b^2$.

**Lemma 6.1.** *For $n, D \in \mathbb{Z}$, there is a binary quadratic form $f(x,y) \in \mathbb{Z}[x,y]$ with determinant $D$ such that $f(x,y) = n$ has a primitive solution if and only if the congruence equation $-z^2 \equiv D \bmod n$ has a solution.*

*Proof.* (if part) We set $z_0, C_0 \in \mathbb{Z}$ to be $D = -z_0^2 + C_0 n$. Then $f_0(x,y) = nx^2 + 2z_0xy + C_0y^2$, whose determinant is $D$ satisfies $f_0(1,0) = n$.
(only if part) Suppose $f(x,y) = Ax^2 + 2Bxy + Cy^2$ has a primitive solution $(x,y) = (x_0, y_0)$ for

$f(x,y)=n$. Fix integers $\bar{x},\bar{y}$ such that $x_0\bar{x}-y_0\bar{y}=1$. Then $g(x,y)=f\left(U\begin{pmatrix} x \\ y\end{pmatrix}\right)=f(x_0,y_0)x^2+2B'xy+C'y^2$, where $U=\begin{pmatrix} x_0 & \bar{y} \\ y_0 & \bar{x}\end{pmatrix}$, and $B',C'$ are some integers, has the same determinant $D$ as $f(x,y)$. Thus $D=C'n-B'^2$. This implies $-B'^2\equiv D \bmod n$. □

By the above lemma, the existence of a solution of (2) is equivalent to the existence of integers $k$ and $z'$, non-zero integer $w$ and $\epsilon'$, which is $+1$ or $-1$ satisfying the following equation:

$$\begin{cases} -z'^2 \equiv \epsilon'(s+kq) \mod & -\epsilon' \frac{r+kp}{w^2} \\ w^2 \mid r+kp \end{cases}$$

Note that since $(s+kq)$ and $(r+kp)$ are coprime, $z'^2$ and therefore $z'$ are prime to $(r+kp)$. This equation is equivalent to the following by setting $z=z'^{-1}$ and $\epsilon=\epsilon'$:

$$\begin{cases} p\equiv \epsilon z^2 \mod \frac{r+kp}{w^2} \\ w^2 \mid r+kp \end{cases} \tag{4}$$

Since $L(p,q)\cong L(p,r)$, we can replace $r$ in (4) with $q$.
We will show that for all $(p,q)$, there exist integers satisfying (4) by using the following two facts, which and whose use are fully explained in Section 3 of [17]:
Put $K_1=\mathbb{Q}(\zeta_p)$ and $K_2=\mathbb{Q}(\sqrt{-1})$, where $\zeta_p=\exp(\frac{2\pi}{p}\sqrt{-1})$.

**Fact 6.1.** *(A special case of the Chebotarev density theorem, for example Theorem* 10 *of* [13] *) Suppose there exists $\sigma\in Gal(K_1K_2/\mathbb{Q})$ satisfying:*

- *$\sigma|_{K_1}$ is $[\zeta_p\longmapsto \zeta_p^m]$, where $m$ is prime to $p$.*
- *$\sigma|_{K_2}$ is $[\sqrt{-1}\longmapsto -\sqrt{-1}]$.*

*Then there exist infinitely many integers $k$ satisfying:*

- *$m+kp$ is prime.*
- *$m+kp\equiv -1$ mod 4.*

**Fact 6.2.** *(See Corollary* 4.5.4 *of* [20] *for example) $\sqrt{-1}\in\mathbb{Q}(\zeta_p)$ if and only if $4\mid p$.*

From these we have the following:

**Lemma 6.2.** *There exists an integer $k$ satisfying at least one of the following:*

- *$q+kp$ is prime and $q+kp\equiv -1$ mod 4.*
- *$r+kp$ is prime and $r+kp\equiv -1$ mod 4.*

*Proof.* (i) the case when $4\nmid p$
In this case, $K_1\cap K_2=\mathbb{Q}$ by Fact 6.2. Thus in $Gal(K_1K_2/\mathbb{Q})$, $[\zeta_p\longmapsto\zeta_p^q]$ and $[\sqrt{-1}\longmapsto -\sqrt{-1}]$ are coexistable. By Fact 6.1, we have an integer $k$ such that $q+kp$ is prime and $q+kp\equiv -1$ mod 4.
(ii) the case when $4\mid p$.
We write $p=4p'$. In this case, $K_1K_2=K_1$ by Fact 6.2. Note that $\zeta_p^{p'}=\sqrt{-1}$. If $\zeta_p$ is mapped to

$\zeta_p^m$ ($m$ is $q$ or $r$), then $\sqrt{-1}$ is mapped to $\zeta_p^{mp'}$, this is $\sqrt{-1}$ when $mp' \equiv p' \bmod p$ and $-\sqrt{-1}$ when $mp' \equiv -p' \bmod p$. Note that $mp' \equiv \pm p' \bmod p$ is equivalent to $m \equiv \pm 1 \bmod 4$. Since $ps - qr = 1$, one of $q$ and $r$ is congruent to $-1$ modulo 4. Therefore, one of $[\zeta_p \longmapsto \zeta_p^q]$ and $[\zeta_p \longmapsto \zeta_p^r]$ maps $[\sqrt{-1} \longmapsto -\sqrt{-1}]$. By Fact 6.1, there exists an integer $k$ such that $q + kp$ is prime and $q + kp \equiv -1 \bmod 4$, or $r + kp$ is prime and $r + kp \equiv -1 \bmod 4$. □

For $L(p, q)$, there exists an integer $q'$, which is $q + kp$ or $r + kp$ such that $L(p, q) \cong L(p, q')$, $q'$ is prime, and $q' \equiv -1 \bmod 4$ by Lemma 6.2. By using the Legendre symbol and Euler's criterion,
$\left(\frac{-p}{q'}\right) \equiv \left(\frac{-1}{q'}\right)\left(\frac{p}{q'}\right) \equiv (-1)^{\frac{q'-1}{2}}\left(\frac{p}{q'}\right) \equiv -\left(\frac{p}{q'}\right) \bmod q'$
This implies there exists $\epsilon \in \{\pm 1\}$ such that $\left(\frac{\epsilon p}{q'}\right) = 1$, i.e. $\epsilon p$ is a quadratic residue modulo $q'$. Therefore, the condition (3) is satisfied.

## 6.2 Planar case

In this section, we consider whether $M$ has a homologically fibered link whose homologically fibered surface is homeomorphic to $\Sigma_{0,n+1}$.
Suppose $M$ has such a link $L$ and its homologically fibered surface $F$. Put $F$ on the surgery diagram in Figure 38 so that it is disjoint from $U_i$'s. On $F$, we fix a spine $K_1 \cup v_1 \cup \cdots \cup v_{n-1} \cup K_n$ of $F$ as in Figure 46. These $K_i$'s represent a bisis of $H_1(F)$. These $K_i$ and $v_j$ have the framing coming from $F$. Note that $v_j$ may have a half-integer framing. Conversely, for given framed (i.e. twisted) knots $K_i$'s and half framed (i.e. half integer twisted) bands $v_j$'s connecting each other like the spine as in Figure 46, we can construct an embedded $\Sigma_{0,n+1}$ in $M$. Therefore we regard the surface $F$ as a set of framed knots $K_i$'s and half framed bands $v_j$'s connecting as in Figure 46. Note that parallel copy of $K_i$ with respect to the framing corresponds to the push-up or the push-down of $K_i$ on $F$. The meridian curves of $U_k$, $K_i$ and $v_j$ are denoted by $m_{U_k}$, $m_{K_i}$ and $m_{v_j}$ respectively. Note that $H_1(M \setminus F) \cong H_1(M \setminus (\cup_{i=1}^n K_i))$ and the push-up and the push-down of $K_i$ on $F$ represent the same element in $H_1(M \setminus F)$ since $m_{v_j}$ is null-homologus in $M \setminus F$. Thus by setting $x_{i,j} = lk(U_i, K_j)$ and $y_{i,j} = lk(K_i, K_j)$, we see $H_1(M \setminus F) \cong \langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle / (p_i m_{U_i} - q_i \Sigma_{k=1}^n x_{i,k} m_{K_k})_{i=1,\ldots,m}$. Let $A_i \in \langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle$ be $p_i m_{U_i} - q_i \Sigma_{k=1}^n x_{i,k} m_{K_k}$.

**Claim 6.1.** *$\{A_1, \ldots, A_m\}$ can be a subset of a basis of $\langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle$.*

*Proof.* Let $\mathbb{B}$ be a basis $\{m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n}\}$. Represent $A_1$ in $\mathbb{B}$. Let $d_1$ be the positive greatest common divisor of the coefficients of $A_1$. Then, $\bar{A}_1 = \frac{A_1}{d_1}$ can be a member of a basis. Let $m_1 = \bar{A}_1$ and $\mathbb{B}^1$ be this new basis, which contains $m_1$. Represent $A_2$ in $\mathbb{B}^1$. Let $A_2'$ be a result of deleting the $m_1$-component from $A_2$. Let $d_2$ be a positive the greatest common divisor of the coefficients of $A_2'$. Then $\bar{A}_2 = \frac{A_2'}{d_2}$ can be a member of a basis of the module generated by $\mathbb{B}^1 \setminus \{m_1\}$. Let $m_2 = \bar{A}_2$ and $\mathbb{B}^2$ be this new basis, which contains $m_2$. Repeat this procedure. As a result, we get a basis of $\langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle$, $\{m_1, \ldots, m_{m-1}\} \cup \mathbb{B}^m$, where $\mathbb{B}^m$ contains $m_m$ and has rank $n+1$. Note that $\bar{A}_i = m_i +$ (a linear combination of $m_j$ for $j < i$). This implies $\langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle / (A_i)_{i=1,\ldots,m}$, which is isomorphic to $H_1(M \setminus F)$, has a torsion subgroup of order $d_1 \cdots d_m$. Therefore $d_i = 1$ for all $i = 1, \ldots, m$ since $F$ is a homologically fibered surface. This implies $A_i = \bar{A}_i = m_i +$ (a linear combination of $m_j$ for $j < i$). Hence $\{A_1, \ldots, A_m\}$ can be a subset of a basis of $\langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle$. □

By using some integer $t_i$, the framed $K_i$ represents $[K_i^{fr}] = t_i m_{K_i} + \Sigma_{k=1}^m x_{k,i} m_{U_k} + \Sigma_{j \neq i} y_{i,j} m_{K_j}$ in $\langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle$ for every $1 \leq i \leq n$. Since these form a basis of
$H_1(M \setminus F) = \langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle / (A_k)_{k=1,\ldots,m}$,
$\{A_1, \ldots, A_m, [K_1^{fr}], \ldots, [K_n^{fr}]\}$ is a basis of $\langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_n} \rangle$. Let $X$ be an $(m \times n)$-matrix whose $(k, j)$-entry is $x_{k,j}$ and $Y$ a symmetric $(n \times n)$-matrix whose $(i, i)$-entry is $t_i$, $(i, j)$-entry is $y_{i,j}$ for $i \neq j$. Then the equation (1) holds.

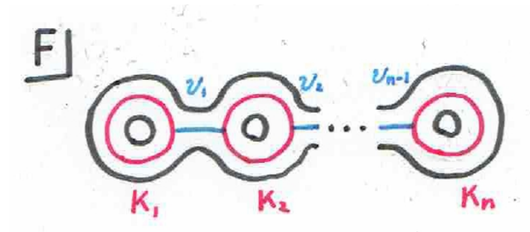


Figure 46: A spine of $F$

Conversely, suppose the solution of the equation (1) exists. Then we can construct framed knots $K_i$ for $1 \leq i \leq n$ such that the framing number of $K_i$ is $t_i$, $lk(U_k, K_j)$ is $(X)_{k,j}$, and $lk(K_i, K_j) = (Y)_{i,j}$. Take annuli which is a neighborhood of $K_i$'s in $M$ with respect to the framings and connect them by any bands (corresponding to $v_i's$). Then we get a surface homeomorphic to $\Sigma_{0,n+1}$. This is a homologically fibered surface by the computation.

## 6.3 Connected boundary case

In this section, we consider whether $M$ has a homologically fibered knot whose homologically fibered surface is homeomorphic to $\Sigma_{g,1}$.
Suppose $M$ has such a knot $K$ and its homologically fibered surface $F$. Put $F$ on the surgery diagram in Figure 38 so that it is disjoint from $U_i$'s. On $F$, we fix a spine $K'_1 \cup K'_2 \cup w_1 \cup \cdots \cup w_{g-1} \cup K'_{2g-1} \cup K_{2g}$ of $F$ as in the top of Figure 47. Note that these $K'_i$'s represent a bisis of $H_1(F)$. Since the meridian of $v'_i$ is null-homologus in $H_1(M \setminus F)$, we consider $g$-surfaces $\{F_1, \ldots, F_g\}$ of genus one to be in the bottom of Figure 47. $H_1(M \setminus (\cup_{i=1}^g F_i)) = \mathbb{Z}^{2g}$ and the push-ups of $K'_{2i-1}$ and $K'_{2i}$ (or push-downs of $K'_{2j-1}$ and $K'_{2j}$) form a basis of $H_1(M \setminus (\cup_{i=1}^g F_i))$. Let $z_i$ be a point $K'_{2i-1} \cap K'_{2i}$. We identify $M \setminus (\cup_{i=1}^g F_i)$ and $M \setminus (\cup_{i=1}^g F_i \times [0,1])$. Let $K_{2i-1}$ be $K'_{2i-1} \times \{0\}$ and $K_{2i}$ be $K'_{2i} \times \{1\}$ and $v_i$ be $\{z_i\} \times [0,1]$. Note that $K_{2i-1} \cup v_i \cup K_{2i}$ is a spine of $F_i \times [0,1]$. We can assign the suture from framed $K_{2i-1}$, framed $K_{2i}$ and framed $v_i$ as in Figure 48: Slide $K_{2i-1}$ along the band $v_i$. Then we get $K'_{2i-1} \times \{1\}$. The boundary of a regular neighborhood of $(K'_{2i-1} \times \{1\}) \cup (K'_{2i} \times \{1\})$ in $\partial(F_i \times [0,1])$ is the suture. Note that $K'_{2i-1} \times \{1\}$ represents $K_{2i-1} \pm (\textit{the meridian of } K_{2i})$ as an element of $H_1(M \setminus (\cup_{i=1}^g F_i))$, the sign depends on whether $v_i$ contains half-twists or not. Thus we regard $\cup_{i=1}^g F_i$ as framed $K_{2i-1}$, $K_{2i}$, $v_i$ connecting as in Figure 48 (, $v_i$ may be twisted). The meridian curves of $U_k$, $K_i$ and $v_j$ are denoted by $m_{U_k}$, $m_{K_i}$ and $m_{v_j}$ respectively. Since $m_{v_j}$ is null-homologus in $M \setminus (\cup_{i=1}^g F_i)$, by setting $x_{i,j} = lk(U_i, K_j)$ and $y_{i,j} = lk(K_i, K_j)$, we see $H_1(M \setminus (\cup_{i=1}^g F_i)) \cong \langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_{2g}} \rangle / \left(p_i m_{U_i} - q_i \Sigma_{k=1}^{2g} x_{i,k} m_{K_k}\right)_{i=1,\ldots,m}$.
Let $A_i \in \langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_{2g}} \rangle$ be $p_i m_{U_i} - q_i \Sigma_{k=1}^{2g} x_{i,k} m_{K_k}$. Similarly to Claim 6.1, $\{A_1, \ldots, A_m\}$ can be a subset of a basis of $\langle m_{U_1}, \ldots, m_{U_m}, m_{K_1}, \ldots, m_{K_{2g}} \rangle$. The framed $K_i$ represents $[K_i^{fr}] = t_i m_{K_i} + \Sigma_{j=1}^m x_{j,i} m_{U_j} + \Sigma_{k \neq i} y_{k,i} m_{K_k}$ in $H_1(M \setminus (\cup_{i=1}^g F_i))$ by introducing the integer $t_i$. As mentioned before, the push-ups of $K'_{2i-1} \times \{1\}$ and $K'_{2i} \times \{1\}$ on $F_i$ represent $[K_{2i-1}^{fr}] + \epsilon_i m_{K_{2i}}$ and $[K_{2i}^{fr}]$, $\epsilon_i \in \{\pm 1\}$. Set $X$ be a $(m \times 2g)$-matrix whose $(k,j)$-entry is $x_{k,j}$ and $Y$ a symmetric $(2g \times 2g)$-matrix whose $(i,i)$-entry is $t_i$, $(i,j)$-entry is $y_{i,j}$ for $i \neq j$ and $Z$ be a $(2g \times 2g)$-matrix whose $(2i-1, 2i)$-entry is $\epsilon_i$, the others 0. Then the equation (2) holds.

Conversely, suppose we have a solution of the equation (2). Then we can construct the framed knots and bands $\{K_1, \ldots, K_{2g}\}$ and $v_1, \ldots, v_g$ such that $lk(U_i, K_j) = (X)_{i,j}$, $lk(K_i, K_j) = (Y)_{i,j}$ and $v_k$ has half-twisted framing if and only if $(Z)_{2k-1,2k} = -1$. Moreover, we can construct $g$-surfaces $\{F_1, \ldots F_g\}$ of genus one by the operation as in Figure 48. Then by connecting these surfaces by arbitrary bands as in the top of Figure 47, we get a surface homeomorphic to $\Sigma_{g,1}$. This is a homologically fibered surface by the computation.

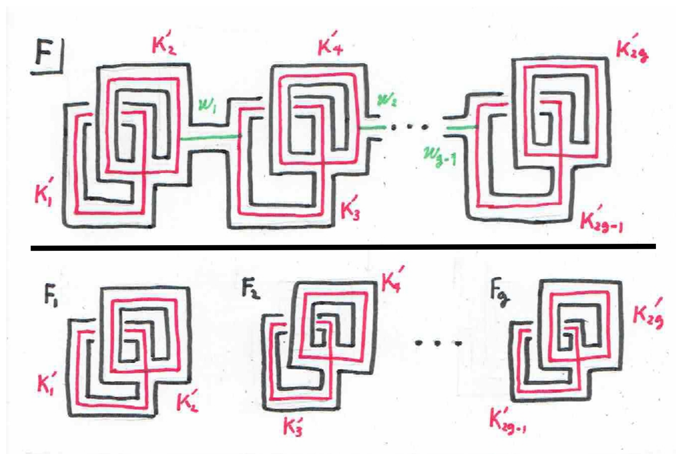


Figure 47: top: A spine of $F$ bottom: Spines of $F_i$'s

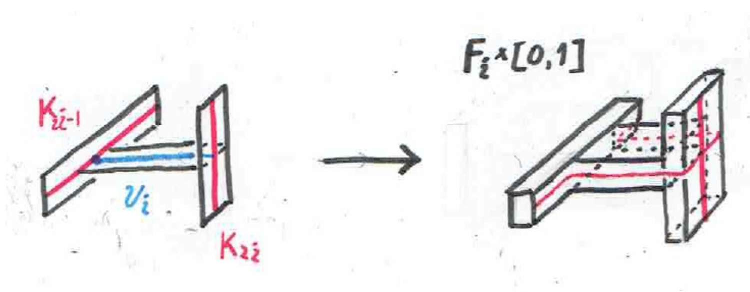


Figure 48: Construct a suture

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